EIGENVALUE PROBLEMS IN SHELL THEORY

by

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A THESIS IN MATHEMATICS

Submitted to the University of Madras in partial fulfillment of
the requirement for the degree of Doctor of Philosophy

MAY 1999

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CERTIFICATE

This is to certify that the Ph.D. thesis submitted by Sabu N. to the University of Madras, entitled **Eigenvalue Problems in Shell Theory**, is a record of bonafide research work done during the period 1994-1999 under my supervision. The research work presented in this thesis has not formed the basis for the award to the candidate of any Degree, Diploma, Associateship, Fellowship or other similar titles.

It is further certified that the thesis represents independent work by the candidate and collaboration when existed was necessitated by the nature and scope of problems dealt with.

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Abstract

In this thesis, we study the justification of eigenvalue problems for classical lower dimensional models of linear elastic shells and rods. More precisely, it is to show that the eigensolutions of the lower-dimensional problem is the limit, in some suitable topology, of the eigensolutions of the three dimensional problem when the thickness of the shell goes to zero.

In chapter 2, we study the case for shallow shells. The techniques used here for proving the convergence rely on those used by Ciarlet and Miara [18] for the justification of the two-dimensional equations of a linearly elastic shallow shell. Here we consider an eigenvalue problem in three-dimensional elasticity posed over a shell, with thickness ϵ having a specific geometry and clamped on a portion of its lateral surface. By suitable scalings on the domain, eigensolutions etc., we transform this problem into a domain which is independent of ϵ. Our main result then consists of showing that the scaled eigensolutions are bounded and the scaled eigenfunctions converge in some suitable topology to a limit which is the solution of a two-dimensional eigenvalue problem. We also show that all eigensolutions of the two-dimensional problem occur in this way, i.e. each eigensolution of the two-dimensional problem is the limit of a sequence of eigensolutions of the three-dimensional problem as the thickness tends to zero.

In chapter 3, we study the case for rods. The techniques used here for proving the convergence rely on those used by Le Dret [28] for convergence of displacement and stresses for linearly elastic rods. Here we consider an eigenvalue problem in three-dimensional elasticity posed over linearly elastic rod of thickness 2ϵ, clamped on both ends. By suitable scalings on the domain, eigensolutions, etc; we transform this problem into a domain which is independent of ϵ. Our main result then consists of showing that the eigensolutions are bounded and converge in some suitable topology to the eigensolutions of a one dimensional model. We also show that the eigensolutions obtained as the limit of the three-dimensional model problem consist of all the eigenvalues of the one-dimensional problem.

In chapter 4, we study the case of flexural shells. The techniques used here for proving the convergence rely on those used by Ciarlet, Lods and Miara [17] for Asymptotic analysis of flexural shells and Ciarlet and Lods [14] for Asymptotic analysis of membrane shells. Here we consider an eigenvalue problem in three-dimensional elasticity posed over a linear elastic shell of thickness 2ϵ, clamped on a portion of its lateral surface, under a geometric assumption on the middle surface of the shell that the space of inextensional displacements is non-zero. Our main result then consists of showing that if the above mentioned space is infinite dimensional, then for each positive integer l, the eigenvalues are of order $O(\epsilon^2)$ and the corresponding scaled eigensolutions converge in some suitable topology to the eigensolution of the two-dimensional problem for flexural shells. In this case, we also show that the eigensolutions obtained as the limit of the three-dimensional problem consist of all the eigensolutions of the two-dimensional problem. If the space is finite dimensional, say N, then we show that the first N eigenvalues are of order $O(\epsilon^2)$ and the corresponding scaled eigensolutions converge to the N eigensolutions of the two-dimensional flexural shell problem and for $l > N$ the eigenvalues are bounded and that either the corresponding eigensolutions converge to the eigensolutions of the two-dimensional membrane equations or that the eigenfunctions converge to zero weakly in $(H^1(\Omega))^2 \times L^2(\Omega)$.

In chapter 5, we study the case of membrane shells. The techniques used here for proving the convergence rely on those used by Ciarlet and Lods [14]. Here we consider an eigenvalue problem in three dimensional elasticity posed over a linear elastic shell of thickness 2ϵ, clamped along its entire lateral surface. We make an essential geometric assumption on the middle surface of the shell that the shell is "uniformly elliptic". By suitable scalings on the domain, eigensolutions, etc., we transform this problem into a domain which is independent of ϵ. Our main result then consists of showing that the eigenvalues are bounded and that either the corresponding eigensolutions converge to the eigensolutions of the two-dimensional problem for membrane shell or that the eigenfunctions converge to zero weakly in $(H^1(\Omega))^2 \times L^2(\Omega)$. We also show that the eigenvalues of the two dimensional problem obtained as limit of the eigenvalues of the three-dimensional problem lies in a bounded subset of $\mathbb{R}$ whereas the two dimensional problem has a sequence of eigenvalues which is unbounded.

Acknowledgments

I am indebted to my thesis supervisor Prof.S.Kesavan not only for introducing me to the field of shell theory, but also for the patience he has shown in guiding me through these five years. He has always given his full attention and has been of great support in many ways.

I have had the pleasure of having many discussions with Dr. Krishna Maddaly on spectral theory of partial differential equations. I am very grateful to him.

I thank Prof.R.Balasubramanian and Prof.Geetha Srinivasa Rao for agreeing to be part of my monitoring committee and for the advice and encouragement they have given me.

I am thankful to the faculty of the TIFR centre, Bangalore for having invited me to visit it and to Dr.A.K.Nandakumaran for many fruitful discussions.

I thank the organizers of the three schools on Nonlinear Functional Analysis and Applications to Differential Equations held at the ICTP, Trieste, Italy, for inviting me to these schools. It gave me exposure to the latest techniques and frontiers of current research in Partial Differential Equations.

I also thank all members of the Mathematics group at the IMSc and the members of the IMSc hostel for making my stay there a pleasant one. Special thanks to S. S. Rao and S. V. Nagaraj for help in Texing this thesis.

I am grateful to the office and library staff for being helpful at various stages of my work at Matscience.

Finally, I thank all the members of my family for the support and encouragement they have given me all along.

Contents

Chapter 1

Introduction

Elastic bodies like plates, shells, rods etc are three-dimensional bodies. However, often, one or more of their dimensions say, the "thickness", is "small" compared to the others. In such cases lower dimensional theories have been proposed as approximations of the usual three-dimensional theory.

One reason for preferring lower-dimensional theories is their simpler mathematical structure which permits one to obtain a richer variety of results. The other, is that these theories are more amenable to numerical computations.

Most of the lower-dimensional theories proposed by Koiter, Nagdhi and others rely on *a priori* assumptions of a mechanical or geometric nature. Further, it is not evident which is the model most suited to a particular case in hand. The answer to this question is of great importance, for it makes no sense to devise accurate methods of computation for the solution of an inappropriate model.

Consequently, before approximating the exact solution of a given lower-dimensional model, we should first know whether it is "close enough" to the exact solution of the three-dimensional model it is intended to approximate.

Thus one is led to the question of mathematically justifying a lower-dimensional model starting from the three-dimensional model.

One way of doing it is by a formal asymptotic method. In a formal asymptotic method, the three-dimensional solution (the displacement field and in some cases the stress field) is first scaled in an appropriate manner so as to be defined on a fixed domain, then expanded as a formal series expansion in terms of a small parameter ϵ, which is the dimensionless half-thickness of a plate or a shell or the dimensionless half-diameter of the cross-section of the rod.

The formal series expansion of the scaled solution is then inserted into the three-dimensional boundary-value problem, and sufficiently many factors of the successive powers of ϵ found in this fashion are equated to zero until the leading term of the expansion can be computed and, hopefully, identified with the scaled solution of a known lower-dimensional problem. Such a method is "formal" in that the successive terms of the expansion, except the leading one, cannot usually "fully satisfy" the boundary conditions of the three-dimensional problem.

Ciarlet and Destuynder (cf.[9] and [10]) applied this method to the weak, or variational formulation of the boundary-value problem of three-dimensional linearly and non-linearly elastic plates. Without making any *a priori* assumptions, they justified, in this fashion, the linear and non-linear Kirchoff-Love plate theories; only the magnitude of the components of the applied loads and of the Lamé constants must be scaled as appropriate powers of the thickness but, as shown in a systematic way by Miara (cf. [36] and [37]), such scalings are unavoidable. The approach of Ciarlet and Destuynder was then extended to von-Karman plates by Ciarlet [6], to Margurre-von Karman shallow shells by Ciarlet and Paumier [19].

The most noticeable virtue of the asymptotic method applied to the weak formulation of elasticity problems is its amenability to a rigorous asymptotic analysis which shows that the three-dimensional scaled solution converges in some suitable topology to the leading term of the formal asymptotic expansion. Such convergence theorems have been established by Ciarlet and Kesavan [11] for plates, Ciarlet and Miara [18] for linearly elastic shallow shells, Ciarlet and Lods [14] for membrane shells, Ciarlet, Lods and Miara [17] for flexural shells and Le Dret [28] for linearly elastic beams.

Convergence theorems can also be obtained from Γ-convergence theory, as in Bourquin, Ciarlet, Geymonat and Raoult [4] for linear elastic plates. Nonlinear "membrane" models that are invariant and valid for large deformation have also been obtained in this fashion by Le Dret and Raoult (cf. [30], [31] and [32]).

Again, for nonlinearly elastic shells, the formal asymptotic method has been successfully applied by Rao [40] to spherical shells, and to general shells by Miara (cf. [38] and [39]), Lods and Miara [35] who showed that the leading term of the formal asymptotic expansion can be identified with the solution of a non-linear two-dimensional membrane or flexural equation according to specific geometrical or kinematical assumption as in the linear case. A convergence theorem has also been obtained by Le Dret and Raoult (cf. [33] and [34]), who also used Γ- convergence theory to obtain non-linear "membrane" shell models that are invariant and valid for large deformations.

Our purpose is to study the corresponding eigenvalue problems, which are important in the analysis of vibrations of the shell. In particular, starting with the assumptions made for stationary problems and transforming the problems as described above, we wish to obtain the limiting lower-dimensional models for vibrations of shells and compare them with the models obtained from the stationary problem.

In Chapter 2, we study the eigenvalue problem for "shallow shells" and in Chapter 3, we study the eigenvalue problem for "rods". The eigenvalue problem for "flexural shells" is studied in Chapter 4 and, in Chapter 5, we study the eigenvalue problem for "membrane shells".

Chapter 2

Shallow Shells

2.1 Introduction

In recent years, a lot of work has been done on the mathematical justification of various classical lower-dimensional models for the study of thin linearly elastic shells. Ciarlet and Miara [18] have considered the case of the shallow shell while Ciarlet and Lods (cf. [14], [15] and [16]) have studied the membrane shell model, Koiter's model and the generalized membrane shell model. Ciarlet, Lods and Miara [17] have justified the flexural shell model. For a general reference on shells and their models, see Ciarlet [7].

The main idea in the above mentioned works is to pose the problem over a three-dimensional domain of the form $\omega \times (-\varepsilon, \varepsilon)$, where $\omega \subset \mathbb{R}^2$ is a bounded domain and $\varepsilon > 0$ is a parameter representing the thickness of the shell and which eventually tends to zero. The problem is then transformed to one over a domain independent of ε, say $\omega \times (-1, 1)$, and the parameter ε now appears in the various bilinear forms constituting the variational formulation of the problem. The limiting problem obtained as $\varepsilon \to 0$ is then shown to be one of the classical lower-dimensional models. All the problems considered so far are of the stationary type, *viz.* the study of the deformation of the shell under the action of body and/or surface forces.

Our purpose is to study the corresponding eigenvalue problems, which are important in the analysis of vibrations of the shell. In particular, starting with the assumptions made for stationary problems and transforming the problems as described above, we wish to obtain the limiting lower-dimensional models for vibrations of shells and compare them with the models obtained from the stationary problems. In this chapter, we confine our attention to the case of the shallow shell.

Apart from the techniques of the authors cited above, we will also need the methods used by Ciarlet and Kesavan [11] (for the study of eigenvalues of clamped plates).

We now briefly outline the problem studied in this chapter and describe the results obtained.

Let $\omega \subset \mathbb{R}^2$ be a bounded domain and let $\varepsilon > 0$ be a parameter (which eventually tends to zero). Let $\theta^\varepsilon : \omega \to \mathbb{R}$ be a smooth mapping. Then the set of points $P^\varepsilon = (x_1, x_2, \theta^\varepsilon(x_1, x_2))$, for $(x_1, x_2) \in \omega$, constitutes the 'middle surface' of the shell. Let $\boldsymbol{d}^\varepsilon(x_1, x_2)$ denote the normal to this surface at P^ε. Then the reference configuration of the shell is given by $\widehat{\Omega}^\varepsilon = \Theta^\varepsilon(\Omega^\varepsilon)$, where, $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ and for each $x^\varepsilon = (x_1, x_2, x_3^\varepsilon) \in \Omega^\varepsilon$, we have

$$\Theta^\varepsilon(x^\varepsilon) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) + x_3^\varepsilon \boldsymbol{d}^\varepsilon(x_1, x_2). \tag{2.1.1}$$

If $\Gamma^\varepsilon = \partial\omega \times [-\varepsilon, \varepsilon]$, then $\widehat{\Gamma}^\varepsilon = \Theta^\varepsilon(\Gamma^\varepsilon)$ stands for the lateral surface of the shell.

Note that if $\theta^\varepsilon \equiv 0$, then we get a plate.

In the sequel, we assume, for simplicity, that the shell is clamped along its lateral surface. Thus, if $\hat{\boldsymbol{v}}^\varepsilon$ is an admissible displacement vector, then $\hat{\boldsymbol{v}}^\varepsilon = \mathbf{0}$ on $\widehat{\Gamma}^\varepsilon$. For such a displacement vector, we define the linearized strain tensor $\widehat{\boldsymbol{e}}^\varepsilon = (\widehat{e}^\varepsilon_{ij}(\hat{\boldsymbol{v}}^\varepsilon))$ (for $1 \le i, j \le 3$) by

$$\widehat{e}^\varepsilon_{ij}(\hat{\boldsymbol{v}}^\varepsilon) = \frac{1}{2}(\widehat{\partial}^\varepsilon_j \widehat{v}^\varepsilon_i + \widehat{\partial}^\varepsilon_i \widehat{v}^\varepsilon_j). \tag{2.1.2}$$

(Here, and throughout the sequel, $\widehat{\partial}_i^\varepsilon$ will stand for $\partial/\partial\widehat{x}_i^\varepsilon$). The eigenvalue problem consists of finding pairs $(\widehat{\xi}^\varepsilon, \hat{\boldsymbol{u}}^\varepsilon)$, where $\widehat{\xi}^\varepsilon \in \mathbb{R}$ and $\hat{\boldsymbol{u}}^\varepsilon$ is an admissible displacement, such that, for $1 \leq i \leq 3$,

$$\left.\begin{array}{rcl} -\widehat{\partial}_j^\varepsilon(\lambda^\varepsilon \widehat{e}_{pp}^\varepsilon(\widehat{\boldsymbol{u}}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon \widehat{e}_{ij}^\varepsilon(\widehat{\boldsymbol{u}}^\varepsilon)) & = & \widehat{\xi}^\varepsilon \widehat{u}_i^\varepsilon \quad \text{in } \widehat{\Omega}^\varepsilon \\ \widehat{\boldsymbol{u}}^\varepsilon & = & 0 \quad \text{on } \widehat{\Gamma}^\varepsilon, \end{array}\right\} \tag{2.1.3}$$

where λ^ε and μ^ε are positive constants depending only on the material the shell is made of. (The Latin indices take the values 1,2 and 3 and we use the convention of summation over repeated indices).

The problem (2.1.3) can be put in variational form. We define the space of admissible displacements as

$$\widehat{\boldsymbol{V}}^\varepsilon = \{\widehat{\boldsymbol{v}}^\varepsilon \in (H^1(\widehat{\Omega}^\varepsilon))^3 | \widehat{\boldsymbol{v}}^\varepsilon = \boldsymbol{0} \text{ on } \widehat{\Gamma}^\varepsilon\}. \tag{2.1.4}$$

Then, the problem is to find pairs $(\widehat{\xi}^\varepsilon, \widehat{\boldsymbol{u}}^\varepsilon) \in \mathbb{R} \times \widehat{\boldsymbol{V}}^\varepsilon \backslash \{\boldsymbol{0}\}$ such that

$$\int_{\widehat{\Omega}^\varepsilon} (\lambda^\varepsilon \widehat{e}_{pp}^\varepsilon(\widehat{\boldsymbol{u}}^\varepsilon)\widehat{e}_{qq}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon) + 2\mu^\varepsilon \widehat{e}_{ij}^\varepsilon(\widehat{\boldsymbol{u}}^\varepsilon)\widehat{e}_{ij}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon))d\widehat{x}^\varepsilon = \widehat{\xi}^\varepsilon \int_{\widehat{\Omega}^\varepsilon} \widehat{u}_i^\varepsilon \widehat{v}_i^\varepsilon d\widehat{x}^\varepsilon \tag{2.1.5}$$

for every $\widehat{\boldsymbol{v}}^\varepsilon \in \widehat{\boldsymbol{V}}^\varepsilon$. It can be shown that there exists a sequence of eigenpairs $\{(\widehat{\xi}^{\varepsilon,l}, \widehat{\boldsymbol{u}}^{\varepsilon,l})\}_{l=1}^\infty$ such that

$$0 < \widehat{\xi}^{\varepsilon,1} \leq \widehat{\xi}^{\varepsilon,2} \leq \cdots \leq \widehat{\xi}^{\varepsilon,l} \leq \cdots \to \infty \tag{2.1.6}$$

and $\{\widehat{\boldsymbol{u}}^{\varepsilon,l}\}$ is a complete orthonormal basis for $(L^2(\widehat{\Omega}^\varepsilon))^3$.

We then transform (2.1.5) into an equivalent problem posed over $\omega \times (-1, 1)$ after suitable transformation and rescaling of the variables $\widehat{\xi}^\varepsilon, \widehat{\boldsymbol{u}}^\varepsilon$, the Lamé constants $\lambda^\varepsilon, \mu^\varepsilon$ and the function θ^ε.

In this fashion, we obtain scaled eigenpairs $(\xi^l(\varepsilon), \boldsymbol{u}^l(\varepsilon)) \in \mathbb{R} \times \boldsymbol{V}$, where $\Omega = \omega \times (-1, 1)$, $\Gamma = \partial\omega \times [-1, 1]$ and

$$\boldsymbol{V} = \{\boldsymbol{v} \in (H^1(\Omega))^3 | \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\}, \tag{2.1.7}$$

which satisfy variational equations in which ε occurs as a parameter. We show that, as $\varepsilon \to 0$, we have $\xi^l(\varepsilon) \to \xi^l$ and $\boldsymbol{u}^l(\varepsilon) \to \boldsymbol{u}^l$ (in $(H^1(\Omega))^3$) for each l for a suitable subsequence. We also show that $u_3^l = \zeta^l$ is a function independent of x_3. In fact $\zeta^l \in H_0^2(\omega)$. The other components are given by

$$u_\alpha^l = \zeta_\alpha^l - x_3 \partial_\alpha u_3^l, \text{ for } \alpha = 1, 2. \tag{2.1.8}$$

The $\zeta_\alpha^l \in H_o^1(\omega)$ and can be uniquely determined in terms of ζ^l. The pair (ξ^l, ζ^l) is an eigenpair for a fourth order elliptic problem posed over ω. We also show that ξ^l is the l-th eigenvalue of this problem for each positive integer l, *i.e.* there are no other eigenvalues for the limit problem, and that the $\{\zeta^l\}$ forms a basis for $L^2(\omega)$. These are shown using the methods of Ciarlet and Kesavan [11].

There is an important difference between the two-dimensional shell model obtained by Ciarlet and Miara [18] and the eigenvalue problem studied in this chapter. The former is a system of coupled fourth-order equations involving all the components of the limit of $\boldsymbol{u}(\varepsilon)$; the latter involves only the third component. Thus, for the eigenvalue problem of shallow shells, the limiting situation is similar to that of plates. We will comment about this in greater detail later (cf. Remark 2.5.3 below).See also Remarks 2.5.2 and 2.7.1 for a discussion of some other minor differences in the model obtained.

This chapter is organized as follows. In Section 2 below, we describe the three-dimensional problem and in Section 3, we transform the problem to one over a fixed domain. In Section 4, we obtain the necessary *a priori* estimates needed to pass to the limit, which is done in Section 5. In Section 6, we study the limiting eigenvalue problem. In Section 7, we translate the results back to the original setting by descaling the variables. Section 8 is reserved for concluding remarks.

2.2 The Three - Dimensional Problem

Throughout the sequel, the Latin indices will vary over the set $\{1, 2, 3\}$ and Greek indices over the set $\{1, 2\}$ for the components of vectors and tensors. The convention of summation over repeated indices will be used in conjunction with the above rules.

Let $\omega \subset \mathbb{R}^2$ be a bounded domain. We assume that the boundary $\partial\omega$ is Lipschitz continuous and that ω lies locally on one side of $\partial\omega$. For each $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)\ ,\ \Gamma_+^\varepsilon = \omega \times \{\varepsilon\}\ ,\ \Gamma_-^\varepsilon = \omega \times \{-\varepsilon\}. \tag{2.2.1}$$

Let $x^\varepsilon = (x_1, x_2, x_3^\varepsilon)$ be a generic point in $\overline{\Omega^\varepsilon}$ and let $\partial_\alpha = \partial_\alpha^\varepsilon = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\varepsilon = \frac{\partial}{\partial x_3^\varepsilon}$.

We assume that, for each $\varepsilon > 0$, we are given a function $\theta^\varepsilon \in \mathcal{C}^3(\omega)$. The middle surface of the shell is then given by

$$\{(x_1, x_2, \theta^\varepsilon(x_1, x_2)) | (x_1, x_2) \in \omega\}.$$

At each point of the middle surface, the normal lies in the direction

$$(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1)$$

and we denote the unit normal vector obtained by normalizing this by $\boldsymbol{d}^\varepsilon(x_1, x_2)$. Then, we define the mapping $\Theta^\varepsilon : \overline{\Omega^\varepsilon} \to I\!R^3$ by

$$\Theta^\varepsilon(x^\varepsilon) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) + x_3^\varepsilon \boldsymbol{d}^\varepsilon(x_1, x_2). \tag{2.2.2}$$

We assume that $\Theta^\varepsilon : \Omega^\varepsilon \to \Theta^\varepsilon(\Omega^\varepsilon)$ is a $\mathcal{C}^1$- diffeomorphism. The set $\widehat{\Omega}^\varepsilon = \Theta^\varepsilon(\Omega^\varepsilon)$ is the reference configuration of the shell and we denote a generic point of this set by $\widehat{x}^\varepsilon$.

If $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$ are the Lamé constants associated with the material the shell is made of, then the eigenvalue problem describing the vibrations of the shell takes the variational form given by (2.1.5).

The $\widehat{\boldsymbol{V}}^\varepsilon$ - ellipticity of the bilinear form appearing on the left-hand side of (2.1.5) follows from Körn's inequality. Hence for each $\widehat{\boldsymbol{f}}^\varepsilon \in (L^2(\widehat{\Omega}^\varepsilon))^3$, there exists a unique $\widehat{\boldsymbol{w}}^\varepsilon \in \widehat{\boldsymbol{V}}^\varepsilon$ satisfying

$$\int_{\widehat{\Omega}^\varepsilon} (\lambda^\varepsilon \widehat{e}_{pp}^\varepsilon(\widehat{\boldsymbol{w}}^\varepsilon)\widehat{e}_{qq}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon) + 2\mu^\varepsilon \widehat{e}_{ij}^\varepsilon(\widehat{\boldsymbol{w}}^\varepsilon)\widehat{e}_{ij}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon))d\widehat{x}^\varepsilon = \int_{\widehat{\Omega}^\varepsilon} \widehat{f}_i^\varepsilon \widehat{v}_i^\varepsilon d\widehat{x}^\varepsilon, \tag{2.2.3}$$

for every $\widehat{\boldsymbol{v}}^\varepsilon \in \widehat{\boldsymbol{V}}^\varepsilon$. We denote $\widehat{\boldsymbol{w}}^\varepsilon = \widehat{G}^\varepsilon(\widehat{\boldsymbol{f}}^\varepsilon)$ and thus $\widehat{G}^\varepsilon : (L^2(\widehat{\Omega}^\varepsilon))^3 \to \widehat{\boldsymbol{V}}^\varepsilon$ defines a bounded linear operator. Since the inclusion $\widehat{\boldsymbol{V}}^\varepsilon \hookrightarrow (L^2(\widehat{\Omega}^\varepsilon))^3$ is compact, we can consider $\widehat{G}^\varepsilon$ as a compact linear operator of $(L^2(\widehat{\Omega}^\varepsilon))^3$ into itself. It also follows (from the symmetry of the associated bilinear form) that $\widehat{G}^\varepsilon$ is self-adjoint. Thus, the problem (2.1.5) then reduces to finding $\widehat{\boldsymbol{u}}^\varepsilon \in \widehat{\boldsymbol{V}}^\varepsilon \backslash \{\boldsymbol{0}\}$ and $\widehat{\xi}^\varepsilon \in I\!R$ such that

$$\widehat{\boldsymbol{u}}^\varepsilon = \widehat{\xi}^\varepsilon \widehat{G}^\varepsilon(\widehat{\boldsymbol{u}}^\varepsilon). \tag{2.2.4}$$

From the spectral theory of compact, self-adjoint linear operators, it follows that there exist eigenpairs $\{(\widehat{\xi}^{\varepsilon,l}, \widehat{\boldsymbol{u}}^{\varepsilon,l})\}_{l=1}^\infty$ with the properties announced in the previous section. The eigenvectors can be normalized in any way and, in view of the transformations we wish to effect in the sequel, we choose them such that

$$\int_{\widehat{\Omega}^\varepsilon} \widehat{u}_i^{\varepsilon,l} \widehat{u}_i^{\varepsilon,m} \, d\widehat{x}^\varepsilon = \varepsilon^3 \delta_{lm}. \tag{2.2.5}$$

The eigenvalues $\{\widehat{\xi}^{\varepsilon,l}\}$ can be characterized intrinsically via the min-max principle for the corresponding Rayleigh quotient which is given by

$$\widehat{R}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon) = \frac{\int_{\widehat{\Omega}^\varepsilon} (\lambda^\varepsilon \widehat{e}_{pp}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon) \widehat{e}_{qq}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon) + 2\mu^\varepsilon \widehat{e}_{ij}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon) \widehat{e}_{ij}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon)) d\widehat{x}^\varepsilon}{\int_{\widehat{\Omega}^\varepsilon} \widehat{v}_i^\varepsilon \widehat{v}_i^\varepsilon \, d\widehat{x}^\varepsilon} \tag{2.2.6}$$

for $\widehat{\boldsymbol{v}}^\varepsilon \in \widehat{\boldsymbol{V}}^\varepsilon \backslash \{\boldsymbol{0}\}$. We have

$$\widehat{\xi}^{\varepsilon,l} = \min_{\boldsymbol{W} \in \widehat{\mathcal{V}}^\varepsilon{}_l} \max_{\widehat{\boldsymbol{v}}^\varepsilon \in \boldsymbol{W}} \widehat{R}^\varepsilon(\widehat{\boldsymbol{v}}^\varepsilon) \tag{2.2.7}$$

where $\widehat{\mathcal{V}}^\varepsilon{}_l$ denotes the collection of all l-dimensional subspaces of $\widehat{\boldsymbol{V}}^\varepsilon$.

2.3 Transformation of the Problem

Since the mappings $\Theta^\varepsilon : \overline{\Omega^\varepsilon} \to \overline{\widehat{\Omega}^\varepsilon}$ are assumed to be $\mathcal{C}^1$-diffeomorphisms, the correspondence that associates with every element $\widehat{\boldsymbol{v}}^\varepsilon \in \widehat{\boldsymbol{V}}^\varepsilon$ the vector

$$\boldsymbol{v}^\varepsilon = \widehat{\boldsymbol{v}}^\varepsilon \circ \Theta^\varepsilon : \Omega^\varepsilon \to I\!R^3 \tag{2.3.1}$$

induces a bijection between the spaces $\widehat{\boldsymbol{V}}^\varepsilon$ and $\boldsymbol{V}^\varepsilon$, where

$$\boldsymbol{V}^\varepsilon = \{\boldsymbol{v}^\varepsilon \in (H^1(\Omega^\varepsilon))^3 | \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma^\varepsilon\}. \tag{2.3.2}$$

For each $\varepsilon > 0$ and $x^\varepsilon \in \Omega^\varepsilon$, consider the Jacobian matrix $(\partial_i^\varepsilon \Theta_j^\varepsilon(x^\varepsilon))$. Let $(b_{ij}^\varepsilon(x^\varepsilon))$ denote the inverse of the Jacobian and let $\delta^\varepsilon(x^\varepsilon)$ denote the determinant of the Jacobian. We assume, henceforth, that all the mappings Θ^ε are orientation preserving, *i.e.*

$$\delta^\varepsilon(x^\varepsilon) > 0 \text{ for all } x^\varepsilon \in \Omega^\varepsilon. \tag{2.3.3}$$

By the chain rule, we have

$$\widehat{\partial}_j^\varepsilon \widehat{v}_i^\varepsilon(\widehat{x}^\varepsilon) = b_{kj}^\varepsilon \partial_k^\varepsilon v_i^\varepsilon(x^\varepsilon). \tag{2.3.4}$$

Then we can transform (2.1.5) into a problem over the domain Ω^ε. From (2.3.4) and the change of variable formula for integrals, it follows that if $(\widehat{\xi}^\varepsilon, \widehat{\boldsymbol{u}}^\varepsilon) \in I\!R \times \widehat{\boldsymbol{V}}^\varepsilon \backslash \{\mathbf{0}\}$ is a solution of (2.1.5), then $(\widehat{\xi}^\varepsilon, \boldsymbol{u}^\varepsilon) \in I\!R \times \boldsymbol{V}^\varepsilon \backslash \{\mathbf{0}\}$ satisfies

$$N^\varepsilon(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon) = \widehat{\xi}^\varepsilon \int_{\Omega^\varepsilon} u_i^\varepsilon v_i^\varepsilon \delta^\varepsilon dx^\varepsilon \tag{2.3.5}$$

for all $\boldsymbol{v}^\varepsilon \in \boldsymbol{V}^\varepsilon$, where $\boldsymbol{u}^\varepsilon = \widehat{\boldsymbol{u}}^\varepsilon \circ \Theta^\varepsilon$ and

$$N^\varepsilon(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon) = \int_{\Omega^\varepsilon} [\lambda^\varepsilon b_{lp}^\varepsilon \partial_l^\varepsilon u_p^\varepsilon \delta_{ij} + \mu^\varepsilon (b_{mj}^\varepsilon \partial_m^\varepsilon u_i^\varepsilon + b_{ni}^\varepsilon \partial_n^\varepsilon u_j^\varepsilon)] b_{kj}^\varepsilon \partial_k^\varepsilon v_i^\varepsilon \delta^\varepsilon dx^\varepsilon. \tag{2.3.6}$$

We now set $\Omega = \omega \times (-1,1), \Gamma = \partial\omega \times [-1,1]$ and with each point $x \in \overline{\Omega}$, we associate the point $x^\varepsilon \in \overline{\Omega^\varepsilon}$ through the bijection

$$\pi^\varepsilon : x = (x_i) \in \overline{\Omega} \mapsto x^\varepsilon = (x_i^\varepsilon) = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega^\varepsilon}.$$

Given $\boldsymbol{v}^\varepsilon \in \boldsymbol{V}^\varepsilon$, we associate the *scaled* functions $\boldsymbol{v}(\varepsilon) \in \boldsymbol{V}$, where

$$\boldsymbol{V} = \{\boldsymbol{v} \in (H^1(\Omega))^3 | \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma\}, \tag{2.3.7}$$

via the relations

$$\left.\begin{aligned} v_\alpha^\varepsilon(x^\varepsilon) &= \varepsilon^2 v_\alpha(\varepsilon)(x) \\ v_3^\varepsilon &= \varepsilon v_3(\varepsilon)(x) \end{aligned}\right\} \tag{2.3.8}$$

for all $x^\varepsilon = \pi^\varepsilon(x)$.

We also scale the eigenvalues as follows:

$$\widehat{\xi}^{\varepsilon} = \varepsilon^2 \xi(\varepsilon). \tag{2.3.9}$$

Finally, we make the following assumptions on the data: we assume that there exist constants $\lambda > 0, \mu > 0$ and a function $\theta \in \mathcal{C}^3(\overline{\omega})$, all independent of ε, such that

$$\lambda^{\varepsilon} = \lambda > 0, \;\; \mu^{\varepsilon} = \mu > 0 \tag{2.3.10}$$

and

$$\theta^{\varepsilon}(x_1, x_2) = \varepsilon\theta(x_1, x_2) \text{ for all } (x_1, x_2) \in \overline{\omega}. \tag{2.3.11}$$

Remark 2.3.1: The assumptions (2.3.10) are justified if we assume that the shells (for each $\varepsilon > 0$) are all made up of the same material. Thus the Lamé constants would not depend on the thickness (ε) of the shell. The assumption (2.3.11) is the crucial shallow shell assumption of Ciarlet and Miara [18]. ∎

If $(e_{ij}(\boldsymbol{v}))$ is the standard strain tensor of $\boldsymbol{v} \in \boldsymbol{V}$, *i.e.*

$$e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i), \tag{2.3.12}$$

we define the tensor $(e^{\theta}_{ij}(\boldsymbol{v}))$ as follows:

$$\left.\begin{array}{rcl} e^{\theta}_{\alpha\beta}(\boldsymbol{v}) & = & e_{\alpha\beta}(\boldsymbol{v}) - \frac{1}{2}(\partial_\beta\theta\partial_3 v_\alpha + \partial_\alpha\theta\partial_3 v_\beta), \\ e^{\theta}_{\alpha 3}(\boldsymbol{v}) & = & e^{\theta}_{3\alpha}(\boldsymbol{v}) = e_{\alpha 3}(\boldsymbol{v}) - \frac{1}{2}\partial_\alpha\theta\partial_3 v_3, \\ e^{\theta}_{33}(\boldsymbol{v}) & = & e_{33}(\boldsymbol{v}). \end{array}\right\} \tag{2.3.13}$$

In the following three Lemmas we state various results that will be needed in the proof of convergence.

Lemma 2.3.1 *Let the function θ^ε be of the form (2.3.11), with $\theta \in C^3(\overline{\omega})$. Then there exists $\varepsilon_0 = \varepsilon_0(\theta) > 0$ such that the Jacobian matrix $(\partial_i^\varepsilon \Theta_j^\varepsilon(x^\varepsilon))$ is invertible for all $x^\varepsilon \in (\overline{\Omega^\varepsilon})$ and all $\varepsilon \leq \varepsilon_0$. For $\varepsilon \leq \varepsilon_0$, let the functions $b_{ij}(\varepsilon) : \overline{\Omega} \to I\!R$ and $\delta(\varepsilon) : \overline{\Omega} \to I\!R$ be defined by*

$$b_{ij}(\varepsilon)(x) = b^\varepsilon{}_{ij}(x^\varepsilon) \tag{2.3.14}$$

$$\delta(\varepsilon)(x) = \delta^\varepsilon(x^\varepsilon) \tag{2.3.15}$$

where the functions $b^\varepsilon{}_{ij} : (\overline{\Omega^\varepsilon}) \to I\!R$ and $\delta^\varepsilon : (\overline{\Omega^\varepsilon}) \to I\!R$ denotes the inverse and determinant of the Jacobian. Then

$$b_{\alpha\beta}(\varepsilon) = \delta_{\alpha\beta} + \varepsilon^2 b^{\#}_{\alpha\beta}(\varepsilon, \theta) \tag{2.3.16}$$

$$b_{\alpha 3}(\varepsilon) = \varepsilon \partial_\alpha \theta + \varepsilon^2 b^{\#}_{\alpha 3}(\varepsilon, \theta) \tag{2.3.17}$$

$$b_{3\beta}(\varepsilon) = -\varepsilon \partial_\beta \theta + \varepsilon^2 b^{\#}_{3\beta}(\varepsilon, \theta) \tag{2.3.18}$$

$$b_{33}(\varepsilon) = 1 + \varepsilon^2 b^{\#}_{33}(\varepsilon, \theta) \tag{2.3.19}$$

$$\delta(\varepsilon) = 1 + \varepsilon^2 \delta^{\#}(\varepsilon, \theta) \tag{2.3.20}$$

and there exists a constant $C_0(\theta)$ such that

$$\sup_{0<\varepsilon\leq\varepsilon_0} \max_{i,j} \max_{x\in\overline{\Omega}} |b^{\#}_{ij}(\varepsilon,\theta)(x)| \leq C_0(\theta) \tag{2.3.21}$$

$$\sup_{0<\varepsilon\leq\varepsilon_0} \max_{x\in\overline{\Omega}} |\delta^{\#}(\varepsilon,\theta)(x)| \leq C_0(\theta) \tag{2.3.22}$$

Proof. See the proof of Lemma 1 in Ciarlet and Miara [18]. ∎

In what follows, $|.|_{0,\Omega}$ and $||.||_{1,\Omega}$ denote the $L^2(\Omega)$ norm and $H^1(\Omega)$ norm, respectively, for both the scalar and vector-valued functions.

Lemma 2.3.2 *Let the assumptions and notations be as in Lemma 2.3.1, and let the functions $\hat{v}_i^\varepsilon : \overline{\Omega}^\varepsilon \to I\!R$ be related to the scaled functions $v_i : \overline{\Omega} \to I\!R$ through the bijections (2.3.1) and (2.3.8). Then*

$$\widehat{e}^{\varepsilon}{}_{\alpha\beta}(\widehat{\boldsymbol{v}}^{\varepsilon})(\widehat{x}^{\varepsilon}) = \varepsilon^2\{e^{\theta}_{\alpha\beta}(\boldsymbol{v}) + \varepsilon^2 e^{\#}_{\alpha\beta}(\varepsilon,\theta;\boldsymbol{v})\}(x) \tag{2.3.23}$$

$$\widehat{e}^{\varepsilon}{}_{\alpha 3}(\widehat{\boldsymbol{v}}^{\varepsilon})(\widehat{x}^{\varepsilon}) = \varepsilon\{e^{\theta}_{\alpha 3}(\boldsymbol{v}) + \varepsilon^2 e^{\#}_{\alpha 3}(\varepsilon,\theta;\boldsymbol{v})\}(x) \tag{2.3.24}$$

$$\widehat{e}^{\varepsilon}{}_{3\alpha}(\widehat{\boldsymbol{v}}^{\varepsilon})(\widehat{x}^{\varepsilon}) = \varepsilon\{e^{\theta}_{3\alpha}(\boldsymbol{v}) + \varepsilon^2 e^{\#}_{3\alpha}(\epsilon,\theta;\boldsymbol{v})\}(x) \tag{2.3.25}$$

$$\widehat{e}^{\varepsilon}{}_{33}(\widehat{\boldsymbol{v}}^{\varepsilon})(\widehat{x}^{\varepsilon}) = \{e^{\theta}_{33}(\boldsymbol{v}) + \varepsilon^2 e^{\#}_{33}(\varepsilon,\theta;\boldsymbol{v})\}(x) \tag{2.3.26}$$

$$= \{e^{\theta}_{33}(\boldsymbol{v}) + \varepsilon^2(\partial_\alpha\theta\partial_\alpha v_3 + b^{\#}_{33}(\varepsilon,\theta)\partial_3 v_3) + \varepsilon^4 e^{b}_{33}(\varepsilon,\theta;\boldsymbol{v})\}(x). \tag{2.3.27}$$

and there exists a constant $C_1(\theta)$ *such that*

$$\sup_{0<\varepsilon\le\varepsilon_0}\max_{i,j}|e^{\#}_{ij}(\varepsilon,\theta;\boldsymbol{v})|_{0,\Omega} + \sup_{0<\varepsilon\le\varepsilon_0}|e^{b}_{33}(\varepsilon,\theta;\boldsymbol{v})|_{0,\Omega} \le C_1(\theta)||\boldsymbol{v}||_{1,\Omega} \tag{2.3.28}$$

Proof. See the proof of Lemma 2 in Ciarlet and Miara [18]. ∎

Lemma 2.3.3 *Let* $\theta \in C^3(\overline{\omega})$ *be a given function, and let the functions* $e^{\theta}_{\alpha\beta}(\boldsymbol{v})$ *be defined as in (2.3.13). Then the mapping*

$$v \to \left\{\sum_{i,j}|e^{\theta}_{ij}(\boldsymbol{v})|^2_{0,\Omega}\right\}^{\frac{1}{2}} \tag{2.3.29}$$

is a norm over the space $\boldsymbol{V}$ *which is equivalent to the norm* $||.||_{1,\Omega}$.

ie, there exists constants $C_2(\theta)$ *and* $C_3(\theta)$ *such that*

$$||\boldsymbol{v}||_{1,\Omega} \le C_2(\theta)\left\{\sum_{ij}|e^{\theta}_{ij}(\boldsymbol{v})|^2_{0,\Omega}\right\}^{\frac{1}{2}} \le C_3(\theta)||\boldsymbol{v}||_{1,\Omega}. \tag{2.3.30}$$

Proof. See the proof of Lemma 3 in Ciarlet and Miara [18]. ∎

Before we present the transformed problem, we need to introduce some more notation. For each $\boldsymbol{v} \in \boldsymbol{V}$, we define the symmetric tensor $\boldsymbol{K}^{\theta}(\varepsilon)(\boldsymbol{v}) = (K^{\theta}_{ij}(\varepsilon)(\boldsymbol{v})) \in (L^2(\Omega))^9$ by means of the following relations:

$$\left.\begin{aligned} K^{\theta}_{\alpha\beta}(\varepsilon)(\boldsymbol{v}) &= e^{\theta}_{\alpha\beta}(\boldsymbol{v}), \\ K^{\theta}_{\alpha 3}(\varepsilon)(\boldsymbol{v}) &= \varepsilon^{-1}e^{\theta}_{\alpha 3}(\boldsymbol{v}), \\ K^{\theta}_{33}(\varepsilon)(\boldsymbol{v}) &= \varepsilon^{-2}e^{\theta}_{33}(\boldsymbol{v}) + \partial_\alpha\theta\partial_\alpha v_3. \end{aligned}\right\} \tag{2.3.31}$$

We introduce the bilinear forms $N(\varepsilon)$ and $D(\varepsilon)$ on $\boldsymbol{V} \times \boldsymbol{V}$ by

$$\left.\begin{aligned} N(\varepsilon)(\boldsymbol{u},\boldsymbol{v}) \;=\; & \\ & \int_\Omega [\lambda K^\theta_{pp}(\varepsilon)(\boldsymbol{u})\delta_{\alpha\beta} + 2\mu K^\theta_{\alpha\beta}(\varepsilon)(\boldsymbol{u})][\partial_\alpha v_\beta - \frac{1}{2}(\partial_\beta\theta\partial_3 v_\alpha + \partial_\alpha\theta\partial_3 v_\beta)]dx \\ & + \int_\Omega [\lambda K^\theta_{\alpha\alpha}(\varepsilon)(\boldsymbol{u})(\partial_\beta\theta\partial_\beta v_3 + b^\#_{33}(\varepsilon,\theta)\partial_3 v_3) + \lambda e^\#_{\alpha\alpha}(\varepsilon,\theta,\boldsymbol{u})\partial_3 v_3]dx \\ & + \int_\Omega 2\mu e^\#_{\alpha3}(\varepsilon,\theta,\boldsymbol{u})[\partial_3 v_\alpha + \partial_\alpha v_3 - \partial_\alpha\theta\partial_3 v_3]dx \\ & + \int_\Omega (\lambda+2\mu)e^\flat_{33}(\varepsilon,\theta,\boldsymbol{u})\partial_3 v_3 dx \\ & + \int_\Omega (\lambda+2\mu)K^\theta_{33}(\varepsilon)(\boldsymbol{u})(\partial_\beta\theta\partial_\beta v_3 + b^\#_{33}(\varepsilon,\theta)\partial_3 v_3)dx \\ & + \varepsilon^{-1}\int_\Omega 2\mu K^\theta_{\alpha3}(\varepsilon)(\boldsymbol{u})(\partial_3 v_\alpha + \partial_\alpha v_3 - \partial_\alpha\theta\partial_3 v_3)dx \\ & + \varepsilon^{-2}\int_\Omega [\lambda K^\theta_{\beta\beta}(\varepsilon)(\boldsymbol{u}) + (\lambda+2\mu)K^\theta_{33}(\varepsilon)(\boldsymbol{u})]\partial_3 v_3(1+\varepsilon^2\delta^\#(\varepsilon,\theta))dx \\ & + \varepsilon B^\#_1(\varepsilon,\theta,\boldsymbol{K}^\theta(\varepsilon)(\boldsymbol{u}),\boldsymbol{v}) + \varepsilon^2 B^\#_2(\varepsilon,\theta,\boldsymbol{u},\boldsymbol{v}) \end{aligned}\right\} \tag{2.3.32}$$

and

$$D(\varepsilon)(\boldsymbol{u},\boldsymbol{v}) = \varepsilon^2\int_\Omega u_\alpha v_\alpha \delta(\varepsilon)dx + \int_\Omega u_3 v_3 \delta(\varepsilon)dx. \tag{2.3.33}$$

If $(\widehat{\xi}^\varepsilon, \boldsymbol{u}^\varepsilon) \in I\!R \times \boldsymbol{V}^\varepsilon\backslash\{\boldsymbol{0}\}$ satisfies (2.3.5), then the pair $(\xi(\varepsilon),\boldsymbol{u}(\varepsilon)) \in I\!R \times \boldsymbol{V}\backslash\{\boldsymbol{0}\}$ obtained via the transformations (2.3.8) - (2.3.9) satisfies the following equation:

$$N(\varepsilon)(\boldsymbol{u}(\varepsilon),\boldsymbol{v}) = \xi(\varepsilon)D(\varepsilon)(\boldsymbol{u}(\varepsilon),\boldsymbol{v}) \tag{2.3.34}$$

for every $\boldsymbol{v} \in \boldsymbol{V}$, where

$$\left.\begin{aligned} \sup_{0<\varepsilon\le\varepsilon_0} |B^\#_1(\varepsilon,\theta,\boldsymbol{K},\boldsymbol{v})| &\le C_4(\theta)|\boldsymbol{K}|_{0,\Omega}\|\boldsymbol{v}\|_{1,\Omega} \\ \sup_{0<\varepsilon\le\varepsilon_0} |B^\#_2(\varepsilon,\theta,\boldsymbol{u},\boldsymbol{v})| &\le C_4(\theta)|\boldsymbol{u}|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} \end{aligned}\right\} \tag{2.3.35}$$

for all $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}$ and $\boldsymbol{K} \in (L^2(\Omega))^9$. The constant $C_4(\theta) > 0$ is independent of ε. The quantity $|\boldsymbol{u}|_{1,\Omega}$ stands for the usual Sobolev semi-norm of $\boldsymbol{u}$, *i.e.* the $L^2(\Omega)$-norm of the gradient of $\boldsymbol{u}$.

Thus if $\{(\tilde{\xi}^{\varepsilon,l}, \hat{\boldsymbol{u}}^{\varepsilon,l})\}_{l=1}^{\infty}$ are the eigensolutions of (2.1.5) normalized as in (2.2.5), then the transformed elements $\{(\xi^l(\varepsilon), \boldsymbol{u}^l(\varepsilon))\}_{l=1}^{\infty}$, transformed as in (2.3.8)-(2.3.9), verify (2.3.34) and are normalized by the condition

$$\varepsilon^2 \int_\Omega u_\alpha^l(\varepsilon) u_\alpha^m(\varepsilon)\delta(\varepsilon)dx + \int_\Omega u_3^l(\varepsilon)u_3^m(\varepsilon)\delta(\varepsilon)dx = \delta_{lm}. \tag{2.3.36}$$

Our aim is to pass to the limit in (2.3.34). To this end, we first need to obtain *a priori* estimates for $\{\xi^l(\varepsilon)\}$, which we do in the next section.

2.4 *A Priori* Estimates

In this section we show that, for each positive integer l, the scaled eigenvalues $\{\xi^l(\varepsilon)\}$ are bounded uniformly with respect to ε. To this end, we consider some special classes of elements of $\boldsymbol{V}$.

Let $\varphi \in H_o^2(\omega)$. Then

$$\boldsymbol{v}_\varphi := (-x_3\partial_1\varphi, -x_3\partial_2\varphi, \varphi) \in \boldsymbol{V}. \tag{2.4.1}$$

Then it is immediate to see that (cf. (2.3.13))

$$e_{\alpha\beta}^\theta(\boldsymbol{v}_\varphi) = -x_3\partial_{\alpha\beta}\varphi + \frac{1}{2}(\partial_\beta\theta\partial_\alpha\varphi + \partial_\alpha\theta\partial_\beta\varphi). \tag{2.4.2}$$

We now compute the expression for $N(\varepsilon)(\boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi)$. First of all, observe that $\partial_3(\boldsymbol{v}_\varphi)_3 = 0$ and that $(\partial_\alpha(\boldsymbol{v}_\varphi)_3 + \partial_3(\boldsymbol{v}_\varphi)_\alpha) = 0$. Thus, $e_{\alpha 3}^\theta(\boldsymbol{v}_\varphi) = e_{33}^\theta(\boldsymbol{v}_\varphi) = 0$. Further (cf. (2.3.31)),

$$K_{\alpha\beta}^\theta(\varepsilon)(\boldsymbol{v}_\varphi) = e_{\alpha\beta}^\theta(\boldsymbol{v}_\varphi), K_{\alpha 3}^\theta(\varepsilon)(\boldsymbol{v}_\varphi) = e_{\alpha 3}^\theta(\boldsymbol{v}_\varphi) = 0$$

and

$$K_{33}^\theta(\varepsilon)(\boldsymbol{v}_\varphi) = \partial_\alpha\theta\partial_\alpha\varphi.$$

Thus, we get

$$\left.\begin{aligned} N(\varepsilon)(\boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi) \;=\; & \\ & \lambda \int_\Omega [-x_3 \Delta\varphi + \partial_\gamma\theta\partial_\gamma\varphi]^2 dx \\ & +2\mu \int_\Omega \sum_{\alpha,\beta} [-x_3\partial_{\alpha\beta}\varphi + \frac{1}{2}(\partial_\beta\theta\partial_\alpha\varphi + \partial_\alpha\theta\partial_\beta\varphi)]^2 dx \\ & +\lambda \int_\Omega (-x_3\Delta\varphi + \partial_\alpha\theta\partial_\alpha\varphi)\partial_\beta\theta\partial_\beta\varphi dx \\ & +(\lambda + 2\mu) \int_\Omega (\partial_\alpha\theta\partial_\alpha\varphi)^2 dx \\ & +\varepsilon B_1^{\#}(\varepsilon, \theta, \boldsymbol{K}^\theta(\varepsilon)(\boldsymbol{v}_\varphi), \boldsymbol{v}_\varphi) + \varepsilon^2 B_2^{\#}(\varepsilon, \theta, \boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi). \end{aligned}\right\} \quad (2.4.3)$$

Before we prove our main result we need the following preliminary estimates.

Lemma 2.4.1 *There exists a constant $C > 0$, depending only on θ such that*

$$\left.\begin{aligned} |B_1^{\#}(\varepsilon, \theta, \boldsymbol{K}^\theta(\varepsilon)(\boldsymbol{v}_\varphi), \boldsymbol{v}_\varphi)| &\leq C \textstyle\int_\omega |\Delta\varphi|^2 d\omega \\ |B_2^{\#}(\varepsilon, \theta, \boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi)| &\leq C \textstyle\int_\omega |\Delta\varphi|^2 d\omega. \end{aligned}\right\} \quad (2.4.4)$$

Proof. By virtue of our preceding computations, it is immediate to see that

$$|\boldsymbol{K}^\theta(\varepsilon)(\boldsymbol{v}_\varphi)|_{0,\Omega} \leq C(\theta)\|\varphi\|_{2,\omega}.$$

Similarly,

$$\|\boldsymbol{v}_\varphi\|_{1,\Omega} \leq C\|\varphi\|_{2,\omega}.$$

Since $\varphi \in H_o^2(\omega)$, by Poincaré's inequality,

$$\|\varphi\|_{2,\omega}^2 \leq C|\varphi|_{2,\omega}^2 = C\int_\omega |\Delta\varphi|^2 d\omega.$$

The result now follows from (2.3.35). ∎

Theorem 2.4.1 *For each positive integer l, there exists a constant $K(l) > 0$ (depending also on θ, but independent of ε) such that*

$$\xi^l(\varepsilon) \leq K(l). \quad (2.4.5)$$

Proof. Since the problem (2.3.34) was derived from (2.3.5) after a change of variable and change of scale, we still have the variational characterization of the scaled eigenvalues $\xi^l(\varepsilon)$. Let $\mathcal{V}_l$ denote the collection of all l-dimensional subspaces of $\boldsymbol{V}$. then

$$\xi^l(\varepsilon) = \min_{\boldsymbol{W} \in \mathcal{V}_l} \max_{\boldsymbol{v} \in \boldsymbol{W}} \frac{N(\varepsilon)(\boldsymbol{v}, \boldsymbol{v})}{D(\varepsilon)(\boldsymbol{v}, \boldsymbol{v})}. \tag{2.4.6}$$

Let $\mathcal{W}_l$ denote the collection of all l-dimensional subspaces of $H_o^2(\omega)$. Let $W \in \mathcal{W}_l$. Then define

$$\boldsymbol{W} = \{\boldsymbol{v}_\varphi | \varphi \in W\}.$$

It follows that $\boldsymbol{W} \in \mathcal{V}_l$. Hence, it follows from (2.4.6) that

$$\xi^l(\varepsilon) \leq \min_{W \in \mathcal{W}_l} \max_{\varphi \in W} \frac{N(\varepsilon)(\boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi)}{D(\varepsilon)(\boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi)}. \tag{2.4.7}$$

We now use the expression (2.4.3) for $N(\varepsilon)(\boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi)$. Integrating, with respect to x_3, all the integrals on the right-hand side of (2.4.3), we are left with integrals over ω. It is then immediate to see that, by virtue of Lemma 2.4.1 and Poincaré's inequality,

$$N(\varepsilon)(\boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi) \leq C \int_\omega |\Delta\varphi|^2 d\omega.$$

Now,

$$D(\varepsilon)(\boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi) = \int_\Omega [\varepsilon^2 x_3^2 |\nabla\varphi|^2 + \varphi^2]\delta(\varepsilon) dx.$$

By virtue of the relation

$$\delta(\varepsilon) = 1 + \varepsilon^2 \delta^\#(\varepsilon, \theta),$$

where $\delta^\#(\varepsilon, \theta)$ is bounded uniformly with respect to ε, we have, for ε sufficiently small, that $\delta(\varepsilon) \geq 1/2$. Thus, with a further appeal to Poincaré's inequality, we have

$$D(\varepsilon)(\boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi) \geq C \int_\omega \varphi^2 d\omega.$$

Thus,

$$\xi^l(\varepsilon) \leq C \min_{W \in \mathcal{W}_l} \max_{\varphi \in W} \frac{\int_\omega |\Delta\varphi|^2 d\omega}{\int_\omega \varphi^2 d\omega}.$$

But the min-max expression on the right-hand side of the above relation gives exactly the l-th eigenvalue of the two-dimensional elliptic eigenvalue problem

$$\left.\begin{aligned} \Delta^2 w &= \Lambda w && \text{in } \omega \\ w = \partial_\nu w &= 0 && \text{on } \partial\omega. \end{aligned}\right\} \tag{2.4.8}$$

This completes the proof of the theorem on setting $K(l) = C\Lambda(l)$, where $\Lambda(l)$ is the l-th eigenvalue of the problem (2.4.8). ∎

2.5 Passing to the Limit

Theorem 2.5.1 *(a) For each positive integer l, there exists a subsequence (still indexed by ε for convenience) such that $(\xi^l(\varepsilon), \boldsymbol{u}^l(\varepsilon))$ converges in $I\!R \times \boldsymbol{V}$ to $(\xi^l, \boldsymbol{u}^l)$; further, $u_3^l = \zeta_3^l \in H_o^2(\omega)$ and*

$$u_\alpha^l = \zeta_\alpha^l - x_3\partial_\alpha\zeta_3^l, \ \textit{for } \alpha = 1, 2 \tag{2.5.1}$$

with $\zeta_\alpha^l \in H_o^1(\omega)$.

(b) The pair $(\xi^l, \boldsymbol{\zeta}^l)$ where $\boldsymbol{\zeta}^l = (\zeta_\alpha^l, \zeta_3^l) \in (H_o^1(\omega))^2 \times H_o^2(\omega)$, satisfies the following variational equations:

$$\left.\begin{aligned} -\int_\omega m_{\alpha\beta}(\zeta_3^l)\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega n_{\alpha\beta}^\theta(\boldsymbol{\zeta}^l)\partial_\alpha\theta\partial_\beta\eta_3 d\omega &= \xi^l \int_\omega \zeta_3^l \eta_3 d\omega \\ \int_\omega n_{\alpha\beta}^\theta(\boldsymbol{\zeta}^l)\partial_\beta\eta_\alpha d\omega &= 0 \end{aligned}\right\} \tag{2.5.2}$$

for all $\boldsymbol{\eta} \in (H_o^1(\omega))^2 \times H_o^2(\omega)$, where

$$m_{\alpha\beta}(\zeta_3) = -\left[\frac{2\lambda\mu}{3(\lambda+2\mu)}\Delta\zeta_3\delta_{\alpha\beta} + \frac{2}{3}\mu\partial_{\alpha\beta}\zeta_3\right], \tag{2.5.3}$$

$$n_{\alpha\beta}^\theta(\boldsymbol{\zeta}) = \frac{2\lambda\mu}{\lambda+2\mu}e_{\rho\rho}^\theta(\boldsymbol{\zeta})\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}^\theta(\boldsymbol{\zeta}) \tag{2.5.4}$$

and

$$e_{\alpha\beta}^\theta(\boldsymbol{\zeta}) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha) + \frac{1}{2}(\partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3), \tag{2.5.5}$$

for $\boldsymbol{\zeta} \in (H_o^1(\omega))^2 \times H_o^2(\omega)$.

Remark 2.5.1: On the face of it, it looks as though the tensor $e^{\theta}(\zeta)$ defined by (2.5.5) is different from the one defined by (2.3.13). Indeed, in view of the relation (2.5.1) connecting u^{l} and ζ^{l}, we have

$$e^{\theta}_{\alpha\beta}(u^{l}) = e^{\theta}_{\alpha\beta}(\zeta^{l}) - x_3\partial_{\alpha\beta}\zeta_3.$$

Notice that the tensor e^{θ} defined by (2.3.13) is of order 3 while the one defined above in (2.5.5) is only of order 2. ∎

Remark 2.5.2: The coefficients occurring in the expressions of $m_{\alpha\beta}$ and $n^{\theta}_{\alpha\beta}$ are one half of those prescribed by Ciarlet and Miara [18]. We will defer a detailed comment on this until Section 7 (cf. Remark 2.7.1 below). ∎

Proof. The proof is similar to that of Theorem 1 of Ciarlet and Miara [18]. Throughout the proof, the various generic constants appearing in the inequalities will be independent of ε but could depend on θ.

Step 1: Boundedness of the eigenvectors.
Setting $\widehat{\xi}^{\varepsilon} = \widehat{\xi}^{\varepsilon,l}$ and $\widehat{u}^{\varepsilon} = \widehat{u}^{\varepsilon,l}$ in (2.1.5) and using the assumptions (2.3.10), we get

$$\left.\begin{aligned} 2\mu\int_{\widehat{\Omega}^{\varepsilon}} \widehat{e}^{\varepsilon}_{ij}(\widehat{u}^{\varepsilon,l})\widehat{e}^{\varepsilon}_{ij}(\widehat{u}^{\varepsilon,l})d\widehat{x}^{\varepsilon} &\leq \widehat{\xi}^{\varepsilon,l}\int_{\widehat{\Omega}^{\varepsilon}} \widehat{u}^{\varepsilon,l}_i\widehat{u}^{\varepsilon,l}_i d\widehat{x}^{\varepsilon} \\ &= \widehat{\xi}^{\varepsilon,l}\int_{\Omega^{\varepsilon}} u^{\varepsilon,l}_i u^{\varepsilon,l}_i\delta^{\varepsilon}dx^{\varepsilon} \\ &= \varepsilon^{2}\xi^{l}(\varepsilon)\int_{\Omega}[\varepsilon^{5}u^{l}_{\alpha}(\varepsilon)u^{l}_{\alpha}(\varepsilon) + \varepsilon^{3}(u^{l}_{3}(\varepsilon))^{2}]\delta(\varepsilon)dx \\ &= \varepsilon^{5}\xi^{l}(\varepsilon) \end{aligned}\right\} \quad (2.5.6)$$

in view of the normalization condition (2.3.36).

We next have

$$\left.\begin{aligned}
&2\mu\int_{\widehat{\Omega}^\varepsilon}\widehat{e}^\varepsilon_{ij}(\widehat{\boldsymbol{u}}^{\varepsilon,l})\widehat{e}^\varepsilon_{ij}(\widehat{\boldsymbol{u}}^{\varepsilon,l})d\widehat{x}^\varepsilon\\
&= 2\mu\varepsilon^5\left(\frac{1}{\varepsilon^4}\int_\Omega\left\{e^\theta_{33}(\boldsymbol{u}^l(\varepsilon))+\varepsilon^2 e^\#_{33}(\varepsilon,\theta;\boldsymbol{u}^l(\varepsilon))\right\}^2\left\{1+\varepsilon^2\delta^\#(\varepsilon,\theta)\right\}dx\right.\\
&+\ \frac{2}{\varepsilon^2}\int_\Omega\sum_\alpha\{e^\theta_{\alpha3}(\boldsymbol{u}^l(\varepsilon))+\varepsilon^2 e^\#_{\alpha3}(\varepsilon,\theta;\boldsymbol{u}^l(\varepsilon))\}^2\left\{1+\varepsilon^2\delta^\#\varepsilon,\theta)\right\}dx\\
&+\left.\int_\Omega\sum_{\alpha,\beta}\left\{e^\theta_{\alpha\beta}(\boldsymbol{u}^l(\varepsilon))+\varepsilon^2 e^\#_{\alpha\beta}(\varepsilon,\theta;\boldsymbol{u}^l(\varepsilon))\right\}^2\left\{1+\varepsilon^2\delta^\#(\varepsilon,\theta)\right\}dx\right)
\end{aligned}\right\}\quad(2.5.7)$$

and whence

$$2\mu\int_{\widehat{\Omega}^\varepsilon}\widehat{e}^\varepsilon_{ij}(\widehat{\boldsymbol{u}}^{\varepsilon,l})\widehat{e}^\varepsilon_{ij}(\widehat{\boldsymbol{u}}^{\varepsilon,l})d\widehat{x}^\varepsilon\geq\mu\varepsilon^5\int_\Omega\sum_{i,j}\left\{e^\theta_{ij}(\boldsymbol{u}^l(\varepsilon))+\varepsilon^2 e^\#_{ij}(\varepsilon,\theta;\boldsymbol{u}^l(\varepsilon))\right\}^2dx\quad(2.5.8)$$

for $\varepsilon\leq min\{1,\frac{1}{\sqrt{2C_0(\theta)}}\}$ (cf.(2.3.22)).

The inequalities $(\alpha+\beta)^2\geq\frac{\alpha^2}{2}-\beta^2$, (2.3.28) and (2.3.30) then imply that

$$\left.\begin{aligned}
&\int_\Omega\sum_{ij}\left\{e^\theta_{ij}(\boldsymbol{u}^l\varepsilon)+\varepsilon^2 e^\#_{ij}(\varepsilon,\theta;\boldsymbol{u}^l(\varepsilon))\right\}^2dx\\
&\qquad\geq\ \frac{1}{2}\sum_{i,j}\left|e^\theta_{ij}(\boldsymbol{u}^l(\varepsilon))\right|^2_{0,\Omega}-\varepsilon^4\sum_{i,j}\left|e^\#_{ij}(\varepsilon,\theta;\boldsymbol{u}^l(\varepsilon))\right|^2_{0,\Omega}\\
&\qquad\geq\ \left(\tfrac{1}{2}\{C_2(\theta)\}^{-2}-9\varepsilon^4\{C_1(\theta)\}^2\right)||\boldsymbol{u}^l(\varepsilon)||^2_{1,\Omega}\\
&\qquad\geq\ \tfrac{1}{4}C_2(\theta)^{-2}||\boldsymbol{u}^l(\varepsilon)||^2_{1,\Omega}
\end{aligned}\right\}\quad(2.5.9)$$

for $\varepsilon\leq min\{\varepsilon_0(\theta),\frac{1}{\sqrt{6C_1(\theta)C_2(\theta)}}\}$.

Hence step 1 follows by combining relations (2.5.6) and inequalities (2.5.8)-(2.5.9).

Step 2: If we denote $\boldsymbol{K}^{\theta,l}(\varepsilon)=\boldsymbol{K}^\theta(\boldsymbol{u}^l(\varepsilon))$, then the norms $\left|\boldsymbol{K}^{\theta,l}(\varepsilon)\right|_{0,\Omega}$ are bounded.

To see this, we simply combine the triangular inequalities

$$\begin{aligned}
\left|K^{\theta,l}_{\alpha\beta}(\varepsilon)\right|_{0,\Omega}&\leq\left|e^\theta_{\alpha\beta}(\boldsymbol{u}^l(\varepsilon))+\varepsilon^2 e^\#_{\alpha\beta}(\varepsilon,\theta;\boldsymbol{u}^l(\varepsilon))\right|_{0,\Omega}+\varepsilon^2\left|e^\#_{\alpha\beta}(\varepsilon,\theta;\boldsymbol{u}^l(\varepsilon))\right|_{0,\Omega}\\
\left|K^{\theta,l}_{\alpha3}(\varepsilon)\right|_{0,\Omega}&\leq\frac{1}{\varepsilon}\left|e^\theta_{\alpha3}(\boldsymbol{u}^l(\varepsilon))+\varepsilon^2 e^\#_{\alpha3}(\varepsilon,\theta,\boldsymbol{u}^l(\varepsilon))\right|_{0,\Omega}+\varepsilon\left|e^\#_{\alpha3}(\varepsilon,\theta,\boldsymbol{u}^l(\varepsilon))\right|_{0,\Omega}\\
\left|K^{\theta,l}_{33}(\varepsilon)\right|_{0,\Omega}&\leq\frac{1}{\varepsilon^2}\left|e^\theta_{33}(\boldsymbol{u}^l(\varepsilon))+\varepsilon^2 e^\#_{33}(\varepsilon,\theta,\boldsymbol{u}^l(\varepsilon))\right|_{0,\Omega}+\left|\partial_\alpha\theta\partial_\alpha u^l_3(\varepsilon)\right|_{0,\Omega}\\
&\quad+\left|e^\#_{33}(\varepsilon,\theta;\boldsymbol{u}^l(\varepsilon))\right|_{0,\Omega}
\end{aligned}$$

with relations (2.5.6)-(2.5.7) the boundedness of the family $(\boldsymbol{u}^l(\varepsilon))_{\varepsilon>0}$ in the space $H^1(\Omega)$ established in step 1, and inequalities (2.3.28).

Step 3: By step 1, there exists a subsequence, still indexed by ε for convenience, and there exists an element $\boldsymbol{u}^l \in \boldsymbol{V}$ such that $\boldsymbol{u}^l(\varepsilon) \rightharpoonup \boldsymbol{u}^l$ weakly in $\boldsymbol{V}$. Since the sequence $(\boldsymbol{K}^{\theta,l}(\varepsilon))_{\varepsilon>0}$ is bounded in $L^2(\Omega)$ (cf.step 2), there exists a constant $D_1(\theta)$ such that

$$\left|e^{\theta}_{\alpha\beta}(\boldsymbol{u}^l(\varepsilon))\right|_{0,\Omega} \leq D_1(\theta)\varepsilon, \quad \left|e^{\theta}_{33}(\boldsymbol{u}^l(\varepsilon))\right|_{0,\Omega} \leq D_1(\theta)\varepsilon^2 \tag{2.5.10}$$

by definition of the function $K^{\theta,l}_{i3}(\varepsilon)$. Since a norm is a weakly lower semi-continuous function,

$$\left|e^{\theta}_{i3}(\boldsymbol{u}^l)\right|_{0,\Omega} \leq \liminf_{\varepsilon\to 0}\left|e^{\theta}_{i3}(\boldsymbol{u}^l(\varepsilon))\right|_{0,\Omega} = 0 \tag{2.5.11}$$

and whence $e^{\theta}_{i3}(\boldsymbol{u}^l) = 0$. This in turn implies that $e_{i3}(\boldsymbol{u}^l) = 0$ and a standard argument (cf.[6]) then implies that the components u^l_i of the limit $\boldsymbol{u}^l$ are of the form given by (2.5.1).

Step 4: By step 2, there exists a subsequence, still indexed by ε for convenience, and there exists an element $\boldsymbol{K}^{\theta,l} = \left(K^{\theta,l}_{ij}\right) \in L^2(\Omega)$ such that $\boldsymbol{K}^{\theta,l}(\varepsilon) \rightharpoonup \boldsymbol{K}^{\theta,l}$ weakly in $L^2(\Omega)$ as $\varepsilon \to 0$ (we may assume that the subsequence found in step 3 and 4 are the same).Then

$$K^{\theta,l}_{\alpha\beta} = e^{\theta}_{\alpha\beta}(\boldsymbol{u}^l), \quad K^{\theta,l}_{\alpha 3} = K^{\theta,l}_{3\alpha} = 0, \quad K^{\theta,l}_{33} = -\frac{\lambda}{\lambda+2\mu}e^{\theta}_{\rho\rho}(\boldsymbol{u}^l). \tag{2.5.12}$$

Since $K^{\theta,l}_{\alpha\beta}(\varepsilon) = e^{\theta}_{\alpha\beta}(\boldsymbol{u}^l(\varepsilon))$ and $\boldsymbol{u}^l(\varepsilon) \rightharpoonup \boldsymbol{u}^l$ *in* $H^1(\Omega)$, it first follows that

$$K^{\theta,l}_{\alpha\beta} \rightharpoonup e^{\theta}_{\alpha\beta}(\boldsymbol{u}^l) \text{ in } L^2(\Omega). \tag{2.5.13}$$

We next note the following easily established result: let $w \in L^2(\Omega)$ be given; then

$$\int_{\Omega} w\partial_3 v dx = 0 \text{ for all } v \in H^1(\Omega) : v = 0 \text{ on } \Gamma \Rightarrow w = 0. \tag{2.5.14}$$

Letting $v_3 = 0$ in equation (2.3.34) and multiplying by ε, we get

$$\int_{\Omega} 2\mu \boldsymbol{K}^{\theta,l}(\varepsilon)\partial_3 v_\alpha dx = \varepsilon\xi^l(\varepsilon)\mathcal{R}(\varepsilon,\theta;\boldsymbol{K}^{\theta,l}(\varepsilon),\boldsymbol{u}^l(\varepsilon),\boldsymbol{v}) \tag{2.5.15}$$

with

$$\sup_{0<\varepsilon\leq\varepsilon_0} |\mathcal{R}(\varepsilon, \theta, \boldsymbol{K}, \boldsymbol{u}, \boldsymbol{v}| \leq D_2(\theta)(|\boldsymbol{K}|_{0,\Omega} + ||\boldsymbol{u}||_{1,\Omega} + 1)||\boldsymbol{v}||_{1,\Omega}, \tag{2.5.16}$$

for all $(v_\alpha) \in H^1(\Omega)$ that vanish on Γ. For each such (v_α), the left-hand side of (2.5.15) converges to $\int_\Omega 2\mu K^{\theta,l}_{\alpha 3}\partial_3 v_\alpha dx$ as $\varepsilon\to 0$, by definition of weak convergence, and by (2.5.16) the right-hand side of (2.5.15) converges to 0, since a weakly convergence sequence is bounded. Hence $\int_\Omega K^{\theta,l}_{\alpha 3}\partial_3 v_\alpha dx = 0$ and thus $K^{\theta,l}_{\alpha 3} = 0$ by (2.5.14).

Letting $v_\alpha = 0$ in equations (2.3.34) and multiplying by ε^2, we find that

$$\int_\Omega \left\{\lambda K^{\theta,l}_{\rho\rho}(\varepsilon) + (\lambda + 2\mu)K^{\theta,l}_{33}(\varepsilon)\right\} \partial_3 v_3 dx = \varepsilon\xi^l(\varepsilon)\mathcal{S}(\varepsilon, \theta; \boldsymbol{K}^{\theta,l}(\varepsilon), \boldsymbol{u}^l(\varepsilon), \boldsymbol{v}) \tag{2.5.17}$$

with

$$\sup_{0<\varepsilon\leq\varepsilon_0} |\mathcal{S}(\varepsilon, \theta; \boldsymbol{K}, \boldsymbol{u}, \boldsymbol{v}| \leq D_3(\theta)(|\boldsymbol{K}|_{0,\Omega} + ||\boldsymbol{u}||_{1,\Omega} + 1)||\boldsymbol{v}||_{1,\Omega} \tag{2.5.18}$$

for all $v_3 \in H^1(\Omega)$ that vanishes on Γ. Hence passing to the limit as $\varepsilon\to 0$ gives

$$\int_\Omega \left\{\lambda K^{\theta,l}_{\rho\rho} + (\lambda + 2\mu)K^{\theta,l}\right\} \partial_3 v_3 dx = 0 \tag{2.5.19}$$

and thus the last relation (2.5.12) follows by another application of the implication (2.5.14).

Step 5: The function (ζ_α, ζ_3) solves the variational problem (2.5.2).

To see this we restrict the function $\boldsymbol{v} = (v_i)$ appearing in the variational equations (2.3.34) of the form

$$v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3, v_3 = \eta_3 \tag{2.5.20}$$

with $\eta_\alpha \in H^1_0(\omega)$, $\eta_3 \in H^2_0(\omega)$. A simple computation then shows that the equations (2.3.34) reduce for such functions $\boldsymbol{v}$ to

$$\begin{aligned}
&\int_\Omega \left\{\lambda K^{\theta,l}_{pp}(\varepsilon)\delta_{\alpha\beta} + 2\mu K^{\theta,l}_{\alpha\beta}(\varepsilon)\right\} \left\{\partial_\alpha v_\beta - \frac{1}{2}(\partial_\beta\theta\partial_3 v_\alpha + \partial_\alpha\theta\partial_3 v_\beta)\right\} dx \\
&+ \int_\Omega \left\{\lambda K^{\theta,l}_{\alpha\alpha}(\varepsilon) + (\lambda + 2\mu)K^{\theta,l}_{33}(\varepsilon)\right\} \partial_\beta\theta\partial_\beta v_3 dx \\
&+ \varepsilon B^{\#}_1(\varepsilon, \theta; \boldsymbol{K}^{\theta,l}(\varepsilon), \boldsymbol{v}) + \varepsilon^2 B^{\#}_2(\varepsilon, \theta; \boldsymbol{u}^l(\varepsilon), \boldsymbol{v}) \\
&= \xi^l(\varepsilon)D(\varepsilon)(\boldsymbol{u}^l(\varepsilon), \boldsymbol{v}).
\end{aligned} \tag{2.5.21}$$

Passing to the limit as $\varepsilon \to 0$, and taking into account the relation

$$\lambda K_{\rho\rho}^{\theta,l} + (\lambda + 2\mu) K_{33}^{\theta,l} = 0 \tag{2.5.22}$$

that was found in step 4, we are left with

$$\begin{aligned} &\int_\Omega \left\{ \lambda K_{pp}^{\theta,l} \delta_{\alpha\beta} + 2\mu K_{\alpha\beta}^{\theta,l} \right\} \left\{ \partial_\alpha v_\beta - \frac{1}{2}(\partial_\beta \theta \partial_3 \theta + \partial_\alpha \theta \partial_3 v_\beta) \right\} dx \\ &= \xi^l \int_\Omega u_3^l v_3 dx \end{aligned} \tag{2.5.23}$$

for all $\boldsymbol{v} = (v_i)$ of the form (2.5.20). Once the components u_i^l of $\boldsymbol{u}^l$ have been replaced by their expressions (2.5.1) and the components v_i of $\boldsymbol{v}$ have been replaced by their expression (2.5.20), it is verified that equations (2.5.23) coincide with equations (2.5.2).

Step 6: The family $(\boldsymbol{u}^l(\varepsilon))_{\varepsilon>0}$ converges strongly to $\boldsymbol{u}^l$ in $H^1(\Omega)$.

To show this, it is enough to show that the family $(e^\theta \boldsymbol{u}^l(\varepsilon))_{\varepsilon>0}$, where $e^\theta(\boldsymbol{u}^l(\varepsilon)) := (e_{ij}^\theta(\boldsymbol{u}^l(\varepsilon)))$ converges strongly to $e^\theta(\boldsymbol{u}^l)$ in $L^2(\Omega)$, as the conclusion will then follow from Lemma 2.3.3.

Given two symmetric tensors $S = (s_{ij})$ and $T = (t_{ij})$ in $L^2(\Omega)$, let

$$\int_\Omega AS : T dx := \int_\Omega \{\lambda s_{pp} s_{qq} + 2\mu s_{ij} t_{ij}\} dx, \tag{2.5.24}$$

Then we have:

$$\begin{aligned} 2\mu \left| \boldsymbol{K}^{\theta,l}(\varepsilon) - \boldsymbol{K}^{\theta,l} \right|_{0,\Omega}^2 &\le \int_\Omega A(\boldsymbol{K}^{\theta,l}(\varepsilon) - \boldsymbol{K}^{\theta,l}) : (\boldsymbol{K}^{\theta,l}(\varepsilon) - \boldsymbol{K}^{\theta,l}) dx \\ &= \int_\Omega A\boldsymbol{K}^{\theta,l} : (\boldsymbol{K}^{\theta,l} - 2\boldsymbol{K}^{\theta,l}(\varepsilon)) dx + \int_\Omega A\boldsymbol{K}^{\theta,l}(\varepsilon) : \boldsymbol{K}^{\theta,l}(\varepsilon) dx \end{aligned} \tag{2.5.25}$$

It is easily checked that, when $\boldsymbol{v} = \boldsymbol{u}^l(\varepsilon)$, equation (2.3.34) can be written as

$$\int_\Omega A\boldsymbol{K}^{\theta,l}(\varepsilon) : \boldsymbol{K}^{\theta,l}(\varepsilon) dx + \varepsilon B_3^\#(\varepsilon, \theta; \boldsymbol{K}^{\theta,l}(\varepsilon), \boldsymbol{u}^l(\varepsilon)) = \xi^l(\epsilon) D(\varepsilon)(\boldsymbol{u}^l(\varepsilon), \boldsymbol{u}^l(\varepsilon)) \tag{2.5.26}$$

where

$$\sup_{0<\varepsilon\le\varepsilon_0} |B_3^\#(\varepsilon, \theta, \boldsymbol{K}, \boldsymbol{v}| \le D_4(\theta)(|\boldsymbol{K}|_{0,\Omega}^2 + ||\boldsymbol{v}||_{1,\Omega}^2) \tag{2.5.27}$$

Using the bounds (2.5.27) and the weak convergence of $(\boldsymbol{u}^l(\varepsilon))_{\varepsilon>0}$ and $(\boldsymbol{K}^{\theta,l}(\varepsilon))_{\varepsilon>0}$, we infer from (2.5.26) that

$$\int_\Omega A\boldsymbol{K}^{\theta,l}(\varepsilon) : \boldsymbol{K}^{\theta,l}(\varepsilon)dx \to \xi^l \int_\Omega \zeta_3^l \zeta_3^l dx \text{ as } \varepsilon \to 0 \tag{2.5.28}$$

and thus

$$\begin{aligned}\Big\{\int_\Omega A\boldsymbol{K}^{\theta,l} : (\boldsymbol{K}^{\theta,l} - 2\boldsymbol{K}^{\theta,l}(\varepsilon))dx + \int_\Omega A\boldsymbol{K}^{\theta,l}(\varepsilon) : \boldsymbol{K}^{\theta,l}(\varepsilon)dx\Big\}& \\ \to \Big\{-\int_\Omega A\boldsymbol{K}^{\theta,l} : \boldsymbol{K}^{\theta,l}dx + \xi^l \int_\Omega \zeta_3^l\zeta_3^l dx\Big\} \text{ as } \varepsilon\to 0&\end{aligned} \tag{2.5.29}$$

Using the last three equalities in (2.5.12), and letting $\boldsymbol{v} = \boldsymbol{u}^l$ in (2.5.23), we obtain

$$\begin{aligned}\int_\Omega A\boldsymbol{K}^{\theta,l} : \boldsymbol{K}^{\theta,l}dx &= \int_\Omega \left\{\lambda K_{pp}^{\theta,l} K_{qq}^{\theta,l} + 2\mu K_{ij}^{\theta,l} K_{ij}^{\theta,l}\right\} dx \\ &= \int_\Omega \left\{\lambda K_{pp}^{\theta,l} K_{\beta\beta}^{\theta,l} + 2\mu K_{\alpha\beta}^{\theta,l} K_{\alpha\beta}^{\theta,l} + [\lambda K_{pp}^{\theta,l} + (\lambda+2\mu)K_{33}^{\theta,l}]K_{33}^{\theta,l}\right\} dx \\ &= \int_\Omega \left\{\lambda K_{pp}^{\theta,l} K_{\alpha\beta}^{\theta,l} + 2\mu K_{\alpha\beta}^{\theta,l} K_{\alpha\beta}^{\theta,l}\right\} dx = \xi^l \int_\Omega \zeta_3^l\zeta_3^l dx.\end{aligned} \tag{2.5.30}$$

Hence it follows from (2.5.25) and (2.5.29) and (2.5.30) that $(\boldsymbol{K}^{\theta,l}(\varepsilon))_{\varepsilon>0}$ converges strongly to $\boldsymbol{K}^{\theta,l}$ in $L^2(\Omega)$

Note that $e_{i3}^\theta(\boldsymbol{u}^l) = 0$ and hence

$$\begin{aligned}|e^\theta(\boldsymbol{u}^l(\varepsilon)) - e^\theta(\boldsymbol{u}^l)|_{0,\Omega}^2 &\le \sum_{\alpha,\beta} |K_{\alpha\beta}^{\theta,l}(\varepsilon) - K_{\alpha\beta}^{\theta,l}|_{0,\Omega}^2 \\ &+ 2\varepsilon^2 \sum_\alpha |K_{\alpha 3}^{\theta,l}(\varepsilon)|_{0,\Omega}^2 + \varepsilon^4 |K_{33}^{\theta,l}(\varepsilon)|_{0,\Omega}^2 + \varepsilon^4 |\partial_\alpha\partial_\alpha u_3^l(\varepsilon)|_{0,\Omega}^2\end{aligned} \tag{2.5.31}$$

and the conclusion follows. ∎

Remark 2.5.3: As already mentioned in the introduction, there is an important difference between the limiting equations obtained by Ciarlet and Miara [18] and the equations (2.5.2) obtained above. In the former, the right-hand side of the second equation is a function of the horizontal components of the forces whereas in the latter, we get zero. This is because the horizontal and vertical components of the displacements and the forces have been scaled in different ways by Ciarlet and Miara [18] to balance the different powers of ε occurring on both sides of the equation. In the case of the eigenvalue problem, we can scale only the displacements and the eigenvalues and hence the powers of ε do not get balanced. The equations (2.5.2) obtained do not give an eigenvalue problem in $(H_o^1(\omega))^2 \times H_o^2(\omega)$ corresponding to the problem obtained by Ciarlet and Miara [18]. Instead, as we will show in the following section, we get a two- dimensional fourth order elliptic eigenvalue problem in which the eigenvector is the third component ζ_3^l of the vector $\boldsymbol{\zeta}^l$. ∎

2.6 The Limit Problem

In this section, we will analyze the limit problem (2.5.2). In particular, we will show that it can be considered as an elliptic eigenvalue problem in the third component alone and that the limiting pairs $\{(\xi^l, \zeta_3^l)\}$ completely describe the spectrum of this problem.

Lemma 2.6.1 *Given $\zeta_3 \in H_o^2(\omega)$, there exists a unique vector $(\zeta_\alpha) \in (H_o^1(\omega))^2$ such that, if $\boldsymbol{\zeta} = (\zeta_\alpha, \zeta_3)$,*

$$\int_\omega n_{\alpha\beta}^\theta(\boldsymbol{\zeta})\partial_\beta \eta_\alpha d\omega = 0 \tag{2.6.1}$$

for all $(\eta_\alpha) \in (H_o^1(\omega))^2$.

Proof. Set $\tilde{\boldsymbol{\eta}} = (\eta_\alpha) \in (H_o^1(\omega))^2$ and define the usual strain tensor

$$e_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha).$$

Then consider the bilinear form

$$\begin{aligned} a(\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\eta}}) &= \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\rho\rho}(\tilde{\boldsymbol{\zeta}})\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(\tilde{\boldsymbol{\zeta}})\right] \partial_\beta \eta_\alpha d\omega \\ &= \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\rho\rho}(\tilde{\boldsymbol{\zeta}}) e_{\sigma\sigma}(\tilde{\boldsymbol{\eta}}) + 2\mu e_{\alpha\beta}(\tilde{\boldsymbol{\zeta}}) e_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\right] d\omega. \end{aligned}$$

This is a symmetric and $(H_o^1(\omega))^2$- elliptic bilinear form and hence by the Lax-Milgram Lemma, given $(f_\alpha) \in (H^{-1})^2$, there exists a unique $\tilde{\boldsymbol{\zeta}} \in (H_o^1(\omega))^2$ such that

$$a(\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\eta}}) = < f_\alpha, \eta_\alpha >$$

for every $\tilde{\boldsymbol{\eta}} \in (H^1_o(\omega))^2$. The result now follows by setting

$$< f_\alpha, \eta_\alpha > = -\int_\omega \rho^\theta_{\alpha\beta}(\zeta_3)\partial_\beta\eta_\alpha d\omega$$

where

$$\rho^\theta_{\alpha\beta}(\zeta_3) = \frac{2\lambda\mu}{\lambda+2\mu}(\partial_\sigma\theta\partial_\sigma\zeta_3)\delta_{\alpha\beta} + \mu(\partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3)$$

∎

Thus, given $\zeta_3 \in H^2_o(\omega)$, we denote by $T\zeta_3 \in (H^1_o(\omega))^2 \times H^2_o(\omega)$, the vector (ζ_α, ζ_3) where $(\zeta_\alpha) \in (H^1_o(\omega))^2$ is the solution of (2.6.1). Substituting this in the first equation of (2.5.2), we can rewrite it as

$$b(\zeta^l_3, \eta_3) = \xi^l \int_\omega \zeta^l_3\eta_3 d\omega \tag{2.6.2}$$

for all $\eta_3 \in H^2_o(\omega)$ where

$$b(\zeta_3, \eta_3) = -\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega n^\theta_{\alpha\beta}(T\zeta_3)\partial_\alpha\theta\partial_\beta\eta_3 d\omega. \tag{2.6.3}$$

Lemma 2.6.2 *The bilinear form $b(.,.)$ defined by (2.6.3) is symmetric and $H^2_o(\omega)$-elliptic.*

Proof. Step 1: First of all, we have the following
Claim: The bilinear form

$$\tilde{B}(\boldsymbol{\zeta}, \boldsymbol{\eta}) = -\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega n^\theta_{\alpha\beta}(\boldsymbol{\zeta})[\partial_\alpha\theta\partial_\beta\eta_3 + \partial_\beta\eta_\alpha]d\omega$$

defined for $\boldsymbol{\zeta}, \boldsymbol{\eta} \in (H^1_o(\omega))^2 \times H^2_o(\omega)$, is symmetric.

Assuming the validity of the claim for the moment, we can conclude the proof of the symmetry of $b(.,.)$ as follows. Let $\zeta_3, \eta_3 \in H^2_o(\omega)$. Let $(\zeta_\alpha), (\eta_\alpha) \in (H^1_o(\omega))^2$ be such that $(\zeta_\alpha, \zeta_3) = T\zeta_3$ and $(\eta_\alpha, \eta_3) = T\eta_3$. Then,

$$b(\zeta_3, \eta_3) \;=\; -\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega n^\theta_{\alpha\beta}(T\zeta_3)\partial_\alpha\theta\partial_\beta\eta_3 d\omega$$
$$+\int_\omega n^\theta_{\alpha\beta}(T\zeta_3)\partial_\beta\eta_\alpha d\omega$$

since the last integral on the right-hand side is zero by virtue of the definition of T. Hence,

$$b(\zeta_3, \eta_3) = \tilde{B}(T\zeta_3, T\eta_3) = \tilde{B}(T\eta_3, T\zeta_3) = b(\eta_3, \zeta_3).$$

This proves the symmetry of $b(.,.)$. Thus, we just need to substantiate the claim.

Step 2: Now,

$$-\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3 d\omega \;=\; \int_\omega \left[\frac{2\lambda\mu}{3(\lambda+2\mu)}\Delta\zeta_3\Delta\eta_3 + \frac{2}{3}\mu\partial_{\alpha\beta}\zeta_3\partial_{\alpha\beta}\eta_3\right] d\omega$$
$$= \; -\int_\omega m_{\alpha\beta}(\eta_3)\partial_{\alpha\beta}\zeta_3 d\omega$$

which is symmetric in ζ_3 and η_3.Further,

$$\int_\omega n^\theta_{\alpha\beta}(\boldsymbol{\zeta})\partial_\alpha\theta\partial_\beta\eta_3 d\omega \;+\; \int_\omega n^\theta_{\alpha\beta}(\boldsymbol{\zeta})\partial_\beta\eta_\alpha d\omega$$
$$= \int_\omega \frac{2\lambda\mu}{\lambda+2\mu}\left[\partial_\rho\zeta_\rho + \partial_\rho\theta\partial_\rho\zeta_3\right]\left[\partial_\alpha\theta\partial_\alpha\eta_3 + \partial_\alpha\eta_\alpha\right] d\omega$$
$$+\int_\omega 2\mu\left[e_{\alpha\beta}(\boldsymbol{\zeta}) + \frac{1}{2}(\partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3)\right]\left[\partial_\alpha\theta\partial_\beta\eta_3 + \partial_\beta\eta_\alpha\right] d\omega$$
$$= \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu}e^\theta_{\rho\rho}(\boldsymbol{\zeta})e^\theta_{\alpha\alpha}(\boldsymbol{\eta}) + 2\mu e^\theta_{\alpha\beta}(\boldsymbol{\zeta})e^\theta_{\alpha\beta}(\boldsymbol{\eta})\right] d\omega$$

which is also symmetric in $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$, thus proving the claim.

Step 3: We now prove the ellipticity of the bilinear form. By definition, the integral

$$\int_\omega n^\theta_{\alpha\beta}(T\zeta_3)\partial_\beta(T\zeta_3)_\alpha d\omega$$

is zero and hence, replacing η_3 by ζ_3 in (2.6.3) and adding the above integral to the right-hand side, we get

$$b(\zeta_3, \zeta_3) = B(T\zeta_3, T\zeta_3)$$

where $B(.,.)$ is the bilinear form associated to the left-hand sides of the equations (2.5.2). A simple computation shows that

$$B(\boldsymbol{\eta},\boldsymbol{\eta}) = \int_\omega \frac{4\lambda\mu}{\lambda+2\mu}\left\{\frac{1}{3}(\triangle\eta_3)^2 + (e^\theta_{\alpha\alpha}(\boldsymbol{\eta}))^2\right\} d\omega$$
$$+4\mu\left\{\frac{1}{3}\sum_{\alpha,\beta}|\partial_{\alpha\beta}\eta_3|^2_{0,\omega} + \sum_{\alpha,\beta}|e^\theta_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\right\} \qquad (2.6.4)$$

By Lemma 2.3.3, there exists a constant $C > 0$ such that

$$\sum_{ij}|e^\theta_{ij}(\boldsymbol{v})|^2_{0,\Omega} \geq C|\boldsymbol{v}|^2_{1,\Omega} \text{ for all } \boldsymbol{v} \in \boldsymbol{V}. \qquad (2.6.5)$$

Given arbitrary elements $\boldsymbol{\eta} = (\eta_\alpha, \eta_3) \in (H^1_0(\omega))^2 \times H^2_0(\omega)$, the function $\boldsymbol{v} = (v_i)$ defined by $v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3$ and $v_3 = \eta_3$, belongs to the space $\boldsymbol{V}$. It is then easily verified that, for such functions $\boldsymbol{v}$, inequality (2.6.5) reduces to (note that $e^\theta_{i3} = 0$):

$$\sum_{\alpha,\beta}|e^\theta_{\alpha\beta}(\boldsymbol{v})|^2_{0,\Omega} = 2\sum_{\alpha,\beta}|e^\theta_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} + \frac{2}{3}\sum_{\alpha,\beta}|\partial_{\alpha\beta}\eta_3|^2_{0,\omega}$$
$$\geq C|\boldsymbol{v}|^2_{1,\Omega} = 2C\sum_{\alpha,\beta}|\partial_\alpha\eta_\beta|^2_{0,\omega} + \frac{2C}{3}\sum_{\alpha,\beta}|\partial_{\alpha\beta}\eta_3|^2_{0,\omega} + 4C\sum_{\alpha}|\partial_\alpha\eta_3|^2_{0,\omega} \qquad (2.6.6)$$

Hence the relations (2.6.4) and (2.6.6) together imply that there exists a constant $\beta > 0$ such that

$$B(\boldsymbol{\eta},\boldsymbol{\eta}) \geq \beta\left\{\sum_{\alpha,\beta}|\partial_{\alpha\beta}\eta_3|^2_{0,\omega} + \sum_{\alpha,\beta}|\partial_\alpha\eta_\beta|^2_{0,\omega}\right\} \qquad (2.6.7)$$

Since the mappings

$$(\eta_\beta) \in (H^1_0(\omega))^2 \rightarrow |\eta_\beta|_{1,\omega} = \left\{\sum_{\alpha,\beta}|\partial_\alpha\eta_\beta|^2_{0,\omega}\right\}^{\frac{1}{2}}$$
$$\eta_3 \in H^2_0(\omega) \rightarrow |\eta_3|_{2,\omega} = \left\{\sum_{\alpha,\beta}|\partial_{\alpha\beta}\eta_3|^2_{0,\omega}\right\}^{\frac{1}{2}} \qquad (2.6.8)$$

are respectively equivalent to the norms $\|.\|_{1,\omega}$ and $\|.\|_{2,\omega}$ over the spaces $(H^1_0(\omega))^2$ and $H^2_0(\omega)$ the bilinear form B(.) in elliptic over $(H^1_0(\omega))^2 \times H^2_0(\omega)$. Thus

$$b(\zeta_3,\zeta_3) \geq \beta(\|(T\zeta_3)_\alpha\|^2_{1,\omega} + \|\zeta_3\|^2_{2,\omega}) \geq \beta\|\zeta_3\|^2_{2,\omega}$$

and the proof is complete. ∎

Thus, given $f \in H^{-2}(\omega)$, there exists a unique $\zeta_3 = Af \in H_o^2(\omega)$ such that

$$b(Af, \eta_3) = < f, \eta_3 >$$

for every $\eta_3 \in H_o^2(\omega)$. Hence the system (2.5.2) is now equivalent to finding $(\xi, \zeta_3) \in \mathbb{R} \times H_o^2(\omega) \backslash \{0\}$ such that

$$A(\xi\zeta_3) = \zeta_3.$$

The injection $H_o^2(\omega) \hookrightarrow L^2(\omega)$ is compact and so we have a sequence of eigenvalues tending to infinity and eigenvectors which form an orthonormal basis of $L^2(\omega)$.

Theorem 2.6.1 *Let $\xi^l(\varepsilon) \rightarrow \xi^l$ and let $\boldsymbol{u}^l(\varepsilon) \rightarrow \boldsymbol{u}^l$ in $\boldsymbol{V}$. Then ξ^l is the l-th eigenvalue of the problem(2.6.2) (which is equivalent to the system(2.5.2)) and $\{u_3^l\}$ is an orthogonal basis for $L^2(\omega)$. Thus, all the eigenelements of the limit problem are obtained as limits of $\{(\xi^l(\varepsilon), \boldsymbol{u}^l(\varepsilon))\}_{l=1}^{\infty}$.*

Proof. Step 1: Since we already know that

$$0 < \xi^1(\varepsilon) \leq \xi^2(\varepsilon) \leq \cdots \leq \xi^l(\varepsilon) \leq \xi^{l+1}(\varepsilon) \leq \cdots,$$

and since $b(.,.)$ is elliptic, it follows that

$$0 < \xi^1 \leq \xi^2 \leq \cdots \leq \xi^l \leq \xi^{l+1} \leq \cdots.$$

Since the operator A is compact, its eigenvalues are all of finite multiplicity. Hence it follows that the sequence $\{\xi^l\}$ is unbounded. Thus, $\xi^l \rightarrow \infty$ as $l \rightarrow \infty$. Further the orthogonality condition (2.3.36) yields, on passing to the limit

$$\int_\Omega u_3^l u_3^m \, dx = \delta_{lm}$$

which in turn gives the orthogonality condition

$$\int_\omega u_3^l u_3^m \, d\omega = \frac{1}{2}\delta_{lm}. \tag{2.6.9}$$

Step 2: There are no other eigenvalues of the limit problem. (This will complete the proof of the theorem). Assume the contrary. Let $\xi \in \mathbb{R}$ be an eigenvalue such that $\xi \neq \xi^l$ for all l. Then there exists an eigenfunction ζ_3 such that

$$\int_\omega \zeta_3^2 d\omega = \frac{1}{2} \text{ and } \int_\omega \zeta_3 \zeta_3^l d\omega = 0 \text{ for all } l. \tag{2.6.10}$$

For each $\varepsilon > 0$, let $\boldsymbol{w}(\varepsilon) \in \boldsymbol{V}$ be the unique solution of the problem

$$N(\varepsilon)(\boldsymbol{w}(\varepsilon), \boldsymbol{v}) = \xi \int_\Omega \zeta_3 v_3 dx \tag{2.6.11}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$. Then proceeding as in Theorem 2.5.1, we can show that $\boldsymbol{w}(\varepsilon) \to \boldsymbol{w}$ in $\boldsymbol{V}$ and that $w_\alpha = z_\alpha - x_3 \partial_\alpha z_3$ and that $w_3 = z_3 \in H_o^2(\omega)$. Further, if $\boldsymbol{z} =, (z_\alpha, z_3)$, then $\boldsymbol{z} = T z_3$ and z_3 will be the solution of

$$b(z_3, \eta_3) = \xi \int_\omega \zeta_3 \eta_3 d\omega \tag{2.6.12}$$

for all $\eta_3 \in H_o^2(\omega)$. By the uniqueness of the solution it follows that $z_3 = \zeta_3$.

Step 3: Since the sequence $\{\xi^l\}$ is unbounded, choose l such that

$$\xi < \xi^l. \tag{2.6.13}$$

Consider the vector

$$\boldsymbol{v}(\varepsilon) = \boldsymbol{w}(\varepsilon) - \sum_{k=1}^{l} D(\varepsilon)(\boldsymbol{w}(\varepsilon), \boldsymbol{u}^k(\varepsilon))\boldsymbol{u}^k(\varepsilon).$$

Since $D(\varepsilon)(\boldsymbol{v}(\varepsilon), \boldsymbol{u}^k(\varepsilon)) = 0$ for $1 \leq k \leq l$, it follows, from the variational characterization of the eigenvalues, that

$$\xi^{l+1}(\varepsilon) \leq \frac{N(\varepsilon)(\boldsymbol{v}(\varepsilon), \boldsymbol{v}(\varepsilon))}{D(\varepsilon)(\boldsymbol{v}(\varepsilon), \boldsymbol{v}(\varepsilon))}. \tag{2.6.14}$$

Let us evaluate the numerator and denominator of the right-hand side of (2.6.14) separately.

On one hand, observe that

$$\begin{aligned}
N(\varepsilon)(\boldsymbol{w}(\varepsilon),\boldsymbol{w}(\varepsilon)) &= \xi\int_\Omega \zeta_3 w_3(\varepsilon)dx \\
&\to 2\xi\int_\omega \zeta_3^2 d\omega, \\
N(\varepsilon)(\boldsymbol{w}(\varepsilon),\boldsymbol{u}^k(\varepsilon)) &= \xi^k(\varepsilon)D(\varepsilon)(\boldsymbol{w}(\varepsilon),\boldsymbol{u}^k(\varepsilon)), \\
N(\varepsilon)(\boldsymbol{u}^k(\varepsilon),\boldsymbol{u}^m(\varepsilon)) &= \xi^k(\varepsilon)\delta_{km}, \\
D(\varepsilon)(\boldsymbol{w}(\varepsilon),\boldsymbol{u}^k(\varepsilon)) &= \int_\Omega(\varepsilon^2 w_\alpha(\varepsilon)u_\alpha^k(\varepsilon)+w_3(\varepsilon)u_3^k(\varepsilon))\delta(\varepsilon)dx \\
&\to 2\int_\omega \zeta_3\zeta_3^k d\omega = 0.
\end{aligned}$$

Combining these relations we get

$$N(\varepsilon)(\boldsymbol{v}(\varepsilon),\boldsymbol{v}(\varepsilon))\to 2\xi\int_\omega \zeta_3^2 d\omega.$$

On the other hand, it follows from the above relations that

$$\boldsymbol{v}(\varepsilon)-\boldsymbol{w}(\varepsilon)\to\mathbf{0} \text{ in } \boldsymbol{V}.$$

Thus,

$$\lim_{\varepsilon\to 0} D(\varepsilon)(\boldsymbol{v}(\varepsilon),\boldsymbol{v}(\varepsilon)) = \lim_{\varepsilon\to 0} D(\varepsilon)(\boldsymbol{w}(\varepsilon),\boldsymbol{w}(\varepsilon)) = 2\int_\omega \zeta_3^2 d\omega.$$

Hence, passing to the limit in (2.6.14), we get

$$\xi^{l+1}\le\xi$$

which contradicts (2.6.13) and the proof is complete. ∎

We have used the method introduced by Kesavan [24] in the study of the homogenization of eigenvalue problems and which was also used by Ciarlet and Kesavan [11] in the study of the eigenvalue problem for plates. We can show that for each integer $l\geq 1$, the whole family $\xi^l(\epsilon),\epsilon>0$ converges to ξ^l as ϵ approaches zero. Also if ξ^l is a simple eigenvalue of the limit problem, then $\xi^l(\varepsilon)$ is a simple eigenvalue of (2.3.34) for sufficiently small ε. In this case we can choose the eigenvector $\boldsymbol{u}^l(\varepsilon)$ such that the *entire* family $\{\boldsymbol{u}^l(\varepsilon)\}$ converges, instead of just a subsequence.

2.7 The Two-Dimensional Model

We now "descale" the functions ζ_i^l, u_i^l and the eigenvalues ξ^l obtained in Theorem 2.5.1. to obtain the two-dimensional model approximating the three-dimensional problem. In view of the scalings (2.3.8) - (2.3.9), we now define

$$\zeta_\alpha^{\varepsilon,l} = \varepsilon^2 \zeta_\alpha^l,\ \zeta_3^{\varepsilon,l} = \varepsilon \zeta_3^l \text{ on } \omega \quad \text{and} \quad \xi^{\varepsilon,l} = \varepsilon^2 \xi^l,\ l = 1, 2, 3, \dots \tag{2.7.1}$$

We also define

$$\left.\begin{array}{rcl} \widehat{u}_\alpha^{\varepsilon,l}(0)(\Theta^\varepsilon(x^\varepsilon)) & = & \varepsilon^2 u_\alpha^l(x) \\ \widehat{u}_3^{\varepsilon,l}(0)(\Theta^\varepsilon(x^\varepsilon)) & = & \varepsilon u_3^l(x) \end{array}\right\} \tag{2.7.2}$$

for all $x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega^\varepsilon}$, where $\Theta^\varepsilon : \overline{\Omega^\varepsilon} \to \overline{\widehat{\Omega}^\varepsilon}$ and $\pi^\varepsilon : \overline{\Omega} \to \overline{\Omega^\varepsilon}$ are as defined in Sections 2 and 3. Then we get the following result.

Theorem 2.7.1: (a) The pairs $(\xi^{\varepsilon,l}, \boldsymbol{\zeta}^{\varepsilon,l}) \in I\!R \times H_o^2(\omega) \backslash \{0\}$ satisfy the equations

$$\left.\begin{array}{rcl} -\displaystyle\int_\omega m_{\alpha\beta}^\varepsilon(\zeta_3^{\varepsilon,l}) \partial_{\alpha\beta}\eta_3 d\omega + \int_\omega n_{\alpha\beta}^{\theta^\varepsilon}(\boldsymbol{\zeta}^{\varepsilon,l}) \partial_\alpha \theta^\varepsilon \partial_\beta \eta_3 d\omega & = & \xi^{\varepsilon,l} \displaystyle\int_\omega \zeta_3^{\varepsilon,l} \eta_3 d\omega \\ \displaystyle\int_\omega n_{\alpha\beta}^{\theta^\varepsilon}(\boldsymbol{\zeta}^{\varepsilon,l}) \partial_\beta \eta_\alpha d\omega & = & 0 \end{array}\right\} \tag{2.7.3}$$

for all $\boldsymbol{\eta} = (\eta_\alpha, \eta_3) \in (H_o^1(\omega))^2 \times H_o^2(\omega)$, where

$$m_{\alpha\beta}^\varepsilon(\zeta_3^\varepsilon) = -\varepsilon^2 \left[\frac{2\lambda\mu}{3(\lambda + 2\mu)} \Delta\zeta_3^\varepsilon \delta_{\alpha\beta} + \frac{2}{3}\mu \partial_{\alpha\beta}\zeta_3^\varepsilon \right], \tag{2.7.4}$$

$$n_{\alpha\beta}^{\theta^\varepsilon}(\boldsymbol{\zeta}^\varepsilon) = \frac{2\lambda\mu}{\lambda + 2\mu} e_{\rho\rho}^{\theta^\varepsilon}(\boldsymbol{\zeta}^\varepsilon) \delta_{\alpha\beta} + 2\mu e_{\alpha\beta}^{\theta^\varepsilon}(\boldsymbol{\zeta}^\varepsilon) \tag{2.7.5}$$

and

$$e_{\alpha\beta}^{\theta^\varepsilon}(\boldsymbol{\zeta}^\varepsilon) = \frac{1}{2}(\partial_\alpha \zeta_\beta^\varepsilon + \partial_\beta \zeta_\alpha^\varepsilon) + \frac{1}{2}(\partial_\alpha \theta^\varepsilon \partial_\beta \zeta_3^\varepsilon + \partial_\beta \theta^\varepsilon \partial_\alpha \zeta_3^\varepsilon). \tag{2.7.6}$$

(Recall that λ and μ are the Lamé constants of the material the shell is made of and that $\theta^\varepsilon = \varepsilon\theta$ is the middle surface of the shell.)

(b) The descaled functions $\widehat{\boldsymbol{u}}^{\varepsilon,l}(0)$ approximating the eigenvectors $\widehat{\boldsymbol{u}}^{\varepsilon,l}$ are given by

$$\left.\begin{array}{rcl} \widehat{u}_\alpha^{\varepsilon,l}(0)(\widehat{x}^\varepsilon) & = & \zeta_\alpha^{\varepsilon,l}(x_1, x_2) - x_3^\varepsilon \partial_\alpha \zeta_3^{\varepsilon,l}(x_1, x_2) \\ \widehat{u}_3^{\varepsilon,l}(0)(\widehat{x}^\varepsilon) & = & \zeta_3^{\varepsilon,l}(x_1, x_2) \end{array}\right\} \tag{2.7.7}$$

at all points $\widehat{x}^\varepsilon = \Theta^\varepsilon(x_1, x_2, x_3^\varepsilon) \in \overline{\widehat{\Omega}^\varepsilon}$. ∎

As in the previous section the problem (2.7.3) can be expressed as a fourth order elliptic eigenvalue problem involving the pair $(\xi^{\varepsilon,l}, \zeta_3^{\varepsilon,l})$ alone.

Remark 2.7.1: We already observed (cf. Remark 2.5.2) that the coefficients obtained for $m_{\alpha\beta}$ and $n_{\alpha\beta}^{\theta}$ in this paper differed from those obtained by Ciarlet and Miara [18] by a factor of 2. Now we further observe that in the descaled model, the coefficients of $m_{\alpha\beta}^{\varepsilon}$ and $n_{\alpha\beta}^{\theta^{\varepsilon}}$ differ from those obtained in [18] by a factor of 2ε. Indeed, we can reconcile the two models as follows. Instead of defining p_i^{ε} (in the absence of surface forces) as the integral of f_i^{ε} over $[-\varepsilon, \varepsilon]$, we should rather define it as the *mean* over $[-\varepsilon, \varepsilon]$, *i.e.*

$$p_i^{\varepsilon} = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f_i^{\varepsilon} dx_3^{\varepsilon}.$$

We feel that this is more natural and it also removes the discrepancy between the models obtained for the stationary and the eigenvalue problems. Of course, in the presence of surface forces, we will then get terms $\frac{1}{2\varepsilon}(g_i^{+\varepsilon} + g_i^{-\varepsilon})$ in p_i^{ε} and $\frac{1}{2}(g_{\alpha}^{+\varepsilon} - g_{\alpha}^{-\varepsilon})$ in s_{α}^{ε} (cf. [18]) which have to be suitably interpreted. ∎

2.8 Conclusions

We have started with the three-dimensional eigenvalue problem for a thin shell and obtained a two-dimensional model based on the shallow shell assumption as in Ciarlet and Miara [18].

The principal difference between the models obtained for the stationary and eigenvalue problems is that in the latter case, it is possible to express the two-dimensional problem as a problem involving only the third component of the eigenvector.

This situation is reminiscent of plate theory (cf. Ciarlet and Kesavan [11]). Indeed, this is perhaps to be expected for we have assumed that $\theta^{\varepsilon} = \varepsilon\theta$ which tends to zero as $\varepsilon \to 0$. Thus, we can expect the shell to behave like a plate. Further, if we set $\theta = 0$, we recover the results of Ciarlet and Kesavan [11] for plates. In particular, we can see that $\zeta_\alpha^l = 0$ for all l.

The other difference is a minor one, concerned with the presentation of the model. As explained in Remark 2.7.1, the coefficients of the limit problems obtained in the two cases differ by a factor of 2 (2ε in the descaled models) and this difference can be reconciled if we put the mean of the forces in the x_3-direction (rather than just their integrals) on the right-hand side of the limit problem in the stationary case.

Chapter 3

Rods

3.1 Introduction

In this chapter, we will derive a one-dimensional eigenvalue problem that describes the limit behaviour of the three-dimensional eigenvalue problem of a thin linearly elastic rod when the thickness of the rod goes to zero.

The study of lower dimensional approximations of three dimensional eigenvalue problems from the mathematical viewpoint has been done in the works of Ciarlet and Kesavan [11] for plates, Le Dret [29] for folded plates and Kesavan and Sabu [25] for shallow shells (cf. Chapter 2).

In each instance, one or several portions of the whole three-dimensional structure have a small thickness which we denote by ϵ.Then if the various data behave as specific powers of ϵ as $\epsilon \to 0$, one can establish the convergence of the (appropriately scaled) components of the displacement vector field towards the solution of a lower dimensional problem. In this chapter, we likewise establish the convergence of the eigenvalues and the associated eigenfunctions towards the solution of a one-dimensional eigenvalue problem in the case of thin rods. We now briefly outline the problem studied in this chapter and describe the results obtained.

Let $\omega \subset I\!R^2$ be a bounded domain and let

$$\Omega_\epsilon = \epsilon\,\omega \times (0,1), \gamma_{\epsilon_0} = \partial\Omega_\epsilon \cap \{x_3 = 0\}, \gamma_{\epsilon_1} = \partial\Omega_\epsilon \cap \{x_3 = 1\} \text{ and } \gamma_\epsilon = \gamma_{\epsilon_0} \cup \gamma_{\epsilon_1}. \tag{3.1.1}$$

For all 3×3 tensors $\boldsymbol{\zeta}$, we define

$$(\boldsymbol{A\zeta})_{ij} = \lambda\zeta_{ll}\delta_{ij} + 2\mu\zeta_{ij} \tag{3.1.2}$$

where λ and μ are the Lamé constants of the material. We then define the space of admissible displacements as

$$\boldsymbol{V}^\epsilon = \{\boldsymbol{v}^\epsilon \in (H_0^1(\Omega_\epsilon))^3; \boldsymbol{v}^\epsilon = 0 \text{ on } \gamma_\epsilon\}. \tag{3.1.3}$$

For each admissible displacement $\boldsymbol{v}^\epsilon$, we define the linearized stress tensor $\boldsymbol{e}(\boldsymbol{v}^\epsilon) = (e_{ij}(\boldsymbol{v}^\epsilon))$ by

$$e_{ij}(\boldsymbol{v}^\epsilon) = \frac{1}{2}(\partial_i v_j^\epsilon + \partial_j v_i^\epsilon) \tag{3.1.4}$$

for $1 \le i,j \le 3$. Then, the eigenvalue problem consists of finding pairs $(\boldsymbol{u}^\epsilon, \xi^\epsilon) \in \boldsymbol{V}^\epsilon\backslash\{0\} \times I\!R$ such that

$$\int_{\Omega^\epsilon} (\boldsymbol{Ae}(\boldsymbol{u}^\epsilon))_{ij} e_{ij}(\boldsymbol{v}^\epsilon) dx = \xi^\epsilon \int_{\Omega^\epsilon} u_i^\epsilon . v_i^\epsilon dx \tag{3.1.5}$$

for every $\boldsymbol{v}^\epsilon \in \boldsymbol{V}^\epsilon$ with the convention of summation over repeated indices. It can be shown that there exists a sequence of eigenpairs $\{(\boldsymbol{u}^{\epsilon,l}, \xi^{\epsilon,l})\}_{l=1}^\infty$ such that

$$0 < \xi^{\epsilon,1} \le \xi^{\epsilon,2} \le \cdots \le \xi^{\epsilon,l} \le \cdots \to \infty \tag{3.1.6}$$

and $\{\boldsymbol{u}^{\epsilon,l}\}$ forms a complete orthonormal basis for $(L^2(\Omega_\epsilon))^3$.

We then transform (3.1.5) into an equivalent problem over $\Omega = \omega \times (0,1)$ after suitable scalings of the variables ξ^ϵ and $\boldsymbol{u}^\epsilon$.

In this fashion, we obtain scaled eigenpairs $(\boldsymbol{u}^l(\epsilon), \xi^l(\epsilon)) \in \boldsymbol{V}\backslash\{0\} \times I\!R$ where

$$\boldsymbol{V} = \{\boldsymbol{v} \in (H^1(\Omega))^3; \boldsymbol{v} = 0 \text{ on } \gamma_1\} \tag{3.1.7}$$

which satisfy variational equations in which ϵ occurs as a parameter. We show that as $\epsilon \to 0$, $\boldsymbol{u}^l(\epsilon) \to \boldsymbol{u}^l$ in $(H^1(\Omega))^3$ and $\xi^l(\epsilon) \to \xi^l$ for each fixed l for a suitable subsequence. We also show that

$$u^l_\alpha = \zeta^l_\alpha(x_3), \quad u^l_3 = -x_\alpha(\zeta^{l'}_\alpha(x_3)) \tag{3.1.8}$$

for some $\boldsymbol{\zeta}^l = (\zeta^l_1, \zeta^l_2) \in (H^2_0(0,1))^2$. The pair $(\boldsymbol{\zeta}^l, \xi^l)$ is an eigenpair for a fourth order elliptic problem posed over (0,1).We can also prove that every eigensolution of the limit problem is a limit of a subsequence of $(\boldsymbol{u}^l(\epsilon), \xi^l(\epsilon))$ for some integer $l \geq 1$.

There is an important difference between the one-dimensional model obtained by Le Dret [28] and the eigenvalue problem studied in this paper. The former is a system of coupled fourth order equations involving all the components of the limit of $\boldsymbol{u}(\epsilon)$. The latter involves only the horizontal components. We will comment about this in greater detail later.

This chapter is organized as follows. In Section 2 below, we describe the three-dimensional problem. In Section 3, we transform the problem into one posed over a fixed domain and in Section 4, we study the limit problem. Section 5 is devoted to concluding remarks.

3.2 The Three-Dimensional Problem

Throughout this chapter, the Latin indices will vary over the set $\{1, 2, 3\}$ and Greek indices will vary over the set $\{1, 2\}$ for the components of vectors and tensors.The convention of summation over repeated indices will be used in conjunction with the above rules.

We consider a family of three-dimensional, isotropic, homogeneous, linearly elastic bodies whose reference configurations are the sets Ω_ϵ defined for all $\epsilon > 0$ by,

$$\Omega_\epsilon = \{x = (x_1, x_2, x_3) \in I\!R^3, (x_1, x_2) \in \omega_\epsilon, 0 < x_3 < 1\} \tag{3.2.1}$$

where $\omega_\epsilon = \epsilon\,\omega$ and ω is a bounded subset of $I\!R^2$, ie, straight cylinders in $I\!R^3$ with axes in the x_3 direction of length 1 and cross section ω_ϵ in the (x_1, x_2) plane. We refer to ϵ as the thickness of the rod under consideration.

Without loss of generality, we may assume that

$$\int_\omega x_1 dx_1 dx_2 = \int_\omega x_2 dx_1 dx_2 = \int_\omega x_1 x_2 dx_1 dx_2 = 0 \tag{3.2.2}$$

which means that we choose the origin of coordinates at the centre of gravity of ω, and the coordinate axes to be the principal axes of inertia of ω. Let I be the 2×2 inertia tensor of ω whose components are

$$I_{\alpha\beta} = \int_\omega x_\alpha x_\beta dx_1 dx_2. \tag{3.2.3}$$

We let $\gamma_{\epsilon_0} = \partial\Omega_\epsilon \cap \{x_3 = 0\}$ and $\gamma_{\epsilon_1} = \partial\Omega_\epsilon \cap \{x_3 = 1\}$ denote the ends of the rod and $S_\epsilon = \partial\Omega_\epsilon \cap \{0 < x_3 < 1\}$ denote its lateral surface. We assume that the rod is clamped on both ends; if $\boldsymbol{v}^\epsilon$ is an admissible displacement vector, then the stress tensor corresponding to this displacement is given by $\boldsymbol{\tau} = \boldsymbol{A}(\boldsymbol{e}(\boldsymbol{v}^\epsilon))$.

The usual scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ will be denoted by $\boldsymbol{u}.\boldsymbol{v} = u_i.v_i$ and the usual scalar product of two tensors $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ will be denoted by $\boldsymbol{\tau} : \boldsymbol{\sigma} = \tau_{ij}\sigma_{ij}$. With this notation, the eigenvalue problem for the rods under consideration admits the following variational formulation.

Find $(\boldsymbol{u}^\epsilon, \xi^\epsilon) \in \boldsymbol{V}^\epsilon \backslash \{0\} \times I\!R$ such that

$$\int_{\Omega_\epsilon} \boldsymbol{A}\boldsymbol{e}(\boldsymbol{u}^\epsilon) : \boldsymbol{e}(\boldsymbol{v}^\epsilon) dx = \xi^\epsilon \int_{\Omega_\epsilon} \boldsymbol{u}^\epsilon.\boldsymbol{v}^\epsilon dx \tag{3.2.4}$$

for all $\boldsymbol{v}^\epsilon \in \boldsymbol{V}^\epsilon$.

The $\boldsymbol{V}^\epsilon$ ellipticity of the bilinear form appearing in the left-hand side of (3.2.4) follows from Körn's inequality. Hence for each $\boldsymbol{f}^\epsilon \in (L^2(\Omega_\epsilon))^3$, there exists a unique $\boldsymbol{w}^\epsilon \in \boldsymbol{V}^\epsilon$ such that

$$\int_{\Omega_\epsilon} \boldsymbol{A}\boldsymbol{e}(\boldsymbol{w}^\epsilon) : \boldsymbol{e}(\boldsymbol{v}^\epsilon) dx = \int_{\Omega_\epsilon} \boldsymbol{f}^\epsilon.\boldsymbol{v}^\epsilon dx \tag{3.2.5}$$

for all $\boldsymbol{v}^\epsilon \in \boldsymbol{V}^\epsilon$.

We denote $\boldsymbol{w}^\epsilon = G^\epsilon(\boldsymbol{f}^\epsilon)$ and thus $G^\epsilon : (L^2(\Omega_\epsilon))^3 \to \boldsymbol{V}^\epsilon$ defines a bounded linear operator. Since the inclusion $\boldsymbol{V}^\epsilon \hookrightarrow (L^2(\Omega_\epsilon))^3$ is compact, we can consider G^ϵ as a compact linear operator of $(L^2(\Omega_\epsilon))^3$ into itself. It is also clear that G^ϵ is selfadjoint. Thus problem (3.2.4) reduces to finding $\boldsymbol{u}^\epsilon \in \boldsymbol{V}^\epsilon$ and $\xi^\epsilon \in I\!R$ such that $\boldsymbol{u}^\epsilon = \xi^\epsilon G^\epsilon(\boldsymbol{u}^\epsilon)$.

From the spectral theory of compact, selfadjoint, linear operators it follows that there exists a sequence of eigenpairs $\{(\boldsymbol{u}^{\epsilon,l}, \xi^{\epsilon,l})_{l=1}^\infty\}$ such that

$$0 < \xi^{\epsilon,1} \leq \xi^{\epsilon,2} \leq \cdots \leq \xi^{\epsilon,l} \leq \cdots \to \infty \tag{3.2.6}$$

$$\int_{\Omega_\epsilon} u_i^{\epsilon,l}.u_j^{\epsilon,k} dx = \delta_{ij} \tag{3.2.7}$$

The eigenvalues $(\xi^{\epsilon,l})$ can be characterized via the min-max principle for the corresponding Rayleigh quotient by

$$\xi^{\epsilon,l} = \min_{\boldsymbol{W} \in \mathcal{V}_l^\epsilon} \max_{\boldsymbol{v}^\epsilon \in \boldsymbol{W}} \frac{\int_{\Omega_\epsilon} \boldsymbol{A}\boldsymbol{e}(\boldsymbol{v}^\epsilon) : \boldsymbol{e}(\boldsymbol{v}^\epsilon) dx}{\int_{\Omega_\epsilon} \boldsymbol{v}^\epsilon.\boldsymbol{v}^\epsilon dx} \tag{3.2.8}$$

where $\mathcal{V}_l^\epsilon$ denotes the collection of all l-dimensional subspaces of $\boldsymbol{V}^\epsilon$.

3.3 The Rescaled Problem

We set $\Omega = \omega \times (0,1)$ and $\gamma = \partial\Omega \cap \{x_3 = 0, 1\}$ and with each point $x \in \bar{\Omega}$, we associate the point $x^\epsilon \in \bar{\Omega}_\epsilon$ through the bijection

$$\pi^\epsilon : x = (x_i) \in \bar{\Omega} \to x^\epsilon = (\epsilon x_1, \epsilon x_2, x_3) \in \bar{\Omega}_\epsilon. \tag{3.3.1}$$

Given $\boldsymbol{v}^\epsilon \in \boldsymbol{V}^\epsilon$, we associate the scaled function $\boldsymbol{v}(\epsilon) \in \boldsymbol{V}$,where

$$\boldsymbol{V} = \{\boldsymbol{v} \in (H^1(\Omega))^3 : \boldsymbol{v} = 0 \text{ on } \gamma\}, \tag{3.3.2}$$

via the relations

$$v_\alpha^\epsilon(x^\epsilon) = v_\alpha(\epsilon)(x), v_3^\epsilon(x^\epsilon) = \epsilon v_3(\epsilon)(x) \tag{3.3.3}$$

for all $x^\epsilon = \pi^\epsilon(x)$. We also scale the eigenvalues as follows:

$$\xi^\epsilon = \epsilon^2 \xi(\epsilon). \tag{3.3.4}$$

Based on the above scalings and the routine calculations of change of variable in the integrals, we deduce that for a given ϵ, the rescaled eigensolutions of (3.2.4) satisfy the following variational problem.

Find $\{(\boldsymbol{u}^l(\epsilon), \xi^l(\epsilon))\}_{l=1}^{\infty} \in \boldsymbol{V}\backslash\{0\} \times I\!R$ such that

$$\int_\Omega b_\epsilon(\boldsymbol{u}^l(\epsilon), \boldsymbol{v})dx = \xi^l(\epsilon)\int_\Omega (u_\alpha(\epsilon)v_\alpha + \epsilon^2 u_3(\epsilon)v_3)dx \tag{3.3.5}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$, where the bilinear form b_ϵ is given by

$$\begin{aligned} b_\epsilon(\boldsymbol{u}, \boldsymbol{v}) &= \epsilon^{-4}[2\mu e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{v}) + e_{\alpha\alpha}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{v})] \\ &\quad + \epsilon^{-2}[4\mu e_{\alpha 3}(\boldsymbol{u})e_{\alpha 3}(\boldsymbol{v}) + \lambda(e_{\alpha\alpha}(\boldsymbol{u})e_{33}(\boldsymbol{v}) + e_{33}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{v}))] \\ &\quad + (2\mu + \lambda)e_{33}(\boldsymbol{u})e_{33}(\boldsymbol{v}). \end{aligned} \tag{3.3.6}$$

and the rescaled eigenvectors $\{\boldsymbol{u}^l(\epsilon)\}_{l=1}^{\infty}$ satisfy the normalization condition

$$\int_\Omega (u_\alpha^l(\epsilon)u_\alpha^k(\epsilon) + \epsilon^2 u_3^l(\epsilon)u_3^k(\epsilon))dx = \delta_{lk}. \tag{3.3.7}$$

The variational problem (3.3.5) may also be rewritten as

$$\int_\Omega \boldsymbol{\sigma}(\epsilon) : \boldsymbol{e}(\boldsymbol{v})dx = \xi(\epsilon)\int_\Omega (u_\alpha(\epsilon)v_\alpha + \epsilon^2 u(\epsilon)v_3)dx \tag{3.3.8}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ where the rescaled stress tensor $\boldsymbol{\sigma}(\epsilon)$ is defined as

$$\left.\begin{aligned} \sigma_{\alpha\beta}(\epsilon) &= 2\mu\epsilon^{-4}e_{\alpha\beta}(\boldsymbol{u}(\epsilon)) + \lambda[\epsilon^{-4}e_{\gamma\gamma}(\boldsymbol{u}(\epsilon)) + \epsilon^{-2}e_{33}(\boldsymbol{u}(\epsilon))]\delta_{\alpha\beta} \\ \sigma_{\alpha 3}(\epsilon) &= \sigma_{3\alpha}(\epsilon) = 2\mu\epsilon^{-2}e_{\alpha 3}(\boldsymbol{u}(\epsilon)) \\ \sigma_{33}(\epsilon) &= (2\mu + \lambda)e_{33}(\boldsymbol{u}(\epsilon)) + \lambda\epsilon^{-2}e_{\gamma\gamma}(\boldsymbol{u}(\epsilon)). \end{aligned}\right\} \tag{3.3.9}$$

If we introduce the auxiliary tensor $\boldsymbol{\chi}^l(\epsilon)$:

$$\left.\begin{aligned} \chi_{\alpha\beta}(\epsilon) &= \epsilon^{-2}e_{\alpha\beta}(\boldsymbol{u}(\epsilon)) \\ \chi_{\alpha 3}(\epsilon) &= \epsilon^{-1}e_{\alpha 3}(\boldsymbol{u}(\epsilon)) \\ \chi_{33}(\epsilon) &= e_{33}(\boldsymbol{u}(\epsilon)) \end{aligned}\right\} \tag{3.3.10}$$

the rescaled stresses assume the more homogeneous form

$$\left.\begin{aligned} \sigma_{\alpha\beta}(\epsilon) &= \epsilon^{-2}(2\mu\chi_{\alpha\beta}(\epsilon) + \lambda\chi_{ii}(\epsilon)\delta_{\alpha\beta}) \\ \sigma_{\alpha 3}(\epsilon) &= \sigma_{3\alpha}(\epsilon) = 2\mu\epsilon^{-1}\chi_{\alpha 3}(\epsilon) \\ \sigma_{33}(\epsilon) &= 2\mu\chi_{33}(\epsilon) + \lambda\chi_{ii}(\epsilon) \end{aligned}\right\} \tag{3.3.11}$$

i.e.

$$\sigma_{\alpha\beta}(\epsilon) = \epsilon^{-2}(\boldsymbol{A}\boldsymbol{\chi}(\epsilon))_{\alpha\beta}; \quad \sigma_{\alpha 3}(\epsilon) = \epsilon^{-1}(\boldsymbol{A}\boldsymbol{\chi}(\epsilon))_{\alpha 3}; \quad \sigma_{33}(\epsilon) = (\boldsymbol{A}\boldsymbol{\chi}(\epsilon))_{33}. \tag{3.3.12}$$

We denote the tensors $\boldsymbol{\sigma}(\epsilon)$ and $\boldsymbol{\chi}(\epsilon)$ associated to the eigenvector $\boldsymbol{u}^l(\epsilon)$ using the above formulae by $\boldsymbol{\sigma}^l(\epsilon)$ and $\boldsymbol{\chi}^l(\epsilon)$ respectively.

3.4 The Limit Problem

In this section we prove that the various unknowns involved $(\boldsymbol{u}^l(\epsilon), \xi^l(\epsilon))$ satisfy appropriate bounds, which upon extraction of a subsequence will allow us to consider limits for these unknowns as $\epsilon \to 0$.Then, we will identify the one-dimensional problem satisfied by the limits. As these limit problems will turn out to be well-posed eigenvalue problems, we will thus be able to determine precisely the limit unknowns as being the eigenvalues and eigenvectors of the limit problem.

To begin with, let us consider the eigenvalues $\xi^l(\epsilon)$.

Lemma 3.4.1 *For each integer $l \geq 1$, there exists a constant k^l (independent of ϵ) such that $\xi^l(\epsilon) \leq k^l$.*

Proof. Since the problem (3.3.5) was derived from (3.2.4) after a change of variable and change of scale, the eigenvalues can be characterized as follows.

$$\xi^l(\epsilon) = \min_{\boldsymbol{W} \in \mathcal{V}_l} \max_{\boldsymbol{v} \in \boldsymbol{W}} \frac{\int_\Omega b_\epsilon(\boldsymbol{v}, \boldsymbol{v}) dx}{\int_\Omega (v_\alpha^2 + \epsilon^2 v_3^2) dx} \tag{3.4.1}$$

where $\mathcal{V}_l$ denote the collection of all l-dimensional subspaces of $\boldsymbol{V}$.

Let $\mathcal{W}_l$ denote the collection of all l-dimensional subspaces of $H_0^2(0,1)$.

For $\varphi \in W \in \mathcal{W}_l$, we define

$$\boldsymbol{v}_\varphi = \{\varphi(x_3), \varphi(x_3), -(x_1 + x_2)\varphi'(x_3)\}, \mathbf{W} = \{v_\varphi : \varphi \in W\} \tag{3.4.2}$$

Then it is easy to verify that $\mathbf{W} \in \mathcal{V}_l$ and $e_{\alpha\beta}(\boldsymbol{v}_\varphi) = e_{\alpha 3}(\boldsymbol{v}_\varphi) = 0$. Hence it follows from (3.4.1) that

$$\begin{aligned}
\xi^l(\epsilon) &\leq \min_{W \in \mathcal{W}_l} \max_{\varphi \in W} \frac{\int_\Omega b_\epsilon(\boldsymbol{v}_\varphi, \boldsymbol{v}_\varphi) dx}{\int_\Omega ((v_\varphi)_\alpha^2 + \epsilon^2 (v_\varphi)_3^2) dx} \\
&= \min_{W \in \mathcal{W}_l} \max_{\varphi \in W} \frac{\int_\Omega (\lambda + 2\mu) e_{33}(\boldsymbol{v}_\varphi) e_{33}(\boldsymbol{v}_\varphi) dx}{\int_\Omega ((v_\varphi)_\alpha^2 + \epsilon^2 (v_\varphi)_3^2) dx} \\
&\leq C \min_{W \in \mathcal{W}_l} \max_{\varphi \in W} \frac{\int_0^1 (\varphi'')^2 dx_3}{\int_0^1 \varphi^2 dx_3}
\end{aligned}$$

where C is a constant independent of ϵ. The expression on the right-hand side of the above relation gives exactly the l-th eigenvalue of the one-dimensional elliptic eigenvalue problem

$$\left.\begin{array}{rcl} \frac{d^4w}{dx^4} & = & \lambda w \\ w(0) & = & w'(0) = w(1) = w'(1) = 0. \end{array}\right\} \tag{3.4.3}$$

This completes the proof of the lemma by setting $k^l = C\lambda(l)$ where $\lambda(l)$ is the l-th eigenvalue of the problem (3.4.3). ∎

Let us now consider the eigenfunctions. To begin with, we need the following lemma that will be used in several instances in the sequel.

Lemma 3.4.2 *Let $\omega \subset I\!R^n$ be an open, connected, regular subset of $I\!R^n$ and let P be a continuous linear operator from $H^1(\omega; I\!R^k)$ into $L^2(\omega; I\!R^m)$ satisfying the following elliptic estimate:*

$$||u||^2_{H^1(\omega;I\!R^k)} \leq C\left(||u||^2_{L^2(\omega;I\!R^k)} + ||Pu||^2_{L^2(\omega;I\!R^m)}\right) \text{ for all } u \in H^1(\omega; I\!R^k). \tag{3.4.4}$$

Let R be the kernel of P and $\mathcal{R} = L^2(0,1;R)$. Then

$$u \mapsto |||u||| := \left(\int_0^1 ||Pu||^2_{L^2(\omega;I\!R^m)}dt\right)^{\frac{1}{2}} \tag{3.4.5}$$

defines a norm on $L^2(0,1;H^1(\omega;I\!R^k))/\mathcal{R}$ that is equivalent to the quotient norm and there exists a linear continuous mapping $\mathcal{P}$ from $L^2(0,1;H^1(\omega;I\!R^k))$ into $\mathcal{R}$ such that

$$||u - \mathcal{P}u||_{L^2(0,1;H^1(\omega;I\!R^k))} \leq C\int_0^1 ||Pu||^2_{L^2(\omega;I\!R^m)}dt \tag{3.4.6}$$

Moreover, if $u \in H^1((0,1);L^2(\omega;I\!R^k))$ (resp. $H^1_0((0,1);L^2(\omega,I\!R^k))$), then we have $\mathcal{P}u \in H^1((0,1);R)$ (resp. $H^1_0((0,1);R)$)

Proof.: See the proof of Lemma 1 in Le Dret [28]. ∎

Remark 3.4.1: We will use Lemma 3.4.2 in the following two cases: $n = 2$ and either $k = 1$, $m = 2$, P is the gradient operator $Pu = (\partial_1 u, \partial_2 u)^T$ and

$R = I\!R$, or $k = 2$, $m = 4$ and P is the linearized strain tensor operator $(Pu)_{\alpha\beta} = (\partial_\alpha u_\beta + \partial_\beta u_\alpha)/2$, in which case (3.4.6) is just the two dimensional Körn's inequality and R is the space of two dimensional infinitesimal rigid displacements. In both cases, we will use the immediate consequence of Lemma 3.4.2 that, if u_n is a sequence such that $|||u_n|||$ is bounded independently of n, then u_n can be decomposed as $u_n = v_n + w_n$ where v_n is bounded in $L^2(0,1;H^1(\omega;I\!R^k))$ and $w_n \in L^2(0,1;R)$ has the same regularity in t as u_n(just set $w_n = \mathcal{P}u_n$) ∎

Theorem 3.4.3 a) *For each positive integer l, there exists a subsequence (still indexed by ϵ for convenience) such that $(\boldsymbol{u}^l(\epsilon), \xi^l(\epsilon))$ converges in $\boldsymbol{V} \times I\!R$ to $(\boldsymbol{u}^l, \xi^l)$; further there exists $(\zeta^l_\alpha) \in (H^2_0(0,1))^2$ such that*

$$u^l_\alpha = \zeta^l_\alpha(x_3), u^l_3 = -x_\alpha \zeta^{l'}_\alpha(x_3) \tag{3.4.7}$$

b) *The pair (ζ^l_α, ξ^l) satisfies the following variational equation*

$$\int_0^1 \frac{E}{a} I_{\alpha\beta} \zeta^{l''}_\alpha \eta''_\beta dx = \xi^l \int_0^1 \zeta^l_\alpha \eta_\alpha dx \tag{3.4.8}$$

for all $(\eta_\alpha) \in (H^2_0(0,1))^2$, where $E = \frac{\mu(3\lambda+2\mu)}{\mu+\lambda}$ is the Young's modulus of the material, $I_{\alpha\beta}$ is the inertia tensor and $a = \int_\omega dx_1 dx_2$ is the area of ω.

Proof. The proof follows the method used by Le Dret [28]. For sake of clarity, the proof is divided into several steps.

Step 1: Boundedness of $\boldsymbol{u}^l(\epsilon)$ and $\boldsymbol{\chi}^l(\epsilon)$:

Taking $\boldsymbol{v} = \boldsymbol{u}^l(\epsilon)$ in (3.3.5), we get

$$\int_\Omega \boldsymbol{A}\boldsymbol{\chi}^l(\epsilon) : \boldsymbol{\chi}^l(\epsilon) dx = \xi^l(\epsilon) \int_\Omega ((u^l_\alpha(\epsilon))^2 + \epsilon^2 (u^l_3(\epsilon))^2) dx. \tag{3.4.9}$$

Due to positivity of the elasticity tensor $\boldsymbol{A}$, the left-hand side of (3.4.9) is bounded from below by $\mu\|\boldsymbol{\chi}^l(\epsilon)\|_{L^2(\Omega,M)}$. Therefore using Körn's inequality, we have

$$\mu\|\boldsymbol{u}^l(\epsilon)\|^2_{H^1(\Omega,I\!R^3)} \le \mu\|e(\boldsymbol{u}^l(\epsilon))\|^2_{L^2(\Omega,M^3)} \le \mu\|\boldsymbol{\chi}^l(\epsilon)\|^2_{L^2(\Omega,M)} \le \xi^l(\epsilon) \le k^l \tag{3.4.10}$$

for $\epsilon \le 1$. This completes the proof of step 1.

Step 2: It follows from step 1 that (for a subsequence) $\xi^l(\epsilon) \to \xi^l$ in $I\!R$,$\boldsymbol{u}^l(\epsilon) \rightharpoonup \boldsymbol{u}^l$ weakly in $\boldsymbol{V}$ and $\boldsymbol{\chi}^l(\epsilon) \rightharpoonup \boldsymbol{\chi}^l$ in $L^2(\Omega, M)$ and hence it follows from (3.3.10) that

$$\chi^l_{33} = e_{33}(\boldsymbol{u}^l) \tag{3.4.11}$$

and there exists a constant C (independent of ϵ) such that

$$\|e_{\alpha\beta}(\boldsymbol{u}^l(\epsilon))\|_{L^2(\Omega)} \leq C\epsilon^2, \quad \|e_{\alpha 3}(\boldsymbol{u}^l(\epsilon))\|_{L^2(\Omega)} \leq C\epsilon \tag{3.4.12}$$

Therefore $e_{\alpha i}(\boldsymbol{u}^l(\epsilon)) \to 0$ strongly in $L^2(\Omega)$ for all α and i and as $\boldsymbol{u}^l(\epsilon) \rightharpoonup \boldsymbol{u}^l$ weakly in $H^1(\Omega)$, we have $e_{\alpha i}(\boldsymbol{u}^l(\epsilon)) \rightharpoonup e_{\alpha i}(\boldsymbol{u}^l)$.Therefore $e_{\alpha i}(\boldsymbol{u}^l) = 0$ for all α and i. Then a standard argument (cf. Le Dret [28]) shows that there exists $(\zeta^l_\alpha) \in (H^2_0(0,1))^2$, $\zeta^l_3 \in H^1_0(0,1)$ such that

$$u^l_\alpha = \zeta^l_\alpha(x_3), u^l_3 = \zeta^l_3(x_3) - x_\alpha \zeta^{l'}_\alpha(x_3). \tag{3.4.13}$$

Step 3: We have

$$\begin{aligned}
\chi^l_{11} &= \chi^l_{22} = \frac{-\lambda(\zeta^{l'}_3(x_3) - x_\alpha \zeta^{l''}_\alpha(x_3))}{2(\lambda+\mu)}, \quad \chi^l_{12} = 0 \\
\chi^l_{33} &= \zeta^{l'}_3(x_3) - x_\alpha \zeta^{l''}_\alpha(x_3)
\end{aligned} \tag{3.4.14}$$

(We do not identify the $\chi^l_{\alpha 3}$ components as they do not play any role in identification of the limit functions ζ^l_i.)

Let $\boldsymbol{w} = (w_1, w_2) \in H^1(\omega; I\!R^2)$ and $\theta \in D((0,1))$.

We consider equation (3.3.5) with admissible test function

$$\boldsymbol{v}(x_1, x_2, x_3) = (w_1(x_1, x_2)\theta(x_3), w_2(x_1, x_2)\theta(x_3), 0)^T \tag{3.4.15}$$

and multiply it by ϵ^2. This yields:

$$\int_\Omega [(A\chi^l(\epsilon))_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{w})\theta + \epsilon(A\chi^l(\epsilon))_{\alpha 3} w_\alpha \theta']dx = \epsilon^2 \xi^l(\epsilon) \int_\Omega u^l(\epsilon) w_\alpha \theta dx. \tag{3.4.16}$$

Passing to the limit as $\epsilon \to 0$ gives

$$\int_\Omega (A\chi^l)_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{w})\theta dx = 0 \tag{3.4.17}$$

Now, as this is true for all $\theta \in D((0,1))$, it follows that

$$\int_{\omega_z} (A\chi^l)_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{w}) dx_1 dx_2 = 0 \tag{3.4.18}$$

for almost all $z \in [0,1]$, where $\omega_z = \omega \times \{x_3 = z\}$ is the cross section at height z. Choosing a countable family $w_n \in H^1(\omega; I\!R^2)$ that is dense in $H^1(\omega; I\!R^2)$, we see that (3.4.18) holds almost everywhere for all such w_n. Let N be the subset of zero measure of [0,1] on which (3.4.18) fails. For $z \notin N$,the density of w_n in $H^1(\omega; I\!R^2)$ implies that (3.4.18) actually holds for all $w \in H^1(\omega; I\!R^2)$.

Recall now that

$$(A\chi^l)_{\alpha\beta} = 2\mu\chi^l_{\alpha\beta} + \lambda(\chi^l_{\gamma\gamma} + \chi^l_{33})\delta_{\alpha\beta} \tag{3.4.19}$$

with

$$\chi^l_{33} = \zeta_3^{l'}(x_3) - x_\alpha \zeta^{l''}(x_3). \tag{3.4.20}$$

Because of estimate (3.4.10) on $\chi^l_{\alpha\beta}(\epsilon)$, it follows from Lemma 3.4.2 applied to $(u^l_1(\epsilon), u^l_2(\epsilon))$ with the two-dimensional strain tensor operator that $u^l_\alpha(\epsilon)$ can be decomposed into

$$u^l_\alpha(\epsilon) = \epsilon^2 \overline{u}^l_\alpha(\epsilon) + r^l_\alpha(\epsilon) \tag{3.4.21}$$

where $\overline{u}^l_\alpha(\epsilon)$ is bounded independently of ϵ in $L^2(0,1; H^1(\omega; I\!R^2))$ and $r^l_\alpha(\epsilon) \in L^2(0,1; R_2)$ where R_2 is the space of two-dimensional infinitesimal rigid displacements. Hence we see from equation (3.4.21) that $\chi^l_{\alpha\beta}(\epsilon) = e_{\alpha\beta}(\overline{\boldsymbol{u}}^l(\epsilon))$, so that by extracting if needed a further subsequence, we may conclude that there exists $\overline{u}^l_\alpha \in L^2(0,1; H^1(\omega))$ such that

$$\chi^l_{\alpha\beta} = e_{\alpha\beta}(\overline{\boldsymbol{u}}^l) \tag{3.4.22}$$

Let us now define, for almost all $z \in [0,1]$, a two-dimensional displacement $\boldsymbol{y}^l_z$ by

$$\boldsymbol{y}^l_z(x_1, x_2) = \begin{pmatrix} \frac{x_2^2 - x_1^2}{2}\zeta_1^{l''}(z) - x_1 x_2 \zeta_2^{l''}(z) + x_1 \zeta_3^{l'}(z) \\ \frac{x_1^2 - x_2^2}{2}\zeta_2^{l''}(z) - x_1 x_2 \zeta_1^{l''}(z) + x_2 \zeta_3^{l'}(z) \end{pmatrix} \tag{3.4.23}$$

This displacement is such that $e_{\alpha\beta}(\boldsymbol{y}^l_z) = \chi^l_{33}\delta_{\alpha\beta}$ for $x_3 = z$.

Therefore,

$$(A\chi^l)_{\alpha\beta} = (Ae(\overline{\boldsymbol{u}}^l + \frac{\lambda}{2(\lambda+\mu)}\boldsymbol{y}^l_z))_{\alpha\beta} \tag{3.4.24}$$

Using (3.4.24) in (3.4.18), we obtain:

$$\int_{\omega_z} (Ae(\overline{\boldsymbol{u}}^l + \frac{\lambda}{2(\lambda+\mu)}\boldsymbol{y}^l_z))e_{\alpha\beta}(\boldsymbol{w})dx_1 dx_2 = 0 \tag{3.4.25}$$

for almost all $z \in [0,1]$. If we choose

$$\boldsymbol{w}^l = \overline{\boldsymbol{u}}^l + \frac{\lambda}{2(\lambda+\mu)}\boldsymbol{y}^l_z, \tag{3.4.26}$$

then, the positivity of the elasticity tensor A implies that

$$e_{\alpha\beta}(\overline{\boldsymbol{u}}^l + \frac{\lambda}{2(\lambda+\mu)}\boldsymbol{y}^l_z) = 0 \tag{3.4.27}$$

on almost all cross section w_z, which is to say in Ω. This proves the claim, since $\chi^l_{\alpha\beta} = e_{\alpha\beta}(\overline{\boldsymbol{u}}^l)$.

Step 4: We define

$$\begin{aligned} \boldsymbol{V}_{BN}(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{V}; \exists \eta_\alpha \in H^2_0(0,1), \exists \eta_3 \in H^1_0(0,1), \\ & v_\alpha(x_1,x_2,x_3) = \eta_\alpha(x_3), v_3(x_1,x_2,x_3) = \eta_3(x_3) - x_\alpha \eta'(x_3)\} \end{aligned} \tag{3.4.28}$$

For test functions $\boldsymbol{v} \in \boldsymbol{V}_{BN}$, the equation (3.3.8) becomes

$$\int_\Omega \sigma^l_{33}(\epsilon)e_{33}(\boldsymbol{v})dx = \xi^l \int_\Omega (u^l_\alpha(\epsilon)v_\alpha + \epsilon^2 u^l_3(\epsilon)v_3)dx. \tag{3.4.29}$$

We know from (3.3.11) that $\sigma^l(\epsilon)_{33} \to \sigma^l_{33}$ weakly in $L^2(\Omega)$ with

$$\sigma^l_{33} = 2\mu\chi^l_{33} + \lambda\chi^l_{ii}. \tag{3.4.30}$$

We can thus pass to the limit in (3.4.29) and obtain

$$\int_\Omega (2\mu\chi^l_{33} + \lambda\chi^l_{ii})e_{33}(\boldsymbol{v})dx = \xi^l \int_\Omega u^l_\alpha v_\alpha dx \tag{3.4.31}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_{BN}(\Omega)$. Taking $\boldsymbol{v} = (0,0,\eta(x_3))$ and substituting the values of χ^l_{ii} we get

$$\int_0^1 \zeta_3^{l'} \eta_3' dx_3 = 0 \text{ for all } \eta_3 \in H_0^1(0,1). \tag{3.4.32}$$

Taking $\eta_3 = \zeta_3^l$ in equation (3.4.32), we get

$$\int_0^1 (\zeta_3^{l'})^2 dx_3 = 0. \tag{3.4.33}$$

This implies that $(\zeta_3^{l'}) = 0$ and since $\zeta_3^l(0) = \zeta_3^l(1) = 0$, we have

$$\zeta_3^l = 0 \text{ on } (0,1). \tag{3.4.34}$$

Hence

$$u_\alpha^l = \zeta_\alpha^l(x_3), u_3^l = -x_\alpha \zeta_\alpha^{l'}(x_3) \tag{3.4.35}$$

$$\chi_{11}^l = \chi_{22}^l = \frac{\lambda x_\alpha(\zeta_\alpha^{l''}(x_3))}{2(\lambda+\mu)}, \chi_{12}^l = 0, \chi_{33}^l = -x_\alpha \zeta^{l''}(x_3). \tag{3.4.36}$$

and the equation (3.4.8) follows by using the above values of u_i^l and χ_{ii}^l and expressing the test function v_i in terms of their associated one-dimensional functions η_i in (3.4.31).

Step 5: For each integer $l > 0$, the sequence $(\boldsymbol{u}^l(\epsilon))$ strongly converges to $\boldsymbol{u}^l$ in $\boldsymbol{V}$.

Due to Körn's inequality in $H^1(\Omega; I\!R^3)$ and due to the clamping condition satisfied by the rescaled displacements, the strong convergence of the displacements is a consequence of the strong L^2-convergence of the associated strain tensors $e_{\alpha\beta}(\boldsymbol{u}^l(\epsilon))$. The latter convergence is in turn implied by the strong convergence of the tensors $\boldsymbol{\chi}^l(\epsilon)$.

We define

$\tau_{\alpha\beta}^l = \chi_{\alpha\beta}^l, \tau_{33}^l = \chi_{33}^l$ and $\tau_{\alpha 3}^l = 0$

The positivity of the elasticity tensor implies that

$$\begin{aligned}
\mu \|\boldsymbol{\chi}^l(\epsilon) - \boldsymbol{\tau}^l\|_{L^2(\Omega, M^3)} &\leq \int_\Omega A(\boldsymbol{\chi}^l(\epsilon) - \boldsymbol{\tau}^l) : (\boldsymbol{\chi}^l(\epsilon) - \boldsymbol{\tau}^l) dx \\
&= \int_\Omega [A\boldsymbol{\chi}^l(\epsilon) : \boldsymbol{\chi}^l(\epsilon) - A\boldsymbol{\tau}^l : (2\boldsymbol{\chi}^l(\epsilon) - \tau^l)] dx \\
&= \int_\Omega [\xi^l(\epsilon)(\boldsymbol{u}^l(\epsilon).\boldsymbol{u}^l(\epsilon)) - A\boldsymbol{\tau}^l : (2\boldsymbol{\chi}^l(\epsilon) - \boldsymbol{\tau}^l)] dx.
\end{aligned} \tag{3.4.37}$$

We can pass to the limit in the right-hand side of this inequality, by using all the already known weak convergences, and thus obtain

$$\mu \limsup_{\epsilon \to 0} \|\chi^l(\epsilon) - \tau^l\|^2_{L^2(\Omega;M^3)} \leq \int_\Omega [\xi^l(u^l_\alpha, u^l_\alpha) - A\tau^l : (2\chi^l - \tau^l)] dx. \qquad (3.4.38)$$

To conclude, we just remark that $A\tau^l : (2\chi^l - \tau^l) = A\tau^l : \tau^l$ since the $\alpha 3$ component do not contribute to this scalar product.By using equations (3.4.8) with $\eta_\alpha = \zeta^l_\alpha$, it follows that the right-hand side of (3.4.38) is zero. ∎

Remark 3.4.2 : As already mentioned in the introduction, there is an important difference between the limit equations obtained in Le Dret [28] and the equations obtained above. In the former, the right-hand side of the second equation is a function of the vertical components of the forces whereas in the latter we get zero. This is because the horizontal and vertical components of the displacement and forces have been scaled in different ways in Le Dret [28] to balance the different powers of ϵ occurring on both sides of the equation. In the case of eigenvalue problem, we can scale only the displacements and the eigenvalues and hence the powers do not get balanced. This leads to vanishing of the vertical component of the displacement and we get a one-dimensional fourth order elliptic eigenvalue problem in which the eigenvector is made up of the horizontal components of the vector $\boldsymbol{u}^l$. ∎

Lemma 3.4.4 *The problem (3.4.8) is a well posed eigenvalue problem which has a sequence of eigenvalues and the corresponding eigenvectors form an orthonormal basis for $(L^2(0,1))^2$ and a basis for $(H^2_0(0,1))^2$.*

Proof. The result follows from the ellipticity of the bilinear form appearing in the left-hand side of the equation (3.4.8) over $(H^2_0(0,1))^2$. ∎

Though we have proved that each subsequence $(\boldsymbol{u}^l(\epsilon), \xi^l(\epsilon))_{\epsilon>0}$, $l \geq 1$ strongly converges in $(H^1_0(\Omega))^3 \times I\!R$ to a solution (u^l_α, ξ^l) of the limit problem (3.4.8) (cf. Theorem 3.4.3), nothing tells us so far whether ξ^l is precisely the l-th eigenvalue of the problem (3.4.8). We shall answer this question affirmatively in the next lemma using the ideas developed by Kesavan [24].

Lemma 3.4.5 *The sequence $\{\xi^l\}_{l=1}^{\infty}$ comprises all the eigenvalues of the problem (3.4.8) and the corresponding eigenvectors form an orthogonal basis for $(L^2(0,1))^2$ and a basis for $(H_0^2(0,1))^2$.*

Proof. Since we already know that

$$0 < \xi^1(\epsilon) \leq \xi^2(\epsilon) \leq \cdots \leq \xi^l(\epsilon) \leq \xi^{l+1}(\epsilon) \cdots \to \infty \tag{3.4.39}$$

and since the Green's operator associated with the limit problem is compact,it follows that each ξ^l is of finite multiplicity and thus

$$0 < \xi^1 \leq \xi^2 \cdots \leq \xi^l \leq \xi^{l+1} \cdots \to \infty \tag{3.4.40}$$

Passing to the limit in the orthogonality relation (3.3.7), we get

$$\int_0^1 u_\alpha^l u_\alpha^m dx = \frac{1}{a}\delta_{lm}. \tag{3.4.41}$$

Suppose that there exists $\xi \in R$ such that $\xi \neq \xi^l$ for all l and ξ is an eigenvalue of the problem (3.4.8). Then there exists an eigenfunction ζ_α such that

$$\int_0^1 \zeta_\alpha \zeta_\alpha dx = \frac{1}{a}, \quad \int_0^1 \zeta_\alpha \zeta_\alpha^l dx = 0, \text{ for all } l. \tag{3.4.42}$$

For each $\epsilon > 0$, let $\boldsymbol{w}(\epsilon) \in \boldsymbol{V}$ be the unique solution of

$$b(\epsilon)(\boldsymbol{w}(\epsilon), \boldsymbol{v}) = \xi \int_\Omega \zeta_\alpha v_\alpha dx, \text{ for all } \boldsymbol{v} \in V. \tag{3.4.43}$$

Then proceeding as in Theorem 3.4.3, we can show that $\boldsymbol{w}(\epsilon) \to \boldsymbol{w} \in \boldsymbol{V}$ and $w_\alpha = z_\alpha(x_3)$, $w_3 = -x_\alpha z_\alpha'(x_3)$ for some $z_\alpha \in (H_0^2(0,1))^2$. Further (z_α) will satisfy

$$\int_0^1 \frac{E}{a} I_{\alpha\beta} z_\alpha'' \eta_\alpha'' dx_3 = \xi \int_0^1 \zeta_\alpha \eta_\alpha dx_3, \text{ for all } \eta \in (H^2(0,1))^2. \tag{3.4.44}$$

By the uniqueness of the solution, it follows that $z_\alpha = \zeta_\alpha$.

Since the sequence $\{\xi^l\}$ is unbounded, choose l such that

$$\xi < \xi^l \tag{3.4.45}$$

For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$, define

$$D(\epsilon)(\boldsymbol{u}, \boldsymbol{v}) = \int u_\alpha v_\alpha dx + \epsilon^2 \int u_3 v_3 dx. \tag{3.4.46}$$

Consider the vector

$$\boldsymbol{v}(\epsilon) = \boldsymbol{w}(\epsilon) - \Sigma_{k=1}^{l} D(\epsilon)(\boldsymbol{w}(\epsilon), \boldsymbol{u}^k(\epsilon))\boldsymbol{u}^k(\epsilon). \tag{3.4.47}$$

Then

$$D(\epsilon)(\boldsymbol{v}(\epsilon), \boldsymbol{u}^k(\epsilon)) = 0 \text{ for all } 1 \leq k \leq l. \tag{3.4.48}$$

Therefore it follows from the variational characterization of the eigenvalues that

$$\xi^{l+1}(\epsilon) \leq \frac{\int_\Omega b(\epsilon)(\boldsymbol{v}(\epsilon), \boldsymbol{v}(\epsilon))}{D(\epsilon)(\boldsymbol{v}(\epsilon), \boldsymbol{v}(\epsilon))}. \tag{3.4.49}$$

Passing to the limit in the above inequality, it can be shown that

$$\xi^{l+1} \leq \xi \tag{3.4.50}$$

which contradicts (3.4.45) and the proof is complete. ∎

Lemma 3.4.6 *A smooth enough solution* (ζ_α^l, ξ^l) *of the variational equation (3.4.8) solves the following equation.*

$$\begin{aligned} &\frac{E}{a}(I_{\alpha\beta}\zeta_\alpha^{l''}(x_3))'' = \xi^l \zeta_\alpha^l \ \textit{in} \ (0,1) \\ &\zeta_\alpha^l(0) = \zeta_\alpha^{l'}(0) = \zeta_\alpha^l(1) = \zeta_\alpha^{l'}(1) = 0. \end{aligned} \tag{3.4.51}$$

Proof. It follows easily from integration by parts in equation (3.4.8). ∎

3.5 Conclusions

By rescaling the variables and posing the three dimensional eigenvalue problem for thin rods over a fixed domain, we have been able to show that the eigensolutions converge towards those of a one-dimensional fourth order eigenvalue problem. It is possible to now effect a descaling and obtain a one-dimensional model approximating the original problem.

The main difference between the model obtained for the eigenvalue problem studied here and the stationary problem studied by Le Dret [28] is that in the former we get a one-dimensional eigenvalue problem involving only the horizontal components of the limit eigenvectors and the vertical component converges to zero, while in the latter, we have a coupled fourth order system involving all three components. This has been commented upon in detail earlier (cf. Remark 3.4.2 in Section 4) and, in some sense, our model is more intrinsic since it does not involve special kinds of scalings of the force components which the stationary problem needed.

Another minor difference is in the presence of the coefficient $\frac{1}{a}$, a being the area of the cross section ω, in the bilinear form of the one-dimensional model. Again this is natural. Even the stationary problem should have this coefficient (cf. Equation (35) of Le Dret [28]). The right-hand side of this equation would then have the average of the forces over a cross-section rather than just the integral.

Both these phenomena were also observed in the case of shallow shells in chapter 2. In that case it is showed that shallow vibrating shells, in the limit, behaved in a manner similar to vibrating plates.

Chapter 4

Flexural Shells

4.1 Introduction

In this chapter, we study the limiting behaviour of eigenvalues and eigenfunctions describing the vibrations of a thin linearly elastic shell, clamped along its lateral surface, under a geometric assumption on the middle surface of the shell that the space of inextensional displacements (cf. (4.4.1)) is non zero. In the stationary case, under additional assumptions on the order of magnitude of the body forces, this leads to the two-dimensional model of the "flexural shell" as shown by Ciarlet, Lods and Miara [17].

Examples of clamped shells which obey the above geometric condition, thus leading to the flexural model, are plates or, more generally, shells which are 'flat' in some region (cf. Remark 4.4.1 below). Also, if the middle surface of the shell is a cylinder and the shell is clamped along a part of the lateral surface, the middle line of which is contained in a generatrix of the cylinder, the above geometric condition holds. The results of this chapter, though proved for shells clamped along the entire lateral surface, holds for partially clamped case as well.

Our procedure to study the corresponding eigenvalue problem is the standard one. Starting with the three-dimensional eigenvalue problem (corresponding to the one studied by Ciarlet, Lods and Miara [17] in the stationary case), we rescale the variables and obtain a problem posed over a fixed domain where the parameter ϵ (corresponding to the thickness of the shell and the dimension of the three-dimensional domain over which the reference configuration of the shell is defined) now appears in the various bilinear forms. We can then pass to the limit after obtaining suitable *a priori* estimates. Unlike the preceding chapters, where we worked in Cartesian coordinates, we will henceforth work in curvilinear coordinates.

The key to making this procedure work lies in obtaining the suitable *a priori* estimates. This is where the principal mathematical contribution of this chapter lies. It must be observed that in previous works (cf. Ciarlet and Lods [14] and Ciarlet, Lods and Miara [17]) the membrane and flexural models were obtained based on two assumptions. First the nature of the space of inextensional displacements and second, the orders of magnitude of the body forces. If the forces were of the order $O(1)$ and the middle surface of the shell is "uniformly elliptic" in the sense that the two principal radii of curvatures are either both > 0 or both < 0 at all points of the middle surface of the shell, then the above mentioned space reduces to zero and the membrane shell model was obtained in the limit. If the space was non-trivial and the forces were of order $O(\epsilon^2)$, the flexural shell model was obtained in the limit.

In our case, we do not have the body forces and so we cannot make any extra assumption on their size. So how does the shell decide on its limiting behaviour vis-a-vis its vibrations, on the basis of the nature of the space of inextensional displacements? We show in this chapter, that if the space is infinite-dimensional, then the eigenvalues (at each level l, $l = 0, 1, 2, ...$) are of the order $O(\epsilon^2)$ by considering suitable test functions to be used in the variational characterization of the eigenvalues and the corresponding scaled eigensolutions converge to the eigensolutions of the two-dimensional flexural shell problem. We also show using the techniques of Ciarlet and Kesavan [11], that all the eigensolutions of the two-dimensional problem are obtained this way. If the space is of finite dimension, say N, then we show that the first

N eigenvalues are of order $O(\epsilon^2)$ and the corresponding scaled eigensolutions of the three-dimensional problem converges to the N eigensolutions of the two-dimensional flexural shell model (which now reduces to a N-dimensional algebraic eigenvalue problem) and that either the other eigenfunctions of the three-dimensional problem converge to zero weakly in $(H^1(\Omega))^2 \times L^2(\Omega)$ or the eigensolutions converge to a solution of the two-dimensional eigenvalue problem for membrane shells.

As in the case of the shallow shell, there will be a difference of a factor of 2 (2ϵ after descaling) between the coefficients obtained here and those obtained on passing to the limit in stationary problems. This is natural and has been discussed in Chapter 2. The difference can be reconciled if, in the stationary model, the modified forces are the means of the body forces over the interval [-1,1] (resp; $[-\epsilon, \epsilon]$ in the descaled version) rather than just the integrals.

This chapter is organized as follows. Section 2 describes the principal notations and the formulation in curvilinear co-ordinates, of the three dimensional problem and its scaled version over a fixed domain. In Section 3, we study the rescaled problem and Section 4 is devoted to the derivation of suitable *a priori* bounds which will be needed to pass to the limit. In Section 5, we study the limit problem and Section 6 is devoted to concluding remarks.

4.2 Statement of the Problem

Let $\omega \subset I\!R^2$ be a bounded domain with Lipschitz continuous boundary γ, such that the domain lies locally on one side of its boundary. Let $y = (y_\alpha)$ denote a generic point in ω. (Greek indices will vary on the set {1,2} and the Latin indices will vary on {1,2,3}. The summation convention will be used for repeated indices in conjunction with the above mentioned rule). Let $\partial_\alpha = \frac{\partial}{\partial y_\alpha}$. Let $\phi : \overline{\omega} \to I\!R^3$ be an injective mapping of class C^3 such that the two vectors

$$\boldsymbol{a}_\alpha(y) = \partial_\alpha \phi(y)$$

are linearly independent vectors for all $y \in \overline{\omega}$, thus forming a covariant basis of the tangent plane to the surface

$$S = \phi(\overline{\omega})$$

at the point $\phi(y)$. The dual basis (contravariant basis) is denoted by $\boldsymbol{a}^\alpha(y)$. We define

$$\boldsymbol{a}^3(y) = \boldsymbol{a}_3(y) = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}.$$

Then we can define

$$\left.\begin{array}{ll} a_{\alpha\beta} := \boldsymbol{a}_\alpha . \boldsymbol{a}_\beta & a^{\alpha\beta} := \boldsymbol{a}^\alpha . \boldsymbol{a}^\beta \\ b_{\alpha\beta} := a^3 . \partial_\beta \boldsymbol{a}_\alpha & b^\beta_\alpha := a^{\beta\sigma} b_{\sigma\alpha} \\ \Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma . \partial_\beta \boldsymbol{a}_\alpha & \end{array}\right\} \tag{4.2.1}$$

in covariant, contravariant or mixed components as the case may be. These verify the usual symmetry relations. We also define

$$b^\sigma_\beta|_\alpha \;=\; \partial_\alpha b^\sigma_\beta - \Gamma^\sigma_{\alpha\tau} b^\tau_\beta - \Gamma^\tau_{\beta\alpha} b^\sigma_\tau \tag{4.2.2}$$

$$c_{\alpha\beta} \;=\; b^\sigma_\alpha b_{\sigma\beta}. \tag{4.2.3}$$

The area element along S is $\sqrt{a}dy$, where

$$a := det(a_{\alpha\beta}). \tag{4.2.4}$$

By the continuity of the functions defined above, there exists $a_0 > 0$ such that

$$0 < a_0 \leq a(y) \text{ for all } y \in \overline{\omega}. \tag{4.2.5}$$

Given $\epsilon > 0$, we define the sets

$$\Omega^\epsilon = \omega \times (-\epsilon, \epsilon), \Gamma^\epsilon_\pm = \omega \times \{\pm\epsilon\}, \Gamma^\epsilon_0 = \gamma \times [-\epsilon, \epsilon] \tag{4.2.6}$$

where $\Gamma_+^\epsilon \cup \Gamma_-^\epsilon \cup \Gamma_0^\epsilon$ defines a partition of the boundary of Ω^ϵ and Γ_0^ϵ is the lateral surface. Let $x^\epsilon = (x_i^\epsilon)$ denote a generic point in $\overline{\Omega}^\epsilon$ and set $\partial_i^\epsilon = \frac{\partial}{\partial x_i^\epsilon}$. Thus $x_\alpha^\epsilon = y_\alpha$ and so $\partial_\alpha^\epsilon = \partial_\alpha$.

Define $\Phi : \overline{\Omega}^\epsilon \to R^3$ by

$$\Phi(x^\epsilon) = \phi(y) + x_3^\epsilon \boldsymbol{a}^3(y), \text{ for all } x^\epsilon = (y, x_3^\epsilon) \in \overline{\Omega}^\epsilon. \tag{4.2.7}$$

It can be shown that, there exists an $\epsilon_0 > 0$, such that the vectors

$$g_i^\epsilon(x^\epsilon) = \partial_i^\epsilon \Phi(x^\epsilon)$$

are linearly independent at all points $x^\epsilon \in \overline{\Omega}^\epsilon$ for $0 < \epsilon \leq \epsilon_0$ and that the mapping Φ is injective. These vectors forms a covariant basis of the tangent space of $\Phi(\Omega^\epsilon)$ (which is $\mathbb{R}^3$) at $\Phi(x^\epsilon)$ and one can, as usual, define the contravariant basis $\{g^{i,\epsilon}(x^\epsilon)\}$ by duality. The covariant and contravariant metric tensors are given respectively by

$$g_{ij}^\epsilon = g_i^\epsilon . g_j^\epsilon \text{ and } g^{ij,\epsilon} = g^{i,\epsilon} . g^{j,\epsilon} \tag{4.2.8}$$

The Christoffel symbols are defined by

$$\Gamma_{ij}^{p,\epsilon} = g^{p,\epsilon} . \partial_i^\epsilon g_j^\epsilon \tag{4.2.9}$$

The volume element is now given by $\sqrt{g^\epsilon} dx$ on $\Phi(\Omega^\epsilon)$ where

$$g^\epsilon = det(g_{ij}^\epsilon). \tag{4.2.10}$$

It can be shown that for $0 < \epsilon \leq \epsilon_0$,

$$0 < g_0 \leq g^\epsilon \leq g_1 \tag{4.2.11}$$

where g_0 and g_1 are constants independent of ϵ.

The set $\Phi(\overline{\Omega}^\epsilon)$ is the reference configuration of a shell of thickness 2ϵ with middle surface $\phi(\overline{\omega})$. We assume that the shell is clamped along its lateral surface Γ_0^ϵ. (Note : The last assumption is just for simplicity of the exposition. The results remain valid even in the partially clamped case as mentioned in the introduction.)

Assuming that the material of the shell is homogeneous and isotropic and that $\Phi(\overline{\Omega}^\epsilon)$ is natural state, the material is characterized by its Lamé constants $\lambda^\epsilon > 0$ and $\mu^\epsilon > 0$. Then the contravariant component of the three-dimensional elasticity tensor are given by

$$A^{ijkl,\epsilon} = \lambda^\epsilon g^{ij,\epsilon} g^{kl,\epsilon} + \mu^\epsilon (g^{ik,\epsilon} g^{jl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}). \tag{4.2.12}$$

Expressed in terms of the curvilinear co-ordinates (x^ϵ) of the reference configuration $\Phi(\overline{\Omega}^\epsilon)$ of the shell, we define the space of admissible displacements by

$$\boldsymbol{V}(\Omega^\epsilon) = \{\boldsymbol{v}^\epsilon = (v_i^\epsilon) \in H^1(\Omega^\epsilon) | \boldsymbol{v}^\epsilon = 0 \text{ on } \Gamma_0^\epsilon\}. \tag{4.2.13}$$

For a displacement vector $\boldsymbol{v}^\epsilon \in \boldsymbol{V}(\Omega^\epsilon)$, we define the covariant components of the linearized strain tensor by

$$e_{i\|j}^\epsilon(\boldsymbol{v}^\epsilon) = \frac{1}{2}(\partial_i^\epsilon v_j^\epsilon + \partial_j^\epsilon v_i^\epsilon) - \Gamma_{ij}^{p,\epsilon} v_p^\epsilon. \tag{4.2.14}$$

Then the eigenvalue problem consists in finding pairs $(\xi^\epsilon, \boldsymbol{u}^\epsilon) \in I\!R \times \boldsymbol{V}(\Omega^\epsilon) \backslash \{0\}$ such that

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e_{k\|l}^\epsilon(\boldsymbol{u}^\epsilon) e_{i\|j}^\epsilon(\boldsymbol{v}^\epsilon) \sqrt{g^\epsilon} dx^\epsilon = \xi^\epsilon \int_{\Omega^\epsilon} u_i^\epsilon v_i^\epsilon \sqrt{g^\epsilon} dx^\epsilon \tag{4.2.15}$$

for all $\boldsymbol{v}^\epsilon \in \boldsymbol{V}(\Omega^\epsilon)$. By classical arguments, we can show that there exists a sequence of eigenvalues

$$0 < \xi^{\epsilon,1} \leq \xi^{\epsilon,2} \leq \ldots \leq \xi^{\epsilon,l} \leq \ldots \to \infty \tag{4.2.16}$$

and we can choose a corresponding family of eigenfunctions $\{\boldsymbol{u}^{\epsilon,l}\}$ such that

$$\int_{\Omega^\epsilon} u_i^{\epsilon,l} u_i^{\epsilon,m} \sqrt{g^\epsilon} dx^\epsilon = \delta_{lm}. \tag{4.2.17}$$

The sequence $\{\boldsymbol{u}^{\epsilon,l}\}$ form a basis in the weighted space

$$(L^2(g_\epsilon; \Omega^\epsilon))^3 = \left\{\boldsymbol{u}^\epsilon | \int_{\Omega^\epsilon} u_i^\epsilon u_i^\epsilon \sqrt{g^\epsilon} dx^\epsilon < \infty\right\} \tag{4.2.18}$$

with the obvious inner-product. (However, in view of the inequalities (4.2.11), it follows that $(L^2(g_\epsilon; \Omega^\epsilon))^3 = (L^2(\Omega^\epsilon))^3$ and that the two topologies are equivalent).

4.3 The Rescaled Problem

We now scale this problem to one posed over a domain independent of ϵ. We set

$$\Omega = \omega \times (-1, 1), \Gamma_{\pm} = \omega \times \{\pm 1\}, \Gamma_0 = \gamma \times [-1, 1]. \tag{4.3.1}$$

If $x = (x_i) \in \Omega$ is a generic point, we set $\partial_i = \frac{\partial}{\partial x_i}$ and with $x^\epsilon = (x_i^\epsilon) \in \overline{\Omega}^\epsilon$, we associate $x \in \overline{\Omega}$ by

$$x_\alpha = x_\alpha^\epsilon = y_\alpha,\ x_3 = \frac{1}{\epsilon} x_3^\epsilon. \tag{4.3.2}$$

Thus, $\partial_\alpha^\epsilon = \partial_\alpha$ and $\partial_3^\epsilon = \frac{1}{\epsilon}\partial_3$.

Given a vector $\boldsymbol{v}^\epsilon \in \boldsymbol{V}(\Omega^\epsilon)$, we associate the vector $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ where

$$\boldsymbol{V}(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^3 | \boldsymbol{v} = 0 \text{ on } \Gamma_0\} \tag{4.3.3}$$

by

$$v_i(x) = v_i^\epsilon(x^\epsilon) \tag{4.3.4}$$

where x and x^ϵ have the correspondence mentioned above. Given an eigenvector $\boldsymbol{u}^{\epsilon,l}$, we denote the corresponding vector obtained via (4.3.4) by $\boldsymbol{u}^l(\epsilon)$. We assume further that the material properties of the shell do not depend on the thickness, and so we set

$$\lambda^\epsilon = \lambda > 0, \mu^\epsilon = \mu > 0 \tag{4.3.5}$$

where λ and μ are independent of ϵ.

Finally, given an eigenvalue $\xi^{\epsilon,l}$, we associate with it the "scaled" eigenvalue $\xi^l(\epsilon)$ by

$$\xi^{\epsilon,l} = \epsilon^2 \xi^l(\epsilon). \tag{4.3.6}$$

Remark 4.3.1: In the case of the shallow shell, the horizontal and vertical components of the vectors were scaled differently. In the present case we have uniform treatment of all components. For the Lamé constants and eigenvalues, we have the same scaling as for shallow shells. Another point to note is that in the case of shallow shells, the shape of the middle surface and its thickness change with ϵ, while now the middle surface is of fixed shape. ∎

With the functions $\Gamma_{ij}^{p,\epsilon}, g^\epsilon, A^{ijkl,\epsilon} : \overline{\Omega}^\epsilon \to I\!R$ appearing in (4.2.9),(4.2.10) and (4.2.12), we associate the functions $\Gamma_{ij}^p(\epsilon), g(\epsilon), A^{ijkl}(\epsilon) : \overline{\Omega} \to I\!R$ defined by

$$\Gamma_{ij}^p(\epsilon)(x) := \Gamma_{ij}^{p,\epsilon}(x^\epsilon) \text{ for all } x^\epsilon \in \overline{\Omega}^\epsilon, \tag{4.3.7}$$

$$g(\epsilon)(x) := g^\epsilon(x^\epsilon) \text{ for all } x^\epsilon \in \overline{\Omega}^\epsilon, \tag{4.3.8}$$

$$A^{ijkl}(\epsilon)(x) := A^{ijkl,\epsilon}(x^\epsilon) \text{ for all } x^\epsilon \in \overline{\Omega}^\epsilon. \tag{4.3.9}$$

Given $(\boldsymbol{v}) = (v_i) \in (H^1(\Omega))^3$, we associate the symmetric tensor $(e_{i\|j}(\epsilon)(\boldsymbol{v}))$ by

$$\left.\begin{array}{rcl} e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) &=& \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma_{\alpha\beta}^\sigma(\epsilon) v_\sigma \\ e_{\alpha\|3}(\epsilon)(\boldsymbol{v}) &=& \frac{1}{2}(\partial_\alpha v_3 + \frac{1}{\epsilon}\partial_3 v_\alpha) - \Gamma_{\alpha 3}^\sigma(\epsilon) v_\sigma \\ e_{3\|3}(\epsilon)(\boldsymbol{v}) &=& \frac{1}{\epsilon}\partial_3 v_3 \end{array}\right\}. \tag{4.3.10}$$

Then if $(\xi^\epsilon, \boldsymbol{u}^\epsilon) \in I\!R \times \boldsymbol{V}(\Omega^\epsilon)\backslash\{0\}$ is a solution of (4.2.15), the scaled variables $(\xi(\epsilon), \boldsymbol{u}(\epsilon)) \in I\!R \times \boldsymbol{V}(\Omega)\backslash\{0\}$ is a solution of the problem

$$\int_\Omega A^{ijkl}(\epsilon) e_{k\|l}(\epsilon)(\boldsymbol{u}(\epsilon)) e_{i\|j}(\epsilon)(\boldsymbol{v}) \sqrt{g(\epsilon)}\, dx = \epsilon^2 \xi(\epsilon) \int_\Omega u_i(\epsilon) v_i \sqrt{g(\epsilon)}\, dx \tag{4.3.11}$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$. Once again, it is clear that $\{\xi^l(\epsilon)\}$ corresponding to $\xi^{l,\epsilon}$ via (4.3.6) are the only eigenvalues of (4.3.11) and that the corresponding eigenvectors $\{\boldsymbol{u}^l(\epsilon)\}$ are complete in $(L^2(\Omega))^3$ and satisfy the orthogonality conditions

$$\int_\Omega u_i^l(\epsilon) u_i^m(\epsilon) \sqrt{g(\epsilon)}\, dx = \delta_{lm}. \tag{4.3.12}$$

Further, we have the following variational characterization of the eigenvalues.

Define the Rayleigh quotient $R(\epsilon)(\boldsymbol{v})$ for $\boldsymbol{v} \in \boldsymbol{V}(\Omega)\backslash\{0\}$ by

$$R(\epsilon)(\boldsymbol{v}) = \frac{\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\epsilon)(\boldsymbol{v}) e_{i\|j}(\epsilon)(\boldsymbol{v}) \sqrt{g(\epsilon)}\, dx}{\epsilon^2 \int_\Omega v_i v_i \sqrt{g(\epsilon)}\, dx}. \tag{4.3.13}$$

Let $\mathcal{V}_l$ be the collection of all l-dimensional subspaces of $\boldsymbol{V}(\Omega)$. Then

$$\xi^l(\epsilon) = \min_{\boldsymbol{W} \in \mathcal{V}_l} \max_{\boldsymbol{v} \in \boldsymbol{W}\backslash\{0\}} R(\epsilon)(\boldsymbol{v}) \tag{4.3.14}$$

Our first result gathers all the properties needed in the sequel concerning the behaviour of the functions $\Gamma^p_{ij}(\epsilon), g(\epsilon), A^{ijkl}(\epsilon)$ as $\epsilon \to 0$. If $w \in \mathcal{C}^0(\overline{\Omega})$, we define

$$||w||_{0,\infty,\overline{\Omega}} := \sup_{x\in\overline{\Omega}} |w(x)|. \tag{4.3.15}$$

Lemma 4.3.1 *The functions $\Gamma^p_{ij}(\epsilon), g(\epsilon), A^{ijkl}(\epsilon)$ are defined for $\epsilon > 0$ as in (4.3.7)-(4.3.9) and the functions $a^{\alpha\beta}, b_{\alpha\beta}, b^\sigma_\alpha, \Gamma^\sigma_{\alpha\beta}, b^\sigma_\beta|_\alpha, c_{\alpha\beta}$ are defined as in (4.2.1)-(4.2.3). Then*

$$||\Gamma^\sigma_{\alpha\beta}(\epsilon) - (\Gamma^\sigma_{\alpha\beta} - \epsilon x_3 b^\sigma_\beta|_\alpha)||_{0,\infty,\overline{\Omega}} \leq C\epsilon^2, \tag{4.3.16}$$

$$\Gamma^3_{\alpha\beta}(\epsilon) = b_{\alpha\beta} - \epsilon x_3 c_{\alpha\beta}, \tag{4.3.17}$$

$$||\Gamma^\sigma_{\alpha 3}(\epsilon) + b^\sigma_\alpha||_{0,\infty,\overline{\Omega}} \leq C\epsilon \tag{4.3.18}$$

$$\Gamma^3_{\alpha 3}(\epsilon) = \Gamma^p_{33}(\epsilon) = 0 \tag{4.3.19}$$

$$||g(\epsilon) - a||_{0,\infty,\overline{\Omega}} \leq C\epsilon \tag{4.3.20}$$

$$||A^{ijkl}(\epsilon) - A^{ijkl}(0)||_{0,\infty,\overline{\Omega}} \leq C\epsilon \tag{4.3.21}$$

$$A^{\alpha\beta\sigma 3}(\epsilon) = A^{\alpha 333}(\epsilon) = 0 \tag{4.3.22}$$

for all $0 \leq \epsilon \leq \epsilon_0$, where

$$A^{\alpha\beta\sigma\tau}(0) := \lambda a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}) \tag{4.3.23}$$

$$A^{\alpha\beta 33}(0) := \lambda a^{\alpha\beta}, A^{\alpha 3\sigma 3}(0) = \mu a^{\alpha\sigma}, \tag{4.3.24}$$

$$A^{3333}(0) = \lambda + 2\mu, A^{\alpha\beta\sigma 3}(0) = A^{\alpha 333} = 0 \tag{4.3.25}$$

and, finally there exists a constant C such that

$$t_{ij} t_{ij} \leq C A^{ijkl}(\epsilon)(x) t_{kl} t_{ij} \tag{4.3.26}$$

for all $0 \leq \epsilon \leq \epsilon_0$, $x \in \overline{\Omega}$, and for all symmetric tensors (t_{ij})

Proof. See the proof of Lemma 3.1 in Ciarlet and Lods [14]. ∎

In the next lemma, we analyze the asymptotic behaviour as $\epsilon \to 0$ of the functions $e_{\alpha||\beta}(\epsilon)(\boldsymbol{v})$. To this end, we are naturally led to introduce the three dimensional analogs (cf. (4.3.27)-(4.3.28)) of the two dimensional change of metric tensor and change of curvature tensor, which play a fundamental role in the definition of the limit eigenvalue problem.

Lemma 4.3.2 *For any $\boldsymbol{v} = (v_i) \in H^1(\Omega)$, let the functions $\gamma_{\alpha\beta}(\boldsymbol{v}) \in L^2(\Omega)$ and $\rho_{\alpha\beta}(\boldsymbol{v}) \in H^{-1}(\Omega)$ be defined by:*

$$
\begin{aligned}
\gamma_{\alpha\beta}(\boldsymbol{v}) &= \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta} v_\sigma - b_{\alpha\beta} v_3, & (4.3.27)\\
\rho_{\alpha\beta}(\boldsymbol{v}) &= \partial_{\alpha\beta} v_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma v_3 + b^\sigma_\beta(\partial_\alpha v_\sigma - \gamma^\tau_{\alpha\sigma} v_\tau)\\
&+ b^\sigma_\alpha(\partial_\beta v_\sigma - \Gamma^\tau_{\beta\sigma} v_\tau) + b^\tau_\alpha|_\beta v_\sigma - c_{\alpha\beta} v_3 & (4.3.28)
\end{aligned}
$$

Then the functions $e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v})$ defined in (4.3.10) satisfy

$$
\|\frac{1}{\epsilon} e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) - e^1_{\alpha\|\beta}(\epsilon)(\boldsymbol{v})\|_{0,\Omega} \le C\epsilon \sum_\alpha \|v_\alpha\|_{0,\Omega} \tag{4.3.29}
$$

where

$$
e^1_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) := \frac{1}{2\epsilon}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \frac{1}{\epsilon}(\Gamma^\sigma_{\alpha\beta} v_\sigma + b_{\alpha\beta} v_3) + x_3 b^\sigma_\beta|_\alpha v_\sigma + x_3 c_{\alpha\beta} v_3 \tag{4.3.30}
$$

and

$$
\|\frac{1}{\epsilon}\partial_3 e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) + \rho_{\alpha\beta}(\boldsymbol{v})\|_{-1,\Omega} \le \left\{ \sum_i \|e_{i\|3}(\epsilon)(\boldsymbol{v})\|_{0,\Omega} + \epsilon \sum_\alpha \|v_\alpha\|_{0,\Omega} + \epsilon \|v_3\|_{1,\Omega} \right\}. \tag{4.3.31}
$$

Proof. See the proof of Lemma 3.2 in Ciarlet, Lods and Miara [17]. ∎

The next two lemmas are crucial as they play essential role in the proof of the boundedness of the scaled eigenvalues as $\epsilon \to 0$.

Lemma 4.3.3 *Let the two-dimensional elasticity tensor $a^{\alpha\beta\sigma\tau}$ be defined by*

$$
a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}). \tag{4.3.32}
$$

Then there exists a constant $C > 0$ such that

$$
a^{\alpha\beta\sigma\tau} t_{\sigma\tau} t_{\alpha\beta} \ge C (t_{\alpha\beta})^2 \qquad \text{for all symmetric tensors } (t_{\alpha\beta}). \tag{4.3.33}
$$

Proof. See the proof of Lemma 2.1 in Bernadou, Ciarlet and Miara [3]. ∎

Lemma 4.3.4 *For $\boldsymbol{\eta} \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$, let the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta}), \rho_{\alpha\beta}(\boldsymbol{\eta})$ be defined as in (4.3.27) and (4.3.28). Then there exists a constant C such that*

$$\left(\sum_{\alpha,\beta} ||\gamma_{\alpha\beta}(\boldsymbol{\eta})||_{0,\omega}^2 + \sum_{\alpha,\beta} ||\rho_{\alpha\beta}(\boldsymbol{\eta})||_{0,\omega}^2\right) \geq C\left(\sum_{\alpha} ||\eta_\alpha||_{1,\omega}^2 + ||\eta_3||_{2,\omega}^2\right) \qquad (4.3.34)$$

for all $\boldsymbol{\eta} = (\eta_i) \in (H_0^1(\omega))^2 \times H_0^2(\omega)$.

Proof. See the proof of Lemma 3.2 in Bernadou, Ciarlet and Miara [3]. ∎

4.4 *A Priori* Estimates

In this section, we will show that if the space $V_F(\omega)$ is infinite dimensional, then the scaled eigenvalues $\xi^l(\epsilon)$ are bounded uniformly with respect to ϵ for each fixed positive integer l. If the dimension of $V_F(\omega)$ is finite, say N, then we show that for $1 \leq l \leq N, \xi^l(\epsilon)$ is uniformly bounded with respect to ϵ, and for $l > N$, we will show that $\epsilon^2\xi^l(\epsilon)$ is uniformly bounded with respect to ϵ and the limits of $\epsilon^2\xi^l(\epsilon), l > N$ lies in a bounded subset of $\mathbb{R}$.

We henceforth denote by C, a constant which does not depend on both ϵ and l but its value varies from place to place.

First of all, we need to define the space of inextensional displacements. Following earlier works (cf.[17]), we define the space of inextensional displacements $V_F(\omega)$ by

$$V_F(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in (H_0^1(\omega))^2 \times H_0^2(\omega) | \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\} \qquad (4.4.1)$$

As observed by Ciarlet and Lods[14], Ciarlet, Lods and Miara[17], this space may or may not be trivial.

Assumption: We assume henceforth that $V_F(\omega) \neq \{0\}$

Remark 4.4.1: It is not clear whether the space $V_F(\omega)$ is always infinite dimensional if it is non-zero. But if the functions $b_{\alpha\beta}$ defined in (4.2.1) vanishes in a neighbourhood ω' of a point in ω, then the space $0 \times 0 \times H_0^2(\omega')$ is contained in

$V_F(\omega)$ and hence $V_F(\omega)$ is infinite dimensional. For example, in the case of plates, $b_{\alpha\beta} = 0$ and $V_F(\omega) = \{0\} \times \{0\} \times H_0^2(\omega)$. ∎

It follows from Lemma 4.3.4 that,

$$\boldsymbol{\eta} \to \left(\sum_{\alpha,\beta} ||\rho_{\alpha\beta}(\boldsymbol{\eta})||_{0,\omega}^2 \right)^{\frac{1}{2}} \tag{4.4.2}$$

will be a norm on the space $V_F(\omega)$ equivalent to the $(H_0^1(\omega))^2 \times H_0^2(\omega)$ norm.

Let $\boldsymbol{\eta} = (\eta_i) \in (H_0^1(\omega))^2 \times H_0^2(\omega)$. Then we define, following an idea of Miara and Sanchez-Palencia, $v_\epsilon(\boldsymbol{\eta}) \in \boldsymbol{V}(\Omega)$ by

$$(v_\epsilon(\boldsymbol{\eta}))_\alpha = \eta_\alpha - \epsilon x_3(\partial_\alpha \eta_3 + 2b_\alpha^\sigma \eta_\sigma) \tag{4.4.3}$$

$$(v_\epsilon(\boldsymbol{\eta}))_3 = \eta_3. \tag{4.4.4}$$

For brevity, we will set

$$\theta_\alpha = \partial_\alpha \eta_3 + 2b_\alpha^\sigma \eta_\sigma \tag{4.4.5}$$

With these notations we have the following result.

Lemma 4.4.1 *Let $\boldsymbol{\eta} \in V_F(\omega)$. Then,*

$$\epsilon^{-1} e_{\alpha||\beta}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) \to -x_3 \rho_{\alpha\beta}(\boldsymbol{\eta}) \text{ in } L^2(\Omega) \text{ as } \epsilon \to 0 \tag{4.4.6}$$

$$\epsilon^{-1} e_{\alpha||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) \text{ is bounded in } L^2(\Omega) \tag{4.4.7}$$

$$e_{3||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) = 0 \text{ for all } \epsilon > 0 \tag{4.4.8}$$

$$v_\epsilon(\boldsymbol{\eta}) \to \boldsymbol{\eta} \text{ in } V(\Omega) \text{ as } \epsilon \to 0. \tag{4.4.9}$$

Proof. Relations (4.4.8) and (4.4.9) are obvious.

A simple computation shows that

$$\epsilon^{-1} e_{\alpha||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) = -\epsilon^{-1}(\Gamma_{\alpha 3}^\sigma(\epsilon) + b_\alpha^\sigma)\eta_\sigma + x_3 \Gamma_{\alpha 3}^\sigma(\epsilon)\theta_\sigma. \tag{4.4.10}$$

Combining with the relation (4.3.18) proves (4.4.7).

We finally prove (4.4.6). To start with, by the Lemma 4.3.2, we have that for $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$

$$||\epsilon^{-1}e_{\alpha||\beta}(\epsilon)(\boldsymbol{v}) - e^1_{\alpha||\beta}(\epsilon)(\boldsymbol{v})||_{0,\Omega} \leq C\epsilon \sum_\alpha ||v_\alpha||_{0,\Omega} \tag{4.4.11}$$

where

$$e^1_{\alpha||\beta}(\epsilon)(\boldsymbol{v}) = \epsilon^{-1}\gamma_{\alpha\beta}(\boldsymbol{v}) + x_3 b^\sigma_\beta|_\alpha v_\sigma + x_3 c_{\alpha\beta} v_3. \tag{4.4.12}$$

Observing that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$, we find (after a tedious computation) that

$$e^1_{\alpha||\beta}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) = -x_3\rho_{\alpha\beta}(\boldsymbol{\eta}) - \epsilon x_3^2 b^\sigma_\beta|_\alpha \theta_\sigma. \tag{4.4.13}$$

Thus from (4.4.11) and (4.4.13) and the definition of $v_\epsilon(\boldsymbol{\eta})$ given by (4.4.3)-(4.4.4), we get

$$||\epsilon^{-1}e_{\alpha||\beta}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) + x_3\rho_{\alpha\beta}(\boldsymbol{\eta})||_{0,\Omega} \leq C\epsilon(||\eta_\alpha||_{0,\omega} + ||\eta_3||_{1,\omega}) \tag{4.4.14}$$

which proves (4.4.6).

∎

Theorem 4.4.2 *Assume that $V_F(\omega)$ is infinite dimensional subspace of $V(\Omega)$. Then for each $l \geq 1$, the sequence $\xi^l(\epsilon)$ is bounded uniformly with respect to ϵ.*

Proof. Let $\mathcal{W}_l$ denotes the collection of all l-dimensional subspaces of $V_F(\omega)$.

Consider the map

$T_\epsilon : V_F(\omega) \to \boldsymbol{V}(\Omega)$ defined by

$$T_\epsilon(\boldsymbol{\eta}) = v_\epsilon(\boldsymbol{\eta}). \tag{4.4.15}$$

For sufficiently small ϵ, T_ϵ is one-one. Thus if $W \in \mathcal{W}_l$, then $T_\epsilon(W) \in \mathcal{V}_l$. Consequently, we have

$$\xi^l(\epsilon) \leq \min_{\boldsymbol{W} \in \mathcal{W}_l} \max_{\boldsymbol{\eta} \in \boldsymbol{W} \setminus (0)} R_\epsilon(v_\epsilon(\boldsymbol{\eta})). \tag{4.4.16}$$

We now proceed to estimate $R_\epsilon(v_\epsilon(\boldsymbol{\eta}))$ for $\boldsymbol{\eta} \in V_F(\omega)$. On one hand

$$\begin{aligned}\int_\Omega (v_\epsilon(\boldsymbol{\eta}))_i (v_\epsilon(\boldsymbol{\eta}))_i \sqrt{g(\epsilon)}dx &\geq g_0 \int_\Omega (v_\epsilon(\boldsymbol{\eta}))_i (v_\epsilon(\boldsymbol{\eta}))_i dx && (4.4.17)\\ &= 2g_0 \int_\omega \eta_3^2 d\omega + g_0 \sum_\alpha \int_\Omega (\eta_\alpha - \epsilon x_3 \theta_\alpha)^2 dx && (4.4.18)\end{aligned}$$

Since $\int_\Omega x_3 \eta_\alpha \theta_\alpha dx = 0$, we get

$$\int_\Omega (v_\epsilon(\boldsymbol{\eta}))_i (v_\epsilon(\boldsymbol{\eta}))_i \sqrt{g(\epsilon)}dx \geq 2g_0 \int_\omega \eta_i \eta_i d\omega \tag{4.4.19}$$

On the other hand,

$$\begin{aligned}&\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) e_{i||j}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\sqrt{g(\epsilon)}dx\\ &\leq g_1^{\frac{1}{2}} \left\{ \int_\Omega A^{\alpha\beta\sigma\tau}(\epsilon)\left[\frac{1}{\epsilon} e_{\sigma\tau}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\right]\left[\frac{1}{\epsilon} e_{\alpha||\beta}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\right] dx \right.\\ &\left. + \ 4\int_\Omega A^{\alpha 3\sigma 3}(\epsilon)\left[\frac{1}{\epsilon} e_{\sigma||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\right]\left[\frac{1}{\epsilon} e_{\alpha||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\right] dx\right\}\end{aligned} \tag{4.4.20}$$

using the symmetries of $A^{ijkl}(\epsilon)$, the fact that $A^{\alpha\beta\sigma3}(\epsilon) = A^{\alpha333}(\epsilon) = 0$, relations (4.2.11) and (4.4.8). By virtue of the relation (4.3.21)-(4.3.26), relations (4.4.6)-(4.4.9) of Lemma 4.4.1 above and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) e_{i||j}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\sqrt{g(\epsilon)}dx\\ &\leq C\left[\sum_{\alpha,\beta} ||\rho_{\alpha\beta}(\boldsymbol{\eta})||_{0,\omega} + \epsilon\left(\sum_\alpha ||\eta_\alpha||_{0,\omega} + ||\eta_3||_{1,\omega}\right)\right]^2\\ &+ \left(\sum_\alpha ||\eta_\alpha||_{0,\omega} + ||\eta_3||_{1,\omega}\right)^2\\ &\leq C\left[\sum_{\alpha,\beta} ||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega} + \sum_\alpha ||\eta_\alpha||^2_{0,\omega} + ||\eta_3||^2_{1,\omega}\right].\end{aligned} \tag{4.4.21}$$

for $\epsilon \leq 1$. But from (4.4.2) it follows that, since $\boldsymbol{\eta} \in V_F(\omega)$,

$$\begin{aligned}\left(\sum_\alpha ||\eta_\alpha||^2_{0,\omega} + ||\eta_3||^2_{1,\omega}\right) &\leq C\left(\sum_\alpha ||\eta_\alpha||^2_{1,\omega} + ||\eta_3||^2_{2,\omega}\right)\\ &\leq C\sum_{\alpha,\beta} ||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega}.\end{aligned} \tag{4.4.22}$$

Thus

$$\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) e_{i||j}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\sqrt{g(\epsilon)}dx \leq C\sum_{\alpha,\beta} ||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega}. \tag{4.4.23}$$

It follows from (4.4.19) and (4.4.23) that

$$R_\epsilon(v_\epsilon(\boldsymbol{\eta})) \leq C\frac{\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega}}{\sum_i||\eta_i||^2_{0,\omega}}. \tag{4.4.24}$$

From Lemma 4.3.3 it follows that there exists $C > 0$ such that

$$\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})\rho_{\sigma\tau}(\boldsymbol{\eta})\sqrt{a}dy \geq C\sum_\alpha ||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega} \tag{4.4.25}$$

for all $\boldsymbol{\eta} \in V_F(\omega)$. Thus, we have

$$R_\epsilon(v_\epsilon(\boldsymbol{\eta})) \leq C\frac{\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})\rho_{\sigma\tau}(\boldsymbol{\eta})\sqrt{a}d\omega}{\int_\omega \eta_i\eta_i\sqrt{a}d\omega} \tag{4.4.26}$$

and hence, from (4.4.16) and (4.4.26) it follows that

$$\xi^l(\epsilon) \leq C\Lambda^l \tag{4.4.27}$$

where Λ^l is the l^{th}-eigenvalue of the two-dimensional problem:

Find $(\Lambda, \boldsymbol{\zeta}) \in I\!R \times V_F(\omega)\backslash\{0\}$ such that

$$\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}d\omega = \Lambda\int_\omega \zeta_i\eta_i\sqrt{a}d\omega \tag{4.4.28}$$

for all $\boldsymbol{\eta} \in V_F(\omega)$. This completes the proof. ∎

Theorem 4.4.3 *Assume that $dim(V_F(\omega)) = N$. Then for $1 \leq l \leq N, \xi^l(\epsilon)$ is uniformly bounded with respect to ϵ and for each positive integer $l > N$, there exists constants C(independent of ϵ and l) and k^l(independent of ϵ) such that $\epsilon^2\xi^l(\epsilon) \leq C(1+\epsilon^2k^l)$.*

Proof. The proof that for $1 \leq l \leq N$, $\xi^l(\epsilon)$ is bounded follows from Theorem 4.4.2.

Let W_l denote the collection of all l-dimensional subspaces of $H^2_0(\omega)$.

For $\eta \in W_l$, define $w_\epsilon(\eta) \in \boldsymbol{V}(\Omega)$ by

$$(w_\epsilon(\eta))_\alpha = -\epsilon x_3\partial_\alpha\eta \tag{4.4.29}$$

$$(w_\epsilon(\eta))_3 = \eta. \tag{4.4.30}$$

Then a simple computation shows that

$$e_{\alpha||\beta}(\epsilon)(w_\epsilon(\eta)) = -\epsilon x_3(\partial_{\alpha\beta}\eta + \Gamma^\sigma_{\alpha\beta}(\epsilon)\partial_\sigma\eta) - \Gamma^3_{\alpha\beta}(\epsilon)\eta \tag{4.4.31}$$

$$e_{\alpha||3}(\epsilon)(w_\epsilon(\eta)) = -\epsilon x_3\Gamma^\sigma_{\alpha 3}\partial_\sigma\eta \tag{4.4.32}$$

$$e_{3||3}(w_\epsilon(\eta)) = 0. \tag{4.4.33}$$

For $W \in W_l$, define

$$\boldsymbol{W} = \{w_\epsilon(\eta) : \eta \in W\}. \tag{4.4.34}$$

Then $\boldsymbol{W} \in \mathcal{V}_l$ and hence it follows from (4.3.14) that

$$\xi^l(\epsilon) \le \min_{W\in W_l} \max_{\eta\in W} R_\epsilon(w_\epsilon(\eta)). \tag{4.4.35}$$

We now proceed to calculate $R_\epsilon(w_\epsilon(\eta))$. On one hand

$$\begin{aligned}\int_\Omega (w_\epsilon(\eta))_i(w_\epsilon(\eta))_i\sqrt{g(\epsilon)}dx &\ge g_0\int_\Omega (w_\epsilon(\eta))_i(w_\epsilon(\eta))_i dx\\ &= 2g_0\int_\Omega \eta^2 dx.\end{aligned} \tag{4.4.36}$$

On the other hand

$$\begin{aligned}&\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(w_\epsilon(\eta))e_{i||j}(\epsilon)(w_\epsilon(\eta))\sqrt{g(\epsilon)}dx\\ &\le g_1^{\frac{1}{2}}\left\{\int_\Omega A^{\alpha\beta\sigma\tau}(\epsilon)[e_{\sigma||\tau}(\epsilon)(w_\epsilon(\eta))][e_{\alpha||\beta}(\epsilon)(w_\epsilon(\eta))]dx\right.\\ &+ \left. 4\int_\Omega A^{\alpha 3\sigma 3}[e_{\sigma||3}(\epsilon)(w_\epsilon(\eta))][e_{\alpha||3}(\epsilon)(w_\epsilon(\eta))]dx\right\}\end{aligned} \tag{4.4.37}$$

using the symmetries of $A^{ijkl}(\epsilon)$, the fact that $A^{\alpha 3\sigma 3}(\epsilon) = A^{\alpha 333}(\epsilon) = 0$, the relations (4.2.11) and (4.4.33). By virtue of relations (4.3.21)-(4.3.26), (4.4.31), (4.4.32) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(w_\epsilon(\eta))e_{i||j}(\epsilon)(w_\epsilon(\eta))\sqrt{g(\epsilon)}dx\\ &\le C\left[\epsilon\left(\sum_{\alpha,\beta}||\partial_{\alpha\beta}(\eta)||_{0,\omega} + \sum_\alpha||\partial_\alpha\eta||_{0,\omega}\right) + ||\eta||_{0,\omega}\right]^2\\ &+ C\epsilon^2\sum_\alpha||\partial_\alpha\eta||^2\\ &\le C\left[\epsilon^2\sum_{\alpha,\beta}||\partial_{\alpha\beta}\eta||^2_{0,\omega} + ||\eta||^2_{0,\omega}\right].\end{aligned} \tag{4.4.38}$$

It follows from (4.4.36) and (4.4.38) that

$$R_\epsilon(w_\epsilon(\eta)) \leq C\frac{\left(\epsilon^2 \sum_{\alpha,\beta} ||\partial_{\alpha\beta}\eta||^2_{0,\omega} + ||\eta||^2_{0,\omega}\right)}{\epsilon^2||\eta||^2_{0,\omega}}. \tag{4.4.39}$$

Hence from (4.4.35) and (4.4.39), it follows that

$$\epsilon^2\xi^l(\epsilon) \leq C(\epsilon^2 k^l + 1) \tag{4.4.40}$$

where k^l is the l-th eigenvalue of the two-dimensional problem:

Find $(k,\zeta) \in I\!R \times H_0^2(\omega)$ such that

$$\left.\begin{array}{l}\Delta^2\zeta = k\zeta \text{ in } \omega \\ \zeta = 0 \text{ on } \partial\omega\end{array}\right\}. \tag{4.4.41}$$

This completes the proof. ∎

4.5 The Limit Problem

In this section we show that if the space $V_F(\omega)$ is infinite dimensional, then for each fixed integer $l \geq 1$, the scaled eigensolutions $(\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ converge towards a limit $(\xi^l, \boldsymbol{u}^l)$ which can be identified with the eigensolution of the two-dimensional "flexural shell" problem posed over the set ω. If the dimension of the space $V_F(\omega)$ is finite, say N, then we will show that the first N scaled eigensolutions converge to the N eigensolutions of the two-dimensional "flexural shell" problem and other eigensolutions either converge to the solution of the two-dimensional "membrane shell" problem or the eigenvectors converge weakly to zero in $(H^1(\Omega))^2 \times L^2(\Omega)$.

The next three lemmas are crucial; they play an important role in the proof of the convergence of the scaled unknowns as $\epsilon \to 0$. In the following statement, the symbols $\to$ and $\rightharpoonup$ denote the strong and weak convergences respectively.

Lemma 4.5.1 *Let* $\boldsymbol{V}(\Omega)$ *be the space defined in (4.3.3) and the functions* $e_{i\|j}(\epsilon)(\boldsymbol{v}) \in L^2(\Omega)$, $\gamma_{\alpha\beta}(\boldsymbol{v}) \in L^2(\Omega), \rho_{\alpha\beta}(\boldsymbol{v}) \in H^{-1}(\Omega)$ *be defined for any function* $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ *as in (4.3.10) ,(4.3.27) and (4.3.28). Let* $(\boldsymbol{v}(\epsilon))_{\epsilon>0}$ *be a sequence of functions in* $\boldsymbol{V}(\Omega)$ *such that*

$$\boldsymbol{v}(\epsilon) \rightharpoonup \boldsymbol{v} \text{ in } H^1(\Omega) \tag{4.5.1}$$

$$\frac{1}{\epsilon} e_{i\|j}(\epsilon)(\boldsymbol{v}(\epsilon)) \rightharpoonup e^1_{i\|j} \text{ in } L^2(\Omega) \tag{4.5.2}$$

as $\epsilon \to 0$. *Then*

$\boldsymbol{v} = (v_i)$ *is independent of the transverse variable* x_3,

$$\overline{\boldsymbol{v}} = (\overline{v_i}) = \frac{1}{2}\int_{-1}^{1} v dx_3 \in (H^1_0(\omega))^2 \times H^2_0(\omega) \tag{4.5.3}$$

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = 0 \tag{4.5.4}$$

$$\rho_{\alpha\beta}(\boldsymbol{v}) \in L^2(\Omega) \text{ and } \rho_{\alpha\beta}(\boldsymbol{v}) = -\partial_3 e^1_{\alpha\|\beta}. \tag{4.5.5}$$

If in addition to (4.5.1)-(4.5.2), there exists a function $\chi_{\alpha\beta} \in H^{-1}(\Omega)$ *such that*

$$\rho_{\alpha\beta}(\boldsymbol{v}(\epsilon)) \to \chi_{\alpha\beta} \text{ in } H^{-1}(\Omega) \text{ as } \epsilon \to 0 \tag{4.5.6}$$

Then

$$\boldsymbol{v}(\epsilon) \to \boldsymbol{v} \text{ in } H^1(\Omega) \text{ as } \epsilon \to 0 \tag{4.5.7}$$

$$\rho_{\alpha\beta}(\boldsymbol{v}) = \chi_{\alpha\beta} \text{ and thus } \chi_{\alpha\beta} \in L^2(\Omega) \tag{4.5.8}$$

Proof. See the proof of Lemma 3.3 of Ciarlet, Lods and Miara[17]. ∎

The key to the convergence theorem (Theorem 4.5.4) is the generalized Körn's inequality (4.5.9), which involves the functions $e_{i\|j}(\epsilon)(\boldsymbol{v})$ defined in (4.3.10) instead of the traditional function $e_{ij}(\boldsymbol{v})$. This generalized Korn's inequality is valid for an arbitrary surface $S = \phi(\omega)$ (the only requirements are that the set ω and the mapping ϕ satisfy the assumptions of Section 2), irrespective of whether the space $V_F(\omega)$ defined in (4.4.1) reduces to zero or not.

Lemma 4.5.2 *Let the space $\boldsymbol{V}(\Omega)$ be defined as in (4.3.3). Then there exists $0 < \epsilon \leq \epsilon_1$ and $C > 0$ such that for all $0 < \epsilon \leq \epsilon_1$*

$$||\boldsymbol{v}||_{1,\Omega} \leq \frac{C}{\epsilon}\left(\sum_{i,j} ||e_{i||j}(\epsilon)(\boldsymbol{v})||_{0,\Omega}^2\right)^{\frac{1}{2}} \text{ for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega) \qquad (4.5.9)$$

where the tensor $e_{i||j}(\epsilon)(\boldsymbol{v})$ is defined as in (4.3.10).

Proof. See the proof of Theorem 4.1 of Ciarlet, Lods and Miara [17]. ∎

Theorem 4.5.3 *There exists a positive constant ϵ_1 such that for all $0 < \epsilon \leq \epsilon_1$*

$$\left\{\sum_{\alpha} ||v_\alpha||_{1,\Omega}^2 + ||\epsilon v_3||_{1,\Omega}^2\right\} \leq C\left\{\sum_{i,j} ||e_{i||j}(\epsilon)(\boldsymbol{v})||_{0,\Omega}^2 + \sum_i ||v_i||_{0,\Omega}^2\right\} \qquad (4.5.10)$$

for all $\boldsymbol{v} = (v_i) \in (H^1(\Omega))^3$

Proof. Given $v \in (H^1(\Omega))^3$, let $\boldsymbol{v}(\epsilon) = (v_1, v_2, \epsilon v_3) \in (H^1(\Omega))^3$.

Then

$$e_{\alpha\beta}(\boldsymbol{v}(\epsilon)) = e_{\alpha||\beta}(\epsilon)(\boldsymbol{v}) + \Gamma^p_{\alpha\beta}(\epsilon)v_p \qquad (4.5.11)$$

$$e_{\alpha 3}(\boldsymbol{v}(\epsilon)) = \epsilon e_{\alpha||3}(\epsilon)(\boldsymbol{v}) + \epsilon\Gamma^\sigma_{\alpha 3}(\epsilon)v_\sigma \qquad (4.5.12)$$

$$e_{33}(\boldsymbol{v}(\epsilon)) = \epsilon^2 e_{3||3}(\epsilon)(\boldsymbol{v}) \qquad (4.5.13)$$

where $e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$, and consequently by (4.3.16)-(4.3.18)

$$\left\{\sum_{i,j} ||e_{ij}(\boldsymbol{v}(\epsilon))||_{0,\Omega}^2\right\} \leq C\left\{\sum_{i,j} ||e_{i||j}(\epsilon)(\boldsymbol{v})||_{0,\Omega}^2 + \sum_i ||v_i||_{0,\Omega}^2\right\} \qquad (4.5.14)$$

for $\epsilon \leq 1$. By the classical Körn's inequality,

$$\begin{aligned} ||\boldsymbol{v}(\epsilon)||_{1,\Omega}^2 &= \sum_\alpha ||v_\alpha||_{1,\Omega}^2 + ||\epsilon v_3||_{1,\Omega}^2 \\ &\leq C\left\{\sum_{i,j} ||e_{ij}(\boldsymbol{v}(\epsilon))||_{0,\Omega}^2 + ||\boldsymbol{v}(\epsilon)||_{0,\Omega}^2\right\} \qquad (4.5.15) \end{aligned}$$

and the theorem follows from inequalities (4.5.14) and (4.5.15). ∎

Theorem 4.5.4 *Assume that the space $V_F(\omega)$ is infinite dimensional. Then*

a) For each integer $l \geq 1$, there exists a subsequence (still indexed by ϵ for convenience) such that $(\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ converges in $I\!R \times H^1(\Omega)$ to $(\xi^l, \boldsymbol{u}^l)$; further $\boldsymbol{u}^l$ is independent of the transverse variable x_3 and $\overline{\boldsymbol{u}}^l \in V_F(\omega)$.

b) The pair $(\xi^l, \overline{\boldsymbol{u}}^l)$ solves the two-dimensional eigenvalue problem for the flexural shell, viz;

Find $(\xi, \boldsymbol{\zeta}) \in I\!R \times V_F(\omega)$ such that

$$\frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy = \xi\int_\omega \zeta_i\eta_i\sqrt{a}dy \text{ for all } \boldsymbol{\eta} = (\eta_i) \in V_F(\omega) \quad (4.5.16)$$

where $a^{\alpha\beta\sigma\tau}$ and $\rho_{\alpha\beta}(\boldsymbol{v})$ are defined as in (4.3.32) and (4.3.28).

Proof. The proof is similar to that of Theorem 4.1 of Ciarlet, Lods and Miara [17]. For clarity, the proof is divided into several steps.

Step 1: Boundedness of the eigenvectors in $H^1(\Omega)$:

From the variational equation (4.3.11), relation (4.3.12), inequality (4.2.11), the boundedness of the eigenvalues $\xi^l(\epsilon)$, the generalized Korn's inequality (4.5.9) and by virtue of relation (4.3.26), we infer that

$$\begin{aligned}
\epsilon^2 C^{-2}\|\boldsymbol{u}^l(\epsilon)\|^2_{1,\Omega} &\leq \sum_{i,j}\|e_{i\|j}(\epsilon)(\boldsymbol{u}^l(\epsilon))\|^2_{0,\Omega} \\
&\leq Cg_0^{\frac{-1}{2}}\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon)(\boldsymbol{u}^l(\epsilon))e_{i\|j}(\epsilon)(\boldsymbol{u}^l(\epsilon))\sqrt{g(\epsilon)}dx \\
&= \epsilon^2 Cg_0^{\frac{-1}{2}}\xi^l(\epsilon)\int_\Omega \boldsymbol{u}^l_i(\epsilon)\boldsymbol{u}^l_i(\epsilon)\sqrt{g(\epsilon)}dx \\
&\leq \epsilon^2 Cg_0^{\frac{-1}{2}}\Lambda^l \qquad (4.5.17)
\end{aligned}$$

Hence the assertion follows.

Step 2: It follows from Step 1 that there exists functions $\boldsymbol{u}^l \in H^1(\Omega)$ and $e^{1,l}_{i\|j} \in L^2(\Omega)$ such that

$$\boldsymbol{u}^l(\epsilon) \rightharpoonup \boldsymbol{u}^l \text{ in } H^1(\Omega) \quad (4.5.18)$$

$$\frac{1}{\epsilon}e_{i\|j}(\epsilon)(\boldsymbol{u}^l(\epsilon)) \rightharpoonup e^{1,l}_{i\|j} \text{ in } L^2(\Omega). \quad (4.5.19)$$

Hence it follows from Lemma 4.5.1 that $\boldsymbol{u}^l$ is independent of x_3 and $\gamma_{\alpha\beta}(\boldsymbol{u}^l) = 0$, ie, $\overline{\boldsymbol{u}}^l \in V_F(\omega)$.

Step 3: The limit functions $e^{1,l}_{i\|j}$ are related to the limit function $\boldsymbol{u}^l$ by

$$-\partial_3 e^{1,l}_{\alpha\|\beta} = \rho_{\alpha\beta}(\boldsymbol{u}^l) \tag{4.5.20}$$

$$e^{1,l}_{\alpha\|3} = 0 \tag{4.5.21}$$

$$e^{1,l}_{3\|3} = \frac{-\lambda}{\lambda + 2\mu} a^{\alpha\beta} e^{1,l}_{\alpha\|\beta}. \tag{4.5.22}$$

Relation (4.5.20) follows from Lemma 4.5.1. Let $\boldsymbol{v} = (v_i)$ be an arbitrary function in the space $\boldsymbol{V}(\Omega)$. Then

$$\epsilon e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) \to 0 \in L^2(\Omega), \tag{4.5.23}$$

$$\epsilon e_{\alpha\|3}(\epsilon)(\boldsymbol{v}) \to \frac{1}{2}\partial_3 v_\alpha \text{ in } L^2(\Omega), \tag{4.5.24}$$

$$\epsilon e_{3\|3}(\epsilon)(\boldsymbol{v}) = \partial_3 v_3 \text{ for all } \epsilon > 0. \tag{4.5.25}$$

Using the variational equation (4.3.11) and relation (4.3.22), we may write

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\epsilon) e_{k\|l}(\epsilon)(\boldsymbol{u}^l(\epsilon)) e_{i\|j}(\epsilon)(\boldsymbol{v}) \sqrt{g(\epsilon)}dx \\
&= \int_\Omega \left\{ A^{\alpha\beta\sigma\tau}(\epsilon)[\frac{1}{\epsilon} e_{\sigma\|\tau}(\epsilon)(\boldsymbol{u}^l(\epsilon))] \right\} \left\{ \epsilon e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) \right\} \sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left\{ A^{\alpha\beta33}(\epsilon)[\frac{1}{\epsilon} e_{3\|3}(\epsilon)(\boldsymbol{u}^l(\epsilon))] \right\} \left\{ \epsilon e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) \right\} \sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left\{ 4A^{\alpha3\sigma3}(\epsilon)[\frac{1}{\epsilon} e_{\sigma\|3}(\epsilon)(\boldsymbol{u}^l(\epsilon))] \right\} \left\{ \epsilon e_{\alpha\|3}(\epsilon)(\boldsymbol{v}) \right\} \sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left\{ A^{33\sigma\tau}(\epsilon)[\frac{1}{\epsilon} e_{\sigma\|\tau}(\epsilon)(\boldsymbol{u}^l(\epsilon))] \right\} \left\{ \epsilon e_{3\|3}(\epsilon)(\boldsymbol{v}) \right\} \sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left\{ A^{3333}(\epsilon)[\frac{1}{\epsilon} e_{3\|3}(\epsilon)(\boldsymbol{u}^l(\epsilon))] \right\} \left\{ \epsilon e_{3\|3}(\epsilon)(\boldsymbol{v}) \right\} \sqrt{g(\epsilon)}dx \\
&= \epsilon^2 \xi^l(\epsilon) \int_\Omega u_i^l(\epsilon) v_i \sqrt{g(\epsilon)}dx.
\end{aligned} \tag{4.5.26}$$

Keep $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ fixed and let $\epsilon \to 0$. Using relations (4.3.21),(4.3.23)-(4.3.25), (4.5.23)-(4.5.25) and the weak convergences (4.5.19), we obtain

$$\int_\Omega \left\{ 2\mu a^{\alpha\sigma} e^{1,l}_{\sigma\|3} \partial_3 v_\alpha + [\lambda a^{\sigma\tau} e^{1,l}_{\sigma\|\tau} + (\lambda + 2\mu) e^{1,l}_{3\|3}] \partial_3 v_3 \right\} \sqrt{a} dx = 0. \tag{4.5.27}$$

Letting $\boldsymbol{v}$ vary in $\boldsymbol{V}(\Omega)$ then yields relations (4.5.21)-(4.5.22) (we use the fact that if $w \in L^2(\Omega)$ and $\int_\Omega w\partial_3 v dx = 0$ for all $\boldsymbol{v} \in H^1(\Omega)$ that vanish on Γ_0, then $w = 0$).

Step 4: The pair $(\xi^l, \overline{\boldsymbol{u}}^l)$ satisfies the variational equations (4.5.16).

Keep the function $\boldsymbol{\eta} \in V_F(\omega)$ fixed, let $\boldsymbol{v} = (v_\epsilon(\boldsymbol{\eta}))$ in the variational equation (4.3.11), where $(v_\epsilon(\boldsymbol{\eta}))$ is defined in (4.4.3)-(4.4.4), and let $\epsilon \to 0$. Using relations (4.3.20), (4.3.21)-(4.3.25), (4.4.6)-(4.4.9), (4.5.21), (4.5.22) and the weak convergences (4.5.19), we obtain:

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon)(\boldsymbol{u}^l(\epsilon))e_{i\|j}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\sqrt{g(\epsilon)}dx \\
&= \int_\Omega \left\{A^{\alpha\beta\sigma\tau}(\epsilon)[\frac{1}{\epsilon}e_{\sigma\|\tau}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\frac{1}{\epsilon}e_{\alpha\|\beta}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\right\}\sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left\{A^{\alpha\beta 33}(\epsilon)[\frac{1}{\epsilon}e_{3\|3}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\frac{1}{\epsilon}e_{\alpha\|\beta}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\right\}\sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left\{4A^{\alpha 3\sigma 3}(\epsilon)[\frac{1}{\epsilon}e_{\sigma\|3}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\frac{1}{\epsilon}e_{\alpha\|3}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\right\}\sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left\{A^{33\sigma\tau}(\epsilon)[\frac{1}{\epsilon}e_{\sigma\|\tau}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\frac{1}{\epsilon}e_{3\|3}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\right\}\sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left\{A^{3333}(\epsilon)[\frac{1}{\epsilon}e_{3\|3}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\frac{1}{\epsilon}e_{3\|3}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\right\}\sqrt{g(\epsilon)}dx \\
&= -\frac{1}{2}\int_\Omega x_3 a^{\alpha\beta\sigma\tau}e^{1,l}_{\sigma\|\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy \\
&= \lim_{\epsilon\to 0}\xi^l(\epsilon)\int_\Omega u_i^l(\epsilon)(v_\epsilon(\boldsymbol{\eta}))_i\sqrt{g(\epsilon)}dx = \xi^l\int_\Omega u_i^l\eta_i\sqrt{a}dx \qquad (4.5.28)
\end{aligned}$$

We have yet to take into account relation (4.5.20), viz; $\rho_{\alpha\beta}(\boldsymbol{u}^l) = -\partial_3 e^{1,l}_{\alpha\|\beta}$ in $L^2(\Omega)$. Since the function $\boldsymbol{u}^l$ is independent of x_3 (cf;step 2), relation (4.5.20) implies that

$$e^{1,l}_{\alpha\|\beta} = \theta^l_{\alpha\beta} - x_3\rho_{\alpha\beta}(\overline{\boldsymbol{u}}^l), \text{ with } \theta^l_{\alpha\beta} \in L^2(\omega) \qquad (4.5.29)$$

Therefore

$$\begin{aligned}
-\frac{1}{2}\int_\Omega x_3 a^{\alpha\beta\sigma\tau}e^{1,l}_{\sigma\|\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dx &= \frac{1}{2}\int_\Omega x_3^2 a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^l)\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dx \\
&= \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{\boldsymbol{u}}^l)\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy. \qquad (4.5.30)
\end{aligned}$$

and thus we have established that $(\xi^l, \overline{\boldsymbol{u}}^l)$ satisfies the variational equation (4.5.16).

Step 5: The weak convergence (4.5.19) is in fact strong; i.e.

$$\frac{1}{\epsilon} e_{i\|j}(\epsilon)(\boldsymbol{u}^l(\epsilon)) \to e^{1,l}_{i\|j} \text{ in } L^2(\Omega) \tag{4.5.31}$$

Using inequalities (4.3.26) and (4.2.11), and letting $\boldsymbol{v} = \boldsymbol{u}^l(\epsilon)$ in the variational equations (4.3.11), we obtain

$$C^{-1} g_0^{\frac{1}{2}} \sum_{i,j} \|\frac{1}{\epsilon} e_{i\|j}(\epsilon)(\boldsymbol{u}^l(\epsilon)) - e^{1,l}_{i\|j}\|^2_{0,\Omega} \leq \Lambda^l(\epsilon), \tag{4.5.32}$$

where

$$\begin{aligned}
\Lambda^l(\epsilon) &:= \int_\Omega A^{ijkl}(\epsilon)\left(\frac{1}{\epsilon} e_{k\|l}(\epsilon)(\boldsymbol{u}^l(\epsilon)) - e^{1,l}_{k\|l}\right)\left(\frac{1}{\epsilon} e_{i\|j}(\epsilon)(\boldsymbol{u}^l(\epsilon)) - e^{1,l}_{i\|j}\right)\sqrt{g(\epsilon)}dx \\
&= \xi^l(\epsilon)\int_\Omega (u_i^l(\epsilon))^2\sqrt{g(\epsilon)}dx + \int_\Omega A^{ijkl}(\epsilon)\left(e^{1,l}_{k\|l} - \frac{2}{\epsilon} e_{k\|l}(\epsilon)(\boldsymbol{u}^l(\epsilon))\right) e^{1,l}_{i\|j}\sqrt{g(\epsilon)}dx
\end{aligned}$$

The convergences (4.3.20), (4.3.21), together with the convergences $u_i^l(\epsilon) \to u_i^l$ in $L^2(\Omega)$ and the weak convergences (4.5.19) imply that

$$\Lambda^l := \lim_{\epsilon\to 0} \Lambda^l(\epsilon) = \xi^l \int_\Omega (u_i^l)^2\sqrt{a}dx - \int_\Omega A^{ijkl}(0) e^{1,l}_{k\|l} e^{1,l}_{i\|j}\sqrt{a}dx \tag{4.5.33}$$

Using (4.3.23)-(4.3.25),(4.5.21)-(4.5.22) and the fact that $\boldsymbol{u}^l$ is independent of x_3 we further infer that

$$\Lambda^l := 2\xi^l \int_\omega (\overline{u}_i^l)^2\sqrt{a}dy - \frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau} e^{1,l}_{\sigma\|\tau} e^{1,l}_{\alpha\|\beta}\sqrt{a}dx \tag{4.5.34}$$

By (4.5.29)

$$\int_\Omega a^{\alpha\beta\sigma\tau} e^{1,l}_{\sigma\|\tau} e^{1,l}_{\alpha\|\beta}\sqrt{a}dx = 2\int_\omega a^{\alpha\beta\sigma\tau}\theta^l_{\sigma\tau}\theta^l_{\alpha\beta}\sqrt{a}dy + \frac{2}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{\boldsymbol{u}}^l)\rho_{\alpha\beta}(\overline{\boldsymbol{u}}^l)\sqrt{a}dy \tag{4.5.35}$$

and thus, since $\overline{\boldsymbol{u}}^l$ satisfies the variational equations (4.5.16)

$$\Lambda^l = -\int_\omega a^{\alpha\beta\sigma\tau}\theta^l_{\sigma\tau}\theta^l_{\alpha\beta}\sqrt{a}dy. \tag{4.5.36}$$

Since $\Lambda^l \geq 0$ (cf. inequality (4.5.32) and definition (4.5.33)) and since the tensor $(a^{\alpha\beta\sigma\tau})$ is uniformly positive definite (with respect to $y \in \omega$ (cf.Lemma 4.3.3), we conclude that $\theta^l_{\alpha\beta} = 0$. These relations have two consequences. First,

$$\Lambda^l = 0 \tag{4.5.37}$$

and thus the strong convergence (4.5.31) holds; secondly, the functions $e^{1,l}_{\alpha\|\beta}$ are given by

$$e^{1,l}_{\alpha\|\beta} = -x_3\rho_{\alpha\beta}(\boldsymbol{u}^l) \qquad (4.5.38)$$

Step 6: The weak convergence (4.5.18) is in fact strong.

$$\boldsymbol{u}^l(\epsilon) \to \boldsymbol{u}^l \text{ in } H^1(\Omega) \qquad (4.5.39)$$

By Lemma 4.5.1, it remains to how that each family $(\rho_{\alpha\beta}(\boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ strongly converges in $H^{-1}(\Omega)$. Since

$$\frac{1}{\epsilon}\partial_3 e_{\alpha\|\beta}(\boldsymbol{u}^l(\epsilon)) \to \partial_3 e^{1,l}_{\alpha\|\beta} \text{ in } H^{-1}(\Omega)$$

as a consequence of Step 5, and since

$$\rho_{\alpha\beta}(\boldsymbol{u}^l(\epsilon)) + \frac{1}{\epsilon}\partial_3 e_{\alpha\|\beta}(\epsilon)(\boldsymbol{u}^l(\epsilon)) \to 0 \text{ in } H^{-1}(\Omega)$$

as a consequence of inequality (4.3.31), it follows that

$$\rho_{\alpha\beta}(\boldsymbol{u}^l(\epsilon)) \to \{-\partial_3 e^{1,l}_{\alpha\|\beta}\} \text{ in } H^{-1}(\Omega)$$

and thus the strong convergences (4.5.39) are established. ∎

Though we have proved that each subsequence $(\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}, l \geq 1$, strongly converges in $\mathbb{R} \times H^1(\Omega)$ to a solution $(\xi^l, \overline{\boldsymbol{u}}^l)$ of the two-dimensional eigenvalue problem for the flexural shells, nothing tells us so far whether ξ^l is precisely the l^{th} eigenvalue (counting multiplicities) of (4.5.16), nor whether the set $(\overline{\boldsymbol{u}}^l)_{l=1}^{\infty}$ forms a complete set in the space $V_F(\omega)$. We shall answer these questions in the affirmative in the next lemma using the ideas developed by Kesavan[24].

Lemma 4.5.5 *Let $(\xi^l, \overline{\boldsymbol{u}}^l), l \geq 1$, be the eigensolutions of problem (4.5.16) found as limits of the subsequence $(\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}, l \geq 1$ of eigensolutions, orthonormalized as in (4.3.12) of problem (4.3.11). Then the sequence $(\xi^l)_{l=1}^{\infty}$ comprises all the eigenvalues, counting multiplicities, of problem (4.5.16) and the associated sequence $(\overline{\boldsymbol{u}}^l)_{l=1}^{\infty}$ of eigenfunctions forms a complete orthonormal set in the space $V_F(\omega)$.*

Proof. Passing to the limit in the orthogonality relation (4.3.12), we get

$$\int_\omega \overline{u}_i^l \overline{u}_i^m \sqrt{a} d\omega = \frac{1}{2}\delta_{lm} \tag{4.5.40}$$

We first show that

$$0 < \xi^1 \leq \xi^2 \leq ... \leq \xi^l \leq ... \to \infty \tag{4.5.41}$$

Since $0 < \xi^1(\epsilon) \leq \xi^2(\epsilon) \leq ... \leq \xi^l(\epsilon) \leq ... \to \infty$, it follows that $0 \leq \xi^1 \leq \xi^2 \leq ...$; since the bilinear form associated with the left-hand side of equation (4.5.16) is coercive over $V_F(\omega)$, it follows that $\xi^1 > 0$. If the sequence were bounded, the eigenvalue problem could have only a finite number of linearly independent eigensolutions, since its associated operator is compact over the space $L^2(\omega)$; but this would contradict the orthonormalization condition (4.5.40). Hence the relation (4.5.41) holds.

We next show that if ξ is any eigenvalue of the problem (4.5.16), there exists an integer $l \geq 1$ such that $\xi = \xi^l$.

Suppose the contrary holds; ie, that $\xi \neq \xi^l$ for all $l \geq 1$ and let $\boldsymbol{\zeta}$ denotes an associated eigenfunction, which satisfies

$$\frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}d\omega = \xi \int_\omega \zeta_i \eta_i \sqrt{a} d\omega \text{ for all } \boldsymbol{\eta} \in V_F(\omega) \tag{4.5.42}$$

$$\int_\omega \zeta_i\zeta_i \sqrt{a} d\omega = \frac{1}{2}, \int_\omega \zeta_i \overline{u}_i^l \sqrt{a} d\omega = 0 \text{ for all } l \tag{4.5.43}$$

For each $\epsilon > 0$, let $\boldsymbol{w}(\epsilon)$ be the unique solution of

$$\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon)(\boldsymbol{w}(\epsilon))e_{i\|j}(\epsilon)(\boldsymbol{v})\sqrt{g(\epsilon)}dx = \epsilon^2\xi \int_\Omega \zeta_i v_i \sqrt{g(\epsilon)}dx \text{ for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega) \tag{4.5.44}$$

Then proceeding as in Theorem 4.5.4, we can show that $\boldsymbol{w}(\epsilon) \to \boldsymbol{w}$ in $\boldsymbol{V}(\Omega)$ and $\overline{\boldsymbol{w}} \in V_F(\omega)$. Further $\overline{\boldsymbol{w}}$ satisfies

$$\frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{\boldsymbol{w}})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}d\omega = \xi \int_\omega \overline{w}_i \eta_i \sqrt{a} d\omega \text{ for all } \boldsymbol{\eta} \in V_F(\omega) \tag{4.5.45}$$

By the uniqueness of the solution, it follows that $\overline{\boldsymbol{w}} = \boldsymbol{\zeta}$. Since the sequence ξ^l is unbounded, we can choose an l such that

$$\xi < \xi^l \tag{4.5.46}$$

For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(\Omega)$, define

$$D(\epsilon)(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega u_i v_i \sqrt{g(\epsilon)} dx \tag{4.5.47}$$

Consider the vector

$$\boldsymbol{v}(\epsilon) = \boldsymbol{w}(\epsilon) - \sum_{k=1}^{l} D(\epsilon)(\boldsymbol{w}(\epsilon), \boldsymbol{u}^k(\epsilon))\boldsymbol{u}^k(\epsilon)$$

Then

$$D(\epsilon)(\boldsymbol{v}(\epsilon), \boldsymbol{u}^k(\epsilon)) = 0 \text{ for all } 1 \leq k \leq l. \tag{4.5.48}$$

Therefore it follows from the variational characterization of the eigenvalues that

$$\xi^{l+1}(\epsilon) \leq \frac{\int_\Omega A^{ijkl}(\epsilon) e_{k\|l}(\epsilon)(\boldsymbol{v}(\epsilon)) e_{i\|j}(\epsilon)(\boldsymbol{v}(\epsilon))\sqrt{g(\epsilon)}dx}{\epsilon^2 D(\epsilon)(\boldsymbol{v}(\epsilon), \boldsymbol{v}(\epsilon))} \tag{4.5.49}$$

Passing to the limit in the above inequality, it can be shown that

$$\xi^{l+1} \leq \xi \tag{4.5.50}$$

which contradicts (4.5.46) and the proof is complete. ∎

Theorem 4.5.6 *Assume that $dim(V_F(\omega)) = N$ and let the space $V_m(\omega)$ be defined by*

$$V_m(\omega) = \{\boldsymbol{\eta} = (\eta_i) : \eta_\alpha \in H_0^1(\omega), \eta_3 \in L^2(\omega)\}. \tag{4.5.51}$$

Then,

a)For $1 \leq l \leq N$, $(\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ converges strongly in $I\!R \times H^1(\Omega)$ to the N eigensolutions of the "flexural shell" problem, viz;

Find $(\xi, \boldsymbol{\zeta}) \in I\!R \times V_F(\omega)$ such that

$$\frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy = \xi \int_\omega \zeta_i \eta_i \sqrt{a} dy \text{ for all } \boldsymbol{\eta} \in V_F(\omega). \tag{4.5.52}$$

b)For each integer $l > N$, there exists a subsequence (still denoted by ϵ) such that

$$u^l_\alpha(\epsilon) \rightharpoonup u^l_\alpha \ in\ H^1(\Omega) \tag{4.5.53}$$

$$u^l_3(\epsilon) \rightharpoonup u^l_3 in\ L^2(\Omega) \tag{4.5.54}$$

$$\epsilon^2\xi^l(\epsilon) \to \xi^l \tag{4.5.55}$$

$\boldsymbol{u}^l = (u^l_i)$ *is independent of the transverse variable* x_3

c)The pair $(\xi^l, \overline{\boldsymbol{u}}^l)$ solves the two-dimensional eigenvalue problem for "membrane shell",viz;

Find $(\xi, \boldsymbol{\zeta}) \in I\!R \times V_m(\omega)$ such that

$$\frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy = \xi\int_\omega \zeta_i\eta_i\sqrt{a}dy \ for\ all\ \boldsymbol{\eta} \in V_m(\omega). \tag{4.5.56}$$

Proof. For clarity, it is divided into several steps.

Step 1: The proof that for $1 \le l \le N, (\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ converges strongly in $I\!R \times H^1(\Omega)$ to the N eigensolutions of (4.5.52) follows from Theorem 4.5.4.

Step 2: We now proceed as in Ciarlet and Lods [14]. From the variational equation (4.3.11), relation (4.3.12), inequalities (4.2.11), (4.3.26), (4.4.40), (4.5.10) and the boundedness of the eigenvalues, it follows that

$$\begin{aligned}
\sum_\alpha ||u^l_\alpha(\epsilon)||^2_{1,\Omega} &+ ||\epsilon u^l_3(\epsilon)||^2_{1,\Omega} \\
&\le C\left\{\sum_{i,j}||e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon))||^2_{0,\Omega} + ||\boldsymbol{u}^l(\epsilon)||^2_{0,\Omega}\right\} \\
&\le Cg_0^{-\frac{1}{2}}\left\{\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\boldsymbol{u}^l(\epsilon))e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon))\sqrt{g(\epsilon)}dx + 1\right\} \\
&\le Cg_0^{-\frac{1}{2}}\left\{\epsilon^2\xi^l(\epsilon)\int_\Omega u^l_i(\epsilon)u^l_i(\epsilon)\sqrt{g(\epsilon)}dx + 1\right\} \\
&\le Cg_0^{-\frac{1}{2}}(\epsilon^2k^l + 1)
\end{aligned} \tag{4.5.57}$$

Hence the norms $||e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon))||_{0,\Omega}, ||u^l_\alpha(\epsilon)||_{1,\Omega}, ||u^l_3(\epsilon)||_{0,\Omega}$ are bounded independent of ϵ. Consequently, there exists a subsequence (still indexed by ϵ for convenience) and there exists functions $e^l_{i||j} \in L^2(\Omega), u^l_\alpha \in H^1(\Omega)$, satisfying $u^l_\alpha = 0$ on Γ_0 and $u^l_3 \in L^2(\Omega)$ such that

$$e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon)) \rightharpoonup e^l_{i||j} \text{ in } L^2(\Omega) \tag{4.5.58}$$

$$u^l_\alpha(\epsilon) \rightharpoonup u^l_\alpha \text{ in } H^1(\Omega) \tag{4.5.59}$$

$$u^l_3(\epsilon) \rightharpoonup u^l_3 \text{ in } L^2(\Omega) \tag{4.5.60}$$

Step 3: The limit function u^l_i found in (4.5.59)-(4.5.60) are independent of x_3.

By (4.3.18) and Step 2,

$$\partial_3 u^l_\alpha(\epsilon) + \epsilon\partial_\alpha u^l_3(\epsilon) = 2\epsilon\left\{e_{\alpha||3}(\epsilon)(\boldsymbol{u}^l(\epsilon)) + \Gamma^\sigma_{\alpha 3}(\epsilon)u^l_3(\epsilon)\right\} \to 0 \text{ in } L^2(\Omega) \tag{4.5.61}$$

Let $\phi \in \mathcal{D}(\Omega)$; since $u^l_\alpha(\epsilon) \rightharpoonup u^l_\alpha$ in $H^1(\Omega)$ and since $(u^l_3(\epsilon))_{\epsilon>0}$ is bounded in $L^2(\Omega)$ by Step 2,

$$\int_\omega \partial_3 u^l_\alpha \phi dx = \lim_{\epsilon\to 0}\int_\Omega \partial_3 u^l_\alpha(\epsilon)\phi dx \tag{4.5.62}$$

$$\lim_{\epsilon\to 0}\int_\Omega \epsilon\partial_3 u^l_3 \phi dx = -\lim_{\epsilon\to 0}\int_\Omega \epsilon u^l_3(\epsilon)\partial_\alpha\phi dx = 0 \tag{4.5.63}$$

whence $\int_\Omega \partial_3 u^l_\alpha \phi dx = 0$. Therefore $\partial_3 u^l_\alpha = 0$ in $L^2(\Omega)$.

Also by Step 2,

$$\partial_3 u^l_3(\epsilon) = \epsilon e_{3||3}(\epsilon)(\boldsymbol{u}^l(\epsilon)) \to 0 \text{ in } L^2(\Omega) \tag{4.5.64}$$

Let $\phi \in \mathcal{D}(\Omega)$; since $u^l_3(\epsilon) \rightharpoonup u^l_3$ in $L^2(\Omega)$ by Step 2

$$\int_\Omega u^l_3\partial_3\phi dx = \lim_{\epsilon\to 0}\int_\Omega u^l_3(\epsilon)\partial_3\phi dx = -\lim \epsilon \to 0\int_\Omega \partial_3 u^l_3(\epsilon)\phi dx = 0 \tag{4.5.65}$$

whence $\partial_3 u_3 = 0$ in the sense of distribution. Hence it follows that u_3 is independent of x_3

Step 4: The limit functions $e^l_{i||j}$ found in (4.5.58) are independent of x_3, moreover they are related to the limit function (u^l_i) by

$$e^l_{\alpha||\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}^l) \tag{4.5.66}$$

$$e^l_{\alpha||3} = 0 \tag{4.5.67}$$

$$e^l_{3||3} = \frac{-\lambda}{\lambda+2\mu}a^{\alpha\beta}e^l_{\alpha||\beta}. \tag{4.5.68}$$

The convergences (4.5.58)-(4.5.60) and relations (4.3.16)-(4.3.17) imply that

$$e_{\alpha\|\beta}(\epsilon)(\boldsymbol{u}^l(\epsilon)) = \frac{1}{2}(\partial_\alpha u^l_\beta(\epsilon) + \partial_\beta u^l_\alpha(\epsilon)) - \Gamma^p_{\alpha 3}(\epsilon)u^l_p(\epsilon) \rightharpoonup \gamma_{\alpha\beta}(\boldsymbol{u}^l) = e^l_{\alpha\|\beta} \text{ in } L^2(\Omega) \tag{4.5.69}$$

which shows that the functions $e^l_{\alpha\|\beta}$ satisfy (4.5.66) and are independent of x_3 (the functions u^l_i are independent of x_3).

Let $\boldsymbol{v} = (v_i)$ be an arbitrary function in the space $\boldsymbol{V}(\Omega)$ .The following relations are immediate consequences of definition (4.3.10) of the functions $e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v})$.

$$\epsilon e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) \to 0 \text{ in } L^2(\Omega) \tag{4.5.70}$$

$$\epsilon e_{\alpha\|3}(\epsilon)(\boldsymbol{v}) \to \frac{1}{2}\partial_3 v_3 \text{ in } L^2(\Omega) \tag{4.5.71}$$

$$\epsilon e_{3\|3}(\boldsymbol{v}) = \partial_3 v_3 \text{ for all } \epsilon > 0. \tag{4.5.72}$$

Using the variational equations (4.3.11) of the scaled three-dimensional problem and relations (4.3.22), we get

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon)(\boldsymbol{u}^l(\epsilon))e_{i\|j}(\epsilon)(\boldsymbol{v})\sqrt{g(\epsilon)}dx \\
&\quad= \int_\Omega \left\{A^{\alpha\beta\sigma\tau}(\epsilon)[e_{\sigma\|\tau}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\epsilon e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v})\right\}\sqrt{g(\epsilon)}dx \\
&\quad+ \int_\Omega \left\{A^{\alpha\beta 33}(\epsilon)[e_{3\|3}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\epsilon e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v})\right\}\sqrt{g(\epsilon)}dx \\
&\quad+ \int_\Omega \left\{4A^{\alpha 3\sigma 3}(\epsilon)[e_{\sigma\|3}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\epsilon e_{\alpha\|3}(\epsilon)(\boldsymbol{v})\right\}\sqrt{g(\epsilon)}dx \\
&\quad+ \int_\Omega \left\{A^{33\sigma\tau}(\epsilon)[e_{\sigma\|\tau}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\epsilon e_{3\|3}(\epsilon)(\boldsymbol{v})\right\}\sqrt{g(\epsilon)}dx \\
&\quad+ \int_\Omega \left\{A^{3333}(\epsilon)[e_{3\|3}(\epsilon)(\boldsymbol{u}^l(\epsilon))]\right\}\left\{\epsilon e_{3\|3}(\epsilon)(\boldsymbol{v})\right\}\sqrt{g(\epsilon)}dx \\
&\quad= \epsilon^3\xi^l(\epsilon)\int_\Omega u^l_i(\epsilon)^2 v_i\sqrt{g(\epsilon)}dx.
\end{aligned} \tag{4.5.73}$$

Keep $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ fixed and let $\epsilon \to 0$. Using relations (4.3.20),(4.3.21),(4.3.23)-(4.3.25),(4.5.70)-(4.5.72) and the weak convergence (4.5.58), we obtain

$$\int_\Omega \left\{2\mu a^{\alpha\sigma}e^l_{\sigma\|3}\partial_3 v_\sigma + [\lambda a^{\sigma\tau}e^l_{\sigma\|\tau} + (\lambda+2\mu)e^l_{3\|3}]\partial_3 v_3\right\}\sqrt{a}dx = 0. \tag{4.5.74}$$

Letting $\boldsymbol{v}$ vary in $\boldsymbol{V}(\Omega)$ then yields relations (4.5.67)-(4.5.68) (if $w \in L^2(\Omega)$ and $\int_\Omega w\partial_3 v dx = 0$ for all $v \in H^1(\Omega)$ that vanishes on Γ_0 then $w = 0$).

Step 5: The pair $(\xi^l, \overline{\boldsymbol{u}}^l)$ solves the variational equations (4.5.56).

Let $\boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega)$ be independent of the variable x_3; then (cf. inequality (4.3.16)-(4.3.18))

$$e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) \to \gamma_{\alpha\beta}(\boldsymbol{v}) := \left\{\frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta} v_\sigma - b_{\alpha\beta} v_3\right\} \text{ in } L^2(\Omega), \quad (4.5.75)$$

$$e_{\alpha\|3}(\epsilon)(\boldsymbol{v}) \to \left\{\frac{1}{2}\partial_\alpha v_3 + b^\sigma_\alpha v_\sigma\right\} \text{ in } L^2(\Omega), \quad (4.5.76)$$

$$e_{3\|3}(\epsilon)(\boldsymbol{v}) = 0, \quad (4.5.77)$$

as $\epsilon \to 0$. Keep such a function $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ fixed in the variational equation (4.3.11) and passing to the limit as $\epsilon \to 0$ and taking into account of the relations (4.3.16)-(4.3.25), the strong convergence (4.5.75)-(4.5.77) and the weak convergence (4.5.58) we get,

$$\frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\overline{\boldsymbol{u}}^l)\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dy = \xi^l \int_\omega \overline{u}^l_i v_i \sqrt{a}dy \quad (4.5.78)$$

Given $\boldsymbol{\eta} = (\eta_i) \in (H^1_0(\omega))^3$, let $\boldsymbol{v} = (v_i)$ be defined by

$$\boldsymbol{v}(y, x_3) = \boldsymbol{\eta}(y) \text{ for } (y, x_3) \in \Omega$$

Then $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, $\boldsymbol{v}$ is independent of x_3 and thus equation (4.5.78) are satisfied with $\overline{\boldsymbol{v}} = \boldsymbol{\eta}$. Since both sides of equation (4.5.78) are continuous, linear forms with respect to $\eta_3 \in L^2(\omega)$ and $H^1_0(\omega)$ is dense in $L^2(\omega)$, these equations are valid for all $\eta \in V_m(\omega)$.

∎

Remark 4.5.1: Observe that we *cannot* conclude from the above theorem that $(\xi^l, \overline{\boldsymbol{u}}^l)$ is an eigensolution of the two-dimensional membrane shell problem. This is because we do not know if $\overline{\boldsymbol{u}}^l \neq 0$. In previous cases, we found that $\overline{\boldsymbol{u}}^l \neq 0$ by passing to the limit in the orthogonality relations. Also, Ciarlet and Lods [14] prove the strong convergence of the solution in $(H^1(\Omega))^2 \times L^2(\Omega)$ in the stationary case under the assumption that the body force $f(\epsilon) \to f$ strongly in $L^2(\Omega)$. We do not have that situation here and we only know that $u^l_3(\epsilon) \rightharpoonup u^l_3$ weakly in $L^2(\Omega)$. Thus we can only say that either $\overline{u}^l = 0$ or $(\xi^l, \overline{u}^l)$ is an eigenfunction of the membrane problem for $l > N$. ∎

Remark 4.5.2: Note that if $V_F(\omega)$ is finite dimensional of dimension, say N, then $\{\xi^l(\epsilon)\}$ for $l > N$ **cannot** be of order $O(\epsilon^2)$. For, if this were the case, we can get convergence of $\{u^l(\epsilon)\}$ in $\boldsymbol{V}(\Omega)$ to $\overline{u}^l$, an eigenvector of the flexural shell problem. This $\overline{u}^l(l > N)$ will be orthogonal to $\overline{u}^i, 1 \leq i \leq N$ and will contradict the fact that dim $(V_F(\omega)) = N$. ∎

4.6 Conclusions

As mentioned in the introduction, we have investigated the behaviour of eigensolutions of a thin shell based uniquely on the non-trivial nature of the space of inextensional displacements $V_F(\omega)$.

In the stationary case, if $V_F(\omega)$ were nontrivial and the body forces were of order $O(\epsilon^2)$,one got the flexural shell model. Here we have no supplementary assumption. If $V_F(\omega)$ were infinite dimensional, all the eigenvalues were shown to be of order $O(\epsilon^2)$ and they converge, for each fixed level l, to those of the flexural shell model. Further, all the eigenvalues of the flexural shell are obtained this way. The eigenvectors converge strongly.

If the dimension of $V_F(\omega)$ were finite, the above results hold only up to the level equal to that dimension. Higher eigenvalues are bounded but are not of order $O(\epsilon^2)$. These higher eigenvalues converge to eigenvalues of the membrane shell model, unless the corresponding eigenvectors converge weakly to zero.

Sanchez Palencia[42], when discussing, the eigenvalues of the shells via the Koiter's model, says that when $V_F(\omega) \neq 0$, the eigenvalues are "low frequency" type and converge to the flexural eigenvalues while when dim $V_F(\omega) = 0$ one could get (for instance under the additional assumption that the shell is "uniformly elliptic") the eigenvalues of the membrane shell in the limit. Such eigenvalues are said to be of "medium frequency".

Here we observe that if dim $V_F(\omega) = N < \infty$, then both kinds of eigenvalues - low and medium frequency- may be present.

Of course, we do not know if the eigenvectors for $l > N$ converge weakly to zero or not. If they all converge weakly to zero, then no medium frequency eigenvalues exists. It will be nice to know if this is indeed the case. If so, it will also be nice to know how to characterize the limits of $\epsilon^2\xi^l(\epsilon)$ for $l > N$.

Of course, to the best of our knowledge, we do not know of any examples of shells for which if $V_F(\omega) \neq 0$, then it is finite dimensional. Sanchez Palencia states that, in general, $V_F(\omega)$ is infinite dimensional. This is yet another open question.

Chapter 5

Membrane Shells

5.1 Introduction

In this chapter, we study the limiting behaviour of eigenvalues and eigenfunctions describing the vibrations of a thin linearly elastic shell, clamped along its lateral surface, under a geometric assumption on the middle surface of the shell that it is "uniformly elliptic"(cf. (5.2.13)). In the stationary case, under additional assumptions on the body forces, this leads to the two-dimensional model of the "membrane shell" as shown by Ciarlet and Lods [14].

Our procedure to study the corresponding eigenvalue problem is the standard one. Starting with the three-dimensional eigenvalue problem (corresponding to the one studied by Ciarlet and Lods[14] in the stationary case), we rescale the variables and obtain a problem posed over a fixed domain where the parameter ϵ (corresponding to the thickness of the shell and the dimension of the three-dimensional domain over which the reference configuration of the shell is defined) now appears in the various bilinear forms. We can then pass to the limit after obtaining suitable *a priori* estimates.

It must be observed that in previous work of Ciarlet and Lods [14], the membrane model was obtained based on two assumptions. If the forces were strongly convergent in $L^2(\Omega)$ and the middle surface of the shell is "uniformly elliptic" in the sense that the two principal radii of curvatures are either both > 0 or both < 0 at all points of the middle surface of the shell, then the membrane shell model was obtained in the limit.

In our case, we do not have the body forces and so we cannot make any extra assumption on their convergence. So how does the shell decide on its limiting behaviour vis-a-vis its vibrations, on the basis of the assumption that the shell is "uniformly elliptic"?. We show in this chapter, that the eigenvalues (at each level l, $l = 1, 2, ...$) are of the order $O(1)$ by considering suitable test functions to be used in the variational characterization of the eigenvalues and either an eigensolution of the three-dimensional problem converges a solution of the two-dimensional eigenvalue problem for membrane shells or the corresponding eigenfunction converges to zero weakly in $(H^1(\Omega))^2 \times L^2(\Omega)$.

The assumption on the set ω, the notation and the geometrical and mechanical nature of the shell are the same as in Chapter 4 ; for this reason, they shall not be repeated here.

Then the eigenvalue problem consists in finding pairs $(\xi^\epsilon, \boldsymbol{u}^\epsilon) \in \mathbb{R} \times \boldsymbol{V}(\Omega^\epsilon)\backslash\{0\}$ such that

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(\boldsymbol{u}^\epsilon) e^\epsilon_{i\|j}(\boldsymbol{v}^\epsilon)\sqrt{g^\epsilon}dx^\epsilon = \xi^\epsilon \int_{\Omega^\epsilon} u^\epsilon_i v^\epsilon_i \sqrt{g^\epsilon}dx^\epsilon \tag{5.1.1}$$

for all $\boldsymbol{v}^\epsilon \in \boldsymbol{V}(\Omega^\epsilon)$. By classical arguments, we can show that there exists a sequence of eigenvalues

$$0 < \xi^{\epsilon,1} \leq \xi^{\epsilon,2} \leq ... \leq \xi^{\epsilon,l} \leq ... \to \infty \tag{5.1.2}$$

and we can choose a corresponding family of eigenfunctions $\{\boldsymbol{u}^{\epsilon,l}\}$ such that

$$\int_{\Omega^\epsilon} u^{\epsilon,l}_i u^{\epsilon,m}_i \sqrt{g^\epsilon}dx^\epsilon = \delta_{lm}. \tag{5.1.3}$$

The sequence $\{\boldsymbol{u}^{\epsilon,l}\}$ form a basis in the weighted space

$$(L^2(g_\epsilon; \Omega^\epsilon))^3 = \{\boldsymbol{u}^\epsilon | \int_{\Omega^\epsilon} u^\epsilon_i u^\epsilon_i \sqrt{g^\epsilon}dx^\epsilon < \infty\} \tag{5.1.4}$$

with the obvious inner-product.

5.2 The Rescaled Problem

We now scale this problem to one posed over a domain independent of ϵ. We set

$$\Omega = \omega \times (-1, 1), \Gamma_{\pm} = \omega \times \{\pm 1\}, \Gamma_0 = \gamma \times [-1, 1]. \tag{5.2.1}$$

If $x = (x_i) \in \Omega$ is a generic point, we set $\partial_i = \frac{\partial}{\partial x_i}$ and with $x^\epsilon = (x_i^\epsilon) \in \overline{\Omega}^\epsilon$, we associate $x \in \overline{\Omega}$ by

$$x_\alpha = x_\alpha^\epsilon = y_\alpha, \; x_3 = \frac{1}{\epsilon} x_3^\epsilon. \tag{5.2.2}$$

Thus, $\partial_\alpha^\epsilon = \partial_\alpha$ and $\partial_3^\epsilon = \frac{1}{\epsilon}\partial_3$.

Given a vector $\boldsymbol{v}^\epsilon \in \boldsymbol{V}(\Omega^\epsilon)$, we associate the vector $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ where

$$\boldsymbol{V}(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^3 | \boldsymbol{v} = 0 \text{ on } \Gamma_0\} \tag{5.2.3}$$

by

$$v_i(x) = v_i^\epsilon(x^\epsilon) \tag{5.2.4}$$

where x and x^ϵ have the correspondence mentioned above. Given an eigenvector $\boldsymbol{u}^{\epsilon,l}$, we denote the corresponding vector obtained via (5.2.4) by $\boldsymbol{u}^l(\epsilon)$. We assume further that the material properties of the shell do not depend on the thickness, and so we set

$$\lambda^\epsilon = \lambda > 0, \mu^\epsilon = \mu > 0 \tag{5.2.5}$$

where λ and μ are independent of ϵ.

Finally, given an eigenvalue $\xi^{\epsilon,l}$, we associate with it the "scaled" eigenvalue $\xi^l(\epsilon)$ by

$$\xi^{\epsilon,l} = \xi^l(\epsilon). \tag{5.2.6}$$

Remark 5.2.1: Upto now we have always scaled the eigenvalues as $\xi^{l,\epsilon} = \epsilon^2 \xi^l(\epsilon)$, but now we do not scale them at all. This is the same situation which accured for eigenvalues $\xi^{l,\epsilon}$ in the flexural shell case when $l > N = dim V_F$.

Given $\boldsymbol{v} = (v_i) \in (H^1(\Omega))^3$, we associate the symmetric tensor $(e_{i\|j}(\epsilon)(\boldsymbol{v}))$ by

$$\left.\begin{array}{rcl} e_{\alpha\|\beta}(\epsilon)(\boldsymbol{v}) & = & \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta}(\epsilon)v_\sigma \\ e_{\alpha\|3}(\epsilon)(\boldsymbol{v}) & = & \frac{1}{2}(\partial_\alpha v_3 + \frac{1}{\epsilon}\partial_3 v_\alpha) - \Gamma^\sigma_{\alpha 3}(\epsilon)v_\sigma \\ e_{3\|3}(\epsilon)(\boldsymbol{v}) & = & \frac{1}{\epsilon}\partial_3 v_3 \end{array}\right\}. \tag{5.2.7}$$

Then if $(\xi^{\epsilon,l}\boldsymbol{u}^{\epsilon,l}) \in I\!R \times \boldsymbol{V}(\Omega^\epsilon)\backslash\{0\}$ is a solution of (5.1.1), the scaled variables $(\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon)) \in I\!R \times \boldsymbol{V}(\Omega)\backslash\{0\}$ is a solution of the problem

$$\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon)(\boldsymbol{u}^l(\epsilon))e_{i\|j}(\epsilon)(\boldsymbol{v})\sqrt{g(\epsilon)}dx = \xi^l(\epsilon)\int_\Omega u_i^l(\epsilon)v_i\sqrt{g(\epsilon)}dx \tag{5.2.8}$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$. Once again, it is clear that $\{\xi^l(\epsilon)\}$ corresponding to $\xi^{l,\epsilon}$ via (5.2.6) are the only eigenvalues of (5.2.8) and that the corresponding eigenvectors $\{\boldsymbol{u}^l(\epsilon)\}$ are complete in $(L^2(\Omega))^3$ and satisfy the orthogonality conditions

$$\int_\Omega u_i^l(\epsilon)u_i^m(\epsilon)\sqrt{g(\epsilon)}dx = \delta_{lm}. \tag{5.2.9}$$

Further, we have the following variational characterization of the eigenvalues. Define the Rayleigh quotient $R(\epsilon)(\boldsymbol{v})$ for $\boldsymbol{v} \in \boldsymbol{V}(\Omega)\backslash\{0\}$ by

$$R(\epsilon)(\boldsymbol{v}) = \frac{\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\epsilon)(\boldsymbol{v})e_{i\|j}(\epsilon)(\boldsymbol{v})\sqrt{g(\epsilon)}dx}{\int_\Omega v_i v_i\sqrt{g(\epsilon)}dx}. \tag{5.2.10}$$

Then

$$\xi^l(\epsilon) = \min_{\boldsymbol{W}\in\mathcal{V}_l}\ \max_{\boldsymbol{v}\in\boldsymbol{W}\backslash\{0\}} R(\epsilon)(\boldsymbol{v}) \tag{5.2.11}$$

where $\mathcal{V}_l$ is the collection of all l-dimensional subspaces of $\boldsymbol{V}(\Omega)$

Define the space

$$V_m(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in (H_0^1(\omega))^2 \times L^2(\omega)\} \tag{5.2.12}$$

The shell is said to be "uniformly elliptic" if its middle surface S is uniformly elliptic in the following sense; there exists a constant $b > 0$ such that

$$b_{\alpha\beta}(y)\xi^\alpha\xi^\beta \geq b|\xi|^2 \tag{5.2.13}$$

for all $\eta \in \omega$ and all $\xi = (\xi^\alpha) \in I\!R^2$.

Under this assumption, it is shown in Ciarlet and Lods [14] that the function

$$\eta \to (\Sigma_{\alpha,\beta}||\gamma_{\alpha\beta}(\eta)||^2)^{\frac{1}{2}} \tag{5.2.14}$$

is a norm in $V_m(\omega)$ which is equivalent to the $(H_0^1(\omega))^2 \times L^2(\omega)$ norm and hence it follows from Theorem 4.1 of Ciarlet and Lods [14] that there exists a constant ϵ_1 and a constant $C > 0$ such that for all $0 < \epsilon \leq \epsilon_1$,

$$\left\{\sum_\alpha ||v_\alpha||_{1,\Omega}^2 + ||v_3||_{0,\Omega}^2\right\} \leq C\left\{\sum_{i,j} ||e_{i||j}(\epsilon)(\boldsymbol{v})||_{0,\Omega}^2\right\} \tag{5.2.15}$$

for all $\boldsymbol{v} = (v_i) \in (H^1(\Omega))^3$

5.3 The Limit Problem

In this section we show that for each integer $l \geq 1$, the scaled eigenvalues are bounded uniformly with respect to ϵ and the limits of $\{\xi^l(\epsilon)\}, l \geq 1$ lies in a bounded subset of $I\!R$ and either the eigensolutions converge to the solutions of the two-dimensional eigenvalue problem for the membrane shell or the corresponding eigenfunction converges to zero weakly in $(H^1(\Omega))^2 \times L^2(\Omega)$.

Theorem 5.3.1 *i)For each integer $l \geq 1$, there exists a subsequence (still denoted by ϵ) such that*

$$u_\alpha^l(\epsilon) \rightharpoonup u_\alpha^l \text{ in } H^1(\Omega) \qquad weakly \tag{5.3.1}$$

$$u_3^l(\epsilon) \rightharpoonup u_3^l \text{in } L^2(\Omega) \qquad weakly \tag{5.3.2}$$

$$\xi^l(\epsilon) \to \xi^l \tag{5.3.3}$$

$\boldsymbol{u}^l = (u_i^l)$ *is independent of the transverse variable* x_3

ii)Either the pair $(\xi^l, \overline{\boldsymbol{u}}^l)$ solves the two-dimensional eigenvalue problem for "membrane shell",viz;

Find $(\xi, \boldsymbol{\zeta}) \in I\!R \times (H_0^2(\omega))^2 \times L^2(\omega)$ *such that*

$$\frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy = \xi\int_\omega \zeta_i\eta_i\sqrt{a}dy \text{ for all } \boldsymbol{\eta} \in (H_0^1(\omega))^2 \times L^2(\omega). \quad (5.3.4)$$

or, $\boldsymbol{u}^l(\epsilon) \rightharpoonup 0$ *weakly in* $(H^1(\Omega))^2 \times L^2(\Omega)$.

iii) There exist positive constants C_1 *and* C_2, *independent of* ϵ *and* l, *such that* $0 < C_1 < \xi^l \le C_2$ *for all* $l \ge 1$.

Proof. Taking $\boldsymbol{v} = \boldsymbol{u}^l(\epsilon)$ in (5.2.8), it follows from the positivity of the tensor $A^{ijkl}(\epsilon)$, inequality (5.2.15) and the orthogonality condition (5.1.3) that

$$\xi^l(\epsilon) \ge C_1 \text{ for all } l \ge 1 \quad (5.3.5)$$

and hence $\xi^l \ge C_1$ for all $l \ge 1$.

It follows from Theorem 4.4.3 that

$$\xi^l(\epsilon) \le C_2(1 + \epsilon^2 k^l) \quad (5.3.6)$$

and hence $\xi^l \le C_2$ for all $l \ge 1$.

Also the proof that $(u_\alpha^l(\epsilon), u_3^l(\epsilon)) \rightharpoonup (u_\alpha^l, u_3^l)$ weakly in $(H^1(\Omega))^2 \times L^2(\Omega)$ and $(\overline{\boldsymbol{u}}^l, \xi^l)$ satisfies (5.3.4) follows from part b) of theorem 4.5.6. ∎

In the case of the stationary problem, Ciarlet and Lods [14] prove the strong convergence of the solution under the additional hypothesis that the body forces converge strongly in $L^2(\Omega)$. In the case of eigenvalue problem we are unable to prove the strong convergence of $\{u_3^l(\epsilon)\}$ in $L^2(\Omega)$ for each l. For this reason we are unable to check that $\boldsymbol{u}^l$ is non-zero and hence ξ^l may or may not be an eigenvalue of the limit problem (cf. Remark 4.5.1).

Let us denote $\boldsymbol{u}(\epsilon) = G^\epsilon(\boldsymbol{f})$ and $\boldsymbol{u} = G(\boldsymbol{f})$ where

$$\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\boldsymbol{u}(\epsilon))e_{i\|j}(\boldsymbol{v})\sqrt{g(\epsilon)}dx = \int_\Omega f_i v_i\sqrt{g(\epsilon)}dx \text{ for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega) \quad (5.3.7)$$

and

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy = \int_\omega\left\{\int_{-1}^1 f_i dx_3\right\}\eta_i\sqrt{a}dy \text{ for all } \boldsymbol{\eta} \in V_m(\omega). \quad (5.3.8)$$

Under the assumption (5.2.13), it has been shown in Ciarlet and Lods [14] that the Green's operator G^ϵ converges strongly to the Green's operator G.

i.e,

$$||G^\epsilon(f) - G(f)||_{(H^1(\Omega))^2 \times L^2(\Omega)} \to 0$$

Thus we have the following result.

Theorem 5.3.2 *i)For each $\xi \in \sigma(G)$, the spectrum of G, there exist $\xi(\epsilon) \in \sigma(G^\epsilon)$ such that $\xi(\epsilon) \to \xi$.*

ii)If $\xi(\epsilon)$ is a sequence in the discrete spectrum $\sigma_d(G^\epsilon)$ of G^ϵ such that $\xi(\epsilon) \to \xi$, where ξ does not belong to $\sigma(G)$, then the normalized eigenvectors associated with the eigenvalues $\xi(\epsilon)$ converges weakly to zero in $(H^1(\Omega))^2 \times L^2(\Omega)$.

Proof. See the proof of Proposition 2.1.11 of Allaire and Conca [1]. ∎

Observe that the Green's operator associated to the limit problem , i.e. G is *not* compact. Hence the structure of its spectrum is complicated.

Let A be the unbounded operator whose inverse is G. (Thus if the bilinear form in the left-hand side of (5.3.8) is $a(u, v)$, we define A by $a(u, v) = (Au, v)_{(L^2(\Omega))^3}$). The spectrum of G contains the reciprocals of the spectral elements of A. Then apart from isolated eigenvalues of finite multiplicity, A also has an essential spectrum which may consists of accumulation points of eigenvalues, eigenvalues of infinite multiplicity, continuous spectrum and so on. As the essential spectrum is bounded, and as A is unbounded, there exists eigenvalues of A, of finite multiplicity, tending to infinity (cf. [41])

We saw in Theorem 5.3.1 that the limits $\{\xi^l\}$ form a *bounded set*. Thus there exist an infinity of eigenvalues of the limit problem which cannot be expressed as limits of $\{\xi^l\}$ for fixed l, as $\epsilon \to 0$. This case is thus in vivid contrast to all the cases considered hitherto where *all* eigenvalues of the limit problem could be obtained as limits as described above.

Neverthless, by virtue of Theorem 5.3.2, every spectral element is the limit of a sequence of eigenvalues of the given family of problems. Thus if λ is in the spectrum of A, we have

$$\lambda = \lim_{\epsilon \to 0} \xi^{l(\epsilon)}(\epsilon) \tag{5.3.9}$$

i.e. the l now varies with ϵ. In view of this, let

$$\underline{\lambda} = \liminf_{\epsilon \to 0} \xi^1(\epsilon) \tag{5.3.10}$$

Then it is clear that $\sigma(A) \cap [0, \underline{\lambda}) = \phi$. Also if C is the constant appearing in Theorem 4.4.3 such that $\xi^l \leq C$ for all l, it follows that the eigenvalues of A which are greater than C can only be approximated as in (5.3.9) and *not* as a limit of $\{\xi^l(\epsilon)\}$ for some fixed l.

We are unable to specify the behaviour of $\{\xi^l(\epsilon)\}$ for fixed l. The limit may fall in the resolvent set, i.e. $\boldsymbol{u}^l(\epsilon) \rightharpoonup 0$ weakly in $(H^1(\Omega))^2 \times L^2(\Omega)$ or ξ^l may be an eigenvalue of finite or infinite multiplicity.

Concluding Remarks

We summarize below the important conclusions obtained and open problems raised by our study of eigenvalue problems of thin elastic shells.

- Shallow shells behave like plates. The solutions of the limiting eigenvalue problem correspond to the third component of the limits of eigenvectors of the three-dimensional problem and the other components of the limit can be obtained in terms of this component.

- Similarly, in the case of rods, the eigenvectors of the limit problem give the horizontal components of the limit of the three-dimensional eigenvectors while the third component tends to zero.

- In the case of flexural shells if $dim V_F = \infty$, we get the eigenvalue problem corresponding to the flexural model obtained in the stationary case.

- In all the above cases, *all* the eigenvalues of the limit problem can be obtained as limits of 'vertical' sequence of eigenvalues, i.e. limits of $\{\xi^l(\epsilon)\}$ for fixed l.

- In all the cases, there is a difference of a factor $\frac{1}{2}$ in the coefficients of the limit problem compared with those presented in the literature for the corresponding stationary cases. However, this discrepancy is easily removed if the limit models for stationary cases are presented with x_3-averages of the body forces on the right-hand side rather than mere integrals with respect to the x_3-variable. This is even more important when we descale the problems to describe the two-dimensional limit model in terms of the thickness 2ϵ of the shell. The power of ϵ which occurs will vary by one in the two models, but if we present the body body forces as x_3-averages, then both models will be the same. (cf. for example, Remark 2.7.1).

- In case $dimV_F < \infty$ for flexural shells and in the case of membrane shells we have an ambiguity regarding the behaviour of limits of 'vertical' sequences $\{\xi^l(\epsilon)\}$ for fixed l.

  This leads to some open questions.

- If $V_F \neq \{0\}$ then is it always infinite-dimensional. If not construct examples of shells with $dimV_F < \infty$.

- If $dimV_F = N$ and $l > N$, does $\boldsymbol{u}^l(\epsilon) \rightharpoonup 0$ weakly in $(H^1(\Omega))^2 \times L^2(\Omega)$. (If not, the flexural shell will have two kinds of eigenvalues in the limit- the 'flexural' eigenvalues for $l \leq N$ and 'membrane' eigenvalues for $l > N$. We conjecture that this will not be the case.

- In case of membrane shells, describe the limits of $\{\xi^l(\epsilon)\}$ for l fixed.

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