HOMEWORK IV

LOCALLY COMPACT ABELIAN GROUPS

(1) If A is a discrete divisible torsion-free abelian group, show that A is isomorphic to $\mathbf{Q}^{\oplus n}$ for some cardinal n.
(2) If A is a discrete abelian group then A is isomorphic to $\mathbf{Q}^{\oplus n} \oplus R$, where n is a cardinal number, $D(R) \subset R_{\text{tor}}$ (here $D(R)$ is the maximal divisible subgroup of R and R_{tor} is its maximal torsion subgroup).
(3) Show that $\mathbf{Q}$ is not projective in the category of abelian groups.
(4) (Primary Decomposition for compact abelian groups) Let G be a compact abelian group. For each prime p let
$$G_p = \{x \in G : \lim_{n\to\infty} p^n x = 0\}.$$
Show that $G = \prod_p G_p$ (a product over all primes) if and only if $\hat{G}$ is torsion (such groups are called topological torsion compact abelian groups; if $G = G_p$ then G is called a topological p-torsion group).
(5) Let A be a discrete p-torsion group. Show that A is finitely generated if and only if $A[p] := \{x \in A : px = 0\}$ is finite. The dimension of $A[p]$ (which can, in general, be infinite) as a vector space over $\mathbf{Z}/p\mathbf{Z}$ is called the p-rank of A.
(6) Let G be a compact topological p-group. Show that the dimension of G/pG as a vector space over $\mathbf{Z}/p\mathbf{Z}$ equals the p-rank of $\hat{G}$.
(7) For what $t \in T$ is the sum
$$\sum_{n\in\mathbf{N}} \frac{1}{n} e^{2\pi i n t}$$
convergent? What values does this sum take?
(8) Show that every continuous function on a compact abelian group G is a uniform limit of trigonometric polynomials (trigonometric polynomials are finite linear combinations of functions of the form $e^{2\pi i\chi}$, where $\chi : G \to T$ is a continuous homomorphism).
(9) Let L be a locally compact abelian group. Let $\hat{L}_d$ denote the group of all continuous homomorphisms $L \to T$, endowed with the discrete topology. Let $\bar{L}$ be the Pontryagin dual of $\hat{L}_d$. Define $\phi : L \to \bar{L}$ by $\phi(x)(\chi) = \chi(x)$ for all $x \in L$ and $\chi \in \hat{L}_d$. Show that ϕ is injective, continuous, and that ϕ has dense image in $\bar{L}$. $\bar{L}$ is called the Bohr compactification of L (named after Harald Bohr, who had a brother named Niels).
(10) Show that a continuous function $f : L \to \mathbf{C}$ has a continuous extension $\bar{f} : \bar{L} \to \mathbf{C}$ (meaning that $\bar{f} \circ \phi = f$) if and only if it is a uniform limit of trigonometric polynomials. Such functions are called almost periodic.

Date: due on Tuesday, September 2, 2009 (before class).