- Permutation reps & intertwiners
- Partition reps. & intertwiners
- Example : S_3
- RSK correspondence
- Classification of simple reps.
- How to go further.

G gp., X : G-set, K : field

$(\rho_X, K[X])$ - permutation rep.

$K[X] = \{\text{functions } X \to K\}$

$\rho_X(g) f(x) = f(g^{-1} \cdot x)$

ρ_X is a homom. $G \to GL(K[X])$

Example : $\lambda \vdash n$ (λ partition of n)

i.e., $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\lambda_1 \geq \dots \geq \lambda_\ell$, $\lambda_1 + \dots + \lambda_\ell = n$.

$X_\lambda = \{(S_1, \dots, S_\ell) \mid \underset{\{1,\dots,n\}}{\underline{n}} = S_1 \sqcup \dots \sqcup S_\ell, |S_i| = \lambda_i\}$

$G = S_n$. X_λ is an S_n-set.

$K[X_\lambda]$ is a representation of S_n.

(2)

$(\rho, V), (\sigma, W)$: representations of G

$T: V \longrightarrow W$ is called an <u>intertwiner</u> if

$T(\rho(g)v) = \sigma(g)T(v) \;\forall\, v \in V, g \in G.$

Notation: $T \in \mathrm{Hom}_G(V, W)$ [Category of reps.

Let's calculate: $\mathrm{Hom}_G(K[Y], K[X])$

X, Y : G-sets.

$$\mathrm{Hom}_k(K[Y], K[X]) = \{T_k \mid k \in K[X \times Y]\}$$

$$T_k f(x) = \sum_{y \in Y} k(x,y) f(y)$$ Integral operator with kernel k

$T_k \in \mathrm{Hom}_G(K[Y], K[X])$ ~~iff~~ $\cancel{\rho_X(g)\, k(x,y)\, \rho_Y(g)}$

$$\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g) = T_k$$

Now: $\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g) f(x)$

$$= T_k \circ \rho_Y(g) f(gx)$$

$$= \sum_{y \in Y} k(gx, y)\, \cancel{f(y)}\, \cancel{\rho_Y(g) f(y)}\; f(g^{-1}y)$$

$$= \sum k(gx, gy) f(y)$$

$$= T_{k^g} f(x).$$

where $k^g(x, y) = k(gx, gy)$.

Conclusion: $T_k \in \mathrm{Hom}_G(K[Y], K[X])$ iff

$$k : G \backslash X \times Y \longrightarrow K.$$

<u>Relative position</u>: (x,y) & (x',y') have the same relative position if $\exists\ g \in G \ni$

$x' = g\cdot x$ and $y' = g\cdot y$.

Take: $G = S_n$, $\lambda, \mu \vdash n$. $\quad \lambda = (\lambda_1, \dots, \lambda_\ell)$
$\mu = (\mu_1, \dots, \mu_m)$

<u>Thm</u>: $S = (S_1, \dots, S_\ell)$, $S' = (S'_1, \dots, S'_\ell) \in X_\lambda$
$T = (T_1, \dots, T_m)$, $T' = (T'_1, \dots, T'_m) \in X_\mu$.
Then (S,T) and (S',T') have the same relative position iff. $|S_i \cap T_j| = |S'_i \cap T'_j|\ \forall (i,j)$.

<u>Pf:</u> $\Rightarrow$ is clear.

Let $A_{ij} = S_i \cap T_j \quad A'_{ij} = S'_i \cap T'_j$.

$|A_{ij}| = |A'_{ij}|$, and $\underline{n} = \coprod_{i,j} A_{ij} = \coprod_{i,j} A'_{ij}$

So, mapping $A_{ij} \to A'_{ij}$ bijectively gives

$g \in S_n \ni g(A_{ij}) = A'_{ij}$

Since $S_i = \coprod_j A_{ij}$, & $T_j = \coprod_i A_{ij}$

$$g(S_i) = \coprod_j g(A_{ij}) = \coprod_j A'_{ij} = S'_i$$

$$g(T_j) = T'_j.$$

QED

Let $r_{ij} = |S_i \cap T_j| \qquad r = (r_{ij})$

$$\sum_j r_{ij} = \lambda_i \qquad \sum_i r_{ij} = \mu_j$$

r is called a $\lambda \times \mu$ matrix. $\begin{cases} \text{non-neg. integer entries} \\ \text{row sums}: \lambda \\ \text{col. sums}: \mu. \end{cases}$

(4) Let $M_{\lambda\mu} = \#\{\lambda \times \mu \text{ matrices}\}$.

<u>Thm</u>: $\dim \operatorname{Hom}_{S_n}(K[X_\mu], K[X_\lambda]) = M_{\lambda\mu}$.

<u>Example</u>: 3 has partitions (3), (2,1), (1,1,1).

$\dim \operatorname{Hom}_{S_n}(K[X_\mu], K[X_\lambda])$:

	(3)	(2,1)	(1,1,1)
(3)	1	1	1
(2,1)	1	2	3
(1,1,1)	1	3	6

Maschke's thm: $\operatorname{char} K > n$ then $K[X_\lambda], K[X_\mu]$ are sums of simple reps.

If $V = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$

$W = V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k}$

$V_1, \dots, V_k$ - pairwise non-isomorphic simples.

$\dim \operatorname{Hom}_G(V, W) = \sum m_i n_i$

$\longrightarrow \dim \operatorname{End}_G V = 1 \iff V$ is simple

$\longrightarrow$ ~~V simple~~ $\dim \operatorname{Hom}_G(V_i, V) = m_i$