HOMEWORK III

FUNCTIONAL ANALYSIS

(1) If f and g are functions on $\mathbf{R}^n$ such that $f * g$ is defined, show that
$$\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g).$$

(2) Suppose that $f, g, h \in C_0(\mathbf{R}^n)$. Show that
$$f * g = g * f \text{ and}$$
$$(f * g) * h = f * (g * h).$$

(3) For each $a > 0$, let $g_a(x) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{x^2}{2a}}$. Show that $g_a * g_b = g_{a+b}$.

(4) Consider the function $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$. Compute $f * f(x)$.

(5) Define $g : \mathbf{R} \to \mathbf{R}$ by
$$g(s) = \begin{cases} e^{-\frac{1}{s}} & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$
Check that the nth derivative
$$g^{(n)}(s) = \begin{cases} P_n(1/s) e^{-\frac{1}{s}} & \text{if } s > 0 \\ 0 & \text{if } s \leq 0. \end{cases},$$
where P_n is a polynomial (special attention should be paid to $s = 0$; the mean value theorem or L'Hôpital's rule is needed).

(6) Suppose that X is a vector space with subspaces $\{X_\alpha\}$ such that $X = \cup_\alpha X_\alpha$, each X_α is a locally convex linear topological space, and the topology on X is the inductive limit of the topologies in the X_α's. Show that a linear functional $T : X \to K$ is continuous if and only if $T_{|X_\alpha}$ is continuous for every α.

(7) Suppose $f \in L^1_{\text{loc}}(\Omega)$. For $\phi \in C_0(\Omega)$ define
$$T_f \phi = \int_\Omega f(x) \overline{\phi(x)} dx.$$
Show that $T_f : \mathcal{D}(\Omega) \to \mathbf{C}$ is continuous.

(8) Recall that a function $\phi : (a, b) \to \mathbf{R}$, $-\infty \leq a < b \leq \infty$ is said to be *convex* if
$$\phi(sx + ty) \leq s\phi(x) + t\phi(y) \text{ for all } s + t = 1.$$
   (a) Prove that the supremum of any collection of convex functions on (a, b) is convex, and that the pointwise limit of a sequence of convex functions is convex.
   (b) Show, by example, that the infimum of a collection of convex functions need not be convex.
   (c) If ϕ is convex on (a, b) and ψ is convex and non-decreasing on the range of ϕ then $\psi \circ \phi$ is convex on (a, b).
   (d) Suppose $\phi : (a, b) \to \mathbf{R}$ is continuous and
$$\phi\left(\frac{x + y}{2}\right) \leq \frac{\phi(x) + \phi(y)}{2}.$$
Show that ϕ is convex. Show that the conclusion does not follow if continuity is omitted from the hypothesis.

Date: due on Monday, January 28, 2008 (before class).