TOWARDS A SEMICLASSICAL THEORY FOR QUANTUM HALL SKYRMIONS

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by

Dutta Sreedhar Babu

Institute of Mathematical Sciences

Madras

UNIVERSITY OF MADRAS

MADRAS 600 005

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DECLARATION

I declare that the thesis entitled " Towards a Semiclassical Theory for Quantum Hall Skyrmions" submitted by me for the Degree of Doctor of Philosophy is the record of work carried out by me during the period from November 1996 to October 2000 under the guidance of Dr. R. Shankar and has not formed the basis for the award of any degree, diploma, associateship, fellowship, titles in this or any other University or other similar Institution of Higher Learning.

Sreedhar

(Dutta Sreedhar Babu)

CERTIFICATE FROM THE SUPERVISOR

I certify that the thesis entitled " Towards a Semiclassical Theory for Quantum Hall Skyrmions" submitted for the Degree of Doctor of Philosophy by Mr. Dutta Sreedhar Babu is the record of research work carried out by him during the period from November 1996 to October 2000 under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

Shankar R

(R. Shankar)

Reader, Theoretical Physics.

Institute of Mathematical Sciences.

Madras 600-113.

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The world I live in, where order is a quest, randomness a routine and probability almost always precipitates into murphy, six years back offered me, upon mild provocation, a neatly set new deck of cards. For a change, fortune and jokers didn't team up and I had the best possible time ever since.

Had I not met Shankar, I believe, I would have been a victim of the butterfly effect. When the chips were down he didn't let me fall into the bottom of the valley. I thank-you him a zillion. I also acknowledge him for teaching me Physics, Patience and Clarity. I apologize to him for my tendency to digress, for missing deadlines and for all those avoidable mistakes.

Contents

Chapter 1

Introduction

1.1 Some significant developments of FQHE

The last two decades, beginning with the discovery of quantum Hall effect by von Klitzing, Dorda and Pepper[1], witnessed the emergence of QHE, both integral and fractional, as one of the major branches of Condensed matter Physics.

The experimental observations on two dimensional electron systems subjected to a strong magnetic field are - a nearly vanishing longitudinal resistance and a quantized Hall conductance, σ_H to an extremely high degree of accuracy independent of the magnetic field and the microscopic details like geometry and purity of the sample.

$$\sigma_H = \nu \frac{e^2}{h} \tag{1.1}$$

The plateaus observed in the hall resistance is the hallmark of IQHE. Our understanding of this phenomenon comes from adiabatically deforming interacting electrons in presence of disorder into noninteracting electron states. Further disorder localizes almost all states except those that are peaked around landau energy levels of an electron in a magnetic field. Hence the hall conductance remains flat when electrons get added to the localized states.

Fractional quantum Hall effect(FQHE) was discovered in 1982 by Tsui, Stormer and Gossard [2]. Its phenomenology is same as that of integral quantum Hall state in almost every detail. There is a plateau. Both longitudinal resistance and conductance in the plateau is zero. The Hall conductance in the plateau is quantized in units of $\frac{e^2}{h}$. The only qualitative difference between the two effects is the quantum of Hall conductance, which is a fraction ($\nu = \frac{1}{3}$ for the one observed by Tsui et. al.) instead of an integer. Yet the fractional quantum Hall state is not adiabatically deformable to any non-interacting electron state but is an unprecedented strongly correlated new state of matter.

Laughlin [3] proposed the following variational ground state to describe a correlated incompressible electron liquid at all filling factors $\nu = \frac{1}{2n+1}$, and not only succeeded in explaining the $\frac{1}{3}$ effect observed by Tsui et. al., but also predicted a fractional quantum Hall state at $\frac{1}{5}$, which was later found experimentally by Chang et. al[4].

$$\psi_L(\{z_i\}) = \prod_{i>j} f(z_i - z_j)\, e^{-\frac{1}{4}\sum_i |z_i|^2} \tag{1.2}$$

The lowest Landau level dictates f to be a analytic function, electrons force f to antisymmetrize and conservation of angular momentum makes $\prod_{i>j} f(z_i - z_j)$ a homogeneous polynomial of degree M, where M is the total angular momentum. $f(z)$ then becomes $z^{(2n+1)}$ and is left really with no variational parameters. When the norm of Laughlin wavefunction is written as a partition function of a classical statistical mechanics problem

$$|\psi_L(\{z_i\})|^2 \; = \; e^{-\beta V(\{z_i\})} \tag{1.3}$$

where $\beta = \frac{1}{2n+1}$, then $V(\{z_i\})$ becomes the classical potential energy for plasma in two dimensions; particles of charge $(2n+1)$ interacting with a logarithmic potential in a uniform neutralizing background. Plasma costs a huge amount of energy unless neutral everywhere. Neutrality condition determines the density of electrons, $\bar{\rho} = \frac{1}{2\pi(2n+1)}$. Hence any deviation from $\bar{\rho}$ makes the norm of Laughlin wavefunction almost vanish and thus in turn makes the electrons to be in an incompressible liquid state.

The $\frac{1}{2n+1}$ states also support quasiparticle excitations which are obtained by adiabatically piercing or removing an infinitesimally thin solenoid carrying one flux quantum. The wavefunctions for these quasiparticles, given by Laughlin[3], are found to carry fractional charge. The construction of quasihole suggested not only a different interpretation of Laughlin wavefunction but also generalizations valid for other filling factors.

An important feature of Laughlin state is presence of the gap in its excitation spectrum. Girvin et. al. [6] obtained the gap for the neutral intra-Landau collective modes of FQHE (magnetophonons) using single-mode approximation projected onto Lowest Landau Level(LLL). This gap agreed with the excitation gap evaluated from numerical diagonalization by Haldane and Rezayi [7]. Also the sum of the energies of quasihole and quasielectron is nearly equal to that of the collective mode, which can be viewed as a quasiparticle-quasihole bound state.

Girvin and MacDonald[5] observed that the Laughlin states have algebraic off-diagonal long range order (ODLRO). Performing a singular gauge transformation on the Laughlin wavefunction, they obtained a bosonic wavefunction and showed that it has ODLRO. Since ODLRO is the hallmark of superfluidity, its existence suggested the idea that there should be a Landau-Ginzburg theory for FQHE. Read[9] constructed an order parameter which is a composite of electron creation operator and $\frac{1}{\nu}$ Laughlin quasihole operator, and showed that this operator has a non-vanishing expectation value between the Laughlin ground states and also proposed a classical Landau-Ginzburg theory for FQHE. Zhang, Hanson and Kivelson [8] mapped the interacting electrons in an external magnetic field onto an interacting boson problem with an additional gauge interaction described by Chern-Simons field and constructed a Landau-Ginzburg theory and also derived Laughlin's wavefunction and Girvin-MacDonald's ODLRO.

An electron in two dimensions can be viewed as a composite boson which is a composite of charged boson with odd number of flux quanta, $\phi_0 = \frac{hc}{e}$, attached to it. When two composite bosons are moved around each other, each boson sees the flux attached to the other and picks up Aharonov-Bohm phase which accounts for the statistics of the original electrons. Incompressibility of electrons at $\nu = \frac{1}{2n+1}$ has a nice picturization in composite boson theory. If we attach $(2n+1)\phi_0$ flux to each boson in the direction opposite to the external magnetic field and uniformly smear the flux tubes, which is taking a mean-field approximation, then the net effect is charged bosons see no magnetic field and form a Bose-Einstein condensate. Any change in the local density causes flux penetration into this charged Bose superfluid and gets expelled out due to Meissner effect. Hence the particle density remains constant and electron fluid is incompressible.

Jain[10] introduced the composite fermions wherein an even number ($2n$) of flux quanta are attached to an electron. The composite fermions then experience, to a mean-field approximation, a magnetic field $B^* = B - 2n\rho\phi_0$ instead of the external field, B. This led to adiabatically deforming a ground state at filling factor ν to that at ν^* which are related as

$$\nu = \frac{\nu^*}{2n\nu^* + 1} \tag{1.4}$$

and to an understanding of FQHE at other filling factors.

1.2 Spin and Skyrmions

1.2.1 Theory

The Zeeman energy of a free electron (gyromagnetic ratio $g = 2$) is exactly equal to the cyclotron energy and hence, the spins get frozen when the electrons are restricted to LLL and their kinetic energy quenched. However in GaAs, materials in which quantum Hall systems are realised, the small effective mass in the conduction band increases the cyclotron energy by a factor of 14. Also the effective coupling of the spin to the external magnetic field is reduced by a factor of -5 (i.e., $g = -0.4$) making the Zeeman energy about 70 times smaller than the cyclotron energy and turning spin into a relevant degree of freedom.

Initial calculation by Chakraborty and Zhang[11] showed that the low energy quasiparticles in experimental systems are spin reversed. The incompressible ground states (at filling factor $\nu = \frac{1}{2n+1}$) are strong ferromagnets and, hence, the low lying spin excitations could be described by the spin field in Non-linear Sigma model (NLSM). Apart from the spin wave excitations there exist higher energy topologically non-trivial textures in the spin field known as Skyrmions. Lee and Kane[12] extended the Landau-Ginzburg Chern-Simons theory to include the spin of electrons and then argued that the quasiparticles could be extended spin textures that are described by topological solitons of the non-linear sigma model (NLSM): skyrmions. Sondhi et. al. [13] studied the quasiparticles at $\nu = 1$ by varying the ratio of Zeeman energy to interaction energy(g^*) and found a change in their character. When g^* is large the quasiparticles are spin flip excitations and for small g^* the relevant quasiparticles are skyrmions. Unlike the spin wave excitations (magnons) which are neutral, skyrmions are charged low-lying spin excitations and exhibit a nontrivial spin order. At the boundary of a skyrmion the local spin takes the ground state value while it is reversed at the center and along any radius it interpolates smoothly between these two limits.

The effective NLSM used by Sondhi et. al [13] is

$$\mathcal{L}_{eff}(x) = \alpha\mathcal{A}(\vec{n})\cdot\partial_t\vec{n} + \alpha'\nabla\vec{n}\cdot\nabla\vec{n} + g\bar{\rho}\mu\vec{n}(x)\cdot\vec{B} - \frac{1}{2}\int_{x'} q(x)V(x-x')q(x') \quad (1.5)$$

where $\vec{n}(x)$ is a unit vector pointing in the direction of spin polarization, $\mathcal{A}$ is the vector potential of a unit monopole, $q(x)$ is the topological charge density and α, α' are parameters fixed by requiring that it yields the correct spin wave dispersion and that it describes correctly the uniform precession of the ferromagnet in a tilted magnetic field. The scale invariant second term is due to the exchange part of the Coulomb interaction. The first three terms would be present for any ferromagnet; however the last term, which is responsible for the macroscopic character of the skyrmion, is specific to the quantum Hall problem and is obtained from the Coulomb term upon replacing the density fluctuation $\delta\rho(x)$ by the topological charge density $q(x)$. This spin charge relation is also verified by Moon et. al. [14] with more than heuristic arguments. The size of the skyrmion is dictated by the competition between the Zeeman term and the Coulomb term. The gap for creating a skyrmion-antiskyrmion pair at $g^* = 0$ is half the gap for creating a single-particle excitations [13]; hence the skyrmions are the relevant quasiparticles in GaAs semiconductors where g^* is small.

The skyrmions in terms of electrons, $c(x)$, can be constructed by rotating spins locally into a hegde-hog $\hat{n}(x) \equiv (\theta(x), \phi(x))$ where θ depends only on $|x|$ while $\phi(x) = \phi$, $|x|$ and ϕ are polar coordinates of x. Under such a rotation

$$c_{\sigma'}(x) \to c'_{\sigma}(x) = U_{\sigma\sigma'}c_{\sigma'}(x) \;\; ; \qquad U = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2})e^{-i\phi} \\ -\sin(\frac{\theta}{2})e^{i\phi} & \cos(\frac{\theta}{2}) \end{pmatrix}$$

When restricted to the lowest Landau level we get

$$c'_{l\sigma} = u_l c_{l\uparrow} + v_l c_{l+1\downarrow}$$

where

$$u_l = \int_x \varphi_l^*(x)\cos(\frac{\theta}{2})\varphi_l(x) \qquad v_l = \int_x \varphi_l^*(x)\sin(\frac{\theta}{2})e^{-i\phi}\varphi_{l+1}(x)$$

and $\varphi_l(x)$ are the LLL eigenfunctions. Fertig. et. al.[15] took the following state $|\psi_-\rangle$ with u_l and v_l as the variational parameters and showed by

Hartree-Fock calculations that the skyrmion energy is always smaller than the localized spin $\frac{1}{2}$ quasiparticle energy suggesting that adding or removing charge from a filled Landau level rapidly degrades its spin polarization.

$$|\psi_-\rangle \equiv \prod_{l=0}^{\infty} c'^{\dagger}_{l\uparrow} |0\rangle = \prod_{l=0}^{\infty} \left(u_l c^{\dagger}_{l\uparrow} + v_l c^{\dagger}_{l+1\downarrow} \right) |0\rangle \tag{1.6}$$

Abolfath et. al. [19] have compared energies of single skyrmion states obtained from the classical NLSM with Hartree-Fock approximation, exact diagonalization calculations and many-body trial wavefunctions. They have numerically evaluated saddle point solution of NLSM, including the Coulomb and the Zeeman terms, to obtain the energy. They have also calculated the energy from exact diagonalization of Hamiltonian for 8 particles in a spherical geometry containing 9 LLL orbitals. They calculated energies using Hartree-Fock approximation on $|\psi_-\rangle$ states given in equation (1.6) taking a few thousand particles. Finally they considered modified variational $|\psi_-\rangle_{mod}$, so as to make spin and not just total angular momentum a good quantum number, and estimated the energies. Their calculations using various techniques reveal that for large skyrmion states (with spin quantum numbers greater than 10) NLSM approach agrees well with the results obtained by the other methods and raises questions which are stated in the next section.

1.2.2 Experiment

Barrett et. al.[16] measured the Knight shift of ^{71}Ga nuclei located in n-doped GaAs quantum wells using optically pumped NMR, for filling factor $0.66 < \nu < 1.76$ and temperatures $1.55K < T < 20K$. The nuclear resonance frequency depends on effective magnetic field as seen by the nuclear moment. These nuclear moments when coupled to the conducting electrons, through hyperfine interaction, see an additional magnetic field proportional to the polarization of the electrons. This displacement in resonance, called knight shift, measures the spin polarization. The knight shift data was the first direct measurement of the electron spin polarization in a quantum hall system, as a function of ν and temperature, and shows that the quasiparticles carry a spin more than 3.5 times that of an electron which agrees well with Fertig. et. al.[15] results.

Schmeller et. al.[18] employed tilted field magneto-transport measurements of the energy gap at $\nu = 1$, 3 and 5 and determined the spin of thermally excited quasielectron-quasihole pairs. At $\nu = 1$ their data show that as many as 7 electron spin flips accompany such excitations while at $\nu = 3$ and 5 only a single spin flips. Aifer et. al. [17] also provided the experimental evidence for the presence of skyrmions by magnetoabsorption spectroscopic observation of the rapid decay in spin alignment over small changes in magnetic field. More recently ([20], [21]) skyrmions with spin as large as 50 have been observed by drastically reducing g^*. All these experiments support the prediction that the charged excitations of the $\nu = 1$ ground state are finite-size skyrmions, with an effective spin reflecting the competition between the Coulomb energy and the Zeeman energy.

1.3 An overview of the thesis

In this thesis we address the following questions and answer them.

1. What is the limit in which this classical approximation is exact?

2. When does the theory itself become Classical (i.e., not just the energetics but the kinematical description itself)?

3. What is that Classical theory?

4. How do we get the LLL Classical theory?

5. How do we give a semiclassical description to quantum skyrmions?

6. What is the effect of Landau level mixing on skyrmions?

In chapter 2, we work in the composite boson formalism [8] as it is the obvious choice of the formalism to use to address the question of the classical limit. Since we want to impose the LLL condition, which is a condition on the quantum states, we develop a coherent state formalism for composite bosons. We systematically do the bosonization wherein the electronic theory is written in terms of bosons attached to fluxes of Chern-Simons fields. The electron operators and the procedure of flux attachment becomes very transparent in terms of coherent states. So we first define the Hilbert space of the composite bosons and construct the coherent state basis for it. We then construct gauge invariant anticommuting operators that create and annihilate flux carrying bosons which are then used to represent the electron creation and annihilation operators and, thus, obtain the explicit mapping between the electronic Hilbert space and observables and the gauge invariant states and observables in the bosonic theory. We also derive the path-integral representation of the evolution operator and obtain the standard Chern-Simons gauge field theory coupled to matter field.

In chapter 3, we project the composite boson coherent states on to the gauge invariant sector and calculate their wave functions. The wavefunctions unveil many interesting aspects. The coherent states which are labeled by the values of the bosonic spinor field and the Chern-Simons gauge field when projected to the gauge invariant sector depend only on gauge invariant combinations of these fields. The wavefunctions reveal another local invariance that relates the transverse part of gauge field to the magnitude of the bosonic fields. Though this transformation is not unitarily realised, it nevertheless implies that the physical states can be labeled by a single bosonic spinor field. We also find LLL condition is equivalent to an analyticity condition on the parameters labeling the coherent states.

In chapter 4, we evaluate the matrix elements of the observables between the projected coherent states in the limit of small amplitude, long-wavelength density fluctuations, which we refer to as the hydrodynamic limit. We find that electric charge density is proportional to the topological charge density if and only if the LLL condition is satisfied. We also study the effect of Landau level mixing on the spin charge relation.

In chapter 5, we examine the quasiparticle states which can be described by projected coherent states. We show (in the hydrodynamic limit) that the projected coherent states can be labeled by physical observables, that they can be resolved into unity and that their overlaps are nonorthogonal and continuous in the parameters labeling them. Thus we find the projected coherent states to satisfy the properties of generalized coherent states parameterized by the values of physical observables. This gauge invariant coherent state basis enables a gauge invariant bosonization.

In chapter 6, we give a classical theory for composite bosons. We get NLSM with a Hopf term upon imposing the LLL condition in this classical theory. We show that if we consider the set of states corresponding to classical configurations characterized by a length scale-λ, then they become orthogonal in the limit of $\lambda \to \infty$. This implies that a system of Skyrmions will behave classically in the limit of their sizes going to infinity and thus we are able to prove that the classical description of the Skyrmion is exact in the limit of large Skyrmions. We make use of classical-quantum correspondence of the coherent states to derive the ground states of quantum hall system and get them to be Laughlin states.

In chapter 7, we give a complete kinematical description of the LLL skyrmions in terms of their collective coordinates. We determine the overlaps for these skyrmions and also construct finite spin skyrmions.

We conclude by summarizing our work presented in the thesis in chapter 8.

Chapter 2

Composite Bosons

In this chapter, we do the usual bosonization wherein the electronic theory is written in terms of bosons attached to fluxes of Chern-Simons fields. The electron operators and the procedure of flux attachment is very transparent in terms of coherent states. So we first define the Hilbert space of the composite bosons and construct the coherent state basis for it. We derive the path-integral representation of the time evolution operator in gauge-invariant subspace of this composite boson Hilbert space to see the equivalence with the standard Chern-Simons Lagrangian. We then construct the electron operators and obtain the explicit mapping between the electronic Hilbert space and observables and the gauge invariant states and observables in the bosonic theory.

2.1 The Composite Boson Hilbert Space

The bosonic degrees of freedom are described by the spinor field operators $\hat{\varphi}_\sigma(x)$, $\hat{\varphi}^\dagger_\sigma(x)$, $\sigma = 1,2$ and the Chern-Simons gauge fields, $a_i(x)$, $i = 1,2$. The $\hat{\varphi}$ operators act on the Hilbert space, $\mathcal{H}_B$ and satisfy the canonical commutation relations,

$$\begin{aligned} \left[\hat{\varphi}_\sigma(x), \hat{\varphi}^\dagger_{\sigma'}(y)\right] &= \delta_{\sigma\sigma'}\delta^2(x-y) \\ \left[\hat{\varphi}_\sigma(x), \hat{\varphi}_{\sigma'}(y)\right] &= \left[\hat{\varphi}^\dagger_\sigma(x), \hat{\varphi}^\dagger_{\sigma'}(y)\right] = 0 \end{aligned} \tag{2.1}$$

The Chern-Simons gauge fields act on the Hilbert space $\mathcal{H}_{CS}$ and satisfy,

$$[a_i(x), a_j(y)] = \sqrt{\frac{\hbar c}{\kappa}}\, \epsilon_{ij}\delta^2(x-y) \tag{2.2}$$

where $\kappa = \frac{e^2}{2\pi\hbar c(2n+1)}$. If we define the complex fields, $a(x)$ and $\bar{a}(x)$ as,

$$\begin{aligned} a(x) &\equiv \frac{a_2(x) + i a_1(x)}{\sqrt{2}}\sqrt{\frac{\kappa}{\hbar c}} \\ \bar{a}(x) &\equiv \frac{a_2(x) - i a_1(x)}{\sqrt{2}}\sqrt{\frac{\kappa}{\hbar c}} \end{aligned} \tag{2.3}$$

then the commutation relation between them is given by,

$$[a(x), \bar{a}(y)] = \delta^2(x-y) \tag{2.4}$$

The full Hilbert space of the composite boson theory is the direct product of the above two spaces and we denote it by,

$$\mathcal{H}_{CB} = \mathcal{H}_B \otimes \mathcal{H}_{CS} \tag{2.5}$$

We denote the gauge invariant sector of this space by $\mathcal{H}_{phy} \subset \mathcal{H}_{CB}$. $\mathcal{H}_{phy}$ consists of the states which respect the Chern-Simons Gauss law constraint,

$$\hat{G}(x)\ |\psi\rangle_{phy} = 0 \tag{2.6}$$

where $\hat{G}(x)$ are the generators of gauge transformations given by,

$$\hat{G}(x) = \kappa \nabla \times \vec{a}(x) - e\hat{\varphi}^{\dagger}(x)\hat{\varphi}(x) \tag{2.7}$$

We will refer to the gauge invariant observables, the operators that commute with $\hat{G}(x)$ as physical observables.

2.2 Coherent State Basis

In this section, we construct the coherent state basis for $\mathcal{H}_{CB}$. The displacement operators are defined to be,

$$\begin{aligned} D(\alpha) &\equiv e^{\int_x [\alpha(x)\bar{a}(x) - \bar{\alpha}(x)a(x)]} \\ U(\varphi) &\equiv e^{\int_x [\varphi(x)\hat{\varphi}(x)^{\dagger} - \bar{\varphi}(x)\hat{\varphi}(x)]} \end{aligned} \tag{2.8}$$

where, $\alpha(x) \equiv \frac{\alpha_2(x)+i\alpha_1(x)}{\sqrt{2}}\sqrt{\frac{\kappa}{\hbar c}}$. The coherent states $|\alpha, \varphi\rangle$, parameterized by the gauge field $\alpha(x)$ and the spinor field $\varphi(x)$ are then given by,

$$|\alpha, \varphi\rangle \equiv U(\varphi)D(\alpha)|0\rangle \tag{2.9}$$

where,

$$a(x)|0\rangle = \hat{\varphi}_{\sigma}(x)|0\rangle = 0 \tag{2.10}$$

The states defined in equation(2.9) can be interpreted as gaussian wave packets peaked around the classical field configuration $(\alpha(x), \varphi(x))$. They satisfy the three standard properties of coherent states [23] namely,

1. Resolution of unity:

$$\int \mathcal{D}[\alpha, \varphi]\ |\alpha, \varphi\rangle\langle\alpha, \varphi| = I \tag{2.11}$$

where $\mathcal{D}[\alpha,\varphi] = \prod_{x,\sigma} \frac{d\alpha(x)d\bar{\alpha}(x)}{2\pi i}\frac{d\varphi_\sigma(x)d\bar{\varphi}_\sigma(x)}{2\pi i}$

2. Continuity of overlaps:

$$\begin{aligned}\langle\alpha_1,\varphi_1|\alpha_2,\varphi_2\rangle &= e^{-\frac{1}{2}\frac{\kappa}{\hbar c}\int_x \vec{\alpha}_1(x)\times\vec{\alpha}_2(x)}e^{-\frac{1}{4}\frac{\kappa}{\hbar c}\int_x(\vec{\alpha}_1(x)-\vec{\alpha}_2(x))^2} \\ &\quad .e^{\frac{1}{2}\int_x[\bar{\varphi}_1(x)\varphi_2(x)-\varphi_1(x)\bar{\varphi}_2(x)]}e^{-\frac{1}{2}\int_x|\varphi_1(x)-\varphi_2(x)|^2} \qquad (2.12)\end{aligned}$$

3. Values of Observables:

$$\langle\alpha,\varphi| : O(a,\bar{a},\hat{\varphi},\hat{\varphi}^\dagger) : |\alpha,\varphi\rangle = O(\alpha,\bar{\alpha},\varphi,\varphi^\dagger) \qquad (2.13)$$

The coherent states are not gauge invariant. Under gauge transformations,

$$\begin{aligned}|\alpha,\varphi\rangle &\to e^{\frac{i}{\hbar c}\int \hat{G}(x)\Omega(x)}\,|\alpha,\varphi\rangle \\ &= e^{\frac{i}{2}\frac{\kappa}{\hbar c}\int_x \vec{\alpha}(x)\times\nabla\Omega(x)}\,|\alpha-\nabla\Omega,\varphi e^{-\frac{ie}{\hbar c}\Omega}\rangle \qquad (2.14)\end{aligned}$$

We will now construct a projection operator that projects any state into the gauge invariant subspace, $\mathcal{H}_{phy}$. Consider,

$$P \equiv \frac{1}{V_G}\int_\Omega e^{\frac{i}{\hbar c}\int_x \Omega(x)\hat{G}(x)} \qquad (2.15)$$

where $\hat{G}(x)$ is the generator of gauge transformations, as given in equation(2.7) and $V_G = \int_\Omega$, is the volume of the gauge group. Shifting the integration variable Ω by β in the projection operator P,

$$e^{\frac{i}{\hbar c}\int_x \beta(x)\hat{G}(x)}P = P \qquad (2.16)$$

Taking $\beta \to 0$,

$$\hat{G}(x)P = 0 \;\Rightarrow\; \hat{G}(x)P|\psi\rangle = 0 \qquad (2.17)$$

This proves that P is an operator that projects any state into $\mathcal{H}_{phy}$.

The above three properties (2.11 - 2.13) and the projection operator defined in equation(2.15) can be used to derive the path integral representation of the gauge invariant evolution operator.

2.3 The path integral representation

In this section we derive the path integral representation of the partition function by splitting the time interval t into N segments of length ϵ and take the limit $\epsilon \to 0$, $N \to \infty$ such that $\epsilon N = t$. And at each intermediate step we insert the resolution of identity (3.2) of $\mathcal{H}_{phy}$,

$$Z = Tr e^{\frac{-i}{\hbar}Ht} = Tr[e^{\frac{-i}{\hbar}\epsilon H}]^N \tag{2.18}$$

$$Z = \prod_{n=0}^{N} \int_{\alpha_n, \varphi_n, \Omega_n} \prod_{n=0}^{N} \langle \alpha_{n+1}\varphi_{n+1} | P e^{\frac{-i}{\hbar}H\epsilon} P | \alpha_n \varphi_n \rangle \tag{2.19}$$

where $(\alpha_{N+1}, \varphi_{N+1}) \equiv (\alpha_0, \varphi_0)$

Since H commutes with P and $P^2 = P$, after explicitly acting P on $|\alpha_n \varphi_n\rangle$ and making use of gauge invariance of $|\alpha\varphi\rangle\langle\alpha\varphi|$, we get

$$Z = \prod_{n=0}^{N} \int_{\alpha_n, \varphi_n, \Omega_n} \prod_{n=0}^{N} t_n \tag{2.20}$$

where,

$$t_n = e^{\frac{i}{2}\frac{\kappa}{\hbar c}\int_x \vec{\alpha}_n(x) \times \nabla\Omega_n(x)} \langle \alpha_{n+1} - \nabla\beta_{n+1}, \varphi_{n+1} e^{-\frac{i\kappa}{\hbar c}\beta_{n+1}} | e^{\frac{-i}{\hbar}\epsilon H} | \alpha_n - \nabla\Omega_n - \nabla\beta_n, \varphi_n e^{-\frac{i\kappa}{\hbar c}(\Omega_n + \beta_n)} \rangle \tag{2.21}$$

To the order ϵ: $\alpha_{n+1} = \alpha_n + \epsilon\dot{\alpha}_n$, $\Omega_{n+1} = \Omega_n + \epsilon\dot{\Omega}_n$ and $\varphi_{n+1} = \varphi_n + \epsilon\dot{\varphi}_n$. And if we choose $\beta_{n+1} = \beta_n + \Omega_{n+1}$ then to the order $O(\epsilon)$ t_n is:

$$t_n \;=\; e^{\frac{i}{2}\frac{\kappa}{\hbar c}\int_x \vec{\alpha}_n(x) \times \nabla\Omega_n(x)}$$

$$\langle \alpha_n - \nabla\beta_n - \nabla\Omega_n + \epsilon(\dot{\alpha}_n - \nabla\dot{\Omega}_n), \{\varphi_n + \epsilon(\dot{\varphi}_n - \frac{ie}{\hbar c}\varphi_n\dot{\Omega}_n)\}e^{-\frac{ie}{\hbar c}(\beta_n+\Omega_n)}| e^{\frac{-i}{\hbar}\epsilon H}$$
$$|\alpha_n - \nabla\beta_n - \nabla\Omega_n, \varphi_n e^{-\frac{ie}{\hbar c}(\Omega_n+\beta_n)}\rangle \qquad (2.22)$$

Using the fact that $\epsilon\dot{\beta}_n = \beta_n - \beta_{n-1} + O(\epsilon^2)$ and

$$\langle\alpha+\delta\alpha, \varphi+\delta\varphi| e^{\frac{-i}{\hbar}\epsilon H}|\alpha,\varphi\rangle = \langle\alpha+\delta\alpha,\varphi+\delta\varphi|\alpha,\varphi\rangle[1 - i\frac{\epsilon}{\hbar}\langle\alpha\varphi|H|\alpha\varphi\rangle] + O(\epsilon^2) \qquad (2.23)$$

where $\delta\alpha \sim O(\epsilon)$ and $\delta\varphi \sim O(\epsilon)$ the above expression for t_n , after defining $\alpha_{0n} \equiv \dot{\Omega}_n/c$, and making use of gauge invariance of H, we get

$$t_n = \exp[\frac{i\epsilon}{\hbar}\int_x \{-\frac{\kappa}{2}\epsilon_{\mu\nu\lambda}\alpha_n^\mu(x)\partial^\nu\alpha_n^\lambda(x) + e\alpha_{0n}(x)\bar{\varphi}_n(x)\varphi_n(x) - i\hbar\frac{1}{2}[\varphi_n(x)\dot{\bar{\varphi}}_n(x) - \bar{\varphi}_n(x)\dot{\varphi}_n(x)] - H(\varphi_n, \alpha_n)\}] \qquad (2.24)$$

If we now take the limit $\epsilon \to 0$ the partition function becomes (after calling α by a)

$$Z = \int \mathcal{D}[a_0(x,t)]\mathcal{D}[a_i(x,t)]\mathcal{D}[\varphi(x,t)]e^{\frac{i}{\hbar}\int dt d^2x \mathcal{L}(x,t)} \qquad (2.25)$$

where $\mathcal{L}(x,t)$ is the standard lagrangian of matter fields coupled to Chern-Simons gauge fields. This confirms the equivalence of our formalism to the standard lagrangian formalism.

2.4 Bosonization

We will now construct gauge invariant anticommuting operators that create and annihilate flux carrying bosons. These operators satisfy the fermionic canonical anticommutation relations and can hence be used to represent the electron creation and annihilation operators in $\mathcal{H}_{CB}$. We will thus be able to map the gauge invariant sector of the composite boson Hilbert space, $\mathcal{H}_{phy}$, to the Hilbert space of the electronic system, $\mathcal{H}_{el}$. The mapping is then used to map the observables of the electronic system to gauge invariant operators in $\mathcal{H}_{CB}$.

We define $c^\dagger_\sigma(x)$ as

$$c^\dagger_\sigma(x) \equiv D(x)\hat{\varphi}^\dagger_\sigma(x)K(x) \tag{2.26}$$

We have used $D(x)$ as short notation for $D(\alpha^v_x)$. α^v_x is the classical configuration of a vortex with a delta function flux density at the point x.

$$\kappa\nabla \times \vec{\alpha}^v_x(z) = e\delta^2(z-x) \tag{2.27}$$

$D(x)$ therefore creates a gaussian wave packet peaked around this classical vortex configuration. When $c^\dagger(x)$ acts on a state, $\hat{\varphi}^\dagger_\sigma(x)$ creates a bosonic particle at x and $D(x)$ attaches Chern-Simons flux to it. The operator $K(x)$ gives the Aharanov-Bohm phase corresponding to all the other particles already present in the state. It is defined as,

$$K(x) \equiv e^{i(2n+1)\int_z \theta(x-z)\hat{\varphi}^\dagger(z)\hat{\varphi}(z)} \tag{2.28}$$

where $\theta(x)$ is the angle the vector, x, makes with the x-axis.

Using the commutation relations given in equations(2.1 and 2.2), it can be verified that the following canonical anti-commutation relations hold good.

$$\{c_\sigma(x), c^\dagger_{\sigma'}(y)\} \;=\; \delta_{\sigma\sigma'}\delta^2(x-y) \tag{2.29}$$

$$\{c_\sigma(x), c_{\sigma'}(y)\} = \{c^\dagger_\sigma(x), c^\dagger_{\sigma'}(y)\} = 0 \tag{2.30}$$

Hence $c_\sigma^\dagger(x)$ and $c_\sigma(x)$ provide a representation of the electron creation and annihilation operators in $\mathcal{H}_{CB}$.

Under gauge transformations,

$$\begin{aligned}\hat{\varphi}_\sigma(x) \to e^{i\frac{e}{\hbar c}\Omega(x)}\hat{\varphi}_\sigma(x) \quad &, \quad \hat{\varphi}_\sigma^\dagger(x) \to e^{-i\frac{e}{\hbar c}\Omega(x)}\hat{\varphi}_\sigma^\dagger(x) \\ a_i(x) \to a_i(x) + \partial_i\Omega(x) \quad &, \quad D(x) \to e^{i\frac{e}{\hbar c}\Omega(x)}D(x)\end{aligned} \tag{2.31}$$

We see that $c_\sigma(x)$ and $c_\sigma^\dagger(x)$ are gauge invariant.

We are now in a position to map $\mathcal{H}_{el}$ into $\mathcal{H}_{phy}$. We map the state with 0 number of electrons, $|0\rangle_{el}$, to the vacuum state of $\mathcal{H}_{CB}$, defined in equation(2.10), projected to $\mathcal{H}_{phy}$.

$$|0>_{el} \;\; \to \;\; P|0\rangle \tag{2.32}$$

Since $c_\sigma(x)$ are gauge invariant, they commute with P. Then from equation(2.10) it follows that,

$$c_\sigma(x)P|0\rangle \; = \; 0 \tag{2.33}$$

The state with N electrons at $(x_1, x_2, ...x_N)$ with spins $(\sigma_1, \sigma_2, ..., \sigma_N)$, $|\{x_n, \sigma_n\}_N\rangle$, is then mapped onto,

$$|\{x_n, \sigma_n\}_N\rangle \;\; \to \;\; \prod_{n=1}^{N} c_{\sigma_n}^\dagger(x_n)P|0\rangle \tag{2.34}$$

Since the states in the RHS of equations(2.32) and (2.34) form a basis for $\mathcal{H}_{el}$, these equations specify the explicit mapping of $\mathcal{H}_{el}$ into $\mathcal{H}_{phy}$.

It is now easy to map the observables as well. The density is given by,

$$\hat{\rho}(x) = c_\sigma^\dagger(x)c_\sigma(x) = \hat{\varphi}_\sigma^\dagger(x)\hat{\varphi}_\sigma(x) \tag{2.35}$$

The spin density is,

$$\hat{S}^a(x) = \frac{1}{2} c_\sigma^\dagger(x) \tau_{\sigma\sigma'} c_{\sigma'}(x) = \frac{1}{2} \hat{\varphi}_\sigma^\dagger(x) \tau_{\sigma\sigma'} \hat{\varphi}_\sigma(x) \tag{2.36}$$

The current density is,

$$\begin{aligned}
\hat{J}_i(x) &= \frac{1}{2}(c_\sigma^\dagger(x)[-i\hbar\partial_i - \frac{e}{c}A_i(x)]c_\sigma(x) + h.c) \\
&= \frac{1}{2}(\hat{\varphi}_\sigma^\dagger(x)[-i\hbar\partial_i - \frac{e}{c}a_i(x) - \frac{e}{c}A_i(x)]\hat{\varphi}_\sigma(x) - \\
&\quad \frac{1}{c}\hat{\varphi}_\sigma^\dagger(x)\hat{\varphi}_\sigma(x)\int_z \alpha_{xi}^v(z)\hat{G}(z) + h.c)
\end{aligned} \tag{2.37}$$

The last term acting on physical states is zero. Thus for matrix elements between physical states, we have,

$$\hat{J}_i(x) = \frac{1}{2}\hat{\varphi}_\sigma^\dagger(x)[-i\hbar\partial_i - \frac{e}{c}a_i(x) - \frac{e}{c}A_i(x)]\hat{\varphi}_\sigma(x) + h.c \tag{2.38}$$

Similarly, the kinetic energy density, $\hat{\mathcal{T}}(x)$ is computed to be,

$$\hat{\mathcal{T}}(x) = \frac{1}{2m}\hat{\varphi}_\sigma^\dagger(x)[-i\hbar\partial_i - \frac{e}{c}a_i(x) - \frac{e}{c}A_i(x)]^2\hat{\varphi}_\sigma(x) \tag{2.39}$$

Chapter 3

Coherent State Wavefunctions

In this chapter, we study the coherent states projected onto $\mathcal{H}_{phy}$. We show that these states form a basis of $\mathcal{H}_{phy}$. Their wavefunctions and expectation values of observables are computed. The LLL condition can then be seen to be equivalent to an analyticity condition on the parameters. We then discuss the relation between the charge density and the topological charge density. Finally, we describe the parametrization of the projected coherent states in terms of a single complex spinor field $W_\sigma(x)$ discussed in the end of section 1 and derive expressions for the observables in terms of $W_\sigma(x)$

3.1 Projected coherent states

Consider the set of coherent states, projected to $\mathcal{H}_{phy}$,

$$|\alpha, \varphi\rangle_p \equiv P|\alpha, \varphi\rangle \tag{3.1}$$

Using the fact that $P^2 = P$ and equation(2.11), we have,

$$\int \mathcal{D}[\alpha]\mathcal{D}[\varphi^\dagger]\mathcal{D}[\varphi] \; |\alpha,\varphi\rangle_{pp}\langle\alpha,\varphi| \;=\; P \tag{3.2}$$

P is the identity operator in $\mathcal{H}_{phy}$ so the projected coherent states form a basis for it.

The coherent states are not eigenstates of the number operator. Thus they have a non-zero overlap with states containing any number of particles. The wavefunction in the N particle sector is the overlap with the states given in equation(2.34),

$$\psi_N(\{x_i,\sigma_i\}) \;=\; \langle\{x_i,\sigma_i\}_N|\alpha,\varphi\rangle_p \tag{3.3}$$

Using equations (2.15), (2.26) and (2.34) we get,

$$\psi_N(\{x_i,\sigma_i\}) = \frac{1}{V_G}\int_\Omega e^{\frac{1}{2}\frac{\kappa}{\hbar c}\int_x \vec{a}(x)\times\nabla\Omega(x)}\langle 0|\prod_{i=1}^{N}[K(x_i)\hat{\varphi}_{\sigma_i}(x_i)D^\dagger(x_i)]|\alpha-\nabla\Omega,\varphi e^{-\frac{ie}{\hbar c}\Omega}\rangle \tag{3.4}$$

Using the fact that,

$$\hat{\varphi}_{\sigma_i}(x_i)K(x_j) = e^{i(2n+1)\theta(x_i-x_j)}K(x_j)\hat{\varphi}_{\sigma_i}(x_i) \tag{3.5}$$

and $\langle 0|K(x) = 0$, we can pull all the K's to the left and rewrite equation (3.4) as,

$$\begin{aligned}\psi_N(\{x_i,\sigma_i\}) \;=\;& e^{-\frac{1}{2}\int_x|\varphi(x)|^2}\prod_{i>j}e^{i(2n+1)\theta(x_i-x_j)}\frac{1}{V_G}\int_\Omega e^{\frac{1}{2}\frac{\kappa}{\hbar c}\int_x\Omega(x)(\nabla\times\vec{a}(x))}\\ &\prod_{i=1}^{N}\varphi_{\sigma_i}(x_i)e^{-\frac{ie}{\hbar c}\sum_{i=1}^{N}\Omega(x_i)}e^{-\frac{1}{2}\frac{\kappa}{\hbar c}\int_x(\sum_{i=1}^{N}\vec{a}_i^v(x))\times(\vec{a}(x)-\nabla\Omega(x))}\\ &e^{-\frac{1}{4}\frac{\kappa}{\hbar c}\int_x(\vec{a}(x)-\nabla\Omega(x)-\sum_{i=1}^{N}\vec{a}_i^v(x))^2}\end{aligned} \tag{3.6}$$

We now write,

$$\alpha_m^i(x) = \epsilon^{ij}\partial_j f_m(x) \tag{3.7}$$

where,

$$-\kappa\nabla^2 f_m(x) = e\delta(x - x_m) \tag{3.8}$$

for $m = 1 - N$.

The zero momentum mode of the Ω integral will make the wavefunction vanish unless the total number of flux quanta equals the total number of particles. i.e.

$$\kappa \int_x \nabla \times \vec{\alpha}(x) = eN \tag{3.9}$$

The Ω integral is gaussian for the other modes and can be done exactly to give,

$$const \times exp[-\frac{1}{4}\frac{\kappa}{\hbar c}\int_x (\vec{\alpha}(x) - \sum_{i=1}^{N} \vec{\alpha}_i(x))^2] \tag{3.10}$$

We then write the wavefunction as,

$$\begin{aligned}
\psi_N(\{x_i, \sigma_i\}) \quad = \quad & const \times e^{-\frac{1}{2}\int_x |\varphi(x)|^2} \prod_{i>j} [e^{i\theta(x_i - x_j)}]^{2n+1} \\
& \prod_{i=1}^{N} [\varphi_{\sigma_i}(x_i) e^{-i\frac{e}{hbarc}\Omega_L(x)}] e^{-\frac{1}{2}\frac{\kappa}{\hbar c}\int_x (\vec{\alpha}(x) - \sum_{i=1}^{N} \vec{\sigma}_i(x))^2}
\end{aligned} \tag{3.11}$$

We also have,

$$\begin{aligned}
\int_x \vec{\alpha}(x).\vec{\alpha}_m(x) \quad &= \quad \int_x \nabla\Omega_T(x).\nabla f_m(x) \\
&= \quad -\int_x \Omega_T(x)\nabla^2 f_m(x) \\
&= \quad \frac{e}{\kappa}\Omega_T(x_m)
\end{aligned} \tag{3.12}$$

Using the fact that the solution of equation (3.8) is,

$$f_m(x) = -\frac{e}{\kappa}\frac{1}{2\pi} \ln|x - x_m| \tag{3.13}$$

and proceeding as in equation (3.12), we have,

$$
\begin{aligned}
\int_x \vec{\alpha}_m(x).\vec{\alpha}_n(x) &= \frac{e}{\kappa} f_m(x_n) \\
&= -\frac{e^2}{\kappa^2}\frac{1}{2\pi}\ln|x_m - x_n| \qquad (3.14)
\end{aligned}
$$

So we have the result,

$$
\begin{aligned}
\int_x (\vec{\alpha}(x) - \sum_{i=1}^{N}\vec{\alpha}_i(x))^2 &= \int_x \nabla\Omega_T(x).\nabla\Omega_T(x) - \frac{e^2}{\kappa^2}\frac{1}{2\pi}\sum_{m\neq n}^{N}\ln|x_m - x_n| \\
&\quad -2\frac{e}{\kappa}\sum_{m=1}^{N}\Omega_T(x_m) + const \qquad (3.15)
\end{aligned}
$$

The (infinite) *const* comes from the $m = n$ terms

Finally the wavefunction is written as,

$$
\begin{aligned}
\psi_N(\{x_i,\sigma_i\}) &= const \times e^{-\frac{1}{2}\int_x |\varphi(x)|^2} e^{-\frac{1}{2}\frac{\kappa}{\hbar c}\int_x \nabla\Omega_T(x)\cdot\nabla\Omega_T(x)} \\
&\quad \times \prod_{i>j}(z_i - z_j)^{2n+1}\prod_{i=1}^{N}[\varphi_{\sigma_i}(x_i)e^{\frac{e}{\hbar c}\{\Omega_T(x_i) - i\Omega_L(x)\}}] \qquad (3.16)
\end{aligned}
$$

where $z = (x_1 + ix_2)$. We express the above equation in a convenient form.

$$
\begin{aligned}
\psi_N(\{x_i,\sigma_i\}) &= const.e^{-\frac{1}{2}\int_x |\varphi(x)|^2} e^{-\frac{1}{2}\frac{\kappa}{\hbar c}\int_x \nabla\Omega_T(x)\cdot\nabla\Omega_T(x)} \\
&\quad \times \prod_{i=1}^{N}[\varphi_{\sigma_i}(x_i)e^{\frac{e}{\hbar c}\{\Omega_T(x_i) - \bar{\Omega}_T(x_i) - i\Omega_L(x_i)\}}]\psi_L(\{x_i\}) \qquad (3.17)
\end{aligned}
$$

where $\psi_L(\{x_i\})$ is the Laughlin wavefunction,

$$
\psi_L(\{x_i\}) = \prod_{i>j}(z_i - z_j)^{2n+1} e^{-\frac{1}{4l_c^2}\sum_i |z_i|^2} \qquad (3.18)
$$

α has been written as

$$
\alpha_i(x) = \epsilon_{ij}\partial_j\Omega_T(x) + \partial_i\Omega_L(x) \qquad (3.19)
$$

and $\bar{\Omega}_T(x) = -\frac{\hbar c}{e}\frac{|x|^2}{4l_c^2}$. Note that when $\varphi_\sigma(x) = constant$ and $\Omega_T(x) = \bar{\Omega}_T(x) \Rightarrow \nabla \times \vec{\alpha} = B$, the wavefunction reduces to the Laughlin wavefunction. Thus the "mean field" state is the Laughlin state. In this case the Ω integral in equation(??) is equivalent to an N vertex operator correlation function in a c=1 conformal field theory. These wavefunctions are exactly of the form written down by Ezawa [24].

3.2 Parameterization and LLL condition

Apart from an overall factor that only affects the norm, the wavefunction in equation(3.17) depends on the parameters α and φ through a spinor field $W_\sigma(x)$ defined as,

$$W_\sigma(x) \equiv \varphi_{\sigma_i}(x_i) e^{\frac{e}{\hbar c}(\Omega_T(x_i) - \bar{\Omega}_T(x_i) - i\Omega_L(x_i))} \tag{3.20}$$

$W_\sigma(x)$ and hence the wavefunction is gauge invariant (as it should be), since under gauge transformations,

$$\begin{aligned} \Omega_T(x) &\rightarrow \Omega_T(x) \\ \Omega_L(x) &\rightarrow \Omega_L(x) + \Omega(x) \\ \varphi_\sigma(x) &\rightarrow \varphi_\sigma(x) e^{\frac{ie}{\hbar c}\Omega(x)} \end{aligned} \tag{3.21}$$

α and φ have a total of 6 real field components. The gauge invariance of the wavefunctions reduces the number of parameters to 5. There is another local invariance of W, i.e.

$$\Omega_T(x) \rightarrow \Omega_T(x) + \chi(x)$$

$$\begin{aligned}\Omega_L(x) &\rightarrow \Omega_L(x)\\ \varphi_\sigma(x) &\rightarrow \varphi_\sigma(x)e^{-\frac{e}{\hbar c}\chi(x)}\end{aligned} \tag{3.22}$$

Only the norm of the state changes under this transformation and the physical state remains the same. Clearly this transformation is not unitarily implemented in $\mathcal{H}_{CB}$. Nevertheless it reduces the number of independent real fields that parameterize the states to 4, the components of the spinor field W. Thus we can define the normalized projected coherent states, that are parameterized by W as,

$$|W\rangle = \frac{1}{\mathcal{N}}|\alpha,\varphi\rangle_p \tag{3.23}$$

where, $\mathcal{N} = {}_p\langle\alpha,\varphi|\alpha,\varphi\rangle_p$, is the norm of $|\alpha,\varphi\rangle_p$.

From equations (3.17) and (3.20) it is clear that the LLL condition is equivalent to the condition that W is analytic,

$$\partial_{\bar{z}}W_\sigma(x) = 0 \tag{3.24}$$

Thus the LLL condition is easily implemented in this formalism as it is equivalent to an analyticity condition on the parameters.

Chapter 4

Hydrodynamic Limit

In this chapter, we evaluate the expectation value of observables in the limit of long-wavelength and small density fluctuations, an approximation we refer to as hydrodynamic limit. We then discuss the spin charge relation and the regime of its validity.

4.1 Observables

We will now compute the expectation values of gauge invariant operators in the projected coherent states. This is given by,

$$\langle \hat{O} \rangle = \langle W|\hat{O}|W \rangle \tag{4.1}$$

where $\hat{O}$ is a gauge invariant observable.

We do all our calculations in the limit of $W_\sigma(x)$ being a slowly varying function of x (over a length scale of l_c). As we will see, this is also the limit of small density fluctuations. We refer to this limit as the hydrodynamic limit. We note that this is also the limit in which the analytic calculations of Murthy and Shankar [25] are done.

Just as in the case of the Laughlin wavefunction, the computation of $\mathcal{N}$ reduces to the computation of the partition function of a classical 2-d plasma problem. Except that here, the plasma density is coupled to an external field which is a function of W. We first evaluate the norm, $\mathcal{N}[W, \Omega_T]$, in the hydrodynamic approximation. It is given by,

$$\begin{aligned} \mathcal{N}[W, \Omega_T] &= \prod_{i=1}^{N} \int_{x_i} \sum_{\sigma_i} |\psi_N(\{x_i, \sigma_i\})|^2 \\ &= const \times e^{-\int_x \bar{W}_\sigma(x) W_\sigma(x) e^{-\frac{2e}{\hbar c}(\Omega_T(x) - \bar{\Omega}_T(x))}} e^{-\frac{e}{\hbar c}\int_x \nabla\Omega_T(x)\cdot\nabla\Omega_T(x)} \\ &\times \prod_{i=1}^{N} \int_{x_i} e^{\sum_i \ln \bar{W}_\sigma(x_i) W_\sigma(x_i) + 2\frac{e}{\hbar c}\sum_i \bar{\Omega}_T(x_i) + (2n+1)\sum_{i,j} \ln|x_i - x_j|} \end{aligned} \quad (4.2)$$

As in the case of the Laughlin wave function, the norm has the form of a classical partition function of a 2D plasma. Here, there is also an "external potential" which is a function of $W^\dagger(x)W(x)$ and $\Omega_T(x)$. In the hydrodynamic limit, we write this partition function as a functional integral over the density field and evaluate it using the saddle point approximation.

When we change the variables from $\{x_i\} \to \tilde{\rho}$, where,

$$\tilde{\rho}(x) = \sum_{i=1}^{N} \delta(x - x_i) \quad (4.3)$$

then for any function $\mathcal{F}$ of $\{x_i\}$, we get

$$\frac{1}{N!} \prod_{i=1}^{N} \int dx_i \mathcal{F}(\{x_i\}) = \int \mathcal{D}[\tilde{\rho}] J[\tilde{\rho}] \mathcal{F}[\tilde{\rho}] \quad (4.4)$$

where the jacobian of the transformation is the entropy factor,

$$J[\tilde{\rho}] = e^{\int_x [\tilde{\rho}(x) - \tilde{\rho}(x) \ln \tilde{\rho}(x)]} \quad (4.5)$$

Hence the norm can be written as,

$$\begin{aligned}
\mathcal{N}[W,\Omega_T] &= const \times e^{-\int_x \bar{W}_\sigma(x) W_\sigma(x) e^{-\frac{2e}{\hbar c}\{\Omega_T(x)-\bar{\Omega}_T(x)\}}}\, e^{-\frac{\kappa}{\hbar c}\int_x \nabla\Omega_T(x)\cdot\nabla\Omega_T(x)} \\
&\quad \times \int \mathcal{D}[\tilde{\rho}] e^{-\mathcal{S}[\tilde{\rho};W]} \qquad (4.6)
\end{aligned}$$

where,

$$\begin{aligned}
\mathcal{S}[\tilde{\rho};W] &= \int_x [-\tilde{\rho}(x) + \tilde{\rho}(x)\ln\tilde{\rho}(x) - \tilde{\rho}(x)\ln\{\bar{W}_\sigma(x)W_\sigma(x)\} \\
&- 2\frac{e}{\hbar c}\tilde{\rho}(x)\bar{\Omega}_T(x) - 2\pi(2n+1)\tilde{\rho}(x)\frac{1}{\nabla^2}\tilde{\rho}(x)] \qquad (4.7)
\end{aligned}$$

We evaluate this functional integral in the saddle point limit. Dropping $\tilde{\rho}\ln\tilde{\rho}$ term when compared to $\tilde{\rho}\frac{1}{\nabla^2}\tilde{\rho}$ and substituting the solution of the saddle point equation,

$$\tilde{\rho}(x) = \bar{\rho} - \frac{1}{4\pi(2n+1)}\nabla^2\ln\{\bar{W}_\sigma(x)W_\sigma(x)\} \qquad (4.8)$$

we get equation,

$$\begin{aligned}
\mathcal{N}[W,\Omega_T] &= const \times e^{-\int_x W^\dagger(x)W(x)e^{-\frac{2e}{\hbar c}\{\Omega_T(x)-\bar{\Omega}_T(x)\}}}\, e^{-\frac{\kappa}{\hbar c}\int_x \nabla\Omega_T(x)\cdot\nabla\Omega_T(x)} \\
&\quad e^{-\frac{1}{8\pi(2n+1)}\int_x[\ln(W^\dagger(x)W(x))+2\frac{e}{\hbar c}\bar{\Omega}_T(x)]\nabla^2[\ln(W^\dagger(x)W(x))+2\frac{e}{\hbar c}\bar{\Omega}_T(x)]} \qquad (4.9)
\end{aligned}$$

Expectation values of observables similarly reduce to the computation of expectation values in the plasma problem. We evaluate the values of density and spin density by proceeding along similar calculational steps employed in evaluating the norm. The density is given by,

$$\begin{aligned}
\rho(x) &\equiv \langle W|\hat{\rho}(x)|W\rangle \\
&= \frac{1}{\mathcal{N}(W,\Omega_T)}\prod_{i=1}^{N}\left(\int dx_i \sum_{\sigma_i}\right)\bar{\psi}_N(\{x_i,\sigma_i\})\sum_{i=1}^{N}\delta(x-x_i)\psi_N(\{x_i,\sigma_i\}) \\
&= \frac{1}{\mathcal{Z}}\int \mathcal{D}[\tilde{\rho}]\tilde{\rho}(x)e^{-\mathcal{S}[\tilde{\rho};W]}
\end{aligned}$$

$$
\begin{aligned}
&\equiv \langle \tilde{\rho}(x) \rangle \\
&= \frac{-1}{4\pi(2n+1)} \nabla^2 \ln\{\bar{W}_\sigma(x) W_\sigma(x)\} + \bar{\rho} \qquad (4.10)
\end{aligned}
$$

where $\mathcal{Z} \equiv \int \mathcal{D}[\tilde{\rho}] e^{-\mathcal{S}[\tilde{\rho};W]}$ and $\bar{\rho}$ is the mean density.

The spin density is,

$$
\begin{aligned}
s^a(x) &\equiv \langle W | \hat{s}^a(x) | W \rangle \\
&= \frac{1}{\mathcal{N}(W,\Omega_T)} \prod_{i=1}^{N} \left(\int dx_i \sum_{\sigma_i} \right) \bar{\psi}_N(\{x_i,\sigma_i\}) \sum_{i=1}^{N} \frac{1}{2} \tau^a(i) \delta(x-x_i) \psi_N(\{x_i,\sigma_i\}) \\
&= \frac{1}{2} \frac{\bar{W}_\sigma(x) \tau^a_{\sigma\sigma'} W_{\sigma'}(x)}{\bar{W}_{\sigma''}(x) W_{\sigma''}(x)} \langle \tilde{\rho}(x) \rangle \qquad (4.11)
\end{aligned}
$$

where τ^a are Pauli spin matrices.

The current density is,

$$
\begin{aligned}
J_j(x) &\equiv \langle W | \hat{J}_j(x) | W \rangle \\
&= \frac{1}{\mathcal{N}(W,\Omega_T)} \prod_{i=1}^{N} \left(\int_{x_i} \sum_{\sigma_i} \right) \bar{\psi}_N(\{x_i,\sigma_i\}) \sum_{i=1}^{N} \frac{1}{2} \delta(x-x_i) \\
&\qquad [-i\hbar \partial_{x_{i_j}} - \frac{e}{c} A_j(x_i)] \psi_N(\{x_i,\sigma_i\}) \\
&+ \quad h.c \\
&= \rho(x) [\hbar L^3_j(x) - \frac{e}{c}(A_j(x) - \alpha_j(x))] \qquad (4.12)
\end{aligned}
$$

where i is the particle index and j is the coordinate index. α is defined through the relation $\kappa \nabla \times \vec{\alpha}(x) = e\rho(x)$ and $L^3_j \equiv \frac{1}{2i}(Z^\dagger \partial_j Z - h.c)$ for $j = 1$ and 2.

The kinetic energy density is,

$$
\begin{aligned}
\mathcal{T}(x) &\equiv \langle W | \hat{\mathcal{T}}(x) | W \rangle \\
&= \frac{1}{\mathcal{N}(W,\Omega_T)} \prod_{i=1}^{N} \left(\int dx_i \sum_{\sigma_i} \right) \sum_{i=1}^{N} | D_i \psi_N(\{x_i,\sigma_i\}) |^2
\end{aligned}
$$

$$\doteq \quad \hbar\omega_c \frac{\partial_z \bar{W}_\sigma(x)\partial_{\bar{z}} W_\sigma(x)}{\bar{W}_{\sigma'}(x)W_{\sigma'}(x)}\langle\bar{\rho}(x)\rangle \tag{4.13}$$

where $D_i = \partial_{\bar{z}_i} + \frac{1}{2}z_i$ and $z = \frac{x_1 + ix_2}{l_c\sqrt{2}}$

To summarize, the following are the expectation values of some of the observables. The deviation of density from mean density $\bar{\rho}$ is,

$$\rho(x) - \bar{\rho} = \frac{-1}{4\pi(2n+1)}\nabla^2 \ln(W^\dagger(x)W(x)) \tag{4.14}$$

The spin density is computed to be,

$$s^a(x) = \frac{\rho(x)}{2}Z^\dagger(x)\tau^a Z(x) \tag{4.15}$$

where we have denoted the normalized spinor by Z,

$$Z_\sigma(x) \equiv \frac{W_\sigma(x)}{\sqrt{W^\dagger(x)W(x)}} \tag{4.16}$$

The current density is, computed to yield,

$$J_i(x) = \rho(x)[\hbar L_i^3(x) - \frac{e}{c}(A_i(x) - \alpha_i(x))] \tag{4.17}$$

where $L_i^3 \equiv \frac{1}{2i}(Z^\dagger\partial_i Z - h.c)$ and $\kappa\nabla \times \vec{\alpha}(x) = e\rho(x)$.

The kinetic energy density is,

$$\mathcal{T}(x) = \hbar\omega_c\rho(x)\frac{\partial_z \bar{W}_\sigma(x)\partial_{\bar{z}} W_\sigma(x)}{\bar{W}_{\sigma'}(x)W_{\sigma'}(x)} \tag{4.18}$$

Note that the kinetic energy density is zero when W is analytic.

4.2 Charge and Topological Charge Densities

The effective NLSM, equation (1.5), for quantum Hall systems has a term which is obtained by replacing, in the Coulomb term, the deviation of the charge density from its mean value by the topological charge density. This equivalence of charge and topological charge densities was first pointed out by Sondhi et. al. [13]. A proof of this result is given by Moon et. al. [14] about which, in this section, we briefly comment and then provide an elegant alternate proof which has the added advantage of being able to study the effect of Landau level mixing on the charge-topological charge relation.

In general, spin and charge are independent degrees of freedom and hence spin density and charge density operators commute. But when these operators are projected to the LLL they cease to commute and thus the dynamics of spin and charge gets entangled. When we rotate spin, charge gets moved and as a consequence of which spin textures carry charge. Moon et. al. defined the spin texture states as $|\psi[\hat{n}(x)]\rangle \equiv e^{-i\bar{\mathcal{O}}}|\psi_0\rangle$ where $\mathcal{O}$ is an operator which reorients the local spin at x from the $\hat{z}$-axis to an axis along the direction $\hat{n}(x)$ and $\bar{\mathcal{O}}$ is $\mathcal{O}$ projected onto the LLL, $|\psi_0\rangle$ is the $S^z = \frac{N}{2}$ member of the spin multiplet. $\bar{\mathcal{O}}$ is assumed to be small and $\hat{n}(x)$ is assumed to be slowly varying. For such states they find $\delta\rho(x) = -\nu\times$ topological charge density. Working with LLL projected operators can obscure certain aspects regarding Landau level mixing. We derive the charge density - topological charge density relation and also find charge - topological charge relation when there is Landau level mixing.

The topological charge density is given by

$$q(x) = \frac{1}{8\pi}\epsilon_{ij}\hat{n}(x)\cdot\partial_i\hat{n}(x)\times\partial_j\hat{n}(x) \tag{4.19}$$

where $\hat{n}(x)$ is the local direction of spin polarization, $\vec{s}(x) = \frac{1}{2}\rho(x)\hat{n}(x)$. In terms of Z, it is given by,

$$q(x) = \frac{1}{2\pi i}\epsilon_{ij}\partial_i Z^\dagger(x)\partial_j Z(x) \tag{4.20}$$

As can be seen from equations (4.14) and (4.20), the topological charge density is not necessarily proportional to the electrical charge density. In fact, in general, they are independent of each other since $W^\dagger(x)W(x)$ and $Z_\sigma(x)$ are independent variables. However, if the LLL condition is satisfied, then the analyticity of $W_\sigma(x)$ relates the modulus and the phase of each component. Then $W^\dagger(x)W(x)$ and $Z_\sigma(x)$ are no longer independent. In fact if we use the analyticity condition, $\partial_i W_\sigma(x) = -i\epsilon_{ij}\partial_j W(x)$, in the RHS of equation (4.14), we get,

$$\rho(x) - \bar{\rho} = -\frac{1}{2n+1}q(x) \tag{4.21}$$

Thus the topological charge density is proportional to the electrical charge density if and only if the LLL condition is satisfied. The relation (4.21) will therefore not be true in presence of Landau level mixing.

When the densities are proportional, the total excess charge, Q, will of course be proportional to the total topological charge, Q_{top}. However, the total charges could be proportional without the densities being so. We will now investigate this possibility. Integrating equation (4.14) over all space, we have,

$$Q = \frac{1}{4\pi(2n+1)}\oint dx^i \epsilon_{ij}\partial_j \ln(W^\dagger(x)W(x)) \tag{4.22}$$

where the contour is at infinity. If W is analytic at infinity, then the RHS of equation (4.22) can be written as,

$$Q = -\frac{1}{2\pi(2n+1)}\oint dx^i \frac{1}{2i}(Z^\dagger(x)\partial_i Z(x) - \partial_i Z^\dagger(x)Z(x))$$

$$= \quad -\frac{1}{2n+1} Q_{top} \tag{4.23}$$

Thus if there is no Landau level mixing in the ground state, the total charge is always proportional to the topological charge. Note that Z and hence $q(x)$ is well defined only if $\rho(x)$ is non-zero everywhere. So all our considerations are true only in this case. They will not hold for polarized vortices where $\rho(x)$ will vanish at some point.

Chapter 5

Gauge Invariant Bosonization

In chapter 3, we saw that the projected coherent states are labeled by a spinor field W, and that the expectation values of observables could be computed in the hydrodynamic limit in terms of W. The states can therefore be labeled by the values of the physical observables, the density $\rho(x)$ and the normalized spinor $Z_\sigma(x)$. In this chapter, we will show that these states themselves satisfy the generalized coherent state properties [23] in $\mathcal{H}_{phy}$. Namely, the resolution of unity and continuity of overlaps. This implies that, in the hydrodynamic limit, the original electronic theory can be expressed completely in terms of bosonic field operators corresponding to $\rho(x)$ and $Z_\sigma(x)$. Thus in this limit, the theory can be bosonized in a gauge invariant way with no redundant degrees of freedom.

5.1 Gauge Invariant Bosonization

5.1.1 Resolution of Unity

The fact that the identity operator in $\mathcal{H}_{phy}$ can be resolved in terms of the projected coherent states has already been shown in equation (3.2). Here we express this same equation in terms of the gauge invariant parameters. We perform the following change of variables in equation (3.2),

$$\alpha_i(x), \varphi_\sigma(x) \;\rightarrow\; \Omega_L(x), \Omega_T(x), W_\sigma(x) \tag{5.1}$$

Further, using equations (3.23) and (4.9), we get

$$\begin{aligned} I &= \int \mathcal{D}[\alpha,\varphi]\; |\alpha,\varphi\rangle_{p\;p}\langle\alpha,\varphi| \\ &= const \int \mathcal{D}[\Omega_T(x)]\mathcal{D}[\Omega_L(x)]\mathcal{D}[W] e^{-\int_x \frac{4e}{\hbar c}\Omega_T(x)} \mathcal{N}[W,\Omega_T] |W\rangle\langle W| \\ &= const \int \mathcal{D}[W]\mathcal{G}[W] |W\rangle\langle W| \end{aligned} \tag{5.2}$$

where the factor $e^{-\frac{4e}{\hbar c}\Omega_T(x)}$ is the Jacobian due to the change of variables $\varphi \rightarrow W$,

$$\mathcal{D}[W] = \prod_{x,\sigma} \frac{dW_\sigma(x) d\bar{W}_\sigma(x)}{2\pi i} \tag{5.3}$$

and

$$\mathcal{G}[W] \equiv \int \mathcal{D}[\Omega_T] e^{-\int_x \frac{4e}{\hbar c}\Omega_T(x)} \mathcal{N}[W,\Omega_T] \tag{5.4}$$

We evaluate $\mathcal{G}[W]$ by doing the Ω_T integral in the hydrodynamic limit. The saddle-point approximation of the integral gives

$$\mathcal{G}[W] = const \times e^{-\int_x \frac{4e}{\hbar c}\bar{\Omega}_T(x)} \mathcal{N}[W,\bar{\Omega}_T] \tag{5.5}$$

where $\tilde{\Omega}_T$ is the solution to the saddle-point equation which, in long wavelength limit ($\nabla^2 \ln \bar{W}W \ll \ln \bar{W}W$) is,

$$\tilde{\Omega_T}(x) = \tilde{\Omega_T}(x) + \frac{\hbar c}{2e} \ln\{\bar{W}_\sigma(x) W_\sigma(x)\} \tag{5.6}$$

When this value for $\tilde{\Omega_T}$ is substituted in equation (5.5) we get,

$$\mathcal{G}[W] \;=\; const \prod_x \frac{1}{[W^\dagger(x) W(x)]^2} \tag{5.7}$$

We now make another change of variables from $W_\sigma(x)$ to $\rho(x), Z_\sigma(x)$, defined by equations (4.14) and (4.16), to get,

$$I \;=\; const \int \mathcal{D}[\rho]\mathcal{D}[Z]\; |\rho, Z\rangle\langle\rho, Z| \tag{5.8}$$

where,

$$\mathcal{D}[\rho] \;=\; \prod_x d\rho(x) \quad \mathcal{D}[Z] \;=\; \prod_x sin^2\theta(x) sin\phi(x) d\theta(x) d\phi(x) d\psi(x) \tag{5.9}$$

Z has been parameterized as

$$Z = (cos\frac{\theta}{2} e^{i(\frac{\psi+\phi}{2})}, sin\frac{\theta}{2} e^{i(\frac{\psi-\phi}{2})}) \tag{5.10}$$

5.1.2 Overlaps

The overlap of two gauge invariant coherent states $|W_1\rangle$ and $|W_2\rangle$, obtained by proceeding with steps similar to those involved in evaluating the norm, is

$$\begin{aligned}\langle W_1|W_2\rangle \;&=\; \frac{1}{\sqrt{\mathcal{N}(W_1, \Omega_{T1})\mathcal{N}(W_2, \Omega_{T2})}} \prod_{i=1}^{N} \left(\int dx_i \sum_{\sigma_i}\right) \bar{\psi}_1(\{x_i, \sigma_i\})\psi_2(\{x_i, \sigma_i\}) \\ &=\; e^{-\frac{1}{8\pi(2n+1)}\int_x [f_{12}(x)\nabla^2 f_{12}(x) - \frac{1}{2} f_{11}(x)\nabla^2 f_{11}(x) - \frac{1}{2} f_{22}(x)\nabla^2 f_{22}(x)]}\end{aligned} \tag{5.11}$$

where

$$f_{ab}(x) = \ln\{\bar{W}_{a\sigma}(x)W_{b\sigma}(x)\} + 2\frac{e}{\hbar c}\bar{\Omega}_T(x) \tag{5.12}$$

If we express W in terms of ρ and Z we get the overlap to be

$$\begin{aligned} \langle W_1|W_2\rangle &= e^{-\frac{1}{8\pi(2n+1)}\int_x \ln\{Z_1^\dagger(x)Z_2(x)\}\nabla^2\ln\{Z_1^\dagger(x)Z_2(x)\}} \\ &\quad \times e^{\frac{1}{2}\int_x(\rho_1(x)+\rho_2(x))\ln\{Z_1^\dagger(x)Z_2(x)\}} \\ &\quad e^{\frac{1}{2}\pi(2n+1)\int_x(\rho_1(x)-\rho_2(x))\frac{1}{\nabla^2}(\rho_1(x)-\rho_2(x))} \end{aligned} \tag{5.13}$$

Making use of the relation, $\bar{Z}_{1\sigma}Z_{2\sigma} = e^{\frac{i}{2}\Phi(\vec{n}_1,\vec{n}_2)}(\frac{1+\vec{n}_1\cdot\vec{n}_2}{2})^{\frac{1}{2}}$ where $\Phi(\vec{n}_1,\vec{n}_2)$ is the area of the spherical triangle with vertices at $\vec{n}_1, \vec{n}_2$ and a third point on the unit sphere, in the above equation we get the following equation.

$$\begin{aligned} \langle W_1|W_2\rangle &= e^{-F[\rho_1,Z_1,\rho_2,Z_2]} \\ F &= \frac{1}{32\pi(2n+1)}\{\Phi(\hat{n}_1,\hat{n}_2)+\ln(\frac{1+\hat{n}_1\cdot\hat{n}_2}{2})\}\nabla^2\{\Phi(\hat{n}_1,\hat{n}_2)+\ln(\frac{1+\hat{n}_1\cdot\hat{n}_2}{2})\} \\ &\quad -\frac{i}{4}\int_x(\rho_1(x)+\rho_2(x))\Phi(\hat{n}_1,\hat{n}_2) - \frac{1}{4}\int_x(\rho_1(x)+\rho_2(x))\ln(\frac{1+\hat{n}_1\cdot\hat{n}_2}{2}) \\ &\quad -\frac{\pi}{2}(2n+1)\int_{x,y}(\rho_1(x)-\rho_2(x))\langle x|\frac{1}{\nabla^2}|y\rangle(\rho_1(y)-\rho_2(y)) \end{aligned} \tag{5.14}$$

where $\Phi(\hat{n}_1,\hat{n}_2)$ is the solid angle subtended by the geodesic triangle with $\hat{n}_1, \hat{n}_2$ and some third point on the unit sphere as vertices. Note that the overlap smoothly goes to 1 as $(\rho_1, Z_1) \to (\rho_2, Z_2)$.

In the following chapter, we construct the classical theory for Composite bosons and shall see the importance of gauge invariant coherent states. The coherent states we constructed inherit the classical-quantum correspondence property which tells us that classical theory completely determines the quantum dynamics.

Chapter 6

Classical Theory

A classical description of a system is specified by introducing a phase space and Poisson brackets (i.e., a manifold along with a Symplectic structure). While a quantum description is effected by introducing a Hilbert space and operators acting on that Hilbert space. A correspondence between the quantum theory and the classical theory can be given by constructing a coherent state basis for the Hilbert space which are labeled by classically interpretable variables. The correspondence relates the label space of the coherent states and the phase space manifold. Operators in the Hilbert space, through the matrix elements in the coherent state basis, get related to the functions on the phase space. Commutators of operators are related to the Poisson brackets of corresponding functions on the manifold. The Poisson brackets (or the Symplectic structure) can be extracted from the overlap of two nearby coherent states.

For simple Hamiltonians, a coherent state remains a coherent state under time evolution. The parameters labeling the coherent states will evolve according to the classical equations of motion. In addition to the parameter evolution, the states accumulate a phase equal to the exponential of the classical action. Hence, the classical dynamics completely determines the evolution of the quantum states.

In the following sections, we give a classical theory for the composite bosons. We also give a classical theory when the composite bosons are restricted to the LLL. We then discuss the limit where the classical theory becomes exact for quantum Hall skyrmions. Finally we make use of classical-quantum correspondence in coherent states and show that the ground states of quantum hall systems are the Laughlin states.

6.1 Classical theory of Composite bosons

The path integral representation of the time evolution operator is obtained by splitting the time t into N segments of length ϵ and taking the limit $N \to \infty$, $\epsilon \to 0$ while their product remains fixed, $N\epsilon = t$. Between every splitting when we insert the resolution of unity, equation (5.8),

$$I = const \int \mathcal{D}[W] \, |W\rangle\langle W|$$

where, $W \equiv (\rho, Z)$ and $\mathcal{D}[W]$ is the integral measure, we get

$$\begin{aligned} Z &= Tr\, e^{-iHt} = Tr\, [e^{-i\epsilon H}]^N \\ &= \lim_{\epsilon\to 0, N\to\infty; \epsilon N = t} \prod_n \int \mathcal{D}[W_n] \prod_n \langle W_{n+1}|e^{-iH\epsilon}|W_n\rangle \\ &= \lim_{\epsilon\to 0, N\to\infty; \epsilon N = t} \prod_n \int \mathcal{D}[W_n] \prod_n \langle W_{n+1}|W_n\rangle[1 - \epsilon\langle W_n|H|W_n\rangle] \end{aligned} \quad (6.1)$$

We keep only the terms linear in ϵ as we shall be taking the limit $\epsilon \to 0$. Since $W_{n+1} = W_n + \epsilon\partial_t W_n$ we get

$$\epsilon\langle \partial_t W_n | W_n \rangle = \ln\langle W_{n+1} | W_n \rangle \tag{6.2}$$

Finally we write the formal path integral as

$$Z = \int \mathcal{D}[W] \; e^{i\int dt[-i\langle\partial_t W|W\rangle - H(W)]} \tag{6.3}$$

where $H(W) = \langle W|H|W\rangle$.

The classical action, $S[W]$, on the manifold parametrized by W gets defined as

$$S[W] = \int dt[-i\langle\partial_t W|W\rangle - H(W)] \tag{6.4}$$

The expression for the first term is obtained from the overlap of nearby states, equation(6.2), while the Hamiltonian function $H(W)$ is given by the matrix element $\langle W|H|W\rangle$. In the expression for overlap, equation (5.13), we put $\rho_2(x) = \rho(x)$, $\rho_1(x) = \rho(x) + \epsilon\partial_t\rho(x)$, $Z_{2\sigma}(x) = Z_\sigma(x)$ and $Z_{1\sigma}(x) = Z_\sigma(x) + \epsilon\partial_t Z_\sigma(x)$, so as to get the overlap for nearby states

$$\langle W + \epsilon\partial_t W|W\rangle = e^{-i\epsilon\int_x \rho(x)L_t^3(x) + O(\epsilon^2)} \tag{6.5}$$

or

$$\langle\partial_t W|W\rangle = -i\epsilon\int_x \rho(x)L_t^3(x) \tag{6.6}$$

where $L_t^3 \equiv \frac{1}{2i}(Z^\dagger\partial_t Z - h.c)$. We make use of the above expression in equation (6.4) and obtain, after rewriting Z in terms of ψ, θ, ϕ as given in equation (5.10), the following action

$$\begin{aligned} S[W] &= \int dt\,[-\frac{1}{2}\int_x \rho(x)(\dot\psi(x) + \dot\phi(x)\cos\theta(x)) \;-\; H(W)] \\ &= \int dt\,[\int_x \frac{1}{4}(\psi + \phi\cos\theta)\dot\rho - \frac{1}{4}\rho\dot\psi - \frac{1}{4}\rho\phi\sin\theta\dot\theta - \frac{1}{4}\rho\cos\theta\dot\phi \;-\; H(W)] \end{aligned} \tag{6.7}$$

We get the second step after doing integration by parts and neglecting the total derivative terms.

The classical action is first order in time derivatives and is an example of the form

$$S = \int_{\gamma=\partial\Gamma} dt \, [a_\mu \dot{x}^\mu - H(x)] = \int_\Gamma \omega - \int_\gamma H(x)dt \qquad (6.8)$$

where $\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu$, $\omega_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. The least action principle produces the following equations of motion

$$\partial_\mu H = \omega_{\mu\nu}\dot{x}^\nu \qquad (6.9)$$

and the Poisson brackets can be defined by

$$\{f, g\} = \omega^{\mu\nu}\partial_\mu f \partial_\nu g \qquad (6.10)$$

where $\omega^{\mu\nu}$ is the matrix inverse of $\omega_{\mu\nu}$.

The antisymmetric matrix $\omega_{\mu\nu}$ is a 4×4 matrix for the action given in equation (6.7).

$$\omega_{\mu\nu} = \begin{bmatrix} \omega_{\rho\rho} & \omega_{\rho\psi} & \omega_{\rho\theta} & \omega_{\rho\phi} \\ \omega_{\psi\rho} & \omega_{\psi\psi} & \omega_{\psi\theta} & \omega_{\psi\phi} \\ \omega_{\theta\rho} & \omega_{\theta\psi} & \omega_{\theta\theta} & \omega_{\theta\phi} \\ \omega_{\phi\rho} & \omega_{\phi\psi} & \omega_{\phi\theta} & \omega_{\phi\phi} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & -\frac{1}{2}\cos\theta \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\rho\sin\theta \\ \frac{1}{2}\cos\theta & 0 & -\frac{1}{2}\rho\sin\theta & 0 \end{bmatrix}$$

The determinant of this matrix $det\omega_{\mu\nu} \neq 0$ and hence the inverse matrix $\omega^{\mu\nu}$ exists.

$$\omega^{\mu\nu} = \begin{bmatrix} \omega^{\rho\rho} & \omega^{\rho\psi} & \omega^{\rho\theta} & \omega^{\rho\phi} \\ \omega^{\psi\rho} & \omega^{\psi\psi} & \omega^{\psi\theta} & \omega^{\psi\phi} \\ \omega^{\theta\rho} & \omega^{\theta\psi} & \omega^{\theta\theta} & \omega^{\theta\phi} \\ \omega^{\phi\rho} & \omega^{\phi\psi} & \omega^{\phi\theta} & \omega^{\phi\phi} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -\frac{2\cot\theta}{\rho} & 0 \\ 0 & \frac{2\cot\theta}{\rho} & 0 & -\frac{2}{\rho\sin\theta} \\ 0 & 0 & \frac{2}{\rho\sin\theta} & 0 \end{bmatrix}$$

The Poisson brackets of the functions $f \equiv f(\rho, \psi, \theta, \psi)$ and $g \equiv g(\rho, \psi, \theta, \psi)$ are obtained by substituting $\omega^{\mu\nu}$ in equation (6.10). The Poisson brackets take the familiar form for conjugate variables in terms of $\Phi \equiv \sqrt{\rho} Z$ and $\Phi^\dagger \equiv \sqrt{\rho} Z^\dagger$ when substituted in equation(6.5). But then the observables are complicated functions of Φ and $\Phi^\dagger$.

In general, the resolution of identity of the coherent state basis is sufficient to define the classical action though this does not necessarily imply that the coherent states inherit the classical-quantum correspondence property [26]. The classical dynamics will not be able to capture the complete quantum dynamics if there is a "gauge invariance" in the parameters labeling the coherent states[27]. $\omega_{\mu\nu}$ will not have an inverse in which case the solutions to the equations of motion are not unique. Since the gauge invariant coherent states we constructed have no redundant variables in their label space, they do inherit the classical-quantum correspondence property.

6.2 LLL Classical theory

If we now impose the LLL condition, then the charge density fluctuations get tied up to the the spin density fluctuations. i.e.,

$$\begin{aligned} \rho(x) &= \bar{\rho} - \frac{1}{2n+1} q(x) \\ &= \bar{\rho} - \frac{1}{2\pi(2n+1)} \epsilon_{ij} \partial_i L_j^3(x) \end{aligned} \tag{6.11}$$

The theory can then be expressed in terms of spin fluctuations alone. The expression for the overlap in equation (6.5) then gets written as,

$$\langle W + \epsilon \partial_t W | W \rangle = e^{-i\epsilon\bar{\rho} \int_x L_t^3(x) + i\epsilon \frac{1}{4\pi(2n+1)} \int_x \epsilon_{\mu\nu\lambda} L_\mu^3(x) \partial_\nu L_\lambda^3(x) + O(\epsilon^2)} \tag{6.12}$$

The first term in the exponent is the solid angle term given in equation (1.5). The second term in the exponent in the RHS of equation (6.12) is the Hopf term. Thus the theory, when restricted to the lowest Landau level is a NLSM with a Hopf term in the action.

6.3 The classical limit

We will now show that for large skyrmions, the theory becomes classical. Consider the set of states corresponding to configurations characterized by a size parameter, d. We parameterize them as,

$$\begin{aligned} \rho^d(x) &= \bar{\rho} + \frac{1}{d^2}\Delta\rho(\frac{x}{d}) \\ Z^d(x) &= Z(\frac{x}{d}) \end{aligned} \qquad (6.13)$$

Substituting $\rho_1^d(x)$, $Z_{1\sigma}^d(x)$ and $\rho_2^d(x)$, $Z_{2\sigma}^d(x)$ in equation (5.14) and changing the variable $x \to dx$ we get,

$$\langle W_1 | W_2 \rangle = e^{\int_x [d^2 \bar{\rho} \ln(\frac{1+\hat{n}_1 \cdot \hat{n}_2}{2}) + 0(d^0)]} \qquad (6.14)$$

$\frac{1}{2}\ln(\frac{1+\hat{n}_1 \cdot \hat{n}_2}{2})$ is zero $\hat{n}_1 = \hat{n}_2$ and negative otherwise. Thus for $W_1 \neq W_2$,

$$\lim_{d \to \infty} \langle W_1 | W_2 \rangle \to 0 \qquad (6.15)$$

The coherent states thus become orthogonal when $d \to \infty$. It can also be shown that the off-diagonal matrix elements of the observables in the coherent state basis, vanish in this limit. Hence the set of states corresponding to a system of skyrmions will behave classically in the limit of the skyrmion sizes tending to infinity.

6.4 Laughlin Ground states

In this section we shall derive the ground state of quantum hall system using coherent states and their classical limit.

Let $|\psi\rangle$ be a normalized state which in the coherent state basis is given as

$$|\psi\rangle = \int_W a(W)\,|W\rangle \tag{6.16}$$

The expectation value of Hamiltonian in state $|\psi\rangle$ is

$$\langle\psi|H|\psi\rangle = \int_{W'}\int_W a(W)\bar{a}(W')\,\langle W'|H|W\rangle \tag{6.17}$$

In the classical limit when the overlap $\langle W'|W\rangle$ becomes orthogonal only the diagonal matrix elements of the operators are non-zero. Hence we get

$$\begin{aligned}\langle\psi|H|\psi\rangle &= \int_{W'}\int_W a(W)\bar{a}(W')\,H(W',W)\,\delta(W-W')\\ &= \int_W |a(W)|^2\,H(W,W)\end{aligned} \tag{6.18}$$

where $H(W,W)=\langle W|H|W\rangle$. Since the above expression is sum of positive quantities the ground state corresponds to the coherent state $|W\rangle$ for which $H(W,W)$ is minimum.

The expectation value of coulomb energy in the classical limit is

$$\begin{aligned}\langle W|V_c|W\rangle &= \int_{x,y}\langle W|\Delta\hat{\rho}(x)\,V(x-y)\,\Delta\hat{\rho}(y)|W\rangle\\ &= \int_{W'}\int_{x,y}\langle W|\Delta\hat{\rho}(x)|W'\rangle\,V(x-y)\,\langle W'|\Delta\hat{\rho}(y)|W\rangle\\ &= \int_{x,y}\Delta\rho_W(x)\,V(x-y)\,\Delta\rho_W(y)\end{aligned} \tag{6.19}$$

The second step in the above equation is obtained by inserting resolution of identity of the coherent states and the third step is obtained upon taking classical limit and integrating W'. $V(x-y)$ is the interaction potential and $\Delta\rho$, given in equation (4.14), is

$$\Delta\rho_W(x) = \rho(x) - \bar{\rho} = \frac{-1}{4\pi(2n+1)}\nabla^2 \ln(W^\dagger(x)W(x)) \tag{6.20}$$

We get the total energy from equations (4.18) and (6.19) as

$$H(W,W) \;=\; \hbar\omega_c \int_x \rho(x)\frac{\partial_z \bar{W}_\sigma(x)\partial_{\bar{z}} W_\sigma(x)}{\bar{W}_{\sigma'}(x)W_{\sigma'}(x)} + \int_{x,y} \Delta\rho_W(x)\; V(x-y)\; \Delta\rho_W(y) \tag{6.21}$$

The kinetic part of the energy vanishes for any analytic W and when $W = const$ the coulomb energy too becomes zero. Since the Hamiltonian is positive definite $W = const$ gives the minimum energy. We have already seen in section (3.1) that the constant W state is the Laughlin state. Thus we obtain the ground state at all $\nu = \frac{1}{2n+1}$ to be the Laughlin state independent of the form of interaction.

Chapter 7

Skyrmions

In this chapter we give a complete kinematical description of lowest Landau level skyrmions in terms of their collective coordinates. We study the overlap of skyrmions and also construct finite spin skyrmions.

7.1 LLL Skyrmion States

We characterize the skyrmions in the LLL by the set of states $\{|\lambda, R\rangle\}$ labeled by two complex parameters, λ and R, related to the size, orientation and position of the skyrmions. These states are a subset of the gauge invariant coherent states $\{|W\rangle\}$ we constructed with the following choice for W.

$$W = \begin{pmatrix} \lambda \\ z - R \end{pmatrix} \tag{7.1}$$

In the next two subsections we evaluate the overlap for the states $|\lambda, R\rangle$ in order to study the collective coordinates of a skyrmion.

7.1.1 Skyrmions positioned at origin

We shall first consider a special case where R coordinates are frozen. The overlap of $|\lambda_1\rangle \equiv |\lambda_1, 0\rangle$ and $|\lambda_2\rangle \equiv |\lambda_2, 0\rangle$ is obtained by substituting $W_1 = \begin{pmatrix} \lambda_1 \\ z \end{pmatrix}$ and $W_2 = \begin{pmatrix} \lambda_2 \\ z \end{pmatrix}$ in the expression for coherent state overlap given in equation (5.11). We get the overlap as

$$\begin{aligned} \langle\lambda_1|\lambda_2\rangle &= e^{-\frac{1}{8\pi(2n+1)}\int_x [g_{12}(x)\nabla^2 g_{12}(x) - \frac{1}{2}g_{11}(x)\nabla^2 g_{11}(x) - \frac{1}{2}g_{22}(x)\nabla^2 g_{22}(x)]} \\ &\times \; e^{\frac{1}{2\pi(2n+1)}\int_x [g_{12}(x) - \frac{1}{2}g_{11}(x) - \frac{1}{2}g_{22}(x)]} \end{aligned} \tag{7.2}$$

where

$$g_{ab}(x) = \ln[\bar{\lambda}_a \lambda_b + \bar{z}z] \tag{7.3}$$

We do the above integrals in the polar coordinates since the integrands are angle independent and put an upper cut-off, L, for the radial coordinate. We then get the following expression,

$$\langle\lambda_1|\lambda_2\rangle = e^{-\frac{1}{2(2n+1)}(\ln L)[|\lambda_1 - \lambda_2|^2 + \bar{\lambda}_2\lambda_1 - \bar{\lambda}_1\lambda_2]} \tag{7.4}$$

The exponential of the overlap contains a divergence when $L \to \infty$. We shall now trace the origin of this divergence by evaluating the overlap at $\nu = 1$ without any approximation. The wavefunction of the coherent states as given in equation (3.17) is

$$\langle\{x_i\sigma_i\}|W\rangle = const \times \prod_{i=1}^{N} W_{\sigma_i}(x_i)\ \psi_L(\{x_i\}) \tag{7.5}$$

where $\psi_L(\{x_i\})$ is the Laughlin wavefunction. For $\nu = 1$ the Laughlin wavefunction is the slater determinant of the analytic LLL wavefunctions $\varphi_l(z)$

$$\psi_L(\{x_i\}) = \sum_P (-1)^P \prod_l \varphi_l(x_{Pl}) \tag{7.6}$$

where the sum is over all the permutations P. Hence the coherent state overlap for $\nu = 1$ becomes

$$\begin{aligned}
\langle W_1|W_2\rangle &= const\ \prod_i \left(\sum_{\sigma_i}\int_{x_i}\right) \langle W_1|\{x_i\sigma_i\}\rangle\ \langle\{x_i\sigma_i\}|W_2\rangle \\
&= const\ \prod_i \int_{x_i} \prod_i [W^\dagger_{\sigma_i}(x_i)W_{\sigma_i}(x_i)]\ |\psi_L(\{x_i\})|^2 \\
&= const\ \prod_i \int_{x_i} \prod_i [W^\dagger_{\sigma_i}(x_i)W_{\sigma_i}(x_i)]\ \sum_{PQ}(-1)^P(-1)^Q \prod_l \varphi^*_l(x_{Pl})\varphi_l(x_{Ql}) \\
&= const'\ \prod_i \int_{x_i} \prod_l [W^\dagger_{\sigma_l}(x_l)W_{\sigma_l}(x_l)]\varphi^*_l(x_l)\varphi_l(x_l) \qquad (7.7)
\end{aligned}$$

Thus we obtain the overlap for $|\lambda_1\rangle$ and $|\lambda_2\rangle$ after using the LLL wavefunction property $z\varphi_l(z) = \sqrt{l+1}\varphi_{l+1}(z)$ as

$$\begin{aligned}
\langle\lambda_1|\lambda_2\rangle &= const\ \times \prod_l \int_z [\bar\lambda_1\lambda_2 + \bar z z]\varphi^*_l(z)\varphi_l(z) \\
&= const\ \times \prod_l [\bar\lambda_1\lambda_2 + (l+1)] \qquad (7.8)
\end{aligned}$$

Therefore we get the overlap for the normalized $|\lambda_1\rangle$ and $|\lambda_2\rangle$ as

$$\langle\lambda_1|\lambda_2\rangle = e^{\sum_l [\ln\{\bar\lambda_1\lambda_2+(l+1)\} - \frac{1}{2}\ln\{|\lambda_1|^2+(l+1)\} - \frac{1}{2}\ln\{|\lambda_2|^2+(l+1)\}]} \qquad (7.9)$$

The sum in the exponential diverges and hence the long-wavelength approximation where sum is replaced by integral is not the cause for this divergence. In fact, the spin also carries the same log divergence which can be seen from the following expression.

$$\langle\lambda|\int_x \frac{1}{2}(\hat s^3(x)+1)|\lambda\rangle = \sum_l \frac{|\lambda|^2}{|\lambda|^2+(l+1)} \qquad (7.10)$$

From the equation(7.4) for $\lambda_1 \neq \lambda_2$ due to the logarithmic divergence in the exponent

$$\lim_{L\to\infty}\langle\lambda_1|\lambda_2\rangle \to 0 \qquad (7.11)$$

Hence the λ-coordinate can be completely treated classically. This limit acquires a further physical significance if we consider a modified state $|\lambda;\kappa\rangle$ with W now given as

$$W = \begin{pmatrix} \lambda \\ zf(r;\kappa) \end{pmatrix} \tag{7.12}$$

where $f(r;\kappa)$ is a function chosen so as to remove the log-divergence in the g_{ab} integrals and in the limit $\kappa \to 0$ the states $|\lambda;\kappa\rangle \to |\lambda\rangle$. Hence for finite κ we are allowing mixing of higher Landau levels and thus κ specifies the amount of Landau level mixing. We can thus conclude that the λ-coordinate of skyrmion goes classical when restricted to LLL. The phase of the overlap tells us $\bar{\lambda}$ and λ are conjugate variables and hence behave like the guiding center coordinates of a charge particle in a magnetic field.

7.1.2 Overlap for a more general case

We shall now examine a more general case for the overlap between $|\lambda_1, R_1\rangle$ and $|\lambda_2, R_2\rangle$. We assume the phase difference between λ_1 and λ_2 to be zero. We also shift the origin in the overlap integrals to the center of mass(CM) of R_1 and R_2. Although for complex integrals of multivalued functions (log-function) one should make sure whether one can analytically continue the origin to the CM, we shall not do so here. The price we pay is lose out information on the phase part of the overlap. We shall instead evaluate the phase separately from the overlap of infinitesimally nearby states which is sufficient to identify the conjugate variables.

We notice that one of the integrals in equation(7.2) can be written

$$\begin{aligned}\int_x g_{12}(x)\nabla^2 g_{12}(x) &= 4\int_x g_{12}(x)\partial_{\bar z}\partial_z g_{12}(x) = -4\int_x \partial_{\bar z} g_{12}(x)\partial_z g_{12}(x) \\ &= 4\int_x \left[\frac{\bar\lambda_1\lambda_2}{D_{12}^2} - \frac{1}{D_{12}}\right] \qquad (7.13)\end{aligned}$$

where $D_{12} = \bar\lambda_1\lambda_2 + (\bar z - \frac{R}{2})(z + \frac{R}{2})$. It is then sufficient to evaluate $\int \frac{1}{D_{12}}$ since a derivative w.r.t $(\bar\lambda_1\lambda_2)$ will give us $\int \frac{1}{(D_{12})^2}$ and an integration w.r.t $(\bar\lambda_1\lambda_2)$ will give us $\int_x g_{12}(x)$. After cumbersome but straight forward integration in polar coordinates with a upper cut-off L in the radial coordinate we get

$$\begin{aligned}\int_x \frac{1}{D_{12}} &= \int_x \frac{1}{\bar\lambda_1\lambda_2 + (\bar z - \frac{R}{2})(z + \frac{R}{2})} \\ &= \pi \ln\left[\frac{(L^2 + \frac{R^2}{4} + \lambda_1\lambda_2) + \sqrt{(L^2 + \frac{R^2}{4} + \lambda_1\lambda_2)^2 - \lambda_1\lambda_2 R^2}}{\frac{R^2}{4} + \lambda_1\lambda_2 + |\frac{R^2}{4} - \lambda_1\lambda_2|}\right] \qquad (7.14)\end{aligned}$$

$$\int_x \frac{1}{|\lambda_1|^2 + |z - \frac{R}{2}|^2} = \pi \ln\left[\frac{(L^2 + {\lambda_1}^2 - \frac{R^2}{4}) + \sqrt{(L^2 + {\lambda_1}^2 - \frac{R^2}{4})^2 + {\lambda_1}^2 R^2}}{2{\lambda_1}^2}\right] \qquad (7.15)$$

and making use of above equations and their derivative w.r.t $(\lambda_1\lambda_2)$ and integrals w.r.t $(\lambda_1\lambda_2)$ we obtain

$$\begin{aligned}&\int_x [g_{12}(x) - \frac{1}{2}g_{11}(x) - \frac{1}{2}g_{22}(x)] \\ = &-\pi\left[\{\lambda_1\lambda_2\ln(2\lambda_1\lambda_2) - \lambda_1\lambda_2 + \lambda_1\lambda_2\frac{R^2}{4}\}\,\theta(\lambda_1\lambda_2 - \frac{R^2}{4}) + \ln\frac{R^2}{2}\,\theta(\frac{R^2}{4} - \lambda_1\lambda_2)\right] \\ - &\pi(\lambda_1 - \lambda_2)^2\ln L + \frac{1}{2}\pi\left[{\lambda_1}^2\ln(2{\lambda_1}^2) - {\lambda_1}^2 + {\lambda_2}^2\ln(2{\lambda_2}^2) - {\lambda_2}^2\right] \qquad (7.16)\end{aligned}$$

and

$$\int_x \quad [g_{12}(x)\nabla^2 g_{12}(x) - \frac{1}{2}g_{11}(x)\nabla^2 g_{11}(x) - \frac{1}{2}g_{22}(x)\nabla^2 g_{22}(x)]$$

$$= 4\pi\left[\ln\{\frac{R^2}{4}+\lambda_1\lambda_2+|\frac{R^2}{4}-\lambda_1\lambda_2|\}-\frac{1}{2}\ln(2\lambda_1{}^2)-\frac{1}{2}\ln(2\lambda_2{}^2)\right]$$
$$+ 4\pi\left[\theta(\lambda_1\lambda_2-\frac{R^2}{4})-\frac{1}{2}\theta(\lambda_1{}^2-\frac{R^2}{4})-\frac{1}{2}\theta(\lambda_2{}^2-\frac{R^2}{4})\right] \qquad (7.17)$$

where $\theta(t)$ is the theta function which is 1 when the argument is positive and 0 otherwise.

In the above expression if we put $R = 0$ we get the expression (7.4) we derived earlier. For $\lambda_1 = \lambda_2 = \lambda$ we get the following expression

$$|\langle\lambda_1, R_1|\lambda_2, R_2\rangle| = e^{-\frac{1}{2(2n+1)}\lambda^2}\left(\frac{4\lambda^2}{R^2}\right)^{\frac{1}{2(2n+1)}\lambda^2}\theta(\frac{R^2}{4}-\lambda^2)$$
$$+ e^{-\frac{1}{8(2n+1)}R^2}\theta(\lambda^2-\frac{R^2}{4}) \qquad (7.18)$$

where $R = |R_1 - R_2|$.

λ is a length scale associated to the size of the skyrmions and R to the distance between two skyrmions. When the skyrmions are close ($R < 2\lambda$) the overlap falls off exponentially while for $R > \lambda$ there is a power-law decay.

We shall now determine the phase for two infinitesimally nearby states $|\lambda, R\rangle$ and $|\lambda+\delta\lambda, R+\delta R\rangle$. Expanding $\int g_{12}$ and $\int g_{12}\nabla^2 g_{12}$ to linear order in $\delta\lambda$ and δR we get the phase of the overlap

$$\frac{1}{(2n+1)}\bar{\lambda}\delta\lambda\left[\frac{1}{2\pi}\int\frac{1}{D}-\frac{|\lambda|^2}{\pi}\int\frac{1}{D^3}\right]$$
$$-\frac{1}{(2n+1)}\delta R\left[\frac{1}{2\pi}\int\frac{(\bar{z}-\bar{R})}{D}-\frac{|\lambda|^2}{\pi}\int\frac{(\bar{z}-\bar{R})}{D^3}\right] \qquad (7.19)$$

where $D = |\lambda|^2+|z-R|^2$. The value of first integral is given in equation (7.15) if we replace R by $2R$. The second integral is obtained by differentiating the first integral twice w.r.t. $|\lambda|^2$. The other two integrals are evaluated from the following identity.

$$\frac{\partial}{\partial \bar{R}} \int \frac{(\bar{z} - \bar{R})}{D} \;=\; - |\lambda|^2 \int \frac{1}{D^2} \tag{7.20}$$

Integrating the right hand side w.r.t $\bar{R}$ will give us the third integral. While differentiating twice w.r.t $|\lambda|^2$ and integrating w.r.t $\bar{R}$ will give us the fourth integral. Finally we get the phase to be

$$\frac{1}{(2n+1)} \bar{\lambda} \delta \lambda \; \ln L \;+\; \frac{1}{2(2n+1)} \bar{R} \delta R \tag{7.21}$$

In the corresponding classical theory the Poisson brackets of functions of $\lambda, \bar{\lambda}, R$ and $\bar{R}$ as given in equation (6.10)

$$\{f, g\} \;=\; \omega^{\mu\nu} \partial_\mu f \partial_\nu g \tag{7.22}$$

where the non-zero $\omega^{\mu\nu}$ are

$$\omega^{\lambda\bar{\lambda}} = -\omega^{\bar{\lambda}\lambda} = \frac{2n+1}{\ln L} \qquad \omega^{R\bar{R}} = -\omega^{\bar{R}R} = 2(2n+1) \tag{7.23}$$

Hence $\bar{\lambda}$ is conjugate to λ and $\bar{R}$ is conjugate to R.

7.2 Finite spin skyrmions

In this section we construct eigenstates of spin operator which describe finite spin skyrmions. We shall determine the LLL skyrmion coherent states evolution under rotation in ordinary space and spin space and determine the eigenvectors as a linear combination of $\{|\lambda\rangle\}$ states with frozen R.

$\mathcal{R}_\theta$ is the unitary operator associated to a rotation in the two dimensional space by an angle θ and its action on the basis states in the N-particle sector is given as

$$\mathcal{R}_\theta \,|\{x_i, \sigma_i\}\rangle \;=\; e^{i\chi}\,|\{x_i e^{i\theta}, \sigma_i\}\rangle \tag{7.24}$$

where χ is an arbitrary phase as symmetries are implemented by unitary operators only up to a phase. We choose χ such that

$$\mathcal{R}_\theta \,|\psi_L\rangle \;=\; |\psi_L\rangle \tag{7.25}$$

where $|\psi_L\rangle$ is the Laughlin state with all the spins pointing down.

$\mathcal{U}_\varphi$ is the unitary operator associated to a rotation about the z-axis in the spin space by an angle φ and its action on basis states is defined as

$$\mathcal{U}_\varphi \,|\{x_i, \sigma_i\}\rangle \;=\; e^{i\chi'} \prod_{i=1}^{N} \left(e^{-\frac{i}{2}\varphi\tau_3}\right)_{\sigma_i \sigma_i'} |\{x_i, \sigma_i'\}\rangle \tag{7.26}$$

The phase χ is set such that

$$\mathcal{U}_\varphi \,|\psi_L\rangle \;=\; |\psi_L\rangle \tag{7.27}$$

We can now find the action of $\mathcal{R}_\theta$ on the LLL skyrmion state $|\lambda\rangle$.

$$\begin{aligned}
\mathcal{R}_\theta|\lambda\rangle &= \mathcal{R}_\theta \prod_i \left(\sum_{\sigma_i}\int_{x_i}\right) \Psi_\lambda(\{x_i\})\,|\{x_i, \sigma_i\}\rangle \\
&= \mathcal{R}_\theta \int_{\{x_i\}} \prod_i \begin{pmatrix} \lambda \\ z_i \end{pmatrix}_{\sigma_i} \psi_L(\{x_i\})\,|\{x_i, \sigma_i\}\rangle \\
&= \int_{\{x_i\}} \prod_i e^{-i\theta} \begin{pmatrix} \lambda e^{i\theta} \\ z_i \end{pmatrix}_{\sigma_i} \psi_L(\{x_i\})\,|\{x_i, \sigma_i\}\rangle \\
&= |\lambda e^{i\theta}\rangle
\end{aligned} \tag{7.28}$$

Similarly we find the action of $\mathcal{U}_\theta$ on $|\lambda\rangle$

$$\mathcal{U}_\varphi|\lambda\rangle \;=\; |\lambda e^{-i\varphi}\rangle \tag{7.29}$$

The skyrmion states $|\lambda\rangle$ are unaltered by the combined action-$\mathcal{U}_\varphi \mathcal{R}_\varphi$ and hence as expected are eigenstates of total momentum operator. We now construct a linear superposition of the skyrmion states as an eigenstate of spin. Consider the following state

$$|s\rangle = \int_\lambda \varphi_s(\lambda)\, |\lambda\rangle \tag{7.30}$$

$$\mathcal{U}_\varphi |s\rangle = \int_\lambda \varphi_s(\lambda)\, |\lambda e^{-i\varphi}\rangle = \int_\lambda \varphi_s(\lambda e^{i\varphi})\, |\lambda\rangle \tag{7.31}$$

For the following choice of normalized wavefunctions

$$\varphi_s(\lambda) = \frac{1}{\sqrt{2(2n+1)\kappa}} \frac{1}{\sqrt{2\pi s!}} \frac{\bar{\lambda}^s}{(2(2n+1)\kappa)^{\frac{s}{2}}}\, e^{-\frac{|\lambda|^2}{4(2n+1)\kappa}} \tag{7.32}$$

we get

$$\mathcal{U}_\varphi |s\rangle = e^{-is\varphi}\, |s\rangle \tag{7.33}$$

The states $|s\rangle$ are thus eigenstates of spin $\hat{s}^3$ and give us finite spin. These states are also orthogonal $\langle s'|s\rangle = \delta_{ss'}$.

Chapter 8

Conclusion

The motivation of this work was to examine the microscopic basis of the semiclassical NLSM for skyrmions at $\nu = 1/(2n+1)$. Therefore, we have developed a coherent state formalism of the composite boson theory. Specifically, we have addressed the questions stated in 1.3. Of particular importance is the question of the existence of a classical theory for Skyrmions and finding their phase space. It is also important to know how to impose the LLL condition in the classical phase space because, in the systems of interest, the energy scales are such that the dynamics predominantly takes place in the LLL.

We showed that the coherent state basis of $\mathcal{H}_{CB}$, when projected to the physical subspace $\mathcal{H}_{phy}$, can be parameterized by a spinor field that we denoted by $W_\sigma(x)$. In the hydrodynamic limit we have shown that these states, $|W\rangle$ themselves satisfy the coherent state properties of the resolution of unity and continuity of overlaps. The LLL condition is equivalent to the condition that $W_\sigma(x)$ are analytic functions.

The charge density is determined by the modulus of W i.e $W^\dagger(x)W(x)$ and the spin density by the normalized CP_1 spinor, $Z_\sigma(x)$. In general these are independent quantities and therefore the charge density is independent of the spin density. However if $W(x)$ is analytic, the modulus and phase of each of its components get tied up. We showed, that consequently, the excess charge density becomes proportional to the topological charge density which is determined by the spin density. Thus this proportionality will cease to hold in presence of Landau level mixing. We also showed that the condition for the total charge density to be proportional to the topological charge is weaker. It is sufficient if $W(x)$ is analytic at infinity. i.e. that the ground state does not have Landau level mixing.

We have given a classical theory for composite bosons. We showed that if we consider the set of states corresponding to classical configurations characterized by a length scale d, then they become orthogonal in the limit of $d \to \infty$. This implies that a system of skyrmions will behave classically in the limit of their sizes going to infinity. We have exploited the classical-quantum correspondence of the gauge invariant coherent states and derived the ground states for quantum hall systems. We have finally given a complete kinematical description of the LLL skyrmions in terms of their collective coordinates and constructed finite spin skyrmions. We find the collective coordinate λ becomes classical in LLL.

Our results show that the coherent state formalism we have developed can be used to study large skyrmions classically. While the LLL condition can be imposed easily, the formalism can also deal with the states where this is not done. The computation of energies etc. does not pose any additional calculational complication. Thus it can be used to study the effects of Landau level mixing in skyrmions. It could also be used to study the system in an external (slowly varying) potential. We will be addressing these problems in future.

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