HOMEWORK I

FUNCTIONAL ANALYSIS

(1) Show that $C[0,1]$ (with the supremum norm) is a separable metric space (it has a countable dense subset).

(2) Let f and g be continuous real function defined on $[0,1]$. Suppose that for each $n = 0, 1, 2, \dots$

$$\int_0^1 f(x)x^n dx = \int_0^1 g(x)x^n dx.$$

Show that $f(x) = g(x)$ for all $x \in [0,1]$.

(3) Exhibit a complex-valued continuous function on the complex unit disc $\{z \in \mathbf{C} \ : \ |z| \leq 1\}$ which is not a uniform limit of polynomials. Which functions on the unit disc are uniform limits of polynomials?

(4) Suppose X is any locally compact, but non-compact Hausdorff topological space. Construct an algebra of real-valued continuous functions on X which separates points, but is not dense in $C(X)$ with respect to the supremum norm.

(5) Let $C(X;\mathbf{C})$ denote the algebra of continuous complex-valued functions on a compact Hausdorff topological space X. Prove the complex analogue of the Stone-Weierstrass theorem: a subalgebra of $C(X;\mathbf{C})$ is dense with respect to the supremum norm if and only if it separates points, and is closed under complex conjugation (i.e., when $f(x)$ is in the algebra, so is $\overline{f(x)}$).

(6) A trigonometric polynomial is a finite complex linear combination of the functions e^{inx}, $n \in \mathbf{Z}$. Prove the Weierstrass trigonometric approximation theorem: every complex-valued continuous periodic function with period 2π is a uniform limit of trigonometric polynomials.

(7) Let X and Y be compact Hausdorff topological spaces. Show that every continuous function on $X \times Y$ is a uniform limit of functions of the form $\sum_{i=1}^n f_i g_i$, where the f_i's are in $C(X)$ and the g_i's are in $C(Y)$.

(8) Show that the Cantor set is nowhere dense in $[0,1]$.

(9) Show that, for all $x_1, \dots, x_n \in \mathbf{C}$,

$$\lim_{p\to\infty} (|x_1|^p + \cdots + |x_n|^p)^{1/p} = \max\{|x_1|, \dots, |x_n|\}.$$

(10) For $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbf{C}^n$, show that

$$(|x_1 + y_1|^2 + \cdots + |x_n + y_n|^2)^{1/2} \leq (|x_1|^2 + \cdots + |x_n|^2)^{1/2} + (|y_1|^2 + \cdots + |y_n|^2)^{1/2}.$$

Date: due on Monday, January 14, 2008 (before class).