HOMEWORK IX

FUNCTIONAL ANALYSIS

(1) Use the Poisson summation formula to prove the theta relation:

$$\sum_{-\infty}^{\infty} e^{-n^2\pi z} = z^{-\frac{1}{2}} \sum_{-\infty}^{\infty} e^{-k^2\pi/z}.$$

(2) Evaluate $\sum_1^\infty (n^2+1)^{-1}$ using the Poisson summation formula.

(3) Let $\hat{\mathbf{R}}$ denote the set of all continuous functions $\chi : \mathbf{R} \to \mathbf{C}$ such that $|\chi(x)| = 1$ for all $x \in \mathbf{R}$, and $\chi(x+y) = \chi(x)\chi(y)$. Note that for each $\xi \in \mathbf{R}$, if $\chi_\xi(x) = e^{ix\xi}$ then $\chi_\xi \in \hat{\mathbf{R}}$. Show that $\xi \mapsto \chi_\xi$ is a bijection $\mathbf{R} \to \hat{\mathbf{R}}$. Furthermore, $\xi_n \to \xi$ if and only if χ_{ξ_n} converges to χ_ξ uniformly on all compact sets.

(4) Prove that $C_0^\infty(\mathbf{R}^n)$ is dense in $\mathcal{O}_M(\mathbf{R}^n)$.

(5) For a tempered distribution T, show that $\widehat{\frac{\partial T}{\partial x_j}} = ix_j\hat{T}$, and that $\widehat{ix_jT} = -\frac{\partial \hat{T}}{\partial x_j}$.

(6) Suppose $\phi \in \mathfrak{D}(\mathbf{R})$ with support contained in $[-a, a]$. Show that

$$\hat{\phi}(\xi) = \int \phi(x)e^{-i\xi x}dx; \quad \xi \in \mathbf{C}$$

is an entire (holomorphic) function, and that for every $q \geq 0$, there exists a constant C_q (independent of ξ) such that

$$|\xi^q\hat{\phi}(\xi)| \leq C_q e^{a|\tau|}, \text{ for all } \xi = \sigma + i\tau \in \mathbf{C};\ \sigma, \tau \in \mathbf{R}.$$

Hint: For $q = 0$, the proof is relatively straightforward. For higher q, work with the Fourier transform of the qth derivative of ϕ[1].

(7) Show that whenever $T \in \mathcal{S}(\mathbf{R}^n)'$ is homogeneous of degree m, $\hat{T}$ is homogeneous of degree $-n-m$.

(8) Show that whenever $T \in \mathcal{S}(\mathbf{R}^2)'$ is invariant under the transformation $(x, y) \mapsto (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$ for all $\theta \in [0, 2\pi]$, then so is $\hat{T}$.

(9) Note that the Fourier transform of a function in $L^1(\mathbf{R}^n)$ is always a well-defined function. For $f \in L^1(\mathbf{R}^n)$, let V_f be the closed subspace of $L^1(\mathbf{R}^n)$ spanned by all the translates of f (i.e., functions of the form $f_y(x) = f(x-y)$). Suppose that $\hat{f}(\xi_0) = 0$ for some $\xi_0 \in \mathbf{R}^n$, show that $\hat{h}(\xi_0) = 0$ for all $h \in V_f$. Deduce that if $V_f = L^1(\mathbf{R}^n)$, then $\hat{f}$ never vanishes[2].

(10) If $f \in L^2(\mathbf{R}^n)$, show that

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{\|x\|\leq R} f(x)e^{-i(x,\xi)}dx,$$

where the limit is taken in $L^2(\mathbf{R}^n)$.

Date: due on Monday, March 24, 2008 (before class).

[1] The converse of this result is the Paley-Wiener theorem: if ψ is an entire function such that for every $q \geq 0$, there exists C_q such that $|\xi^q\psi(\xi)| \leq C_q e^{a|\tau|}$, then ψ is the Fourier transform of a smooth function which vanishes for $|x| > a$.

[2] The converse of this result is the Tauberian theorem of Norbert Wiener.