HOMEWORK III

LOCALLY COMPACT ABELIAN GROUPS

(1) Let V be a finite dimensional vector space over $\mathbf{C}$. Let S be a set of pairwise commuting linear endomorphisms of V. Then there exists a basis of V with respect to which all the matrices in S are upper triangular, and all the semisimple matrices in S are diagonal.

(2) Let G be a compact abelian group. For each $x \in G$, let $T_x f(y) = f(y - z)$. For each $\chi \in \hat{G}$, let $M_\chi f(y) = e^{2\pi i \chi(y)} f(y)$. Show that $L^2(G)$ has no invariant subspaces which are invariant under all the operators T_x $(x \in G)$ and M_χ $(\chi \in \hat{G})$.

(3) Solve the previous problem when G is a discrete group (results of this type are part of the Mackey-Stone-von Neumann theorem, which we will revisit later in this course).

(4) Let A be a discrete abelian group. Show that A is torsion-free if and only if $\hat{A}$ is divisible [Hint: to prove that $\hat{A}$ is divisible, show that $n\hat{A}$ separates points in A].

(5) Suppose G is a compact abelian group such that $\hat{G}$ has torsion elements, then show that G is not connected (or in other words, if G is connected, then $\hat{G}$ is torsion-free) [Hint: if $n\chi = 0$ for some $\chi \in \hat{G}$, partition G by the values that χ takes].

(6) Show that a compact abelian group is connected if and only if it is divisible.

(7) An element x in a abelian group G is said to be infinitely divisible if, for every $n \in \mathbf{N}$, there exists $y \in A$ such that $ny = x$. Show that the set of infinitely divisible elements of G form a subgroup $D(G)$ and $G/D(G)$ has no infinitely divisible elements.

(8) If G is a compact abelian group show that $D(G) = G_0$, the identity component of G.

(9) Show that $\prod_p \mathbf{Z}_p$ has a dense cyclic subgroup.

Date: due on Tuesday, August 26, 2009 (before class).