HOMEWORK II

FUNCTIONAL ANALYSIS

(1) Describe all the convex subsets of $\mathbf{R}$.
(2) Describe all the nowhere dense convex subsets of $\mathbf{R}^2$.
(3) Suppose $f : [0, \infty) \to [0, \infty)$ is continuous on $[0, \infty)$ and twice continuously differentiable on $(0, \infty)$ such that $f'(x) \geq 0$ for all $x > 0$, and $f''(x) \leq 0$ for all $x > 0$. Show that $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$.
(4) Show that any locally convex topological vector space where the topology is given by a countable sufficient family of seminorms is a metric space.
(5) Show that the topology of any metrizable locally convex topological vector space is given by a countable sufficient family of seminorms.
(6) Show that any linear isomorphism of finite dimensional locally convex topological spaces is a homeomorphism.
(7) Let l^2 denote the vector space consisting of sequences $x = \{x_n\}_{n=1}^{\infty}$ in K such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$, topologised by the norm $\|x\|_2 = (\sum_{n=1}^{\infty} |x_n|^2)^{-\frac{1}{2}}$. Show that the subset consisting of all sequences $\{x_n\}$ satisfying the condition
$$\sum_{n=1}^{\infty} n^2 |x_n|^2 < 1$$
is a convex balanced set which is not absorbing.
(8) Let l^{∞} denote the vector space consisting of bounded sequences $x = \{x_n\}_{n=1}^{\infty}$ in K, topologised by the norm $\|x\|_{\infty} = \sup_n \{|x_n|\}$. Show that the subspace consisting of sequences that vanish at all but finitely many points is a convex balanced set which is not absorbing.
(9) Let G be a compact abelian metric topological group (i.e., G is a metric space with an abelian group structure such that the multiplication map $G \times G \to G$ given by $(g, g') \mapsto gg'$ and the inversion map $G \to G$ given by $g \mapsto g^{-1}$ are continuous) such that every open subset of G containing the identity element contains a compact, open subgroup. Suppose that for every $g \in G$ there exists a positive integer n_g such that $g^{n_g} = e$ (here, e denotes the identity element of G). Show that there exists a positive integer n such that $g^n = e$ for every $g \in G$; in other words every compact torsion metric group has bounded order [Hint: For each positive integer m, consider the subgroup of elements such that $g^m = 1$. Apply Baire's theorem to this countable family of subgroups]. Why does this argument break down when G is not abelian? [1]

Date: due on Monday, January 21, 2008 (before class).

[1] I learned this from George Willis. He mentioned that the corresponding problem remains open for groups which are not abelian.