HOMEWORK VII

FUNCTIONAL ANALYSIS

(1) Prove the *Riesz-Fischer theorem* for Fourier series: if $\{a_n\}_{-\infty}^{\infty}$ is a sequence of complex numbers such that $\sum_{-\infty}^{\infty} |a_n|^2 < \infty$ then there exists $f \in L^2[0,1]$ such that

$$f(x) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n x}.$$

(2) A function $f \in C^\infty(\mathbf{R})$ is called *rapidly decreasing* if, for every polynomial p and every multi-index s, $|p(x)D^s f(x)| \to 0$ as $\|x\| \to \infty$. The *Schwartz space* $\mathcal{S}$ is the space of all rapidly decreasing functions.
   (a) If $f \in \mathcal{S}$, then $D^j f \in \mathcal{S}$ for every multi-index j.
   (b) $\mathcal{S} \subset L^q(\mathbf{R})$ for every $1 \leq q \leq \infty$.
   (c) Show that $\mathcal{S}$ is a Fréchet space in such a way that $f_n \to f$ if and only if for every polynomial p and every multi-index j, $pD^j f_n \to pD^j f$ uniformly.

(3) $T \in \mathcal{D}(\mathbf{R}^n)'$ is called a *tempered distribution* if its restriction to $\mathcal{S}$ is continuous with respect to the topology defined in the previous exercise.
   (a) Give an example of a distribution which is not tempered.
   (b) Show that every compactly supported distribution is tempered.

(4) If $T \in \mathcal{D}(\mathbf{R}^n)'$ is homogeneous of degree α for some $\alpha \in \mathbf{R}$, then $\frac{\partial}{\partial x_i} T$ is homogeneous of degree $\alpha - 1$.

(5) Let $P(\xi_1, \ldots, \xi_n)$ be a homogeneous polynomial of degree m in n variables. Show that the distribution $P(D)\delta_0$ is homogeneous on $\mathbf{R}^n$ of degree $-n-m$.

(6) Find all $T \in \mathcal{D}(\mathbf{R})'$ which are invariant under the group of translations $x \mapsto x - a$, $a \in \mathbf{R}$.

(7) What are the compactly supported distributions on $\mathbf{R}^2$ which are invariant under all transformations of the form

$$(x, y) \mapsto (ax + by, cx + dy); \quad a, b, c, d \in \mathbf{R},\ ad - bc = 1?$$

(8) For $T \in \mathcal{D}(\mathbf{R})'$ and $f \in C_0^\infty(\mathbf{R})$ define the convolution $T * f$ as the function

$$(T * f)(x) = (T, f_x), \quad \text{where } f_x(u) = f(x - u).$$

Show that $T * f \in C^\infty(\mathbf{R})$, and that if T is compactly supported, then so is $T * f$.

Date: due on Monday, March 10, 2008 (before class).