HOMEWORK II

LOCALLY COMPACT ABELIAN GROUPS

(1) An abelian (or non-abelian) group A is said to have bounded order if there exists $N \in \mathbf{N}$ such that $Nx = 0$ for all $x \in A$. If A is a discrete or compact abelian group and has bounded order, show that $\hat{A}$ has bounded order.

(2) An abelian group is said to be torsion if, for every $x \in A$ there exists $N = N(x) \in \mathbf{N}$ such that $Nx = 0$. Find a torsion discrete abelian group such that $\hat{A}$ is not torsion.

(3) A Hausdorff topological space is said to be zero-dimensional if its topology has a base consisting of open and closed sets. Show that a zero-dimensional Hausdorff topological space is totally disconnected (the only connected subsets are singleton).

(4) If A is a discrete abelian torsion group, then $\hat{A}$ is zero-dimensional.

(5) If A is a discrete abelian group with an element $x \in A$ such that $nx \neq 0$ for all $n \in \mathbf{N}$ (in other words, A is not torsion), show that $\hat{A}$ is not zero-dimensional.

(6) If A is a discrete torsion group show that every neighborhood of 0 in $\hat{A}$ contains a compact open subgroup.

(7) Let p be a prime. An abelian group A is said to be p-torsion if, for every $x \in A$, there exists $n \in N$ such that $p^n x = 0$.
   (a) Show that the set of p-torsion elements of any abelian group A form a subgroup.
   (b) If A is torsion, then $A = \oplus_p A_p$, the direct sum being over all primes.

(8) If A is discrete p-torsion then $\lim_{n\to\infty} p^n \chi = 0$ for all $\chi \in \hat{A}$.

(9) Show that $(\mathbf{Q}/\mathbf{Z})_p$ is isomorphic to the Prüfer group $\mathbf{Z}(p^\infty)$ (defined in the previous assignment). Conclude that $\mathbf{Q}/\mathbf{Z} = \oplus_p \mathbf{Z}(p^\infty)$.

(10) Let p be a prime. The p-adic norm of an integer x is defined as

$$|x|_p = p^{-v} \quad \text{where } v = \max\{n \geq 0 : p^n | x\}.$$

   (a) Show that $d(x, y) = |x - y|_p$ is a metric on $\mathbf{Z}$ with the ultrametric triangle inequality: $d(x, z) \leq \max\{d(z, y), d(y, z)\}$ for all $x, y, z \in \mathbf{Z}$.
   (b) Let $\mathbf{Z}_p$ denote the completion of $\mathbf{Z}$ with this metric (the space of equivalence classes of Cauchy sequences, where two Cauchy sequences are equivalent if their difference converges to 0). Show that $d(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n)$ is a well-defined matric on $\mathbf{Z}_p$.
   (c) Show that $\mathbf{Z}_p$ is isomorphic to $\widehat{\mathbf{Z}(p^\infty)}$ as a topological group (so there is an isomorphism of groups which is also a homeomorphism).
   (d) Show that $\widehat{\mathbf{Q}/\mathbf{Z}} = \prod_p \mathbf{Z}_p$.

Date: due on Wednesday, August 19, 2009 (before class).