HOMEWORK V

LOCALLY COMPACT ABELIAN GROUPS

(1) Let H denote the Heisenberg group associated to a discrete abelian group A:
$$H = \{e^{2\pi it} T_x M_\chi : t \in T,\ x \in A,\ \chi \in \hat{A}\} \subset U(L^2(A))$$
Show that the centre of H ($\{z \in H : zh = hz \text{ for all } h \in H\}$) is the subgroup $\{e^{2\pi it} : t \in T\}$.

(2) Let A be a discrete abelian group. Show that the map $T \times A \times \hat{A} \to U(L^2(A))$ defined by
$$(t, x, \chi) \mapsto e^{2\pi it} T_x M_\chi$$
is a homeomorphism onto its image ($T \times A \times \hat{A}$ is given the product topology, and $U(L^2(A))$ is given the strong operator topology).

(3) Let $G = \hat{A}$, where A is a discrete abelian group. Let H_A be the Heisenberg group associated to A, as in class, and let H_G be the subgroup of $U(L^2(A))$ consisting of operators of the form $e^{2\pi it} T_x M_\chi$ where $T_x f(u) = f(u - x)$ for $x \in G$ and $M_\chi f(u) = \chi(u) f(u)$ for $\chi \in A$. Show that H_A and H_G are isomorphic as topological groups.

(4) Let A and G be as in the previous problem. Let $K = A \times G$. Show that $\nabla : K \to \hat{K}$ defined by
$$\nabla(x, \chi)(y, \lambda) = \chi(y) - \lambda(x) \text{ for } x, y \in A\ \chi, \lambda \in G$$
is an isomorphism of topological groups.

(5) Let A be a finite abelian group. Let $G = \hat{A}$ and $K = A \times G$. For each $k = (x, \lambda) \in K$, let W_k denote the Weyl operator
$$W_k = T_x M_\chi.$$
Show that $\mathrm{Trace}(W_k) = 0$ unless $k = 0$.

(6) With the notation of the previous problem, show that $\{W_k : k \in K\}$ is an orthonormal basis of $\mathrm{End}_{\mathbf{C}} L^2(A)$, where the Hermitian inner product on $\mathrm{End}_{\mathbf{C}} L^2(A)$ is given by
$$\langle A, B \rangle = \mathrm{Trace}(A^* B).$$

(7) Let N be the group of unipotent upper-triangular 3×3 matrices over a finite field. Fix any non-trivial character χ of its centre. Show that there is a unique (up to isomorphism) irreducible representation of N with central character χ.

Date: due on Tuesday, September 9, 2009 (before class).