HOMEWORK II

ANALYSIS I

(1) Let $p(x) = \alpha_0 x^m + \alpha_1 x^{m-1} + \cdots \alpha_m$. Show that the sequence $p(n)$ converges to ∞ if $\alpha_0 > 0$ and to $-\infty$ if $\alpha_n < 0$.
(2) For which real numbers does the sequence $\sin(n\theta\pi)$ converge? For which real numbers does it not converge? Give complete proofs in each case.
(3) Prove that $\sin(n!\theta\pi)$ converges to 0 for all rational values of θ. What can you say about this sequence when θ is irrational?
(4) If $N \geq 1753$ then the number of days in the year N A .D. is
$\lim_{n\to\infty} \{365 + (\cos^2(N\pi/4))^n - (\cos^2(N\pi/100))^n + (\cos^2(N\pi/400))^n\}$.
(5) Euler gave the sum of the series
$$\cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$$
as zero on the ground that
$$z + z^2 + z^3 + \cdots = \frac{z}{1-z}$$
and
$$1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \frac{z}{z-1}.$$
What was the error in his reasoning?
(6) Prove the triangle inequality for complex numbers, i.e., that for any two complex numbers z_1 and z_2,
$$|z_1 + z_2| \leq |z_1| + |z_2|.$$
(7) Show that every real number with a recurring decimal expansion is rational.
(8) Let $r = \frac{p}{q}$ and assume that q and 10 have no common factors.
   (a) Show that there exists a positive integer n such that $10^n - 1$ is divisible by q.
   (b) Conclude that r can be expressed in the form
   $$\frac{P}{10^n - 1} = \frac{P}{10^n} + \frac{P}{10^{2n}} + \cdots ,$$
   i.e., a pure recurring decimal with n figures.
(9) Show that every rational number has either a finite decimal expansion or a recurring decimal expansion.
(10) Prove that
$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Date: due on 24 August 2004.