HOMEWORK V

FUNCTIONAL ANALYSIS

(1) If $(\ ,\ ) : X \times X \to K$ satisfies $(\alpha x, y) = \alpha(x, y)$, $(x + y, z) = (x, z) + (y, z)$, $(y, z) = \overline{(x, y)}$ and $(x, x) = \|x\|^2$, for all $x, y \in X$ and $\alpha \in K$, where $\|x\|$ is a norm, then the norm satisfies the parallelogram identity (i.e., X is a pre-Hilbert space).

(2) Let $\Omega \subset \mathbf{R}^n$ be an open domain, and suppose that $f_n : \Omega \to \mathbf{R}$ is a sequence of C^1 functions such that $f_i \to f$ uniformly on Ω. Show that f is also a C^1 function and that $D^i f_n \to D^i f$ uniformly on Ω.

(3) Show that $\mathcal{C}^k(\Omega)$ is complete for $k \leq \infty$ (recall that this space of smooth functions has the topology of uniform convergence on compact sets).

(4) Show that $\mathcal{D}(\Omega)$ is complete.

(5) Show that $L^\infty(S, \mathfrak{B}, m)$ is complete.

(6) Show that, for an open domain $\Omega \subset \mathbf{R}^n$, for every positive integer k, the space $\hat{H}^k(\Omega)$ of C^k functions x on Ω such that

$$\|x\| := \int_\Omega \Big(\sum_{|\alpha| \leq k} \int_\Omega \big| D^j x(t) \big|^2 \Big)^{1/2} ds < \infty$$

is not complete when its topology is given by the norm $\|x\|$ defined above.

(7) Give $\mathbf{R}^n$ the L^∞ norm for each n. Compute the operator norm of the linear operator $\mathbf{R}^n \to \mathbf{R}^m$ which is given by the matrix $((a_{ij}))_{m\times n}$.

(8) Show that the operators $L^p(\mathbf{R}) \to L^p(\mathbf{R})$ defined by

$$T_x f(y) = f(y - x), \quad M_\xi f(y) = e^{2\pi i \xi y} f(y)$$

are continuous for every $x \in \mathbf{R}$ and $\xi \in \mathbf{R}$ and $1 \leq p \leq \infty$. Show that T_x and M_ξ do not commute if $x\xi \neq 0$.

(9) Let X and Y be normed linear spaces and $T : X \to Y$ be a linear operator. Show that T admits a continuous inverse if and only if there exists a positive constant γ such that

$$\|Tx\| \geq \gamma \|x\| \text{ for every } x \in X.$$

(10) Suppose X and Y are normed linear spaces such that X is infinite dimensional and $Y \neq \{0\}$. Show that there is a discontinuous linear operator $X \to Y$.

Date: due on Monday, February 11, 2008 (before class).