COLOR INVARIANT ADDITIVE FLUXES FOR LATTICE GAUGE THEORY

THESIS

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by

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DEDICATION

To Dr.S.Rajkumar
Psychiatrist

and friend

who has helped me through

all my problems over the years.

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CERTIFICATE

This is to Certify that the Ph.D thesis submitted by **Mr.G.H.Gadiyar**, to the University of Madras, entitled : Color Invariant Additive Fluxes for Lattice Gauge Theory is a record of bonafide research work done by him, during 1987-1992, under my supervision. The research work presented in this thesis has not formed the basis for the award to the candidate of any Degree, Diploma, Associateship, Fellowship or other similar title. It is further certified that the thesis represents independent work on the part of the candidate, and collaboration was necessitated by the nature and scope of the problems dealt with.

N. D. Hari Dass

Prof.N.D.Haridass
Supervisor

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INTRODUCTION

This thesis deals with a reformulation of lattice gauge theory in a new language. This reformulation seems to be suitable for further analytical and numerical analysis. The basis for this reformulation is the fact that the physical subspace of the Hilbert space of pure lattice gauge theory can be explicitly labelled using certain gauge invariant local operators[1].

In 2+1 dimensions in the SU(2) pure lattice gauge theories the Gauss law is explicitly solved[2]. This leads to a basis for physical states which is given by all triangulations with half integer sides with coordination number six. This arises from the fact that the solution of the Gauss law is related to the Clebsch-Gordan formula for rewriting the direct product of two Lie group representations as a direct sum. For SU(2) this is just the triangle rule. Next the pure SU(2) theory on a square lattice is shown to be exactly equivalent to an Abelian gauge theory on a Kagome lattice.

In order to do the same for other groups the equivalent of the triangle rule

needs to be found. This is carried out. The starting point is the Little wood - Richardson rule for the Clebsch-Gordon series. There is for SU(3) however a better algorithm. This is what is used and extended for SU(N). This helps us to obtain color invariant fluxes for SU(3). Pure SU(3) lattice gauge theory on a square lattice is shown to be equivalent to a certain abelian gauge theory with a $U(1) \times U(1)$ local gauge invariance on a Kagome lattice. Further, degrees of freedom which generate a Y-type string interaction emerge[3].

The ideas are not restructed to SU(2) and SU(3) only. The solution of the Little wood - Richardson rule in the SU(N) case in favour of certain random variables makes possible the generalization of the triangle rule to all SU(N)[4].

Thus color-invariant flux description of all pure lattice gauge theories is carried out. This could lead to better numerical and analytical understanding of the large N limit of lattice gauge theories and the confinement phase in particular.

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CHAPTER 1

The analysis carried out in this thesis is based on the Hamiltonian approach to lattice gauge theory[1]. Kogut and Susskind work with a discrete space lattice but with time continuous. They work in the gauge $Ao = 0$ and are able to define a Hamiltonian for the gauge degrees of freedom. In this approach one deals with a quantum mechanics with a well-defined Hamiltonian operator. The advantages of this will be seen as we go along.

Consider the partition function approach to field theory. Here a lattice is used to regularize the theory. The continuum theory has Lagrangian

$$L = -\frac{1}{4}\int F^{\alpha}_{\mu\nu}F^{\alpha}_{\mu\nu}d^3x\,.$$

To go over to a lattice gauge theory with lattice spacing a one has to make a formal replacement given by

$$L = -\frac{1}{4}\int F^{\alpha}_{\mu\nu}F^{\alpha}_{\mu\nu} \rightarrow \frac{1}{4g^2a^4}TrPexp[igC\oint_c A^{\alpha}_{\mu}\frac{\sigma^{\alpha}}{2}dx^{\mu}] \rightarrow \frac{1}{4g^2a^4}TrUUUU\,. \tag{1}$$

To go over to the Hamiltonian theory the t direction is placed on a special footing. Further the temporal gauge $Ao = 0$ is chosen. It must be noted that in this case gauge invariance is imposed as a constraint on the Hilbert space of states. This is the Gauss law

$$G_\alpha(\overrightarrow{n})/phys >= 0.$$

where $G_\alpha(\overrightarrow{n})$ is the generator of local rotations in color space at the lattice site labelled by $\overrightarrow{n}$.

Consider plaquettes with t in them. For these the formal replacement made in equation (1) simplified to

$$TrU^+(t_{i+1})U(t_i) + H.C = -Tr[U^+(t_{i+1}) - U^+(t_i)] \times [U(t_{i+1}) - U(t_i)] + const.$$

This makes it possible to construct the velocity term in the Lagrangian. The plaquettes with t in them contribute

$$-\frac{1}{4g^2a^2}\, Tr\left[\frac{U^+(t_{i+1}) - U^+(t_i)}{a_t}\right]\left[\frac{U(t_{i+1}) - U(t_i)}{a_t}\right]$$

where a_t is the lattice spacing in the t direction. Taking the continuum limit

in the t direction and noting that the $\int d^3x$ goes over to $\sum a^3$ the conventional quantum mechanics picture of lattice gauge theory is got.

$$L = \sum_{links} \frac{a}{4g^2} Tr\dot{U}^+\dot{U} + \sum_{plaq} \frac{1}{4ag^2} TrUUUU + H.C.$$

The Hamiltonian is next formed by going from velocity to momentum variables by the canonial procedures.

$$\begin{aligned} H &= \sum\sum \left[\dot{U}^+_{ij} \frac{\partial L}{\partial U_{ij}} + \dot{U}_{ij} \frac{\partial L}{\partial \dot{U}_{ij}} \right] - L \\ &= \sum_{links} \frac{a}{4g^2} Tr\dot{U}^+\dot{U} - \sum_{plaq} \frac{1}{4ag^2} (TrUUUU + HC) \end{aligned}$$

This is a system which consists of coupled tops. To go over to the quantized form and for future convenience the $\dot{U}$ are eliminated and the generators of local gauge transformations are used. Let $E^a(\overrightarrow{n}, i)$ be the quantum generator of the transformation

$$U(\overrightarrow{n}, i) \rightarrow \left(1 + i\epsilon \frac{\sigma^a}{2}\right) U(\overrightarrow{n}, i)$$

Here i labels the direction of the link coming from site $\overrightarrow{n}$. The commutation relations which are obeyed are given as

$$[E^a(n, i), U(m, j)] = \frac{1}{2} \sigma^a U(m, j) \delta_{ij} \delta_{nm} .$$

$$\left[E^{\alpha}(n,i),E^{\beta}(m,j)\right]=-i\epsilon^{\alpha\beta\gamma}E^{\gamma}(n,i)\delta_{ij}\delta_{nm}\,.$$

Now $\dot{U}^{+}$ and $\dot{U}$ are got rid of in favour of the infinitesimal generators E^{α}.

It is noted that E^{α} generates a local gauge rotation which is a symmetry of L. So it follows that

$$\begin{aligned}E^{\alpha} &= \frac{\partial L}{\partial \dot{U}_{ij}}\left(i\frac{\sigma^{\alpha}}{2}U\right)_{ij}+\frac{\partial L}{\partial \dot{U}^{+}_{ij}}\left(-i\frac{\sigma^{\alpha}}{2}U\right)^{+}_{ij}\\ &= i\frac{a}{g^{2}}\left(Tr\dot{U}^{+}\frac{\sigma^{\alpha}}{2}U-H.C.\right).\end{aligned}$$

To compute $E^{\alpha}E^{\alpha}$ certain identities have to be used. These are

$$U^{+}U=1 \qquad \dot{U}^{+}U+U^{+}\dot{U}=0$$

and

$$\sigma^{\alpha}_{ij}\sigma^{\alpha}_{kl}=2\delta_{il}\delta_{jk}-1\delta_{ij}\delta_{kl}\,.$$

This gives

$$E^{\alpha}E^{\alpha}=\frac{a^{2}}{2g^{4}}Tr\dot{U}^{+}\dot{U}\,.$$

Thus the final form of the hamiltonian in terms of the E^{α} and U variables is obtained.

$$H=\sum_{links}\frac{g^{2}}{2a}E^{\alpha}E^{\alpha}-\frac{1}{4ag^{2}}\sum_{plaq}(TrUUUU+H.C.)\,.$$

The physical states are defined as those annihilated by generators of local rotations at each site. This is given by

$$\sum E^{\alpha}(n,j)/phys>=0.$$

This summarizes the Kogut-Susskind approach to achieve a Hamiltonian for pure lattice gauge theory.

We now specialize the case of SU(2) and 2+1 dimensional theory. The physics of the Hamiltonian is given by the physics of coupled symmetric tops. We now proceed to show that the Gauss's law constraint can be explicitly and locally solved to get a basis for gauge invariant states[2].

The Hamiltonian is

$$H=\sum_{n,i}E(n,i)^2-K\sum_{n,i>j}TrU(ni)U(n+i,j)U^{+}(n+j,i)U^{+}(n,j)+H.C.$$

n is a label for sites of the lattice whereas i labels the i^{th} direction. Each link (ni) is associated with an SU(2) symmetric top whose configuration is given by an SU(2) matrix: this is the rotation matrix from space fixed to body

fixed axes.

To understand the gauge degree of freedom on a single link, it is sufficient to understand the quantum mechanics of a rigid rotator. To specify the configuration of the rigid rotator the rotation from body fixed to space fixed axes needs to be specified. The rotation may be represented in the form.

$$U_j = exp(i\vec{T}_j\vec{\Omega}) .$$

Here $T_{j\alpha}(\alpha = 1,2,3)$ are representation matrices of the generators of the rotation group for angular momentum j. Introduce a notation for matrices in which lower components refer to space axes and upper components to body axes. If Vi are components of a vector in space fixed frame, then in the body fixed frame the corresponding components are given by $(U_1)^l_i Vi = V^l$.

The body and space axes for the rigid rotator bear the same relationship as the indices on the two ends of a link in SU(2) Yang-Mills theory. The action of a rotation of space axes on U is given by left multiplication by the matrix V and the rotation of body axes relative to the body is given by right

multiplication. The statement of local gauge invariance in Yang Mills theory translates into invariance under separate body rotations and space rotations. This makes us consider the spherical rotator as only this case has invariance under rotations of the body axes.

The [redacted] rotator has angular velocity vector given by

$$\vec{\omega} = \frac{d}{dt}\vec{\Omega}$$

The generator of space rotations, that is the angular momentum is given by $I\vec{\omega}$ where I is the moment of inertia of the rigid rotator. The Hamiltonian is given by

$$H = \frac{J^2}{2I} = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\Omega}^2 .$$

The system has moment of inertia tensor diagonal, thus the Hamiltonian is invariant under individual body and space rotations as should be true if gauge invariance is to hold. The Hamiltonian can also be written as

$$H = \frac{J'^2}{2I}$$

where $\overrightarrow{J}' = U_1 \overrightarrow{J}$ is the generator of body rotations.

To define the quantum mechanics of the spherical rotator the eigenvectors are classified as simultaneous eigenvectors of the operators J^2, J_z, J'^2, J'_z. Further $J^2 = J'^2$ so to label states it suffices to use quantum numbers $J^2 = J'^2, J_z$ and J'_z. To summarize, there is a correspondence between the rigid rotator and SU(2) Yang-Mills theory:

global color rotation	$\rightarrow$	simultaneous body and space rotation.
local gauge transformation	$\rightarrow$	separate body and space rotations.
final end of link	$\rightarrow$	body index.
beginning end of link	$\rightarrow$	space index.

The body fixed and space fixed angular momenta correspond to the generators of gauge transformations which rotate one end of the link and leave unaffected the other end.

To return to the lattice gauge theory, the angular momenta with respect

to space fixed and body fixed axes are $E_+(ni)$ and $E_-(n+i\,,i)$ respectively. These are the generators of left and right SU(2) rotations on $U(ni)$. From the analysis of the rotator and looking at the analogy to SU(2) Yang-Mills theory, it follows that $E_+(ni)$ and $E_-(n+i\,,i)$ commute with each other and further that $E_+(ni)^2 = E_-(n+i\,,i)^2$. This says that the total angular momentum is the same in both frames and is denoted as $E(ni)^2$ in the Hamiltonian. The z-components of the angular momentum in the body and space fixed frames need to be also specified. Denote these eigenvalues of $E_+^3(ni)$ and $E_-^3(n+i\,,i)$ by $m_+(ni)$ and $m_-(n+i\,,i)$. So the labels corresponding to $J^2 = J'^2\,, J'_z$ and J_z are $j(ni)\,, m_+(ni)$ and $m_-(n+i\,,i)$.

Thus the basis for the Hilbert space is $\mid \{j(ni)\,, m_+(ni)\,, m_-(n+i\,,i)\} >$ where $j\,, m_+$ and m_- are all integral multiples of $\frac{1}{2}$ with $j(ni) \geq \mid m_+(ni) \mid$ and $j(ni) \geq \mid m_-(n+i\,,i) \mid$. This is before the implementation of the Gauss-law constraint.

The non-abelian Gauss's law giving the physical states is $\sum_i [E_+(ni) +$

$E_-(ni)] = 0$ at each site. This says the physical states are invariant under local gauge transformations. The advantage of using the basis where $E^3_+(ni)$ and $E^3_-(ni)$ are diagonal is that Gauss's law can be explicitly and locally solved to get a basis for gauge-invariant states.

To solve the Gauss's law constraint a change of basis is made where the following operators involving sums of angular momenta are diagonal,

$$\Big\{E(n1)^2\,, E(n2)^2\,, [E_+(n1) + E_+(n2)]^2\,,$$

$$[E_-(n1) + E_-(n2)]^2\,, [E^3_+(n1) + E^3_+(n2)]\,, [E^3_-(n1) + E^3_-(n2)]\Big\}$$

Next a change of basis is again made

$$\Big\{E(n1)^2\,, E(n2)^2\,, [E_+(n1) + E_+(n2)]^2\,, [E_-(n1) + E_-(n2)]^2\,,$$

$$[E_+(n1) + E_+(n2) + E_-(n1) + E_-(n2)]^2\,,$$

$$[E^3_+(n1) + E^3_+(n2) + E^3_-(n1) + E^3(n2)]\Big\}\,.$$

The gauge invariant states are those where for each n, the last two operators of the above set have zero eigenvalues. This makes the eigenvalues of

$[E_+(n1) + E_+(n2)]^2$ and $[E_-(n1) + E_-(n2)]^2$ equal and a single label $j(n12)$ can be used. Thus the basis for gauge invariant states is

$$| \{j(n1), j(n2), j(n12)\} >$$

Thus three half integers are associated with each site. This tallies with the number of physical degrees of freedom for gluons. Such counting also holds also for higher dimensions and other gauge groups also.

The basis $| \{j(n1), j(n2), j(n12)\} >$ has the following property: the labels are not mutually independent. $j(n12)$ can be obtained in two possible ways by adding angular momenta $j(n1)$ and $j(n2)$ or $j(n-1,1)$ and $j(n-2,2)$. The labels satisfy the triangle inequalities. This property can be used to develop a geometric representation. The constraints can be conviently represented on the dual lattice. Associate $j(ni)(i = 1,2)$ with the link dual to (ni) and $j(n12)$ with the diagonal completing the corresponding dual links into a triangle. So a half-integer is associated with each link of the dual lattice where one set of diagonals is drawn. This means a basis for gauge invariant

states is given by all triangulations with half-integer sides and coordination number six.

In this gauge invariant subspace the gauge invariant local dynamics can be written down. $E(ni)^2$ when acting on the basis of the form given by $|\{j(n1), j(n2), j(n12)\} >$ gives simply $j(ni)[j(ni)+1]$. The plaquette term with the U's has the effect of changing the lengths of the six lines emanating from the corresponding dual site by $\pm\frac{1}{2}$. By applying the Wigner-Eckart theorem repeatedly the transition - matrix elements from the lengths $\{j_1, j_2, j_3, j_4, j_{15}, j_{39}\}$ to the corresponding primed values can be computed. The matrix element depends on the lengths of the edges of the hexagon $j_{14}, j_{23}, j_5, j_6, j_9, j_{10}$ through these do not change under the action of the plaquette centered at the corresponding dual site. Refer Fig 1 and Fig 2.

The matrix element is given by

$$< M >= \left\{ \begin{matrix} j_{14} & j_1 & j_4 \\ \frac{1}{2} & j'_4 & j'_1 \end{matrix} \right\} \left\{ \begin{matrix} j_{23} & j_2 & j_3 \\ \frac{1}{2} & j'_3 & j'_2 \end{matrix} \right\} \left\{ \begin{matrix} j_5 & j_1 & j_{15} \\ \frac{1}{2} & j'_{15} & j'_1 \end{matrix} \right\} \times$$

$$\times \left\{ \begin{matrix} j_6 & j_2 & j_{15} \\ \frac{1}{2} & j'_{15} & j'_2 \end{matrix} \right\} \left\{ \begin{matrix} j_9 & j_3 & j_{39} \\ \frac{1}{2} & j'_{39} & j'_3 \end{matrix} \right\} \left\{ \begin{matrix} j_{10} & j_4 & j_{39} \\ \frac{1}{2} & j'_{39} & j'_4 \end{matrix} \right\} \times$$

$$\times \sqrt{2j_1+1}\sqrt{2j_2+1}\sqrt{2j_3+1}\sqrt{2j_4+1}\sqrt{2j_{15}+1}\sqrt{2j_{39}+1}$$

$$\times \sqrt{2j'_1+1}\sqrt{2j'_2+1}\sqrt{2j'_3+1}\sqrt{2j'_4+1}\sqrt{2j'_{15}+1}\sqrt{2j'_{39}+1}$$

$$\times (-1)^{j_5+j_{14}+j_{23}-j_9-j_6+j_{10}+j_2-j'_2-j_1+j'_1}$$

There is a 6-j symbol denoted by $\{\ldots\}$ associated with each of the six triangles of the hexagon. The angular momenta involved in each 6-j symbol are the original and new lengths of the two arms of the corresponding triangle denoted by j and j' respectively, the edge of the hexagon and $\frac{1}{2}$

corresponding to the tensorial property of the operator U. Thus we see that the time-evolution of the system has been mapped to local fluctuations of the triangulated surface.

To repeat the Gauss's law constraint for non-abelian lattice gauge theory has been solved locally and explicitly and the dynamics on the physical states obtained. The dynamics is local and is related to a quantum theory of a discretized membrane.

To make progress, the constraint coming from the Gauss's law can be solved in a different way: now an algebraic solution of the constraints in contrast to the geometric solution outlines earlier is implemented[3]. This makes it possible to rewrite the SU(2) theory on a square lattice as an abelian gauge theory on a Kagome lattice, thus getting rid of the non-abelian algebra. To do this new dynamical variables, which create or annihilate a unit of an additive color electric flux without reference to color content, are introduced. Using these new dynamical variables it is possible to obtain a complete basis

for gauge invariant states in the loop space without all the complications that the collection of all Wilson loops have. This also shows in a precise sense how a U(1) gauge theory is relevant for the confinement mechanism in SU(2) gauge theory. Thus 't Hoofts' conjecture that a gauge theory of the abelian subgroup is relevant for the confinement mechanism is established.

Given any triangle with half-integer sides j_1 , j_2 and j_3 corresponding to the addition of angular momenta the combinations

$$\begin{aligned} N_1 &= j_2 + j_3 - j_1 \\ N_2 &= j_3 + j_1 - j_2 \\ N_3 &= j_1 + j_2 - j_3 \end{aligned}$$

are always integers and non negative. Infact three arbitrary independent non-negative integers

$$N_1 \geq 0, \quad N_2 \geq 0 \quad \& \quad N_3 \geq 0$$

uniquely generate all such triangles. These integers are associated with the sides of an inscribed triangle as in Fig 3. The inscribed triangle is for figurative purposes only and does not satisfy the triangle inequality constraints

in general.

This gives us an alternate way of characterizing all triangulations. The j's have been associated (Fig 2) with the links of a triangular lattice obtained by drawing one set of diagonals on the lattice dual to the original lattice. For convenience of representation this lattice is deformed to a regular triangular lattice (Fig 3). In addition equilateral triangles are inscribed in each triangle of this dual lattice. These inscribed triangles form a Kagome lattice. The integers N_1 , N_2 and N_3 are associated with the links of this Kagome lattice. For any inscribed triangle, the sum of N_I on any two sides gives twice the j on the link of the triangular lattice on which they impinge. Since each dual link is common to two inscribed triangles, a constraint exists (Fig 3) such as

$$N_1 + N_2 = N_4 + N_5$$

That is the sum of the weights on the two sides of any triangle of the Kagome lattice should equal the sum on the two sides of the other triangle meeting at the common vertex. An alternative way for specifying a basis for the

physical states is now available. This is done by assigning arbitrary non-negative integers to the links of the Kagome lattice, subject to the constraint at every vertex, which generates all triangulations uniquely.

The next step is to describe the dynamics in this new basis. The effect of a plaquette operator is to independently change by $\pm\frac{1}{2}$ the j's associated with the six links of the triangular lattice incident on the vertex dual to the plaquette. This corresponds to change by ± 1 of certain N's on a 'star' of this Kagome lattice centered on this dual vertex. The links on which N increases are denoted by a solid line and those on which N decreases by a jagged line. The plaquette operator corresponds to the diagrams of Fig 4. Several other diagrams have to be included. These are got by rotating these diagrams by multiples of 60° and/or by interchanging the solid and jagged lines. All these result in distinct diagrams.

The matrix elements for the amplitudes of various processes can be written down. These amplitudes have a six fold symmetry corresponding to the

notations of the 'star' by multiples of 60°. This is in the representation where the $\hat{2}$ axis is inclined to the $\hat{1}$ axis at 60°. This is inspite of the fact that the links, on which the sum of the color spins on the $\hat{1}$ and $\hat{2}$ links are represented are altogether on a different footing. The other term in the Hamiltonian which corresponds to the electric field does not respect this symmetry.

To represent the Hamiltonian in this basis, we associate a harmonic oscillator creation operator a^* and an annihilation operator a with each link of the Kagome lattice. The eigenvalue of the corresponding number operator is related to the weight N_1 on the link. The conservation law at every vertex is interpreted as the Gauss law constraint associated with a U(1) local gauge invaraince in the following way. The triangles of the Kagome lattice can be consistently assigned positive (+) or negative (-) signatures alternately depending on whether they point 'up' or 'down'. This signature is also ascribed to the links which correspond to the sides of the triangles. A local charge of +1 or -1 is associated with both ends of a link depending on whether

the signature of the link is positive or negative respectively. This gives a U(1) gauge theory on the Kagome lattice. There are differences with the usual U(1) gauge theory in the following ways. On each link the variable is an oscillator instead of a planar rotator, the U(1) charge depends on the signature of the link and the possible gauge invariant interactions appear with very specific amplitudes. The additive constraints at the vertices can be solved by using closed loops as is done in the usual U(1) lattice gauge theory[4]. Consider a closed loop of the Kagome lattice which everywhere goes straight or takes 60° turns. To assign $N = 1$ for the links of this loop is consistent with the constraints. Such a loop is called an allowed loop. As the values of N are only allowed to be positive these loops are not assigned any orientation. All the allowed configurations can be generated by an arbitrary collection of such allowed loops. The N for the link is given by the total number of transits along that particular link. In case of loops which intersect or overlap either partially or completely different ways of forming

closed loops are not distinguished. All this differs from the usual U(1) lattice gauge theory. This procedure gives a complete basis for physical states in the loop space without the redundancies and constraints that the set of all Wilson loops have.

An allowed loop increases j on the links of the dual lattice it intersects by $\frac{1}{2}$. Thus the total color flux on the links of a closed loop of the original lattice is increased by $\frac{1}{2}$. This differs from the usual Wilson loop operator which has a more complicated action. Thus these allowed loops are more fundamental entities.

Although the theory differs from usual U(1) lattice gauge theory, most of the concepts and techniques can be taken over. The operators $u = aN^{-\frac{1}{2}}$ and $u^+ = N^{-\frac{1}{2}}a^+$ are analogs of the usual U(1) link variables $U = e^{i\theta}$ and $U^+ = e^{i\theta}$. Further the number operator N is the analog of the electric field E. In the phase where the compactness of the link variable, or equivalently the discreteness of the conjugate variable, is irrelevant, there are massless

vector bosons. This can be seen by ignoring the fact that the spectrum of θ is compact and of N is discrete can considering quadratic fluctuations about their expectation values. The calculation is very complicated as instead of the U U U U -term now a set of terms involving products of upto twelve u variables with coefficients depending on the conjugate varaibles N is involved. Because of the Kagome lattice and the specific interactions involved even though there is one kind of link variable u, there can be three massless excitations of the SU(2) gauge theory. The effects of compactness of θ can be interpreted in terms of the monopole degrees of freedom. Confinement is a consequence of the monopoles forming a condensate.

Thus the non-abelian lattice gauge theory is mapped exactly into an abelian gauge theory on a Kagome lattice. New dynamical variables create or annihilate a unit of an additive color invariant electric flux. These give a complete basis for physical states in the form of non-oriented allowed closed loops of unit flux. The conjecture of 't Hooft that topological excitation of the

abelian subgroup of SU(N) determine the confinement mechanism of SU(N) gauge theory is thereby placed on a firm footing[5]. Thus now numerical and analytic techniques can be developed to understand gauge theories on a new footing.

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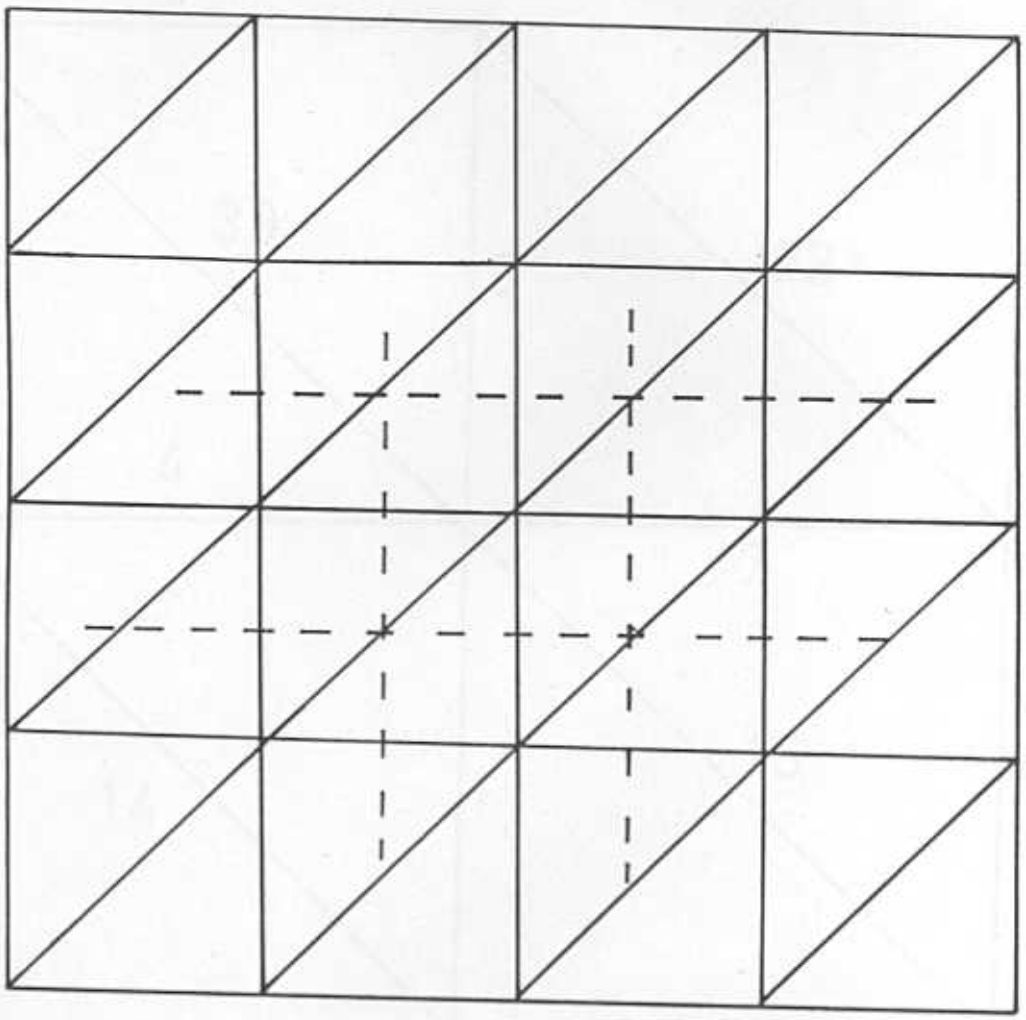


Fig 1

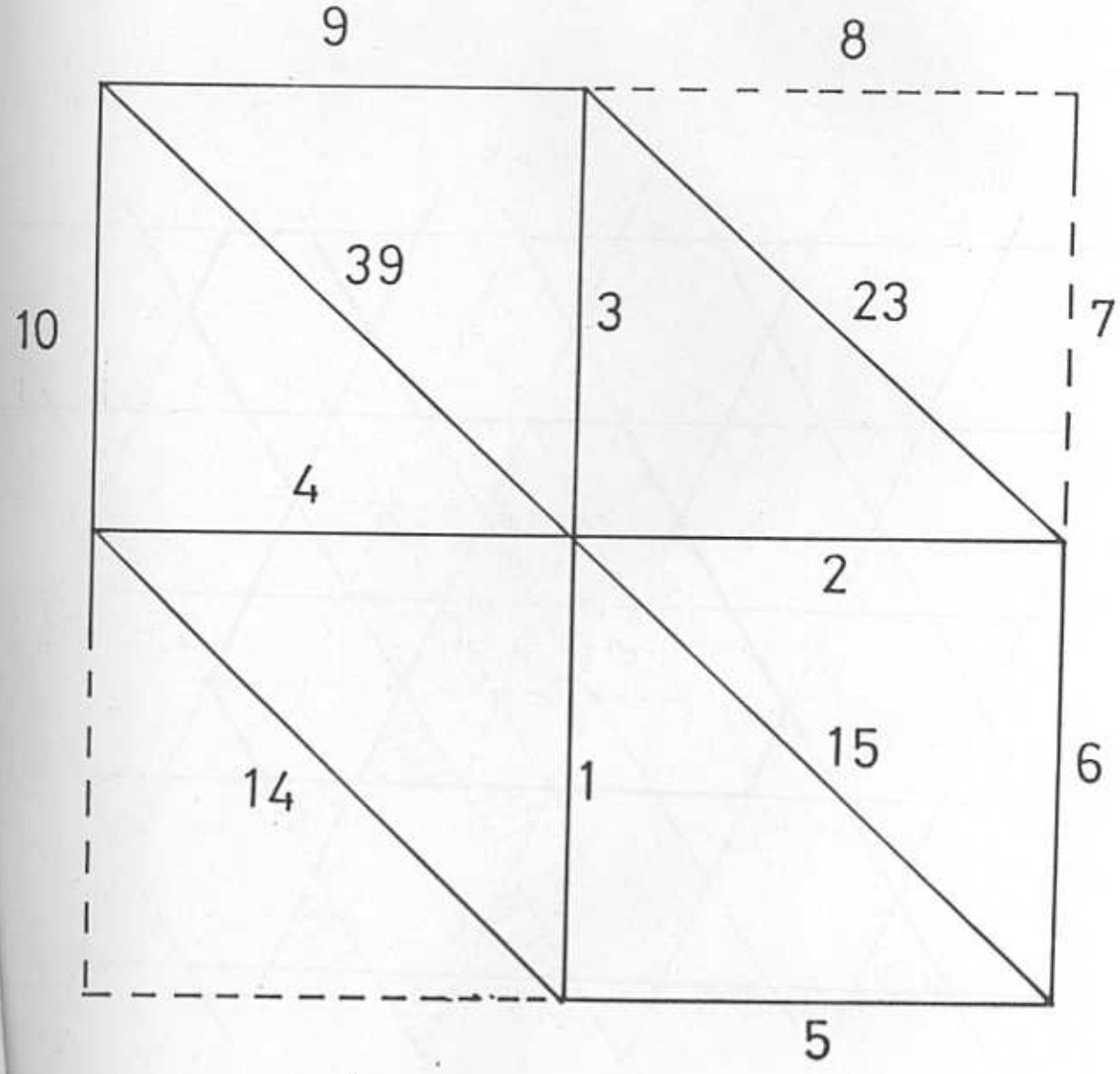


Fig 2

Fig 3

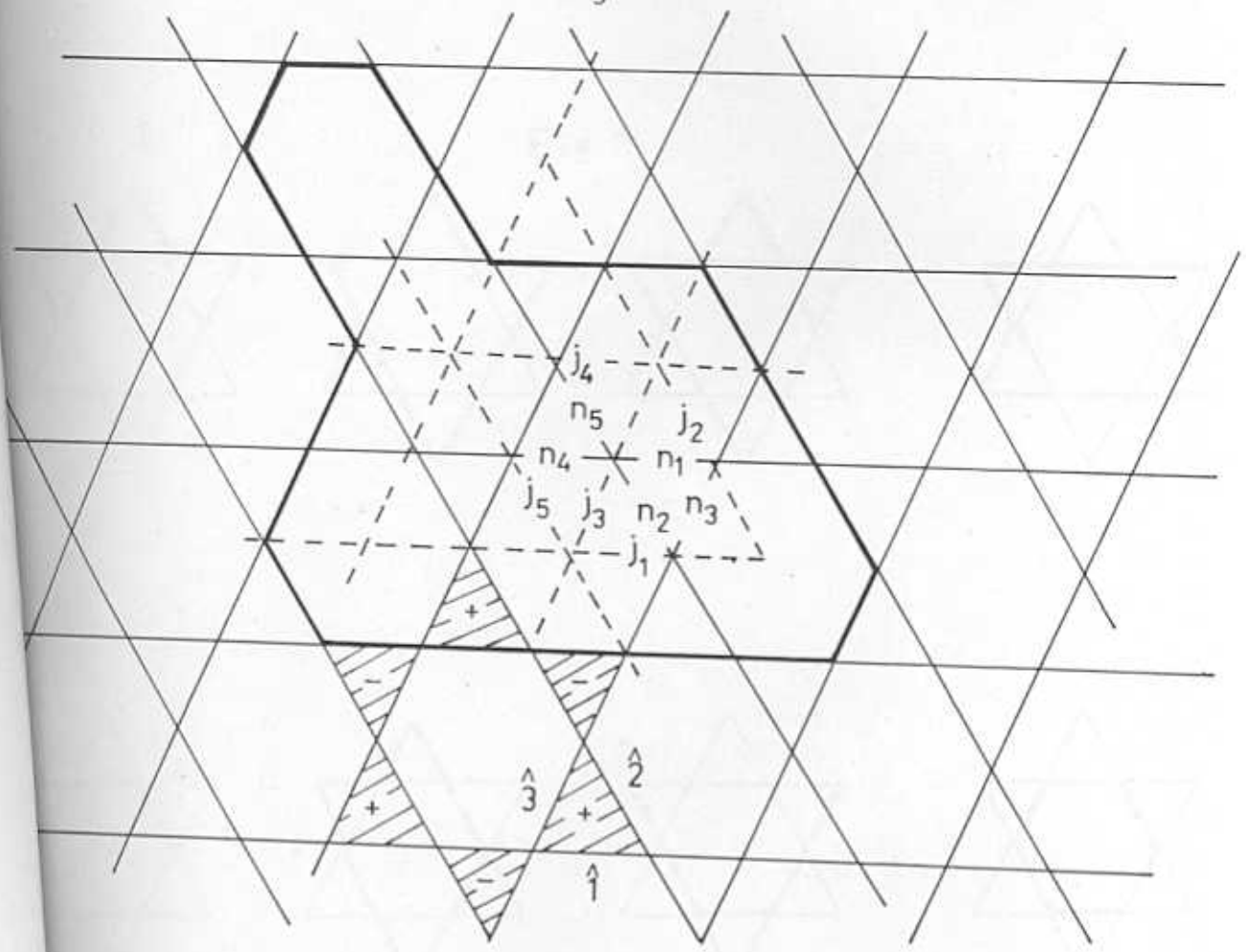
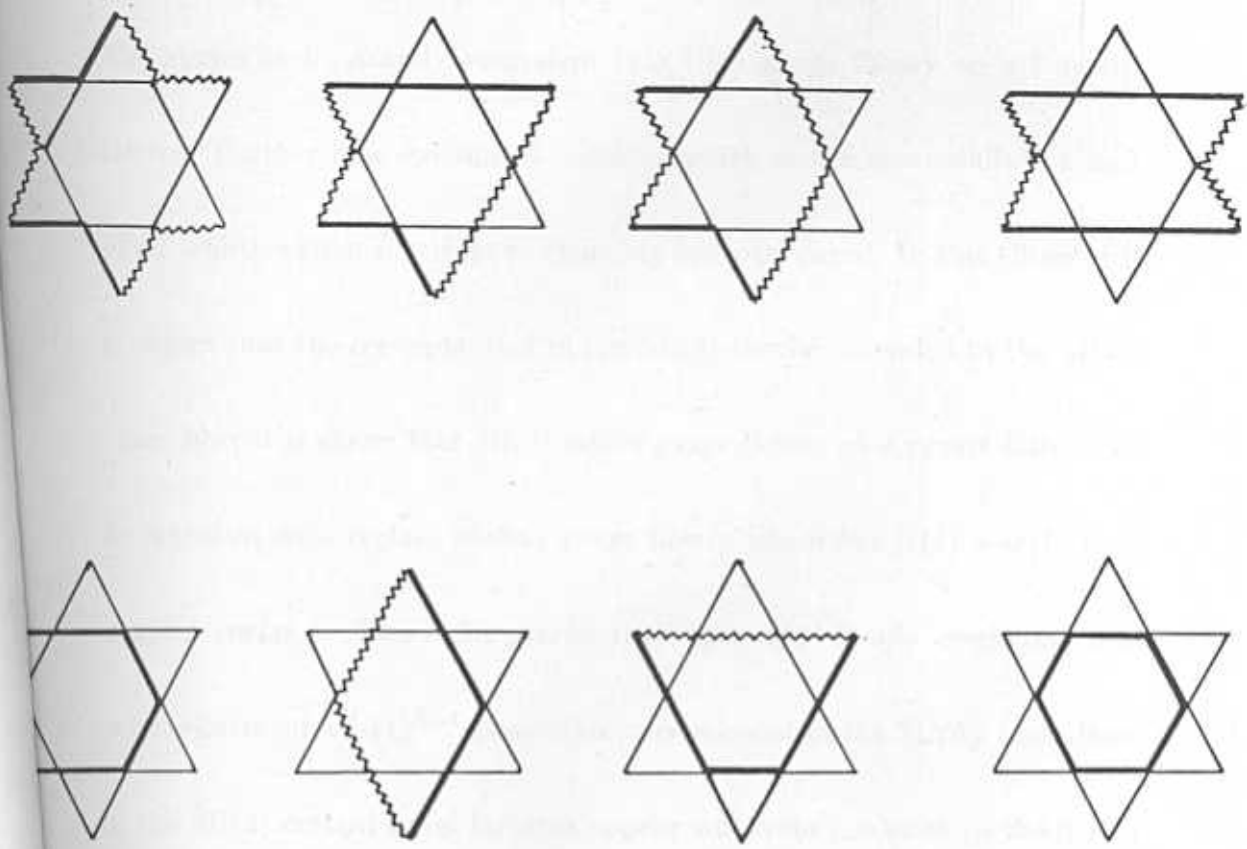


Fig 4



CHAPTER 2

In the first Chapter the SU(2) lattice gauge theory on a square lattice was shown to be exactly equivalent to a U(1) gauge theory on a Kagome lattice. Further new dynamical variables which create or annihilate a unit of an additive color invariant electric flux are introduced. In this Chapter it is shown that the concepts used in the SU(2) can be extended to the SU(3) case. *Here it is shown that SU(3) lattice gauge theory on a square lattice can be rewritten as a certain abelian gauge theory which has* $U(1) \times U(1)$ *local gauge invariance. This is the precise realization of t' Hoofts' conjecture that for confinement a* $U(1)^{N-1}$ *gauge theory is relevant in the SU(N) case.* Here in the SU(3) certain novel features appear which do not exist in the SU(2) case. For example a Y-type string interaction which is needed for holding a baryon together emerges from the analysis in a natural way.

The key idea in the reformulation is that the physical subspace of the Hilbert space of lattice gauge theories can be explicitly labelled using certain

gauge invariant local operators. In the SU(2) case in 2+1 diemensions it was seen that the basis for the physical states is given by triangulations with half-integer sides. This arises from the triangle rule in the addition of angular momentum. Such triangle inequality constraints are solved in favour of certain non-negative integers. This gives a certain abelian gauge theory.

To extend the results from $SU(2)$ to $SU(3)$ involves tackling a group theoretical problem. The problem is to get an analog of the triangle rule for addition of angular momenta. The SU(3) case differs from SU(2) in the following ways: firstly the representations are not real and secondly in the decomposition of the direct product of two irreducible representations as a direct sum (the Clebsch-Gordan series) the same irreducible representation may appear more than once. The Clebsch-Gordan series is usually derived by using the machinery of Young tableaux and the Littlewood-Richardson rule, but this does not prove useful for our purposes. A closed formula for the Clebsch-Gordan series is needed. To do this for the SU(3) case a

better algorithm exists. This has been developed by several groups[1]. This algorithm is not very widely known. This algorithm is now described. Then the algorithm is translated into a form and language suitable for this work and later we apply the results to lattice gauge theroy.

The method has two main steps. In the first stage the direct product of two irreducible representations is written as the direct sum of certain special reducible representations. In the next stage the special reducible representations are decomposed to the direct sum of irreducible representations.

It is well known that the irreducible representations of SU(3) are characterised as the transformations induced on irreducible tensorial sets by unitary unimodular transformations of a three dimensional complex vector space. Denote by (n, m) the irreducible representation with basis the set of tensors with n upper indices and m lower indices, which are completely symmetric in upper indices, completely symmetric in lower indices and traceless. Denote by $(n, n'; m, m')$ the representation with basis the set of all tensors

with $n + n'$ upper indices, and $m + m'$ lower indices, which are completely symmetric among the first n upper indices, completely symmetric among the last n' upper indices, completely symmetric among the first m lower indices, completely symmetric among the last m' lower indices and traceless. $(n, n'; m, m')$ can be throught of as the direct product of (n, m) and (n', m') with all traces removed but without any symmetrization.

To decompose the direct product of irreducible representations into these special reducible representations is the first stage. This is done by separating out all tensors that can be obtained by contracting, in all possible ways, indices from the set of n with indices from the set of m', and indices from the set of n' with indices from the set of m.

$$(n, m) \otimes (n', m') = \sum_{i=0}^{\min(n,m')} \sum_{j=0}^{\min(n',m)} (n - i, m - j, n' - j, m' - i)$$

where the summation indicates a direct sum.

The next stage is to decompose an arbitrary tensor from the basis of $(n, m; n', m')$ into a sum of linear combinations of completely symmetric

traceless tensors. Take for example the tensor which is the basis of $(n, m; n', m')$ denoted by $T \begin{matrix} i_1 \cdots & i_n\, i_{n+1} \cdots & i_{n+n'} \\ j_1 \cdots & j_m\, j_{m+1} \cdots & j_{m+m'} \end{matrix}$. Take a pair of upper indices, say i, and i_{n+1} without any loss of generality. This tensor can be decomposed as a symmetric and antisymmetric tensor under interchange of i_1 and i_{n+1}. Using the $\in$ tensor the antisymmetric part can be written as a tensor of lower rank.

$$S \begin{matrix} i_2 \cdots & i_n i_{n+2} \cdots & i_{n+n'} \\ k j_1 \cdots & j_m j_{m+1} \cdots & j_{m+m'} \end{matrix} = \in_{k i_1 i_{n+1}} T \begin{matrix} i_1 \cdots & i_{n+n'} \\ j_1 \cdots & j_{m+m'} \end{matrix}$$

This tensor is already completely symmetric in its lower indices. The proof is as follows. Take, say, j_1 and j_{m+1}.

$$\in^{r j_1 j_{m+1}} S \begin{matrix} i_2 \cdots & i_{n+n'} \\ k \cdots & j_{m+m'} \end{matrix}$$

$$= \in^{r j_1 j_{m+1}} \in_{k i_1 i_{n+1}} \; T \begin{matrix} i_1 \cdots & i_{n+n'} \\ j_1 \cdots & j_{m+m'} \end{matrix}$$

$$= \left(\delta^r_k \delta^{j_1}_{i_1} \delta^{j_{m+1}}_{i_{n+1}} - \delta^{j_1}_k \delta^r_{i_1} \delta^{j_{m+1}}_{i_{n+1}} + \text{cyclic} \right) T \begin{matrix} i_1 \cdots i_{n+n'} \\ j_1 \cdots j_{m+m'} \end{matrix}$$

$$= 0 (\text{from tracelessness of T})$$

So it follows that the tensor is symmetric under interchange of j_1 and j_{m+1} as its contraction with the $\in$ tensor is zero.

To decompose the special reducible representation follows as a consequence. The tensors are symmetrized as follows: remove a pair of upper indices and add a lower index ; or remove a pair of lower indices and add an upper index. The process terminates when one runs out of indices. In the basis space of the representation the process is written as

$$
\begin{aligned}
(n, n'; m, m') &= (n + n' , m + m') \\
&\oplus \sum_{i=1}^{\min(n ,n')} (n + n' - 2i , m + m' + i) \\
&\oplus \sum_{j=1}^{\min(m ,m')} (n + n' + j , m + m' - 2j) .
\end{aligned}
$$

In order to use this it is convenient to first restate the results. An irreducible representation of SU(3) may be conveniently labelled by an ordered pair (j^- , j^+) of two arbitrary non-negative integers which represent the number of 2-column and 1-column boxes in the corresponding Young's tableaux. The Casimirs can be expressed in terms of j^- and j^+. The algorithm derived in the preceding paragraphs can be reinterpreted in the following way. The

two stage process given above can be now interpreted as follows.

Let the irreducible representation (j_1^-, j_1^+) have j_1^- boxes labelled '-1' and j_1^+ boxes labelled '+1'. Similarly the irreducible representation (j_2^-, j_2^+) consists of j_2^- boxes labelled '-2' and j_2^+ boxes lebelled '+2'. Remove some '-1' boxes and an equal number of '+2' boxes. Denote this number as $N^-(2,1)$. Remove also some '-1' boxes and an equal number of '-2' boxes. Denote this number by $N^+(2,1)$. Of the remaining boxes (i) either remove a number say $|L|$ of '-1' boxes and an equal number of '-2' boxes and introduce $|L|$ number of new boxes lbelled '+3'. (ii) or remove a number say $|L|$ of '+1' boxes and an equal number of '+2' boxes and introduce $|L|$ number of new boxes labelled '-3'. The differentiate between the two possibilities (i) and (ii) a sign is attached to L. Denote the number of ± 1 boxes left over after the removal by $N^{\pm}(3,1)$ and the ± 2 boxes remaining are denoted by $N^{\pm}(3,2)$.

This means

$$j_1^{\pm} = N^{\pm}(2,1) + N^{\pm}(3,1) + |L|\theta(\pm L)$$
$$j_2^{\pm} = N^{\pm}(1,2) + N^{\pm}(3,2) + |L|\theta(\pm L)\,. \tag{1}$$

By definition $N^{\pm}(i\,,j) = N^{\mp}(j\,,i)$ and $\theta(L)$ is defined as

$$\theta(L) \;=\; +1 \quad L > 0$$
$$\;=\; 0 \quad\;\; L \leq 0\,.$$

The above operations are connected with one irreducible representation in the Clebsch-Gordan series. To get a consistent notation the complex conjugate of this irreducible representation is labelled by $(j_3^-\,, j_3^+)$. This is given by the total number of '-' and '+' boxes, including '$\pm$ 3' boxes remaining. Therefore

$$j_3^{\pm} = N^{\pm}(1,3) + N^{\pm}(2,3) + |L|\theta(\pm L)\,. \tag{2}$$

All possible irreducible representations are got by carrying out the procedure given above in all possible ways. The notations are chosen to get a uniform formula for all the $j_1^{\pm}\,, j_2^{\pm}$ and $j_3^{\pm}$. In all six arbitrary non-negative integers $N^{\pm}(1,2)\,, N^{\pm}(2,3)$ and $N^{\pm}(3,1)$ and one integer L carrying a sign

uniquely label the representations resulting from the decomposition. However these integers are subject to the constraints (1). This means that for given initial representations (j_1^+, j_1^-) and (j_2^+, j_2^-) three remaining integers label the irreducible representations of the decomposition. This means that there is a label in addition to (j_3^+, j^-) available. This extra integer distinguishes the repeating irreducible representations.

The equations (1) and (2) are analogs of the triangle rule satisfied in the addition of angular momenta. This is seen as follows. Let a non-negative integer j represent an irreducible representation of SU(2). j is twice the angular momentum J. If j_3 is an irreducible representation in the Clebsch-Gordan series for the direct product of j_1 and j_2, by the triangle rule,

$$\begin{aligned} N_1 &= \tfrac{1}{2}(j_2 + j_3 - j_1) \\ N_2 &= \tfrac{1}{2}(j_3 + j_1 - j_2) \\ N_3 &= \tfrac{1}{2}(j_1 + j_2 - j_3) \end{aligned} \tag{3}$$

are all non-negative integers.

Moreover if we rewrite these equations as

$$j_1 = N_2 + N_3 \qquad j_2 = N_3 + N_1 \qquad j_3 = N_1 + N_2 \tag{4}$$

all non-negative integers N_1, N_2, N_3 subject to the constraints (4) give all the irreducible representations in the decomposition uniquely. Comparing equations (1) and (2) with equation (4), we can see that an analog of the triangle rule is now available for SU(3).

In the SU(2) case in Chapter I, the j's are represented as sides of a triangle,, and the N's on the sides of an inscribed triangle. For the SU(3) case, in a similar way, each pair $(j_1^-, j_1^+), (j_2^-, j_2^+)$ and (j_3^-, j_3^+) are represented on a side of a triangle and each pair $N^{\pm}(i, j)$ on the side of an inscribed triangle. j_i^+ is obtained by adding up $N^+(\ldots, i)$ on the two sides of the inscribed triangle which impinge on side i and also $\mid L \mid$ if L is positive. j^- is obtained by adding up $N^-(\ldots, i)$ on the two sides of the inscribed triangle which impinge on the side i and also $\mid L \mid$ if L is negative. Refer Fig 1 and Fig 2.

To apply these results to lattice gauge theory is the next task. Consider

the subspace of the Hilbert space of 2+1, pure SU(3) lattice gauge theory satisfying Gauss' law constraint. This may be described as follows. To every link of a 2-d square lattice is associated an irreducible representation (j^-, j^+) of SU(3). This is represented on the corresponding dual link as in the SU(2) case. Also, at every site, with the pairs of links in $+\hat{1}$ and $+\hat{2}$ directions is associated any irreducible representation (j_{21}^-, j_{21}^+) which is an element in the Clebsch-Gordan series of athe direct product of the irreducible representations on the two links. In a similar way with the pair of links in $-\hat{1}$ and $-\hat{2}$ directions at the site an irreducible representation (j_{12}^-, j_{12}^+) is associated which is in the Clebsch-Gordan series of the direct product of irreducible representations on the two links. To satisfy the Gauss law the two irreducible representations associated with the pairs of links at each site must be conjugates of each other. If this is satisfied, that is if $j_{12}^{\pm} = j_{21}^{\mp}$, they can be combined into the identity representation. It is convenient to represent (j_{12}^-, j_{12}^+) on the diagonal of the dual lattice. This is indicated in

the Figure 3. There is a difference with the SU(2) case, in the SU(2) case the representations are real in the SU(3) case they are complex. Now which of the two conjugate irreducible representations is on the diagonal has to be specified.

We now have pairs of non-negative integers associated with the links of a triangular lattice. Thse pairs of integers satisfy certain constraints arising from group theory. These constraints are given by the equations (1) and (2) given. As in Chapter 1 a triangle is inscribed into each triangle of the lattice. These inscribed triangles form a Kagome lattice. With the sides of the triangle are associated the variables $N^{\pm}(I,J) = N^{\mp}(J,I)$. With the face of the triangle the variable L is associated. The equations (1) and (2) given (j^-, j^+) in terms of $N^{\pm}$ and L. Each link of the triangular lattice on which (j^-, j^+) is represented is common to two triangles of the Kagome lattice. This gives a pair of conservation laws at each vertex.

Next, two oscillators are associated with each link of the Kagome lattice.

$N^{\pm}$ on the link are identified with the eigenvlaues of the number operators of the two oscillators. Further a planar rotator is associated with each triangle of the Kagome lattice. L for the triangle is identified with the planar angular momentum of this rotator. The two conservation laws are regarded as the Gauss' law constraints arising from a $U(1)\times U(1)$ local gauge invariance. The generators of the two transformations involve $\hat{N}^{\pm}$ and $\hat{L}$ operators and hence commute with each other. However, the oscillator and the rotator variables carry charges of both gauge groups in an intricate fashion as seen below.

The conservation law matches $j^{+}(j^{-})$ an one side with $j^{-}(j^{+})$ on the other at a diagonal link of the triangular lattice, whereas $j^{+}(j^{-})$ is matched with $j^{+}(j^{-})$ at the other links. In order to get a uniform version of the conservation law at every vertex, we proceed as follows. We give up the identification $N^{\pm}(I,J) = N^{\mp}(I,J)$. Instead we temporarily associate a favoured direction for each link of the Kagome lattice as shown in Figure 4. Now $N^{\pm}$ are defined to be $N^{\pm}(I,J)$ with $I \rightarrow J$ in the favoured direction. We now express the

conservation law in terms of 'allowed' loops as in Chapter 1.

An allowed loop is one which at any vertex of the Kagome lattice passes on to a side of the other triangle at the vertex instead of turning back into the same triangle. Now it is easily checked that for any allowed loop, the following assignment of the weights satisfies the conservation laws. $N^+ = 1$ for the links of any one straight section of the allowed loop, after a bend $N^- = 1$ for the links of the next straight section, $N^+ = 1$ again on the following straight section and so on. Interchanging N^+ and N^- in the assignment given above again satisfies the conservation laws. In these case L = 0 for the triangles touched by the loop. If $L = \pm 1$ at a triangle, the conservation law requires three allowed loops to emanate from this triangle, one along each of the three triangles touching this triangle. This is a novel feature not present in the SU(2) case. This can be traced back to the singlet that can be formed from three triplets of SU(3). This reminds one of the Y-type string interactions for holding quarks together within a baryon. Thus, even in a pure gauge

theory, in SU(3), thre are color invariant degrees of freedom associated with the faces of triangles which behave like the Y-type source of color invariant flux.

Let L = +1 on a triangle. We now specify the rule for the $N^{\pm}$ weights to satisfy the conservation law. Imagine a clock hand centered at the triangle. Then for each of the three triangles touching this triangle, either $N^{+} = 1$ for the link which is first touched by the clock hand, or $N^{-} = 1$ for the other link which is touched alter. If L = -1 the assignments of N^{+} and N^{-} have to be interchanged.

The rest of the analysis is same as the SU(2) case. Thus in this Chapter the SU(3) lattice gauge theory on a square lattice has been analysed and shown equivalent to a certian abelian theory with a $U(1) \times U(1)$ symmetry on a Kagome lattice. Degrees of freedom which generate Y-type string interaction emerge[2].

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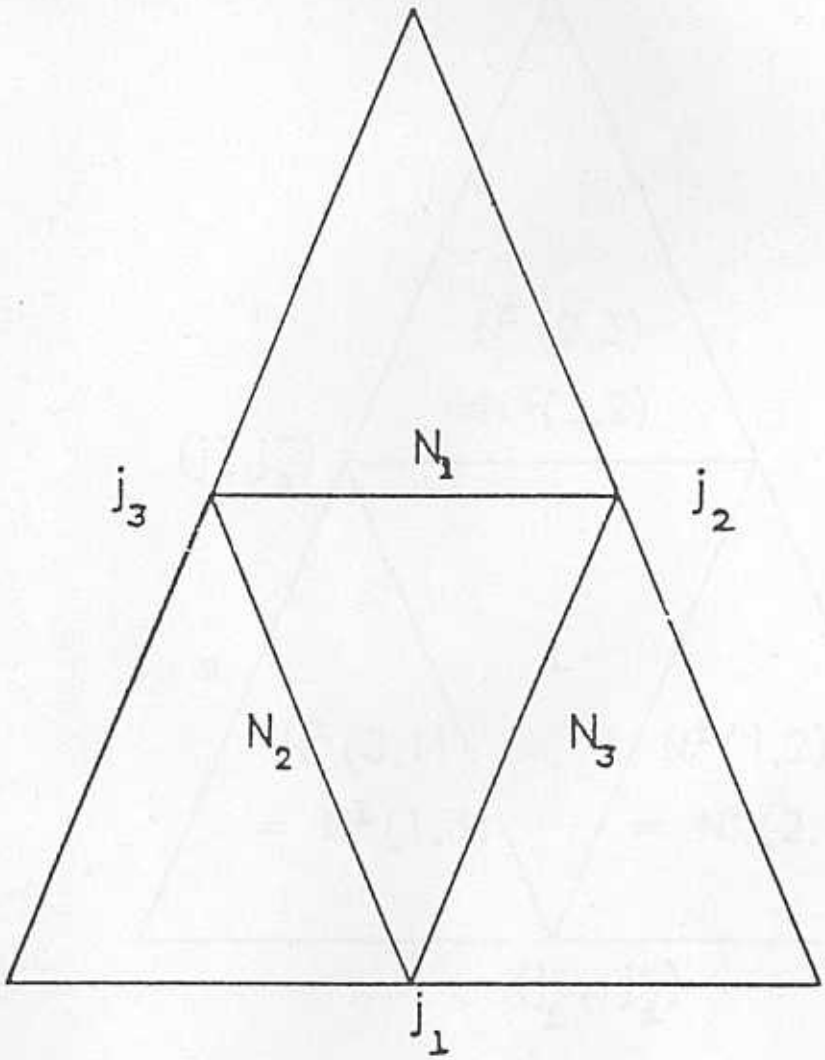
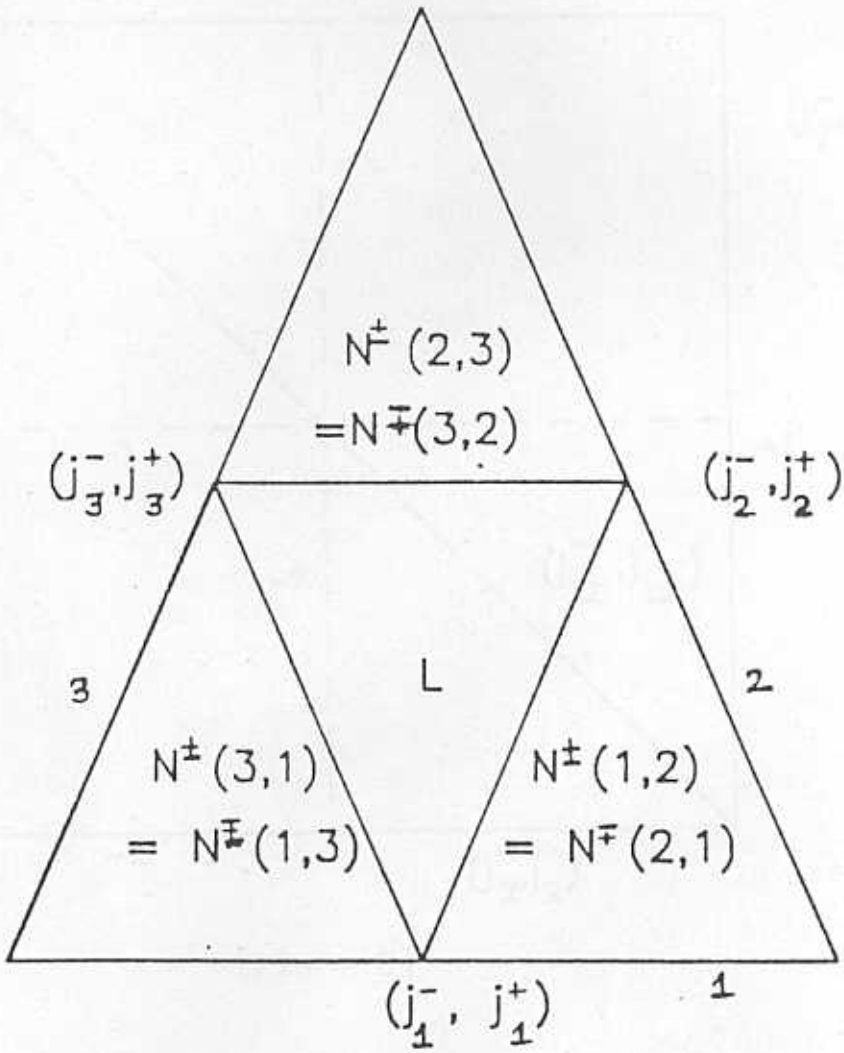
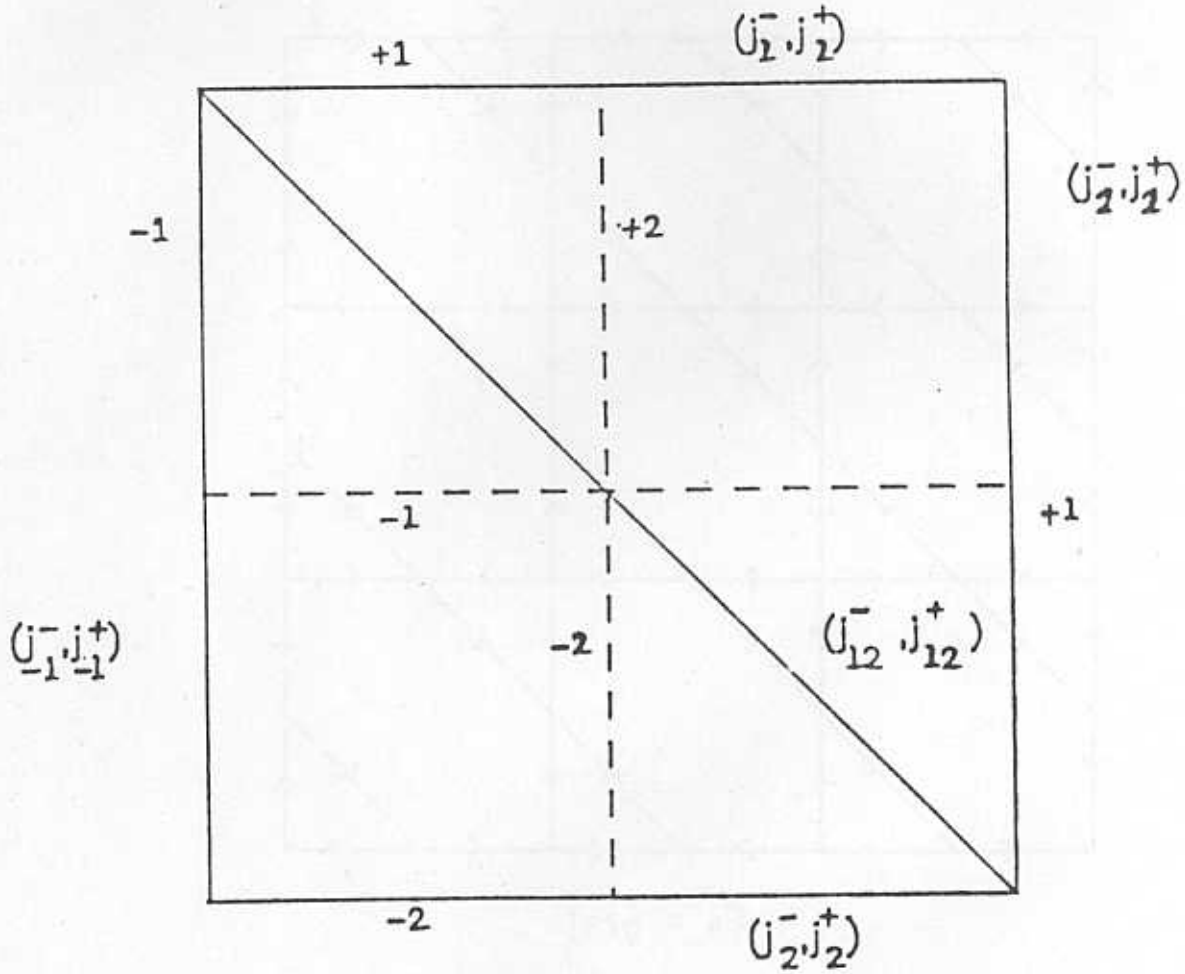


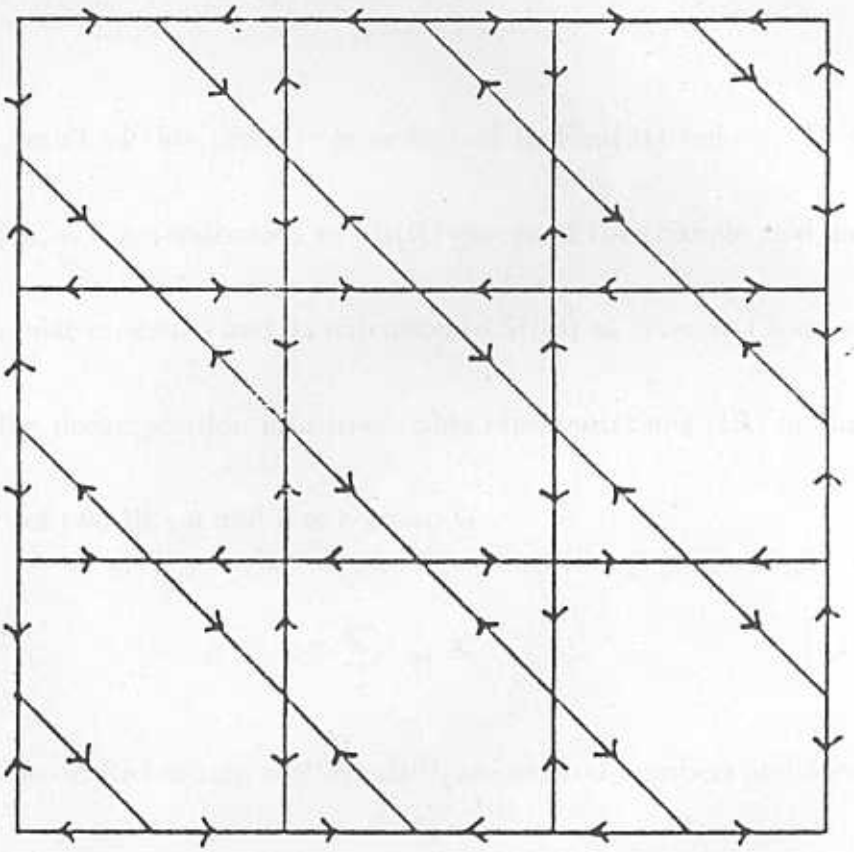
[Fig - 1]



[Fig - 2]



[Fig - 3]



[Fig - 4]

CHAPTER 3

The main result of this Chapter is embodied in Eqn(34) below. This can be looked at as a generalization to SU(N) groups of the triangle rule for addition of angular momenta and its extension to SU(3) as given in Chapter 2. Consider the decomposition into irreducible representations (IR) of the direct product of two IR's $\underline{\mu}$ and $\underline{\nu}$ of a group G.

$$\underline{\mu} \otimes \underline{\nu} = \sum_{\lambda} C^{\lambda}_{\mu\nu} \, \underline{\lambda}. \tag{1}$$

$C^{\lambda}_{\mu\nu}$, the Littlewood-Richardson coefficients[1], are natural numbers and convey which IR's $\underline{\lambda}$ appear in the decomposition and the number of times they appear. They are of relevance to other branches of mathematics[2] also.

There are different kinds of algorithms for $C^{\lambda}_{\mu\nu}$ (in case of compact Lie groups). Well known examples are,

(i) Steinberg's rule which uses weight lattice and Weyl group.

(ii) Littlewood-Richardson's (L-R) rule, codified into an operation

on the Young tableaux for the IR's.

As yet, in the general case, there is no explicit algebraic formula or a geometric rule for $C^{\lambda}_{\mu\nu}$. It is only in the case of SU(2) that the solution is well known and expressed as the triangle rule. There are modified algorithms[3],[4],[5] for SU(3), which can be formulated as an algebraic formula and interpreted as a variant of the triangle rule. One of these was used in Chapter 2.

Here we obtain an algebraic formula for $C^{\lambda}_{\mu\nu}$ for any SU(N). Our solution is, as in the case of N = 2 or 3, in terms of some random integers which are subject only to linear constraints by the given IR's, $\underline{\mu}$ and $\underline{\nu}$. Our techniques can be readily extended to other groups.

Different approaches to our problem are possible, as in the SU(3) case:

(i) Use of irreducible tensors[3].

(ii) L-R rule[4].

(iii) Method of invariants[5].

Of these, the L-R rule is already halfway to the solution and moreover reduces

the problem to jugglery with inequalities. Therefore we adopt this approach. The other approaches, though more difficult, would give a deeper explanation of our formulae.

The L-R rule is algorithmic. To carry out derivation of a generalized triangle rule we make an algebraic formulation of the L-R rule.

The L-R rule for SU(N) involves the following steps[1].

(1) The IR's are uniquely represented by Young tableaux having not more than (N-1) rows. The tableaux is left-justified and any row does not extend beyond the previous row. Let the i-th row extend beyond (i+1)-th row by j_i number of boxes. (j_{N-1} means the number of boxes in (N-1)-th row.) Thus every IR is uniquely labelled by an ordered set $(j_1, j_2, \cdots, j_{N-1})$ of independent natural numbers. The Casimirs for the IR can be computed using these j_i's.

(2) Lable all boxes of the first IR, $\underline{\mu}_{,,}$ by 'o'. We denote j_i for this IR alternately by n_{io}. For the second IR, $\underline{\nu}_{,,}$ label all boxes of the first row by

'1', second row by '2' and so on, for all the (N-1) rows.

(3) Choose a number n_{i1} of 1-boxes and append them to the right end of the i-th row, $i = 1, 2, \ldots (N-1)$ of the first tableau $\underline{\nu}$. Use the remaining boxes, n_{N1} in number, $(n_{N1} \geq 0)$, to start the N-th row. To this resulting tableau, repeat the process, appending n_{i2} number of 2-boxes to the i-th row and, again appending all remaining boxes to the N-th row. Repeat the process sequentially with 3- , 4- ,... (N-1) - boxes, until all boxes of the second IR are exhausted. Of such tableaux, only those that satisfy the following constraints are kept.

(4) The Right Rule: Count the boxes successively from the first row onwards, always from the right end to the left end in each row. At any stage of this process, the total number of (i+1) - boxes counted shouldnot exceed the total number of i-boxes. This means, in particular, that the first row can have only 1-boxes, the second row, only 1-and 2-boxes and so on. i-th row $(i = 1, 2, \ldots N-1)$, cannot have any (i+1) -, (i+2) -, ... (N-1) - boxes. Also

there are no N-boxes in the N-th row. Therefore,

$$n_{ij} = 0 \quad unless \quad i \geq j, \quad i = 1, 2, ..., N \qquad j = 1, 2, ..., (N-1) \tag{2}$$

This rule further gives a set of inequalities for these n_{ij}. Maximal constraints from this rule are obtained by requiring the number of j - boxes upto the i-th row be atleast equal to the number of (j+1) - boxes upto the $(i+1)^{th}$ row. (Here and henceforth, by the word 'upto' we mean 'upto and including'.) We write these constraints as a set of equalities by using slack variables β_{ij} which are non-negative integers. Each β_{ij} is equal to difference between the larger and the smaller side of an inequality. We label the equations so obtained as R_{ij}.

$$R_{ij} : \sum_{k=j}^{i} n_{kj} = \sum_{k=j+1}^{i+1} n_{k,j+1} + \beta_{ij} \tag{3}$$

with

$$\beta_{ij} \geq o \tag{4}$$

The last equation of this sequence requires that the number of (N-2) - boxes upto the (N-1)-th row is atleast equal to the number of (N-1)-boxes upto the

N-th row. Thus the range of the indices in these equations is,

$$\begin{aligned} i \geq j, \quad i = 1, 2, ..., (N-1), \\ j = 1, 2, ..., (N-2), \end{aligned} \tag{5}$$

(5) The Left Rule : The j-boxes in the (i+1)-th row should stop short of the first j-box in the previous, i.e. the i-th row. Again we write these constraints as equalities using slack variables α_{ij}, which are non-negative integers and label the equations as L_{ij}. Remembering that n_{io} corresponds to the extension of the i-th row beyond (i+1)-th row of the IR $\underline{\mu}$,

$$L_{ij} : \sum_{k=0}^{j-1} n_{ik} = \sum_{k=1}^{j} n_{i+1,k} + \alpha_{ij} \tag{6}$$

$$\alpha_{ij} \geq o \tag{7}$$

We have j-boxes in (i+1)-th row only if $j \leq i+1$, so that we have the above equations only in such cases. The last equation of the sequence (6) corresponds to the (N-1)-boxes in the N-th row falling short of the first (N-1)-box in (N-1)th row. Thus the range of indices are,

$$\begin{aligned} i+1 \geq j, \quad i = 1, 2, ..., (N-1) \\ j = 1, 2, ..., (N-1) \end{aligned} \tag{8}$$

Having made the formulation of the L-R rule algebraic we now analyse the equations (3) and (6) to clarify all the constraints.

The elements $\alpha_{i,i+1}$ above the main diagonal in the α_{ij} matrix are not independent objects. Infact from

$$\begin{aligned} L_{i,i+1} &: \textstyle\sum_{k=0}^{i} n_{ik} = \sum_{k=1}^{i+1} n_{i+1,k} + \alpha_{i,i+1} \\ L_{ii} &: \textstyle\sum_{k=0}^{i-1} n_{ik} = \sum_{k=1}^{i} n_{i+1,i} + \alpha_{ii} \\ R_{ii} &: n_{ii} = n_{i+1,i+1} + \beta_{ii} \end{aligned}$$

we see that

$$\alpha_{i,i+1} = \alpha_{ii} + \beta_{ii}, i = 1, 2, \ldots, (N-2). \tag{9}$$

which gives $\alpha_{i,i+1}$ as a sum of two other non-negative integers. This means the inequality corresponding to $L_{i,i+1}$ simply follows from L_{ii} and R_{ii} inequalities, and may be ignored.

Also the equations,

$$L_{i1} : n_{io} = n_{i+1,1} + \alpha_{i1}, i = 1, 2, ..., (N-1) \tag{10}$$

$$R_{ii} : n_{ii} = n_{i+1,i+1} + \beta_{ii}, i = 1, 2, ..., (N-2) \tag{11}$$

present n_{io} and n_{ii} as sums of other non-negative integers, and may be ignored.

Note that $n_{N-1,N-1}$ is still an independent variable.

We may use $L_{i+1,j}$ in $L_{i+1,j+1}$ to get

$$n_{i+1,j} - n_{i+2,j+1} = \alpha_{i+1,j+1} - \alpha_{i+1,j} \tag{12}$$

For each i = 1,2, ..., (N-2) the last equation in this chain corresponds to j = i (as we are not using $L_{i,i+1}$ equation). Therefore the range of indices are,

$$i \geq j, \quad i,j = 1,2,...,(N-2) \tag{13}$$

Similarly we may use R_{ij} in $R_{i+1,j}$ to get,

$$n_{i+1,j} - n_{i+2,j+1} = \beta_{i+1,j} - \beta_{ij} \tag{14}$$

Now for each i = 1,2, ..., (N-2), the last equation in this chain has j = i. Therefore the range of indices is again as in (13).

The chain of equations (12) and (14) may be compactly combined into,

$$\begin{aligned} l_{ij} &= n_{i+1,j} \quad -n_{i+2,j+1} \\ &= \alpha_{i+1,j+1} \quad -\alpha_{i+1,j} \\ &= \beta_{i+1,j} \quad -\beta_{ij} \end{aligned} \tag{15}$$

The range of indices is as in (13). Thus l_{ij} is a lower triangular (N-2) x (N-2) matrix.

Loss of information in the chain of substitutions (12) and (14) could only be in the very first in each chain i.e. L_{i1} or R_{jj}. But we have already decided that these are not independent equations. Therefore the set (15) contains all the constraints following from the L-R rule. This means that if we regard l_{ij} as independent random variables (taking both positive or negative integer values) and compute n_{ij}, α_{ij} and β_{ij} using equations (15), then the L-R rule is automatically satisfied - provided the values so computed are non-negative. In each chain of these difference equations, a boundary condition has to be specified for an unique solution. However, we have to choose boundary conditions very different from initial value problems. Otherwise we are not guaranteed non-negative solutions for n_{ij}, α_{ij} and β_{ij} because l_{ij}'s are not given to be non-negative. In the following section, we obtain a general solution for such a chain, parametrized by a random variable. Such

parameters together with the set l_{ij} are the random variables in terms of which the L-R rule is solved.

Consider the chain of equations,

$$n_{i+1} - n_i = l_i, i = 1, 2, ..., N \tag{16}$$

where the n_i's are required to be non-negative integers, but l_i's given are positive or negative integers. In order to write a general solution, valid for arbitrary l_i's, we proceed as follows.

Equations (16) may be rewritten as,

$$n_i - n_1 = \sum_{p=1}^{i-1} \ell_p, i = 2, 3, ..., (N+1). \tag{17}$$

Denote the least of the n_i's by n, a non-negative integer. We have,

$$n - n_1 = min_{q=1}^{N} \left(0, \sum_{p=1}^{q} \ell_p \right) \tag{18}$$

where the argument zero on the r.h.s. corresponds to the situation where n_1, is the least of the n_i's. From (17) and (18), we get,

$$n_i = n + \sum_{p=1}^{i-1} \ell_p - min_{q=1}^{N} \left(0, \sum_{p=1}^{q} \ell_p \right), i = 1, 2, ..., (N+1). \tag{19}$$

In this solution, the second term on the r.h.s. is taken to be zero for i = 1 case.

It turns out that the first (i.e. n_1) and the last (i.e. n_{N+1}) variables in the chain (15) are of special interest. We have,

$$n_1 = n - min_{q=1}^{N} \left(0, \sum_{p=1}^{q} \ell_p \right) \tag{20}$$

Also,

$$\begin{aligned} n_{N+1} &= n + \textstyle\sum_{p=1}^{N} \ell_p - min_{q=1}^{N} \left(0, \textstyle\sum_{p=1}^{q} \ell_p \right) \\ &= n + max_{q=1}^{N} \left(\textstyle\sum_{p=1}^{N} \ell_p, \textstyle\sum_{p=1}^{N} \ell_p - \textstyle\sum_{p=1}^{q} \ell_p \right) \end{aligned}$$

so that,

$$n_{N+1} = n + max_{q=1}^{N} \left(0, \sum_{p=q}^{N} \ell_p \right) \tag{21}$$

where the argument zero corresponds to the $q = N$ case.

We may now apply this technique to our chains of equations(15). The first chain in (15) connects n_{ij}'s along lines parallel to the diagonal. The second chain connects α_{ij}'s along a row and the third chain, β_{ij}'s along a

column. We get,

$$\begin{aligned} n_{i+1,j} &= P_{i+1-j} - \sum_{p=1}^{j-1} \ell_{i-j+p,p} + \max_{q=1}^{N-i-2+j} \left(0, \sum_{p=1}^{q} \ell_{i-j+p,p}\right) \\ \alpha_{ij} &= Q_i + \sum_{p=1}^{j-1} \ell_{i-1,p} - \min_{q=1}^{i-1} \left(0, \sum_{p=1}^{q} \ell_{i-1,p}\right) \\ \beta_{ij} &= R_j + \sum_{p=1}^{i-1} \ell_{pj} - \min_{q=j}^{N-2} \left(0, \sum_{p=j}^{q} \ell_{pj}\right) \end{aligned} \tag{22}$$

In these equations, it is to be understood that a $\sum$ or a min operation on the r.h.s. yields zero if it is over a forbidden range. This convention gives,

$$\begin{aligned} n_{N1} &= P_{N-1} \\ \alpha_{11} &= Q_1 \\ n_{N-1,N-1} &= \beta_{N-1,N-1} = R_{N-1}, \end{aligned} \tag{23}$$

(where $n_{N-1,N-1}$ has been relabelled as $\beta_{N-1,N-1}$). This is just a renaming of the variables on the l.h.s of the eqns. (23). Note that these variables are not connected by the chain of equations (15), but are nevertheless independent variables. Thus the range of indices in (22) may be taken to be,

$$i \geq j, \quad i,j = 1,2\ldots,(N-1) \tag{24}$$

Now the set (22) expresses all variables of the L-R rule in terms of certain random variables

$$\text{(i)} \quad P_i, Q_i, R_i, \quad i = 1,2,\ldots,(N-1) \tag{25}$$

which are non-negative integers and

$$\text{(ii)} \quad \ell_{ij}, i \geq j, i, j = 1, 2, \ldots, (N-2) \tag{26}$$

which can be positive or negative integers.

Our main objects of interest are the j_i's for the given IR's and for the IR's in the decomposition. Let j_i^a label the two initial IR's for a = 1,2 and an IR in the decomposition for a = 3. We may express them in terms of the variables n_{ij}, α_{ij} and β_{ij} and hence in terms of P_i, Q_i, R_i and l_{ij} variables. We have,

$$j_i^1 = n_{io}$$

and using (10),

$$j_i^1 = n_{i+1,1} + \alpha_{i1} \tag{27}$$

Also,

$$j_i^2 = \sum_{k=i}^{N} n_{ki} - \sum_{k=i+1}^{N} n_{k,i+1}$$

because all i-boxes of the second IR are distributed among the $i-, (i+$

1)−, ... $N-th$ rows of the resulting young tableau. Now using $R_{N-1,i}$ in (3), we get,

$$j_i^2 = n_{Ni} + \beta_{N-1,i} \tag{28}$$

This equation is valid as it is only for $i = 1, 2, \ldots, (N-2)$. For $i = (N-1)$,

$$j_{N-1}^2 = n_{N,N-1} + n_{N-1,N-1},$$

so that with our relabelling of $n_{N-1,N-1}$ by $\beta_{N-1,N-1}$ (see eqn.(23)), eqn(28) is valid for all $i = 1, 2, \ldots, (N-1)$.

Further,

$$\begin{aligned} j_i^3 &= \textstyle\sum_{k=0}^{i} n_{ik} - \sum_{k=1}^{i+1} n_{i+1,k} \\ &= \alpha_{i,i+1} \end{aligned}$$

using $L_{i,i+1}$ in (6). Further using (9),

$$j_i^3 = \alpha_{ii} + \beta_{ii} \tag{29}$$

Again, as it is, this is valid for $i = 1, 2, \ldots, (N-2)$. For $i = N-1$,

$$j_{N-1}^3 = \sum_{k=0}^{N-1} n_{N-1,k} - \sum_{k=1}^{N-1} n_{Nk}$$

because these are no N-boxes. Now we may use $L_{N-1,N-1}$ and again with $n_{N-1,N-1} \equiv \beta_{N-1,N-1}$ we see that eqn.(29) is valid for all $i = 1, 2, \ldots, (N-1)$.

We notice that the variables on the r.h.s. of eqns.(27),(28) and (29) are either the first or the last in the three chains of eqns.(15). Therefore we may apply the solutions (20) and (21). We get,

$$\begin{aligned} j_i^1 &= P_i + Q_i + \max_{q=1}^{N-i-1}\left(0, \textstyle\sum_{p=1}^{q} \ell_{i-1+p,p}\right) \\ &\quad - \min_{q=1}^{i-1}\left(0, \textstyle\sum_{p=1}^{q} \ell_{i-1,p}\right) \end{aligned} \tag{30a}$$

$$j_i^2 = P_{N-i} + R_i + \max_{q=i}^{N-2}\left(0, \sum_{p=q}^{N-2} \ell_{pi}\right) - \min_{q=1}^{i-1}\left(0, \sum_{p=q}^{i-1} \ell_{N-1-i+p,p}\right) \tag{30b}$$

$$j_i^3 = Q_i + R_i + \max_{q=1}^{i-1}\left(0, \sum_{p=q}^{i-1} \ell_{i-1,p}\right) - \min_{q=i}^{N-2}\left(0, \sum_{p=i}^{q} \ell_{pi}\right) \tag{30c}$$

These equations give the solution of the L-R rule in terms of the random variables given in (25) and (26). Given any two IR's, one has to obtain all values of these random variables which when used in eqns.(30 a) and (30 b) give the corresponding numbers $\{j_i^1\}$ and $\{j_i^2\}$. Then, for each set of values of these random variables, there is an IR $\{j_i^3\}$ in the CG series, given by the

set of eqns. (30 c). Thus our random variables not only give the allowed IR's of the decomposition as many times as its multiplicity, but also serve to label this multiplicity[6].

Equations (30) may be interpreted as an analogue of the triangle rule for the addition of angular momenta. For $N = 2$ and $N = 3$, these equations reduce to triangle rules considered in Chapter 1 and Chapter 2. Infact, the rule for a general N is similar to the $N = 3$ case. Notice that the set $\{j_i^1, j_i^2, j_i^3, j_{N-i}^1, j_{N-i}^2, j_{N-i}^3\}$ for a given i involve only a subset of the random variables, viz., $\{P_i, Q_i, R_i, P_{N-i}, Q_{N-i}, R_{N-i}\}$ and only the ℓ-type variables in the $(i-1)-$ and $(N-i-1)-$ rows, $i-$ and $(N-i)-$ columns and along the diagonals, $i-th$ and $(N-i)-th$ from the main diagonal. This suggests a relabelling of the variables, in close analogy with the $N = 3$ case, as follows:

(i) The solution is symmetric in the three IR's if the complex conjugate of the third IR is used. This is to be expected, because, we are symmetrically treating all the three IR's on an equal footing in building a singlet. Thus,

we use,

$$j_i^{3\star} = j_{N-i}^3, i = 1, 2, ..., (N-1) \tag{31}$$

in place of j_i^3 in our formulae.

(ii) Define

$$\begin{aligned} P_i &= n^{12}(i), \\ Q_i &= n^{31}(N-i), \\ R_i &= n^{23}(i) \end{aligned} \tag{32}$$

in analogy with the SU(3) case.

(iii) Relabel the variables ℓ_{ij} (which were displayed as a lower triangular matrix) using three coordinates,

$$\ell(I, J, K), \quad I + J + K = (N-2) \tag{33}$$

Here the position of the variable in the triangle is characterized in a symmetric way by specifying its distance as counted from the sides 1, 2 and 3 (Figs.). Distance I from side 1 is just the column index j, distance J from side 2 is the row index i. Distance K from side 3 is as measured along a row or a column and is therefore $K = N - 2 - I - J$. Ofcourse, it is sufficient to

specify just two of the coordinates, as the third can be obtained from (33).

Using the new labels, the solution of the L-R rule is:

$$j_i^1 = n^{12}(i) + n^{31}(N-i) + \max_{q=1}^{N-i-1}(0, \sum_{p=1}^{q} \ell(p,-,i))$$

$$-\min_{q=1}^{i-1}(0, \sum_{p=1}^{q} \ell(p, N-i, -))$$

$$j_i^2 = n^{23}(i) + n^{12}(N-i) + \max_{p=q}^{N-2}(0, \sum_{p=q}^{N-2} \ell(i,p,-))$$

$$-\min_{q=1}^{i-1}(0, \sum_{p=q}^{i-1} \ell(-,p,N-i))$$

$$j_i^{3\star} = n^{31}(i) + n^{23}(N-i) + \max_{q=1}^{N-i-1}(0, \sum_{p=q}^{N-i-1} \ell(-,i,p))$$

$$-\min_{q=N-i}^{N-2}(0, \sum_{p=N-i}^{q} \ell(N-i,-,p)) \qquad (34)$$

These formulae can be represented diagrammatically as in Fig.1 (The Figure may be deformed into an equilateral triangle for displaying manifest symmetry in 1,2 and 3 lables).

It can be noticed that the SU(N) case has striking similarity with the SU(3) case. To apply the result to lattice gauge theory one follows the

arguments as given in Chapter 2 for the SU(3) case. The $N^{\pm}$ and L variables in the SU(3) case now go over to the n(i) and ℓ variables respectively.

However there is a major obstacle to the program. To calculate the Hamiltonian in the new basis one needs more understanding of the analogues of 3-j, 6-j, 9-j symbols in the general SU(N) case. The color invariant fluxes however can be easily constructed. One can also see that the (N-1) variables n(i) give a $U_{(1)}^{N-1}$ theory and the ℓ variables lead to a interaction of the same sort as in the SU(3) case. This is in accordance with t'Hooft's conjecture.

It is extremely important to complete the program initiated in this thesis as one hopes that the $N \to \infty$ limit of SU(N) gauge theories can be understood. This is being carried out and will be published in the near future.

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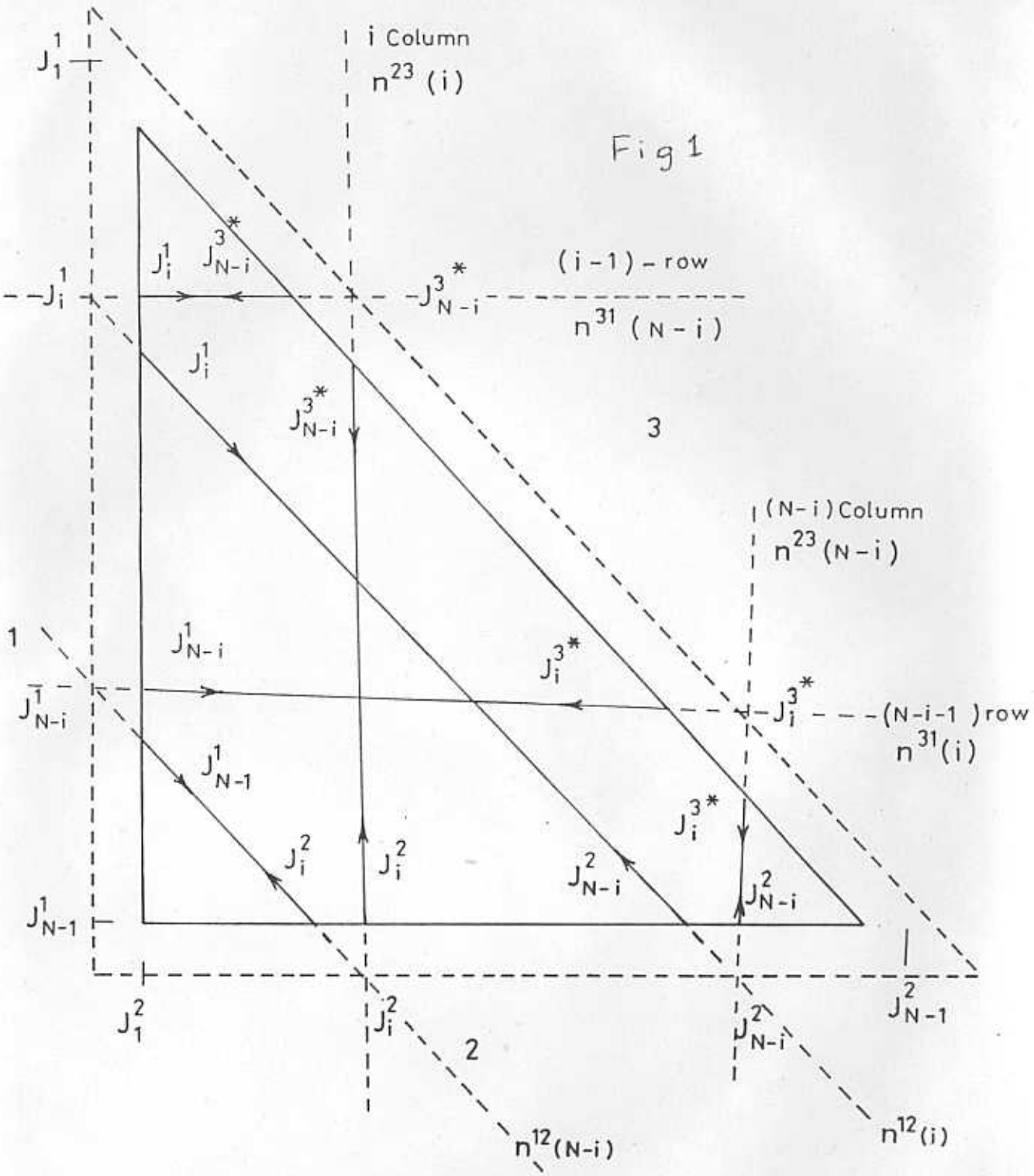
[6] This Chapter is based on *Algebraic Solution of the Littlewood - Richardson Rule: Multiplicity in SU(N) Clebsch-Gordan Series* by G.H.Gadiyar and H.S.Sharatchandra. I.M.Sc. preprint.


i Column
$n^{23}(i)$
Fig 1
J_1^1
J_i^1
J_{N-i}^{3*}
J_i^1
J_{N-i}^{3*}
(i-1) - row
J_{N-i}^{3*}
$n^{31}(N-i)$
3
(N-i) Column
$n^{23}(N-i)$
1
J_{N-i}^1
J_{N-i}^1
J_i^{3*}
J_i^{3*}
(N-i-1) row
$n^{31}(i)$
J_{N-1}^1
J_i^{3*}
J_i^2
J_i^2
J_{N-i}^2
J_{N-i}^2
J_{N-1}^1
J_1^2
J_i^2
J_{N-i}^2
J_{N-1}^2
2
$n^{12}(N-i)$
$n^{12}(i)$