MIDSEMESTER EXAM

FUNCTIONAL ANALYSIS

(1) Prove that the real line, equipped with the distance
$$d(x, y) = |\arctan x - \arctan y|,$$
is not complete.

(2) If X is a Hilbert space, show that $\dim X < \infty$ if and only if there exists a compact set in X which contains an open neighborhood of 0.

(3) We say that a sequence of distributions T_n converges to a distribution T if $\lim_{n\to\infty} T_n(\phi) \to T(\phi)$ for every $\phi \in C_0^\infty(\Omega)$. Show that if T_n converges to T, then $\partial T_n/\partial x_j$ converges to $\partial T/\partial x_j$ for each j.

(4) Suppose that the topology of a locally convex topological vector space is given by a countable sufficient family of seminorms. Show that there exist continuous seminorms $\{p_n(x)\}$ on X satisfying
$$p_1(x) \le p_2(x) \le p_3(x) \le \cdots \text{ for every } x \in X$$
such that any neighborhood of 0 in X contains a subset of the form
$$\{x \mid p_n(x) < \epsilon\}$$
for some $n \in \mathbf{N}$ and some $\epsilon > 0$.

(5) Suppose X and Y are Banach spaces and $\mathcal{A}$ is a collection of bounded operators $X \to Y$. Suppose that for all $x \in X$, $\{\|Tx\| : T \in \mathcal{A}\}$ is bounded. Show that $\{\|T\| : T \in \mathcal{A}\}$ is bounded. Hint: Let $\mathcal{A}_N = \{x \in X : \|Tx\| \le N \text{ for all } T \in \mathcal{A}\}$. Then use the Baire category theorem.

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