HOMEWORK XII

FUNCTIONAL ANALYSIS

(1) Show that $\pi : \mathbf{R}^n \mapsto U(L^2(\mathbf{R}^n))$ defined by
$$(\pi(\xi)f)(x) = e^{i\xi\cdot x}f(x)$$
is continuous when $U(L^2(\mathbf{R}^n))$ (the set of unitary operators on $L^2(\mathbf{R}^n)$) is given the strong operator topology.

(2) Let (X, μ) be a measure space, and let $\phi \in L^\infty(X)$. Define $M_\phi : L^p(X) \to L^p(X)$ by
$$(M_\phi f)(x) = \phi(x)f(x).$$
Show that, for $1 \le p \le \infty$, M_ϕ is bounded, and that $\|M_\phi\| = \|\phi\|_\infty$.

(3) Let $\phi \in L^1(\mathbf{R}^n)$. For every $f \in L^p(\mathbf{R}^n)$ show that $\phi * f \in L^p(\mathbf{R}^n)$. Define $C_\phi : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ by
$$C_\phi f = \phi * f.$$
Show that, for $1 \le p \le \infty$, C_ϕ is bounded, and that $\|C_\phi\| = \|\phi\|_1$.

(4) For $A \in GL_n(\mathbf{R})$, let T_A be the corresponding translation operator on functions on R^n: $T_A f(x) = f(Ax)$ for any $x \in \mathbf{R}^n$.
   (a) Show that T_A defines a bounded operator on the Sobolev space $W^{k,p}(\mathbf{R}^n)$ for each $1 \le p \le \infty$ and $k = 1, 2, \ldots$.
   (b) Show that T_A defines an isometry of the Hilbert space $W^{k,2}(\mathbf{R}^n)$ if and only if A is orthogonal.

(5) Let R^+ denote the multiplicative group of positive real numbers. Show that the measure $\mu(dx) = \frac{1}{x}dx$ is invariant under the action of R^+ on itself given by $\phi_a(x) = ax$ for all $a \in \mathbf{R}^+$.

(6) Let B denote the group of non-singular upper triangular 2×2 matrices with real entries. Identify B with $\mathbf{R}^* \times \mathbf{R} \times \mathbf{R}^*$ using
$$(a, b, c) \mapsto \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$
Show that the measure $d\mu(a, b, c) = \frac{1}{a^2c}\, da\, db\, dc$ is invariant under the action of B on itself given by $\phi_{\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} xa & xb + yc \\ 0 & zc \end{pmatrix}$.

(7) Show that the measure in the previous exercise is not invariant under the action of B on itself given by $\phi_{\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} ax & ay + bz \\ 0 & cz \end{pmatrix}$.

Date: due on Monday, April 14, 2008 (before class).