- Permutation reps & intertwiners
- Partition reps. & intertwiners
- Example : S_3
- RSK correspondence
- Classification of simple reps.
- How to go further.

G gp., X : G-set, K : field

$(\rho_X, K[X])$ - permutation rep.

$K[X] = \{\text{functions } X \to K\}$

$\rho_X(g) f(x) = f(g^{-1} \cdot x)$

ρ_X is a homom. $G \to GL(K[X])$

Example: $\lambda \vdash n$ (λ partition of n)

i.e., $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\lambda_1 \geqslant \dots \geqslant \lambda_\ell$, $\lambda_1 + \dots + \lambda_\ell = n$.

$$X_\lambda = \{(S_1, \dots, S_\ell) \mid \underbrace{\underline{n}}_{\{1,\dots,n\}} = S_1 \sqcup \dots \sqcup S_\ell,\ |S_i| = \lambda_i\}$$

$G = S_n$. X_λ is an S_n-set.

$K[X_\lambda]$ is a representation of S_n.

②

(ρ, V), (σ, W) : representations of G

$T : V \longrightarrow W$ is called an <u>intertwiner</u> if
$T(\rho(g)v) = \sigma(g)T(v) \;\forall v \in V, g \in G.$

Notation: $T \in \mathrm{Hom}_G(V, W)$ [Category of reps.

Let's calculate: $\mathrm{Hom}_G(K[Y], K[X])$
X, Y : G-sets.

$$\mathrm{Hom}_k(K[Y], K[X]) = \{T_k \mid k \in K[X \times Y]\}$$

$$T_k f(x) = \sum_{y \in Y} k(x,y) f(y)$$ Integral operator with kernel k

$T_k \in \mathrm{Hom}_G(K[Y], K[X])$ iff $\cancel{\rho_X(g)\, k(x,y)\, \rho_Y(g)}$

$$\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g) = T_k$$

Now: $\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g) f(x)$

$$= T_k \circ \rho_Y(g) f(gx)$$

$$= \sum_{y \in Y} k(gx, y) \cancel{f(y)}\, \cancel{\rho_Y(g) f(y)}\; f(g^{-1}y)$$

$$= \sum k(gx, gy) f(y)$$

$$= T_{k^g} f(x).$$

where $k^g(x,y) = k(gx, gy)$.

Conclusion: $T_k \in \mathrm{Hom}_G(K[Y], K[X])$ iff
$k : G \backslash X \times Y \longrightarrow K$.

Relative position: (x,y) & (x',y') have the same relative position if $\exists\ g \in G \ni$

$x' = g\cdot x$ and $y' = g \cdot y$.

Take: $G = S_n$, $\lambda, \mu \vdash n$. $\lambda = (\lambda_1, \ldots, \lambda_\ell)$
$\mu = (\mu_1, \ldots, \mu_m)$

Thm: $S = (S_1, \ldots, S_\ell)$, $S' = (S'_1, \ldots, S'_\ell) \in X_\lambda$
$T = (T_1, \ldots, T_m)$, $T' = (T'_1, \ldots, T'_m) \in X_\mu$.

Then (S,T) and (S',T') have the same relative position iff. $|S_i \cap T_j| = |S'_i \cap T'_j|\ \forall (i,j)$.

Pf: $\Rightarrow$ is clear.

Let $A_{ij} = S_i \cap T_j \quad A'_{ij} = S'_i \cap T'_j$.

$|A_{ij}| = |A'_{ij}|$, and $\underline{n} = \coprod_{i,j} A_{ij} = \coprod_{i,j} A'_{ij}$

So, mapping $A_{ij} \to A'_{ij}$ bijectively gives

$g \in S_n \ni g(A_{ij}) = A'_{ij}$

Since $S_i = \coprod_j A_{ij}$, & $T_j = \coprod_i A_{ij}$

$$g(S_i) = \coprod_j g(A_{ij}) = \coprod_j A'_{ij} = S'_i$$

$$g(T_j) = T'_j.$$

QED

Let $r_{ij} = |S_i \cap T_j| \qquad r = (r_{ij})$

$$\sum_j r_{ij} = \lambda_i \qquad \sum_i r_{ij} = \mu_j$$

r is called a $\lambda \times \mu$ matrix. $\begin{cases} \text{non-neg. integer entries} \\ \text{row sums : } \lambda \\ \text{col. sums : } \mu. \end{cases}$

④ Let $M_{\lambda\mu} = \#\{\lambda \times \mu \text{ matrices}\}$.

<u>Thm</u>: $\dim \operatorname{Hom}_{S_n}(K[X_\mu], K[X_\lambda]) = M_{\lambda\mu}$.

<u>Example</u>: 3 has partitions (3), (2,1), (1,1,1).

$\dim \operatorname{Hom}_{S_n}(K[X_\mu], K[X_\lambda])$:

	(3)	(2,1)	(1,1,1)
(3)	1	1	1
(2,1)	1	2	3
(1,1,1)	1	3	6

Maschke's thm: char $K > n$ then $K[X_\lambda], K[X_\mu]$ are sums of simple reps.

If $V = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$

$W = V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k}$

$V_1, \dots, V_k$ - pairwise non-isomorphic simples.

$\dim \operatorname{Hom}_G(V, W) = \sum m_i n_i$

$\longrightarrow \dim \operatorname{End}_G V = 1 \iff V$ is simple

$\longrightarrow$ ~~V simple~~ $\dim \operatorname{Hom}_G(V_i, V) = m_i$

From the table:

— $K[X_{(3)}]$ is simple. $V_{(3)} := K[X_{(3)}]$

— $V_{(3)}$ has mult. 1 in $K[X_\lambda]$ $\forall \lambda \vdash 3$.

Define: $K[X_\lambda]_{(3)}$ by: $K[X_\lambda] = V_{(3)} \oplus K[X_\lambda]_{(3)}$

$$\dim \mathrm{Hom}_{S_3}(K[X_\mu]_{(3)}, K[X_\lambda]_{(3)})$$
$$= \dim \mathrm{Hom}_{S_3}(K[X_\mu], K[X_\lambda]) - 1$$

	(2,1)	(1,1,1)
(2,1)	1	2
(1,1,1)	2	5

— $K[X_{(2,1)}]_{(3)}$ is simple. $V_{(2,1)} := K[X_{(2,1)}]_{(3)}$

— $V_{(2,1)}$ has multiplicity 2 in $K[X_\lambda]_{(3)}$

Define: $K[X_{(1,1,1)}]_{(2,1)}$ by $K[X_{(1,1,1)}] = K[X_\lambda]_{(3)} \oplus V_{(2,1)}^{\oplus 2}$

$$\dim \mathrm{End}_{S_3}(K[X_{(1,1,1)}]_{(2,1)}) = 5 - 2 \times 2 = 1$$

$$V_{(1,1,1)} := K[X_{(1,1,1)}]_{(2,1)}.$$

For every $\lambda \vdash 3$, we have constructed a simple rep. of S_3.

⑥

$K[X_{(1,1,1)}]$ - reg. rep.

dim = multiplicity

This works for all n.

Get an algorithm to ~~show that~~ construct all simple reps, and to compute their characters.

<u>Why?</u>

<u>Theorem</u> (Combinatorial Resolution Theorem)

$(P, \leq)$: poset.

$\forall\ \lambda \in P$, have rep. of G: U_λ.

$M_{\mu\lambda} := \dim \mathrm{Hom}_G(U_\lambda, U_\mu)$.

Suppose $\exists\ K_{\mu\lambda} \geq 0 \ni$

① $K_{\mu\lambda} > 0 \Rightarrow \mu \leq \lambda$

② $K_{\lambda\lambda} = 1 \quad \forall\ \lambda \in P$

③ $M_{\mu\lambda} = \sum\limits_{\nu \leq \lambda, \nu \leq \mu} K_{\nu\lambda} K_{\nu\mu}$

Then $U_\lambda = \bigoplus\limits_{\mu \leq \lambda} V_\mu^{\oplus K_{\mu\lambda}}$

for a family of pairwise non-isomorphic simple reps. $\{V_\mu\}_{\mu \in P}$ of G.

$\wp$

Pf: Induct on $|P|$.

Take $\lambda \in P$ minimal

(2) + (3) $\Rightarrow M_{\lambda\lambda} = 1$

So $V_\lambda := U_\lambda$ is simple.

(3) $\Rightarrow M_{\lambda\mu} = K_{\lambda\lambda} K_{\lambda\mu} = K_{\lambda\mu}$

So V_λ has multiplicity $K_{\lambda\mu}$ in V_μ.

Define: $U'_{\lambda\mu}$ by $U_{\lambda\mu} = U'_{\lambda\mu} \oplus V_\lambda^{\oplus K_{\lambda\mu}}$

Then $\dim \mathrm{Hom}(U'_\mu, U'_\eta) = M_{\mu\eta} - K_{\lambda\mu} K_{\lambda\eta} =: M'_{\mu\eta}$

Let $P' = P - \{\lambda\}$

$$M'_{\mu\eta} = \sum_{\substack{\nu \le \mu,\ \nu \le \eta \\ \nu \in P'}} K_{\nu\mu} K_{\nu\eta}$$

So $\{U'_\lambda\}_{\lambda \in P'}$ satisfies the hypotheses. QED.

~~The~~ Φ SSYT:

$$\begin{matrix} 1 & 1 & 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 3 & 4 & \\ 3 & 4 & 6 & & & \\ 5 & 5 & 7 & & & \end{matrix}$$

shape $(6,5,3,3)$

type $(4,3,3,3,2,1,1)$

$K_{\mu\lambda} = \#$ SSYT of shape μ & type λ.

$K_{\mu\lambda} > 0 \Leftrightarrow \mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i \quad \forall i$.

So if we define $\mu \le \lambda$ if $K_{\mu\lambda} > 0$ then "$\le$" is a partial order on Partitions of n.

(8) Knuth: Permutations, Matrices & Generalized Young Tableaux, Pac. J. Math., 1970

used Schensted's insertion algorithm:

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 3 & 4 & \\ 3 & 4 & 6 & & & \\ 5 & 5 & 7 & & & \end{array} \quad \leftarrow 1$$

creates a new box.

As follows: start with a matrix $A \in M_{\ell \times m}(\mathbb{Z}_{\geq 0})$.

$$\begin{pmatrix} 2 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 1 & 1 & 1 & 3 & 1 & 2 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

Start with the empty tableau $\emptyset = Y_0$

$Q_i = Y_{i-1} \leftarrow v_i$

$P_i = P_{i-1} + u_i$ inserted into the new box

<u>Thm</u>: RSK is a bijective correspondence between $\lambda \times \mu$ matrices and pairs (P, Q) of SSYT $\ni$ type$(P) = \lambda$ type$(Q) = \mu$, shape(P) = shape(Q),

i.e., $$M_{\lambda\mu} = \sum_{\nu \leq \lambda,\, \nu \leq \mu} K_{\nu\lambda} K_{\nu\mu}.$$

$K_{\lambda\lambda} = 1$ b/c there is only one SSYT of shape & type λ:

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & \\ 3 & 3 & 3 & & & \\ 4 & 4 & 4 & & & \end{array}$$

The RSK correspondence serves as an organizing principle in the study of Symmetric fns.

m_λ = monomial symm. fn $= \sum_{shape(\alpha)=\lambda} x^\alpha$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

shape(α) : partition obtained by rearranging α_i

$\vec{m} = (m_\lambda)_{\lambda \in P}$

e_λ = elementary sym. fns.

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}.$$

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$$

$\vec{e} = (e_\lambda)$

h_λ = complete sym. fns.

$$h_n = \sum_{|\alpha| = n} x^\alpha$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$$

$\vec{h} = (h_\lambda)$

$K = K_{\mu\lambda}$ $\quad J = (\delta_{\lambda\mu'})$ (perm. matrix)

$\vec{e} = \vec{m} K' J K$ (Dual RSK)

$\vec{h} = \vec{m} K' K$ (RSK)

$\vec{p} = \vec{m} P$ where $P_{\mu\lambda} = \mathrm{trace}(\omega_\mu, K[X_\lambda])$

$= \vec{m} K' X$ $\left(\because K[X_\lambda] = \sum_{\mu \leq \lambda} V_\mu^{\oplus K_{\mu\lambda}}\right)$

So if you define $\vec{s} = \vec{m} K'$,

$\vec{p} = \vec{s} X$ (Frobenius char. fmla.)