Automorphism Orbits of Subgroups in Finite Abelian Groups

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My Institute

My collaborators

  

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$$\text{Abelian groups of order } p^n \leftrightarrow \Lambda(n).$$

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If the above inequalities hold, we write:

$$\mu \subset \lambda$$

"containment order" on partitions

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$$(3,1),\ (2,1),\ (1,1),\ (3),\ (2),\ (1).$$

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Its subgroups are of type (1^k) for $0 \leq k \leq n$.

Counting subgroups

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We denote this number by

$$\binom{\lambda}{\mu}_p$$

Example

$$\binom{(1^n)}{(1^k)}_p = \text{no. of } k\text{-dimensional subspaces in } (\mathbf{Z}/p\mathbf{Z})^n$$
$$= \binom{n}{k}_p \text{ the Gaussian binomial coefficient}$$
$$= \frac{(p^n-1)(p^n-p)\cdots(p^n-p^{n-k+1})}{(p^k-1)(p^k-p)\cdots(p^k-p^{k-1})}$$
$$= \sum_{\lambda\subset((n-k)^k)} p^{|\lambda|}$$
$$= \sum_{D(\pi)\subset\{k\}} p^{\mathrm{inv}(\pi)}.$$

Note: $D(\pi)$ is the descent set of π.

Example

$$\binom{(5,4,3,2,1)}{(3,2,1)}_p = p^{12} + 3p^{11} + 6p^{10} + 7p^9 + 6p^8 + 3p^7 + p^6$$

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In general

- $\binom{\lambda}{\mu}_p$ is always a polynomial in p with non-negative integer coefficients.
- combinatorial interpretations are available (Butler - 1994).
- theory is well developed (related to Hall polynomials etc.)

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If A is a p-group of type λ, then $G_\lambda(p) = \mathrm{Aut}(A)$ is a group of order

$$p^{\sum_{i,j} \min(\lambda_i, \lambda_j)} \prod_{i=1}^{\infty} \prod_{j=1}^{m_i} (1 - p^{-j}),$$

where $\lambda = (1^{m_1} 2^{m_2} \cdots)$ (exponential notation).

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Or better still, develop a theory of G-orbits of structures in A

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Then $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ lie in the same G orbit if and only if

$$h(x_1, \ldots, x_k) = h(y_1, \ldots, y_k).$$

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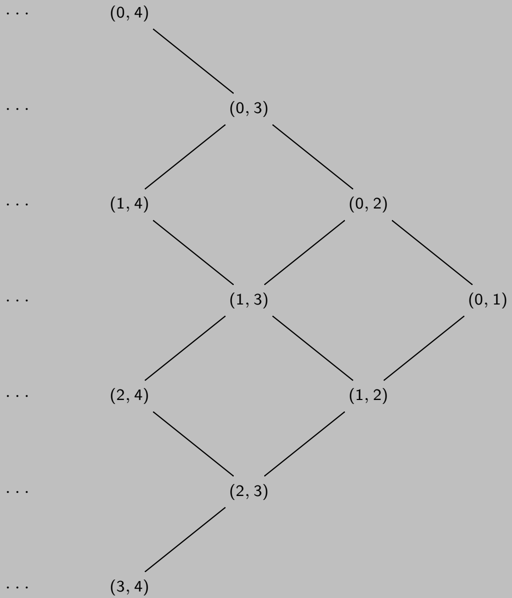
· · · (0, 4)
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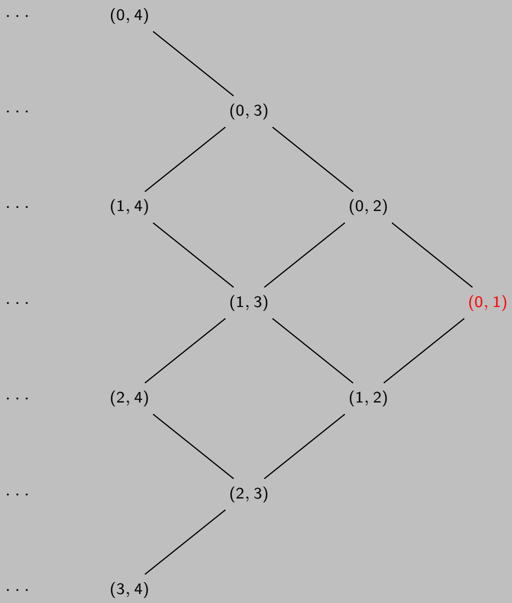

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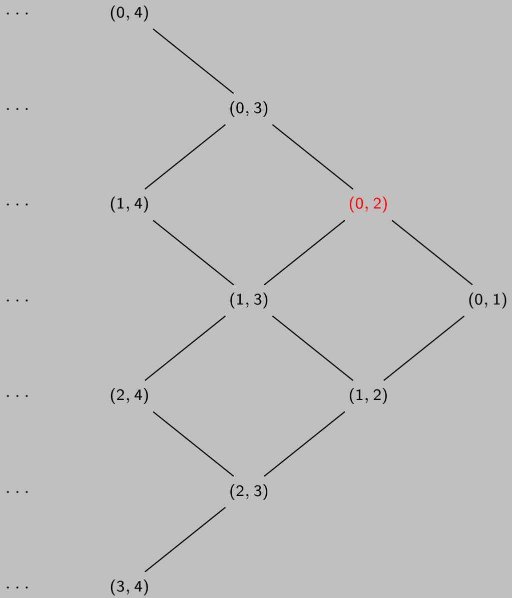

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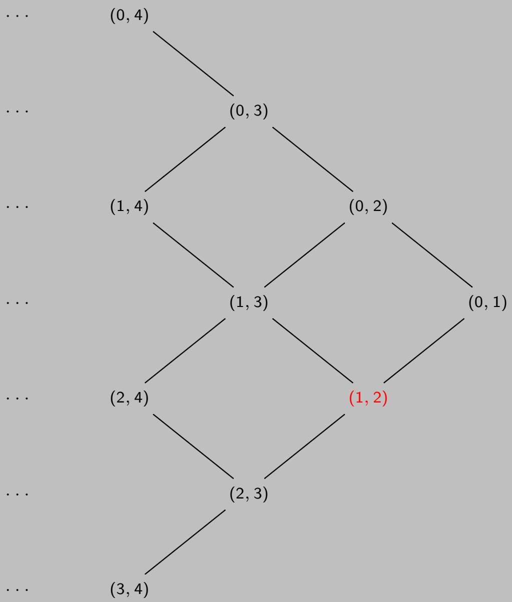

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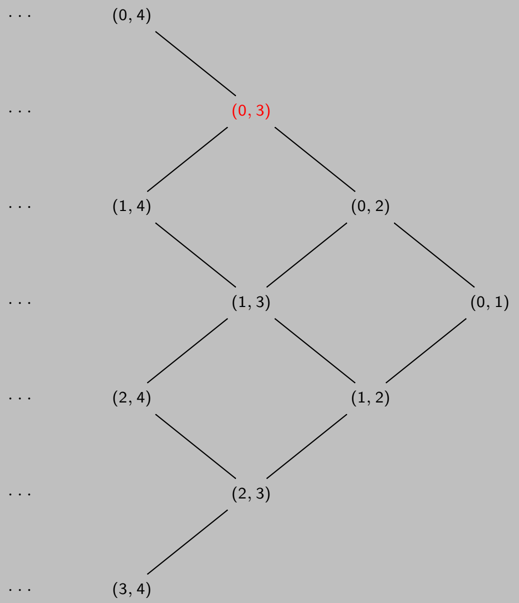

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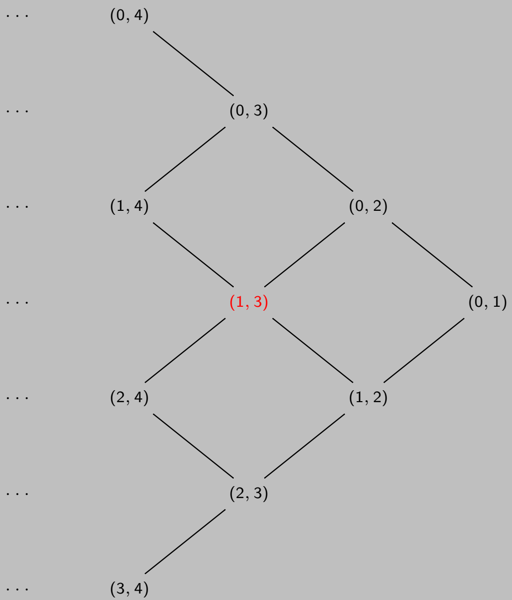

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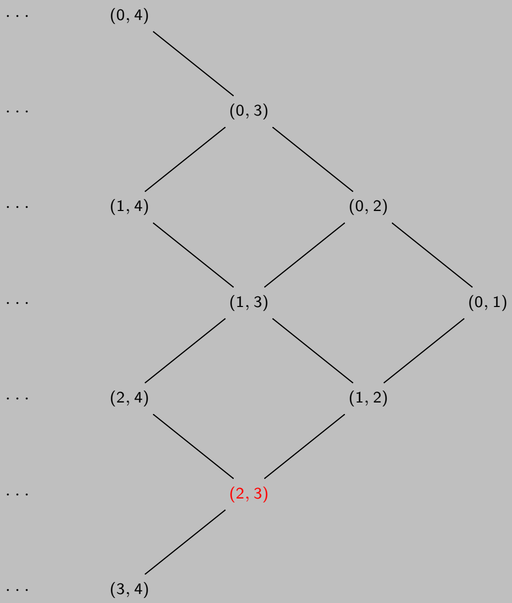

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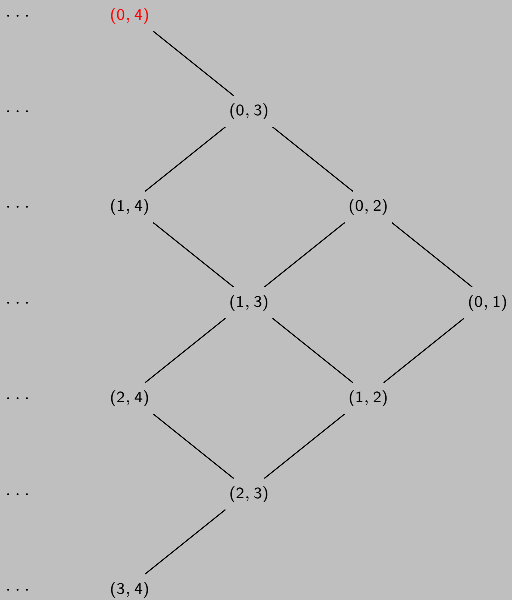

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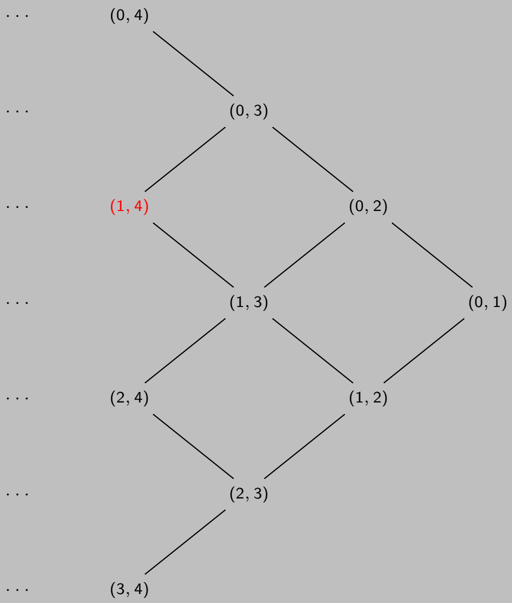

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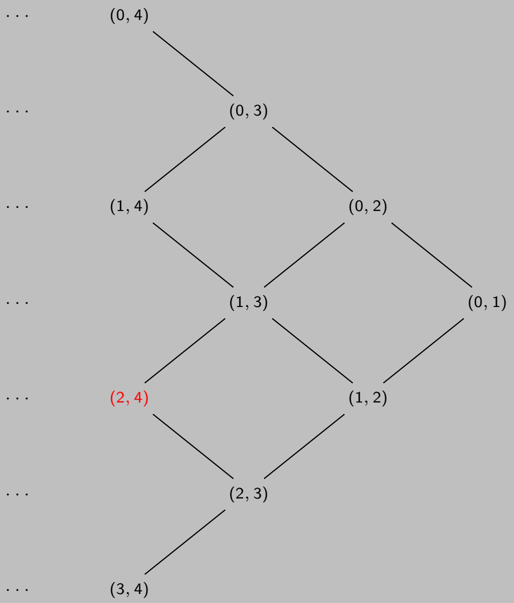
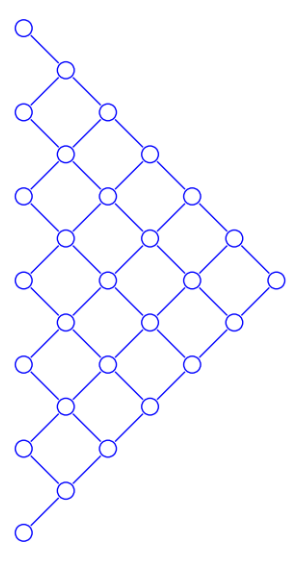


Ideal of an element

$$(p^2, 0, p, p^2, p, 1)$$
$$\lambda = (7, 6, 4, 3, 3, 1)$$



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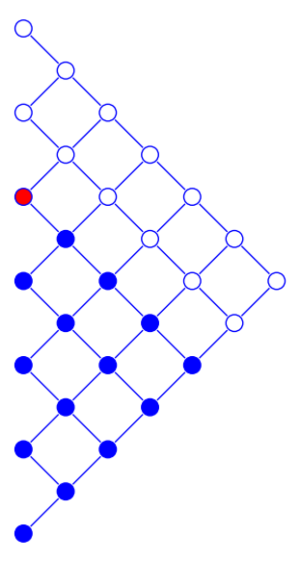

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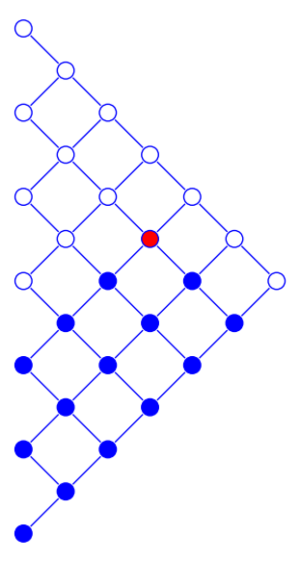

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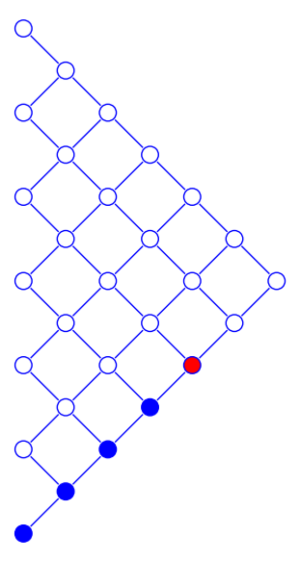

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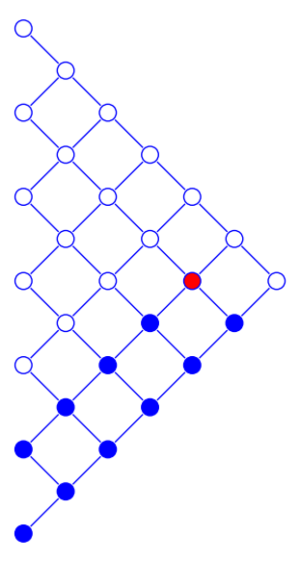

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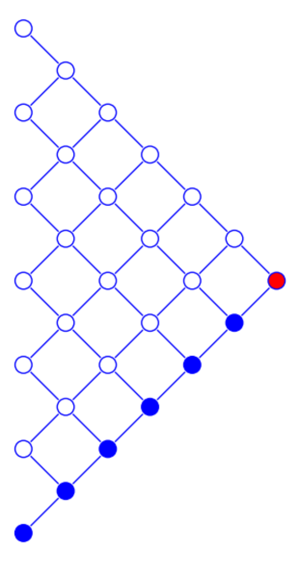

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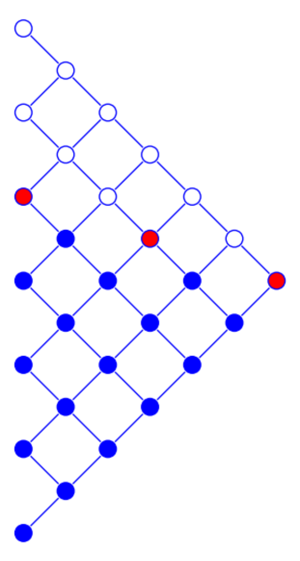

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Theorem (with Kunal Dutta)

Two elements $a, b \in A$ lie in the same G-orbit if and only if $I(a) = I(b)$.

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where $m_\lambda(k)$ is the number of times k occurs in λ. It is easy to see that (for A of type λ)

$$\beta_I := \#\{a \in A \mid I(a) \subset I\} = [I]_\lambda.$$

Using Möbius inversion on the lattice of order ideals in P, we get the cardinality of a G-orbit in A:

$$\alpha_I := \#\{a \in A \mid I(a) = I\} = [I]_\lambda \prod_{(v,k)\in \max I} (1 - p^{-m_\lambda(k)}).$$

In the example

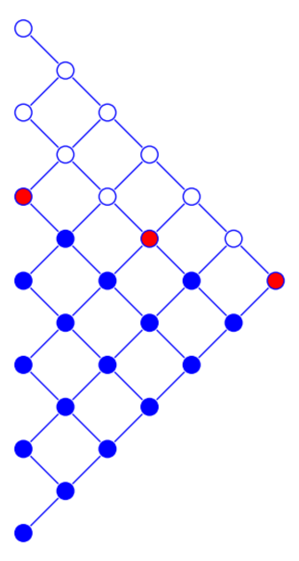
$$a = (p^2, 0, p, p^2, p, 1)$$

$$\lambda = (7, 6, 4, 3, 3, 1)$$

$$[I]_\lambda = 5 + 4 + 3 + 2 \times 2 + 1$$

$$\max I = \{(2,7), (1,4), (0,1)\}$$

$$\alpha_I = p^{17}(1 - p^{-1})^2(1 - p^{-2})$$



Orbits of pairs

Theorem (with C. P. Anilkumar)

For every partition λ, there exists a monic polynomial $n_\lambda(t)$ with integer coefficients with degree λ_1 such that, for any prime p, if A is an abelian p-group of type λ, then the number of G-orbits in $A \times A$ is given by

$$|G \backslash (A \times A)| = n_\lambda(p).$$

(1)	$t+2$
(2)	t^2+2t+2
$(1,1)$	$t+3$
(3)	t^3+2t^2+2t+2
$(2,1)$	t^2+5t+5
$(1,1,1)$	$t+3$
(4)	$t^4+2t^3+2t^2+2t+2$
$(3,1)$	t^3+5t^2+7t+4
$(2,2)$	t^2+3t+5
$(2,1,1)$	t^2+5t+6
$(1,1,1,1)$	$t+3$
(5)	$t^5+2t^4+2t^3+2t^2+2t+2$
$(4,1)$	$t^4+5t^3+7t^2+6t+4$
$(3,2)$	$t^3+5t^2+10t+7$
$(3,1,1)$	t^3+5t^2+8t+6
$(2,2,1)$	t^2+6t+8
$(2,1,1,1)$	t^2+5t+6
$(1,1,1,1,1)$	$t+3$

Open Conjecture

For every partition λ, $n_\lambda(t)$ has non-negative integer coefficients.

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To access raw data visit:
`http://www.imsc.res.in/~amri/pairs/`

A general qualitative result

Let A be a finite abelian group.

Definition

Say that a tuple $\mathbf{a} = (a_1, \ldots, a_k)$ *degenerates to* $\mathbf{b} = (b_1, \ldots, b_k)$ *(denoted* $\mathbf{a} \to \mathbf{b}$ *if there exists an endomorphism* $\phi : A \to A$ *such that*

$$\phi(a_i) = b_i \text{ for } i = 1, \ldots, k.$$

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Clearly, if two tuples or subgroups lie in the same G-orbit, then they degenerate to each other.

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Theorem (with Wesley Calvert and Kunal Dutta)

If tuples $\mathbf{a} \to \mathbf{b}$ *and* $\mathbf{b} \to \mathbf{a}$, *then* $\mathbf{a}$ *and* $\mathbf{b}$ *lie in the same* G*-orbit.*

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The proof is based on a proof of Mackey and Kaplansky of Ulm's theorem, which provides a classification of countable reduced torsion abelian groups. The result therefore holds in their more general setting. Nevertheless, it seems to be non-trivial even for finite abelian groups.

Consequence

Degeneration descends to a poset structure on the set of G-orbits of tuples or subgroups.

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