# ASPECTS OF D- BRANE PHYSICS AND QUANTUM GRAVITY

by

TAPOBRATA SARKAR

A THESIS IN PHYSICS

Presented to the University of Madras in Partial Fulfillment of

the Requirements for the Degree of Doctor of Philosophy

SEPTEMBER 1999

The Institute of Mathematical Sciences

C.I.T. Campus, Taramani,

Chennai 600113 INDIA

THE INSTITUTE OF MATHEMATICAL SCIENCES
CIT CAMPUS, TARAMANI, CHENNAI 600 113, INDIA

## CERTIFICATE

This is to certify that the Ph.D. thesis titled "ASPECTS OF D- BRANE PHYSICS AND QUANTUM GRAVITY" submitted by **Tapobrata Sarkar** is a record of bonafide research work done under my supervision. The research work presented in this thesis has not formed the basis for the award to the candidate of any Degree, Diploma, Associateship, Fellowship or other similar titles. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

T. Jayaraman
Thesis Supervisor

September 1999

## ABSTRACT

In this thesis, we have studied some aspects of D-brane physics and quantum gravity. The main results obtained are summarised below.

In part one of the thesis, we have considered D-branes wrapped around supersymmetric cycles of Calabi-Yau manifolds from the viewpoint of $N = 2$ Landau-Ginzburg models on world sheets with boundary, as well as by consideration of boundary states in the corresponding Gepner models. Using the Landau-Ginzburg approach, we provide a target space interpretation for the boundary states. In our method, the boundary states are obtained by applying Cardy's procedure to combinations of characters in the Gepner models which are invariant under spectral flow. We relate the two descriptions using the common discrete symmetries of the two descriptions. We thus provide an extension to the boundary of the bulk correspondence between Landau-Ginzburg orbifolds and the corresponding Gepner models.

We have studied D-branes on Calabi-Yau manifolds from the point of view of gauged linear sigma models. We obtain an appropriate set of boundary conditions on the fields of the theory from the variation of the action under ordinary and supersymmetric variation. These boundary conditions, that define the D-brane are studied in the gauged linear sigma model as well as its infra-red limit. We find that we can obtain a consistent set of boundary condition describing D0 branes and also D-branes wrapping middle cycles of Calabi-Yau, the description of D2, D4 and D6 branes on Calabi Yau manifolds appear to be more difficult.

Next, we have studied an application of D-branes to a certain class of string theoretic black holes. We study the emission of scalar particles from a class of near-extremal five dimensional black holes and the corresponding D-brane configuration at high energies, and show that the distribution functions and the black hole grey-body factors are modified in the high energy regime of the Hawking spectrum in such a way that the emission rates exactly match in both descriptions. We extend the results to charged scalar emission in five dimensions and to neutral and charged scalar emission in four dimensions.

Finally, we have studied the application of holography, a generic principle of quantum gravity and hence of string theory, to inflationary cosmology. We have

applied the holographic principle during the inflationary stage of our universe. We have illustrate the analysis in the case of new and extended inflation which, together, typify generic models of inflation. We find that in the models of extended inflation type, and perhaps of new inflation type also, a naive application of the holographic principle leads to its violation, whereas the correct procedure, that restores the holographic principle, leads to a lower bound on the density fluctuations.

## PUBLICATIONS

Publications used in the thesis

1. S. Govindarajan, T. Jayaraman and Tapobrata Sarkar, *Worldsheet approaches to D-branes on supersymmetric cycles*, `hep-th/9907131`

2. Saurya Das, Arundhati Dasgupta and Tapobrata Sarkar, *High energy effects on D-brane and black hole emission rates*, Phys.Rev. **D55** (1997) 7693

3. S. Kalyana Rama and Tapobrata Sarkar, *Holographic principle during inflation and a lower bound on density fluctuations*, Phys.Lett. **B450** (1999) 55

Other publications

1. Saurya Das, Arundhati Dasgupta, P. Ramadevi and Tapobrata Sarkar, *Planckian scattering of D-branes*, Phys. Lett. **B428** (1998) 51

2. Saurya Das, Arundhati Dasgupta, Parthasarathi Majumdar and Tapobrata Sarkar, *Black Hole fermionic radiance and D-brane decay*, (`hep-th/9707124`)

3. Tapobrata Sarkar and Rahul Basu, *Validity of double scaling analysis in semi-inclusive processes: $J/\psi$ production at HERA*, Eur. Phys. J **C5** (1998) 493

4. Tapobrata Sarkar and Rahul Basu, *The asymptotic behaviour of $F(L)$ in the double scaling limit*, (`hep-ph/9607232`)

# ACKNOWLEDGMENTS

*I would like to thank my family for their understanding and encouragement throughout the course of my Ph.D work.*

*I would like to sincerely thank my thesis supervisor, T. Jayaraman for his patience, advice and constant encouragement during the course of this work, and for being a good friend. I am indebted to him for guiding me during this work, for the several things that he has taught me, and for introducing me to some of the most interesting aspects of superstring theory.*

*My sincere thanks to Suresh Govindarajan for his advice and encouragement, and for collaboration and stimulating discussions during my work, and the painstaking effort taken by him in helping me with the proofreading of this thesis.*

*I would like to thank S. Kalyana Rama for discussions and collaboration during part of the work presented in this thesis.*

*I am thankful to my colleagues Saurya and Arundhati for many fruitful discussions and for collaboration during part of the work reported in this thesis.*

*Thanks are due to Parthasarathi Majumdar and Rahul Basu for discussions and collaboration in the course of my work at IMSc.*

*I would like to express my thanks to Geetanjali, for her help and support, and for being what she is.*

*Thanks are due to Ash for helping to maintain a very spirited atmosphere.*

*I would like to thank Subrata for being a good friend and helping me out in times of need.*

*Thanks to Neil, Shasikant, Roy, Ganesh and Bonny for providing the much needed non-academic breaks.*

*Finally, I would like to thank all the academic and non-academic members of IMSc without whose help and support this work could not have been done.*

# Contents

# Chapter 1

# Introduction

String theory [1][2] is the leading candidate for a fundamental theory of nature that unifies all known interactions. While at observable distance scales, quantum field theory (QFT) provides an accurate description of natural phenomena, it becomes unsatisfactory at very short distances, i.e at extremely large energies. This is because in such regimes, quantum effects of gravity become important, and it has proved to be extremely difficult to formulate a consistent theory of quantum gravity, using known methods of QFT.

The underlying reason for this difficulty is that QFT is essentially a theory of point particles, whose interactions diverge at small distances. Whereas for the electromagnetic and for the strong and weak interactions, these divergences can be handled by the standard method of renormalisation, this procedure fails for a theory of gravity, which can be shown to be non-renormalisable under usual QFT techniques. This inability to formulate a consistent, finite theory of gravity in the framework of QFT leads us to string theory. In this theory, the basic entities are one-dimensional extended objects, called strings, rather than point particles. These extended objects, embedded in space, trace out a world sheet with time. Consistency conditions require that the dimension of this embedding space-time (often referred to as the target space) is ten, for a string theory that describes both bosons and fermions. Further, perturbatively consistent string theories appear to require space-time supersymmetry.

Due to the extended nature of strings, short distance divergences that arise in usual QFT because of the point-like structure of its basic entities, are absent. Oscillations of the string give rise to an energy spectrum that can be interpreted as particle excitations. Among the several types of particles that arise in the spectrum of the oscillating string are the massless spin two particle, the graviton, that is present in all consistent string theories. This makes string theory a natural candidate for a quantum theory of gravity. Among its various other features, string theory can give rise to abelian and non-abelian gauge interatcions that can include the standard model, and is thus a leading candidate for a grand unified theory, that includes gravity.

There were, however, a number of drawbacks in the initial formulations of string theory. First of all, in these theories, gravity was treated perturbatively, i.e the metric was assumed to be a small perturbation over flat space. This was not entirely satisfactory, since it was not apriori clear how to describe, for example, objects like black holes, in the framework of string theory. Further, there was no unique string theory, and five known consistent string theories could be formulated. In the past few years, there has been a great deal of progress in our understanding of these issues. From the ideas of string duality, it has been possible to relate certain string theories at small values of the string coupling to other string theories at strong coupling. All the consistent theories, formulated perturbatively, turn out to be limits in the space of vacua of a single theory, called the M-theory. (Some perhaps would prefer a description in terms of two basic theories, M and F-theory). Duality conjectures have also motivated in part the discovery of extended objects in string theory, called D-branes. These objects, which are solitonic in nature, are playing a very important role in the development of our understanding of physics at strong gravitational coupling.

In this thesis, we have studied some aspects of D-brane physics and quantum gravity. The thesis is organised as follows. In chapter 2, we will study D-branes on curved manifolds [3]. In particular, we consider D-branes wrapped around supersymmetric cycles of Calabi-Yau manifolds. We use two different approaches for this. First, we treat these objects from the viewpoint of supersymmetric Landau-Ginzburg models on a world sheet with boundary. Secondly, we formulate a description of these wrapped D-branes by considering their boundary states, using Gepner models. This method uses Cardy's general prescription for the construction of boundary states in boundary conformal field theories. The Landau-Ginzburg approach enables us to provide a target space interpretation for the boundary states. We will relate the two approaches by using the common discrete symmetries of the two descriptions.

In chapter 3, we will consider D-branes on curved manifolds, using the formalism of supersymmetric gauged linear sigma models on a world sheet with boundary [4]. We attempt to construct a consistent set of boundary conditions on the fields of this model, that describes D-branes, that is successful for some cases. We show how the consistency of boundary conditions on the various fields demands the addition of new surface terms in the theory.

In chapter 4, we turn to the application of D-branes in the physics of black holes. We study the process of scalar Hawking radiation from a class of near-extremal five and four dimensional black holes and the corresponding D-brane configurations, at high energies [5]. We show how the correspondence between black-holes and D-branes, one aspect of which is the similarity of the Hawking radiation spectrum, is preserved even for high-energy particle emission in five dimensions. We also point out a possible discrepency for four dimensional black holes.

In chapter 5, we study the application of a general principle of quantum gravity (and hence of string theory), namely holography, in the context of cosmology [6]. We discuss how to apply the holographic principle (initially proposed for standard cosmology) to the scenario of the inflationary universe. We discuss how naive application of the holographic principle in this case leads to its violation, while the correct procedure that does not violate the principle in the inflationary universe leads to a lower bound on density fluctuations of the universe, that is close to observed values.

Finally, we conclude with some remarks on possible further directions of study. In the rest of this introductory chapter, we review some of the background necessary for the work reported in this thesis. We begin with a brief review of string theory and its solitonic solutions, the D-branes.

## 1.1 String theory and D-branes

The worldsheet action for a free string theory propagating in ten dimensional flat space-time (in conformal gauge) is given by

$$S = \frac{1}{4\pi\alpha'}\int d\sigma d\tau\left[\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \partial \psi_\mu \ + \ c.c\right] \tag{1.1}$$

where $\alpha$ denotes the world sheet coordinates $\sigma$ and $\tau$, the $X^\mu$s and $\psi^\mu$s are the bosonic and fermionic degrees of freedom, with $\mu = 0, 1, \cdots 9$ and the string tension is given by $T = \frac{1}{2\pi\alpha'}$. The bosonic part of the action, when written in the Nambu-Goto form, can be seen to be proportional to the area of the world sheet in ten dimensional space-time. Consistency of the quantum theory forces the restriction that the target space is ten dimensional. There are five different types of consistent string theories in ten space-time dimensions, namely the Type I, Type IIA, Type IIB, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic strings. Details of these can be found in the standard literature [1][2]. In this thesis, we will be concerned mostly with Type II superstrings, and, for the sake of completeness, we will briefly discuss Type II strings in ten dimensional flat space-time.

Type II string theory is a theory of purely closed strings. The world sheet theory in these cases is a free field theory that contains ten scalars and ten Majorana fermions. The fermionic degrees of freedom living on the world sheet can have periodic or anti-periodic boundary conditions on a spatial slice of the world sheet (which is a circle in this case). Periodic boundary conditions are usually referred to as Ramond (R) boundary conditions, while anti-periodic ones are called Neveu-Schwarz (NS) boundary conditions. The scalars satisfy the usual periodic boundary condition on the world sheet.

Since there are two sets of fermions, namely the left moving and the right moving ones, we can specify four different sectors in the theory. These are the NS-NS, R-R, NS-R and R-NS sectors respectively. Space-time bosons are obtained from the NS-NS and R-R sectors, and space-time fermions from the other two sectors. Two different Type II theories can be defined, by projecting the full spectrum onto states that contain only an even number of left moving and right moving fermions, a procedure called the GSO projection. In the Type IIA theory, opposite GSO projections are taken in the left and right sectors, so that the target-space fermionic spectrum is nonchiral, while similar GSO projection in the two sectors defines the Type IIB theory, whose target-space fermionic spectrum is chiral.

As we have remarked, the space-time bosons in the two theories come from the NS-NS and the R-R sectors. Type IIA and IIB theories differ only in the Ramond sector, and hence the NS-NS bosonic states are the same for both theories. These are the metric, a rank two antisymmetric tensor, and the dilaton. The massless states in the R-R sector, are, however, different in the two theories. While in the Type IIA theory, these consist of a one form vector field and a rank three antisymmetric tensor, in the Type IIB theory, the massless R-R spectrum consists of a scalar, a rank two antisymmetric tensor, and a rank four self-dual anti-symmetric tensor field.

Generically, a $(p+1)$ form potential coming from the R-R sector would couple to a $p - brane$, which is an extended object in $p$ space dimensions, generalising the one form vector potential coupling to the electric charge in electromagnetism. In fact Type II supergravities, that arise as the low energy limit of Type II string theories, are known to have black p-brane solutions that carry such charges [7]. In perturbative closed string theory, however, there are no objects that can carry R-R charge. This is because in the space of states of perturbative string theory, the R-R fields appear through the field strengths, rather than the potentials. Most non-perturbative string dualities, on the other hand, require the existence of such solitonic R-R charged objects. Consider for example, M-theory, whose low energy limit is 11 dimensional supergravity. Massless R-R and NS-NS fields of the Type IIA can be obtained by dimensional reduction of 11-D supergravity. The dimensional reduction of 11-D supergravity however also gives Kaluza-Klein states, which are charged under a R-R field in the Type IIA theory. Since, perturbative string states do not carry R-R charges, this implies the existence of states that carry this charge, outside the usual perturbative spectrum. These R-R charged states can in fact be shown to be solitonic zero-branes of Type IIA theory. Similarly, the $SL(2, Z)$ self duality of the Type IIB string theory can be shown to require the existence of objects that carry a R-R 2-form charge, and thus requires the existence of a solitonic 1-brane, namely a solitonic string. Higher (solitonic) p-branes carrying R-R charges can similarly be shown to be required under certain duality conjectures.

Solitonic solutions in string theory carrying NS-NS charge were discussed earlier in the context of solitons that couple to the dual of the NS-NS rank 2 antisymmetric tensor field. The Type IIA or Type IIB fundamental string itself couples to the NS-NS 2-form field, while its Hodge dual, a 6-form potential, couples to a 5-brane, commonly known as the NS 5-brane. The NS 5-brane breaks half of the spacetime supersymmetry and hence is a BPS saturated object. An important feature of this soliton is that the dilaton field blows up at the centre of the solution, which signals the presence of a region of strong string coupling, called the core. Away from the core, the five-brane is exactly described by a superconformal field theory. The SCFT in transverse coordinates is the tensor product of an $N = 4$ supersymmetric $SU(2)$ WZW model and a free scalar field with a background charge.

In contrast, solitons carrying R-R charges have a simpler CFT description. Initial efforts of realising the solitonic states involved the construction of these solutions as (singular) solutions of the low energy supergravity equations of motion. However, a means of realising these solitonic objects in a CFT description (with an appropriate world sheet) as in the case of perturbative string states was still lacking. This situation was remedied by Polchinski [8], who suggested a novel way to construct these solitonic solutions, by CFT methods. These solutions are known as the D(irichlet)-branes, a Dirichlet p-brane being charged by one unit under the R-R $(p+1)$ form gauge field. We may note here that the difference between these objects and the usual quantum field theory solitons is that the mass of D-branes go as $\frac{1}{g_s}$, $g_s$ being the string coupling, as opposed to the usual inverse square relationship of the mass to the coupling constant for ordinary field theory solitons as well as NS 5-branes. These D-branes saturate the Bogomolny-Prasad-Sommerfield (BPS) bound, i.e the mass per unit volume $m$ of a D-brane in appropriate units equals its R-R charge, $q$. They also break half the space-time supersymmetry.

The existence of D-branes as solitons can be understood in the following simple way. Consider the bosonic part of the open string action, in eq. (1.1)

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\,[\partial_\alpha X^\mu \partial^\alpha X_\mu]$$

The variation of this action is

$$\delta S = -\frac{1}{2\pi\alpha'}\int d^2\sigma \delta X^\mu \partial^2 X_\mu + \frac{1}{2\pi\alpha'}\int d\sigma \delta X^\mu \partial_n X_\mu$$

In order to make the surface term vanish, we might impose either Neumann or Dirichlet boundary conditions on the coordinates, Imposing Neumann boundary condition on the field $X^\mu$ would imply $\partial_n X^\mu = 0$ with $n$ denoting the normal derivative to the open string world sheet. This boundary condition is Poincare invariant (in the target space) in the subset of $\mu$'s on which the condition is imposed. We can also impose Dirichlet boundary conditions $\partial_\tau X^\mu = 0$ on some of the coordinates, where $\partial_\tau$ denotes the tangential derivative along the world sheet. Imposing Dirichlet boundary conditions on some or all of the coordinates defines a hyper-plane in the theory, whose world volume lies along the directions transverse to the ones on which Dirichlet boundary conditions have been imposed. Further, open string endpoints can lie on this hyperplane. This extended object is called a D-brane. In a ten dimensional superstring theory, a D-p brane, i.e a p-dimensional extended object is defined in terms of the boundary conditions on the bosonic fields as:

$$\begin{aligned} \partial_n X^\alpha(\sigma = 0, \pi) = 0 \quad 0 < \alpha < p \\ X^\beta(\sigma = 0, \pi) = x^\alpha \quad (p+1) < \beta < 9 \end{aligned} \tag{1.2}$$

Boundary conditions on the world sheet fermions are determined from the above, using various consistency conditions. For example, world sheet supersymmetry transformations, that relate these bosonic fields to the corresponding fermions impose various conditions on the fields, if one requires that the boundary conditons on the bosons as given above are to be compatible with these transformations. As we have mentioned earlier, we consider only Type II theories in this thesis. Type IIA theory contains D-p branes for even values of $p$, while for type IIB strings, there exists D-p branes for odd $p$. An important property of these D-branes is that they are BPS saturated objects. Due to the presence of boundaries in the open string world sheet that defines the D-brane, the left and right moving space-time supercurrents get related at these boundaries, as a consequence of which only a linear combination of the corresponding supercharge is conserved. Thus, the D-brane breaks half the space-time supersymmetry and is BPS saturated. As a consequence of this, parallel D-branes do not exert any force on each other. This can be seen as follows. Consider the interaction of two parallel D-p branes via the exchange of a closed superstring. This is a tree-level process in the closed string channel, but can also be viewed as a one-loop open string process. The one loop amplitude in the open string channel can be evaluated using the Coleman-Weinberg formula [9] and the result is

$$\mathcal{A} = 2V_{p+1} \int \frac{dt}{t} \left(8\pi\alpha' t\right)^{-\frac{p+1}{2}} e^{\frac{-t}{2\pi\alpha'}x.x} \times \frac{1}{2\eta(q)^4} \left[\theta_3(q)^4 - \theta_2(q)^4 - \theta_4(q)^4\right] \qquad (1.3)$$

where $V_{p+1}$ denotes the volume of the D-p brane, $x$ is a vector denoting its separation in transverse space, and $q = e^{-\pi t}$, where $\infty < t < 0$ is the modular parameter. $\eta$ is defined as $\eta(q) = q^{1/6}\prod_{n=1}^{\infty}(1-q^{2n})^2$ and the $\theta_i$ are the usual Jacobi theta functions. The expression for $\mathcal{A}$ vanishes due to the Jacobi identity, and hence the net static force between two parallel D-p branes is seen to be zero. From the point of view of the closed string channel, this is interpreted as the cancellation of forces between the NS-NS and R-R sectors reflecting the fact that the D-branes are BPS saturated.

Action principles for D-branes, describing its low-energy degrees of freedom, were first formulated in [10] in the context of bosonic D-branes. Here, a world sheet non-linear sigma model corresponding to mixed Dirichlet-Neumann boundary conditions (defining the D-brane) was written down for bosonic string theory, and it was shown that the equations of motion of the background fields computed from the sigma model (via usual renormalization group techniques) agreed with those obtained from a proposed world-volume Dirac-Born-Infield (DBI) action of the D-brane. Supersymmetric generalisations of DBI actions, corresponding to branes in Type II theories were considered by several authors. In [11], DBI actions that consistently incorporated R-R gauge field couplings to the D-brane were considered. Let us briefly state this result. It was shown in [11] that the action for a D-1 brane, that couples to a R-R 2-form field is given by

$$\begin{aligned} S_2 &= T\int d^2\sigma e^{-\phi(X)}\sqrt{det}(G_{mn}(X) + B_{mn}(X) + F_{mn}(X)) \\ &+ \frac{1}{2}i\epsilon^{mn}\left(C_{mn} + C(F_{mn} + B_{mn})\right) + \cdots \end{aligned} \qquad (1.4)$$

Where $T$ is related to the D-string tension by $T^D = e^{-\phi/2}T$, and $G_{mn}$ $B_{mn}$ and $C_{mn}$ are the pullbacks to the world sheet of the target space metric, the NS-NS 2-form field strength, and the R-R two form field strength, and are given by

$$G_{mn} = G_{\mu\nu}(X)\partial_m X^\mu \partial_n X^\nu \quad C_{mn} = C_{\mu\nu}(X)\partial_m X^\mu \partial_n X^\nu \quad B_{mn} = B_{\mu\nu}(X)\partial_m X^\mu \partial_n X^\nu$$

where $\mu, \nu = 0, 1, \cdots 9$. The dots indicate the fermionic part of the action. The $\sigma$'s are world sheet coordinates. $F_{mn}$ is the world sheet gauge field strength. Actions for higher dimensional branes can also be written down similarly.

## 1.2 D-branes on curved manifolds

Till now, we have considered superstrings and D-branes in ten dimensional flat space-time. In order to formulate more realistic theories in four dimensional space-time, one has to compactify some of the dimensions of the theory. A string theory with four space-time dimensions can, for example, be constructed by considering the ten dimensional space-time to be an (external) Minkowski four-manifold $\mathcal{M}$ times an "internal manifold" $\mathcal{I}$, which is six-dimensional. In the limit when the dimensions of $\mathcal{I}$ become very small, one expects to obtain a theory in four dimensions. The manifold $\mathcal{I}$ constrains the behaviour of the theory in the four external dimensions, and as it turns out, there are only a few choices for the internal manifold $\mathcal{I}$ that gives rise to phenomenologically interesting theories in $\mathcal{M}$. In such theories, one would like to have some unbroken space-time supersymmetry, which is required to obtain a tachyon-free string spectrum and is also important in resolving issues like the cosmological constant problem. [1] Such supersymmetric compactification has been studied widely in the literature. One can, for example, consider the internal space-time to be a torus of appropriate dimensions. Toroidal compactifications, however, give rise to theories with large supersymmetry, and are not phenomenologically interesting. Theories that preserve lesser space-time supersymmetry were studied in the important work of Candelas et. al [12], in which it was shown that solutions to the classical equations of motion of ten-dimensional $N = 1$ supergravity could be

[1]While recently there has been much discussion of non-supersymmetric string theories without tachyons, it is still not clear that such theories are consistent. The issue of non-perturbative breaking of supersymmetry in string theory remains an unresolved problem.

obtained, preserving $N = 1$ supersymmetry in four-dimensional space time, provided the internal manifold $\mathcal{I}$ was a *Calabi-Yau* manifold.

Let us first briefly describe the basic notions about Calabi-Yau manifolds. A complex manifold is a topological space, with local complex coordinates, denoted by $(z^\mu, z^{\bar{\mu}})$, such that the transition functions which relates the local coordinates in two different patches of the manifold are purely holomorphic i.e a function of $z^\mu$ only. A complex manifold is called Hermitian if it is has a metric that can be written in the form

$$ds^2 = g_{\mu\bar{\nu}} dz^\mu dz^{\bar{\nu}}$$

with $g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$. In a hermitian manifold, it is possible to write down a $(1,1)$ Kähler form, using the hermitian metric,

$$J = i g_{\mu\bar{\nu}} dz^\mu \wedge dz^{\bar{\nu}}$$

and the complex manifold is called a Kähler manifold if this $(1,1)$ form is closed, i.e if $dJ = 0$. A Calabi-Yau (CY) manifold in $d$ complex dimensions is defined to be one which is Kähler, and in addition has an $SU(d)$ holonomy. It can be shown that the metric on such a manifold is Ricci-flat. Alternatively, one can define a CY manifold as a Kähler manifold with a vanishing first Chern class, with a theorem due to Yau then implying that there exists a unique Ricci-flat metric on the manifold. A notion that will be useful for us is that of a $(p,q)$ form, which is a completely antisymmetric tensor of rank $(p,q)$, defined as $\Omega^{p,q} = \Omega_{\mu_1,\cdots\mu_p,\bar{\nu}_1\cdots\bar{\nu}_q} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_p} dz^{\bar{\nu}_1} \wedge \cdots \wedge dz^{\bar{\nu}_q}$ A $(p,q)$ form is said to be harmonic if it is annihilated by the Laplacian operator acting on it, and we can define the Hodge numbers $h^{p,q}$ of the Calabi-Yau manifold as the number of independent harmonic $(p,q)$ forms on it.

Now let us briefly recapitulate the spectrum of Type II string theory compactified on a CY manifold of complex dimension 3, i.e with the external space-time being four dimensional Minkowski space-time. The spectrum of these theories can be shown to have the multiplet structure of $N = 2$ supersymmetry in four dimensional space-time. In particular, with $h^{2,1}$ denoting the number of harmonic $(2,1)$ forms, and $h^{1,1}$ the number of harmonic $(1,1)$ forms, it can be shown that the spectrum of Type IIA theory will contain $h^{2,1} + 1$ hypermultiplets (each containing four scalars) and

$h^{1,1}$ vector multiplets. For Type IIB theories on CY manifolds, it can be similarly shown that there are $h^{1,1}+1$ hypermultiplets and $h^{2,1}$ vector multiplets.

The internal part of a string theory compactified on CY can be described in terms of a non-linear sigma model, with a CY manifold as the target space. Let us consider the compactification of a superstring theory on a CY 3-fold. The corresponding $N=2$ supersymmetric non linear sigma model (NLSM) in two dimensions governs the map $\Phi : \Sigma \to \mathcal{I}$ where $\mathcal{I}$ is a Calabi Yau manifold and $\Sigma$ is a two dimensional Riemann surface characterised by local coordinates $(z,\bar{z})$. Denoting the bosons, which are the (complex) co-ordinates of the target space CY as $\phi^{\mu}$ and $\phi^{\bar{\nu}}$, and the left moving and right moving fermions as $\psi^{\mu}_{+}$ and $\psi^{\nu}_{-}$, and the corresponding complex conjugates as $\psi^{\bar{\mu}}_{+}$ and $\psi^{\bar{\nu}}_{-}$, the action for this model can be written as

$$S=\frac{1}{4\pi\alpha'}\int d^2z \quad \Big[\frac{1}{2}g_{\mu\bar{\nu}}\partial_z\phi^{\mu}\partial_{\bar{z}}\phi^{\bar{\nu}}+ig_{\bar{\mu}\nu}\psi^{\bar{\mu}}_{-}D_z\psi^{\nu}_{-}+ig_{\bar{\mu}\nu}\psi^{\bar{\mu}}_{+}D_{\bar{z}}\psi^{\nu}_{+}+ R_{\mu\bar{\mu}\nu\bar{\nu}}\psi^{\mu}_{+}\psi^{\bar{\mu}}_{+}\psi^{\nu}_{-}\psi^{\bar{\nu}}_{-}\Big] \tag{1.5}$$

Gepner [13], on the other hand, provided an exact CFT description of superstrings compactified on CY by using tensor products of minimal models, which are unitary representations of $N=2$ SCFT's, with the central charge of the $k$th minimal model being $c=\frac{3k}{k+2}$. In this construction, the internal SCFT part of the compactified theory is constructed by tensoring together an appropriate number of minimal models so that the central charge of the product theory has the required value, and then projecting the spectrum onto odd integral $U(1)$ charged states. For example, compactification on a Calabi-Yau 3-fold requires an internal SCFT with $c=9$, and one particular way of realising it is to tensor five copies of the $k=3$ minimal model, each having a central charge of $\frac{9}{5}$. Carrying this out, and finally projecting onto states with odd integral $U(1)$ charge, it can be shown that the massless spectrum of this model is identical to that of the superstring spectrum on the quintic hypersurface in $CP^4$.

Gepner's description of Calabi-Yau compactification of superstrings via minimal models may seem surprising, because apriori the minimal models do not have any geometric significance. This connection between minimal models and CY was studied further by Greene *et al* [14], using the fact that the RG fixed point of the $N=2$

Landau-Ginzburg (LG) model for a single chiral superfield $\Phi$, with superpotential $W(\Phi) = \Phi^{k+2}$ is indeed the $k$'th $N = 2$ minimal model. Using a naive path integral approach, these authors showed how the form of the LG superpotential corresponding to a particular combination of minimal models gives rise to a constraint that is exactly similar to the defining equation of the corresponding CY. This was put on a more firm basis by Witten in [15], where it was shown, starting from a gauged linear sigma model, that the CY non-linear sigma model and the LG model are actually describing two different phases of the same theory, with the variation of the Kähler parameter in the theory interpolating between the two. Let us briefly review this argument.

We start with the gauged linear sigma model (LSM) in two dimensions, that has $N = 2$ supersymmetry, and is obtained from dimensional reduction of $N = 1$ supersymmetry in four space-time dimensions. The theory contains a chiral multiplet and a vector multiplet. The chiral multiplet consists of six scalar fields $\phi_i$, their fermionic partners, $\psi_{+i}$ and $\psi_{-i}$, the auxiliary fields $F_i$ and the complex conjugates of these. Here, $+$ and $-$ labels on the fermions label the left moving and the right moving ones respectively. The vector multiplet contains the gauge fields $v_0$ and $v_1$, the real auxiliary field $D$, the scalar $\sigma$, the left and right moving fermions, $\lambda_+$ and $\lambda_-$ and the complex conjugates of the $\sigma$ and $\lambda$. The action, with the simplest gauge group $U(1)$ contains the usual kinetic energy term, gauge term, and a term involving the theta parameter. In addition, it contains a superpotential term and a

Fayet-Iliopaulos term. These terms are given by

$$\begin{aligned}
S_{kin} &= \int d^2y \; -D_\alpha\bar{\phi}_i D^\alpha \phi_i + \frac{i}{2}\left[\bar{\psi}_{-i}(D_0+D_1)\psi_{-i} - \left((D_0+D_1)\bar{\psi}_{-i}\right)\psi_{-i}\right] \\
&\quad + \frac{i}{2}\left[\bar{\psi}_{+i}(D_0-D_1)\psi_{+i} - \left((D_0-D_1)\bar{\psi}_{+i}\right)\psi_{+i}\right] \\
&\quad + |F_i|^2 - 2\bar{\sigma}\sigma Q_i^2\bar{\phi}_i\phi_i - \sqrt{2}Q_i(\bar{\sigma}\bar{\psi}_{+i}\psi_{-i} + \sigma\bar{\psi}_{-i}\psi_{+i}) + DQ_i\bar{\phi}_i\phi_i \\
&\quad - i\sqrt{2}Q_i\bar{\phi}_i(\psi_{-i}\lambda_+ - \psi_{+i}\lambda_-) - i\sqrt{2}Q_i\phi_i(\bar{\lambda}_-\bar{\psi}_{+i} - \bar{\lambda}_+\bar{\psi}_{-i}) \\
S_{gauge} &= \frac{1}{e^2}\int d^2y \; \frac{1}{2}v_{01}^2 + \frac{1}{2}D^2 + \frac{i}{2}\left[\bar{\lambda}_+(\partial_0-\partial_1)\lambda_+ - \left((\partial_0-\partial_1)\bar{\lambda}_+\right)\lambda_+\right] \\
&\quad + \frac{i}{2}\left[\bar{\lambda}_-(\partial_0+\partial_1)\lambda_- - \left((\partial_0+\partial_1)\bar{\lambda}_-\right)\lambda_-\right] - \partial_\alpha\sigma\partial^\alpha\bar{\sigma} \\
S_W &= -\int d^2y \left(F_i\frac{\partial W}{\partial\phi_i} + \frac{\partial^2 W}{\partial\phi_i^2}\psi_{-i}\psi_{+i} + F_i^*\frac{\partial\bar{W}}{\partial\bar{\phi}_i} - \frac{\partial^2\bar{W}}{\partial^2\bar{\phi}_i}\bar{\psi}_{-i}\bar{\psi}_{+i}\right) \\
S_{r,\theta} &= -r\int d^2yD + \frac{\theta}{2\pi}\int d^2y v_{01} \qquad (1.6)
\end{aligned}$$

Here, $D_0 = (\partial_0 + iQ_i v_0)$ and $D_1 = (\partial_1 + iQ_i v_1)$ denote the gauge covariant derivatives, $i$ labels the coordinates on a complex manifold $C^{n+1}$. and $v_{01} = \partial_0 v_1 - \partial_1 v_0$ is the $U(1)$ field strength. The fields $D$ and $F_i$ enter as auxiliary fields, and one can solve for their classical equations of motion, and obtain

$$\begin{aligned}
D &= -e^2\left(\sum_i Q_i|\phi_i|^2 - r\right) \\
F_i &= \frac{\partial W}{\partial\phi_i} \qquad (1.7)
\end{aligned}$$

Where $e$ is the gauge coupling constant and $Q_i$ denotes the $U(1)$ charges of the bosonic fields $= \phi_i$. The potential energy of the dynamical scalar fields $\phi_i$ and $\sigma$ (a scalar field from the gauge multiplet) is given by

$$U(\phi_i,\sigma) = 2|\sigma|^2\sum_i Q_i^2|\phi_i|^2 + \frac{D^2}{2e^2} + \sum_i |F_i|^2 \qquad (1.8)$$

Requiring anomaly cancellation and R-invariance, The superfields are taken to be $n$ fields $S_i$ of charge 1 and 1 field $P$ of charge $-n$, ensuring that the total $U(1)$ charge of the bosonic fields is zero, and in addition we assume a gauge invariant form of the superpotential, $W = P.G(S_i)$ where $G$ is a homogeneous polynomial of degree $n$. Then, the $D$ term in the potential energy is given by

$$D = -e^2\left(\sum_i |s_i|^2 - n|p|^2 - r\right)$$

$p$ and $s_i$ being the bosonic components of $P$ and $S_i$ respectively. Now, minimising the potential energy, and noting the non-trivial dependence of the Fayet-Iliopoulos term $D$ on $r$, one can show that for $r >> 0$, the minimising procedure gives

$$\sum_i \bar{s}_i s_i = r$$

which is precisely a copy of the complex projective space $CP^{n-1}$ with a Kähler class proportional to $r$. On the other hand, in the limit $r << 0$, i.e negative values of the Kähler parameter, one obtains a unique classical vaccum state, with the massless excitations governed by a superpotential that has a degenerate critical point, and hence defines a Landau-Ginzburg theory. Witten's construction thus shows that the Calabi-Yau and Landau-Ginzburg are two different phases of the same theory, the LG being obtained by the analytic continuation of CY to negative Kähler class.

It is clear that the topics that we have discussed above relating to superstring compactification need to be studied in the context of D-branes. A study of D-branes, in string theories compactified on CY, would naturally be a very important part of the study of these solitons. D-branes wrapped on *supersymmetric* cycles of CY, have been, in particular, a topic of interest. In [16] and [17], wrapping of branes on such cycles were discussed and the authors obtained geometric criteria for the wrapping cycles to preserve half the space-time supersymmetry. Wrapping of D-branes on supersymmetric cycles of CY from a CFT point of view was discussed by Ooguri *et al* [18]. They considered boundary states of such D-branes, and showed how the geometric data is encoded in the boundary states. The difference in approach between [16] and [18] is that while the former used the formalism of low energy effective supergravity actions for the p-brane solitons to study the supersymmetric cycles, the latter used the approach of the open string $N = 2$ world sheet superconformal field theory (SCFT) to study these cycles.

In [18], Ooguri *et al* classified the boundary conditions for the world sheet $N = 2$ SCFT that preserves half the space-time supersymmetry, by considering relationships between the fields in a supersymmetric sigma model for Calabi-Yau given in eq. (1.5). This model can be twisted in two different ways to give topological field theories, that are related by mirror symmetry [19]. There are two possible boundary conditions for the model (called the A-type and the B type boundary conditions) which relate to the two different ways of topologically twisting it (the A twist and the B twist). In [18], appropriate Dirichlet and Neumann boundary conditions were imposed on the fields of the model of (1.5), thus defining D-branes in these models. It was shown that the A-type boundary conditions in particular led to D-branes wrapping middle dimensional cycles of the target space Calabi-Yau, which are special Lagrangian submanifolds, and the B type boundary condition corresponds to D-branes wrapping on even dimensional cycles, which are complex holomorphic submanifolds. These authors also analysed mirror symmetry in the presence of D-branes, and some issues regarding open string world sheet instanton corrections.

Analysis of D-branes on supersymmetric cycles of CY from a conformal field theory viewpoint, was further discussed by Recknagel and Schomerus [20], who have used Gepner models in their analysis. Boundary states for Gepner models can be obtained by tensoring together an appropriate number of minimal models, and imposing the A or B type boundary conditions that we have just described. This procedure involves Cardy's construction of boundary states in conformal field theories defined on a manifold with boundary.

In the work presented in chapter 2 of this thesis, we have taken a somewhat different approach in studying boundary states of D-branes in Gepner models. Our method uses the formalism developed in [21], where a space-time supersymmetric modular invariant partition function for the closed string was constructed. This method used certain combination of characters in the $N = 2$ minimal models that are invariant under spectral flow, an operation, which in general, interpolates between the various isomorphic representations associated with the $N = 2$ superconformal algebra. Using the supersymmetric characters associated with spectral flow (which in this case can be thought of as interpolating between the Neveu-Schwarz and Ramond sectors of the theory), we will discuss how to construct cylinder partition functions in a manner in which some space-time supersymmetry is preserved. We will calculate the associated boundary states, and in order to provide a target-space interpretation of these boundary states, we will consider boundary Landau-Ginzburg

(LG) models. The relationship of LG models with Gepner models is provided by the fact that we have stated earlier, namely that an LG model of a scalar superfield with superpotential $\Phi^{k+2}$ flows in the infrared, to a level $k$ $N = 2$ minimal model.

In chapter 3 of this thesis, we will study the CY-LG correspondence, from the point of view of Witten's LSM, in the presence of D-branes. It is expected that D-branes can be defined for some continuous value of the Kähler parameter of the theory, $r$. For example, one may consider the effect of putting a D-brane on a CY space and varying the Kähler parameter $r$. Reducing the value of $r$ and making it negative, one enters the LG phase. It is interesting to see what happens to the D-brane under this change of Kähler parameter, by using the LSM description of the D-brane boundary CFT. In this framework, one can also investigate processes that result in a change of topology of space-time. Aspinwall *et al*, in [22] have argued that there are physcally smooth processes in string theory that result in such changes of topology. These processes are interesting if we keep in mind that in Einstein's general theory of relativity (GTR), the space-time metric is defined with respect to a fixed topology, that does not undergo any changes under smooth processes of general relativity. Of course, it might be expected that large curvature fluctuations might result in such a change, but these processes are hitherto unknown in the context of general relativity. String theory, which is a candidate for a quantum theory of gravity, indeed provides examples of such processes as argued in [22]. One can try and investigate these topology changing processes in the presence of D-branes. In order to begin addressing these issues, it is necessary to formulate a boundary description of Witten's linear sigma model (which describes the open string CFT), and construct D-branes in these models. We study some aspects of such a construction in chapter 3.

## 1.3 Black holes in string theory and their D-brane description

Despite the undoubted success of string theory as a means of understanding perturbative quantum gravity, this is still an insufficient test of the theory as a quantum theory that incorporates gravity. A natural question that arises is whether string theory provides a better understanding of other general relativistic phenomena in the semi-classical or quantum regime. In particular, an interesting question is whether string theory provides a better understanding of black holes.

Black holes, which arise in GTR as singular solutions to the Einstein's equations, are thermal systems obeying laws of thermodynamics. An understanding of the nature and behaviour of black holes has been a long standing problem in GTR. Before the advent of D-branes, these issues were difficult to address in string theory, which was formulated essentially in the perturbative regime, as opposed to black holes, which exist at strong coupling regimes of gravity. This, however, seems to be possible with the discovery of D-branes. Of course, D-branes exist at weak values of the string coupling, but being BPS configurations, they can be used to calculate certain quantities that would remain unchanged even when the coupling is tuned to large values. In this section, we address the issue of D-brane descriptions of black holes.

Two of the most interesting aspects of black holes are entropy and Hawking radiation. Bekenstein and Hawking [23] showed that the entropy of a black hole is proportional to the area of its horizon, i.e

$$S = \frac{A}{4\pi G} \tag{1.9}$$

where $S$ denotes the entropy, $A$ the area of the black hole, and $G$ is the Newton's constant. Efforts have been made to attribute this entropy to the degeneracy of the quantum states of the black hole. These attempts have achieved remarkable success in the case of D-brane description of black holes. Certain combinations of D-branes were shown to have properties that closely resembled black holes. In a description of extremal and near-extremal black holes using such D-brane configurations, certain quantities can be calculated at weak string coupling which can be expected to be valid at all values of the coupling due to the non-renormalisation theorems available for BPS configurations. Hence one can extrapolate black hole results from the D-brane picture of black hole entropy.

The first such calculation was done by Strominger and Vafa [24] who obtained the area dependence of black hole entropy by considering a certain class of string theoretic black holes in five dimensions (that arise out of low energy supergravity solutions for Type II strings), and the corresponding D-brane descriptions of these black holes. As we have remarked earlier, D-branes have the property that open string endpoints can end on them. Counting the degeneracy of such open string states, for a particular configuration of D-branes, Strominger and Vafa reproduced the familiar area law of Bekenstein and Hawking (eq.(1.9)).

The other important aspect of black holes which we will study in chapter 4 of this thesis is Hawking radiation. In classical general relativity, the singularity associated with a black hole is always hidden behind a horizon, which essentially implies that nothing can escape from the area within the horizon. However, Hawking [25] showed that quantum mechanically, a black hole can, indeed, emit particles. He further went on to prove that in the semi-classical approximation, the spectrum of the emitted particles was thermal in nature, and hence is a random distribution containing no information about the state of the black hole. This immediately led to one of the still unsolved paradoxes in the extension to the semi-classical regime of GTR, the *information loss paradox*. If a black hole Hawking radiates, i.e, emits a thermal spectrum, then, since infalling particles into the black hole might carry information, we ultimately end up with a loss of information as the black hole evaporates. A precise understanding of Hawking radiation from black holes, and the information loss paradox, is an essential part of our attempts of understanding general relativity. One would like to address these issues from the string theory point of view as well.

As we remarked earlier, certain configurations of D-branes are seen to have properties that seem to identify them with black holes at strong coupling. The process of Hawking radiation of scalars in this picture can be understood as follows. Let us consider black holes arising out of the low energy effective actions of Type IIB string theory in five dimensions, the internal space being a 5-torus, $T^5$. We start from the low-energy effective action for Type IIB string theory in ten space-time dimensions,

$$S_{IIB} = \frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{12}e^{\phi}H^2\right] \tag{1.10}$$

where $R$ is the Ricci scalar, $\phi$ the dilaton field, and $H$ the R-R three form field strength, and $G_{10}$ denotes the ten-dimensional Newton's constant. All the other fields have been set to zero. Now, on toroidally compactifying this theory to five dimensions, where the internal five space dimensions consist of a circle of radius $R$ and a 4-torus, $T^4$ with volume $V$, and in addition some momemtum along the direction of the circle, there exists the following solution to the ten dimensional metric

$$\begin{aligned}
ds_{10}^2 &= \left(1+\frac{r_0^2\sinh^2\alpha}{r^2}\right)^{-3/4}\left(1+\frac{r_0^2\sinh^2\gamma}{r^2}\right)^{-1/4} \\
&\left[-dt^2+dx_5^2+\frac{r_0^2}{r^2}(\cosh\sigma dt+\sinh\sigma dx_5)^2+\left(1+\frac{r_0^2\sinh^2\alpha}{r^2}\right)dx_i dx^i\right] \\
&+ \left(1+\frac{r_0^2\sinh^2\alpha}{r^2}\right)^{1/4}\left(1+\frac{r_0^2\sinh^2\gamma}{r^2}\right)^{3/4}\times \\
&\left[\left(1-\frac{r_0^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2\right] \qquad (1.11)
\end{aligned}$$

The three charges, corresponding to the three form R-R field strength, its Hodge dual, and the momentum are given by

$$\begin{aligned}
Q_1 &= \frac{V}{4\pi^2 g}\int e^{\phi} * H = \frac{Vr_0^2}{2g}\sinh 2\alpha \\
Q_5 &= \frac{1}{4\pi^2 g}\int H = \frac{r_0^2}{2g}\sinh 2\gamma \\
n &= \frac{R^2Vr_0^2}{2g^2}\sinh 2\sigma \qquad (1.12)
\end{aligned}$$

Where $n$ is related to the momentum $P$ along the circle, by $P=\frac{n}{R}$. $Q_1$ and $Q_5$ are chosen to be large. Now, on dimensionally reducing this metric to five space-time dimensions, we get the five dimensional metric,

$$ds_5^2 = -f^{-2/3}\left(1-\frac{r_0^2}{r^2}\right)dt^2+f^{1/3}\left[\left(1-\frac{r_0^2}{r^2}\right)^{-1}dr^2+r^2d\Omega_3^2\right] \qquad (1.13)$$

with $f$ being a function of the three charges, given by

$$f=\left(1+\frac{r_0^2\sinh^2\alpha}{r^2}\right)\left(1+\frac{r_0^2\sinh^2\gamma}{r^2}\right)\left(1+\frac{r_0^2\sinh^2\sigma}{r^2}\right) \qquad (1.14)$$

This is the five-dimensional Schwarzschild metric, with the event horizon at $r=r_0$ [26] The corresponding D-brane configuration consists of an appropriate number of D-1 branes and D-5 branes, wrapped in the internal compact space time, with the D-1 brane lying along one of the directions of the D-5 brane and there is in addition a net momentum along the D-1 brane direction. This configuration, as we will see in chapter 4, is equivalent to a *long* D-1 brane in the direction of the original 1 brane. Now, open strings can move along the world volume direction of the D-1 brane, and the left and right moving open strings can be taken to constitute two sets of non-interaction one dimensional gases. The interaction between a pair of oppositely moving open string states may result in the formation of a closed string, which cannot reside on the world volume of the D-brane and is hence emitted as scalars in space time.

This picture was studied in detail by Das and Mathur in [27]. The authors studied the emission rate of low energy scalars, in the model described in the last paragraph, by considering the Born-Infield action of the long one brane wrapped along one of the directions of the $T^5$. They then calculated the absorption cross section of such low energy quanta by the corresponding extremal black hole, with the metric described above, and found exact agreement of the two results.

Let us briefly describe their result. Consider the Dirac-Born-Infield action for the D-1 brane, (1.4). Setting the gauge field and the R-R field to zero, we get

$$S_{BI} = T\int d^2\sigma e^{-\phi(X)}\sqrt{det(G_{mn}(X) + B_{mn}(X)} \tag{1.15}$$

Starting from this action, The lowest order interactions are obtained by treating the metric as a small perturbation over the flat metric, and also expanding the transverse coordinates around the brane position. In this approximation, from the action above, in static gauge ($X^0 = \sigma^0$, $X^1 = \sigma^1$), the interaction between two open strings and the metric is given by the term

$$\frac{1}{2}\left(\delta_{ij} + 2G_{10}h_{ij}\right)\partial_\alpha X^i \partial^\alpha X^j$$

Where $i, j = 2, \cdots 9$ denote the coordinates transverse to the D-string and we have expanded the metric as $G_{\mu\nu} = \eta_{\mu\nu} + 2G_{10}h_{\mu\nu}(X)$. From this interaction term, it is possible to do a field theoretic calculation of the absorption cross section of transverse gravitons which can be thought of as a scalars in space-time (with the indices in the internal part). The final result is [27]

$$\Gamma_D = A_H\,\rho\left(\frac{\omega}{T_H}\right)\frac{d^4k}{(2\pi)^4}\,, \tag{1.16}$$

$A_H$ is the horizon area of the corresponding black hole in the supergravity limit, obtained in terms of the D-brane parameters from entropy calculations [27] and $\rho$ is a thermal distribution function with the effective temperature being the Hawking temperature of the black hole. In the above calculation, it was further assumed that the left and right moving open strings on the world volume of the 1-brane, which constitute two sets of non-interacting one dimensional gases, have effective temperatures $T_L$ and $T_R$, with $T_L >> T_R$.

The result in the last paragraph is then compared to the corresponding low energy absorption cross section in the black hole regime with the 5-D black hole metric given by eq. (1.13). Computation of the absorption cross section in the semi-classical approximation is carried out by solving the scalar field equation in the background of this metric. Writing the massless minimally coupled scalar wave-function as $\phi(r,t) = R(r)e^{-i\omega t}$, $\omega$ denoting the energy of the scalar quanta, the field equation reduces to, on rewriting $g(r) = r^{3/2}R(r)$,

$$\left[\frac{d^2g(r)}{dr^2} + \omega^2 f(r)g(r) - \frac{3}{4\pi^2}g(r)\right] = 0 \tag{1.17}$$

Note that in the above, we have restricted only to spherically symmetric wave-functions, since the higher angular momentum components can be shown to have negligible contribution to the absorption cross section in the limit of low $\omega$ that we are interested in. The solution to the above equation will contain constants, that can be evaluated by solving the equation in the regions near and far from the black hole horizon, and demanding that the solutions match at some intermediate region. For the region far from the black hole horizon, the solutions of the wave equation are Bessel functions, while they are Coulomb functions in the region close to the horizon. The arbitrary constants are matched in an intermediate region where both solutions reduce to an inverse square relationship between $R$ and $r$. We will not go into the details of the computation, a similar case will be worked out in chapter 4 of this thesis. For the moment, let us just state that the result for the absorption cross section from the semi-classical calculation is exactly the same as the expression in (1.16). These calculations have also been carried out for other fields, like fermions, fixed scalars etc. and agreement and disagreement of the results from the two sides have been analysed. In refs [28],[29], an ab-initio derivation of the Hawking radiation spectrum from the D1- D5 system has been attempted, starting from the moduli space of the low-energy degrees of freedom of the corresponding gauge theory, and the Hawking emission spectrum of minimal and fixed scalars have been computed.

In chapter 4 of this thesis, we will study neutral and charged scalar Hawking radiation rates from a class of five and four dimensional near-extremal black holes and their corresponding D-brane configurations, in the regime of high energies of the emitted quanta, while relaxing the condition $T_L >> T_R$. We will calculate black hole greybody factors under these conditions, following the methods of [27] and [26]. We will show that in this extended range also the black hole emission rates will match the corresponding results obtained from the D-brane picture. We will also show that for four-dimensional black holes, the black hole and D-brane emission rates match for a more restricted range of energy compared to the five-dimensional case.

## 1.4 The holographic principle in string theory and cosmology

The D-brane black hole correspondence that we just described, brings us to another important issue of application of string theory techniques to the more 'observable' aspects of GTR. A natural question to ask, for example, is how the string theory solitons fit in with the description of our observable universe. The latter is described in terms of the Friedmann equations of cosmology, arising as special cases of the Einstein's equations in GTR. At the same time, one can also ask how the *general* principles of string theory and D-branes translate into the case of our universe. This is important because string theory, is a quantum theory of gravity and the general principles that are applicable to string theory and D-branes, must, somehow play an important role in the description of our universe as well. In this section, we address one such issue, namely, the question of how *holography*, conjectured to be one of the deep and important principles of quantum gravity and hence of string theory manifests itself in the context of cosmology.

Holography, originally proposed in the context of quantum gravity by 'tHooft [30] was extended to string theory by Susskind in [31]. The principle, which states that any macroscopic region of space can be effectively described by a theory which lives on its boundary, with the number of degrees of freedom per Planck area not exceeding unity, has been seen to have wide-ranging consequences. This theory has been put to test in the context of D-branes, via the recently proposed AdS/CFT conjecture, by Susskind and Witten [32]. Stated mathematically, the holographic principle implies that the entropy $S$ contained in a region of volume $V$, will never exceed the area $A$ that bounds this volume.

A motivation for the holographic principle is the following: consider a region of space with volume $V$, bounded by an area $A$ and containing an entropy $S$. Assume that this entropy is greater than that of a black hole with the same surface area. Now, if we add more energy to this region to form a black hole, then, assuming that the entropy of the black hole is given by the Bekenstein-Hawking (BH) formula $S = \frac{A}{4}$ (in units where the Newton's constant $G_N = 1$, we see that the generalised second law of thermodynamics [23] will be violated. In order to avoid this, the holographic principle proposes that the entropy and area are related by $\frac{S}{A} < 1$.

It is expected that the holographic conjecture arising in the context of string theory, would play an important role in the description of the observable universe. Before attempting to address such issues, however, one has to understand the precise meaning of the holographic principle applied to our present universe, i.e in cosmology. As it turns out, a naive application of the holographic principle to cosmology often gives incorrect results, and a refinement of the procedure has to be carred out. The procedure to correctly apply the holographic principle to cosmology has recently been conjectured by Fischler and Susskind (FS)[33]. Let us first briefly review their argument.

We start with the usual Friedmann-Robertson-Walker (FRW) metric, that is assumed to describe the universe,

$$ds^2 = dt^2 - R^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{1.18}$$

The first law of thermodynamics, when applied to this scenario, gives us,

$$d\left[\frac{(p+\rho)R^3}{T}\right] = 0 \tag{1.19}$$

where $p$ and $\rho$ are the equilibrium pressure and energy density respectively. Thus, we define the comoving entropy density, $\sigma = \frac{p+\rho}{T}R^3$, which is a constant in the evolution of the universe. Because of this, the entropy contained in a region of coordinate size $r$, say will be given by $S = \sigma r$. Now, the area of this region can be shown to be $A = Rr^2$ Hence, when $r$ is sufficiently large, the entropy exceeds the area and the holographic principle, which requires that $S$ must always be less than $A$ is violated.

Now, we state the FS proposal that remedies this. According to this proposal, the holographic principle should be restated as follows. If $r_H$ denotes the coordinate size of the cosmological *horizon*, given by the expression

$$r_H(t) = \int_0^t \frac{dt}{R(t)}$$

then, the entropy contained within a volume of coordinate size $r_H$ should not exceed the area of the horizon, in Planck units. Mathematically,

$$\sigma r_H^d < [Rr_H]^{d-1} \tag{1.20}$$

for a theory with $d$ spatial dimensions. That this proposal is along the correct lines, can be seen easily. Consider first the present day universe. The horizon size of the present universe, which is the same as the age of the universe, is numerically equal to $10^{60}$, whereas the entropy of the observable universe is of the order of $10^{86}$. Hence, clearly the holographic principle is satisfied today. Now consider what will happen in the future. Assuming an expansion rate of the form $R(T) \sim t^p$, so that $r_H \sim t^{1-p}$, the holographic principle, from (1.20), dictates that $p > \frac{1}{d}$. This can easily be shown to lead to a bound on the parameter $\gamma$ in the equation relating the pressure and the energy densities, $p = \gamma\rho$, namely, $\gamma < 1$. The bound on $\gamma$ is known to follow from well known facts about special relativity, and hence gives added support to the FS proposal. Finally, Fischler and Susskind were also able to prove that their proposal for the holographic principle was also valid in the past, i.e, upto Planck time.

The FS proposal, is however valid for standard cosmology, arising out of the FRW metric. Standard cosmology, however, has a number of shortcomings, two of them being the horizon problem and the flatness problem. We will deal with these morefully in Chapter 5. For the moment, let us remark that one can take two alternatives as a way out of these problems. The first is the so called inflationary cosmology, where it is assumed that at some point of time during its evolution, the universe passed through a phase of exponential growth, that finally results in a huge release of entropy, and it can be shown that this scenario and its refinements cure many of the problems arising in standard cosmology. The second approach is string cosmology, first proposed by Veneziano, who argued that in string theory, the cosmological evolution of the universe is different from standard cosmology. In particular, along with the post big-bang phase with an initial singularity, it predicts a pre big-bang phase with a final singularity, and the two branches can be connected smoothly. Several problems arising in standard cosmology can be shown to be absent in string theory inspired cosmology. Both these approaches have been successful in explaining several observed features of the present universe. A study of the FS proposal in the context of string cosmology has been carried out in [34]. In chapter 5 of this thesis, we will study the application of the FS proposal in the context of the inflationary universe. We will argue that a naive application of the FS proposal would imply a violation of the holographic bound, but a more careful analysis has to be performed during the process of entropy production in the inflationary stage, which shows that holography is indeed obeyed in this model. This will lead us to a derivation of an interesting lower bound on the density fluctuations of the universe, that is close to observed values.

# Chapter 2

# D-branes on Curved Manifolds : I

In this chapter, we pursue two different worldsheet approaches to understanding D-branes wrapped on supersymmetric cycles in Calabi-Yau manifolds. The two approaches that we use are the boundary $N = 2$ supersymmetric Landau-Ginzburg (LG) formulation and a boundary state construction in terms of the Gepner model. The Landau-Ginzburg formulation of strings on Calabi-Yau manifolds has been very successful in understanding various aspects of such closed string theories. We would extend this by considering the same LG models on worldsheets with boundary, in a manner that preserves a $N = 2$ worldsheet supersymmetry on the boundary. We will discuss how these LG models with boundary provide a natural description of D-branes wrapped on both even and middle-dimensional supersymmetric cycles in the general Calabi-Yau manifold. In the second description, we use the Gepner model construction. However, in contrast to other approaches available in the literature, we would consider linear combinations of characters of the spacetime super conformal field theory (SCFT) and the internal SCFT that are invariant under the operation of spectral flow. With this approach we discuss how to construct the cylinder partition functions in a manner that explicitly demonstrates that some of spacetime supersymmetry is preserved and thus leads to a vanishing partition function. We will discuss the associated boundary states for these partition functions. As specific illustrations, we would consider in this chapter the $1^3$ and $2^2$ Gepner models that describe a $T^2$ compactification. We will relate the boundary state construction to the boundary condition LG description by making use of a common discrete symmetry group occurring in both the Gepner model and its corresponding LG orbifold.

Throughout this chapter, we will, for simplicity, restrict ourselves to the case where all the spatial coordinates are wrapped on the appropriate supersymmetric cycle and hence from the viewpoint of the non-compact spacetime, we have a zero-brane. From this point of view, the world-volume theory describes the moduli of the corresponding D-brane wrapped on the cycle inside the Calabi-Yau manifold [35, 36].

Before we begin, let us first briefly summarise the progress that has been made in providing a conformal field theory description of D-branes wrapped around supersymmetric cycles in Calabi-Yau spaces. The first important step was provided by the work of Ooguri, Oz and Yin[18], who formulated the general boundary conditions on the world-sheet $N = 2$ SCFT that would be necessary to describe such cycles. Subsequently using the work of Cardy on boundary CFT[37], Recknagel and Schomerus[20] described in some generality the boundary states in Gepner models [13],[38], that would be relevant to the description of both even and odd dimensional supersymmetric cycles in the corresponding Calabi-Yau manifolds. Further in refs. [39] some applications of this construction have been pursued. In later work, Recknagel and Schomerus have also studied the role of boundary operators in such constructions[40]. Other approaches have studied the case of D-branes in the context of group manifolds as described by Wess-Zumino-Witten (WZW) models[41, 42, 43]. Finally, we must mention the important work of Brunner, Douglas, Lawrence and Römelsberger[44], that studied in detail the structure and several aspects of D-branes on the quintic, using both Gepner models and other techniques. We consider the techniques of the work presented in this chapter to be complementary to the ideas and results contained therein.

To begin, let us, for future reference, introduce some results that we shall be needing. This will also set the notations and conventions that will be used throughout this chapter.

## 2.1 Background and conventions

The $N = 2$ superconformal algebra can be expressed in terms of the (anti)commutation relations among its generators, which consist of the energy momentum tensor, $T(z)$, its worldsheet superpartners, $G^{\pm}(z)$ of conformal weight 3/2, and a U(1) current, $J(z)$, which is a primary field in the algebra with conformal weight 1. The algebra is given by the following relations that can be derived from the operator product expansions of the generators. Writing the mode expansion of the operators as

$$\begin{aligned} T(z) &= \Sigma_n L_n z^{-n-2} \\ J(z) &= \Sigma_n J_n z^{-n-1} \\ G^{\pm}(z) &= \Sigma_n G^{\pm}_{n\pm a} z^{-(n\pm a)-\frac{3}{2}} \end{aligned} \tag{2.1}$$

where we have rewritten the two supercurrents $G^1$ and $G^2$ as

$$G^{\pm} = \frac{1}{\sqrt{2}}\left(G^1 \pm iG^2\right) \tag{2.2}$$

and the free parameter $a$ in the expansion of the $G$'s has the range $0 \leq a < 1$ The algebra is given by the following relations that can be derived from the operator product expansions of the generators[49].

$$\begin{aligned} [L_m, L_n] &= (m-n)\, L_{m+n} + \frac{c}{12} m\left(m^2-1\right)\delta_{m+n,0} \\ \left[L_m, G^{\pm}_{n\pm\eta}\right] &= \left(\frac{m}{2} - (n \pm a)\right) G^{\pm}_{m+n\pm a} \\ [J_m, J_n] &= \frac{c}{3} m\delta_{m+n,0} \\ [L_n, J_m] &= -mJ_{m+n} \\ \left[J_n, G^{\pm}_{m\pm a}\right] &= \pm G^{\pm}_{m+n\pm a} \\ \left\{G^{+}_{n+a}, G^{-}_{m-a}\right\} &= 2L_{m+n} + (n-m+2a)\, J_{m+n} \\ &\quad + \frac{c}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m+n,0} \end{aligned} \tag{2.3}$$

In terms of the free parameter $a$, $a = 0$ corresponds to the Ramond (R) algebra and $a = \frac{1}{2}$ corresponds to the Neveu-Schwarz (NS) algebra. We shall refer to states in representations of the Ramond algebra as Ramond states and similarly, one obtains

Neveu-Schwarz states. Primary states of the $N = 2$ algebra are labelled by their dimension $h$ and $U(1)$ charge $q$.

A subset of the primary fields of the NS algebra are the chiral primary fields. which create states that are annihilated by the operator $G^+_{-1/2}$, i.e,

$$G^+_{-1/2}|\phi\rangle = 0 \quad , \tag{2.4}$$

Similarly, antichiral primary fields are constructed with the condition that the corresponding highest weight states are annihilated by the operator $G^-_{-1/2}$. This means that there are no poles in the operator product expansion of these fields and $G^\pm$, i.e

$$G^\pm(z)\phi(w) \sim regular \tag{2.5}$$

Considering the relation

$$\langle\phi|\{G^-_{\frac{1}{2}}, G^+_{-\frac{1}{2}}|\phi\rangle \;=\; \langle\phi|\,(2L_0 \;-\; J_0)\,|\phi\rangle \tag{2.6}$$

which follows from the (anti)commutation relations of the $N = 2$ algebra, we can see that since the l.h.s of the above equation vanishes, by the definition of the chiral primary field, we obtain, the following relationship between the dimension and $U(1)$ charge of a chiral primary field:

$$h_\phi = \frac{q_\phi}{2} \quad . \tag{2.7}$$

the more general relation being $h_\phi \geq \frac{q_\phi}{2}$ as can be seen from the positive definiteness of the l.h.s of (2.6). Similarly, anti-chiral fields satisfy $h = -q/2$. In a theory with $(2,2)$ worldsheet supersymmetry, i.e., theories with $N = 2$ supersymmetry in the holomorphic(left-moving) and anti-holomorphic(right-moving) sectors, one can construct four combinations of the chiral and anti-chiral fields. These are $(c,c)$, $(a,a)$, $(c,a)$, and $(a,c)$ states in the theory. Let us consider the chiral primaries once more. From the relationship between the dimension and the $U(1)$ charge, one can easily write down the operator product expansion between these fields, which is of the form:

$$\phi(z)\rho(w) \;=\; \Sigma_i\,(z-w)^{h_{\chi_i}-h_\phi-h_\phi}\,\chi_i(w) \tag{2.8}$$

From the fact that the U(1) charges add on operator product expansion and using the relationship of eq. (2.7), one can easily see that there is no singularity in the above OPE and that for consistency, the only terms surviving on the r.h.s are those for which the field $\chi_i$ is itself a chiral primary. This proves that the chiral primaries form a closed ring under OPE. Further, using the anticommutator

$$\langle\phi|\{G^{-}_{\frac{3}{2}}, G^{+}_{-\frac{3}{2}}\}|\phi\rangle = \langle\phi|\left(2L_0 - 3J_0 + \frac{2}{3}c\right)|\phi\rangle \tag{2.9}$$

where $|\phi\rangle$ is a chiral primary state, we can deduce that the conformal dimension of $\phi$ is bounded from above, i.e $h_\phi \leq \frac{c}{6}$ Hence, we obtain the result that the chiral primaries in $N = 2$ SCFT form a closed finite non- singular ring under OPE.

An important aspect of the $N = 2$ algebra is the existence of a spectral flow isomorphism. One can show that the after the following redefinition:

$$\begin{aligned} L'_n &= L_n + \eta J_n + \frac{1}{6}\eta^2 c\delta_{n,0} \\ J'_n &= J_n + \frac{1}{3}\eta c\delta_{n,0} \\ \left(G^{\pm}_r\right)' &= G^{\pm}_{r\pm\eta} \quad , \end{aligned} \tag{2.10}$$

the redefined operators also satisfy the $N = 2$ algebra with a moding shifted by the parameter $\eta$ $(a \to a + \eta)$. This correspondence can be carries over to the states in the representation of the algebra. This is done by means of the spectral flow operator $U_\eta$ which is defined by the unitary mapping

$$|f_\eta\rangle = U_\eta|\phi\rangle \tag{2.11}$$

$|f\rangle$ defining a state in the representation of the original algebra, and $|\phi_\eta\rangle$ being in the $\eta$ twisted sector of the theory, and the unitary map $U_\eta$ transforms the generators $L_n(J_n)$ as

$$L'_n(J'_n) = U_\eta L_n(J_n) U_\eta^{-1} \tag{2.12}$$

To derive an expression for the spectral flow operator, we first note that the $U(1)$ current can be bosonised as

$$J(z) = i\sqrt{\frac{c}{3}}\partial_z\phi \tag{2.13}$$

where $\phi$ is a free boson, and in terms of this, and field f that creates the state $|f\rangle$ with dimension $h$ and $U(1)$ charge $Q$ can be written as

$$f = ge^{iq\sqrt{\frac{3}{c}}\phi} \tag{2.14}$$

where the field $g$ is neutral, its $U(1)$ charge being zero and the dimension given by

$$h_g = h - \frac{3Q^2}{2c} \tag{2.15}$$

From this, we easily see that if we shift the bosonic exponent in the expression for $f$ as

$$f_\eta = ge^{i\sqrt{\frac{3}{c}}(q+\frac{c}{3}\eta)\phi} \tag{2.16}$$

Then the dimension and $U(1)$ charge of the new field in the twisted sector is given by

$$\begin{aligned} h_\eta &= h + q\eta + \frac{\eta^2}{6}c \\ q' &= q + \eta\frac{c}{3} \end{aligned} \tag{2.17}$$

which is exactly as required, as seen by comparing this with the expressions for the twisted modes. Hence an explicit representation of the spectral flow operator is given by

$$U_\eta = e^{i\sqrt{\frac{c}{3}}\eta\phi} \quad , \tag{2.18}$$

When $\eta = \frac{1}{2}$, the spectral flow operator interpolates between the Neveu-Schwarz and the Ramond sectors. In the context of spacetime supersymmetric string theory, this spectral flow relates spacetime bosons to spacetime fermions.

For a given representation $p$ of the $N = 2$ algebra, the character is defined as

$$\chi_p(\tau, z, u) = e^{-2i\pi u}\,\mathrm{Tr}\, e^{2i\pi z J_0}\, e^{2i\pi\tau(L_0 - \frac{c}{24})} \tag{2.19}$$

where the trace runs over the particular representation denoted by $p$ and $u$ is an arbitrary phase. The explicit formulae for the characters of certain models in terms of the Jacobi theta functions will be written down later. Under spectral flow with parameter $\eta$, the character for the $\eta$-shifted representation is given by

$$\begin{aligned} \chi_p^{\eta}(\tau, z) &= \mathrm{Tr}\, e^{2i\pi J_0'}\, e^{2i\pi\tau(L_0' - \frac{c}{24})} \\ &= \chi_p\left(\tau, z + \eta\tau, -\frac{1}{6}\eta^2\tau c - \frac{1}{3}\eta z c\right) \end{aligned} \tag{2.20}$$

Next, let us briefly discuss boundary states in $N = 2$ SCFT's that we will use in our analysis. A BPS state such as a D-brane wrapped on a supersymmetric cycle will preserve half the spacetime supersymmetry. Using the correspondence between spacetime supersymmetry and the existence of a global $N = 2$ supersymmetry on the worldsheet, the presence of a BPS state will be signalled by the boundary preserving a linear combination of the $(2, 2)$ worldsheet supersymmetry. The analysis of Ooguri et al. shows that there are two possible linear combinations[18].

**A-type boundary condition:**

$$J_L = -J_R \quad , \quad G_L^{+} = \pm G_R^{-} \quad , \quad e^{i\phi_L} = e^{-i\phi_R} \tag{2.21}$$

**B-type boundary condition:**

$$J_L = J_R \quad , \quad G_L^{+} = \pm G_R^{+} \quad , \quad e^{i\phi_L} = (\pm)^d e^{i\theta} e^{i\phi_R} \quad , \tag{2.22}$$

where the $\phi$'s are the scalars associated with the bosonisation of the $U(1)$ current of the $N = 2$ supersymmetry algebra in the left and right-moving sectors. These boundary conditions are for the open string channel.

Boundary states which preserve a $N = 2$ supersymmetry are expected to be related to D-branes wrapping around supersymmetric cycles. The boundary states satisfy the closed string equivalent of the above boundary conditions. In order to do this, we write the boundary conditions in the closed string channel with the replacement $J_R \to -J_R$ and $G_R^{\pm} \to iG_R^{\pm}$ as compared to the open string channel. The A type boundary condition then reads,

$$(J_L - J_R)\,|B\rangle = 0 \quad ; \quad (G_L^{\pm} \pm iG_R^{\mp})\,|B\rangle = 0 \quad , \tag{2.23}$$

where $|B\rangle$ is a boundary state. The condition on the $U(1)$ current picks out a selection rule for the fields of the theory that can contribute to the boundary state, namely for the A-type boundary condition, corresponding to D-branes wrapping around middle dimensional cycles, we have $q_L = q_R$ for the $U(1)$ charge. Thus, the $(c, c)$ and $(a, a)$ states can contribute to the A-type boundary state while the $(a, c)$ and $(c, a)$ states cannot. Similarly, for the B-type boundary condition

$$(J_L + J_R)\,|B\rangle = 0 \quad ; \quad (G_L^{\pm} \,\pm\, iG_R^{\pm})\,|B\rangle = 0 \tag{2.24}$$

implying that the $(c, a)$ and $(a, c)$ states contribute to the boundary state.

Generalising a procedure due to Ishibashi, one can construct solutions of the above conditions for all primary fields which are 'left-right' symmetric[50]. The explicit form of the Ishibashi state associated with such a representation $a$ is given by

$$|a\rangle\rangle \;=\; \sum_N |a; N\rangle \otimes U\overline{|a; N\rangle} \tag{2.25}$$

where $|a, N\rangle$ is an orthonormal basis for the representation $a$ and $U$ is an anti-unitary matrix which preserves the highest weight state $|a\rangle$. For A-type boundary conditions, one has to replace $U$ with $U\Omega$ where $\Omega$ is the mirror automorphism of the $N = 2$ algebra[20]. We shall label the Ishibashi states for the A-type and B-type boundary conditions by $|a\rangle\rangle_A$ and $|a\rangle\rangle_B$ respectively.

Let us now turn to Cardy's construction of boundary states in a conformal field theory formulated on a manifold with boundary. The set of Ishibashi states that we have discussed, form a basis for such boundary states. Thus, any boundary state $|\alpha\rangle$ is given by a linear combination of the Ishibashi states

$$|\alpha\rangle = \sum_a \frac{\psi_\alpha{}^a}{(S_0{}^a)^{\frac{1}{2}}}\,|a\rangle\rangle \quad , \tag{2.26}$$

where $S$ is the modular S-matrix and 0 refers to the identity operator. The $\psi_\alpha{}^a$ are not arbitrary but will have to satisfy a consistency condition which we will now derive. The arguments are due to Cardy[37] but we will follow the discussion in ref. [51]. Consider a conformal field theory associated with a chiral algebra on a cylinder with perimeter $T$ and length $L$ subject to boundary conditions $\alpha$ and $\beta$.. The partition function of the system can be calculated in two ways: One can consider the result as coming from periodic 'time' $T$ evolution with the prescribed boundary conditions. Topologically, this corresponds to an annulus. The annulus partition function is given by

$$\mathcal{A}_{\alpha\beta} = \sum_i n_{i\alpha}{}^{\beta}\, \chi_i(q) \quad , \tag{2.27}$$

where $n_{i\alpha}{}^{\beta}$ denotes the number of times the irreducible representation $i$ occurs in the spectrum of the Hamiltonian $H_{\alpha\beta}$ (which generates the 'time' evolution) and $q = e^{-\pi T/L}$. Another way corresponds to treating the $L$ direction as time and the partition function for time evolution from the boundary state $|\alpha\rangle$ to the boundary state $|\beta\rangle$ is given by

$$\mathcal{C}_{\alpha\beta} = \sum_a \frac{\psi_\alpha{}^a (\psi_\beta{}^a)^\dagger\, \chi_a(\tilde{q})}{S_0{}^a} \quad , \tag{2.28}$$

where $\tilde{q} = e^{-4\pi L/T}$ and the sum is over Ishibashi states. On equating eqn. (2.27) to the modular transformation $\tau \to -1/\tau$ (with $\tau = i2L/T$) of eqn. (2.28), one obtains the following consistency condition:

$$n_{i\alpha}{}^{\beta} = \sum_a \frac{S_i{}^a}{S_0{}^b}\, \psi_\alpha{}^a\, (\psi_\beta{}^a)^\dagger \quad . \tag{2.29}$$

In the above, note that the sum is over Ishibashi states while the index $i$ is over characters of all irreducible representations of the chiral algebra. Note that these two are not necessarily the same except for theories which are 'left-right' symmetric i.e., the toroidal partition function is given by $\mathcal{T} = \sum_i C_{ij}\chi_i(q)\chi_j(\bar{q})$, where $C$ is the charge conjugation matrix. It can be shown[51] that the matrices $n_i = (n_i)_\alpha{}^\beta$ form a representation to the fusion algebra

$$\sum_\beta n_{i\alpha}{}^{\beta} n_{j\beta}{}^{\gamma} = \sum_k N_{ij}{}^k n_{k\alpha}{}^{\gamma} \quad , \tag{2.30}$$

where $N_{ij}{}^k$ is the fusion matrix. Cardy has provided a solution to the consistency equation (2.29) for theories whose toroidal partition function is 'left-right' symmetric. He constructs boundary states (and hence boundary conditions) corresponding to the representations $a$ which appear in the Ishibashi states. Let us label the corresponding boundary states by $|\tilde{a}\rangle$ given by

$$|\tilde{a}\rangle = \sum_b \frac{S_a{}^b}{(S_0{}^b)^{1/2}} |b\rangle\rangle \quad , \tag{2.31}$$

where the sum is over Ishibashi states. This solves eqn. (2.29) for

$$n_{i\tilde{a}}{}^{\tilde{b}} = N_{ia}{}^{b} \quad , \tag{2.32}$$

The consistency condition now turns into the Verlinde formula. Complications can arise in attempting to apply Cardy's results directly. One which we will encounter is that different representations may have the same Virasoro character. This will show up as a multiplicity in the appearance of the characters in the toroidal partition function. In addition, the S-matrix will not have several of its usual properties such as it being symmetric and so on. In such cases, the S-matrix needs to be *resolved.* There is a fairly general procedure due to Fuchs, Schellekens and Schweigert which one uses to obtain a resolved S-matrix which has its usual properties[52]. Sometimes, however there exists some discrete symmetry which distinguishes representations which have the same character. In these cases, one can use the charge under the discrete symmetry to obtain a resolved (or at least a partially resolved) S-matrix. We refer the reader to ref. [52] for the procedure to resolve the S-matrix. We will discover in the case of Gepner models that this is the case generically, and we will need to resolve the S-matrix before using Cardy's solution to eqn. (2.29).

We now briefly review Gepner models and discuss results that will be useful for us. As we have commented before, Gepner models are exactly solvable supersymmetric compactifications of type II string theory, where the internal part of the SCFT is constructed by tensoring together $N = 2$ minimal models. The central charge of the minimal model of level $k$ is given by

$$c = \frac{3k}{k+2} \tag{2.33}$$

A simple construction of the minimal model of level $k$ is realised by adding one free boson to the $Z_k$ parafermionic field theory. This is done as follows: from the free bosonic theory with the field denoted by $\phi$, and the $Z_k$ parafermionic theory with parafermionic fields labelled by $\psi_1$ and its hermitian conjugate, $\psi_1^\dagger$, one can construct

$$\begin{aligned} G^+(z) &= \sqrt{\frac{2k}{k+2}}\psi_1 : e^{i\phi\sqrt{\frac{k+2}{k}}} : \\ G^-(z) &= \sqrt{\frac{2k}{k+2}}\psi_1^\dagger : e^{-i\phi\sqrt{\frac{k+2}{k}}} : \\ J &= i\sqrt{\frac{2k}{k+2}}\partial_z\phi \end{aligned} \tag{2.34}$$

The operator product expansions for these generators satisfy the $N = 2$ super conformal algebra. The primary fields of the theory are labelled by three integers $l, m, s$, and denoted by $\Phi^l_{m,s}$ whose dimension $h$ and $U(1)$ charge $q$ are given by are given by

$$\begin{aligned} h &= \frac{l(l+2)-m^2}{4(k+2)} + \frac{s^2}{8} \\ q &= \frac{m}{k+2} - \frac{s}{2} \end{aligned} \tag{2.35}$$

where $l = 0, 1, \cdots, k$ and $m = -(k+1), -k, \cdots, (k+2) \bmod (2k+4)$ and $s = 0, 2, \pm 1$. The labelling integers satisfy the constraint $l + m + s \in 2Z$. In addition, there is an identification given by $(l, m, s) \sim (k - l, m + k + 2, s + 2)$. The $N = 2$ characters of the minimal models are defined in terms of the usual Jacobi theta functions as:

$$\chi^{l(s)}_m(\tau, z, u) = \sum_{j \bmod k} C^l_{m+4j-s}(\tau)\theta_{2m+(4j-s)(k+2),2k(k+2)}(\tau, 2kz, u) \tag{2.36}$$

where $\theta_{n,m}(\tau, z, u)$ denotes the Jacobi theta function, and the $C^l_m(\tau)$ are the characters of the parafermionic field theory. The characters have the property that $\chi^{l(s)}_m$ is invariant under $s \to s + 4$ and $m \to m + 2(k + 2)$ and is zero if $l + m + s \neq 0$ mod 2. By using the properties of the theta functions, the modular transformation of the minimal model characters is

$$\begin{aligned} \chi^{l(s)}_m\left(-\frac{1}{\tau}, 0, 0\right) &= \sum_{l',m',s'} \sin\left(\frac{\pi(l+1)(l'+1)}{k+2}\right) \exp\left(\frac{i\pi mm'}{k+2}\right) \\ &\quad \times \exp\left(-\frac{i\pi ss'}{k+2}\right) \chi^{l'(s')}_{m'}(\tau, 0, 0) \end{aligned} \tag{2.37}$$

where in the above sum we impose $l' + m' + s' = 0$ mod 2. Gepner constructed compactifications of the heterotic string which had spacetime supersymmetry by representing the internal part by a tensor product of $N = 2$ minimal models. His considerations are equally applicable for compactifications of the type II string. Consider the tensor product of $n$ minimal models of level $k_i$ $(i = 1, \cdots, n)$. The total central charge of the internal model is given by

$$c_{\text{int}} = \sum_{i=1}^{n} \frac{3k_i}{k_i + 2} \quad , \tag{2.38}$$

where $c_{\text{int}} = 15 - 3d/2$, where $d$ is the dimensionality of spacetime. Thus, for $d = 4$, $c_{\text{int}} = 9$. Gepner constructs a spacetime supersymmetric partition function by first projecting onto states for which total $U(1)$ charges in both the left-moving and right-moving sectors is an odd integer. Then, in order to preserve $N = 1$ worldsheet supersymmetry, the NS sector states of each sub-theory are coupled to each other and do not mix with the $R$ sector states. He thus multiplies all the $NS$ sector partition functions in each sub-theory and similarly for other sectors(i.e, $\widetilde{NS}$, $R$ and $\widetilde{R}$). The full partition function is a sum of the contributions from the four sectors. Modular invariance of the full partition function is a consequence of modular invariance in each of the sub-theories. In considering boundary states for D-branes wrapped on cycles of Calabi-Yau manifolds, we consider the internal part of the SCFT to be product of $N = 2$ minimal models. Our method for constructing boundary states for Gepner models uses the formalism (based on Gepner's analysis) developed by Eguchi et al. in ref [21]. A modular invariant partition function is constructed by using spectral flow invariant orbits(these are certain sums of $N = 2$ characters dictated by spectral flow). The type II partition function is written in terms of supersymmetric characters associated with the spectral flow invariant orbits. Spacetime supersymmetry is manifest in this following an argument due to Gepner[38, 13].

Let us now turn to the Landau-Ginzburg description of the Gepner models that we have just discussed. There is a lot of evidence that the level $k$ $N = 2$ minimal model can be obtained as the RG fixed point of a Landau-Ginzburg model (with global $N = 2$ supersymmetry)of a single scalar superfield with superpotential $\Phi^{k+2}$. It has been shown that the central charge of the RG fixed point matches that of the minimal model and more recently, the elliptic genus of the two theories was shown to match[53].

The massless spectra and symmetries of certain Gepner models are in one to one correspondence with those obtained in some Calabi-Yau compactifications[38, 13]. This result was initially shown by Gepner for the quintic hypersurface in $CP^4$ which is equivalent to the $(k=3)^5$ Gepner model. For this example, it was shown in ref. [54] that certain Yukawa couplings between the massless fields also agreed from both sides. The explanation of this phenomenon came first by a path integral argument due to Greene et. al [14]. Using the relationship between the level $k$ $N=2$ minimal model and the the LG theory with superpotential $\Phi^{k+2}$, for the Gepner model given by $(k_1, k_2, ...k_n)$, they chose the LG superpotential $W(\Phi_1, \Phi_2, \cdots \Phi_n) = \Phi_1^{k_1+2} + \Phi_2^{k_2+2} + \cdots \Phi_n^{k_n+2}$. Assuming that the $D$ terms in the theory are irrelevant operators and their effect can be neglected in the path integral for this model, it was shown in ref. [14] that one exactly ends up with the constraint that defines a Calabi-Yau manifold in weighted projective space. There was a need to impose a discrete identification in order to make the argument work. This corresponds to an orbifolding of the LG model and gives rise to the integer projection imposed by Gepner in order to have spacetime supersymmetry. Thus the precise statement is that the Gepner model is equivalent to the LG orbifold. The Calabi-Yau - Landau-Ginzburg correspondence was later proved more rigourously by Witten [15] where it was shown how a varying Kähler parameter interpolates between the geometrical (Calabi-Yau) and the non-geometrical (Landau-Ginzburg) phases.

For instance the string vacuum that corresponds to five copies of the $k=3$, $N=2$ minimal model, is obtained by orbifolding by $\exp[i2\pi J_0]$, where $J_0$ measures the left $U(1)$ charge. Other more complicated orbifolding possibilities exist (and lead to other Calabi-Yau manifolds) but we shall not consider them here. An $N=2$ LG theory which has not be orbifoldized contains only $(c,c)$ and $(a,a)$ states. However in order that a LG description of a $N=2$ super conformal field theory reproduce the string vacuum it is essential that it also include the $(a,c)$ states. These states appear in the twisted sector of the LG orbifold[55, 56].

## 2.2 D-branes in Landau-Ginzburg models

In this section, we will describe D - branes wrapped on supersymmetric cycles using the Landau-Ginzburg description of Calabi-Yau manifolds. We will first generalise the bulk Landau-Ginzburg theory by including boundary terms which preserve part of the worldsheet supersymmetry following the work of Warner[57]. We will obtain the analog of A and B type boundary conditions in this system. For the case of the quintic, we will show that A-type boundary conditions naturally choose a real submanifold which is the supersymmetric three-cycle constructed by Becker et al.[16].

We will consider the massive Euclidean Landau-Ginzburg theory in two dimensions, with complex bosons $\phi_i$ and complex Dirac fermions denoted by $\psi, \overline{\psi}$, with the left and right moving components denoted by the subscripts + and − respectively. The action for the model (in which we have taken the boundary to lie on the line $x^0 \equiv x = 0$ and $x^1 \equiv y$) is given by

$$S = S_{\text{bulk}} + S_{\text{boundary}} \quad , \tag{2.39}$$

where

$$\begin{aligned} S_{\text{bulk}} &= \int_{-\infty}^{0} dx^0 \int_{-\infty}^{\infty} dx^1 \left\{ -\left(\partial_\alpha\overline{\phi}_i \partial_\alpha \phi_i\right) \right. \\ &- \frac{1}{2}\left(\overline{\psi}_{-i}\partial_0\psi_{-i} - \overline{\psi}_{+i}\partial_0\psi_{+i} - \partial_0(\overline{\psi}_{-i})\psi_{-i} + (\partial_0\overline{\psi}_{+i})\psi_{+i}\right) \\ &+ \frac{i}{2}\left(\overline{\psi}_{-i}\partial_1\psi_{-i} + \overline{\psi}_{+i}\partial_1\psi_{+i} - \partial_1(\overline{\psi}_{-i})\psi_{-i} - \partial_1(\overline{\psi}_{+i})\psi_{+i}\right) \\ &+ \left.\left(\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\right)\psi_{+i}\psi_{-j} + \left(\frac{\partial^2 \overline{W}}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\right)\overline{\psi}_{+i}\overline{\psi}_{-j} - \left|\frac{\partial W}{\partial\phi_i}\right|^2 \right\} \\ S_{\text{boundary}} &= \int_{-\infty}^{\infty} dy \left(-\frac{1}{2}\overline{\psi}_i\gamma^* \psi_i\right) \end{aligned} \tag{2.40}$$

In the above $W(\phi)$ is a quasi-homogeneous superpotential. As is usual for theories with boundary, the kinetic energy term for the fermions written in symmetric form. In addition, we have included an explicit boundary term following the work of Warner[57].[1].

[1]The dictionary which relates Warner's notation to ours is as follows: $\lambda_1 = \psi_{+i}$, $\lambda_2 = \psi_{-i}$, $\overline{\lambda}_1 =$

We have used an off diagonal basis where the two dimensional $\gamma$ matrices are defined by

$$\gamma^0 = \begin{pmatrix} 0 & i \\ 0 & 1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.41}$$

The supersymmetry transformations of this model are given explicitly in terms of the components to be

$$\begin{aligned}
\delta\phi_i &= -\left(\psi_{+i}\epsilon_+ + \psi_{-i}\epsilon_-\right) \\
\delta\overline{\phi}_i &= \left(\overline{\psi}_{-i}\overline{\epsilon}_- + \overline{\psi}_{+i}\overline{\epsilon}_+\right) \\
\delta\psi_{+i} &= \left(-\partial_0\phi_i + i\partial_1\phi_i\right)\overline{\epsilon}_+ + \frac{\partial\overline{W}}{\partial\overline{\phi}_i}\epsilon_- \\
\delta\overline{\psi}_{+i} &= \left(\partial_0\overline{\phi}_i - i\partial_1\overline{\phi}_i\right)\epsilon_+ - \frac{\partial W}{\partial\phi_i}\overline{\epsilon}_- \\
\delta\psi_{-i} &= \left(\partial_0\phi_i + i\partial_1\phi_i\right)\overline{\epsilon}_- - \frac{\partial\overline{W}}{\partial\overline{\phi}_i}\epsilon_+ \\
\delta\overline{\psi}_{-i} &= \left(-\partial_0\overline{\phi}_i - i\partial_1\overline{\phi}_i\right)\epsilon_- + \frac{\partial W}{\partial\phi_i}\overline{\epsilon}_+
\end{aligned} \tag{2.42}$$

This action is now varied under ordinary and supersymmetric variation, giving rise to boundary terms, and consistent boundary conditions are imposed in order to cancel these. The boundary terms coming from ordinary variation can be written as

$$\begin{aligned}
\delta_{ord}S = -\int_{-\infty}^{\infty} dy(\partial_0\overline{\phi}_i)\delta\phi \;&+\; (\partial_0\phi)\delta\overline{\phi} + \frac{1}{2}\left(\overline{\psi}_{-i} - \overline{\psi}_{+i}\right)\left(\delta\psi_{+i} + \delta\psi_{-i}\right) \\
&-\; \frac{1}{2}\left(\psi_{+i} - \psi_{-i}\right)\left(\delta\overline{\psi}_{-i} + \delta\overline{\psi}_{+i}\right)
\end{aligned} \tag{2.43}$$

evaluated on the line $x = 0$. Similarly, the boundary terms arising out of supersym-

$\overline{\psi}_{-i}$, $\overline{\lambda}_2 = \overline{\psi}_{+i}$, $\alpha_1 = \overline{\epsilon}_-$, $\alpha_2 = \overline{\epsilon}_+$, $\overline{\alpha}_1 = \epsilon_+$, $\overline{\alpha}_2 = \epsilon_-$.

metric variations of the action can be written as

$$\begin{aligned}\delta_{susy}S &= \int_{-\infty}^{\infty} dy\left[-\frac{1}{2}\partial_0\phi_i\left(\overline{\psi}_{-i}-\overline{\psi}_{+i}\right)+\frac{i}{2}\partial_1\phi_i\left(\overline{\psi}_{-i}+\overline{\psi}_{+i}\right)\right](\overline{\epsilon}_- - \overline{\epsilon}_+)\\ &+ \left[\frac{1}{2}\partial_0\overline{\phi}_i\left(\psi_{+i}-\psi_{-i}\right)+\frac{i}{2}\partial_1\overline{\phi}_i\left(\psi_{+i}+\psi_{-i}\right)\right](\epsilon_+ - \epsilon_-)\\ &+ \frac{1}{2}\left(\frac{\partial W}{\partial\phi_i}\right)(\psi_{+i}+\psi_{-i})(\overline{\epsilon}_- + \overline{\epsilon}_+)\\ &- \frac{1}{2}\left(\frac{\partial\overline{W}}{\partial\overline{\phi}_i}\right)(\overline{\psi}_{-i}+\overline{\psi}_{+i})(\epsilon_+ + \epsilon_-) \qquad (2.44)\end{aligned}$$

Now, following our earlier discussion on the A-type boundary conditions, we will look for the unbroken $N = 2$ supersymmetry to be given by [2]

$$\epsilon_+ = \overline{\epsilon}_- \quad . \qquad (2.45)$$

The above choice is dictated by A-type boundary condition $G_L^+ = \pm G_R^-$ for the supersymmetry generators.

The supersymmetric variation the action $S$ after imposing $\epsilon_+ = \overline{\epsilon}_-$ is

$$\begin{aligned}\delta_{\rm susy}S &= \int_{-\infty}^{\infty} dy\left[-\frac{1}{2}\partial_0\phi_i\left(\overline{\psi}_{-i}-\overline{\psi}_{+i}\right)+\frac{i}{2}\partial_1\phi_i\left(\overline{\psi}_{-i}+\overline{\psi}_{+i}\right)\right](\epsilon_+ - \epsilon_-)\\ &+ \left[\frac{1}{2}\partial_0\overline{\phi}_i\left(\psi_{+i}-\psi_{-i}\right)+\frac{i}{2}\partial_1\overline{\phi}_i\left(\psi_{+i}+\psi_{-i}\right)\right](\epsilon_+ - \epsilon_-)\\ &+ \frac{1}{2}\left[\frac{\partial W}{\partial\phi_i}(\psi_{+i}+\psi_{-i})-\frac{\partial\overline{W}}{\partial\overline{\phi}_i}(\overline{\psi}_{+i}+\overline{\psi}_{-i})\right](\epsilon_+ + \epsilon_-) \qquad (2.46)\end{aligned}$$

Further, let us assume that the fermions also satisfy the following condition:[3]

$$(\psi_{+i} - \overline{\psi}_{-i})|_{x=0} = 0 \qquad (2.47)$$

The following set of boundary conditions on the bosonic fields makes the action invariant under the $N = 2$ supersymmetry. The bosonic boundary conditions are also consistent with the supersymmetric variation of the fermionic boundary

[2]One can also choose $\epsilon_+ = -\overline{\epsilon}_-$ here.

[3]Since $J_L = -J_R$ for A-type boundary conditions, we are not permitted to set $\psi_{+i} + \psi_{-i} = 0$ on the boundary. Thus one has to choose $(\psi_{+i} - \overline{\psi}_{-i}) = 0$ on the boundary.

condition in eqn. (2.47).

$$\begin{aligned} \partial_0\left(\phi_i-\overline{\phi}_i\right)|_{x=0} &= 0 \\ \partial_1\left(\phi_i+\overline{\phi}_i\right)|_{x=0} &= 0 \\ \left(\frac{\partial W}{\partial\phi_i}-\frac{\partial\overline{W}}{\partial\overline{\phi}_i}\right)\Bigg|_{x=0} &= 0 \end{aligned} \tag{2.48}$$

Hence (2.47) and (2.48) give us a set of boundary conditions on the fields such that we have an unbroken $N=2$ supersymmetry on the boundary. The last line of the eqn. (2.48) has to be viewed as a consistency condition on the boundary condition. It has a simple interpretation (in the infrared limit) provided the equation $W=0$ admits a pure imaginary solution. It corresponds to the statement that for directions along the brane, the variation of $W$ has to vanish. For example, for a circle given by $f(x,y)=(x^2+y^2-1)=0$, the analogous statement is that $\partial_\phi f=0$, where $\phi$ is the angle in cylindrical polar coordinates. We will see that similar conditions appear even for B-type boundary conditions whenever a Neumann boundary condition is imposed on fields.

These 'mixed' boundary conditions should correspond to a D- brane wrapped on some cycle of Calabi-Yau given by the equation $W(\phi)=0$. Let us see if this can be substantiated. Notice that, the last of the equations in (2.48) implies that the real part of all the complex scalar fields $\phi_i$ can be chosen to vanish on the boundary at $x=0$. Thus, the target space interpretation is that the cycle corresponds to a submanifold of the Calabi-Yau given by the coordinates becoming imaginary on the boundary. As shown in [16], for the quintic hypersurface defined in $CP^4$ by the equation

$$\Sigma_{i=1}^5\left(\phi_i\right)^5 = 0 \quad , \tag{2.49}$$

imposing the reality (or equivalently pure imaginary)[4] condition on all the $\phi_i$ indeed provides one with a submanifold which is a supersymmetric three-cycle.

Actually, (2.47) and (2.48) are not the most general choice choice of boundary

[4]We will nevertheless refer to this as real submanifold.

conditions. The following set of boundary conditions is more general:

$$\begin{aligned}
(\psi_{+i} - A_i{}^j\, \overline{\psi}_{-j})|_{x=0} &= 0 \\
\partial_0 \left(\phi_i - A_i{}^j\, \overline{\phi}_j\right)|_{x=0} &= 0 \\
\partial_1 \left(\phi_i + A_i{}^j\, \overline{\phi}_j\right)|_{x=0} &= 0 \\
\left.\left(A_i{}^j\, \frac{\partial W}{\partial \phi_j} - \frac{\partial \overline{W}}{\partial \overline{\phi}_i}\right)\right|_{x=0} &= 0 \quad ,
\end{aligned} \tag{2.50}$$

where the symmetric matrix $A$ satisfies $AA^* = 1$ and it is block diagonal i.e., it does not mix fields with different charge under the $U(1)$ of the unbroken $N = 2$ supersymmetry algebra. One simple choice is given by $A = \mathrm{Diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})$ subject to the condition involving the superpotential being satisfied.

Given a matrix $A$ which provides boundary conditions consistent with the superpotential, we can construct other consistent choices. Let us assume that the superpotential is invariant under a discrete group $G$ which acts holomorphically on the fields. Let ${}^g\phi_i = g_i{}^j \phi^j$ be the the action of $g \in G$. The invariance of the superpotential under $G$ implies that $W(\phi) = W({}^g\phi)$. Corresponding to the element $g$, we can construct another $N = 2$ preserving boundary condition on the fields given by $A_g = g^{-1} \cdot A \cdot g^*$. Clearly, if $g$ is a real matrix, then $A$ and $A_g$ belong to the same conjugacy class and we do not obtain new boundary conditions.

Clearly with a LG theory it would be difficult to provide a description of the boundary states in the cylinder channel with the same degree of explicitness that we can associate with free-field theories. However we can notice the following. We can label the boundary states by the primary fields associated with them as in the general case discussed by Cardy and implemented by Recknagel and Schomerus. Since for the A-type boundary condition, one needs equal charges from the left and right moving sectors in the construction of the boundary state, it is clear that the lowest states are associated with the application of the LG fields themselves on the ground state vacuum of the theory. It is clear that this may involve appropriate number of $\phi$ fields, such that the $U(1)$ charge projection condition is satisfied, a similar set of states with the application of $\bar{\phi}$ fields and also states built by application of both $\phi$ and $\bar{\phi}$ fields such that they have integral $U(1)$ charge. Some of these states will be obviously in the massive sector and will not contribute to massless states but as we shall see later such states are required in the general definition of the boundary state. This ties in rather nicely with the method for the construction of boundary states that we will pursue in section IV of the paper. In this connection we note also that so far we have no means yet, strictly within the LG formulation, to determine the normalization of the boundary states as is done by the method of Cardy for the boundary states of an arbitrary minimal model.

We will now discuss B-type boundary conditions. For the B-type boundary conditions, the $N = 2$ supersymmetry is given by

$$\epsilon_+ = -\epsilon_- \quad . \tag{2.51}$$

We will now look for boundary conditions on the fields such that the above supersymmetry is preserved. Under supersymmetry variation of the action, (after setting $\epsilon_+ = -\epsilon_-$ as required), we obtain a boundary term of the form

$$\begin{aligned} \int_{-\infty}^{\infty} dy \qquad & \left\{ \left[ \partial_0 \phi_i \left( \overline{\psi}_{-i} - \overline{\psi}_{+i} \right) - i \partial_1 \phi_i \left( \overline{\psi}_{-i} + \overline{\psi}_{+i} \right) \right] \overline{\epsilon}_+ \right. \\ + & \left. \left[ \partial_0 \overline{\phi}_i \left( \psi_{+i} - \psi_{-i} \right) + i \partial_1 \overline{\phi}_i \left( \psi_{+i} + \psi_{-i} \right) \right] \epsilon_+ \right\} \end{aligned} \tag{2.52}$$

The vanishing of the above boundary term suggests two possible boundary conditions:

1. $\partial_0 \phi_i|_{x=0} = 0$ and $(\psi_{-i} + \psi_{+i})|_{x=0} = 0$. This corresponds to Neumann boundary conditions on the field $\phi_i$ and its complex conjugate $\overline{\phi}_i$. Consistency with supersymmetry imposes the additional condition $\frac{\partial W}{\partial \phi_i}|_{x=0} = 0$. Note that this is a condition in spacetime where it says that the tangential derivative along the boundary vanishes.

2. $\partial_1 \phi_i|_{x=0} = 0$ and $(\psi_{-i} - \psi_{+i})|_{x=0} = 0$. This corresponds to Dirichlet boundary conditions on the field $\phi_i$ and its complex conjugate $\overline{\phi}_i$.

Since the above set of boundary conditions treat both the real and imaginary parts of the complex scalar fields $\phi_i$ in identical fashion, the cycle which is chosen by the boundary conditions will correspond to a holomorphic submanifold of the Calabi-Yau. Thus the cycle is a supersymmetric cycle.

Again, one can construct a general boundary condition. It is specified by a hermitian matrix $B$ which satisfies $B^2 = 1$ and is block diagonal i.e., it does not mix fields with different charge under the $U(1)$ of the unbroken $N = 2$ supersymmetry algebra. The general boundary condition is given by

$$
\begin{aligned}
(\psi_{+i} + B_i^{\ j}\psi_{-j})|_{x=0} = 0 \quad &, \\
\partial_0(\phi_i + B_i^{\ j}\phi_j)|_{x=0} = 0 \quad &, \\
\partial_1(\phi_i - B_i^{\ j}\phi_j)|_{x=0} = 0 \quad &, \\
\left.\left(\frac{\partial W}{\partial \phi_i} + B_i^{*j}\frac{\partial W}{\partial \phi_j}\right)\right|_{x=0} = 0 &
\end{aligned}
\tag{2.53}
$$

Since $B$ squares to one, its eigenvalues are $\pm 1$. An eigenvector of $B$ with eigenvalue of $+1$ corresponds to a Neumann boundary condition and $-1$ corresponds to a Dirichlet boundary condition.

Let us now discuss what the B-type boundary states would look like with Dirichlet or Neumann boundary conditions on the LG fields. With Neumann or Dirichlet boundary conditions it is easy to see that the $U(1)$ current obeys boundary conditions that require all boundary states to have equal and opposite charges in the left and right moving sectors. This implies that all the boundary states for such cycles must come from the twisted sector in the LG theory. It is not immediately clear what difference the Neumann and Dirichlet boundary conditions would make since in the twisted sector the zero-mode of the LG fields are no longer present. However it is nevertheless clear that the even supersymmetric cycles are charged under the Ramond-Ramond ground states of the twisted sector. Before we turn to specific examples we would add that all the massless states could probably be constructed by an extension of the method of Kachru and Witten[58] where they used the cohomology of the $\bar{Q}_+$ charge to define the massless states in the left-moving sector of a (2,2) compactification of the heterotic string. In the case of D-branes in the openstring sector we have only one $L_0$ operator and two supercharges. It is clear that an extension of the methods of ref. [58] will be possible[4].

We now present some explicit examples for the construction of boundary conditions that we have outlined above. From our previous analysis, we have seen that a general A-type boundary condition is parametrised by a matrix $A$ and a matrix

$B$ for B-type boundary conditions. As mentioned earlier, one has to ensure that a particular choice of the matrix $A$ or $B$, the consistency condition involving the superpotential is satisfied. We find that for B-type boundary conditions in all the examples we consider we are unable to impose Neumann boundary conditions on all fields simultaneously. A simple example involving one scalar field (like the LG mode associated with the $N = 2$ minimal model at level $k$) shows that the only condition one can impose on the scalar is the Dirichlet one. This is not inconsistent with the fact that in the models we consider, in the infrared limit, the best one can do is to impose Neumann boundary conditions on all but one of the fields.

1. The $1^3$ model

   This model is described by the superpotential involving three scalar fields given by $W = (\phi_1^3 + \phi_2^3 + \phi_3^3)$. A-type boundary conditions pick out the submanifold (one-cycle) given by

   $$(x_1^3 + x_2^3 + x_3^3) = 0 \quad ,$$

   where $x_i = \mathrm{Im}\phi_i$. The discrete symmetry group of this superpotential is given by $G = (S_3 \times (Z_3)^3)/Z_3$[5]. Other supersymmetric cycles which can be constructed from this cycle are $(ix_1, i\omega^a x_2, i\omega^b x_3)$, where $a$ and $b$ are integers satisfying $a+b = 0 \bmod 3$[6]. These correspond to choosing $A = \mathrm{Diag}(1, \omega^a, \omega^b)$. Thus, we $a = 2, b = 1$ respectively. One can verify that the three one-cycles are non-intersecting.

   There exists another choice for $A$ given by $A_1 = \mathrm{Diag}[1, 1, \exp(i2\pi/3)]$, which leads to the one-cycle given by

   $$(x_1^3 + x_2^3 - x_3^3) = 0 \quad ,$$

   where $x_i = \mathrm{Im}\phi_i$ $(i = 1, 2)$, $x_3 = \mathrm{Im}(\exp(-i\pi/3)\phi_3)$. By studying the action of $S_3$ on this cycle, we will see that this cycle is not chosen in the Gepner model

[5] $S_3$ is the permutation group with three elements (here it permutes the three fields), the three $Z_3$'s are generated by the action $\phi^i = \omega\phi_i$ (for $i = 1, 2, 3$). ($\omega$ is a non-trivial cube root of unity.) The quotient $Z_3$ is the diagonal $Z_3$.

[6] This condition comes from requiring that the discrete symmetry generator commute with the supersymmetry generator.

construction.

Earlier, we had imposed the condition $a+b=0 \bmod 3$ in the matrix $A$. Relaxing this condition, we will get two more sets of one-cycles corresponding to $a+b=1,2 \bmod 3$. Within each set, the one-cycles are non-intersecting. However, if one considers one-cycles from different sets, they can intersect. For example, the one-cycle chosen by $A = \text{Diag}(1,1,1)$ intersects the one-cycle chosen by $A = \text{Diag}(1,\omega,\omega)$ at the point $(0,1,1) \simeq (0,\omega,\omega)$ in homogeneous coordinates. The cylinder amplitude between these two states will not vanish since the two boundary states do not preserve the same supersymmetry generators. Further, one expects to see a tachyon in the open string spectrum.

For B-type boundary conditions, we find the following consistent choices:

(a) Choose $B = \text{Diag}(-1,-1,-1)$ which corresponds to Dirichlet boundary conditions on all scalars. Let $\phi_i = c_i$ where $c_i$ are constants. Presumably, they will have to satisfy $(c_1^3 + c_2^3 + c_3^3) = 0$ given the infrared limit of the bulk theory but this does not follow from the consistency conditions. Clearly $(c_1, c_2, c_3)$ corresponds to a point (in homogeneous coordinates) on the torus and corresponds to a supersymmetric zero-cycle.

(b) For $B = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, the consistency conditions imply that $\phi_1 + \phi_2 = 0$ and $\phi_3 = 0$ with $(\phi_1 - \phi_2)$ being free i.e., satisfying Neumann boundary conditions.

We are unable to find choices for $B$ such that one obtains two Neumann and one Dirichlet boundary condition in addition to the all Neumann case which can be clearly ruled out by analysing the consistency condition involving the superpotential.

2. The $2^2$ model

This model is described by the superpotential $W = \phi_1^4 + \phi_2^4 + \phi_3^2$, where we have included a 'trivial' quadratic piece. For A-type boundary conditions given by

$A = 1$, there are no real solutions. However, choosing $A = \mathrm{Diag}(1, i, 1)$, one obtains the one-cycle given by the equation

$$x_1^4 - x_2^4 + x_3^2 = 0 \quad ,$$

where $x_1 = \mathrm{Im}\phi_1$ , $x_2 = \mathrm{Im}(\phi_2/\sqrt{i})$ and $x_3 = \mathrm{Im}\phi_3$. This equation has solutions for real $x_i$. The discrete symmetry group of this model is given by $G = (S_2 \times (Z_4)^2 \times Z_2)/Z_4$. Choosing an element of $G$ given by $g = (i^a, i^b, (-)^c)$ with $a + b + 2c = 0 \bmod 4$. By following the procedure mentioned earlier we obtain $A_g = \mathrm{Diag}[(-)^a, i(-)^b, 1]$ which provides cycles related to $A = \mathrm{Diag}(1, i, 1)$ by a $Z_2$ subgroup.

There is another choice given by $A_2 = \mathrm{Diag}(1, 1, -1)$, one obtains the one-cycle given by the equation

$$x_1^4 + x_2^4 - x_3^2 = 0 \quad ,$$

where $x_1 = \mathrm{Im}\phi_1$ , $x_2 = \mathrm{Im}\phi_2$ and $x_3 = \mathrm{Re}\phi_3$. Again, this one-cycle is not chosen by the Gepner model model construction. This cycle is invariant under the $S_2$ exchange while the first choice is not invariant.

3. The Quintic

We have already seen the example of a real three-cycle obtained from the A-type boundary conditions with $A = 1$. The Quintic has a discrete symmetry group $G = (S_5 \times (Z_5)^5/Z_5)$. A subgroup is given by the $Z_5$ generated by

$$g : (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \to (\phi_1, \alpha\phi_2, \alpha^2\phi_3, \alpha^3\phi_4, \alpha^4\phi_5) \quad ,$$

where $\alpha$ is any non-trivial fifth root of unity. This boundary condition corresponds to a three-cycle of the quintic which is related to the real three-cycle by the $Z_5$ transformation. It follows trivially that this cycle is a special Lagrangian submanifold of the deformed quintic and hence a supersymmetric cycle. It is clear that this procedure leads to the construction of supersymmetric cycles. Considering the full group $G$, one can generate $G$-related supersymmetric cycles by the choice $g = \mathrm{Diag}(1, \alpha^a, \alpha^b, \alpha^c, \alpha^d)$, where $a, b, c, d$ are integers which satisfy $a + b + c + d = 0 \bmod 5$.

For B-type boundary conditions, we find the following three consistent choices for the matrix $B$: (i) $B_1 = \text{Diag}(-1,-1,-1,-1,-1)$; (ii) $B_2 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & -1_{3\times 3} \end{pmatrix}$ and (iii) $B_3 = \begin{pmatrix} -\sigma_1 & 0 & 0 \\ 0 & -\sigma_1 & 0 \\ 0 & 0 & -\sigma_1 \end{pmatrix}$ where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a Pauli matrix. The first choice is the all Dirichlet one. The second choice has one Neumann and four Dirichlet conditions and the last one has two Neumann and three Dirichlet conditions on some linear combinations of the fields.

4. The Conifold

   The deformed conifold is described by a non-compact Calabi-Yau associated with the superpotential[59, 60]

$$W = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 - \frac{\mu}{\phi_5} \quad ,$$

   where $\mu = 0$ is the conifold limit and $\mu = |\mu|e^{i\phi}$ is complex. Imposing A-type boundary conditions with $A = \text{Diag}(1,1,1,1,e^{2i\phi})$ chooses the three-cycle given by the equation ($x_i = \text{Im}\phi_i$, $i = 1,2,3,4$ and $x_5 = \text{Im}(\phi_5 e^{-i\phi})$)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - \frac{|\mu|}{x_5} = 0 \quad .$$

   Working in inhomogeneous coordinates $y_i = x_i\sqrt{x_5}$, we obtain an $S^3$ of radius $\sqrt{|\mu|}$ which is known to be a special Lagrangian submanifold of the conifold and hence is a supersymmetric cycle.

## 2.3 D-branes in Gepner models

In this section we will be constructing the boundary states associated with cycles of a Calabi-Yau space which can be obtained as a Gepner model. The Calabi-Yau is specified by tensoring together $N = 2$ minimal models and truncating to states with integer charge under the $U(1)$ of the $N = 2$ supersymmetry.

The torus partition function is constructed in terms of supersymmetric characters. The analysis of Gepner showed the relationship between spacetime supersymmetry and spectral flow with $\eta = \frac{1}{2}$ in the $N = 2$ supersymmetry algebra. The supersymmetric character is obtained by first constructing the Virasoro character in the Neveu-Schwarz (NS) sector and then including all characters (whose states are related to the original one by spectral flow in steps of $\eta = \frac{1}{2}$). For example, the graviton character is obtained by first considering the identity operator. Then, one applies the spectral flow operation once to obtain a state in the Ramond sector. The second application leads one back to the NS sector. This procedure is repeated until one returns to the original state after a few iterations. The supersymmetric character (in the lightcone gauge) can be written as

$$X_i = \frac{1}{2}\left\{NS_i\left(\frac{\theta_3}{\eta}\right)^n - \widetilde{NS}_i\left(\frac{\theta_4}{\eta}\right)^n - R_i\left(\frac{\theta_2}{\eta}\right)^n + \widetilde{R}_i\left(\frac{\theta_1}{\eta}\right)^n\right\} \quad , \tag{2.54}$$

where $(\frac{\theta}{\eta})^n$ come from level one SO($2n$) characters associated with the non-compact spacetime of dimension $d$(with $n = (d-2)/2$). The signs reflect the GSO projection required in order to obtain the correct spin-statistics connection. In the above, $NS$ refers to the Virasoro character in the NS sector ($NS = \mathrm{tr}_{NS}\ q^{L_0-c/24}$) while $R$ refers to the Virasoro character in the R sector ($R = \mathrm{tr}_R\ q^{L_0-c/24}$). $\widetilde{NS}$ and $\widetilde{R}$ refer to the Virasoro characters in the appropriate sector with the inclusion of $(-)^F$, where $F$ is the worldsheet fermion number ($\widetilde{NS} = \mathrm{tr}_{NS}\ (-)^F\ q^{L_0-c/24}$). As a consequence of spacetime supersymmetry, each of the supersymmetric characters vanish identically. See ref. [38, 13] for the details of the argument. However, in the cases considered in this chapter, we have also explicitly verified that this is indeed true.

Since the multiplicities $D_i$ is generically not equal to one, one needs to resolve the S-matrix associated with the Gepner model. There is a procedure due to Fuchs, Schellekens and Schweigert which we employ to resolve the S-matrix[52]. The Cardy prescription can then be applied to the resolved S-matrix in order to obtain the boundary states corresponding to D-branes wrapped around cycles of the Calabi-Yau corresponding to the Gepner model. The resolution of the S-matrix for models such as the quintic is computationally complex and hence we will illustrate the procedure for the simple case of the $1^3$ and $2^2$ Gepner models (for A-type boundary conditions). Here, we will see a very nice match with respect to the analysis using the LG description and hence be able to directly achieve a target space interpretation for the boundary states.

We should point out the differences between our approach and that of Recknagel and Schomerus. In their construction, the boundary conditions such as $J_L = J_R$ are imposed separately in each of the minimal models which enters the theory after which they construct boundary states for by tensoring together boundary states of the individual minimal models. Thus, the boundary is forced to preserve the $N = 2$ algebra of each minimal model rather than the diagonal $N = 2$. This seems to ensure that the setting is "rational". In our construction, we work with spectral flow invariant orbits. Given the intimate relationship between spacetime supersymmetry and spectral flow, our restriction may seem natural in the context of D-branes since they are BPS states in spacetime. The supersymmetric characters can be seen to be sums of characters of the extended algebra $\mathcal{W}$, one obtains by including the $\eta = \frac{1}{2}$ spectral flow operator to the $N = 2$ algebra[21]. Thus, our boundary states preserve the extended algebra $\mathcal{W}$ rather than the $N = 2$ of the individual minimal models. "Rationality" is obtained because we work with only a finite number of supersymmetric characters rather than characters of the irreducible representations of $\mathcal{W}$. We believe that these two approaches complement each other and must not be considered to be distinct.

Let us start our analysis with the example of the $(k = 1)^3$ Gepner model. This model is obtained by the tensoring of three copies of the $k = 1$ $N = 2$ minimal model. This is the Gepner model for a torus at its SU(3) point. The characters of the $k = 1$ minimal model in the NS sector will be labelled as follows. $(\chi^l_m \equiv \chi^{l(s=0)}_m + \chi^{l(s=2)}_m)$

| $\chi^l_m$ | q | h | Label |
|---|---|---|---|
| $\chi^0_0$ | 0 | 0 | $A = \theta_{0,3}(\frac{\tau}{2})/\eta(\tau)$ |
| $\chi^1_1$ | 1/3 | 1/6 | $B = \theta_{2,3}(\frac{\tau}{2})/\eta(\tau)$ |
| $\chi^1_{-1}$ | -1/3 | 1/6 | $B_c = \theta_{4,3}(\frac{\tau}{2})/\eta(\tau)$ |

$B$ is associated with a chiral primary state and $B_c$ is associated with an anti-chiral primary state. Under spectral flow(with $\eta = 1$), we have the sequence

$$A \to B \to B_c \to A$$

The spectral flow invariant orbits for this model in the NS sector are

| Label | Orbit | $q$; $h$ |
|---|---|---|
| $NS_0$ | $A^3 + B^3 + B_c^3$ | $q = h = 0$ |
| $NS_1$ | $3ABB_c$ | $q = 0$; $h = \frac{1}{3}$ |

In the above table, the values of $q$ and $h$ correspond to the state with the smallest value of $h$ occurring in the spectral flow invariant $NS$ orbit. $NS_0$ is the graviton orbit and the other orbit is massive i.e., it corresponds to massive states in the non-compact spacetime. The choice of $3ABB_c$ rather than $ABB_c$ as the character for the $NS_1$ state can be understood as follows: Let us assume that the three minimal models are labelled $1, 2, 3$ respectively. Then a spectral flow invariant orbit is given by $(A_1B_2B_{c3} + B_1B_{c2}A_3 + B_{c1}A_2B_3)$, where we have explicitly kept the minimal model label. Getting rid of these labels leads to $3ABB_c$ and hence our choice. The S-matrix for this model is derived to be

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$D_0 = 1$ and $D_1 = 2$. It is sufficient to consider the NS sector to obtain the S-matrix. One can show that this S-matrix is identical to that obtained from the modular transformation of the full supersymmetric character[38, 13]. A modular invariant torus partition function for this model is given by

$$\mathcal{T} = \sum_{i=0}^{1} D_i \, |X_i|^2 \tag{2.55}$$

where $X_i$ are the supersymmetric characters[7].

However, as things stand one cannot apply Cardy's prescription here since the character $X_1$ occurs with multiplicity 2 in the toroidal partition function. In order to obtain the resolved S-matrix, one splits $D_1 = 2 = 1 + 1$. Thus the resolved

[7]The multiplicity of two associated with $NS_1$ again is related to the fact that if we kept track of the minimal model labels, there are two distinct spectral flow invariant orbits given by the even permutation $NS_{1+} = (A_1B_2B_{c3} + B_1B_{c2}A_3 + B_{c1}A_2B_3)$ and the odd permutation $NS_{1-} = (B_1A_2B_{c3} + B_{c1}B_2A_3 + A_1B_{c2}B_3)$. This actually completely resolves the S-matrix here. In more complicated situations, this will enable us to partially resolve the S-matrix.

S-matrix is a $3 \times 3$ matrix. It is

$$\widetilde{S} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{2.56}$$

where $\omega$ is a cube root of unity. One can check that:

- $\widetilde{S}^2 = C$ where $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is the charge conjugation matrix.
- $\widetilde{S}$ is symmetric and unitary.
- $\widetilde{S}^4 = 1$.
- $(\widetilde{S}T)^3 = \widetilde{S}^2$ with $T = \mathrm{Diag}(-i, -i\omega, -i\omega)$.

Now, one can apply Cardy's procedure using the resolved S-matrix. Let $|0\rangle\rangle_A$, $|1^+\rangle\rangle_A$, $|1^-\rangle\rangle_A$ be the Ishibashi states (associated with the characters $X_0$, $X_{1,\pm}$) which satisfy A-type boundary conditions.

$$\begin{aligned} |\tilde{0}\rangle &= 3^{-1/4}\left(|0\rangle\rangle_A + |1^+\rangle\rangle_A + |1^-\rangle\rangle_A\right) && (2.57) \\ |\tilde{1}\rangle &= 3^{-1/4}\left(|0\rangle\rangle_A + \omega|1^+\rangle\rangle_A + \omega^2|1^-\rangle\rangle_A\right) && (2.58) \\ |\tilde{2}\rangle &= 3^{-1/4}\left(|0\rangle\rangle_A + \omega^2|1^+\rangle\rangle_A + \omega|1^-\rangle\rangle_A\right) && (2.59) \end{aligned}$$

Note that if we kept track of the minimal model labels, under the exchange of labels $2 \leftrightarrow 3$, $|1^+\rangle\rangle_A \leftrightarrow |1^-\rangle\rangle_A$. Thus, under the same exchange $|\tilde{1}\rangle \leftrightarrow |\tilde{2}\rangle$ with the state $|\tilde{0}\rangle$ being invariant. These boundary states fit in beautifully with the analysis of the $1^3$ model in the LG description. There we obtained a set of A-type boundary conditions parametrised by $A = \mathrm{Diag}(1, \omega^a, \omega^b)$ with $a + b = 0 \bmod 3$. We make the following correspondence: The $a = b = 0$ boundary condition is identified with the state $|\tilde{0}\rangle$ and $(a, b) = (1, 2), (2, 1)$ with the other two boundary states (using properties of these b.c.'s under the $2 \leftrightarrow 3$ exchange in the LG model)[8]

[8]The other choice of boundary condition given by $A_1 = \mathrm{Diag}(1, 1, \exp[i2\pi/3])$ is ruled out because the equation for the one-cycle is clearly not invariant under the $2 \leftrightarrow 3$ exchange.

A more direct correspondence can be worked out by considering the part of the boundary state involving only the $(c,c)$ and $(a,a)$ states. Following the analysis in the LG orbifold, the boundary condition given by the matrix $A = \mathrm{Diag}(1, \omega^a, \omega^b)$ corresponds to multiplying the $(a,a)$ field by the phases given in $A$. Let $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ be the generators of the $(a,a)$ ring. Multiplying this by $A$, one sees that $|0\rangle\rangle_A \to |0\rangle\rangle_A$, $|1^+\rangle\rangle_A \to \omega^a|1^+\rangle\rangle_A$ and $|1^-\rangle\rangle_A \to \omega^b|1^-\rangle\rangle_A$, where we have used $a+b=0$ mod 3. Thus, these boundary states are related to D1-branes of the type IIB string wrapping around $Z_3$ related supersymmetric one-cycles on the torus at the SU(3) point. This result is in agreement with the analysis of Ooguri et al. using different methods. We can also compare with the result of Recknagel and Schomerus. The three boundary states we obtain are a subset of the nine states they construct. They correspond to special linear combinations of their boundary states as picked out by our requirement of spectral flow invariant orbits.

It is easy to verify that the cylinder partition function $C_{\bar{i}\bar{i}} = 3^{-1/2}(X_0 + 2X_1)$. This reflects the $Z_3$ relationship between the three supersymmetric cycles. Under a modular transformation, the annulus partition function we obtain is given by $\mathcal{A}_{\bar{i}\bar{i}} = X_0$. This implies that $n^{i}_{\bar{0}\bar{0}} = \delta_{i0}$ i.e, only the identity sector propagates in the vacuum channel. Both amplitudes vanish as required by supersymmetry. Let us briefly indicate how this comes about in the cylinder channel. Putting in the appropriate spacetime factors, the partition function can be written as

$$Z \;=\; \frac{1}{2\eta^2}\left[|\theta_3^3 NS_i \;-\; \theta_4^3 \tilde{N}S_i \;-\; \theta_2^3 R_i|^2\right] \tag{2.60}$$

For the massless orbit, the partition function equals

$$Z_{massless} \;=\; \left(\frac{\prod_{j=1}^{\infty}(1-q^{\frac{2j}{3}})^3 \;+\; 3q^{\frac{2}{3}}\prod_{j=1}^{\infty}(1-q^{6j})^3}{\prod_{j=1}^{\infty}(1-q^{2j})}\right)(\theta_3^4 \;-\; \theta_4^4 \;-\; \theta_2^4) \tag{2.61}$$

and that for the massive orbit equals

$$Z_{massive} \;=\; \frac{3q^{\frac{2}{3}}\prod_{j=1}^{\infty}(1-q^{6j})^3}{\prod_{j=1}^{\infty}(1-q^{2j})}\,(\theta_3^4 \;-\; \theta_4^4 \;-\; \theta_2^4) \tag{2.62}$$

As can be seen, both vanish due to Jacobi's abstruse identity. Finally, the annulus amplitude $\mathcal{A}_{\bar{0}\bar{i}} = X_i$. Thus, we see that the massive character $X_1$ is related to off-diagonal D-brane configurations (i.e., a D-brane configuration that begins at one boundary and ends at another).

Next, we come to the $(k=2)^2$ Gepner model, which describes a torus at the SU(2)×SU(2) point. The characters of the $k=2$ minimal model in the NS sector will be labelled as follows. ($\chi^l_m \equiv \chi^{l(s=0)}_m + \chi^{l(s=2)}_m$)

| $\chi^l_m$ | q | h | Label |
|---|---|---|---|
| $\chi^0_0$ | 0 | 0 | $A = \chi_0(\tau)\theta_{0,2}(2\tau) + \chi_{\frac{1}{2}}(\tau)\theta_{2,2}(2\tau)$ |
| $\chi^2_2$ | 1/2 | 1/4 | $B = \chi_0(\tau)\theta_{1,2}(2\tau) + \chi_{\frac{1}{2}}(\tau)\theta_{3,2}(2\tau)$ |
| $\chi^2_{-2}$ | -1/2 | 1/4 | $B_c = \chi_0(\tau)\theta_{3,2}(2\tau) + \chi_{\frac{1}{2}}(\tau)\theta_{1,2}(2\tau)$ |
| $\chi^2_0$ | 0 | 1/2 | $C = \chi_0(\tau)\theta_{2,2}(2\tau) + \chi_{\frac{1}{2}}(\tau)\theta_{0,2}(2\tau)$ |
| $\chi^1_1$ | 1/4 | 1/8 | $D = \chi_{\frac{1}{16}}(\tau)\theta_{1,2}(\tau)$ |
| $\chi^1_{-1}$ | -1/4 | 1/8 | $D_c = \chi_{\frac{1}{16}}(\tau)\theta_{3,2}(\tau)$ |

where

$$\begin{aligned}
\chi_0(\tau) &= \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) , \\
\chi_{\frac{1}{2}}(\tau) &= \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \quad \text{and} \\
\chi_{\frac{1}{16}}(\tau) &= \sqrt{\frac{\theta_2(\tau)}{2\eta(\tau)}}
\end{aligned}$$

are the Ising model characters. Under spectral flow (with $\eta = 1$), we have the sequences

$$A \to B \to C \to B_c \to A$$

$$D \to D_c \to D$$

The spectral flow invariant orbits for this model are

| Label | Orbit | $q; h$ |
|---|---|---|
| $NS_0$ | $A^2 + B^2 + C^2 + B_c^2$ | $q = h = 0$ |
| $NS_1$ | $2(AC + BB_c)$ | $q = 0; h = \frac{1}{2}$ |
| $NS_2$ | $2DD_c$ | $q = 0; h = \frac{1}{4}$ |

$NS_0$ is the graviton orbit and the other two orbits are massive. Again, a straighforward but lengthy calculation shows that the cylinder channel partition function is indeed zero. To state the results,

$$Z_{massless} \;=\; 4\theta_3^2(2\tau)\left(\theta_3^4(\tau) \;-\; \theta_4^4(\tau) \;=\; \theta_2^4(\tau)\right) \tag{2.63}$$

while the massive orbits give rise to

$$\begin{aligned} Z^1_{massive} &= 2\theta_2^2(2\tau)\left(\theta_3^4(\tau) \;-\; \theta_4^4(\tau) \;-\; \theta_2^4(\tau)\right) \\ Z^2_{massive} &= \left(\theta_3(2\tau) \;+\; \theta_2(2\tau)\right)\left(\theta_3^4(\tau) \;-\; \theta_4^4(\tau) \;-\; \theta_2^4(\tau)\right) \end{aligned} \tag{2.64}$$

The S-matrix for this model is derived to be

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

$D_0 = 1$, $D_1 = 1$ and $D_2 = 2$.

In order to resolve the fixed point ambiguity, we need to split the $D_2$ as the sum of squares. $D_2$ can be written as $1+1$ leading to a resolution of $S$ as a $4 \times 4$ matrix. The resolved S-matrix is given by

$$\widetilde{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

As one can see, $\widetilde{S}$ is symmetric and squares to the identity matrix.

Now, one can apply Cardy's procedure using the resolved S-matrix. Let $|0\rangle\rangle_A$, $|1\rangle\rangle_A$ $|2^+\rangle\rangle_A$, $|2^-\rangle\rangle_A$ be the Ishibashi states associated with the characters $X_0, X_1, X_{2,\pm}$ which satisfy A-type boundary conditions. Then Cardy's formula leads to the following four boundary states:

$$|\tilde{0}\rangle \;=\; 2^{-1/2}\left(|0\rangle\rangle_A + |1\rangle\rangle_A + |2^+\rangle\rangle_A + |2^-\rangle\rangle_A\right) \tag{2.65}$$

$$|\tilde{1}\rangle \;=\; 2^{-1/2}\left(|0\rangle\rangle_A + |1\rangle\rangle_A - |2^+\rangle\rangle_A - |2^-\rangle\rangle_A\right) \tag{2.66}$$

$$|\tilde{2}\rangle \;=\; 2^{-1/2}\left(|0\rangle\rangle_A - |1\rangle\rangle_A - |2^+\rangle\rangle_A + |2^-\rangle\rangle_A\right) \tag{2.67}$$

$$|\tilde{3}\rangle \;=\; 2^{-1/2}\left(|0\rangle\rangle_A - |1\rangle\rangle_A + |2^+\rangle\rangle_A - |2^-\rangle\rangle_A\right) \tag{2.68}$$

We thus obtain four boundary states. These four states are related to each other by an $S_2 \times Z_2$ subgroup of the discrete symmetry group. The $Z_2$ is the same one which gave different one-cycles in the LG description. The boundary state $|\tilde{0}\rangle$ can be identified with the boundary condition corresponding to $A = \mathrm{Diag}(1, i, 1)$. Now let us discuss how the other choice for $A$ is ruled out. The character $D$ is associated with the LG field $\phi$. We will thus use $\phi_i$ to represent the corresponding chiral primary in the $i$-th minimal model. Thus the character $B$ is associated with $\phi^2$. The part of the Ishibashi state involving only the chiral primaries associated with $DD_c$ will look something like

$$|2^{\pm}\rangle\rangle = (\phi_1\bar{\phi}_2 \pm \phi_2\bar{\phi}_1)|0\rangle + \cdots \quad ,$$

where there is a sign ambiguity in the definition if we require that it be an eigenstate of the permutation group $S_2$. Both states will be associated with the same character $NS_2 = 2DD_c$. The resolution of the S-matrix distinguishes between these two boundary states. Under $S_2$, we have that $|2^{\pm}\rangle\rangle \to \pm|2^{\pm}\rangle\rangle$. The Ishibashi state associated with $NS_0$ remains invariant under this $S_2$. However for the character associated with $NS_1$, there are two possible Ishibashi states

$$|1^{\pm}\rangle\rangle = (\phi_1^2\bar{\phi}_2^2 \pm \phi_2^2\bar{\phi}_1^2)|0\rangle + \cdots \quad ,$$

where $\pm$ denotes the $S_2$ eigenvalue. Requiring that $S_2$ relate the boundary state $|\tilde{0}\rangle$ to either $|\tilde{2}\rangle$ or $|\tilde{3}\rangle$ chooses the minus sign. Thus, we get under this $S_2$ $|\tilde{0}\rangle \leftrightarrow |\tilde{3}\rangle$ and $|\tilde{1}\rangle \leftrightarrow |\tilde{2}\rangle$.

There is another $Z_2$ subgroup of the discrete symmetry group generated by $\phi_1 \to i\phi_1$ and $\phi_2 \to -i\phi_2$ (This corresponds to $a = 1, b = 3$ using the notation given in the examples section for the $2^2$ model.) One can check that under this $Z_2$, $|2^{\pm}\rangle\rangle \to -|2^{\pm}\rangle\rangle$. One can also see that the states associated with $NS_0$ and $NS_1$ remain invariant under this $Z_2$. Under the action of this $Z_2$ one has $|\tilde{0}\rangle \leftrightarrow |\tilde{1}\rangle$ and $|\tilde{2}\rangle \leftrightarrow |\tilde{3}\rangle$

In order to translate this picture into the LG language let us summarise the effect of the two discrete groups on the LG fields. Under the $S_2$, $\phi_1 \leftrightarrow \phi_2$ and under the $Z_2$, $\phi_1 \to i\phi_1$ and $\phi_2 \to -i\phi_2$. We had discovered two different boundary conditions in the LG model given by $A = \mathrm{Diag}(1, i, 1)$ and $A_2 = \mathrm{Diag}(1, 1, -1)$. Under the $S_2 \times Z_2$,

$A$ gives rise to four different boundary conditions, while $A_2$ boundary condition is invariant under $S_2$. Thus the Gepner model construction seems to choose the $A$ boundary condition.

We will now compare with the results of Gutperle and Satoh (GS) for the $2^2$ model obtained by using the method of Recknagel and Schomerus. One can show that $NS_0 = \theta_3(\tau)[\theta_3^2(\tau) + \theta_4^2(\tau)]/\eta(\tau)$ and $NS_1 = \theta_3(\tau)[\theta_3^2(\tau) - \theta_4^2(\tau)]/\eta(\tau)$. (Here $\eta$ is the Dedekind eta function and $\theta_i$ are the standard theta functions.) The annulus amplitude $\mathcal{A}_{\alpha\alpha} = X_0$ which can be seen to be equal to partition function for $(l'_1, l'_2) = (0,0)$ in the notation of GS (upto factors of $\eta$). The boundary state $|\tilde{0}\rangle + |\tilde{1}\rangle$ gives rise to the annulus amplitude $(2X_0 + 2X_1)$ which is equal to the GS calculation for $(l'_1, l'_2) = (1,1)$. Interestingly, there does not seem to be a consistent boundary state which can give rise to the $(l'_1, l'_2) = (1,0)$. For example, there is a state given by the combination of Ishibashi states $|0\rangle\rangle_A + |1\rangle\rangle_A$ which cannot be written as an integer sum of the four boundary states we have constructed. This state gives the annulus amplitude for $(l'_1, l'_2) = (1,0)$ but is ruled out by its incompatibility with eqn. (2.29).

Finally, In order to illustrate the increase in the degree of complexity, we consider the simplest non-toroidal model: the $1^6$ Gepner model. This corresponds to one of the orbifold points in K3 moduli space. The notation for the $k=1$ characters are as in the $1^3$ model.

| Label | Orbit | Multiplicity |
|---|---|---|
| $NS_0$ | $A^6 + B^6 + B_c^6$ | 1 |
| $NS_1$ | $A^3B^3 + B^3B_c^3 + B_c^3A^3$ | 20 |
| $NS_2$ | $3A^2B^2B_c^2$ | 30 |
| $NS_3$ | $A^4BB_c + AB^4B_c + ABB_c^4$ | 30 |

$NS_0$ corresponds to the graviton orbit, $NS_1$ is a massless orbit and $NS_2$, $NS_3$ are massive orbits. In the above table, by multiplicity we mean the number of distinct orbits which occur if we keep track of the minimal model labels.

The S-matrix is calculated from the S-matrix of the minimal model to be

$$S = \frac{1}{9}\begin{pmatrix} 1 & 20 & 30 & 30 \\ 1 & -7 & 3 & 3 \\ 1 & 2 & 3 & -6 \\ 1 & 2 & -6 & 3 \end{pmatrix}$$

$D_0 = 1$, $D_1 = 20$, $D_2 = 30$ and $D_3 = 30$. The resolved S-matrix is expected to be an $81 \times 81$ matrix which increases the complexity of the operation. However, in this example, if one keeps track of the minimal model labels, one should in principle be able to directly compute the resolved S-matrix. This is because we find that the multiplicity is equal to the $D_i$ associated with the orbit. This is not generically true. This model is presumably tractable if one uses a computer program to automate the process.

As a natural extension of the work presented in this chapter, it would be of interest to investigate D-branes from the point of view of Witten's gauged linear sigma model (GLSM). Techniques used in this chapter can be used to study this case, with the GLSM description of the open string CFT. It would be particularly interesting to see whether the LG-CY correspondence shown by Witten by making use of linear sigma models in the closed string example will go through with open string boundary conditions. In the next chapter, we begin a systemmatic study of this issue. We analyse boundary linear sigma models, and discuss how to define D-branes in the same, and study how the boundary conditions as presented here translate into that case.

# Chapter 3

# D-branes on Curved Manifolds : II

In this chapter, we consider the gauged linear sigma model (GLSM) approach to the construction of D-branes on curved manifolds. As we have remarked before, it was shown by Witten [15] that the Calabi-Yau (CY) nonlinear sigma model and the Landau-Ginzburg (LG) model are two different phases of this theory. The Kähler parameter $r$ for CY manifolds with $h^{1,1} = 1$ interpolates between the two phases. Large positive values of $r$ correspond to the geometric (CY) phase, while for large negative values of $r$, the theory goes to the non-geometric (LG) phase. Construction of D-branes in the GLSM, would help to ascertain the behaviour of D-branes as the theory passes from the geometric to the non-geometric phase. To achieve this, we will first set up a consistent set of boundary conditions starting from the $N = 2$ gauged linear sigma model with a boundary. This model has certain bosonic and fermionic fields arranged in chiral and vector multiplets. We impose boundary conditions such that the boundary terms that arise under ordinary and supersymmetric variation of the action cancel. We verify the consistency of these boundary conditions under supersymmetrc variation. These boundary conditions (Neumann or Dirichlet) will then define the D-brane. In the same way as in Chapter 2, we define A-type and B-type boundary conditions, that would correspond to D-branes wrapping on middle and even dimensional cycles of the CY. Let us begin by describing the notation and conventions to be used in this chapter.

## 3.1 Notation and conventions

We will begin with a description of the $N = 2$ gauged linear sigma model in two dimensions. The model can be obtained by dimensional reduction of $N = 1$ supersymmetry in four dimensions. We would consider a theory with a single $U(1)$ vector multiplet and $n$ charged chiral multiplets. In the Wess-Zumino gauge, the $U(1)$ vector multiplet consists of a vector field $v_\alpha$, $\alpha = (0, 1)$, a complex scalar $\sigma$, two complex chiral fermions $\lambda_\pm$, and a real scalar auxiliary field $D$. Each of the $n$ chiral multiplets consist of a complex scalar $\phi_i$, two complex chiral fermions $\psi_{\pm i}$, and a complex scalar auxiliary field $F_i$. They have charges $Q_i$ under the $U(1)$. The Wess-Zumino gauge breaks supersymmetry, but modified supersymmetries exist which close up to gauge transformations. These transformations are labelled by two Grassmann parameters $\epsilon_\pm$.

The Lagrangian of the GLSM is given as a sum of four terms, the kinetic, (super)potential, gauge and the Fayet- Iliapoulos and theta terms. In addition, we will also add a boundary term to this action, which we refer to as the contact term, $S_{contact}$. As we will argue later, this is necessary in order to impose a consistent set of boundary conditions in the infra-red limit of the GLSM. The action is written in components (in the Wess-Zumino gauge) as

$$\begin{aligned}
S_{kin} &= \int d^2y - D_\alpha\bar{\phi}_i D^\alpha\phi_i + \frac{i}{2}\left[\bar{\psi}_{-i}(D_0+D_1)\psi_{-i} - \left((D_0+D_1)\bar{\psi}_{-i}\right)\psi_{-i}\right] \\
&\quad + \frac{i}{2}\left[\bar{\psi}_{+i}(D_0-D_1)\psi_{+i} - \left((D_0-D_1)\bar{\psi}_{+i}\right)\psi_{+i}\right] \\
&\quad + |F|^2 - 2\bar{\sigma}\sigma Q_i^2\bar{\phi}_i\phi_i - \sqrt{2}Q_i(\bar{\sigma}\bar{\psi}_{+i}\psi_{-i} + \sigma\bar{\psi}_{-i}\psi_{+i}) + DQ_i\bar{\phi}_i\phi_i \\
&\quad - i\sqrt{2}Q_i\bar{\phi}_i(\psi_{-i}\lambda_+ - \psi_{+i}\lambda_-) - i\sqrt{2}Q_i\phi_i(\bar{\lambda}_-\bar{\psi}_{+i} - \bar{\lambda}_+\bar{\psi}_{-i}) \\
S_{gauge} &= \frac{1}{e^2}\int d^2y\frac{1}{2}v_{01}^2 + \frac{1}{2}D^2 + \frac{i}{2}\left[\bar{\lambda}_+(\partial_0-\partial_1)\lambda_+ - \left((\partial_0-\partial_1)\bar{\lambda}_+\right)\lambda_+\right] \\
&\quad + \frac{i}{2}\left[\bar{\lambda}_-(\partial_0+\partial_1)\lambda_- - \left((\partial_0+\partial_1)\bar{\lambda}_-\right)\lambda_-\right] - \partial_\alpha\sigma\partial^\alpha\bar{\sigma} \\
S_W &= -\int d^2y\left(F_i\frac{\partial W}{\partial\phi_i} + \frac{\partial^2 W}{\partial\phi_i^2}\psi_{-i}\psi_{+i} + \bar{F}_i\frac{\partial\bar{W}}{\partial\bar{\phi}_i} - \frac{\partial^2\bar{W}}{\partial^2\bar{\phi}_i}\bar{\psi}_{-i}\bar{\psi}_{+i}\right) \\
S_{r,\theta} &= -r\int d^2yD + \frac{\theta}{2\pi}\int d^2y v_{01} \\
S_{contact} &= \int dy\frac{\theta}{4\pi\sum_j Q_j^2|\phi_j|^2}\sum_i Q_i\left(\bar{\psi}_{+i}\psi_{+i} + \bar{\psi}_{-i}\psi_{-i}\right) \qquad (3.1)
\end{aligned}$$

Here $i$ labels the coordinates on a complex manifold $C^k$. In writing down the action, we have symmetrized the fermion kinetic terms as appropriate for a theory with a boundary. As we have remarked earlier, we have used a single $U(1)$ gauge field in writing down the action. This is appropriate for CY manifolds such as the quintic which have $h^{1,1} = 1$ restricts us to such a choice.

This model is invariant under the supersymmetry transformations that can be deduced from [61], and were listed in [15], which we include here for completeness. Our convention follows that of [15]. The dimensionally reduced formulae for the vector multiplet are

$$
\begin{aligned}
\delta v_0 &= i\left(\bar{\epsilon}_+\lambda_+ + \bar{\epsilon}_-\lambda_- + \epsilon_+\bar{\lambda}_+ + \epsilon_-\bar{\lambda}_-\right),\\
\delta v_1 &= i\left(\bar{\epsilon}_+\lambda_+ - \bar{\epsilon}_-\lambda_- + \epsilon_+\bar{\lambda}_+ - \epsilon_-\bar{\lambda}_-\right),\\
\delta\sigma &= -i\sqrt{2}\bar{\epsilon}_+\lambda_- - i\sqrt{2}\epsilon_-\bar{\lambda}_+,\\
\delta\bar{\sigma} &= -i\sqrt{2}\epsilon_+\bar{\lambda}_- - i\sqrt{2}\bar{\epsilon}_-\lambda_+,\\
\delta D &= -\bar{\epsilon}_+(\partial_0 - \partial_1)\lambda_+ - \bar{\epsilon}_-(\partial_0 + \partial_1)\lambda_-\\
&+ \epsilon_+(\partial_0 - \partial_1)\bar{\lambda}_+ + \epsilon_-(\partial_0 + \partial_1)\bar{\lambda}_-,\\
\delta\lambda_+ &= i\epsilon_+ D + \sqrt{2}(\partial_0 + \partial_1)\bar{\sigma}\epsilon_- - v_{01}\epsilon_+,\\
\delta\lambda_- &= i\epsilon_- D + \sqrt{2}(\partial_0 - \partial_1)\sigma\epsilon_+ + v_{01}\epsilon_-,\\
\delta\bar{\lambda}_+ &= -i\bar{\epsilon}_+ D + \sqrt{2}(\partial_0 + \partial_1)\sigma\bar{\epsilon}_- - v_{01}\bar{\epsilon}_+,\\
\delta\bar{\lambda}_- &= -i\bar{\epsilon}_- D + \sqrt{2}(\partial_0 - \partial_1)\bar{\sigma}\bar{\epsilon}_+ + v_{01}\bar{\epsilon}_-
\end{aligned} \tag{3.2}
$$

The transformation rules for the gauge multiplet is given by

$$
\begin{aligned}
\delta\phi_i &= \sqrt{2}(\epsilon_+\psi_- - \epsilon_-\psi_+),\\
\delta\psi_+ &= i\sqrt{2}(D_0 + D_1)\phi_i\bar{\epsilon}_- + \sqrt{2}\epsilon_+ F_i - 2Q_i\phi_i\bar{\sigma}\bar{\epsilon}_+,\\
\delta\psi_- &= -i\sqrt{2}(D_0 - D_1)\phi_i\bar{\epsilon}_+ + \sqrt{2}\epsilon_- F + 2Q_i\phi_i\sigma\bar{\epsilon}_-,\\
\delta F &= -i\sqrt{2}\bar{\epsilon}_+(D_0 - D_1)\psi_+ - i\sqrt{2}\bar{\epsilon}_-(D_0 + D_1)\psi_-\\
&+ 2Q_i(\bar{\epsilon}_+\bar{\sigma}\psi_- + \bar{\epsilon}_-\sigma\psi_+) + 2iQ_i\phi_i(\bar{\epsilon}_-\bar{\lambda}_+ - \bar{\epsilon}_+\bar{\lambda}_-)
\end{aligned} \tag{3.3}
$$

## 3.2 Variation of the action and surface terms

When the action (3.1) is varied under ordinary, gauge and supersymmetric variations, in addition to giving the equations of motion of the various fields, it produces non-trivial boundary terms. The requirement that these boundary terms vanish gives rise to boundary conditions that relate the different fields and the supersymmetry parameters on the boundary. In order to determine a consistent set of boundary conditions, our strategy is as follows. We start with the A-type or B-type boundary conditions on the boson $\phi_i$. Demanding that this is consistent with supersymmetry then leads to various boundary conditions on other fields of the theory, and finally these will be used to check whether the boundary terms in the variation of the action cancel. These boundary conditions then define a wrapped D-brane in the GLSM.

We now list the boundary terms arising out of the variation of the action in eq. (3.1), under ordinary and supersymmetric variations. Note that the variation of any of the terms in the action under gauge transformations of the $U(1)$ gauge group does not give rise to any boundary term.

1. Boundary terms arising due to ordinary variation are given by

$$\begin{aligned}
\delta S^{ord}_{kin} &= \int dy - \left[(\partial_1\phi_i + iQ_i v_1\phi_i)\delta\bar{\phi}_i + (\partial_1\bar{\phi}_i - iQ_i v_1\bar{\phi}_i)\delta\phi_i\right] \\
&\quad + \frac{i}{2}\left[(\bar{\psi}_-\delta\psi - \psi_+\delta\bar{\psi}_+) - (\bar{\psi}_+\delta\psi_+ - \psi_-\delta\bar{\psi}_-)\right], \\
\delta S^{ord}_{gauge} &= \frac{1}{e^2}\int dy - v_{01}\delta v_0 - \left[(\partial_1\bar{\sigma})\delta\sigma + (\partial_1\sigma)\delta\bar{\sigma}\right] \\
&\quad + \frac{i}{2}\left[(\bar{\lambda}_-\delta\lambda_- - \lambda_+\delta\bar{\lambda}_+) - (\bar{\lambda}_+\delta\lambda_+ - \lambda_-\delta\bar{\lambda}_-)\right] \\
\delta S^{ord}_{r,\theta} &= -\frac{\theta}{2\pi}\int dy\ \delta v_0 \\
\delta S_{contact} &= \delta_{ord}\sum_i \frac{\theta}{4\pi\sum_j Q_j^2|\phi_j|^2}\delta\left[\frac{Q_i}{4\pi r}\left(\bar{\psi}_{+i}\psi_{+i} + \bar{\psi}_{-i}\psi_{-i}\right)\right] \qquad (3.4)
\end{aligned}$$

2. Boundary terms arising due to supervariation are

$$\begin{aligned}
\delta S^{susy}_{kin} &= \int dy \frac{1}{\sqrt{2}}\left[(\partial_0\phi_i)(\bar{\epsilon}_+\bar{\psi}_- + \bar{\epsilon}_-\bar{\psi}_+) - (\partial_0\bar{\phi}_i)(\epsilon_+\psi_- + \epsilon_-\psi_+)\right] \\
&+ \frac{1}{\sqrt{2}}\left[(\partial_1\phi_i)(\bar{\epsilon}_+\bar{\psi}_- - \bar{\epsilon}_-\bar{\psi}_+) - (\partial_1\bar{\phi}_i)(\epsilon_+\psi_- - \epsilon_-\psi_+)\right] \\
&+ \frac{iQ_i}{\sqrt{2}}\left[(v_0\phi_i)(\bar{\epsilon}_+\bar{\psi}_- + \bar{\epsilon}_-\bar{\psi}_+) + (v_0\bar{\phi}_i)(\epsilon_+\psi_- + \epsilon_-\psi_+)\right] \\
&+ \frac{iQ_i}{\sqrt{2}}\left[(v_1\phi_i)(\bar{\epsilon}_+\bar{\psi}_- - \bar{\epsilon}_-\bar{\psi}_+) + (v_1\bar{\phi}_i)(\epsilon_+\psi_- - \epsilon_-\psi_+)\right] \\
&+ iQ_i\left[\bar{\phi}_i\sigma\epsilon_+\psi_+ + \phi_i\bar{\sigma}\bar{\epsilon}_+\bar{\psi}_+ + \bar{\phi}_i\bar{\sigma}\epsilon_-\psi_- + \phi_i\sigma\bar{\epsilon}_-\bar{\psi}_-\right] \\
&+ \frac{i}{\sqrt{2}}\left[(\bar{\epsilon}_+\psi_+ - \bar{\epsilon}_-\psi_-)\bar{F}_i \;+\; (\epsilon_+\bar{\psi}_+ - \epsilon_-\bar{\psi}_-)F_i\right] \\
\delta S^{susy}_{gauge} &= \frac{1}{e^2}\int dy\, \frac{i}{\sqrt{2}}\left[(\partial_0\sigma)(\epsilon_+\bar{\lambda}_- - \bar{\epsilon}_-\lambda_+) - (\partial_0\bar{\sigma})(\epsilon_-\bar{\lambda}_+ - \bar{\epsilon}_+\lambda_-)\right] \\
&+ \frac{i}{\sqrt{2}}\left[(\partial_1\sigma)(\epsilon_+\bar{\lambda}_- + \bar{\epsilon}_-\lambda_+) + (\partial_1\bar{\sigma})(\epsilon_-\bar{\lambda}_+ + \bar{\epsilon}_+\lambda_-)\right] \\
&- \frac{i}{2}v_{01}\left[\epsilon_+\bar{\lambda}_+ + \epsilon_-\bar{\lambda}_- + \bar{\epsilon}_+\lambda_+ + \bar{\epsilon}_-\lambda_-\right] \\
&+ \frac{D}{2}\left[\epsilon_+\bar{\lambda}_+ - \epsilon_-\bar{\lambda}_- + \bar{\epsilon}_-\lambda_- - \bar{\epsilon}_+\lambda_+\right] \\
\delta S^{susy}_{W} &= i\sqrt{2}\int dy\left[\frac{\partial W}{\partial\phi_i}(\bar{\epsilon}_-\psi_- - \bar{\epsilon}_+\psi_+) + \frac{\partial\bar{W}}{\partial\bar{\phi}_i}(\epsilon_-\bar{\psi}_- - \epsilon_+\bar{\psi}_+)\right] \\
\delta S^{susy}_{r,\theta} &= \frac{-i\theta}{2\pi}\int dy\left[\bar{\epsilon}_+\lambda_+ + \epsilon_+\bar{\lambda}_+ + \bar{\epsilon}_-\lambda_- + \epsilon_-\bar{\lambda}_-\right] \qquad (3.5)
\end{aligned}$$

Finally, we have boundary terms arising out of the variation of the contact term,

$$\delta_{susy}S_{contact} = \delta\frac{\theta}{4\pi\sum_j Q_j^2|\phi_j|^2}\sum_i\left[\frac{Q_i}{4\pi r}\left(\bar{\psi}_{+i}\psi_{+i} + \bar{\psi}_{-i}\psi_{-i}\right)\right] \qquad (3.6)$$

In the above, we have not written down explicitly the expressions for $\delta_{ord}S_{contact}$ and $\delta_{susy}S_{contact}$, as for most of our purposes, this will be cancelled by the $\theta$ term in the original action. The boundary terms given above can be used to analyse the boundary conditions that cause (3.4) and (3.5) to vanish consistently.

## 3.3 GLSM in the infra-red limit

In the GLSM, the presence of the $\theta$ term in the action making it difficult to impose a consistent set of boundary conditions. It seems natural, therefore, to consider first the action in the limit of the gauge coupling $e \to \infty$. The GLSM, in this limit, reduces to a non-linear sigma model (NLSM), with the fields in the vector multiplet becoming Lagrange multipliers providing constraints. Let us begin by describing these constraints.

In this limit, the fields of the vector multiplet, $\sigma, \lambda_\pm, v_\alpha, D$ become nonpropagating. The D-term constraint is

$$\sum_i Q_i|\phi^i|^2 - r = 0 \quad . \tag{3.7}$$

$\lambda$ becomes a fermionic Lagrange multiplier in the path integral and enforces the constraint

$$\sum_i Q_i\bar{\phi}_i\psi_{i\pm} = 0 \tag{3.8}$$

which forms the coefficients of the SUSY parameters in the Grassmann variation of the D-term constraint. Further variation of eq. (3.8) under supersymmetry gives us several other constraints. Note that $\sigma$ enters quadratically in the action, and its equation of motion is:

$$\sigma = -\frac{\sum_i Q_i\bar{\psi}^i_+\psi^i_-}{\sqrt{2}\sum_i(Q_i)^2|\phi^i|^2} \tag{3.9}$$

The gauge fields, in this limit provide the Gauss law constraints,

$$\begin{aligned} v_0 &= \frac{1}{\sum_j 2Q_j^2|\phi_j|^2}\sum_i \left[iQ_i\left(\bar{\phi}_i\partial_0\phi_i - \phi_i\partial_0\bar{\phi}_i\right) + Q_i\left(\bar{\psi}_{+i}\psi_{+i} + \bar{\psi}_{-i}\psi_{-i}\right)\right] \\ v_1 &= \frac{1}{\sum_j 2Q_j^2|\phi_j|^2}\sum_i \left[iQ_i\left(\bar{\phi}_i\partial_1\phi_i - \phi_i\partial_1\bar{\phi}_i\right) + Q_i\left(\bar{\psi}_{+i}\psi_{+i} - \bar{\psi}_{-i}\psi_{-i}\right)\right] \end{aligned} \tag{3.10}$$

Also, in this limit, variation of the expression for $\sigma$ gives rise to the constraints,

$$\begin{aligned} &\sum_i Q_i^2\bar{\psi}_{+i}\phi_i = 0 \\ &\sum_i Q_i^2\psi_{-i}\bar{\phi}_i = 0 \\ &\sum_i Q_i^2\psi_{+i}\bar{\phi}_i = 0 \\ &\sum_i Q_i^2\bar{\psi}_{-i}\phi_i = 0 \end{aligned} \tag{3.11}$$

In trying to impose a consistent set of boundary conditions in the infra-red limit, we are faced with a problem because of the nature of the expression for $v_0$. From (3.10), we see that the solution for $v_0$ involves a fermion bilinear, and this in particular, makes it difficult to impose sensible boundary conditions on the fields of the theory. The $\theta$ term, is however, to be interpreted as the NS-NS B-field turned on over a 2-cycle, whose size is controlled by the Kahler parameter $r$. In order to obtain an explicit expression for the NS-NS B-field, we substitute the solutions in (3.10) in the $\theta$ term of the bulk action. The result is a B-field coupling

$$\int d^2x B_{\bar{i}j}(\phi)\left(\partial_0\bar{\phi}^i\partial_1\phi^j - \partial_1\bar{\phi}^i\partial_0\phi^j\right) \tag{3.12}$$

where

$$B_{\bar{i}j} = \frac{i\theta}{2\pi}\left[-\frac{Q_i\eta_{\bar{i}j}}{\sum_k (Q_k)^2|\phi^k|^2} + \frac{\left(Q_i(Q_j)^2 + Q_j(Q_i)^2\right)\bar{\phi}^j\phi^i\eta_{i\bar{i}}\eta_{j\bar{j}}}{\left(\sum_k (Q_k)^2|\phi^k|^2\right)}\right], \tag{3.13}$$

and a fermion bilinear term which is a total derivative:

$$\frac{1}{\sum_k (Q_k)^2|\phi^k|^2}\sum_i \int d^2x\left[(\partial_0 - \partial_1)\bar{\psi}_{+i}\psi_{+i} + -(\partial_0 + \partial_1)\bar{\psi}_{-i}\psi_{-i}\right]$$

Here

$$\eta_{i\bar{i}}\phi^i\bar{\phi}^i = \sum_i |\phi^i|^2 \tag{3.14}$$

is the flat metric in complex coordinates.

The fermion bilinear is cancelled by the contact term of Eq. (3.1). To see that this term has be be cancelled by the introduction of a new term, note that in the ordinary supersymmetric nonlinear sigma model, fermion couplings to the B-field appear only through couplings to the three-form, $H = dB$. In fact if we look at $H$ in the space $\mathbf{C}^N$ it will not vanish, but upon restricting it to the symplectic quotient, it will. The simplest way to see this is as follows. Since $B$ appears with holomorphic and anti-holomorphic indices, the components of $H$ will have the structure $H_{ij\bar{k}}$ and $H_{i\bar{j}\bar{k}}$. The fermion bilinear terms will appear in the action as:

$$\int d^2x\bar{\psi}^j\psi^i\left(\partial\phi^k H_{\bar{j}ik} + \partial\bar{\phi}^k H_{\bar{j}i\bar{k}}\right) \tag{3.15}$$

Given the indicial structure of $B$, $H$ can be written as:

$$H_{ki\bar{j}} = \partial_{[k} B_{i]\bar{j}} \tag{3.16}$$

Inserting this in Eq. (3.15) gives us zero after we impose Eqs. (3.7) and (3.8). Hence, we should not expect any fermion bilinear terms in the action, and have explicitly cancelled the one that arises, by introducing a contact term. Such a term arises in ref. [62] as well. There one finds that in flat space one needs to add a fermion bilinear term proportional to $B$ in order to maintain the two-form gauge invariance and spacetime supersymmetry.

## 3.4 Boundary conditions in the infra-red limit

With the addition of the contact term, let us analyse the boundary conditions for the GLSM in the infra-red limit. From our analysis in the last chapter, we would like to describe consistent boundary conditions on the fields, that describe D-branes wrapping around middle or even dimensional cycles of Calabi-Yau. Let us first consider the A-type boundary conditions.

### A-type boundary conditions

As in the last chapter, for the A-type boundary condition, we impose

$$\epsilon_+ = \bar{\epsilon}_- \tag{3.17}$$

These impose different boundary conditions on the real and the imaginary parts of $\phi$ and are given by

$$\begin{aligned} \phi_i \; - \; \bar{\phi}_i &= 0 \\ \partial_1(\phi_i \; + \; \bar{\phi}_i) &= 0 \end{aligned} \tag{3.18}$$

Supervarying the first of the two equations gives the fermion boundary condition,

$$\psi_{+i} = \bar{\psi}_{-i} \tag{3.19}$$

where we have set Further variation of the fermion boundary condition in (3.19) gives

$$F_i + \bar{F}_i = 0 \tag{3.20}$$

Under the above boundary conditions, it is easy to see from the equation of motion of $\sigma$ in (3.9), that

$$\sigma = 0 \tag{3.21}$$

and the supersymmetric variation of the above equation gives

$$\lambda_+ + \bar{\lambda}_- = 0 \tag{3.22}$$

Note that for this boundary condition, from the expression for $v_0$ in (3.10), it is clear that $v_0 = 0$. This requires $\delta_{susy} v_0 = 0$, which is indeed consistent with (3.22)

**B-type boundary conditions**

**A. Pure Dirichlet boundary conditions**

First let us consider Dirichlet boundary conditions on all the bosonic coordinates. This would correspond to a D0 brane in Calabi-Yau. Let us first consider the boundary conditions for the chiral multiplet. For each direction labelled by $i$, we require, in this case that the bosonic variation vanishes, i.e

$$\delta\phi_i = \delta\bar{\phi}_i = 0 \tag{3.23}$$

On supersymmetric variation, this boundary condition leads to a condition on the fermions, namely,

$$\psi_{+i} = \psi_{-i} \tag{3.24}$$

where, as appropriate for B-type boundary conditions, we have imposed

$$\epsilon_+ = \epsilon_-$$

Supersymmetric variation of (3.24) then leads to the condition

$$D_0\phi_i + \frac{i}{\sqrt{2}} Q_i \left(\sigma + \bar{\sigma}\right) \phi_i = 0 \tag{3.25}$$

This actually implies the following boundary conditions,

$$\begin{aligned} \partial_0\phi_i &= 0 \\ v_0 &+ \frac{i}{\sqrt{2}}(\sigma + \bar{\sigma}) = 0 \end{aligned} \tag{3.26}$$

as can be seen from the expressions for $v_0$ and $\sigma, \bar{\sigma}$ at the boundary. In order to determine the allowed boundary conditions on the other fields of the gauge multiplet, first note that in the infra-red limit, the contact term cancels the boundary term coming from the $\theta$ term. This is because the contribution of the $\theta$ term to the variation of the action is given by $\frac{i\theta}{2\pi}\int dy\delta v_0$, and on substituting the value of $v_0$ from the Gauss law constraint given by eq. (3.10), and using the $\phi_i$ boundary condition, this variation exactly cancels that of the fermionic contact term.

Under supersymmetric variation of the Gauss law constraint, comparing coefficients of $\epsilon_\pm$ and $\bar{\epsilon}_\pm$ in the variation of (3.10) leads to the following expressions for $\lambda_\pm$

$$\begin{aligned}\lambda_+ &= \sum_i \frac{Q_i}{2r}\left[\sqrt{2}\left(D_0 - D_1\right)\phi_i\bar{\psi}_{-i} - i\sqrt{2}\bar{F}_i\psi_{+i} - 2iQ_i\phi_i\bar{\sigma}\bar{\psi}_{+i}\right] \\ \lambda_- &= \sum_i \frac{Q_i}{2r}\left[\sqrt{2}\left(D_0 + D_1\right)\phi_i\bar{\psi}_{+i} - i\sqrt{2}\bar{F}_i\psi_{-i} + 2iQ_i\phi_i\sigma\bar{\psi}_{-i}\right]\end{aligned} \tag{3.27}$$

and the complex conjugates of these equations. Similar expressions can also be obtained from varying the expression for $\sigma$ in (3.9). Now, on imposing the boundary condition (3.25), and the constraints in eq. (3.11), we can see that the boundary condition on $\lambda_\pm$ is

$$\lambda_+ = \lambda_- \tag{3.28}$$

The boundary condition on $\sigma$ can be similarly determined by using its expression in terms of $\psi_\pm$ in (3.9). It is easy to see that in the purely Dirichlet case, with the boundary condition of (3.24), the boundary condition on $\sigma$ is given by

$$\sigma = \bar{\sigma} \tag{3.29}$$

It is easy to see that the boundary terms in the ordinary and supersymmetric variation of the action vanish on imposing the boundary conditions for the gauge multiplet.

### B. Pure Neumann boundary conditions

Let us now turn to the pure Neumann case. In this case, all the coordinates of the CY have Neumann boundary conditions, and thus should describe a six-brane wrapping the entire CY. The imposition of the Neumann boundary conditions is however difficult in the NLSM limit. From eq. (3.1), we can see that the contact term has the bosonic contribution $\sum_i Q_i^2|\phi_i|^2$ in the denominator, whose contribution to the surface terms in the variation of the action makes it difficult to impose sensible boundary conditions on the fields. A simplification occurs if we restrict ourselves to the large volume limit of the CY, where one bosonic field of $U(1)$ charge $n$ is set to zero, and the remaining $n$ bosonic fields have unit $U(1)$ charge. In this case, the contact term is

$$S_{contact} = \int dy \frac{\theta}{4\pi r} \sum_i Q_i \left(\bar{\psi}_{+i}\psi_{+i} + \bar{\psi}_{-i}\psi_{-i}\right)$$

However, even in this case, the variation of the bosonic part of the $\theta \int dy \delta v_0$ term gives rise to rotated boundary conditions on the bosons, with a rotation parameter proportional to $\theta$, even if we consider the case of a single Neumann direction. This clearly poses a problem, because in order to interpret these $\theta$ dependent rotated boundary conditions as an effect arising due to the NS-NS 2-form B-field, we need two Neumann directions defining a D-2 brane.

Rotated boundary conditions are also difficult to interpret for the mixed case, i.e., for the D2-brane and D4-brane. In these cases, the boundary conditions on the vector multiplet coming from the Dirichlet and Neumann directions are seen to be in conflict. While the Neumann directions imply a rotated boundary condition on $\sigma$ and $\lambda_\pm$, the Dirichlet directions impose no such rotated condition.

One possible resolution to this problem is to introduce a Wilson line term in the action that totally cancels the effect of the $\theta$ term. This would make it possible to impose a consistent set of boundary conditions in the pure Neumann and the mixed Dirichlet Neumann cases on the fields of the gauge multiplet. However, the justification of the introduction of a Wilson line term in the action is not apriori clear.

## 3.5 Boundary conditions at finite gauge coupling

We will now carry over the above set of boundary conditions to the case of finite $e^2$.

**A-type boundary conditions**

For A-type boundary conditions, continuing the results that we have presented for the infra-red limit to the regime of finite gauge coupling is simple. It can be seen that all the boundary terms arising in the variation of the action in (3.1) will cancel on imposing the following set of boundary conditions

- The supersymmetry parameters satisfy the condition

$$\epsilon_+ = \bar{\epsilon}_- \tag{3.30}$$

- Boundary conditions on the fields of the chiral multiplet

$$\begin{aligned} (\phi_i - A_i^j \bar{\phi}_i) &= 0 \\ \partial_1(\phi_i + A_i^j \bar{\phi}_j) &= 0 \\ \psi_{+i} &= \bar{\psi}_{-i} \\ \bar{F}_i + A_i^j F_j &= 0 \end{aligned} \tag{3.31}$$

- Boundary conditions on the fields of the vector multiplet

$$\begin{aligned} \sigma &= 0 \\ \lambda_+ + \bar{\lambda}_- &= 0 \end{aligned} \tag{3.32}$$

where we have introduced the matrix $A$ as in chapter 2, with the same conditions on the $A$, i.e $A$ is block diagonal and $AA^* = 1$ In effect, the contact term plays no role in the A-type case, since it is identically zero due to the boundary condition on the $\psi_\pm$. These boundary conditions thus smoothly carry over to the regime of finite coupling, and defines a D-brane wrapping around a middle dimensional cycle of CY.

**B-type boundary conditions**

For B-type boundary conditions, the extrapolation of the results of the infra-red limit to regimes of finite gauge coupling seems to be difficult. For the case of purely Dirichlet boundary conditions, however, one can see that the following set of boundary conditions, cancel all the boundary terms arising in the variation of the action.

- The supersymmetry parameters satisfy the condition

$$\epsilon_+ = \epsilon_- \tag{3.33}$$

- Boundary conditions on the fields of the chiral multiplet

$$\begin{aligned} \psi_{+i} &= \psi_{-i} \\ \partial_0 \phi_i &= 0 \end{aligned} \tag{3.34}$$

- Boundary conditions on the fields of the vector multiplet

$$\begin{aligned} v_0 &+ \frac{1}{\sqrt{2}}(\sigma + \bar{\sigma}) = 0 \\ \lambda_+ &= \lambda_- \\ \sigma &= \bar{\sigma} \\ \partial_1(\sigma &+ \bar{\sigma}) = 0 \\ v_{01} &= 0 \end{aligned} \tag{3.35}$$

The case of Neumann boundary conditions poses a problem similar to the one in the GLSM as in the NLSM limit. As we have mentioned earlier, the contact term gives rise to non-trivial bosonic contributions to the boundary terms in the variation of the action that makes it difficult to impose sensible boundary conditions on the vector multiplet, with one or more Neumann directions. As we have remarked, a possible resolution to this problem is to introduce a Wilson line term in the action that cancels the effect of the $\theta$ term, so that no contact terms are necessary. Such a term would however have non-trivial effects in the GLSM limit. The transition from the geometric (CY) to the non-geometric (LG) phase in the GLSM is controlled by the parameter $r$. A theory with a non-zero value of $\theta$ can be shown to have a smooth transition between the two phases, while for $\theta = 0$, the transition becomes singular at $r = 0$.

We have not yet been able to determine a fully consistent set of boundary conditions for the fields in the GLSM that describes one or more Neumann boundary conditions, and work is in progress in this direction.

# Chapter 4

# D-branes and stringy black holes : Hawking radiation of high energy scalars

In this chapter, we shall study the emission of scalar particles from a class of near-extremal five dimensional black holes and the corresponding D-brane configurations, at high energies.

D-branes have been used to provide a *microscopic* interpretation to the Bekenstein-Hawking entropy of certain stringy black holes. These black holes can be identified with elementary or solitonic string states and the degeneracy of the latter has been has been shown to match with the entropy of the black hole [65, 55, 66]. Hawking radiation of scalars has also been understood in terms of their D-brane description. The Planck distribution function in the Hawking spectrum was obtained in [67, 68] using the distribution functions of open string states residing on the D-branes. Thereafter, it was shown in [27] that the rate of D-branes decaying into low energy scalars perfectly matched the Hawking spectrum from the corresponding black hole.

We shall discuss how, for a certain class of five dimensional stringy black holes, the distribution functions and the black hole greybody factors are modified in the high energy tail of the Hawking spectrum in such way that the emission rates still match exactly. Further, we will extend the results to charged scalar emission and to the case of four dimensional black holes.

Let us first set the notation and conventions used. The black holes to be considered are solutions of the low energy effective action of type IIB string theory compactified on a five dimensional torus which we denote by $T^5$. Their D-brane description consists of $Q_5$ D-5-branes wrapped around $T^5$ and $Q_1$ D-1-branes wrapped around $S^1$ contained in the $T^5$ and a collection of open strings carrying some momentum along $S^1$. The situation is equivalent to a single 'long' D-1-brane wrapped $Q_1Q_5$ times around the $S^1$ [68, 69]. The left and right moving massless open string states on this long brane constitute two non-interacting one dimensional gases, of temperatures $T_L$ and $T_R$, approximated by canonical ensembles at low energies. A pair of oppositely moving states, each carrying energy $\omega/2$, can annihilate to form a closed string state, like the graviton in the internal dimensions, of energy $\omega$, which cannot reside on the D-brane and is emitted as a scalar particle. This is broadly the effective description of Hawking radiation of these black holes in the D-brane picture.

The Hawking radiation rate can be calculated from the Dirac-Born-Infeld action. To leading order, this rate is given by :

$$\Gamma_D = g_{\text{eff}}\, \omega\, \rho\left(\frac{\omega}{2T_L}\right)\, \rho\left(\frac{\omega}{2T_R}\right)\, \frac{d^4k}{(2\pi)^4} \tag{4.1}$$

where $\rho\left(\omega/2T_{L,R}\right) = 1/\left(\exp(\omega/2T_{L,R}) - 1\right)$ and $g_{\text{eff}}$ is related to the parameters of the corresponding black hole. $T_L$ and $T_R$ are the effective temperatures of the left and right moving canonical ensembles. $d^4k$ is the usual phase space factor. In the limit $T_L \gg T_R$, it was shown that [27],

$$\Gamma_D = A_H\, \rho\left(\frac{\omega}{T_H}\right)\, \frac{d^4k}{(2\pi)^4} \tag{4.2}$$

where $T_H$ is the Hawking temperature of the black hole [27]. On the other hand, the Hawking spectrum from the black hole is given by

$$\Gamma_H = \sigma_{\text{abs}}\, \rho\left(\frac{\omega}{T_H}\right)\, \frac{d^4k}{(2\pi)^4} \tag{4.3}$$

where, $\sigma_{\text{abs}}$ is the greybody factor for the black hole, which for low energy emissions, is just the area of the event horizon $A_H$. Substituting, $\sigma_{\text{abs}} = A_H$ in eq. (4.3), we find that the rates (4.2) and (4.3) match exactly.

In [72], the restriction $T_L \gg T_R$ was dropped, while still remaining in the near extremal region, and it was shown in general that for $T_L \sim T_R$,

$$\sigma_{\text{abs}} = g_{\text{eff}}\, \omega \frac{e^{\omega/T_H} - 1}{\left(e^{\omega/2T_L} - 1\right)\,\left(e^{\omega/2T_R} - 1\right)} \tag{4.4}$$

with the Hawking temperature given by

$$\frac{2}{T_H} = \frac{1}{T_L} + \frac{1}{T_R} \tag{4.5}$$

Once again, it is seen from (4.1) and (4.3), that the D-brane and the black hole decay rates match.

In the above analysis, it was strictly assumed that the energy of the emitted scalars was vanishingly small. The emission of high-energy scalars from D-branes and black holes was studied in [70] although confined to the $T_L \gg T_R$ regime. The energy $\omega$ was chosen such that $T_R\ , T_H \ll \omega \ll T_L$. In this regime, the right moving open strings were treated as a microcanonical ensemble and the corresponding distribution function was modified to

$$\rho\left(\frac{\omega}{2T_R}\right) \approx \exp\left[S_R(N_R' - m) - S_R(N_R')\right] = \exp(-\Delta S_R) \tag{4.6}$$

where $\Delta S_R$ is the change in the right moving entropy on removal of a boson at level $m$ with energy $\omega/2$. Here $N_R'$ and $N_L'$ are the left and right moving momenta on the long D-1-brane respectively (the actual momenta on the 1-brane is given in terms the quantum number $N_{L,R} = N_{L,R}'/Q_1Q_5$). Now, the black hole entropy is given by [27, 70]

$$S_{BH} = 2\pi(\sqrt{N_L'} + \sqrt{N_R'}) \tag{4.7}$$

In the limit $T_L \gg T_R$, since $N_L' \gg N_R'$, we get from (4.7), $\Delta S_{BH} \approx \Delta S_R$. Thus,

$$\rho\left(\frac{\omega}{2T_R}\right) \approx \exp\left[S_{BH}(M - \omega) - S_{BH}(M)\right] = \exp(-\Delta S_{BH}) \tag{4.8}$$

where $\Delta S_{BH}$ is the change in the entropy of the black hole of initial mass $M$ after it emits the Hawking particle of energy $\omega$.

On the black hole side, this change in the distribution function has been attributed to the back reaction effects which become important at high energies. In [76, 70] this was studied by modelling the outgoing particle as a spherical shell and quantising it. In the WKB approximation, the Hawking factor $\rho(\omega/T_H)$ turned out to be precisely the right hand side of Eq.(4.8). The left distribution function and the greybody factor remains unchanged and thus, once again the D-brane and black hole emission rates are found to match.

In the rest of this chapter, we will calculate the greybody factor in a more complete analysis that relaxes the condition $T_L \gg T_R$ and we work in the range $\omega \gg T_{L,R,H}$. The gas of open strings is treated as a microcanonical ensemble in both the left and the right sectors. We show that the greybody factor gets significantly modified in the high energy tail of the spectrum. With these conditions, we find that the emission rates match in this extended range also. Finally, we generalize these results to charged scalar emission from five-dimensional black holes and also to neutral and charged scalar emission from 4-dimensional black holes.

## 4.1 D-brane Emission Spectrum at High Energies

Consider a one-dimensional gas of massless open strings in a box of length $L$. The total momentum $P$ of the gas is given in terms of the quantum number $N'$ by $P = 2\pi N'/L$ and the energy of a colliding string by

$$\omega/2 = 2\pi m/L \tag{4.9}$$

For low energy excitations, such that $m \ll \sqrt{N'}$, the gas is well approximated by a canonical ensemble, and the distribution function is of the Bose-Einstein form. However, for higher energies, when

$$\sqrt{N'} \ll m \ll N' \tag{4.10}$$

which amounts to the excitation energy being much greater than the corresponding temperature, the canonical description is inadequate, and the gas should be described by a microcanonical ensemble. Since we are interested in the regime where

$T_L \sim T_R$ and $\omega$ exceeds these temperatures, the microcanonical distribution functions should be invoked in the right as well as the left sectors. This is given by [70]:

$$\rho\left(\frac{\omega}{2T_{L,R}}\right) = \exp\left[-2\pi\left(\sqrt{N'_{L,R}} - \sqrt{N'_{L,R} - m}\right)\right] \tag{4.11}$$

From Eqs.(4.7) and (4.9), we write

$$\begin{aligned} \rho\left(\frac{\omega}{2T_L}\right)\rho\left(\frac{\omega}{2T_R}\right) &= \exp\left[-2\pi\left(\sqrt{N'_L} - \sqrt{N'_L - L(\frac{\omega}{4\pi})}\right)\right] \\ &\quad \times \exp\left[-2\pi\left(\sqrt{N'_R} - \sqrt{N'_R - L(\frac{\omega}{4\pi})}\right)\right] \\ &= \exp(-\Delta S_L - \Delta S_R) = \exp(-\Delta S_{BH}) \end{aligned} \tag{4.12}$$

Thus, Eq.(4.1) can be written as

$$\Gamma_D = g_{\text{eff}}\, \omega\, \exp(-\Delta S_{BH})\, \frac{d^4k}{(2\pi)^4} \tag{4.13}$$

In the black hole side, the Hawking factor becomes, on inclusion of back reaction [70],

$$\rho\left(\frac{\omega}{T_H}\right) \approx \exp\left[S_{BH}(M-\omega) - S_{BH}(M)\right] = \exp(-\Delta S_{BH}) \tag{4.14}$$

which implies

$$\Gamma_H = \sigma_{\text{abs}}\, \exp(-\Delta S_{BH})\, \frac{d^4k}{(2\pi)^4} \tag{4.15}$$

In the next section, we will calculate $\sigma_{\text{abs}}$ and compare the D-brane and black hole emission rates. Note that, in [70] the left sector did not contribute to $\Delta S_{BH}$ and the relation between the distribution functions was

$$\rho\left(\frac{\omega}{2T_R}\right) = \rho\left(\frac{\omega}{T_H}\right) .$$

Here, on the other hand, both the sectors become equally important and contribute to the Hawking factor.

## 4.2 Black Hole Greybody Factors at High Energies

In this section, we calculate the greybody factors for the 5-dimensional black hole under consideration for quanta with high energies. We follow the methods of [71, 72, 73]. This is appropriate in the energy regime where back reaction becomes important. We solve the Klein-Gordon equation in the background of the metric given by [74]:

$$\begin{aligned} ds^2 &= \frac{1}{(f_1 f_2 f_3)^{2/3}}\left[-dt^2\left(1-\frac{r_0^2}{r^2}\right)\right] \\ &+ (f_1 f_2 f_3)^{1/3}\left[\left(1-\frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right] \end{aligned} \tag{4.16}$$

where

$$f = \left(1 + \frac{r_n^2}{r^2}\right)\left(1 + \frac{r_1^2}{r^2}\right)\left(1 + \frac{r_5^2}{r^2}\right)$$

$$\text{and} \quad h = 1 - \frac{r_0^2}{r^2} \tag{4.17}$$

The parameters $r_1$, $r_5$ and $r_n$ can be expressed in terms of the two charges $Q_1$, $Q_5$ and the momentum $n$ along the D-1-brane as follows:

$$r_1^2 = \frac{gQ_1}{V}, \quad r_5^2 = gQ_5, \quad r_0^2\frac{\sinh 2\sigma}{2} = \frac{g^2 n}{R^2 V}, \quad r_n^2 = r_0^2 \sinh^2\sigma\ ,$$

where $\sigma$ is a boost parameter. The radial part of the Klein-Gordon equation for a scalar field $\phi$, in the the $s$-wave approximation, and propagating in the background of the above metric is given by

$$\frac{h}{r^3}\frac{d}{dr}\left(hr^3\frac{dR}{dr}\right) + \omega^2 f R = 0 \tag{4.18}$$

In our calculations, we relax the low energy condition $\omega r_5, \omega r_1 \ll 1$, originally imposed in [72] and solve the above equation by treating the new $\omega$ dependent terms that enter due to this relaxation, as a perturbation over the terms originally present. The following analysis is valid as long as $\omega r_1, \omega r_5 < 1$, although it need not be vanishingly small. Towards the end of this section, we show that this is the relevant range for comparing with the D-brane results. Equation (4.18) is solved by dividing space into two regions, the near and far zones, and then matching the solutions at some intermediate region. We assume the following relation between the various parameters:

$$r_0,\ r_n \ll r_m < r_1, r_5 \tag{4.19}$$

where the near and far solutions are matched at $r = r_m$. In the far region, equation (4.18) becomes

$$\frac{d^2\psi}{dr^2} + \left[\omega^2 + \frac{-3/4 + \omega^2(r_1^2 + r_5^2)}{r^2}\right]\psi = 0 \tag{4.20}$$

where we have substituted $R = \psi r^{-3/2}$ and the restrictions given in (4.19). The term $\omega^2(r_1^2 + r_5^2)$ was absent in [72] because of the low energy condition. Defining $\rho = \omega r$, we obtain,

$$\frac{d^2\psi}{d\rho^2} - \left[-1 + \frac{3/4 - \omega^2(r_1^2 + r_5^2)}{\rho^2}\right]\psi = 0 \tag{4.21}$$

which has the solution

$$\psi = \sqrt{\frac{\pi}{2}\rho}\ \left[\alpha J_{1-\epsilon}(\rho) + \beta N_{1-\epsilon}(\rho)\right] \tag{4.22}$$

where $\epsilon \equiv \omega^2(r_1^2+r_5^2)/2$. Now, in the matching region, we use the small $\rho$ expansion for the Bessel functions, and finally obtain the solution,

$$R = \sqrt{\frac{\pi}{2}}\omega^{3/2}\left[\frac{\alpha}{2}\frac{(\frac{\rho}{2})^{-\epsilon}}{\Gamma(2-\epsilon)} + \frac{\beta}{2}\left(\left(\frac{\rho}{2}\right)^{-\epsilon}\cot\pi(1-\epsilon) - \frac{\epsilon}{\sin\pi(1-\epsilon)}\left(\frac{\rho}{2}\right)^{-2+\epsilon}\right)\right] \tag{4.23}$$

On the other hand, the asymptotic expansions of the Bessel functions yield the solutions

$$J_{1-\epsilon}(\rho) = \sqrt{\frac{2}{\pi\rho}}\cos(\rho - \frac{3\pi}{4} + \frac{\pi\epsilon}{2}) \tag{4.24}$$

$$N_{1-\epsilon}(\rho) = \sqrt{\frac{2}{\pi\rho}}\sin(\rho - \frac{3\pi}{4} + \frac{\pi\epsilon}{2})\,,$$

which are used to compute the incoming flux at infinity, given by

$$\Phi_{\text{in}} = -\frac{\omega}{4}|\alpha|^2 \tag{4.25}$$

In this computation, we have dropped a $\beta$ dependent piece. From equation (4.23), it is clear that the term multiplying $\beta$ is large for small values of the perturbation parameter. This implies that $\beta/\alpha \ll 1$.

In the near zone, Eq.(4.18) can be written as

$$\frac{h}{r^3}\frac{d}{dr}\left(hr^3\frac{dR}{dr}\right) + \omega^2\left[\frac{(r_n r_1 r_5)^2}{r^6} + \frac{(r_1 r_5)^2}{r^4} + \frac{(r_1^2 + r_5^2)}{r^2}\right] R = 0 \tag{4.26}$$

Defining new variables $v$ and parameters $A, B$ as,

$$v = \frac{r_0^2}{r^2}; \quad A = \frac{\omega^2}{4}\left(\frac{r_1 r_5 r_n}{r_0}\right)^2; \quad B = \frac{\omega^2}{4}\left(\frac{r_1 r_5}{r_0}\right)^2 \tag{4.27}$$

equation (4.26) becomes

$$(1-v)\frac{d}{dv}\left((1-v)\frac{dR}{dv}\right) + \left[A + \frac{B}{v} + \frac{\epsilon}{2v^2}\right] R = 0 \tag{4.28}$$

Notice that close to the horizon, $v \to 1^-$. Thus, on writing $v = 1 - \delta$ and expanding the $1/v^2$ term in square brackets, we obtain the equation for the near region as

$$(1-v)\frac{d}{dv}\left((1-v)\frac{dR}{dv}\right) + \left[A + \frac{B \ + \epsilon/2}{v} + \frac{\epsilon\delta}{2}\right] R = 0 \tag{4.29}$$

Hereafter, we drop the $\epsilon\delta/2$ term, which is very small. In order to compute the flux of neutral scalars absorbed into the black hole, we need to know the near region solution very close to the horizon. In equation (4.28), if we make the substitution $y = -\ln(1-v)$, we obtain, in this region, a simple harmonic equation for $R$, namely

$$\frac{d^2R}{dy^2} + (A + B + \frac{\epsilon}{2})R = 0 \tag{4.30}$$

And the incoming solution is given by

$$R_{\text{in}} = K \exp(-i\sqrt{A + B + \frac{\epsilon}{2}}\ \ln(1-v)) \tag{4.31}$$

Substituting $z = (1-v)$, and writing an ansatz for the solution as $R = Kz^{-i(p+q)/2}R_1$, we obtain

$$z(1-z)\frac{d^2R_1}{dz^2} + (1-z)(1-ip-iq)\frac{dR_1}{dz} + pqR_1 = 0 \tag{4.32}$$

This is seen to be a hypergeometric equation in $R$, where we have defined $p$ and $q$ by the equations

$$(p+q)^2 = 4(A+B+\frac{\epsilon}{2}) \;\; ; \;\; pq = B \; + \; \frac{\epsilon}{2} \tag{4.33}$$

The above equation has the solution

$$R_1 = F(-ip, -iq, 1-ip-iq, z) \; , \tag{4.34}$$

where $F$ is the hypergeometric function, and hence the full solution for $R$ is given by

$$R = Kz^{-i(p+q)/2} F(-ip, -iq, 1-ip-iq, z) \tag{4.35}$$

Next, we express $p$ and $q$ in terms of the black hole parameters. Solving the equation (4.33) yields for $p$ and $q$

$$\begin{aligned} p &= \frac{\omega r_1 r_5}{2r_0}\mathrm{e}^{\sigma} + \frac{\omega r_0}{4}\frac{(r_1^2+r_5^2)}{r_1 r_5}\frac{1}{\cosh\sigma} \; , \\ q &= \frac{\omega r_1 r_5}{2r_0}\mathrm{e}^{-\sigma} + \frac{\omega r_0}{4}\frac{(r_1^2+r_5^2)}{r_1 r_5}\frac{1}{\cosh\sigma} \; . \end{aligned} \tag{4.36}$$

Substituting for $T_{L,R}$, namely

$$T_{L,R} = \frac{r_0}{2\pi r_1 r_5} \; e^{\pm\sigma} \tag{4.37}$$

and using $r_1 \sim r_5$, we get

$$p = \frac{\omega}{4\pi T_R} + \frac{\omega r_0}{2\cosh\sigma} \;\; \text{and} \;\; q = \frac{\omega}{4\pi T_L} + \frac{\omega r_0}{2\cosh\sigma} \tag{4.38}$$

In order to proceed to calculate the absorption cross section, let us first match the far and near zone solutions at $r = r_m$. Extrapolating the near solution given by equation (4.35) to the region of small $v$ (large $r$) yields,

$$R = K\left[\frac{\Gamma(1-ip-iq)}{\Gamma(1-ip)\Gamma(1-iq)} + v_m(a \; + \; b \; \ln v_m)\right] \tag{4.39}$$

where $a$ and $b$ are constants depending on $p$ and $q$. Next, we expand the right hand side of Eq.(4.23) in powers of $\epsilon$, and retaining the lowest order terms in $\epsilon$, we find the matching condition at $r = r_m$:

$$\sqrt{\frac{\pi}{2}}\omega^{3/2}\left[\frac{\alpha}{2}\left(1-\epsilon \; \ln(\rho_m/2)\right)\right] = K\left[E + v_m(a + b \; \ln v_m)\right] \tag{4.40}$$

where

$$E = \frac{\Gamma(1 - ip - iq)}{\Gamma(1 - ip)\Gamma(1 - iq)}$$

and we have imposed the condition $z \simeq 1$. The matching region is chosen such that $\omega r_m$ is slightly less than unity. Thus, the second term on the left hand side of Eq.(4.40) can be dropped, and we get the relation as in [72]

$$\sqrt{\frac{\pi}{2}}\, \omega^{3/2}\, \frac{\alpha}{2} = K\ E \tag{4.41}$$

Now, let us calculate the absorption cross-section [27]. The flux into the black hole, from Eq.(4.31) is given by

$$\Phi_{\text{abs}} = -r_0^2(p + q)|K|^2 \tag{4.42}$$

From Eqs.(4.25) and (4.42), we get

$$\sigma_{\text{abs}} = \frac{4\pi}{\omega^3}\frac{\Phi_{\text{abs}}}{\Phi_{\text{in}}} = \frac{2\pi^2 r_0^2}{\omega}(p + q)\frac{1}{|E|^2} \tag{4.43}$$

Using the identity

$$|\Gamma(1 - ix)|^2 = \frac{\pi x}{\sinh \pi x}\ ,$$

we get

$$\frac{1}{|E|^2} = \frac{2\pi pq}{p + q}\frac{\exp(2\pi(p + q))\ -\ 1}{(\exp(2\pi p)\ -\ 1)(\exp(2\pi q)\ -\ 1)} \tag{4.44}$$

Now, recalling the expressions for $p$ and $q$, in Eq. (4.36), we see that in the limit when $\omega/T_L, R \gg 1$, we can ignore the factors of unity in the numerator and denominator and finally we are left with the following expression for the absorption cross-section:

$$\sigma_{\text{abs}} = \sigma^0_{\text{abs}}\ +\ \sigma^1_{\text{abs}} \tag{4.45}$$

where, $\sigma^0_{\text{abs}} = \pi^3 r_1^2 r_5^2 \omega$ , and the correction term $\sigma^1_{\text{abs}} = 4\pi^3 \omega r_0^2 r_1 r_5 \cosh\sigma$. Thus, we see, following the relation between the various parameters that we have considered,

$$\frac{\sigma^1_{\text{abs}}}{\sigma^0_{\text{abs}}} \sim \left(\frac{r_0}{r_1}\right)^2 \ll 1\ .$$

Using the definition $g_{\text{eff}} = \pi^3 r_1^2 r_5^2$, we get,

$$\sigma_{\text{abs}} = g_{\text{eff}}\ \omega \tag{4.46}$$

Now, let us compare the expressions for the black hole and D-brane decay rates at the high energy regime that we are considering. Substituting (4.46) in (4.15), we see that the black hole decay rate becomes

$$\Gamma_H = g_{\text{eff}}\, \omega \, \exp\left(-\Delta S_{BH}\right) \frac{d^4k}{(2\pi)^4} \tag{4.47}$$

which is just the D-brane decay rate (4.13). It may be noted that this matching cannot be obtained by naively ignoring the unity factors in (4.1),(4.3) and (4.4). This is because of the fact that in the regime of high energy particle emission that we are interested in, the Planckian distribution of the Hawking particles is no longer valid and we have to instead resort to Eqs. (4.12) and (4.14). Hence our result (4.46) effects a subtle match between the black hole and D - brane decay rates at high energies. It can be shown that in the special case $T_L \gg T_R$ ($\sigma \to \infty$), the results of [70] are reproduced.

A word about the range of validity of the above result is in order. As stated earlier, microcanonical corrections become important when the condition (4.10) holds. Using (4.9) and substituting the expression for temperature [27], namely

$$T_{L,R} = \sqrt{\frac{8E_{L,R}}{L\pi f}} \tag{4.48}$$

in (4.10), we obtain,

$$\frac{\omega}{T_{L,R}} \gg 1 \tag{4.49}$$

which was the condition under which we had derived (4.46). Hence we see that taking microcanonical corrections into consideration naturally enforces the high energy condition (4.49). In terms of the black hole parameters, this can be written as

$$\omega r_5 \gg \frac{r_0}{r_1} \tag{4.50}$$

Also, our perturbative analysis is valid so long as $\omega r_5 < 1$, which is consistent with the condition $m \ll N'_{L,R}$ of Eq.(4.10). Hence, the range of $\omega$ for which our calculations are valid is

$$\frac{r_0}{r_1} \ll \omega r_5 < 1 \tag{4.51}$$

On the other hand, it is clear that for low energies (canonical distribution), $m \ll \sqrt{N'}$, implying $\omega r_5 \ll r_0/r_1$. Thus, it is sufficient to calculate the greybody factor for $\omega r_5 \ll 1$, as in [72]. However, in our case, it becomes important to look at $\sigma_{\text{abs}}$ for higher $\omega$, and (4.51) exhausts the range over which the D-brane distribution functions follow that of microcanonical ensembles.

## 4.3 Charged Emission Rates Including Back Reaction

The results of the previous sections can be extended to include charged scalar emission. The decay rate for low energy charged scalar emission from D-Branes, has been obtained in [75]. The emitted massless graviton field with polarization along the compact directions now have a net momentum along the compact $S^1$ direction on which the 1-brane is wrapped. The decay rate is given by,

$$\Gamma_D = g_{\text{eff}} \frac{(\omega^2 - e^2)}{\omega} \rho\left(\frac{\omega + e}{2T_L}\right) \rho\left(\frac{\omega - e}{2T_R}\right) \frac{d^4k}{(2\pi)^4} \tag{4.52}$$

Comparing with Eq.(4.1), we find that here the energies and the momenta of the left and right modes are shifted by a factor of $\pm e/2$ respectively. This ensures that there is a net momentum $e$ in the $S^1$ direction, while the energy of the outgoing particle is $\omega$. This net momenta along the compact direction gives rise to a Kaluza-Klein charge $e$ for the space-time scalar and there is also a mass such that $|e| = m$. When $T_L >> T_R$, and $\omega$ is low, the emission rate is,

$$\Gamma_D = \frac{A_H(\omega - e)}{\omega} \rho\left(\frac{\omega - e}{2T_R}\right) \frac{d^4k}{(2\pi)^4} \, . \tag{4.53}$$

For higher energies, however, the decay rate is modified. In the regime $T_L \sim T_R$, $(\omega \pm e)/2T_{L,R} \gg 1$, the density functions are best approximated as a microcanonical distribution in this regime. The expression for the left and right densities are same as that in eq (4.12), however with energies $(\omega + e)/2$ and $(\omega - e)/2$ of the left and right particles respectively. The product of the left and right density functions combine to give

$$\rho\left(\frac{\omega + e}{2T_L}\right) \rho\left(\frac{\omega - e}{2T_R}\right) = \exp(-\Delta S_{BH}) \tag{4.54}$$

where now $\Delta S_{BH}$ is given by $\Delta S_{BH} = S(M,Q) - S(M-\omega, Q-e)$. Here $M$ is the ADM mass of the black hole and $Q$ it's Kaluza-Klein charge, proportional to the momentum $N_L - N_R$. Clearly $\Delta S_{BH}$ is the change in entropy due to the emission of a particle with energy $\omega$ and charge $e$. Then Eq.(4.52) can be written as

$$\Gamma_D = g_{\text{eff}} \frac{(\omega^2 - e^2)}{\omega} \exp(-\Delta S_{BH}) \frac{d^4k}{(2\pi)^4} . \tag{4.55}$$

The microcanonical decay rate thus obtained can be reproduced exactly from field theory, following [70], using the techniques developed in [76]. Charged black holes emit charged particles at a rate given by,

$$\Gamma_H = \frac{\sqrt{\omega^2 - e^2}}{\omega} \sigma_{\text{abs}} \, \rho \left( \frac{\omega - e}{T_H} \right) \frac{d^4k}{(2\pi)^4} \tag{4.56}$$

The density function is evaluated by computing Bogoliubov coefficients. These relate the wave function at the horizon to the normal components of the wave function at $r \to \infty$. Due to the infinite boosts associated with the horizon, the wavefunction $\phi_h$ is well approximated by the WKB value

$$\phi_h = \exp(\imath S) \tag{4.57}$$

The action $S$ is calculated along the trajectory of the charged shell which approximates the outgoing charged scalar wave. The Bogoliubov coefficients are hence,

$$\alpha_{\omega\omega'} = \frac{1}{u(r,e)} \int_{-\infty}^{\infty} e^{\imath\omega t} e^{\imath S} \, dt \quad , \quad \beta_{\omega\omega'} = \frac{1}{v(r,-e)} \int_{-\infty}^{\infty} e^{-\imath\omega t} e^{\imath S} \, dt \tag{4.58}$$

where $u(r,e)$ and $v(r,-e)$ give the radial wavefunction of the positive energy, positively charged particle and negative energy and negatively charged particle respectively. Since $\omega$ and the action diverge near the horizon, the saddle point value of the integral dominates. The saddle point value is determined through the following equation,

$$\frac{\partial S}{\partial t} \pm \omega = 0 \tag{4.59}$$

By the Hamilton-Jacobi equation, the saddle point corresponds to $\mathcal{H}^{\mp e} = \mp\omega$, $\mathcal{H}$ being the Hamiltonian of the outgoing particle. Since, the trajectory of the charged shell is that of a null geodesic in the metric [76]

$$ds^2 = -[N_t(M + \mathcal{H}, Q + e)dt]^2 + [dr + N_r(M + \mathcal{H}, Q + e)dt]^2 \tag{4.60}$$

we have,

$$\dot{r} = N_t(M + \mathcal{H}, Q + e) - N_r(M + \mathcal{H}, Q + e) \tag{4.61}$$

Using this and the fact that $\partial S/\partial r = P$ (P is the canonical conjugate momentum), we find;

$$|\alpha_{\omega\omega'}| = \exp(-\text{Im} \int_{r_{+0}}^{r_f} P_+ dr) \quad |\beta_{\omega\omega'}| = \exp(-\text{Im} \int_{r_{-0}}^{r_f} P_- dr) \tag{4.62}$$

where $P_\pm$ correspond to the positive energy, positively charged particle and negative energy, negatively charged particle trajectories respectively. As in the case of [70], the positive energy trajectory gives a real value of the integration, while there is an imaginary contribution from the other. As $r_0 = R(M - \omega, Q - e) - \epsilon$,

$$\text{Im} \int_{r_o}^{r_f} P_- dr = -\pi \int_0^{-\omega} \frac{d\mathcal{H} - \Phi dQ}{\kappa(M + \mathcal{H}, Q - e)} = -\frac{1}{2} \int_{M,Q}^{M-\omega,Q-e} dS_{BH} \tag{4.63}$$

where we have used $\dot{r}\, dP = d\mathcal{H} - \Phi dQ$ near the horizon and $d\mathcal{H} = (\kappa/2\pi) dS_{BH} + \Phi dQ$, $\Phi$ being the electromagnetic scalar potential at the horizon. Thus

$$|\beta_{\omega\omega'}|^2 = \exp[S_{BH}(M - \omega, Q - e) - S_{BH}(M, Q)] = \exp(-\Delta S_{BH}), \quad |\alpha_{\omega\omega'}| = 1 \tag{4.64}$$

The density function, in the high energy approximation is [70],

$$\rho\left(\frac{\omega - e}{T_H}\right) \approx \frac{|\beta_{\omega\omega'}|^2}{|\alpha_{\omega\omega'}|^2} \tag{4.65}$$

Therefore,

$$\rho\left(\frac{\omega - e}{T_H}\right) = \exp(-\Delta S_{BH}) \tag{4.66}$$

and from equation (4.54), it is seen that this equals the value of $\rho(\omega + e/2T_L)\ \rho(\omega - e/2T_R)$ obtained from the D-brane picture.

The greybody factor, $\sigma_{\text{abs}}$ for high energies is determined using the same methods as in the neutral emission case. The scalar equation is

$$\left(1 + \frac{r_1^2}{r^2}\right)\left(1 + \frac{r_5^2}{r^2}\right)\left[\omega^2 - e^2 + (\omega \sinh\sigma - e\cosh\sigma)^2 \frac{r_0^2}{r^2}\right] R + \frac{h}{r^3}\frac{d}{dr}\left(hr^3 \frac{dR}{dr}\right) = 0 \tag{4.67}$$

By making the following transformations as in [72]

$$\omega'^2 = \omega^2 - e^2, \quad e^{\pm\sigma'} = e^{\pm\sigma}\left(\frac{\omega \mp e}{\omega'}\right), \quad r'_n = r_0 \sinh \sigma' \tag{4.68}$$

we obtain equation (4.18) with $\omega \to \omega'$ and $r_n \to r'_n$. To order $(r_0/r_1)^2$, we again obtain the $\sigma_{abs}$ as in[72] as

$$\sigma_{abs} = g_{\text{eff}}\, \omega' \frac{e^{\frac{\omega - e\Phi}{T_H}} - 1}{\left(e^{\frac{\omega+e}{2T_L}} - 1\right)\left(e^{\frac{\omega-e}{2T_R}} - 1\right)} \tag{4.69}$$

Here $(\omega+e)/T_L + (\omega-e)/T_R = (\omega - e\Phi)/T_H$ and $\Phi = \tanh\sigma$ is the value of the scalar potential at the horizon. Thus clearly as $\omega'/T_{L,R} > 1$, $\sigma_{abs} \to g_{\text{eff}}\, \omega'$. Inserting this value in Eq.(4.56), it is seen to match with Eq.(4.55).

## 4.4 Scalar Emission in Four Dimensions

In this section, we briefly comment on the extension of our results to include high energy emission of scalar particles in the more realistic case of four dimensions. In the five dimensional case that we have analysed, inclusion of the high energy effects did not affect the matching of the black hole and D-brane decay rates, even when the low energy condition of [72] was relaxed, upto leading order of the perturbation parameter that we considered. However, in this case we shall show that the same is not true, and that there is indeed a leading order correction to $\sigma_{\text{abs}}$ which means that the decay rates no longer match exactly as one goes beyond the low energy condition, $\omega r_1 \ll 1$. We shall not indicate the calculations explicitly, which are essentially in the same lines as in Section (III), but rather state the main results. The results for charged scalar emission rates may be obtained by a simple extension [77]. The relevant wave equation whose solution we seek is [77, 78]

$$\frac{h}{r^2}\frac{d}{dr}\left(hr^2\frac{dR}{dr}\right) + \omega^2 f R = 0 \tag{4.70}$$

where $f = \prod_{i=1}^{4}\left(1 + \frac{r_i}{r}\right)$, $r_i$'s being the parameters of the four dimensional black hole. This equation can be expanded in powers of $1/r$ in the near and far regions exactly as we did in the five dimensional case. For the near region, keeping the next to leading order term in $1/r$ leads to the equation

$$\frac{d^2\psi}{dr^2} + \left(1 + \frac{A}{r}\right)\omega^2\psi = 0 \tag{4.71}$$

where

$$A = (r_1 + r_2 + r_3), \quad \psi = rR$$

Interestingly, this equation is formally similar to the Coulomb wave equation. It has a solution in terms of confluent hypergeometric equations, namely

$$\psi = \alpha F(\eta\rho) + \beta G(\eta\rho) \tag{4.72}$$

Where $\rho = \omega r$ and $\eta = \frac{-A\omega}{2}$ is a small parameter. The asymptotic form for this expression for large $r$ is given by

$$F = g\cos\theta + f\sin\theta \qquad G = f\cos\theta - g\sin\theta \tag{4.73}$$

where $f$ and $g$ are constants depending on the black hole parameters, namely, to first order in $\eta$,

$$\begin{aligned} f &= 1 + \frac{\eta}{2\rho} \\ g &= -\frac{\eta}{4\rho^2} \\ \theta &= \rho - \eta\ln 2\rho - \eta\gamma\ , \end{aligned} \tag{4.74}$$

$\gamma$ being Euler's constant. The flux of incoming particles calculated from this form of the wave function at infinity is given by

$$f_{\text{in}} = \frac{|\alpha|^2}{4}\left(-1 - \frac{\eta}{2\omega r} - \frac{\eta}{4\omega^2 r^2}\right)\left(-\omega + \frac{\eta}{r}\right) \tag{4.75}$$

It can be seen that most of the terms in the above expression can be neglected as $r \to \infty$ and we are left with

$$f_{\text{in}} = \frac{|\alpha|^2}{4}\omega \tag{4.76}$$

Note that we have dropped a $\beta$ dependent piece in $f_{in}$ as exactly in the five dimensional calculation it turns out to be extremely small compared to the $\alpha$ dependent term.

In the near region, the equation (4.70) reduces after keeping the leading order terms in descending powers of $\frac{1}{r}$ to

$$\frac{h}{r^2}\frac{d}{dr}\left(hr^2\frac{dR}{dr}\right) + \left(\frac{A}{r^4} + \frac{B}{r^3} + \frac{C}{r^2}\right)\omega^2 R = 0 \tag{4.77}$$

Very close to the horizon we can once again make the substitution $y = -\log(1-v)$ so that the equation (4.77) reduces to the simple harmonic equation

$$\frac{d^2R}{dy^2} + (a + b)\,R = 0 \tag{4.78}$$

where

$$(a+b)^2 = 4\,(A+B+C)\,; \qquad ab = B+C$$

Just as in the five dimensional case, $a$ and $b$ are related to the black hole parameters, the relation being given by

$$\begin{aligned} a &= \frac{\omega}{4\pi T_R} + 2\pi\omega\, T_H\,(r_1 r_3 + r_2 r_3 + r_1 r_2) \\ b &= \frac{\omega}{4\pi T_L} + 2\pi\omega\, T_H\,(r_1 r_3 + r_2 r_3 + r_1 r_2)\,. \end{aligned} \tag{4.79}$$

Again, writing the incoming solution as

$$R = K\exp\left(-i\sqrt{A + B + C}\,\ln(1 - v)\right) \tag{4.80}$$

the incoming flux at the black hole horizon is calculated to be

$$f_{\text{abs}} = |K|^2 r_0 \frac{(a + b)}{2} \tag{4.81}$$

Now, matching the near and the far zone solutions as in [72] gives the relation

$$\frac{K}{\alpha} = \frac{\omega}{E}\left(1 - \frac{\pi\eta}{2}\right) \tag{4.82}$$

where $E$ is given in terms of gamma functions as

$$E = \frac{\Gamma(1-ia-ib)}{\Gamma(1-ia)\Gamma(1-ib)} \tag{4.83}$$

so that the calculation for $\sigma_{\text{abs}}$ finally yields

$$\sigma_{\text{abs}} = \frac{4\pi}{\omega^2}\frac{f_{\text{abs}}}{f_{\text{in}}} = \sigma^0_{\text{abs}}\,[1 - \pi\eta] \tag{4.84}$$

where $\sigma^0_{\text{abs}} = 4\pi^2 r_1 r_2 r_3 \omega$ at high energies. Hence we see from (4.84) that there is a order $\eta$ correction to $\sigma_{\text{abs}}$, in contrast to the five dimensional case where this correction was negligible. This, along with the microcanonical condition implies that the D-brane and black hole decay rates match only in the energy regime given by

$$\sqrt{\frac{r_0}{r_1}} \ll \omega r_1 \ll 1 ,$$

where the emission rates in four dimensions is given by

$$\Gamma_H = \Gamma_D = 4\pi^2 r_1 r_2 r_3 \, \omega \; \exp(-\Delta S_{BH}) \, \frac{d^3 k}{(2\pi)^3} .$$

The difference between the five and four dimensional cases that we have dealt with is also apparent from the general analysis of [78]. The far zone equation that was effectively the source of the difference in the two different dimensions can be written to leading order, for $D$ dimensions as,

$$\frac{d^2\psi}{d\rho^2} + \left[1 - \frac{(D-2)(D-4)}{\rho^2}\right] \psi = 0 \tag{4.85}$$

where $\rho = \omega r$, and $R = r^{-\frac{D-2}{2}} \psi$. The general solution of the above equation is

$$F = \sqrt{\frac{\pi}{2}} \rho^{\frac{1}{2}} J_{(D-3)/2}(\rho) \tag{4.86}$$

In five dimensions, the addition of an interaction terms also of the form $1/\rho^2$) simply modifies the order of the Bessel function. The resulting corrections in $\sigma_{\text{abs}}$ is negligible. However, in four dimensions, there is a new $1/\rho$ term, which gives rise to the Coulomb wave function and a leading order correction in the final result. The fact that in four dimensions the D-brane and black hole rates exactly match only for a restricted range of paremeters have also been pointed out by other authors [78]

In conclusion, let us note that in the calculation of high energy greybody factors presented in this chapter, it was assumed that there were no explicit back reaction effects as in the case of the Hawking spectrum. This can be justified as follows: the modified black hole metric due to back reaction of the shell can be approximated by the original metric with a shift in the ADM mass of the black hole by $\omega$. The corresponding effect on the D-brane is the reduction of the excitation energy of the gas of open strings i.e. $E = E_L + E_R \to E - \omega$. Since $E$ can be written as

$$E = \frac{\pi\, r_0^2\, \cosh 2\sigma}{8G_5} \, ,$$

the parameters $r_0$ and $\sigma$ are changed accordingly. It can be shown that these changes result in a correction also of order $(r_0/r_1)^2$ in $\sigma_{\text{abs}}$ and hence can be ignored.

In this chapter, we have studied an application of D-branes to black holes. We have analysed Hawking radiation rates for high energy scalar particles from D-branes and compared them to corresponding black hole results. It is an interesting question to study further applications of principles of string theory and D-branes in general relativity. One interesting area for the application of such principles is cosmology. It is natural to ask how physical principles, formulated in string theory and D-branes are applicable to our universe. In the next chapter, we study one such issue, namely the application of the holographic principle to cosmology. The holographic principle, originally proposed in the context of quantum gravity and string theory, and tested for D-brane physics, has recently been applied in standard cosmology. We will study the application of this principle in inflationary cosmology.

# Chapter 5

# Holography and inflationary cosmology

## 5.1 The holographic principle: notation and conventions

The *holographic principle*, was first proposed by 't Hooft [30] in the context of black hole physics and later extended by Susskind [31] to string theory. As we have stated in chapter I, it implies that the degrees of freedom in a spatial region are all encoded on its boundary, and that the number of degrees of freedom per Planck area does not exceed unity. Accordingly, the entropy in a spatial region does not exceed its boundary area in Planck units. As a consequence of the holographic principle, for example, the physics of the bulk is describable by the physics on the boundary. In the context of D-brane physics, this principle has been studied recently for some anti de Sitter spaces [79, 80]. An understanding of the holographic principle in the context of cosmology is interesting, since general principles applicable to string theory and D-branes should finally have applications to the present universe. As it turns out, however, a naive application of the holographic principle to the present universe gives erroneous results. Fischler and Susskind (FS) propose to remedy this by arguing that the correct application of holography in the context of cosmology should be to apply it to a region determined by the cosmological horizon.

Before considering the FS proposal in details, let us summarize a few basic facts about standard cosmology, that would set the notation and conventions for the rest of the chapter. The universe is assumed to be homogeneous and isotropic, and described by the Robertson-Walker metric,

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] . \tag{5.1}$$

where $k = +1, -1$ or $0$ for a closed, open or flat universe.The evolution of the scale factor $R(t)$ is governed by the Friedman equations

$$\begin{aligned} \frac{d^2R}{dt^2} &= -\frac{4\pi}{3}G(\rho + 3p)R \\ H^2 + \frac{k}{R^2} &= \frac{8\pi}{3}G\rho \end{aligned} \tag{5.2}$$

where $H \equiv \frac{1}{R}\frac{dR}{dt}$ is the Hubble constant and $\rho$ and $p$ denote the energy density and the pressure of the universe. The Friedmann equations may be shown to imply that for a radiation dominated universe ($p = \frac{1}{3}\rho$), the scale factor evolves with time as $R \propto t^{\frac{1}{2}}$, while for a matter dominated universe ($p = 0$), the relationship is $R \propto t^{\frac{2}{3}}$ and for a vacuum energy dominated universe, it is $R \propto exp^{(Ht)}$. Now, let us consider the flat universe, i.e, $k = 0$. With $\rho$, $p$, and $T$ be the energy density, pressure, and temperature of the universe, let us also define the quantities

$$r_H = \int_0^t \frac{dt}{R} \quad \text{and} \quad d_H = r_H R$$

which are the coordinate size and the physical size of the horizon respectively. The (constant) comoving and the (varying) physical entropy densities, $\sigma$ and $s$ respectively, are then given by

$$\sigma = \frac{\rho + p}{T} R^3 \equiv sR^3 . \tag{5.3}$$

With the above definitions, let us examine the Fischler and Susskind (FS) [81] method of applying the holographic principle to cosmology. Their proposal is as follows. Let $\Gamma$ be a spherical spatial region of coordinate size $r$ with boundary $B$ and let $L$ be the light-like surface formed by past light rays from $B$ towards the center of $\Gamma$. Then according to FS proposal, the holgraphic principle implies that the entropy passing through $L$ never exceeds the area of $B$ [81].

The entropy $S$ contained within $\Gamma$, and the area $A$ of the boundary $B$ are

$$S = \frac{4\pi}{3} s d_H^3 \tag{5.4}$$

$$A = 4\pi d_H^2 . \tag{5.5}$$

According to FS proposal, the holographic principle implies that

$$S \lesssim A . \tag{5.6}$$

The approximate order of magnitude values of various quantities in our universe at different epochs are tabulated below [86]. All quantities, here and in the following, are in Planck units, unless mentioned otherwise.

| $T$ | $T_0$ | $5.5eV$ | $10^{14}GeV$ | $T_{pl}$ |
|---|---|---|---|---|
| $R$ | $6 \times 10^{60}$ | $2 \times 10^{56}$ | $10^{34}$ | $10^{29}$ |
| $r_H$ | 1 | $7 \times 10^{-3}$ | $4 \times 10^{-25}$ | $3 \times 10^{-30}$ |
| $d_H$ | $6 \times 10^{60}$ | $2 \times 10^{54}$ | $5 \times 10^{9}$ | 0.3 |
| $s$ | $10^{-95}$ | $2 \times 10^{-82}$ | $9 \times 10^{-16}$ | 2 |
| $S$ | $10^{88}$ | $3 \times 10^{81}$ | $5 \times 10^{14}$ | 0.3 |
| $A$ | $4 \times 10^{122}$ | $3 \times 10^{109}$ | $3 \times 10^{20}$ | 1.4 |

$T_0 = 2.75K$ is the present temperature of the universe and $T_{pl} = 1.2 \times 10^{19} GeV$ is the Planck temperature. Note that the constant comoving entropy density is given, for our choice of $r_H(\text{present}) = 1$, by

$$\sigma_0 \simeq 10^{87} .$$

It is clear from the above table that the holographic principle is obeyed in our universe from Planckian time upto the present. It is obeyed indefinitely in the future too if our universe is flat or open [81].

The above (FS) analysis was for the standard scenario, based on the Friedmann equations. This standard model, however, has a number of shortcomings. In the next section, we outline two of them, namely the *flatness problem* and the *horizon problem*, and discuss the *inflationary scenario* which was proposed originally by Guth and then extended by several other authors, in order to solve some of the problems of standard cosmology.

## 5.2 Problems with standard cosmology and the Inflationary scenario

From Einstein's equations, one can show that the energy density $\rho$ of the universe at a given time $t$ and the critical energy density corresponding to a flat universe, $\rho_{cr}$ are related by

$$|\Omega - 1| = |\frac{\rho - \rho_{cr}}{\rho_{cr}}| = \left[\frac{dR}{dt}\right]^{-2} \tag{5.7}$$

The present day value of $\Omega$ is known to be given by

$$0.1 \leq \Omega_p \leq 2 \tag{5.8}$$

where the subscript $p$ denotes the present value of $\Omega$. On the other hand, in the early stages of evolution of the universe, $\left[\frac{dR}{dt}\right]^{-2} \simeq t$, so that the quantity $|\Omega - 1|$ was extremely small. It can be shown that for $\Omega$ to lie in the range (5.8) now, the early universe must have had

$$|\Omega - 1| \leq 10^{-59}\frac{M_P^2}{T^2} \tag{5.9}$$

$T$ being the corresponding temperature. Thus, at Planck time, $T \simeq M_P$,

$$|\Omega - 1| \leq 10^{-59} \tag{5.10}$$

The question of why the energy density of the early universe was so close to the critical value is known as the flatness problem.

The second problem, the horizon problem, is the fact that inspite of the assumption that the initial universe was homogeneous and isotropic, it can still be shown to be composed of a huge number ($10^{83}$) of causally disconnected regions. If we consider the physical distance $l$ that a light pulse starting at $t = 0$ would traverse at a time $t$ and the radius $L(t)$ of the region at time $t$ that evolves into the present universe, then a ratio of the volumes gives

$$\frac{l^3}{L^3} = 10^{-83} \tag{5.11}$$

at $T = 10^{17} GeV$. Since $l^3$ would denote the volume of a causally connected region, the universe is thus seen to consist of $10^{83}$ causally disconnected regions. This is the horizon problem.

In order to solve these, and some other problems of standard cosmology, Guth [83] proposed the *inflation* scenario. The basic idea of inflation is to assume that there was a period of time when vacuum energy was the dominant component of the energy density of the universe, and hence the scale factor grew *exponentially*. Generically, in the inflationary scenario, a small, causally connected patch of the universe inflates from say time $t = 0$ to $t = t_e$. The scale factor $R$ grows by a factor of $e^N$ and the universe supercools. At the end of inflation, the universe reheats to a temperature $T_R \lesssim T_b$, releasing an enormous amount of entropy. This scenario can be shown to solve, among others, the two above mentioned problems in standard cosmology.

The actual details of the reheating and the entropy production are model dependent [86, 87, 88]. However, the relevent physical process falls broadly in one of the two categories where the reheating and the entropy production are due to

- (1) the decay of the 'slow rolling' inflaton - typified by new inflation [84] or
- (2) bubble wall collisions - typified by extended inflation [85].

As an illustration, let us briefly discuss the process of new inflation as in (1). Stated in general terms, inflation is understood by the dynamics of a massive scalar field in the expanding universe. This scalar field $\phi$, which weakly interacts with other fields - scalars, fermions, photons, gravitons etc., is assumed to be trapped in a false vacuum charactarised by $\langle\phi\rangle = 0$ with a vacuum energy $V(\phi)$, and, during the course of evolution, rolls to the actual minimum of the potential, denoted by $\langle\phi\rangle = \sigma$. During this process, the field $\phi$ might also encounter a barrier in the potential. However, the actual details of this will not be relevent to our discussion. During the time it takes for $\phi$ to evolve to the minimum of the potential, the universe possesses an enormous amount of energy, $\rho_{vac} \simeq V(\phi = 0) \equiv M^4$. Now, once the temperature of the universe falls below a certain critical temperature $T \simeq T_C \equiv M$, this vacuum energy is the dominant component of the energy density of the universe, and the universe expands exponentially, according to the Friedmann equations. The crucial feature of this model of inflation is the period of *slow rollover* of the field $\phi$, which is typically $\simeq 10^{-34}$ seconds.

The potential becomes steep as $\phi$ nears the minimum of the potential, and on reaching the minimum, $\phi$ oscillates about the value $\phi = \sigma$, in the process releasing the enormous amount of vaccum energy $V(\phi = 0)$ in the form of thermal energy of the particles that it decays into. This decay process will result in the end of inflation, and finally the universe is reheated to a temperatue given by $T_R$ where $T_R^4 \simeq V(\phi = 0) \simeq M^4$. The reheating process is accompanied by an enormous release of entropy that is crucial in solving some of the problems of standard cosmology.

In view of the crucial role played by the amount of entropy in the universe for the application of the holographic principle to cosmology, it is a natural question to ask if this principle is still satisfied during the inflationary stage. Let us consider this question in more detail in the next section.

## 5.3 Holography in the inflationary universe

For the region with $r_H = 1$, the natural values of the entropy $S$ and the area $A$ at the beginning of inflation are

$$\begin{aligned} S_b &\simeq \sigma_b \simeq T_b^3 t_b^3 \simeq T_b^{-3} \;\; (10^{15}) \\ A_b &\simeq t_b^2 \simeq T_b^{-4} \;\; (10^{20}) \,, \end{aligned}$$

where the numbers in the bracket are the values if $T_b = 10^{-5}$. So, $S_b < A_b$ and the holographic principle is obeyed at the beginning of inflation.

At the end of inflation we get, for the region with $r_H = 1$,

$$\begin{aligned} S_e &\simeq \sigma_e \simeq e^{3N} T_b^{-3} \\ A_e &\simeq e^{2N} T_b^{-4} \end{aligned}$$

where we have taken $T_R \simeq T_b$. In order to account for the observed entropy of the universe, we require [83, 86]

$$\sigma_e \gtrsim \sigma_0 \simeq 10^{87} .$$

Hence,

$$e^N \gtrsim \sigma_0^{\frac{1}{3}} T_b \ (10^{24}) .$$

This is the required 60 e-folding of the inflationary scenario [83]. For these values,

$$\frac{S_e}{A_e} \simeq e^N T_b \ (10^{19})$$

which clearly *violates* the holographic principle. The required entropy production will not violate the holographic principle, as applied above, only if $T_b \lesssim 10^{-15} \simeq 10^4 GeV$ with $e^N T_b \lesssim 1$. However, such a low value is unsatisfactory for other reasons [86].

More importantly, the above application of the holographic principle is naive and is precisely the one Fischler and Susskind warned against. The spatial region $\Gamma$ and the boundary $B$, which evolve along the light-like surface $L$ into the present ones, are marked at the end of inflation, when $T = T_R \simeq T_b$, by (Table I)

$$r_H \simeq 3 \times 10^{-30}\, T_b^{-1} \ (10^{-25}) .$$

For such a region, we get

$$\frac{S_e}{A_e} \simeq r_H T_b e^N \simeq 10^{-30} e^N .$$

The holographic principle is then obeyed if the inflation factor

$$e^N \lesssim 10^{30} ,$$

which is sufficient to solve all the problems in Guth's original proposal for inflation [83].

Typically, however, $e^N$ is of the order of $10^{100} - 10^{300}$ in extended inflation [85] and of the order of $10^{10^3} - 10^{10^7}$ in new inflation [84, 86]. So, the above bound is a severe constraint on inflationary models and acheiving it is likely to be unnatural, if possible at all. Also, the above application of holographic principle is in the era immediately following the entropy production, and not when the entropy is actually being produced.

The entropy is produced at the end of inflation during the reheating process and the universe reheats to a temperature $T_R \lesssim T_b$, where $T_b$ is the temperature at the beginning of inflation. The physical entropy density $s_R$ during the entropy production can be taken, on an average, to be [86]

$$s_R \simeq T_R^3 \,.$$

The holographic principle, applied during this process to a suitable region, to be identified below, of physical size $\simeq d$ (hence of volume $d^3$ with boundary area $d^2$), implies that

$$S_R \lesssim A_R \quad \longrightarrow \quad T_R^3 d \lesssim 1 \,. \tag{5.12}$$

The actual details of the reheating and the entropy production are model dependent [86, 87, 88]. However, the relevent physical process falls broadly in one of the two categories where the reheating and the entropy production are due to (1) bubble wall collisions - typified by extended inflation [85], or (2) the decay of the 'slow rolling' inflaton - typified by new inflation [84]. We now identify the size $d$ in each of these cases.

(1) The true vacuum bubbles nucleate during inflation, expand with the speed of light, and eventually percolate the universe, thus ending the inflation. Upon percolation, the bubble walls collide and release the energy and entropy into the interior of the bubbles, thereby reheating the universe to a temperature $T_R \lesssim T_b$. Typically, the reheating time is of the order of the time required for light to cross the bubble [88]. Thus, it is natural to apply the holographic principle to the interior of each bubble. On an average, the time between the bubble nucleation and collision is less than or of the order of $t_e$, the duration of inflation. The interior of the bubble is in

a true vacuum state and, thus, its size $d \simeq t_e$. With no further condition on $T_R$, the holographic principle implies that

$$T_R^3 t_e \lesssim 1 . \tag{5.13}$$

(2) The inflaton slowly rolls down to its minimum and begins to oscillate, thus ending the inflation. The oscillating inflaton decays into other particles, releasing the energy and entropy into the universe and, thereby, reheating it to a temperature $T_R \lesssim T_b$. The entropy is produced simultaneously and everywhere in the inflated region. Thus, it is natural to apply the holographic principle to any region covered by a light ray starting from a point and travelling for a time $\simeq t_R$, where $t_R$ is the duration of reheating. The universe is in a true vacuum state during reheating and, thus, the size of this region $d \simeq t_R$.

The universe reheats within a few Hubble time $H_e^{-1} \simeq t_e$ at the end of inflation. Taking $t_R \lesssim t_e$, and with no further condition on $T_R$, the holographic principle implies the relation (5.13), same as in the previous case.

As we have remarked earlier, the inflaton decay is often modelled by that of a massive scalar field interacting with other fields [86, 87]. In typical models, the reheating time $t_R \simeq \gamma_d^{-1}$, where $\gamma_d$ is the decay rate which, for $T_b = 10^{14} GeV$, is $\mathcal{O}(10^{-6} - 10^{-12})$ in Planck units depending on the model and the decay products. Moreover, the reheating temperature $T_R$ is related to $\gamma_d$ by

$$T_R \simeq \sqrt{\gamma_d}$$

in Planck units. Note that, in these models, the reheating time $t_R$ (the reheating temperature $T_R$) has no relation to the Hubble time $t_e$ at the end (the temperature $T_b$ at the beginning) of inflation. The holographic relation (5.12) then implies

$$\sqrt{\gamma_d} \lesssim 1 , \tag{5.14}$$

a condition well satisfied in these models.

We now explore the consequences of the relation (5.13). The reheating temperature $T_R \lesssim T_b$ depends only on at what temperature the inflation sets in. The duration of inflation $t_e$ then satisfies an upper bound given by (5.13). Such an upper bound on $t_e$ can be expected, among other things, to lead to an upper bound on the inflation factor $e^N$. This is simply because longer the duration of inflation, larger is the expansion factor.

An upper bound on $e^N$ can, in turn, be expected to lead to a lower bound on the density fluctuations in the universe, which seed the large scale structure formation. This is because, essentially the inflation dampens the quantum fluctuations of the fields, which reenter as density fluctuations in the later era. Hence, larger the inflation factor, more the damping of quantum fluctuations, and thus smaller the resulting density fluctuations.

Although the above physical reasoning is direct and simple, the actual calculations of $t_e$ and of the density fluctuations are quite involved. Also, to our knowledge, there is no model independent formula which relates the density fluctuations to the duration or the amount of inflation. Hence, in the next section, we illustrate these consequences explicitly in the context of new and extended inflation. However, following the above reasoning, they are expected to be valid generally.

## 5.4 A lower bound on density fluctuations from the holographic principle

Consider the duration of inflation $t_e$ and the expansion factor $e^N$. (For details about various expressions used below, we use references [84, 86, 87] for new inflation and [85, 88, 89, 90] for extended inflation.)

**New Inflation:** Let the inflaton potential be

$$V = V_0 - \frac{\lambda}{n}\,\phi^n\,, \quad n \geq 4\,, \tag{5.15}$$

where $V_0 \equiv \frac{3H_b^2}{8\pi} = M^4 \simeq T_b^4$, and $\lambda$ is a coupling constant. Equation (5.13) implies that the inflation factor $e^N$ is restricted by an upper bound given by

$$N \simeq H_b t_e \lesssim \left(\frac{T_b}{T_R}\right)^3 T_b^{-1}\,.$$

Note that with $T_b = 10^{14}GeV$ and $T_R \simeq 0.1T_b$, we have $N \lesssim 10^8$, which conforms well with the amount of inflation occuring in these models.

The duration of inflation $t_e$ is related to the coupling constant $\lambda$ by

$$t_e \simeq 4\pi^2 H_b^{\frac{8}{n}-3} \left(\frac{3}{8\pi^2\lambda}\right)^{\frac{2}{n}} .$$

Equation (5.13) then implies a lower bound

$$\lambda \gtrsim T_R^{\frac{3n}{2}} H_b^{4-\frac{3n}{2}} . \tag{5.16}$$

**Extended Inflation:** The model is specified by a parameter $\omega \simeq 10 - 20 \lesssim 25$. Equation (5.13) implies that the inflation factor $e^N$ is restricted by an upper bound

$$e^N \simeq t_e^{\omega+\frac{1}{2}} \lesssim T_R^{-3(\omega+\frac{1}{2})} .$$

Note that with $T_R \simeq T_b = 10^{14}GeV$, and with $\omega = 10$, we have $e^N \lesssim 10^{160}$, which conforms well with the amount of inflation occuring in these models.

Consider the density fluctuations on a scale $\lambda_0 (\simeq 10^{60}$ for the horizon) today. Let $T_0 = 2.75K$ be the present temperature of the universe.

**New Inflation:** The inflaton potential is given by (5.15). The density fluctuations are then given by

$$\frac{\delta\rho}{\rho} \simeq A_{ni} H_b^{\frac{n-4}{n-2}} \lambda^{\frac{1}{n-2}} \tag{5.17}$$

where $n \geq 4$ and

$$A_{ni} \simeq \frac{16}{3}\left(\frac{2}{3}\ln\frac{H_b\lambda_0 T_0}{T_R}\right)^{\frac{n-1}{n-2}} .$$

With $H_b \simeq M^2 \simeq T_b^2$, equation (5.16) then implies a lower bound on the density fluctuations:

$$\frac{\delta\rho}{\rho} \gtrsim A_{ni}\left(\frac{T_R}{T_b}\right)^{\frac{3n}{2(n-2)}} T_b^{\frac{n}{2(n-2)}} . \tag{5.18}$$

For $T_b = 10^{14}GeV$, $T_R \simeq 0.1T_b$, and $n = 4$, $A_{ni} \simeq \mathcal{O}(10^2)$ and the above bound gives

$$\frac{\delta\rho}{\rho} \gtrsim \mathcal{O}(10^{-6}) .$$

Considering the approximations involved, the above lower bound on the density fluctuations implied by equation (5.13) is remarkably close to the observed value $\simeq 10^{-6}$ [91] if inflation takes place at $T_b \simeq 10^{14}GeV$.

**Extended Inflation:** The density fluctuations are given by

$$\frac{\delta\rho}{\rho} \simeq A_{ei}\left(T_0\lambda_0\sqrt{2\omega+1}\right)^{\frac{4}{2\omega-1}}(2\omega+1)^{\frac{3}{2}}t_e^{-\frac{2\omega+1}{2\omega-1}} \tag{5.19}$$

where

$$A_{ei} \simeq \frac{\sqrt{\pi}}{3}\left(\frac{8\pi}{9}\right)^{\frac{2}{2\omega-1}}\left(\frac{6\omega+9}{6\omega+5}\right)^{\frac{2\omega+3}{2\omega-1}} .$$

With $\lambda_0 \simeq 10^{60}$, $T_0 \simeq 10^{-32}$, and $\omega \simeq 10$, equation (5.13) then implies a lower bound on the density fluctuations:

$$\frac{\delta\rho}{\rho} \gtrsim 10^8 A_{ei} T_R^{\frac{3(2\omega+1)}{2\omega-1}} . \tag{5.20}$$

For $T_R \simeq T_b \simeq 10^{14}GeV$ and $\omega \simeq 10$, $A_{ei} \simeq \mathcal{O}(1)$ and the above bound gives

$$\frac{\delta\rho}{\rho} \gtrsim \mathcal{O}(10^{-7}) .$$

Considering the approximations involved, the above lower bound on the density fluctuations implied by equation (5.13) is remarkably close to the observed value $\simeq 10^{-6}$ [91] if inflation takes place at $T_b \simeq 10^{14}GeV$.

Before we end this chapter, let us note that equation (5.13), which led to our lower bounds on density fluctuations, arises as a consequence of the holographic principle in models, typified by extended inflation, where the reheating and the entropy production are due to bubble wall collisions. The reheating temperature is taken to depend only on at what temperature the inflation sets in. Thus, in such models, the lower bound on density fluctuations is a consequence of the holographic principle.

Equation (5.13) also arises as a consequence of the holographic principle in those models, typified by new inflation, where the reheating and the entropy production are due to the decay of the 'slow rolling' inflaton decay, if the reheating time is of the order of a few Hubble time at the end of inflation and if the reheating temperature depends only on at what temperature the inflation sets in. Then, in these models also, the lower bound on the density fluctuations is a consequence of the holographic principle.

Often the inflaton decay, relevent in the models of new inflation type, is modelled by that of a massive scalar field interacting with other fields. In such models, the reheating time (the reheating temperature) has no relation to the Hubble time at the end (the temperature at the beginning) of inflation. Typically, the holographic principle is automatically satisfied in these models with no further consequences.

Perhaps, it is that such models of inflaton decay may be specific possibilities only, while the generic possibilities have reheating times (the reheating temperature) of the order of the Hubble time at the end (the temperature at the beginning) of inflation. If so, then, in the models of new inflation type too, the holographic principle is likely to lead to a lower bound on the denisty fluctuations.

Conversely, and just as likely, the inflaton decay models *are* the generic models of reheating. Moreover, it may also be that similar generic models exist for bubble wall collisions too, in which the reheating time (the reheating temperature) has no relation to the Hubble time at the end (the temperature at the beginning) of inflation. If so then, in the models of extended inflation type also, the holographic principle is likely to be automatically satisfied with no further consequences.

# Chapter 6

# Conclusions

In this thesis, we have reported on the work done on some aspects of D-brane physics and quantum gravity. First, we have studied D-branes wrapped around supersymmetric cycles of Calabi-Yau manifolds using boundary Landau-Ginzburg (LG) theory as well as boundary conformal field theory (CFT) formulations. Next, we have analysed such D-branes from the point of view of gauged linear sigma models on world sheets with boundary. We have then investigated some aspects of application of D-brane physics to the understanding of Hawking radiation of black holes. Finally, we have studied the application of the holographic principle, a generic principle for quantum gravity, in the context of cosmology. We conclude this thesis with comments on the work presented here, and directions for future research.

Our LG formulation of D-branes on curved manifolds is suitable for understanding the associated boundary conditions from the target space viewpoint, while the boundary CFT formulation provides the corresponding boundary state. As we have shown, the common discrete symmetry group associated with both the LG orbifold and the corresponding Gepner model is a useful tool in relating boundary conditions to boundary states. It also suggests that boundary states of D-branes constructed in [20], by tensoring boundary states for the individual minimal models may be further classified by means of charges associated with the discrete symmetry group. In our method, this is also seen through the resolution of the S-matrix of the Gepner model.

It would be an interesting problem to extend the program of studying closed string vacuum for Calabi-Yau compactifications involving the use of LG models and the general structure of N=2 superconformal theories, to the case of D-brane states. Clearly, we would need to extend the use of the Landau-Ginzburg model techniques so that more relevant information can be extracted. As has been noted by other authors, this may involve the extension of the methods of the $N = 2$ topological field theory techniques to the case of boundary $N = 2$ SCFTs. Index calculations of various kinds, for example, may be performed in the LG model using purely free-field techniques by extension of similar techniques used in the closed string case [53]. A calculation of $\mathrm{Tr}(-1)^F$ in the Ramond sector of the openstring was carried out in [44], and it would be interesting to evaluate this by our methods.

The construction that we have used in chapter 2 for the boundary states seems a priori difficult to extend to the case of K3 and Calabi-Yau three-fold compactifications. In particular the fixed point resolution would appear to be hopelessly complicated even in the simplest cases. But since the resolution would involve presumably no more than the use of the full symmetry of the model it might be possible to solve the problem by computer techniques. In such a situation, the results presented for the $T^2$ in Chapter 2 would be extendable to the case of compactifications like the quintic Calabi-Yau. The diagonal partition functions (that is between identical branes) in the cylinder channel and hence in the annulus channel are however known even despite the fixed point resolution even in the complicated cases by our construction. It would be interesting to extend this to non-diagonal cases by our methods.

In chapter 3 of this thesis, we have studied D-branes on Calabi-Yau manifolds from the point of view of gauged linear sigma models (GLSM). We have tried to determine a complete set of boundary conditions defining D-branes in the GLSM description of the open string CFT. This begins an analysis of whether the LG-CY correspondence shown by Witten by making use of linear sigma models will go through for the case of linear sigma models with boundary. We have shown that a careful analysis of the boundary conditions, by demanding consistency in the infrared limit, forces us to introduce a fermion bilinear term in the action, a fact which has been known in other contexts. This, although it seems to produce a complete set of boundary conditions for the D0 brane, still does not resolve ambiguities regarding the boundary conditions of higher dimensional branes. In particular, we have seen that boundary conditions in the gauge multiplet are difficult to define in these cases. Although in this thesis we have not completed the full analysis of D-branes on CY using the GLSM description, it is worth noting that the two matrices $A$ and $B$ that we introduced in Chapter 2 to parametrise general A-type and B-type boundary conditions, might be useful to resolve this issue. Also, the boundary conditions for the D-0 brane that we have written down, seems to be manifestly independent of the Kähler parameter, and seems to imply that the D-0 brane moduli space does not receive any corrections on the transition from the geometric (CY) to the non-geometric (LG) phase. This however, needs to be checked by explicit calculations of open string instanton corrections, which we have not attempted in this thesis. It would be interesting to investigate this in details.

In chapter 4 of this thesis, we have studied an application of D-branes to black holes. We have compared black hole and D-brane decay rates for neutral and charged scalar emission at high energies. Whereas previous studies of these emission rates were confined to the regime of small energies of the emitted quanta, we have generalised these results by treating the one-dimensional gases of the open strings moving along the D1-brane (in the long brane approximation) as a microcanonical ensemble. This naturally incorporated the high energy condition, $\omega/T_{L,R} \gg 1$, which was crucially used in calculating the black hole greybody factor. We have shown that while in five dimensions, the decay rates match for all values of energy consistent with the microcanonical picture, in four dimensions, they match only in a restricted range.

Calculations involving emission rates from D-branes and comparison of these with corresponding black hole results has also been carried out in the context of the recently discovered AdS-CFT correspondence. Several approximations that we have made in our calculation, are not necessary in this picture, although the final results seem to be in agreement. This puts the D-brane black hole correspondence on a more firm footing. We note however, that stronger statements regarding this correspondence can be made by studying the moduli space of the D-brane configurations that we have studied in this thesis, a method that has been used in the calculations of [28]. Calculations of emission rates for other particles like vectors and fermions have also been carried out in the literature, and as an application of the results presented in Chapter 4, it would be interesting to extend these calculations to the regime of high energy emissions, and see whether matching conditions are still valid. It would also be interesting to compare our results from calculations using CFT techniques, in view of the AdS-CFT duality.

In chapter 5, we have moved on to another important aspect of general relativity, namely cosmology, and shown how the holographic principle of quantum gravity (and of string theory) is applicable to the inflationary universe. We have shown that a correct application of the holographic principle to the case of new and extended inflation (which corresponds to generic models of inflation) leads to a lower bound on density fluctuations. Considering the approximations involved, the lower bound on the density fluctuations that we have obtained is remarkably close to the observed values. To our knowledge, this is the first instance where a *lower bound* on the density fluctuations is obtained theoretically. It would be interesting to establish such a bound rigorously, in a model independent way. This could then be taken as a prediction of the holographic principle.

Another interesting point to note is that by an appropriate coordinate transformation [86], the inflating universe can be cast into a static de Sitter one. One can then translate the present analysis and compare the results with those obtained for some anti de Sitter spaces in [80]. This might provide some insights into the holographic principle in static universes.

# Bibliography

[1] M. B. Green, J. Schwarz, E. Witten, *Superstring Theory Vol. I and II*, Cambridge Univ. Press.

[2] J. Polchinski, *String Theory Vol. I and II*, Cambridge University Press.

[3] S. Govindarajan, T. Jayaraman, T. Sarkar, *Worldsheet approaches to D-branes on supersymmetric cycles*, (hep-th/9907131)

[4] S. Govindarajan, T. Jayaraman, T. Sarkar, *in preparation*

[5] S. Das, A. Dasgupta, T. Sarkar, Phys. Rev. **D55** (1997) 7693

[6] S. Kalyana rama, T. Sarkar, Phys. Lett. **B450** (1999) 55

[7] G. Horowitz, A. Strominger, Nucl. Phys. **B360** (1991) 197

[8] J. Dai, R. G. Leigh, J. Polchinski, Mod. Phys. Lett **A4** (1989) 2073
J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724
J. Polchinski, *Tasi Lectures on D- Branes*, **hep-th/9611050** and references therein

[9] S. Coleman, E. Weinberg, Phys. Rev. **D7** (1973) 1888

[10] R. G. Leigh, Mod. Phys. Lett. **A4** (1989) 2767

[11] A. A. Tseytlin, Nucl. Phys. **B469** (1996) 51

[12] P. Candelas, G. Horowitz, A. Strominger, E. Witten, Nucl. Phys. **B258** (1985) 46

[13] D. Gepner, Nucl. Phys. **B296** (1988) 757; Phys. Lett. **B199** (1987) 380

[14] B.R. Greene, C. Vafa, N.P. Warner, Nucl. Phys. **B324** (1989) 371

[15] E. Witten, Nucl. Phys. **B403** (1993) 159 (hep-th/9301042)

[16] K. Becker, M. Becker, A. Strominger, Nucl. Phys. **B 456** (1995) 130

[17] M. Bershadsky, V. Sadov, C. Vafa, Nucl. Phys. **B 463** (1996) 420

[18] H. Ooguri, Y. Oz, Z. Yin, Nucl. Phys. **B477** (1996) 407 (hep-th/9606112)

[19] E. Witten, Mirror Manifolds and Topological Field Theory, hep-th/ 9112056

[20] A. Recknagel, V. Schomerus, Nucl. Phys. **B531** (1998) 185

[21] T. Eguchi, H. Ooguri, A. Taormina, S.K Yang, Nucl. Phys. **B 315** (1989) 193

[22] P.S. Aspinwall, B. Greene, D.R. Morrison, Phys. Lett. **B303** (1993) 249

[23] J. Bekenstein, Phys. Rev. **D7** 2333 (1973)
S. W. Hawking, Phys. Rev. **D13** (1976) 191

[24] A. Strominger, C. Vafa, Phys. Lett. **B379**, 99 (1996) (hep-th/9601029)

[25] S. W. Hawking, Comm. Math. Phys. **43** (1975) 199

[26] J. Maldacena, A. Strominger, Phys. Rev. **D55** (1997) 861

[27] S. R. Das, S. D. Mathur, Nucl. Phys. **B478** (1996) 561

[28] J. R. David, G. Mandal, S. R. Wadia Nucl.Phys. **B544** (1999) 590

[29] J. R. David, G. Mandal, S. R. Wadia (hep-th/9907075)

[30] G. 't Hooft, gr-qc/9310026
C. R. Stephens, G. 't Hooft, B. F. Whiting, Class. Quant. Grav. **11** (1994) 621, (gr-qc/9310006)

[31] L. Susskind, J. Math. Phys. **36** (1995) 6377

[32] L. Susskind, E. Witten, *The Holographic Bound in Anti-de Sitter Space* (hep-th/9805114)

[33] W. Fischler, L. Susskind, *Holography and Cosmology*, **hep-th/ 9806039**

[34] D. Bak, S. -J. Rey, (hep-th/9811008)
A. K. Biswas, J. Maharana, R. K. Pradhan, (hep-th/9811051)

[35] M. Bershadsky, V. Sadov, C. Vafa, Nucl. Phys. **B463** (1996) 398-414 (hep-th/9510225)

[36] M. Bershadsky, V. Sadov, C. Vafa, Nucl. Phys. **B463** (1996) 420-434 (hep-th/9511222)

[37] J. L. Cardy, Nucl. Phys. **B 324** (1989) 581.

[38] D. Gepner, *Lectures on $N = 2$ string theory, High Energy Physics and Cosmology: proceedings* Ed. J.C. Pati, S. Randjbar-Daemi, E. Sezgin, Q. Shafi. World Scientific, 1990 and references therein

[39] M. Gutperle, Y. Satoh, Nucl. Phys. **B543** (1999) 73 hep-th/9808080.

[40] A. Recknagel, V. Schomerus, Nucl. Phys. **B545** (1999) 233 (hep-th/9811237)

[41] S. Stanciu, Nucl. Phys. **B526** (1998) 295 (hep-th/9708166)

[42] S. Stanciu, A. Tseytlin, JHEP **9806** (1998) 010.

[43] M. Kato, T. Okada, Nucl. Phys. **B499** (1997) 583-595 (hep-th/9612148)

[44] I. Brunner, M. R. Douglas, A. Lawrence, C. Römelsberger, (hep-th/9906200)

[45] M. Bianchi, A. Sagnotti, Phys. Lett. **B247** (1990) 517

[46] M. Bianchi, A. Sagnotti, Nucl. Phys. **B361** (1991) 519

[47] M. Bianchi, G. Pradisi, A. Sagnotti, Nucl. Phys. **B376** (1992) 365

[48] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti, Y.S. Stanev, Phys. Lett. **B387** (1996) 743 (hep-th/9607229) and references therein

[49] N. P. Warner, (hep-th/9301088), *High Energy Physics and Cosmology 1992: proceedings* Ed. E. Gava et al., World Scientific (1993)

[50] N. Ishibashi, Mod. Phys. Lett. **A4** (1989) 251

[51] R. E. Behrend, P. A. Pearce, V. A. Petkova, J.-B. Zuber, (hep-th/9809097)

[52] J. Fuchs, A. N. Schellekens, C. Schweigert (hep-th/9601078)

[53] E. Witten, Int. J. Mod. Phys. **A9** (1994) 4783 (hep-th/9304026)

[54] J. Distler, B. Greene, Nucl. Phys. **B309** (1988) 295

[55] C. Vafa, Mod. Phys. Lett. **A4** (1989) 1169

[56] K. Intriligator, C. Vafa, Nucl. Phys. **B339** (1990) 95

[57] N. P. Warner, Nucl. Phys. **B450** (1995) 663 (hep-th/9506064)

[58] S. Kachru, E. Witten, Nucl. Phys. **B407** (1993) 637-666 (hep-th/9307038)

[59] D. Ghoshal, S. Mukhi, Nucl. Phys. **B425** (1994) 173 (hep-th/9312189)

[60] D. Ghoshal, C. Vafa, Nucl. Phys. **B453** (1995) 121 (hep-th/9506122)

[61] J. Wess, J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, 1992

[62] M. Gutperle, Nucl.Phys. **B508** (1997) 133

[63] T. Jayaraman, A. Lawrence, H. Ooguri, T. Sarkar (unpublished)

[64] A. Sen, Mod. Phys. Lett. **A10**, 2081 (1995)

[65] J. M. Maldacena, A. Strominger, Phys. Rev. Lett. **77**, 428 (1996)
C. V. Johnson, R.R. Khuri, R.C. Myers, Phys. Lett. **B378**, 78 (1996)
M. Cvetic, D. Youm, Phys. Rev. **D54**, 2612 (1996)
G. Horowitz, *The Origin of Black Hole entropy in String theory*, (gr-qc/9604051)
A. A. Tseytlin, Nucl. Phys. **B477**, 431 (1996)
N. Hambli, Phys. Rev. **D54**, 5129 (1996)

[66] C. Callan, J. Maldacena, Nucl. Phys. **B475**, (1996) 645

[67] J. Maldacena, L. Susskind, Nucl. Phys. **B475**, (1996) 679

[68] S. R. Das, S. D. Mathur, Phys. Lett. **B375**, (1996) 103

[69] E. Keski-Vakkuri, P. Kraus, Nucl.Phys. **B491** (1997) 249

[70] W. G. Unruh, Phys. Rev. **D 14**, 3251 (1976)

[71] J. Maldacena, A. Strominger, Phys. Rev. **D55** (1997) 861

[72] S. R. Das, G. Gibbons, S. D. Mathur, Phys.Rev.Lett. **78** (1997) 417

[73] G. Horowitz, J. Maldacena, A. Strominger, Phys. Lett. **B383** (1996) 151

[74] S. S. Gubser, I. R. Klebanov, Nucl.Phys. **B482** (1996) 173

[75] P. Kraus, F. Wilczek, Nucl. Phys. **B433** (1994) 665 ; Nucl. phys. **B437** (1994) 231

[76] S. S. Gubser, I. R. Klebanov, Phys. Rev. Lett. **77** (1996) 4491

[77] S. P. de Alwis, K. Sato, Phys.Rev. **D55** (1997) 6181

[78] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253

[79] L. Susskind, E. Witten, *The Holographic Bound in Anti-de Sitter Space* (hep-th/9805114)

[80] W. Fischler, L. Susskind, *Holography and Cosmology* (hep-th/9806039)

[81] M. Maggiore, A. Riotto, *D-branes and Cosmology* (hep-th/9811089)

[82] A. H. Guth, Phys. Rev. **D 23** (1981) 347

[83] A. D. Linde, Phys. Lett. *B 108* (1982) 389
A. Albrecht, P. J. Steinhardt, Phys. Rev. lett. **48** (1982) 1220

[84] D. La, P. J. Steinhardt, Phys. Rev. lett. **62** (1989) 376

[85] A. D. Linde, *Particle Physics and Inflationary Cosmology*, Harwood academic publishers (1990)
E. W. Kolb, M. S. Turner, *The Early Universe*, Addison-Wesley publishing company (1990)

[86] A. D. Linde, *Lectures on Inflationary Cosmology* (hep-th/9410082)

[87] E. J. Weinberg, Phys. Rev. **D 40** (1989) 3950
D. La, P. J. Steinhardt, E. W. Bertschinger, Phys. Lett. *B 231* (1989) 231
See also P. J. Steinhardt, F. Accetta, Phys. Rev. lett. **64** (1990) 2740

[88] E. W. Kolb, D. S. Salopek, M. S. Turner, Phys. Rev. **D 42** (1990) 3925
A. H. Guth, B. Jain, Phys. Rev. **D 45** (1992) 426

[89] P. J. Steinhardt, Class. Quant. Grav. **10** (1993) S33; Int. J. Mod. Phys. **A 10** (1995) 1091.

[90] G. F. Smoot et al, Ap. J. **396** (1992) L1