# HOMEWORK X

## LOCALLY COMPACT ABELIAN GROUPS

(1) For $f \in L^1(\mathbf{R}^2)$, $g \in L^1(\mathbf{R}^2)$ define
$$f\#g(k) = \int e^{\pi i \omega(k,l)} f(k-l)g(l)dl,$$
where for $k = (x, \xi)$ and $l = (y, \eta)$ $\omega(k, l) = \xi y - \eta x$. Show that $\|f\#g\|_1 \leq \|f\|_1 \|g\|_1$ [Hint: look for the proof of a similar statement for ordinary convolution].

(2) Show that $f\#(g\#h) = (f\#g)\#h$ for $f, g, h \in L^1(\mathbf{R}^2)$.

(3) Let $\Phi(k) = V(\phi, \phi)(k)$, where $\phi(u) = 2^{1/4} e^{-\pi u^2}$ and $V$ denotes the Wigner transform:
$$V(f, g)(k) = \langle W_k f, g \rangle; \quad W_k f(u) = e^{2\pi i \xi (u - x/2)} f(u - x),$$
with $k = (x, \xi)$. Show that
$$\Phi(k) = e^{-\pi^2 (x^2 + \xi^2)/2}.$$

(4) What functions can the Gaussian be replaced by in the proof of the Stone-von Neumann theorem for $\mathbf{R}$ (using the Wigner transform) given in class?

(5) For each $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$ show that there exists a unitary operator $U_\sigma : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ such that $U_\sigma \circ W_{(x,\xi)} = W_{(ax+b\xi, cx+d\xi)} \circ U_\sigma$, where $W_k = W_{(x,\xi)}$ is as in Problem 3. Furthermore, for $\sigma, \tau \in SL_2(\mathbf{R})$, $U_{\sigma\tau}$ and $U_\sigma \circ U_\tau$ are multiples of each other by unit complex numbers.

(6) When $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ find a formula for $U_\sigma$ of Problem 5.

(7) For a locally compact abelian group, deduce the Plancherel theorem from the Stone-von Neumann theorem.

*Date*: due on Wednesday, October 28, 2009 (before class).