HOMEWORK IX

LOCALLY COMPACT ABELIAN GROUPS

(1) Let L and L' be topological torsion groups. Find a counterexample to the following statement:

If L_p and L'_p (the p-primary components of L and L' respectively) are isomorphic for each prime p, then L is isomorphic to L'.

(2) Show that the p-primary component of the Pontryagin dual of a topological torsion group is the Pontryagin dual of its p-primary component.

(3) Let L be a locally compact abelian group which admits a compact open subgroup G. Set $A = L/G$. Let G_0 be the identity component of G. Let L' be the pre-image of A_{tor} in L modulo G_0. Show that L' is a topological torsion group which is independent of the choice of G. (L' is called the topological torsion part of L.)

(4) Let L be a locally compact abelian group which admits a compact open subgroup. Show that the topological torsion part of the Pontryagin dual of L is isomorphic to the Pontryagin dual of the topological torsion part of L.

(5) Let G be an abelian group which can be given a compact topological group structure. Show that $E(A, G) = 0$ for all torsion-free groups A (such a group is called a cotorsion group).

(6) Let $\mathbf{A}_f$ denote the restricted direct product over all primes $\prod'_p(\mathbf{Q}_p, \mathbf{Z}_p)$. Show that $\mathbf{A}_f$ is isomorphic to its Pontryagin dual. Thus also the group of adèles of $\mathbf{Q}$, $\mathbf{A} := \mathbf{R} \times \mathbf{A}_f$ is isomorphic to its Pontryagin dual.

(7) An embedding ι of $\mathbf{Q}$ in $\mathbf{A}$ is obtained by mapping each rational number x to the element of $\mathbf{A}$ whose component at p for every p is x and whose real component is also x. Show that the image of $\mathbf{Q}$ in $\mathbf{A}$ under ι is discrete.

(8) Let $\psi : \mathbf{R} \to T$ be the canonical quotient map, and for each prime p let $\psi_p : \mathbf{Q}_p/\mathbf{Z}_p \to T$ be the unique homomorphism which takes $p^{-n} + \mathbf{Z}_p$ to the image of $p^{-n} \in \mathbf{R}$ in T. For $x = (x_{\mathbf{R}}, (x_p)) \in \mathbf{A}$, define $\psi_x : \mathbf{Q} \to T$ by

$$\psi_x(y) = \psi(x_{\mathbf{R}} y) + \sum_p \psi_p(x_p y) \text{ for all } y \in \mathbf{Q}$$

(which is a finite sum since $x_p y \in \mathbf{Z}_p$ for all but finitely many p). Show that $x \mapsto \psi_x$ induces an isomorphism $\mathbf{A}/\mathbf{Q} \to \hat{\mathbf{Q}}$.

Date: due on Wednesday, October 14, 2009 (before class).