HOMEWORK VIII

FUNCTIONAL ANALYSIS

(1) For $f \in \mathcal{S}(\mathbf{R}^n)$, define its translate by y as the function $T_y f(x) = f(x-y)$, and its modulate by ξ as the functions $M_\xi f(x) = e^{i\langle \xi, x\rangle} f(x)$. Compute $\widehat{T_y f}$ and $\widehat{M_\xi f}$ (in terms of x, ξ and $\hat{f}$).

(2) Let V denote the space of all complex valued functions on the ring $\mathbf{Z}/n\mathbf{Z}$. For $f \in V$, define its *discrete Fourier transform* by

$$\hat{f}(\xi) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbf{Z}/n\mathbf{Z}} f(x) e^{-2\pi i x \xi / n}, \quad \xi \in \mathbf{Z}/n\mathbf{Z}.$$

Prove the discrete Fourier inversion formula:

$$\hat{\hat{f}}(x) = f(-x) \text{ for all } f \in V.$$

(3) Show that $e^{-|x|^2/2} \in \mathcal{S}(\mathbf{R}^n)$.

(4) If $f(x) \in \mathcal{S}(\mathbf{R}^n)$, show that the integral

$$\int_{\mathbf{R}^n} f(x) e^{-i\langle \xi, x\rangle} dx$$

is absolutely convergent for all $\xi \in \mathbf{R}^n$.

(5) Deduce Parseval's identity

$$\int \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi = \int f(x)\overline{g(x)} dx \text{ for all } f, g \in \mathcal{S}(\mathbf{R}^n)$$

from the identity

$$\int g(\xi)\hat{f}(\xi) d\xi = \int \hat{g}(x) f(x) dx \text{ for all } f, g \in \mathcal{S}(\mathbf{R}^n).$$

(6) Show that $\mathcal{S}(\mathbf{R}^n)$ is a Fréchet space.

(7) Let D and M denote the operators $\mathcal{S}(\mathbf{R}) \to \mathcal{S}(\mathbf{R})$ given by

$$(Df)(x) = \frac{d}{dx} f(x), \quad (Mf)(x) = x f(x).$$

As usual, let $\mathcal{F}$ denote the Fourier transform operator. Show that

$$\mathcal{F} \circ (D^2 - M^2) = (D^2 - M^2) \circ \mathcal{F}.$$

(8) If $f(x) = W(x) e^{-x^2/2}$ is a solution to the differential equation

$$(D^2 - M^2) f = \lambda f,$$

show that $W(x)$ satisfies the equation

$$W''(x) - 2xW'(x) = (\lambda + 1) W(x).$$

(9) Show that, for each $n \geq 0$, there is a polynomial $W(x)$ of degree n satisfying $W''(x) - 2xW'(x) = (\lambda+1)W(x)$ if and only if $\lambda = -(2n+1)$. Such a polynomial, denoted $P_n(x)$, is unique up to scaling.

(10) Let $\phi_n(x) = P_n(x) e^{-x^2/2}$. Define $\psi_n(x) = \phi_n(x)/\|\phi_n\|_{L^2}$. Show that, for each non-negative integer n, $\hat{\psi}_n(x) = c_n \psi_n(x)$, where c_n is a complex fourth root of unity. Hint: use Exercises (7-9)[1].

Date: due on Monday, March 17, 2008 (before class).

[1] ψ_n is called the *nth Hermite function*. The Hermite functions are rapidly decreasing functions which form an orthonormal basis of $L^2(\mathbf{R})$.