HOMEWORK I

LOCALLY COMPACT ABELIAN GROUPS

(1) If both A and $\hat{A}$ are discrete, show that A is finite.
(2) Describe the invariant probability measure on a finite group.
(3) Describe the invariant probability measure on T.
(4) Show that there is no invariant probability measure on $\mathbf{R}$.
(5) Let $\{A_i\}_{i\in I}$ be a collection of abelian groups endowed with the discrete topology. Let $A = \oplus_{i\in I} A_i$ also endowed with the discrete topology. Show that $\hat{A} = \prod_i \widehat{A_i}$ with the product topology.
(6) Recall that an abelian group A is said to be divisible if for every $n \in \mathbf{N}$ and $y \in A$, there exists $x \in A$ such that $nx = y$. If A is divisible and endowed with the discrete topology, show that $\hat{A}$ is torsion-free.
(7) Think of $\mathbf{Q}$ as a discrete group. What is the cardinality of $\hat{\mathbf{Q}}$?
(8) Does $\mathbf{Q}$ admit a non-discrete topology with respect to which it is a locally compact (Hausdorff) abelian group?
(9) Fix a prime p. Let $\mathbf{Z}(p^\infty)$ (the Prüfer group) be the quotient of the free abelian group generated by symbols x_n, $n \in \mathbf{N}$, modulo the relations $px_1 = 0$, and $px_i = x_{i-1}$ for $i > 1$, endowed with the discrete topology.
   (a) Show that $\mathbf{Z}(p^\infty)$ is divisible.
   (b) Show that for every $x \in \mathbf{Z}(p^\infty)$ there exists $k \in \mathbf{N}$ such that $p^k x = 0$.
   (c) Show that $\widehat{\mathbf{Z}(p^\infty)}$ is not divisible.
   (d) What is the cardinality of $\widehat{\mathbf{Z}(p^\infty)}$?

Date: due on Wednesday, August 12, 2009 (before class).