Experimental Mathematics with Python and Sage

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27 February 2016

Binomial Coefficients

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sage: [len([k for k in range(n+1) if binomial(n, k)%2 ==1])
for n in range(20)]
[1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16, 2, 4, 4, 8]
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Write numbers in binary and count number of 1's

```
0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011,...
0, 1,  1,  2,   1,   2,   2,   4,    1,    2,    2,    3,...
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Let $\nu(n)$ denote the number of 1's in binary expansion of n.

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```

Let $\nu(n)$ denote the number of 1's in binary expansion of n.

Conjecture

No. of odd binomial coefficients of the form $\binom{n}{k}$ for fixed n is $2^{\nu(n)}$.

Binomial coefficients in Pascal's triangle (Meru Prastaara)

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$\binom{n}{k}$ = number of paths from the apex to the kth position in the nth row.

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1 3 3 1

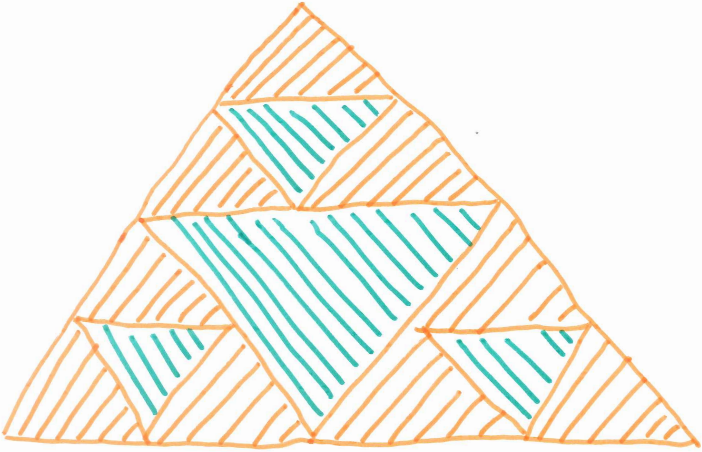
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Binomial coefficients in Pascal's triangle (Meru Prastaara)



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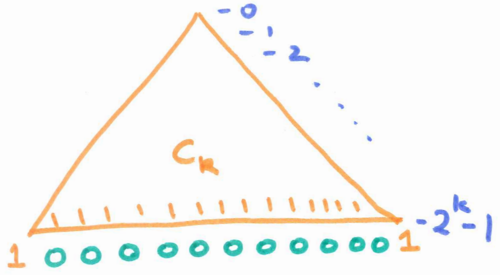

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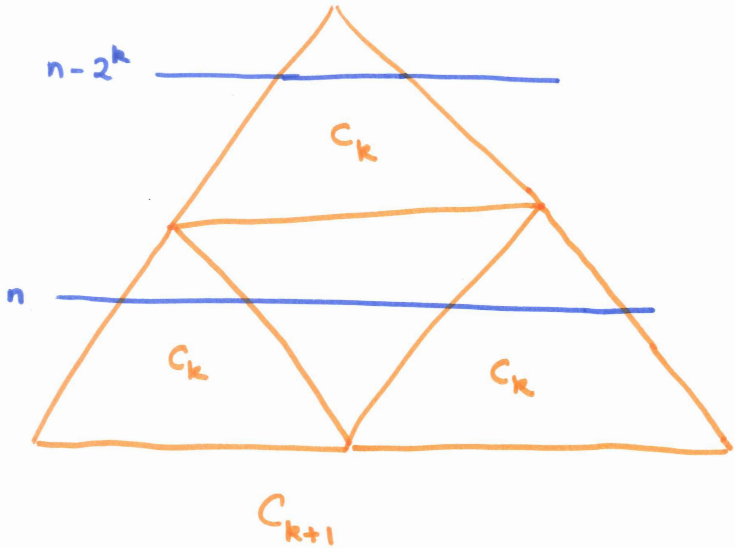

Recurrence Relation

If $2^k \leq n < 2^{k+1}$,

$$a_n = 2a_{n-2^k}.$$

$$a_{42} = 2 \times a_{10} = 2 \times 2 \times a_2 = 2 \times 2 \times 2.$$

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In terms of binary expansions:

```
42 = 101010
10 =   1010
 2 =     10
```

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In terms of binary expansions:

```
42 = 101010
10 =   1010
 2 =     10
```

Total number of times you double is the number of times 1 appears in the binary expansion.

Integer Partition

$$
\begin{aligned}
5 &= 5 \\
&= 4 + 1 \\
&= 3 + 2 \\
&= 3 + 1 + 1 \\
&= 2 + 2 + 1 \\
&= 2 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

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\begin{aligned}
5 &= 5 \\
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&= 2 + 2 + 1 \\
&= 2 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

$$p(5) = 7$$

```
sage: [number_of_partitions(n) for n in range(19)]
[1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231,
 297, 385]
```

Big Open Problem in Mathematics

Find an exact closed formula for $p(n)$

Hardy and Ramanujan

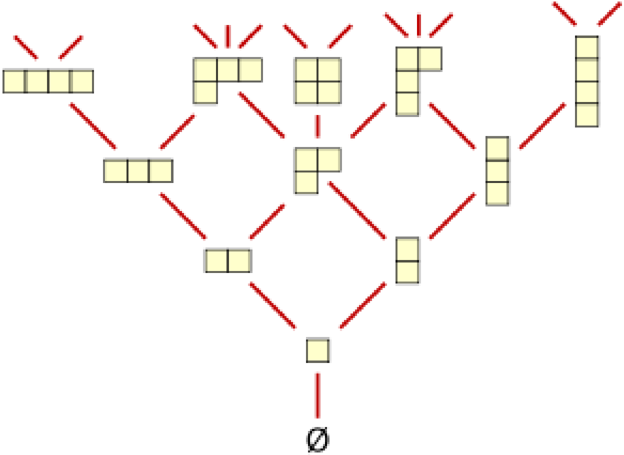

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}) \text{ as } n \to \infty.$$

```
sage: number_of_partitions(100000)
2749351056977569651267751632098635268817342931598005475
8203125984302147328114964173055050741660736621590157844
7742962489404930630702004617927644930335101160793424571
9015571894350972531246610845200636955893446424871682878
9832182345009262853831404597021307130674510624419227311
2389997022844086093709355316296978515695698921961084801
5860056942109851 9
```

is based on the Hardy-Ramanujan-Rademacher formula:

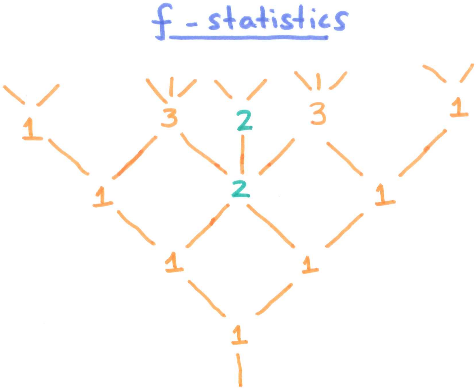
$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k}\, A_k(n)\, \frac{d}{dn} \left(\frac{1}{\sqrt{n - \frac{1}{24}}} \sinh\left[\frac{\pi}{k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)} \right] \right)$$

Young's lattice



The f-statistic of a partition

$$f_\lambda = \text{Number of paths from the apex to } \lambda \text{ in Young's lattice}$$



How often in f_λ odd?

```
sage: [len([la for la in Partitions(n) if la.dimension()%2 == 1])
for n in range(19)]
[1, 1, 2, 2, 4, 4, 8, 8, 8, 8, 16, 16, 32, 32, 64, 64, 16, 16, 32]
```

How often in f_λ odd?

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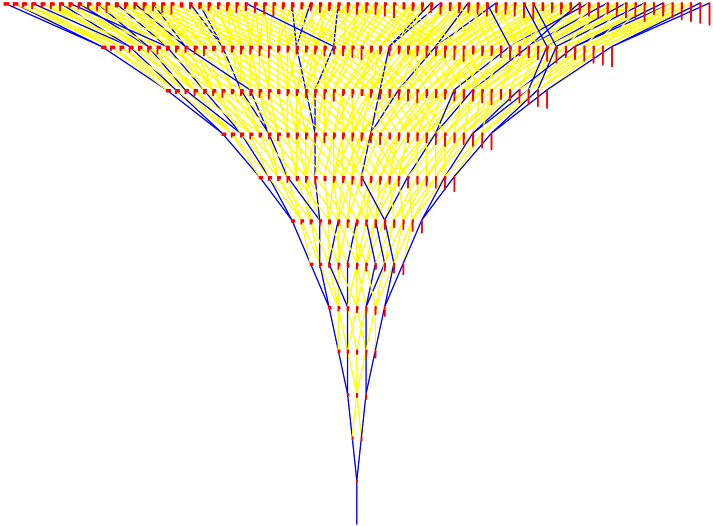
sage: [len([la for la in Partitions(n) if la.dimension()%2 == 1])
for n in range(19)]
[1, 1, 2, 2, 4, 4, 8, 8, 8, 8, 16, 16, 32, 32, 64, 64, 16, 16, 32]
```

Powers of 2 again!



Odd Partitions in Young's lattice form a binary tree

If the binary expansion of n has 1's in the places $k_1, k_2, \ldots, k_r$, number of partitions of n for which f_λ is odd is:

$$c_n = s^{k_1+k_2+\cdots+k_r}$$

Example

$$n = 42 = 101010 = 2^5 + 2^3 + 2^1$$

$$c_{42} = 2^{5+3+1} = 2^8 = 256.$$

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Mathematics > Combinatorics

Odd partitions in Young's lattice

Arvind Ayyer, Amritanshu Prasad, Steven Spallone

(Submitted on 8 Jan 2016)

We show that the Hasse diagram of the subposet of Young's lattice consisting of partitions with an odd number of standard Young tableaux is a binary tree. This tree exhibits self-similarities at all scales, and has a simple recursive description.

Comments:	13 pages, 3 figures
Subjects:	**Combinatorics (math.CO)**; Representation Theory (math.RT)
MSC classes:	05A15, 05A17, 05E10, 20C30
Cite as:	arXiv:1601.01776 [math.CO]
	(or arXiv:1601.01776v1 [math.CO] for this version)

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