HOMEWORK XIII

FUNCTIONAL ANALYSIS

(1) Deduce Hahn-Banach theorem for complex vector spaces from the Hahn-Banach theorem for real vector spaces.

(2) In the proof of the Banach-Alaoglu theorem, where E is a normed linear space over K (K is $\mathbf{R}$ or $\mathbf{C}$), $B_x = \{x \in K \mid |c| \leq \|x\|\}$, check that $i : E^* \to \prod_{x \in E} B_x$ defined by

$$i(\lambda) = (\lambda(x))_{x \in E}$$

is a homeomorphism of E^* onto its image, when E^* has the weak* topology and $\prod B_x$ has the product topology.

(3) Show that there is no translation invariant probability measure on $\mathbf{R}$ (i.e., a probability measure μ on the Borel sigma field for which $\mu(x + E) = \mu(E)$ for every measurable set E and every $x \in \mathbf{R}$.

(4) Let E be a Banach space. Show that any sequence in E that converges in the weak topology is bounded. Also, any sequence in E^* that converges in the weak* topology is bounded (with respect to the operator norm). Hint: use the uniform boundedness principle.

(5) Let X be a finite measure space. For each $\phi \in L^\infty(X)$, let $M_\phi : L^2(X) \to L^2(X)$ be the operator

$$M_\phi(f)(x) = \phi(x)f(x).$$

Show that the weak operator topology on $L^\infty(X)$ is the same as the weak* topology (as the dual of $L^1(X)$).

(6) Let $\phi_1 : [0,1] \to [0,1]$ be the transformation $x \mapsto x^2$. Show that any probability measure on $[0,1]$ invariant under ϕ_1 must be supported on $\{0,1\}$. Now identify ϕ_1 with a homeomorphism of S^1 onto itself by identifying 0 and 1. Let ϕ_2 be any homeomorphism of S^1 which does not fix the image of $0 \in [0,1]$ in S^1. Show that there is no invariant probability measure on S^1 which is invariant under both ϕ_1 and ϕ_2 (thus, when a non-abelian group acts on a compact space, there may be no invariant probability measure),

(7) An extreme point of a convex set A is a point x such that whenever $x = tx_1 + (1-t)x_2$, with $x_1, x_2 \in A$ and $0 < t < 1$, then $x_1 = x_2 = x$. If X is a compact Hausdorff space, show that the extreme points of the set of probability measures on X is the set of probability measures which are supported at a single point (i.e., there is a single point with probability one).

(8) Let A be the set of all doubly stochastic matrices (matrices with positive real entries all of whose rows and columns add up to one). Show that A is convex, and that its extreme points are the permutation matrices (matrices which permute the standard basis vectors).

Date: due on Monday, April 21, 2008 (before class).