HOMEWORK XI

FUNCTIONAL ANALYSIS

(1) Let T_n be the operator on l^2 given by
$$T_n(x_1, x_2, \ldots) = (x_n, x_{n+1}, \ldots).$$
Show that $T_n \to 0$ in the strong operator topology, but T_n^* does not converge in the strong operator topology.

(2) Let $\mathcal{H}$ be a complex Hilbert space. Show that the map $\mathcal{H} \to L(\mathcal{H}, \mathbf{C})$ defined by $x \mapsto \lambda_x$, where
$$\lambda_x(y) = (y, x)$$
is an isometry (i.e., a bijection which preserves the metric) when $L(\mathcal{H}, \mathbf{C})$ is thought of as a metric space with the operator norm. Note that this isometry is not complex linear; it is conjugate linear.

(3) Let $\mathcal{H}$ be a Hilbert space. Suppose a sequence $\{T_n\}$ of bounded operators converges to the bounded operator T in the weak operator topology. Show that the sequence $\{T_n^*\}$ converges to T^* in the weak operator topology.

(4) Show that the composition map
$$L(X, Y) \times L(Y, Z) \to L(X, Z); \quad (T, S) \mapsto S \circ T$$
is continuous in the uniform topology (i.e., it is continuous when the left hand side has the product topology).

(5) Show that the composition map
$$L(X, Y) \times L(Y, Z) \to L(X, Z); \quad (T, S) \mapsto S \circ T$$
is continuous in each variable when the other is fixed and the operator spaces are given the strong operator topology.

(6) Show that the composition map
$$L(X, Y) \times L(Y, Z) \to L(X, Z); \quad (T, S) \mapsto S \circ T$$
is continuous in each variable when the other is fixed and the operator spaces are given the weak operator topology.

(7) Construct sequences $\{S_n\}$ and $\{T_n\}$ of operators such that $S_n \to 0$ and $T_n \to 0$ in the weak operator topology, but $S_n \circ T_n$ does not converge to 0. Conclude that composition of operators is not continuous in the weak operator topology (see Problem 6). Hint: revisit Problem 1.

(8) Let $\mathcal{H}$ be a Hilbert space, and consider the subset $U(\mathcal{H})$ of $L(\mathcal{H}, \mathcal{H})$ consisting of isometries of $\mathcal{H}$ (these are the *unitary operators*). Show that composition $U(\mathcal{H}) \times U(\mathcal{H}) \to U(\mathcal{H})$ given by $(S, T) \mapsto S \circ T$ and inversion $U(\mathcal{H}) \to U(\mathcal{H})$ given by $S \mapsto S^{-1}$ are continuous when $U(\mathcal{H})$ is given the subspace topology of the weak operator topology (in other words, $U(\mathcal{H})$ is a *topological group*).

(9) Show that the weak operator topology and strong operator topology are equivalent on $U(\mathcal{H})$ for every Hilbert space $\mathcal{H}$.

Date: due on Monday, April 7, 2008 (before class).