A
MANIFESTLY LORENTZ COVARIANT LOCAL QUANTUM FIELD THEORY OF DYONS

A THESIS
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by

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CERTIFICATE

This is to certify that the Ph.D thesis submitted by Manu Mathur to the University of Madras entitled :

" A Manifestly Lorentz Covariant , Local Quantum Field Theory Of Dyons "

is a record of bonafide research work done by him under my supervision during 1987 - 1991. The research work presented in this thesis has not formed the basis for the award of any degree, diploma, associateship, fellowship or other similar title. It is further certified that the thesis represents independent work on the part of the candidate.

H.S. SHARATCHANDRA

(Supervisor)

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I am grateful to my supervisor Dr. H.S.Sharatchandra under whose guidance this work was done.

I also feel extremely happy in acknowledging my friends & senior colleagues who have helped me in different ways throughout my career.

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ORGANIZATION OF THE THESIS

The thesis is organized as follows:

In the first chapter we give a brief summary of the earlier works on Monopoles and Dyons. The purpose and motivation for this chapter are very limited.*It is not to present a detailed review on the subject but to highlight and analyze the origin of the numerous problems associated with the earlier attempts to construct a Quantum Field Theory of Dyons.*The most serious problems with these theories were:

The absence of Manifest Lorentz Covariance and / or Manifest Locality.

The origin for this and the various other problems (to be discussed later in the thesis) is the presence of *Dirac String* at the level of dynamics.This aspect can be appreciated even at the level of Classical Mechanics / Quantum Mechanics . Hence in the first chapter we analyze the above problems starting from Classical and Quantum Mechanics of Monopoles / Dyons before going to the Field Theory of these particles ■

In the second chapter we give the path integral formulation of our theory of Dyons. We show that it is possible to construct a

Quantum Field Theory of these particles evading the need of Dirac string completely at the level of partition function. Hence all the problems mentioned above are also evaded. We also show that the dynamics in this theory is governed by certain " gauge symmetry " ■

In the third chapter we discuss the Hamiltonian formulation of our theory. Some of the aspects, like the role of the Dirac strings in our formulation, become more transparent in this language ■

In the last chapter, we start with a careful analysis of the various subtle aspects of the theory. We also discuss a couple of related problems which are yet to be addressed.

Finally we conclude the thesis with a simple analogy and a flow chart illustrating the basic idea involved in our formulation ■

The various ideas and techniques which are not directly essential for the continuity of our discussion but otherwise important for getting a global perspective of our theory are given in the appendices.

CHAPTER I

" MONOPOLES, DIRAC STRINGS & ALL THAT ......"

In 1931 Dirac[1-a] showed that magnetic monopoles fit naturally into the framework of quantum mechanics and lead to a connection between electric charges (e) and magnetic charges (g),

$$eg = 2\pi \times \text{Integer}.$$

Therefore the existence of even a single monopole of strength g_o would imply quantization of all electric charges in the nature, in units of $(2\pi/g_o)$. This in a sense is a purely quantum mechanical phenomenon since classically Maxwell's equations and Lorentz force equations are consistent with all known physical principles for any values of electric and magnetic charges. After Dirac's work there were "successful" attempts to write down non-relativistic quantum mechanics[1] and non-quantized relativistic particle theory[2] with (non-local terms) of these particles. But a consistent relativistic quantum field theory of these particles appeared to be difficult to the extent that some people thought that the concept of a monopole was not compatible with the ideas of quantum mechanics and relativity together.[3-g]

Meanwhile there were a number of attempts to construct a quantum field theory of monopoles[3]. In 1966 Schwinger[3-b] constructed a quantum field theory of these particles. This formulation was not manifestly local and manifestly Lorentz covariant. He explicitly verified that the theory was Lorentz invariant provided

$$eg = 4\pi \times \text{Integer}$$

This condition is more restrictive than the one proposed by Dirac.

It was not until 1971, 40 years after Dirac's first paper on monopoles appeared that Zwanziger[3-d,e] was able to construct a local action formulation of these particles. This theory could be extended to quantum field theory. He showed that the Dirac quantization condition is necessary even for the consistency of quantum field theory. Although this formulation avoided the problem of non-locality present in the previous theories by introducing two vector potentials, it was still not manifestly Lorentz covariant. The theory explicitly involved an arbitrary unit vector associated with the Dirac string.

In this thesis we construct a manifestly Lorentz covariant, local quantum field theory of Dyons using only one vector potential[4]. We show that the principle of minimal coupling along with the dual symmetry of Maxwell's equations uniquely lead to this theory and this turns out to be a gauge theory. This is in contrast with the earlier theories where by inverting one of the Maxwell's equations (in terms of a vector potential), attempts were made to formulate an action so as to reproduce the other Maxwell's equations by a variational principle. In the presence of both electric and magnetic sources , this procedure involves integration of one of these along a line (the Dirac string).This led to the problems of Lorentz covariance , non-locality ,singular gauge transformations etc.

1.1] *CLASSICAL MECHANICS*

The classical mechanics of a system of magnetic monopoles and electric charges particles is described by the generalized Maxwell's equations,

$$\partial_\mu F_{\mu\nu} = J_\nu \tag{1.1-a}$$

$$\partial_\mu \tilde{F}_{\mu\nu} = K_\nu \tag{1.1-b}$$

and the Lorentz force equation,

$$m \frac{d^2 x^i_\mu}{d\tau^2} = \left[e^i F_{\mu\nu}(x^i_\mu) + g^i \tilde{F}_{\mu\nu}(x^i_\mu) \right] \frac{dx^i_\nu}{d\tau} \tag{1.1.-c}$$

for the i^{th} particle which carries an electric charge e_i and a magnetic charge g_i. A particle which has both electric and magnetic charge is called a 'dyon'. In the above equations $F_{\mu\nu}$ is the electromagnetic field strength ($F_{oi} = E_i$ is the electric field and $F_{ij} = \varepsilon_{ijk} B_k$ is the dual of the magnetic field),(τ)

is the proper time and $X^{i}_{\mu}(\tau)$ (i= 1,2,--N) is the world line of the (i^{th}) particle carrying the electric and magnetic charges e_i, g_i respectively. J_μ & K_μ are the electric and magnetic currents,

$$J_\mu(x) = \sum_{i=1}^{N} e_i \int dX^i_\mu \, \delta^4(x-X^i_\mu(\tau)) \qquad (1.1.-d)$$

$$K_\mu(x) = \sum_{i=1}^{N} g_i \int dX^i_\mu \, \delta^4(x-X^i_\mu(\tau)) \qquad (1.1.-e)$$

These equations are invariant under the 'duality transformations',

$$F_{\mu\nu} \longrightarrow \cos\theta \, F_{\mu\nu} + \sin\theta \, \tilde{F}_{\mu\nu} \qquad (1.2\text{-a})$$

$$\tilde{F}_{\mu\nu} \longrightarrow -\sin\theta \, F_{\mu\nu} + \cos\theta \, \tilde{F}_{\mu\nu} \qquad (1.2\text{-b})$$

$$J_\mu \longrightarrow \cos\theta \, J_\mu + \sin\theta \, K_\mu \qquad (1.2\text{-c})$$

$$K_\mu \longrightarrow -\sin\theta \, J_\mu + \cos\theta \, K_\mu \qquad (1.2\text{-d})$$

These symmetry transformations at ($\theta = \pi/2$) will be crucial in our formulation.

These equations ((1.1-a)-(1.1-e)) are manifestly Lorentz covariant, local and internally consistent. So at the level of classical equations of motion, electric charges and magnetic charges can co-exist for any values of (e) and (g).

One can immediately see that the quantization of this system will not be straightforward[1]. The reason is that in quantum mechanics, the electron sees the vector potential A_μ, rather than the electric or magnetic fields, where A_μ is defined by :

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.3}$$

In the presence of magnetic charges, ($\partial_\mu \tilde{F}_{\mu\nu} \neq 0$), it is not possible to define the vector potential A_μ as in (1.3). Dirac's prescription for defining a quantum theory in the presence of magnetic charges is to use an A_μ which is singular along a line in $R^{(3)}$, emanating from the monopole and going to infinity along an arbitrary line ($\mathfrak{L}$). In fact the magnetic field of a monopole of strength g can be written as [Fig.-1],

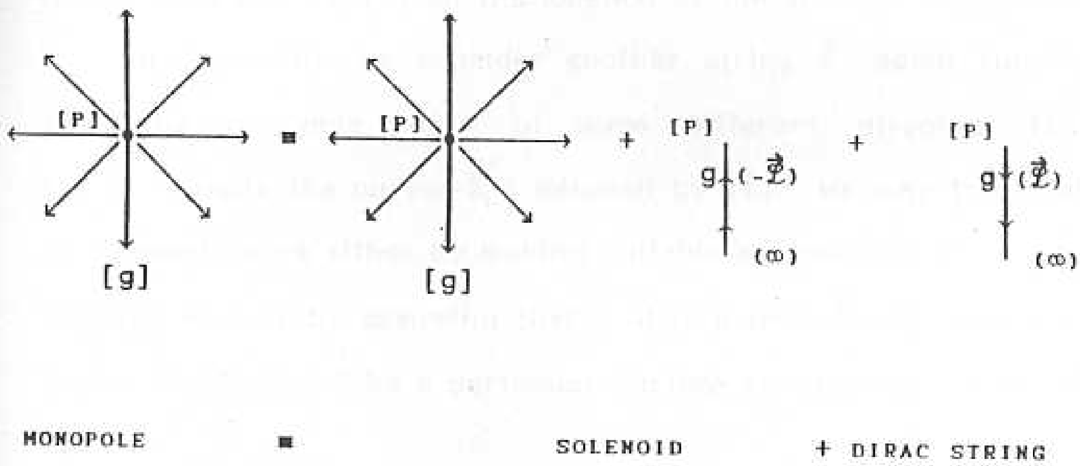
$$\vec{B}_{\mathrm{Monopole}}(\vec{r}) = \vec{\nabla} \times \vec{A}_{\mathrm{SOLENOID}}(\vec{\mathfrak{L}},r) + \vec{h}(\vec{\mathfrak{L}},\vec{r}) \tag{1.4}$$

Here $\vec{\mathfrak{L}}$ is a string directed away from the monopole and going to ∞ in any arbitrary chosen direction. $\vec{A}_{\mathrm{Solenoid}}$ is the vector potential for an infinitely thin solenoid (carrying g unit of flux towards the monopole) lying along $\mathfrak{L}$. $\vec{h}(\vec{\mathfrak{L}},\vec{r})$ is the Dirac string along $\vec{\mathfrak{L}}$ and carries g unit of flux from monopole to ∞ in the direction of $\vec{\mathfrak{L}}$ (Fig-1-a).

Explicitly,

$$\vec{h}(\vec{\mathfrak{L}},r) = g \int_{\mathfrak{L}} \overrightarrow{dx}\ \delta^3(\vec{r}-\vec{x}) \ . \tag{1.5}$$

Where the integration is done along $\vec{\mathfrak{L}}$.



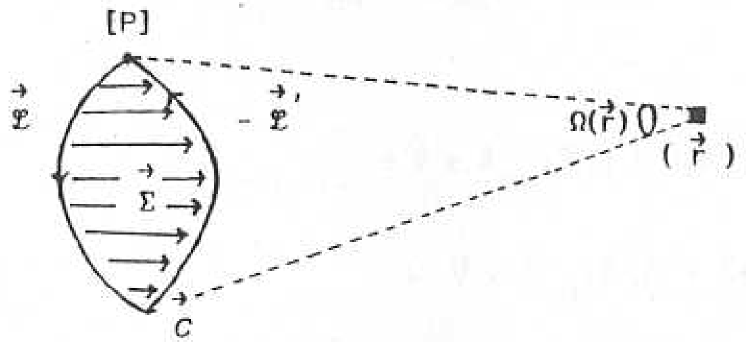
[Fig.1-a]: Defining the vector potential for the monopole. The solenoid and the Dirac string carry g units of flux along $(-\vec{\mathfrak{L}})$ and $(\vec{\mathfrak{L}})$ respectively.

To describe the motion of an electrically charged particle in the field of a monopole, this vector potential of the solenoid (1.4) is used for the minimal coupling. The singular magnetic field associated with this vector potential along the string $\mathfrak{L}$ is made unobservable to the electron by demanding that the magnetic flux (g) along this string should not give rise to Aharanov-Bohm effect. This leads to Dirac quantization condition:

$$eg = 2\pi(\text{Integer}) \tag{1.6}$$

The magnetic field of the monopole then differs from that of the infinitely thin solenoid used to define the vector potential by an unobservable Dirac string. Still one has to show that the theory does not depend on the location of the solenoid along $(-\vec{\mathcal{L}})$.

To show this, we consider another string $\vec{\mathcal{L}}'$ again running from the monopole to ∞ in some different direction $(\vec{\mathcal{L}}')$. Let $\vec{C}$ denote the curve$(-\vec{\mathcal{L}}')$ followed by $(\vec{\mathcal{L}})$. We may treat this as a closed curve either by making suitable assumptions about what happens at ∞ or by assuming that $\vec{\mathcal{L}}$ differs from $\vec{\mathcal{L}}'$ only in a finite region Fig(2). Let $\vec{\Sigma}$ be a particular surface spanned by $\vec{C}$ and $\Omega(\vec{r})$



[Fig. 2]: The ill defined gauge transformation.
($\Lambda = g/4\pi\ \Omega$).

be the solid angle subtended at point $\vec{r}$ by this particular surface. Various choices of spanning $\vec{\Sigma}$ will lead to values of Ω differing by 4π but will yield the same value of $\vec{\nabla}\,\Omega$, except on $\vec{C}$ itself where Ω & $\vec{\nabla}\Omega$ are ill defined. We consider a gauge transformation

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} - g/4\pi\, \vec{\nabla}\Omega \tag{1.7}$$

This transformation is everywhere smooth except on the surface $\vec{\Sigma}$ where it is singular. To see the singularity structure on $\vec{\Sigma}$, we apply Stoke's theorem to a small loop enclosing $\vec{C}$. Using the fact that the solid angle $\Omega(\vec{r})$ has discontinuity of 4π across the surface $\vec{\Sigma}$, we get,

$$\vec{\nabla} \times \left\{ \vec{A}'\ (\mathcal{L}')_{SOL} - \vec{A}(\mathcal{L})_{SOL} \right\} = \vec{h}(\vec{\mathcal{L}},r) - \vec{h}(\vec{\mathcal{L}}',r).$$

or

$$\vec{B}_{Monopole} = \vec{\nabla} \times \vec{A}_{SOL}(\vec{\mathcal{L}},\vec{r}) + \vec{h}(\vec{\mathcal{L}},\vec{r})$$

$$= \vec{\nabla} \times \vec{A}_{SOL}(\vec{\mathcal{L}}',\vec{r}) + \vec{h}(\vec{\mathcal{L}}',\vec{r}) \tag{1.8}$$

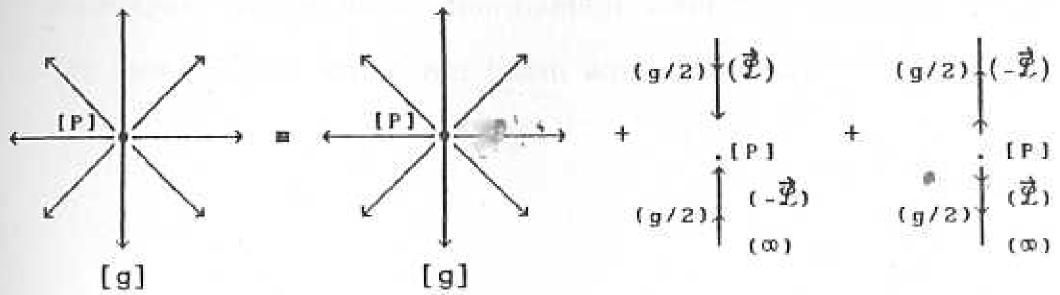
Hence the two strings $\vec{\mathcal{L}}$ and $\vec{\mathcal{L}}'$ are connected by the gauge transformation $\Lambda(r)$, which is singular and multivalued. Since it is defined modulo g, the single valuedness of the wave function again implies the Dirac quantization condition. Thus to incorporate strings in this formulation one has to deal with singular and multivalued gauge transformations

The formulation of Wu & Yang[1-c] copletely avoids the problem of singularities associated with Dirac's prescription by using different vector potentials in different regions of space. In the common region these vector potentials are connected by a gauge transformations $\Lambda(r)$ which are again multivalued as $\Lambda(r)$ is identified with $[\Lambda(r) + (\text{Integer})\, g]$. Requiring the wave function to be single valued again leads to Dirac quantization condition.

Moreover the prescriptions described above do not fix the Dirac quantization condition uniquely. To appreciate this, we consider, e.g., a symmetrical solenoid carrying $(^{g}/_{2})$ unit of magnetic flux towards the monopole from two opposite directions $(\vec{\mathfrak{L}})$ and $(-\vec{\mathfrak{L}})$ [Fig. 1-b].

Now demanding the absence of Aharanov-Bohm effect due to the infinite magnetic field along the symmetrical solenoid (carrying $(^{g}/_{2})$ unit of flux) requires

$$\frac{e.g}{2} = 2\,\pi\ (\text{Integer})$$



MONOPOLE = A SYMMETRICAL SOLENOID + DIRAC STRING

[Fig. 1-b] : Non-uniqueness of Dirac Quantization Condition.

This was essentially the reason for the extra factor of two appearing in the Schwinger's field theory formulation of the monopoles.

So we find that even in the context of quantum mechanics, due to the Dirac string, there are problems of manifest rotational covariance , non-uniqueness of the Dirac quantization condition,singular gauge transformations etc.. In the Wu & Yang formulation though the latter is avoided, the other problems still persist. It is exactly the relativistic version of these problems which showed up in the previous attempts of constructing field theory of monopoles.

Before going to the field theory, we elaborate on these problems in the domain of classical relativistic particle mechanics, where Dirac quantization condition manifests itself in the form of the action not being single valued.[3-9]

1.3] *ACTION FORMULATION AND NONLOCALITY :*

The classical action formulation of monopoles in the presence of electrically charged particles follows from a number of different looking but actually equivalent action principles[2-a, 3-d, 3-h]. For example the action given by Schwinger[3-b] is essentially

$$I = -\sum_i m_i \int_{P_i} dx + \int d^4x \left[1/4\ F^2_{\mu\nu} - 1/2\ F_{\mu\nu}(\partial \wedge A)_{\mu\nu} \right.$$

$$\left. - J_\mu A_\mu - K_\mu B_\mu(\vec{n}) \right] \qquad (1.9\text{-a})$$

Here,

$$B_\mu(\vec{n}) = (n.\partial)^{-1}\ n.F^d_{\mu\nu}(x) \qquad (1.9\text{-b})$$

and n_μ is some arbitrary unit vector. $A \wedge B \equiv A_\mu B_\nu - A_\nu B_\mu$ for any four vectors A_μ and B_μ and $F_{\mu\nu} \equiv 1/2\ \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$.

$(n.\partial)^{-1}$ is the integral operator with the kernel,

$$(n.\partial)^{-1}_{xy} = \int_0^\infty ds \left[\delta^4(x-y-ns) \right] \qquad (1.9\text{-c})$$

and $B_\mu(n)$ can be thought of as the magnetic vector potential.

The equation of motion for F gives the correct field strength tensor :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - G^d_{\mu\nu} \qquad (1.10\text{-a})$$

with

$$G_{\mu\nu} = (n.\partial)^{-1} (n_\mu K_\nu - n_\nu K_\mu) \qquad (1.10\text{-b})$$

$G_{\mu\nu}(x)$ describes the electric and magnetic fields of a string lying along the curve $W^\mu(s) = n^\mu s$ [s:o to ∞] carrying g units of flux along it. Varying the trajectories of the particles, one gets the Lorentz force equation

$$m \frac{d^2 x^i_\mu}{d\tau^2} = \left[e_i(\partial_\mu A_\nu - \partial_\nu A_\mu) + g_i(\partial_\mu B_\nu - \partial_\nu B_\mu) \right] \frac{dx^i_\nu}{d\tau} \qquad (1.11)$$

So this formalism leads to the correct equations of motion only if the trajectories of the charges never intersect the Dirac string of any other particles i.e the "the Dirac Veto". This in some sense is the classical analogue of demanding the absence of

Aharanov Bohm effect in quantum mechanics. Brandt and Primack[2-a] showed that it is possible to avoid the problem of the Dirac Veto completely by modifying the minimal coupling term

$$\int_{\Gamma_e} A_\mu dx_\mu$$

whenever the trajectory (Γ_e) of the electric charge hits the Dirac string of a monopole.

To show the string (n_μ) independence of the theory, consider $n_\mu \longrightarrow n'_\mu$ followed by a gauge transformation $A_\mu \longrightarrow A_\mu + \partial_\mu \lambda$. The invariance of the action under the combined transformation implies

$$\partial \wedge \partial \lambda \;=\; \left\{ \left[(n'.\partial)^{-1} n' - (n.\partial)^{-1} n \right] \wedge K \right\}^d \qquad (1.12)$$

Using Stoke's theorem, we find the equation is satisfied provided λ has a discontinuity of $\pm\, g_j$ across the surface,

$$\Sigma_j = X_j(\tau) \otimes \left\{ s'n' \otimes s\, n : 0 \le s, s' \le \infty \right\}$$

for each j. Here $X_j(\tau)$ is the trajectory of the j^{th} monopole with τ as it's proper time [Fig.3].

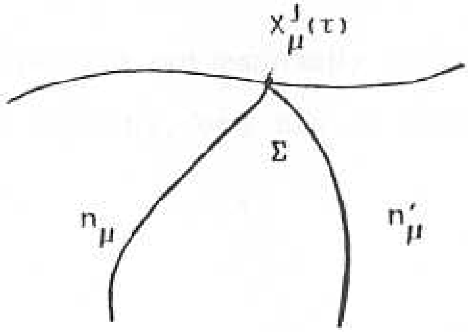


Fig.3] : The gauge transformations changing n to n' are discontinuous on across the surface Σ.

Now the contribution $\int J_\mu A_\mu d^4x$ to the action changes by $\sum_{ij} N_{ij}(e_i g_j - e_j g_i)$, where N_{ij} is the number of times (+ve, -ve or zero) that X_i intersects Σ_j (or equivalently X_j intersects Σ_i). Therefore, the action is invariant modulo $(e_i g_i - e_j g_i)$ under the change of the strings.

The following remarks regarding these and the other equivalent formulations given by others (including Dirac's) are in order :

1) This action can be trivially generalized to field theory of monopoles/dyons if we replace J_μ & K_μ by the corresponding field currents and modify the action accordingly.

2) The theory is not manifestly Lorentz covariant. To show this invariance explicitly, one has to deal with singular gauge transformations.

As mentioned above the action (1.9) can be generalized to field theory by replacing J_μ and K_μ by the corresponding field currents and modifying the kinetic energy term accordingly. This apparently non-local formulation proposed by Schwinger[3-b] involves only one vector potential and is not manifestly Lorentz covariant. The problem of non-locality was avoided by Zwanziger[3-d] by introducing an extra (magnetic) vector potential along with the electric vector potential as follows.

The equations (1.1.-a,b) imply

$$F = -(\partial \wedge B)^d + (n.\partial)^{-1} (n \wedge J). \tag{1.13-a}$$

$$F^d = -(\partial \wedge A)^d + (n.\partial)^{-1} (n \wedge K). \tag{1.13-b}$$

Here n_μ is an arbitrary unit vector associated with the irection of the Dirac string. Both these expressions for F are on-local. To make them local we use the identity

$$F = \left\{ \left[n \wedge (n.F) \right] - \left[n \wedge (n.F^{d}) \right]^{d} \right\} \tag{1.14-a}$$

From (1.13-a,b)

$$n.F = n.(\partial \wedge A) \tag{1.14-b}$$

$$n.F^{d} = n.(\partial \wedge B) \tag{1.14-c}$$

Using (1.14-a,b,c) we get a local expression for F:

$$F = \left\{ n \wedge [n.(\partial \wedge A)] \right\} - \left\{ n \wedge [n.(\partial \wedge B)]^{d} \right\} \tag{1.15}$$

Substituting this back in the Maxwell's equations we get local field equations for electric and magnetic vector potentials A & B :

$$(n.\partial)(n.\partial)\, A_{\mu} - (n.\partial)\, \partial_{\mu}(n.A) - n_{\mu}(\, n.\partial)(\, \partial.A) + n_{\mu}\partial^{2}(n.A)$$

$$- (\, n.\partial)\, \epsilon_{\mu\nu\rho\lambda}\, n^{\nu}\, \partial^{\rho} B^{\lambda} = J_{\mu} \tag{1.16-a}$$

and

$$(n.\partial)(n.\partial)\, B_{\mu} - (n.\partial)\partial_{\mu}(n.B) - n_{\mu}(\, n.\partial)(\partial.B) + n_{\mu}\partial^{2}(n.B)$$

$$- (n.\partial)\, \varepsilon_{\mu\nu k\lambda}\, n^{\nu}\, \partial^{k} A^{\lambda} = K_{\mu} \qquad (1.16\text{-b})$$

Obviously these local equations follow from a local action[3-d], which is given by,

$$I(n) = -1/2 \int d^4x \left\{ [n.(\partial \wedge A)].[n.(\partial \wedge B)] + [n.(\partial \wedge A)]^2 + (A \to B,\ B \to -A) \right\} \qquad (1.17)$$

Starting from this action in the functional integral formulation, Zwanziger was able to demonstrate that all gauge invariant Green functions are independent of the Dirac string provided the generalized Dirac-Schwinger-Zwanziger condition

$$e_i g_j - e_j g_i = 4\pi \times (\text{integer}) \qquad (1.18\text{-a})$$

holds. This is the quantization condition for dyons that carry both electric and magnetic charges.

This theory suffers from the following drawbacks :

1] The theory does not have manifest Lorentz covariance.

2] The quantization condition (1.18-a) follows from the use of the symmetric kernel for the operator $(n.\partial)^{-1}$,

$$(n.\partial)^{-1}_{xy} = \frac{1}{2}\int ds \left[\delta^4(x-y-n\ s) - \delta^4(x-y+n\ s)\right]$$

but we also have the choice of choosing an asymmetric Dirac string corresponding to

$$(n.\partial)^{-1}_{xy} = \int \delta^4(x-y-ns)\ ds.$$

This leads to the weaker quantization condition

$$(e_i g_j - e_j g_i) = 2\pi \times (\text{Integer}) \qquad (1.18\text{-b})$$

Hence the fictitious object, the Dirac string, affects the final physical consequences of the theory. This arbitrariness in the quantization condition is the relativistic version of the arbitrariness discussed in sec.(1.2) in the context of quantum mechanics (Fig. 1-a, 1-b).

So the basic reason for the various problems associated with the previous formulations was essentially the presence of an unphysical object, the Dirac string. In other words the dynamics was described by choosing the 'Wrong Dynamical Variables' leading to various artificial problems. Now we show that just by using the dual symmetry (1.2) of Maxwell's equations along with the principle of minimal action one is led to the choice of appropriate variables which evade the need of the Dirac string completely. Hence the problems of non-locality , Lorentz covariance, etc, automatically get resolved. Moreover, the resulting theory is a gauge theory of monopoles/dyons and the Dirac quantization condition turns out to be a consequence of this gauge invariance ■

CHAPTER II

" A THEORY WITHOUT STRINGS ATTACHED "

In this and the next chapter we describe our formulation of a theory of dyons[4,5]. As mentioned earlier we will be essentially dealing with the choice of correct dynamical variables to construct a theory of monopoles (dyons). We will show that the principle of minimal coupling along with the dual symmetry of Maxwell's equations in the presence of magnetic charges , the two well known principles of physics, uniquely lead to this choice. Having found them we will describe their dynamical and kinematical aspects in detail. We find that their dynamics is governed by a certain "gauge symmetry". This gauge symmetry is the origin of the Dirac quantization condition. This should be contrasted with the earlier approaches[3] where the physical Dirac quantization condition was a consequence of a fictitious object - the Dirac string. Introduction of this unphysical string in all the earlier theories further led to a series of serious but artificial problems like non-locality, loss of manifest Lorentz covariance etc. Our formulation in terms of "dual dynamical variables" completely evades the very need of the Dirac string at the level of dynamics. Hence all the problems associated with the earlier attempts automatically get resolved. We demonstrate our ideas using spin zero fields.

Throughout this chapter we use the functional integral formulation. The Hamiltonian description will be given in the next chapter. We will often appeal to the U(1) lattice gauge theory frame work, which not only provides a natural ultraviolet cut off for our theory and puts our "dual variables" on a much firmer footing, but also provides valuable insights regarding the non-perturbative phase structure of the theory [Appendix.-E]. In fact, we have been motivated by extensive hints contained in the literature on compact U(1) lattice gauge theories, where monopoles arise due to compactness of the dynamical variables [Appendix C].

In formulating the theory, we exploit certain "dual transformations"[9] on the scalar and vector fields along with the principle of minimal coupling. Now we describe each of these ideas separately before putting them together to develop a local manifestly Lorentz co-variant theory of dyons.

DUAL SYMMETRY OF MAXWELL'S EQUATIONS

The Maxwell's equations (1.1) in the presence of electric and magnetic charges,

$$\partial_\mu F_{\mu\nu} = J_\nu$$

and

$$\partial_\mu \tilde{F}_{\mu\nu} = K_\nu \qquad (1.1)$$

have the dual invariance,

$$F_{\mu\nu} \longleftrightarrow \tilde{F}_{\mu\nu}, \quad J_\mu \longleftrightarrow K_\mu \qquad (1.2)$$

This invariance essentially implies that a system containing only electric charges $\left(\partial_\mu F_{\mu\nu} = J_\nu , \partial_\mu \tilde{F}_{\mu\nu} = 0\right)$ can not be distinguished from the one which contains only magnetic charges. $\left(\partial_\mu \tilde{F}_{\mu\nu} = K_\nu , \partial_\mu F_{\mu\nu} = 0\right)$. Defining the electric and magnetic vector potentials by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\tilde{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ in the above two cases respectively , it is clear that the dual invariance of Maxwell's equations corresponds to

$$A_\mu \leftrightarrow B_\mu \text{ and } J_\mu \leftrightarrow K_\mu$$

Hence, if electric charges couple to electric vector potential A_μ by minimal coupling, the magnetic charges will couple to B_μ, the magnetic vector potential, by minimal coupling. This will be the starting point of our theory (equation 2.13).

IIA-2] DUAL TRANSFORMATION ON SCALAR FIELD :

For simplicity we start with a real free massless scalar field $\theta(x)$. The Euclidean functional integral is

$$Z = \int d\theta \, \exp -\int (\partial_\mu \theta)(\partial_\mu \theta) \, d^4x \tag{2.1}$$

We linearize θ in the action by introducing an auxiliary vector field C_μ,

$$Z = \int d\theta \, dC_\mu \, \exp -\int (+i\partial_\mu \theta \, C_\mu + \frac{C_\mu^2}{4}) d^4x \tag{2.2}$$

Integration over θ gives a constraint over C_μ,

$$\partial_\mu C_\mu = 0 \, . \tag{2.3}$$

The solution is

$$C_\mu = \partial_\nu \, \tilde{H}_{\mu\nu} \tag{2.4}$$

Here $H_{\mu\nu}$ is an antisymmetric tensor field & $\tilde{H}_{\mu\nu} = 1/2\, \epsilon_{\mu\nu\rho\sigma} H_{\rho\sigma}$

Putting this solution back in (2.2) we get,

$$Z = \int dH_{\mu\nu} \exp -\int (\partial_\nu \tilde{H}_{\mu\nu})^2 \, d^4x. \qquad (2.5)$$

Therefore a theory of a massless real scalar is equivalent to that of a massless antisymmetric (real) tensor.[9]

Having introduced extra degrees of freedom (Appendix-B), we have a local gauge invariance,

$$H_{\mu\nu} \rightarrow H_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \,. \qquad (2.6)$$

The corresponding Hamiltonian formulation and the exact matching of degrees of freedom in the initial and final theory has been worked out by Witten & Deser. We describe it briefly in Appendix B.

By introducing sources for $(\partial_\mu \theta)$, it can be easily checked that the Green functions for $(\partial_\mu \theta)$ get transformed to the Green functions for $\partial_\nu \tilde{H}_{\mu\nu}$ in the final theory, i.e,.

$$\langle \partial_\mu \theta(1)\ \partial_\nu \theta(2) \ldots \rangle = \langle \partial_\rho \tilde{H}_{\mu\rho}(1)\ \partial_\sigma \tilde{H}_{\nu\sigma}(2) \ldots \rangle \tag{2.7}$$

Later in this chapter we generalize this technique to a complex massive scalar field coupled to the U(1) gauge field A_μ. We call the transformation such as (2.7) which transforms the scalar field ϕ to the antisymmetric tensor field $H_{\mu\nu}$ the duality transformation on matter.

IIA-3] DUAL TRANSFORMATION ON VECTOR POTENTIAL :

In Maxwell's theory, in the absence of sources, the field strength tensor $F_{\mu\nu}$ can be expressed in either of the following ways :

$$[a] \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$[b] \quad \tilde{F}_{\mu\nu} = (\partial_\mu B_\nu - \partial_\nu B_\mu)$$

Here A_μ is the electric vector potential and B_μ is the magnetic vector potential. In the functional integral formulation one can transform A_μ to B_μ as follows.

We start with the functional integral for pure quantum electrodynamics written in terms of the electric vector potential A_μ,

$$Z = \int dA_\mu \exp -1/4 \int (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \, d^4x \tag{2.8}$$

Again we linearize in the A_μ field by introducing an auxiliary antisymmetric tensor field $G_{\mu\nu}$,

$$Z = \int dA_\mu \int dG_{\mu\nu} \exp -\int \left[i(\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{G}_{\mu\nu} - G^2_{\mu\nu} \right] d^4x \qquad (2.9)$$

Integration over A_μ implies,

$$\partial_\nu \tilde{G}_{\mu\nu} = 0 \, . \qquad (2.10)$$

The solution of this constraint is,

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad (2.11)$$

Again, introducing sources for $(\partial_\mu A_\nu - \partial_\nu A_\mu)$, we find that their Green functions get mapped onto that of $\epsilon_{\mu\nu\rho\sigma} (\partial_\rho B_\sigma - \partial_\sigma B_\rho)$. Hence the vector field B_μ obtained in (2.11) is indeed the magnetic vector potential.

Now we introduce minimal coupling of the electric current J^{el}_μ to the photon described in terms of the electric vector potential A_μ. The partition function is

$$Z\left(J^{el}\right) = \int dA_\mu \exp -\int \left\{ \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + J^{el}_\mu A_\mu \right\} d^4x \, .$$

Repeating the above procedure we find that the dual theory in terms of the magnetic vector potential B_μ has a non local coupling to J^{el}_μ, i.e,

$$Z\left[J^{el}\right] = \int dB_\mu \exp -\frac{1}{4}\int\left[[\partial_\mu B_\nu - \partial_\nu B_\mu]^2 + (n.\partial)^{-1}\,(n_\mu J_\nu - n_\nu J_\mu)\right]^2 \tag{2.12}$$

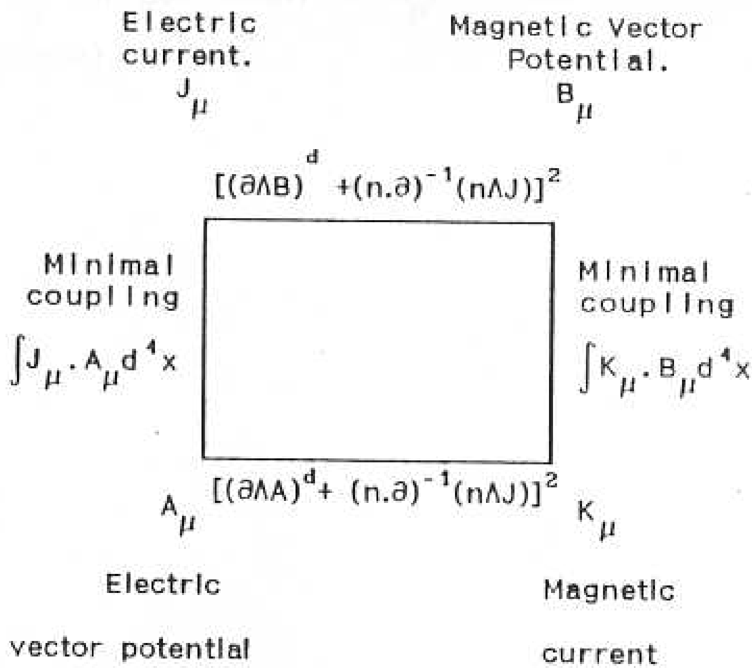


Fig.4]: The non-local $(A_\mu - K_\mu)$ / $(B_\mu - J_\mu)$ interactions involving the Dirac string.

Here n_μ is an arbitrary unit 4 vector and

$$(n.\partial)^{-1} = \frac{1}{2}\int_0^\infty ds \left[\delta^4(x-n\ s) - \delta^4(x+n\ s)\right]$$

As expected, the magnetic vector potential couples to the electric current non-locally via the " Dirac string n_μ". Similarly the magnetic current K_μ couples locally (i.e. minimal coupling) to B_μ but non locally to A_μ. These interactions are described in [Fig-4].

B] FORMULATION OF THE THEORY

Having equipped ourselves with the basic ideas to be used in our formulation we write down the partition function of a scalar monopole φ minimally coupled to B_μ,

$$Z = \int D\varphi \, DB_\mu \, \exp -\int\sum \Big[\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + (\partial_\mu \varphi^* - ig\, B_\mu \varphi^*)$$

$$(\partial_\mu \varphi + ig\, B_\mu \varphi) + V(\varphi^*\varphi)\Big] \qquad (2.13)$$

Here $\int$ in the exponent stands for the integration over $R^{(4)}$. As stressed earlier, if at this stage we convert B_μ to A_μ, the coupling of $\varphi^* \overleftrightarrow{\partial_\mu} \varphi$ with A_μ will not only be non local but we will also lose the manifest Lorentz covariance due to the presence of the Dirac string n_μ(Fig.4). Essentially, this had been the approach of most of the earlier attempts.

We rewrite (2.13) using radial and phase degrees of freedom of $\varphi(x)$ [= $R(x) \exp i\theta(x)$],

$$Z = \int DR\, D\theta\, DB_\mu \exp -\int \left[\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + R^2(x)\,(\partial_\mu\theta + g\, B_\mu)^2 \right.$$

$$\left. + (\partial_\mu R(x)^2 + V(R^2)\right] \tag{2.14}$$

The ultra local Jacobian coming from (ultra-local) the point transformation is absorbed in the definition of the measure. Now the action can be linearized in B_μ and φ by introducing the auxiliary fields C_μ and $G_{\mu\nu}$:

$$\int DR\, D\theta\, DB_\mu\, DG_{\mu\nu}\, DC_\mu \exp -\int \left[\frac{1}{4}\, \epsilon_{\mu\nu\rho\sigma}(\partial_\mu B_\nu - \partial_\nu B_\mu)\, G_{\rho\sigma} + \frac{1}{4}\, G^2_{\mu\nu} \right.$$

$$\left. + 2iR(x)\,(\partial_\mu\theta(x) + g\, B_\mu(x))C_\mu + C^2_\mu + (\partial_\mu R)^2 + V(R^2)\right]$$

$$\tag{2.15}$$

Formally integrating over B_μ & θ, we get the functional δ-functions implying the constraints,(#1)

(#1) *Here to begin with we have taken the range of θ to be $[-\infty,+\infty]$ and not $[0,2\pi]$. We will take the periodicity property of θ into account more carefully later.*

$$\partial_\mu (RC_\mu) = 0 \qquad , \qquad \partial_\mu \tilde{G}_{\mu\nu} + 2g\ R\ C_\nu = 0\ . \qquad (2.16)$$

Here $\tilde{G}_{\mu\nu} \equiv \frac{1}{2}\ \epsilon_{\mu\nu\rho\sigma}\ G_{\rho\sigma}$ is the dual of $G_{\mu\nu}$. The first constraint also follows as a self-consistency condition of the second. These constraints are solved in the form :

$$2RC_\mu = -\partial_\nu \tilde{H}_{\mu\nu} \qquad , \qquad G_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu + g\ H_{\mu\nu}) \qquad (2.17)$$

Thus we can formally rewrite the functional integral (1) as

$$Z = \int DR \int DH_{\mu\nu} \int DA_\mu \ \exp -\int \Big[\frac{1}{4}\ (\partial_\mu A_\nu - \partial_\nu A_\mu + g\ H_{\mu\nu})^2$$

$$+ \ \frac{1}{4R^2}\ (\partial_\mu \tilde{H}_{\mu\nu})^2 \ + \ (\partial_\mu R)^2 + V(R^2) \Big] \qquad (2.18)$$

Hence, if we insist on describing spin zero magnetically charged matter in the conventional way by a scalar field φ, it's coupling to the usual vector potential A_μ is non-local. Instead we describe it by a real antisymmetric tensor $H_{\mu\nu}$ and the radial field R. In this description we completely evade the need of the

Dirac string to describe the interactions of monopoles with electric vector potential A_μ . Hence even the questions regarding the problems of non-locality and the loss of Lorentz co-variance etc do not arise.

One also comes across these specific interactions of the antisymmetric tensor $H_{\mu\nu}$ of our theory in Supergravity, String theory[6] and Compact Lattice Quantum Electrodynamics (appendix-C).

At this stage coupling the electric current J_μ to the photon A_μ can be trivially done by minimal coupling $\int J_\mu A_\mu d^4x$. But before we do this and obtain the Dirac quantization condition, we describe some interesting features of the duality tansformations performed above.

IIB-1] <u>*Features of Duality Transformations :*</u>

A] Having introduced some redundant degrees of freedom in $H_{\mu\nu}$ to describe the magnetic matter, the dual theory has a bigger local gauge invariance,

$$A_\mu \longrightarrow A_\mu + g\Lambda_\mu \tag{2.19-a}$$

$$H_{\mu\nu} \longrightarrow H_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu . \tag{2.19-b}$$

These transformations form a local $R^{(4)}$ gauge group. Here, $R^{(4)}$ is the continuous group of 4-tuples of real numbers under addition. We will refer to this as "m-type" gauge invarianceas it is associated with the magnetic matter $H_{\mu\nu}$. Writing Λ_μ as the sum of longitudinal and transverse parts:

$$\Lambda_\mu = \partial_\mu \Lambda + \Lambda^T_\mu \tag{2.19-c}$$

we notice that $H_{\mu\nu}$ is sensitive only to transverse part of Λ, i.e, Λ^T_μ (with $\partial_\mu\Lambda^T_\mu = 0$) and it removes the redundant degrees of freedom of $H_{\mu\nu}$ (Appendix B). Having fixed Λ^T_μ this way, the longitudinal part of Λ_μ, i.e., $\partial_\mu\Lambda$ reduces the three polarization vectors of A_μ to two transverse polarization vectors.

Under m-type gauge transformations, the matter $H_{\mu\nu}$ plays the role of the gauge field (it transforms with a derivative) and the transformation of electromagnetic potential A_μ ($A_\mu \to A_\mu + g\,\Lambda_\mu$) is analogous to that of $\theta(x)$ under U(1) gauge transformation ($\theta(x) \longrightarrow \theta(x) - g\,\alpha(x)$). Hence the roles of the gauge fields and the matter fields get interchanged. The Noether currents corresponding to (2.19) are now $F_{\mu\nu}(A,H)$, the full field strength tensor of the dual theory given by,

$$F_{\mu\nu}(A,H) = \partial_\mu A_\nu - \partial_\nu A_\mu + gH_{\mu\nu} \tag{2.20-a}$$

The conservation of this Noether current $F_{\mu\nu}$ follows from the equations of motion of A_μ,

$$\partial_\nu\, F_{\mu\nu}(A,H) = 0 \tag{2.20-b}$$

This is the Maxwell's equations in the absence of electric charges.

This conservation can also be looked upon as a consistency condition for the equation of motion of H :

$$\partial_\rho\, H_{\mu\nu\rho} = g\, F_{\mu\nu} \tag{2.20-c}$$

Here,

$$H_{\mu\nu\rho} = \frac{1}{2R^2}\left(\partial_\mu H_{\nu\rho} + \partial_\nu H_{\rho\mu} + \partial_\rho H_{\mu\nu}\right) \qquad (2.20\text{-d})$$

is the field strength tensor associated with the antisymmetric tensor field $H_{\mu\nu}$. These two ways of looking at the conservation laws of Noether current $F_{\mu\nu}(A,H)$ are quite analogous to that of $\varphi^* \overleftrightarrow{D}_\mu \varphi$, the Noether current of the initial theory. The latter again can be looked upon either as a consequence of the equation of motion of the matter field φ or as the consistency condition of the equation of motion for the gauge field B_μ. This comparison makes the interchange of roles of the matter and the gauge degrees of freedom even more transparent . The Bianchi identity of the dual theory takes the form,

$$\partial_\nu \tilde{F}_{\mu\nu} = \partial_\nu \tilde{H}_{\mu\nu} \qquad (2.20\text{-e})$$

implying a non-zero magnetic current which is trivially conserved.

[B] Introducing sources for the Noether current, $\varphi^* \overleftrightarrow{D}_\mu \varphi$ and the topological currents, $\epsilon_{\mu\nu\rho\sigma}(\partial_\rho B_\sigma - \partial_\sigma B_\rho)$, we find that their correlation functions get mapped into the correlation functions of the topological current, $\partial_\nu \tilde{H}_{\mu\nu}$ and the Noether current, $F_{\mu\nu}$ respectively,

$$\langle \phi^* \overleftrightarrow{D}_\mu \phi(1)\ \phi^* \overleftrightarrow{D}_\nu \phi(2)--- \rangle \ = \ \langle \partial_\rho \tilde{H}_{\mu\rho}(1)\ \partial_\sigma \tilde{H}_{\nu\sigma}(2)--- \rangle$$

(2.21-a)

$$\langle \epsilon_{\mu\nu\rho\sigma}(\partial_\rho B_\sigma - \partial_\sigma B_\rho)(1)--- \rangle \ = \ \langle F_{\mu\nu}(A,H)(1)--- \rangle$$

(2.21-b)

Hence, under the duality transformations, not only the roles of matter and the gauge degrees of freedom get interchanged, but the Noether and topological currents also change their roles. The equivalence of $\varphi * \overleftrightarrow{D}_\mu \varphi$ and $\partial_\nu \tilde{H}_{\mu\nu}$ implies that the in-field operators of monopoles in the initial and dual theories are related by non-local transformations. This non-local relationship between the initial and the final dual variables removes the Dirac strings of previous formulations from the dynamics. This non-locality is analogous to the non-local relationship between the Sine-Gordon and the Thirring field in the Sine-Gordon Thirring model in (1+1) dimension[14]. We elaborate on this in the third chapter.

C] . In the presence of many magnetically charged spin zero fields $(R^p, H^p_{\mu\nu})$, $p = 1,2,....,N$ with magnetic charges g^p the interaction term generalizes to $[F_{\mu\nu}(A,H^p)]^2$, where,

$$F_{\mu\nu}(A,H^{P}) = \left(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu} + \sum_{p=1}^{N} g^{P}\, H^{P}{}_{\nu}\right) . \qquad (2.22)$$

We now have a separate m-type gauge invariance for each of the H^{P} fields,

$$H^{P}_{\mu\nu} \rightarrow H^{P}_{\mu\nu} + \partial_{\mu}\, \Lambda^{P}_{\nu} - \partial_{\nu}\, \Lambda^{P}_{\mu} \qquad (2.23\text{-a})$$

$$A_{\mu} \rightarrow A_{\mu} + \sum_{p=1}^{N} g^{P}\, \Lambda^{P}_{\mu} \qquad (2.23\text{-b})$$

IIB-2] The Spectrum

Having discussed the properties of duality transformations in detail, it is interesting to translate the spectrum of the theory described by (2.13) into the language of the dual theory corresponding to (2.18). It is easier and more fruitful to do this first for the case of spontaneously broken U(1) symmetry $\langle\phi\rangle \neq 0$.

a] Higg's Phase

In this case we have Higg's mechanism with A_μ "eating up" the Goldstone boson θ to give a massive boson described by $(A_\mu + \partial_\mu \theta)$. Also there is a massive real scalar, the Higgs, described by $R' = R - R_o$. Moreover, considering configurations which are independent of one of the dimensions (let us say x_3), we also have Nielson- Oleson vortices in the initial theory described by the topological index,

$$N^{(3)} = \frac{1}{2\pi} \int_0^{2\pi} d\psi \, \frac{\partial\theta}{\partial\psi}(x_1, x_2) = \frac{g}{2\pi} \oint \vec{B}.\vec{d\ell} \; .$$

(2.24-a)

Here $\vec{B}$ is the magnetic vector potential and the line integral is over the boundary of the (x,y) plane, ψ is the azimuthal angle in the x-y plane. Using Stokes' theorem, the line integral in (2.24-a) can also be written as a surface integral over the(x_1,x_2) plane,

$$N^{(3)} = \frac{g}{2\pi}\int (\vec{\nabla} \times \vec{B}).\vec{ds} = \frac{g}{2\pi}\int \vec{E}.\vec{ds}$$

(2.24-b)

Here $\vec{E}$ is the electric field. Hence, for each of the three planes we have a conserved topological charge $N^{(i)}$(i=1,2,3).

In the corresponding dual theory, the antisymmetric tensor $H_{\mu\nu}$ "eats up" A_μ to give a theory of massive $H_{\mu\nu}$, equivalent to a massive vector boson [Appendix-B]. The Noether current in the dual theory is given by $F_{\mu\nu}(= \frac{1}{g}\partial_\rho H_{\mu\nu\rho})$. Choosing (e.g.) $\nu = 3$, the conservation law implies,

$$\partial_0 F_{03} = \partial_1 F_{13} + \partial_2 F_{23} .$$

Integrating this over the (xy) plane we get,

$$\int dx_1\, dx_2\, \partial_0 F_{03} = \int dx_1 dx_2 \left[\partial_1 F_{13} + \partial_2 F_{23}\right] = 0 \,.$$

Hence for each plane we have a conserved "Noether charge" $\tilde{N}^{(i)}$,

$$\tilde{N}^{(i)}(i=1,2,3) = \int \vec{E}^i\, \vec{ds}^i \qquad (2.24\text{-c})$$

This is again the net electric flux across each of the three planes in the dual theory. Moreover it is conserved because of the equations of motion and not because of topological reasons.

Thus once again we find that the topological and the Noether properties of the theory get interchanged under duality transformations.

b] *The Coulomb Phase*

Now we consider the coulomb phase $\langle\varphi\rangle = 0$. In (B_μ,φ) language the spectrum has massless photons described by B_μ and charged spin-0 particles and antiparticles described by φ and φ^*.

As a consequence of the $1/R^2$ coefficient in $\frac{1}{R^2}(\partial_\nu \tilde{H}_{\mu\nu})^2$ term in (2.18) there are no excitations directly corresponding to the fields $H_{\mu\nu}$ & R. The charged particles φ correspond to certain non-local combinations of $H_{\mu\nu}$ & R. This is analogous to the Mandlestam construction of the fermion fields from the bosons in the (1 +1) dimension Sine Gordon Thirring model[14]. This equivalence is best described in lattice gauge theory in the Hamiltonian formulation and we describe it in detail in chapter [III].

IIB-3] Gauge Invariance & The Dirac Quantization Condition

Till now we have been discussing the theory with fields carrying only magnetic charges. We now proceed to construct the theory with many dyon fields and show how the Dirac quantization condition arises.

As mentioned earlier having got a local manifestly Lorentz co-variant action for magnetic charges coupled to the usual electric vector potential A_{μ}, we can trivially couple electrically charged matter (Φ^{*q},Φ^{q}) [with electric charges $e^{q}(q=1...M)$] to the photon by minimal coupling. Again we have the option of describing electrically charged spin zero matter Φ^{q} by it's radial degrees of freedom ρ^{q} and the antisymmetric tensor $K^{q}{}_{\mu\nu}$. In the latter representation we have the following two advantages :

a] Magnetic charges $(R^{q}, H^{q}_{\mu\nu})$ and electric charges $(\rho^{q}, K^{q}_{\mu\nu})$ are treated on equal footing and hence the dual symmetry of Maxwell's equations is comparatively more transparent.

b] We can trivially get a theory of dyons just by identifying corresponding R^{p} with ρ^{q} and $H^{p}_{\mu\nu}$ with $K^{q}_{\mu\nu}$.

Henceforth we stick to this antisymmetric tensor description of spin zero fields. Again, because of introducing redundant degrees of freedom to describe electric matter in the form of $K^q_{\mu\nu}$ (q = 1....M) we have another sacred gauge invariance,

$$K^q_{\mu\nu} \longrightarrow K^q_{\mu\nu} + \partial_\mu \lambda^q_\nu - \partial_\nu \lambda^q_\mu , \quad \rho^q \to \rho^q \tag{2.25-a}$$

$$H^p{}_{\mu\nu} \to H^p{}_{\mu\nu} , \quad R^p \to R^p, \quad A_\mu \to A_\mu \tag{2.25-b}$$

Being associated with electric charges $K_{\mu\nu}$, we refer to these as "e-type" gauge transformations[#2]. This invariance is purely due to kinematical reasons. Hence the gauge fields A_μ and magnetic charge fields ($R, H_{\mu\nu}$) remain invariant. Moreover the invariance of A_μ under e-type gauge transformations also follows from the fact that m type gauge transformation already reduce the degrees of freedom of A_μ to two physical (transverse polarizations) degrees of freedom.

[#2]

Here we differ from the nomenclature used in the reference (4) where the original U(1) gauge transformation was called the "e-type" gauge transformation.

$K_{\mu\nu}$, being dual to the phase of the scalar field Φ, couples to the photon A_μ by minimal coupling in its dual form i.e. $\exp i\left[(\partial_\mu \tilde{K}_{\mu\nu})A_\nu\right]$. This coupling is e-type gauge invariant. Under m-type gauge transformations,

$$\delta\left(ie\ A_\mu\ \partial_\nu \tilde{K}_{\mu\nu}\right) = ie\ g\ \Lambda^T_\mu\ \partial_\nu \tilde{K}_{\mu\nu}$$

So this coupling is sensitive only to the transverse part of the m type gauge transformation (2.19-c). It appears that there is no way to make it m-type gauge invariant. To resolve this apparent discrepancy we consider the regulated form of this theory on lattice and show that the problem resolves itself by yielding the Dirac quantization condition. This regularization is most natural in our formulation because it automatically takes periodicity of θ variable in the original theory into account. We come back to this point later. The part of the lattice action involving electromagnetic interaction of electric and magnetic charges is

$$S' = -\frac{1}{4} \sum_{n,\mu,\nu} \left\{\left[\Delta_\mu a_\nu(n) - \Delta_\nu\ a_\mu(n) + \sum_p g^p\ h^p_{\mu\nu}(n)\right]^2 + \frac{1}{2} \sum_q e^q \sum_{\substack{n \\ (\mu\nu\rho\sigma)}} \epsilon_{\mu\nu\rho\sigma}\ K_{\mu\nu}(n)(\Delta_\rho a_\sigma(n) - \Delta_\sigma a_\rho(n))\right\} \qquad (2.26)$$

where n labels the sites of a hypercubic lattice in four dimensions and μ,ν,ρ,σ (=1,2,3,4) label the unit vectors along the corresponding directions. The connection between the lattice variables (lower case latin letters) and the continuum fields (upper case latin letters) is in an evident notation. We have scaled lattice fields (by the lattice spacing a) suitably to make them dimensionless. Because the bare charges e & g in the continuum theory are dimensionless they are not scaled relative to lattice definition. It is the last term in (2.25) which is not m type gauge invariant. This non invariance can be traced to the fact that, while making duality transformation on the compact variable θ $[0:2\pi]$, its range was taken to be $[-\infty,+\infty]$. Hence, in the dual theory, the range of the dynamical variables is more than what corresponds to the scalar field theory we started with. We carefully remove these discrepancies in $H_{\mu\nu}(K_{\mu\nu})$ and the gauge transformations, $\Lambda_{\mu}(\lambda_{\mu})$ by regulating the theory on lattice. This is quite analogous to compact lattice quantum electrodynamics [Appendix-C] or x-y model[8] (in 2 dimension), where the periodic nature of the dynamical variables involved is incorporated by using a lattice cut-off. The resultant $H_{\mu\nu}$ $(K_{\mu\nu})$ and gauge parameters Λ_{μ} (λ_{μ}) turn out to be integer valued fields. In the functional integral formulation on lattice it is easy to see that periodicity of $\theta(n)$ at every lattice site (n) will make $H_{\mu\nu}(n)$ ($K_{\mu\nu}(n)$) integer valued [Appendix- C]. Hence the transverse parts of the m type (e type) gauge transformation $\Lambda^{T}_{\mu}(n)$ $(\lambda^{T}_{\mu}(n))$ also take integer values.

Having handled these discrepancies in the dual theory , the original transverse part of m type (e type) gauge group $R^{(4)}$ (2.19-c) gets reduced to a local $Z^{(4)}$ gauge group for each magnetic (electric) matter. Here $Z^{(4)}$ is the discrete group of 4 tuples of integers under addition. Hence this theory has to be invariant under the following two independent gauge groups :

$(Z^{(4)})^{M}$: Corresponding to the transverse m-type gauge invariance. (M is the # of magnetic matter fields).

$(Z'^{(4)})^{N}$: Corresponding to the transverse e-type gauge invariance (N is the # of electric matter fields).

The longitudinal part of the m type (e type) gauge transformations still take values on the real line and form a group $(R^{(1)})^{M}$ $((R'^{(1)})^{N})$ under addition. The theory is trivially invariant under these longitudinal gauge groups. From (2.19,2.23-b) it is clear that $(R^{(1)})^{M}$ reduces the three polarization vectors of A_{μ} to two transverse polarizations. The longitudinal part of e type gauge transformations $(R'^{(1)})^{N}$ do not play any role in the theory.

The partition function of the theory is also invariant under the transverse gauge groups $(Z'^{(4)})^M$, $(Z^{(4)})^M$ except for the last term in (2.26). This term is not invariant under $(Z^{(4)})^M$ gauge group. Demanding invariance of the partition function under this leads to Dirac quantization condition. This can be easily seen as follows.

We arrange magnetic and electric charges in increasing order :

$$\left[g^{o} < g^{(1)} < g^{(2)} \;--\; < g^{(M)}\right]$$

$$\left[e^{o} < e^{(1)} < e^{(2)} \;--\; < e^{(N)}\right]$$

and make a particular gauge transformation

$$\Lambda^{o}(n) = 1 \qquad \Lambda^{p(\neq 0)} = 0 \qquad \text{for} \qquad p = 1,\ldots,M$$

at a particular site (n). Invariance of the partition function under this implies,

$$e^{q}\, g_{o} = 2\pi\; x(\text{Integer})$$

$$\Rightarrow \; e^q = [2\pi/g_0] \text{ (Integer)} \tag{2.27-a}$$

Hence all electric charges get quantized in the units of $\left({}^{2\pi}/_{g_o}\right)$, with it's minimum possible value $[e_o = 2\pi \,/\, g_o]$.

To see the magnetic charge quantization, we consider the term containing minimum charged field $K^{o}_{\mu\nu}$ and make a gauge transformation,

$$\Lambda^{p} = 1 \quad (p \neq 0) \,.$$

Now the invariance of the partition function implies,

$$e^{o} g^{p} = 2\pi \text{ Integer}$$

$$\Rightarrow g^{p} = g_o \text{(Integer)} \tag{2.27-b}$$

Hence all magnetic charges get quantized in terms of a minimum unit (g_o). The resulting electric and magnetic charge lattice forms a rectangular grid [Fig.5-a].

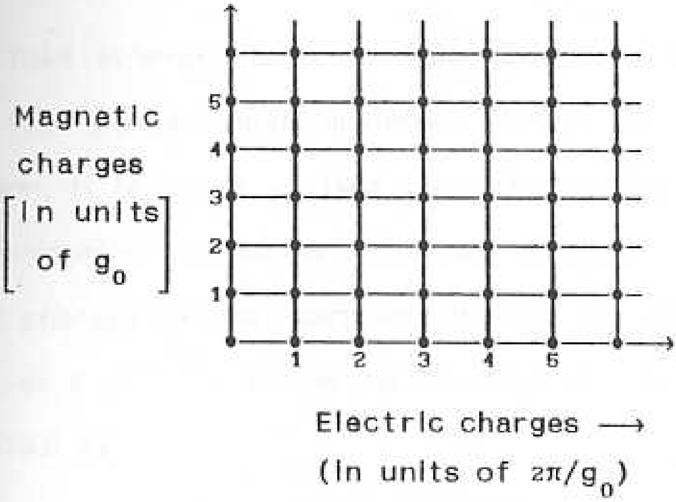


Fig.5-a]: The Lattice of allowed Electric and Magnetic charges. (without θ term)

Thus the m type gauge invariance is truly a quantum invariance. It is the invariance of the partition function and not of the classical action. This is analogous to the gauge invariance of the Wess-Zumino-Witten model.

In this theory we also have the option of introducing the term :

$$\theta \left(\sum_{(n,\mu,\nu)} f_{\mu\nu}(n)\, \tilde{f}_{\mu\nu}(n) \right) \tag{2.28}$$

Here,

$$f_{\mu\nu}(n) = \left(\Delta_\mu \, a_{n,\nu} - \Delta_\nu \, a_{n,\mu} + g \, h_{n,\mu\nu} \right)$$

is the full field strength tensor. This term is invariant under both m & e type gauge transformations. Though this is analogous to the θ term, it is not a surface term due to the presence of $h_{\mu\nu}$. By considering equations of motion for $a_\mu(n)$ we find that all magnetic charges (dyons) carrying charges g^p get (additional) electric charge $\theta \, g^p$ (7). Hence the modified electric charges are given by [Fig.5-b],

$$q^i = e^i + \theta g^i \; . \tag{2.29}$$

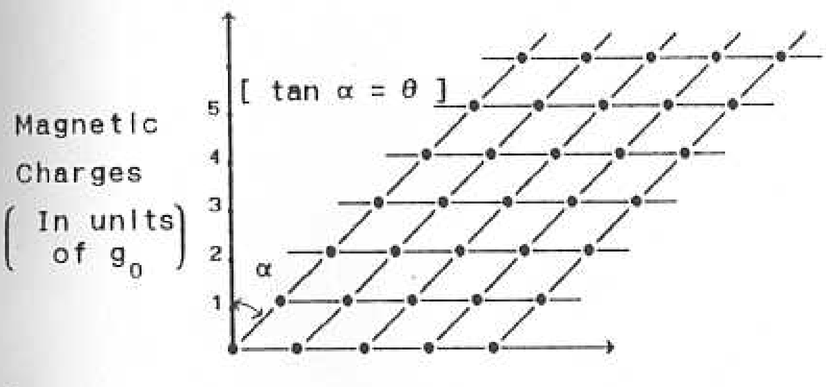


Fig.5-b]: The Lattice of allowed Electric and Magnetic charges. (with θ term)

The equation (2.29) implies a weaker quantization condition,

$$q_i g_j - q_j g_i = 2\pi Z_{ij} \tag{2.30}$$

Here Z_{ij} are sets of integers. This condition is the Dirac Schwinger quantization condition.

The (e,g) grid is now oblique (Fig5-b) with angle of inclination from the original g axis

$$\alpha = \tan^{-1}\theta. \tag{2.31}$$

In the next chapter we describe the Hamiltonian formulation of this theory ■ ■

CHAPTER III

THE HAMILTONIAN FORMULATION

In this section we discuss the Hamiltonian version of the theory on lattice. This approach gives a better intuitive picture of the duality transformations performed in the previous sections and therefore the understanding of certain subtle aspects, like the role of the Dirac string, dual gauge transformations etc in our formulation, is more transparent in this language. In fact, this thesis problem was motivated while studying the duality transformations in the lattice gauge theories.

We start with the Hamiltonian formulation of the duality transformation on the complex scalar field and the vector fields respectively and then combine these ideas together to construct the Hamiltonian formulation of our theory of Dyons. We also show that the charged particles ϕ and ϕ^* correspond to certain non-local combinations of $H_{\mu\nu}$ and R, analogous to the Mandelstam construction of the fermion fields from the bosons in (1+1) dimension[14].

[III.1] DUAL TRANSFORMATION ON SCALAR FIELD

The Hamiltonian for a complex scalar field φ on the lattice is

$$H = \sum_n \left[|\Pi(n)|^2 + \varphi^*(n)\,\varphi(n) \right] \tag{3.1}$$

$$+ \; x \sum_{n,l} \left(\varphi^*(n)\,\varphi(n+l) + \varphi^*(n+l)\,\varphi(n)\right) + V(\varphi^*\varphi)$$

Here we have rescaled the fields suitably and x is the hopping parameter. $\Pi(n)$ is the conjugate momentum of $\varphi^*(n)$ with the following commutation relations

$$\left[\Pi(n)\,,\,\varphi^*(n)\right] = -i = \left[\Pi^*(n),\,\varphi(n)\right]$$

$$\tag{3.2}$$

$$\left[\Pi(n)\,,\,\Pi^*(n)\right] = 0 = \left[\phi^*(n),\,\varphi(n)\right]$$

We define the creation, anhilation operators

$$a^{+}(n) = \frac{1}{\sqrt{2}}\left[\varphi^{*}(n) - i\,\Pi(n)\right] \quad , \quad b^{+}(n) = \frac{1}{\sqrt{2}}\left[\varphi(n) - i\,\Pi^{*}(n)\right]$$

(3.3-a)

$$a(n) = \frac{1}{\sqrt{2}}\left[\varphi(n) + i\,\Pi^{*}(n)\right] \quad , \quad b(n) = \frac{1}{\sqrt{2}}\left[\varphi^{*}(n) + i\,\Pi(n)\right]$$

(3.3-b)

with this the equation (3.2) implies,

$$[a,a^{+}] = 1 = [b,b^{+}]$$

(3.4)

$$[a,b] = 0 = [a,b^{+}]$$

The Hamiltonian in terms of these operators is given by

$$H = \sum_{n}\Big(a^{+}(n)\,a(n) + b^{+}(n)\,b(n)\Big) + \frac{1}{2}\sum_{n,l}\Big(a^{+}(n)\,a(n+l) +$$

(3.5)

$$a^{+}(n)b^{+}(n+l) + b(n)\,a(n+l) + b(n)\,b^{+}(n+l) + h.c\Big) + V(a,a^{+},b,b^{+})$$

The particle and anti-particle number operators, which commute with this Hamiltonian are N_a $(=a^{+}(n)a(n))$, N_b $(=b^{+}(n)b(n))$ with eigenvalues $n_a(n)$ and $n_b(n)$ respectively.

Instead of characterizing the states in the Hilbert space by $| n_a(n), n_b(n) \rangle$, we characterize them by $| \ell(n), n_c(n) \rangle$, where $\ell(n) \equiv n_a(n)-n_b(n)$ is the eigevalue of the charge operator $L(n) \equiv N_a(n)-N_b(n)$ and $n_c(n)$ is the eigenvalue of the operator $N(n) \equiv \mathrm{Min}(N_a(n),N_b(n))$. Here $\mathrm{Min}(N_a,N_b)$ is the the operator with smaller eigenvalue. The operatos $N(n)$ and $L(n)$ have integer eigenvalues with the ranges $(0,\infty)$ and $(-\infty,\infty)$ respectively. The new representation $| \ell(n),n_c(n) \rangle$ defined this way, has the advantage of being charecterized by two integers independent of each other.[#1]

The number operator N can also be written as

$$N(n) \equiv C^{+}(n)\, C(n) \tag{3.6}$$

where C^{+} and C are the new creation and anhilation operators satisfying

$$[C,C^{+}] = 1; \quad [C,C] = [C^{+},C^{+}] = 0 \tag{3.7}$$

These operators are explicitly constructed in terms of the

[#1] *The representation ,e.g, charecterized by the eigevalues of the operators* $L(n) \equiv N_a(n) - N_b(n)$ *and* $N \equiv N_a(n) + N_b(n)$ *has to satisfy the constraint,*

$$N + L = 2N_a \geq 0$$

original operators later in this section.

The operator L is analogous to the angular momentum operator of a two dimensional rotor. We denote the raising and the lowering operators for L by $e^{\pm i\theta}$ respectively. This construction corresponds to going from (φ, φ^*) variables to the radial variables (R,θ) in the previous chapter.

To find the correspondence between the new operators (L,N) and the original operators (a,b), we start with the identification,

$$|N_a, N_b\rangle \equiv |N_c \equiv \mathrm{Min}\,(N_a, N_b)\,,\, L \equiv N_a - N_b\,\rangle \tag{3.8}$$

with this we find,

$$e^{i\theta}|n_c, \ell\,\rangle = |n_c, \ell+1\,\rangle$$

$$= \epsilon(n_a - n_b)\,|n_a + 1, n_b\rangle + \epsilon(n_b - n_a)\,|n_a, n_b - 1\rangle$$

Here $\epsilon(n)$ is the step function

$$\epsilon(n) = 1 \qquad n \geq 0$$

$$= 0 \qquad n < 0$$

Therefore,

$$e^{i\theta} = \frac{\epsilon(N_a - N_b)}{\sqrt{N_a}} a^+ + \frac{\epsilon(N_b - N_a)}{\sqrt{N_b}} b$$

(3.9-a)

Similarly the relations between the other new operators and the original operators are

$$N = \mathrm{Min.}(a^+a\ , b^+b) \quad , \quad L = a^+a - b^+b \qquad (3.9\text{-}b)$$

$$C^+ = \epsilon(a^+a - b^+b)\ b^+ + \epsilon(b^+b - a^+a)\ a^+ \qquad (3.9\text{-}c)$$

$$C = \epsilon(a^+a - b^+b)\ b + \epsilon(b^+b - a^+a)\ a \qquad (3.9\text{-}d)$$

Using the same technique ,we find the inverse relationships

$$a^+ = \sqrt{N+L}\ \exp(i\theta)\ \epsilon(L) + C^+ \exp(i\theta)\ \epsilon(-L) \qquad (3.10\text{-}a)$$

$$a = \epsilon(L)\ \exp(i\theta) \sqrt{N+L} + \epsilon(-L)\ C \exp(-i\theta) \qquad (3.10\text{-}b)$$

$$b^+ = \epsilon(L)\ C^+ \exp(-i\theta) + \epsilon(-L) \sqrt{N-L}\ \exp(i\theta) \qquad (3.10\text{-}c)$$

$$b = \exp(i\theta)\ C\ \epsilon(L) + \exp(-i\theta) \sqrt{N-L}\ \ \epsilon(-L) \qquad (3.10\text{-d})$$

The next step is to formulate this theory in terms of the antisymmetric tensor operators. For this we define the dual field $h_i(n)$ by

$$\ell(n) = \sum_{i=1}^{3} \Delta_i \vec{h}_i(n) \qquad (3.11)$$

where Δ_i is the lattice difference operator:

$$\Delta_i F(n) \equiv F(n+i) - F(n) \qquad (3.12\text{-a})$$

for any arbitrary function F(n). For later use, we define another difference operator $\tilde{\Delta}_i$ on the lattice :

$$\tilde{\Delta}_i F(n) \equiv F(n) - F(n-i) \qquad (3.12\text{-b})$$

The relation (3.11) defining $h_i(n)$ is the lattice Hamiltonian version of the functional integral correspondence (2.21-a) with the identification

$$\vec{h}_i(n) \equiv \tilde{H}_{oi}(n) \qquad (3.13)$$

From (3.11) it is clear that the $h_l(n)$ operators living on lattice links are real and integer valued with range$[-\infty,+\infty]$.

The charge operator in the equation (3.11) is invariant under the dual gauge transformation

$$\vec{h}_l(n) \rightarrow \vec{h}_l(n) + (\vec{\nabla} \times \vec{\Lambda}(n))_l \qquad (3.14)$$

Here $\Lambda_l(n)$ is the gauge parameter and is integer valued. This gauge invariance corresponds to the m-type gauge invariance (2.6) in the path integral formulation.

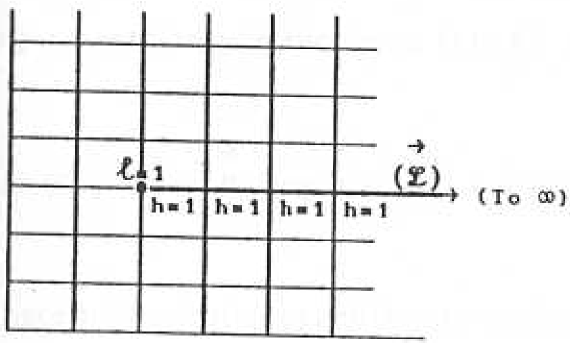
Now we characterize the Hilbert space by the eigenvalues of $\vec{h}_l(n)$ and N operators with the identification,

$$|L, N\rangle = |\sum_{i=1}^{3} \Delta_i h_l(n), N\rangle \qquad (3.15)$$

The equation (3.14) implies that for the physical states we should demand:

$$|\vec{h}_l(n) , N(n)\rangle = |\vec{h}_l(n) + (\vec{\nabla} \times \vec{\Lambda})_{nl} , N(n)\rangle$$

Inverting the relation (3.11) along a fixed line $\vec{\mathfrak{L}}$ in the direction specified by an arbitrary unit vector $\hat{r}$ [Fig. 6],



[Fig. 6]: The Duality Transformation on Lattice. The String $\mathfrak{L}$ can be rotated randomly by the dual gauge transformation.

$$h_l(n) = \hat{r}_1 \sum_{m=0}^{\infty} \ell(n + \hat{r}\, m \;)$$

$$= \sum_{m=0}^{\infty} (a^+ a)\,(n + \hat{r}\, m) + \sum_{m=0}^{\infty} \;(b^+\, b)\,(n + \hat{r}\, m) \tag{3.16}$$

Here m takes integer values.

Let $\chi_l(n)$ be the conjugate variable to $h_l(n)$. To find the relationship between $\chi_l(n)$ and $\theta(n)$, we consider the action of the raising operator on a state $|\, \ell(m)\, ,\, n_c(m) \rangle$

$$e^{i\theta(n)} |\, \ell(m)\, ,\, n_c(m) \rangle = |\, \ell(m) + \delta_{nm}\, ,\, n_c(m) \rangle\, . \tag{3.17}$$

This implies that in the dual description one has to create

$h_l(n)$ quanta along a line starting from the point(n) and going to infinity along an arbitrary direction $\vec{\mathcal{L}}$ [Fig.6]. Therefore,

$$\exp(i\theta(n)) = \prod_{m=0}^{\infty} \exp i [\, \chi_l(n + \hat{r}\, m)] \tag{3.18}$$

Here the subscript l on $\chi(n)$ specifies the direction of $\hat{r}$.

This along with (3.10) gives the creation and the anhilation operators for the spin zero monopoles in the dual description,

$$a = \prod_{m=0}^{\infty} \exp i [\, \chi_l(n + \hat{r}\, m,)] \Big[\epsilon(\Delta_l h_l(n) - 1) \sqrt{((\Delta_l h_l(n) + C^{+}C))} + C\, \epsilon(-\Delta_l h_l(n) + 1\,) \Big] \tag{3.19-a}$$

$$b^{+} = \Big[\epsilon(\Delta_l h_l(n))\, C^{+} + \epsilon(-\Delta_l h_l(n)) \sqrt{(C^{+}C + \Delta_l h_l(n))} \Big] \Big[\prod_{m=0}^{\infty} \exp i [\, \chi_l(n + \hat{r}\, m)] \Big] \tag{3.19-b}$$

Thus the two representations characterized by (N_a, N_b) and $(h_l(n), N)$ are related by non-local transformations. Later in this chapter we show that the non-local matrix elements of the Hamiltonian H describing the dynamics of a spin zero monopole in (N_a, N_b) description turn out to be local in latter representation characterized by $(\, h_l(n),\, N\,)$ operators.

[III.2] DUAL TRANSFORMATION OF VECTOR POTENTIAL

We start with pure U(1) Lattice Hamiltonian

$$H = g^2 \sum_{nl} e_l^2(n) - 1/g^2 \sum_{n,l,j} \cos\left(\theta_l(n) + \theta_j(n+l) - \theta_j(n) - \theta_l(n+j)\right) \tag{3.20}$$

where $\theta_l(n)$ are the angular variables on lattice links and $e_l(n)$ are the corresponding conjugate momenta to be associated with the electric vector potential and the electric field respectively. The argument of the last term

$$\left(\theta_l(n) + \theta_j(n+l) - \theta_j(n) - \theta_l(n+j)\right)$$

is the lattice field strength tensor associated with the plaquette (l,j) at the lattice site (n). (Appendix-A)

The commutation relations are analogous to that of a planar rotor,

$$\left[e_l(n)\ ,\ \theta_j(m)\right] = -i\ \delta_{nm}\ \delta_{lj}\ . \tag{3.21}$$

Thus the electric fields being conjugate to the angular variables have integer eigenvalues.

The naive continuum limit of (3.20) corresponds to the partition function (2.8) with the identification, $A_l(n) = \frac{\theta_l(n)}{a}$

The above Hamiltonian is invariant under the gauge transformation

$$\theta_l(n) \rightarrow \theta_l(n) + \Delta_l \Lambda(n) \qquad (3.22\text{-a})$$

$$E_l(n) \rightarrow E_l(n) \qquad (3.22\text{-b})$$

where $\Lambda(n)$, the gauge parameter is also an angular variable. We use the lattice difference operators defined in the previous section.

The equation (3.21) implies,

$$\left\{\exp - i \sum_{mj}\left[\left(\tilde{\Delta}_j\ e_j(m)\right) \Lambda(m)\right]\right\} \theta_l(n) \left\{\exp\ i \sum_{mj}\left[\left(\tilde{\Delta}_j\ e_j(m)\right) \Lambda(m)\right]\right\}$$

$$= \theta_l(n) + \Delta_l \Lambda(n) \qquad (3.23)$$

Hence $\sum_l \left(\tilde{\Delta}_l\ e_l(n)\right)$ is the generator of the gauge transformation (3.22). Thus the physical states should satisfy the Gauss law constraint

$$\sum_{l} \left(\tilde{\Delta}_l \; e_l(n)\right) | \text{ Physical State } \rangle = 0 \tag{3.24}$$

at every lattice site (n). This constraint should be identified with the equation (2.10) in the corresponding functional integral formulation. On the lattice, it means that the algebraic sum of the electric fields along the links meeting at every vertex (n) is zero.

The Blanchi identity on the lattice is

$$\exp i \sum_{p \subset C} \theta_{ij}(n) \equiv 1 \tag{3.25}$$

Here the sum in the exponent is over the plaquettes belonging to a cube C with the orientations given by the outward normal (Fig 7-a).

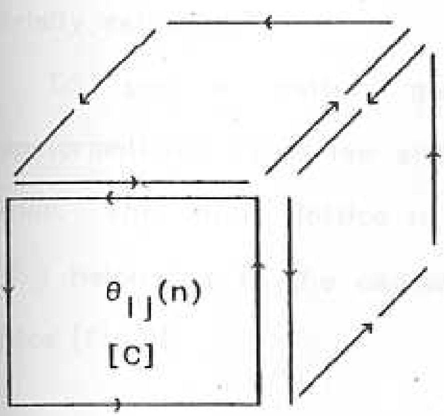


Fig.7-a]: The Bianchi identity and oriented plaquettes belonging to a cube C. It corresponds to the Gauss law of the original theory.

To solve the Gauss law constraint, we associate the "dual antisymmetric plaquette variables" $\ell_{ij}(n)$ to the electric fields $e_i(n)$,

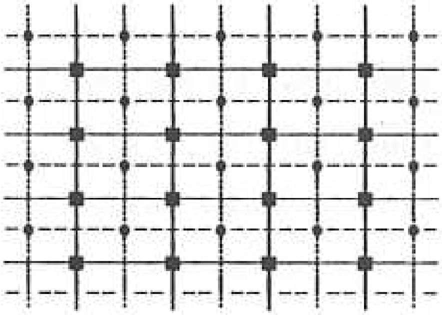
$$e_i(n) = \tilde{\Delta}_j \ell_{ij}(n) \qquad (3.26\text{-a})$$

The equation (3.21) implies that on the gauge invariant states ℓ_{ij} also form a planar rotor algebra with θ_{ij} as it's conjugate variable, i.e.,

$$\left[\theta_{ij}(n), \ell_{kl}(m)\right] = -i\, \delta_{nm}(\delta_{ik}\, \delta_{jl} - \delta_{il}\, \delta_{jk}) \qquad (3.26\text{-b})$$

In terms of these dual variables $\ell_{ij}(n)$ the Gauss law is trivially satisfied.

To get a better geometrical picture of the duality transformations, Gauss law and Bianchi identity, we go to the dual lattice. The dual lattice is defined as : The set of points $\{ \bar{n} \}$ belonging to the centers of the unit cells on the original lattice [Fig.8].



[Fig.8] : The dual lattice in 2 dimensions.

■ Dual lattice sites $(\bar{n})$; • Original lattice sites (n).

The dual variables $\ell_{ij}(n)$ being antisymmetric tensors in three dimensions can be associated with the link vectors $\tilde{\theta}_i(\bar{n})$ on the dual lattice as follows.

$$\ell_{ij}(n) = \epsilon_{ijk} \, \tilde{\theta}_k(\bar{n}) \tag{3.27}$$

Here $(\bar{n})$ denotes the dual lattice sites.

In three dimensions the equation (3.27) associates a dual link variable $\theta_i(\bar{n})$ perpendicular to every plaquette term $\ell_{ij}(n)$ on the original lattice.

The solution of the Gauss law constraint (3.24) now looks like

$$e_i(n) = \epsilon_{ijk} \, \tilde{\Delta}_j \, \tilde{\theta}_k(n) \tag{3.28}$$

This equation defines the magnetic vector potential $\tilde{\theta}_l(n)$ and is the Hamiltonian version of the equation (2.11) in the corresponding path integral formulation. The equation (3.27) implies that on the dual lattice, the original Gauss law corresponds to the Bianchi identity of the dual theory [Fig.7-b].

Thus under the duality transformations the roles of the Gausslaw and the Bianchi identity get interchanged.

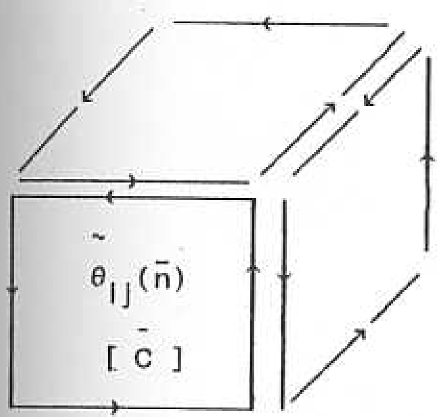


Fig.7-b]: The Bianchi identity of the dual theory corresponds to the Gauss law of the original theory.

In terms of the dual variables the Hamiltonian (3.18) is

$$H = g^2 \left(\tilde{\Delta}_l \, \ell_{ij}(n) \right)^2 + \frac{1}{g^2} \left(\mathcal{L}^{+}_{ij}(n) + \mathcal{L}^{-}_{ij}(n) \right) \qquad (3.29)$$

Here $\mathcal{L}^{\pm}_{ij}(n) = \exp \pm i \left(\theta_{ij}(n) \right)$ are the raising and lowering operators for the planar rotors (ℓ_{ij}).

Compared to (3.20),in the transformed Hamiltonian above, the roles of the terms without coupling and with couplings to the neighbors have interchanged. The coupling constant g has also reversed it's role under the above duality transformations. These are the general features associated with any duality transformation.

[III-3] THE LOCAL DYNAMICS

In this section we describe the lattice Hamiltonian formulation of our theory of dyons discussed in the functional integral language in Chapter II. Essentially we will be combining the ideas discussed in the first two sections of this Chapter.

The Hamiltonian describing the dynamics of a spin zero magnetic monopole φ with photon is

$$H = H_m + H_g + H_{int} \tag{3.30}$$

with

$$H_m = \sum_n \left[|\Pi(n)|^2 + |\varphi(n)|^2 \right] + V(|\varphi|^2) \tag{3.30a}$$

$$H_g = \sum_{n,i} \left(\vec{b}_i(n) \right)^2 - \sum_{n,i,j} \text{Cos}\left[\tilde{\theta}_i(n) + \tilde{\theta}_j(n+i) - \tilde{\theta}_j(n) - \tilde{\theta}_i(n+j) \right] \tag{3.30b}$$

$$H_{int} = x \sum_{n,i} \left(\varphi^*(n)\, U_i(n)\, \varphi(n+i) + \varphi^*(n+i)\, U_i^*(n)\, \varphi(n) \right) \tag{3.30c}$$

Here $\vec{b}_i(n)$ is the integer valued magnetic field with the angular variable $\tilde{\theta}_i(n)$ as it's conjugate momenta which should be identified with the magnetic vector potential. $U_i(n) = \exp i\, g\, \tilde{\theta}_i(n)$ and g is the coupling constant.

In writing down the Hamiltonian (3.30), we have exploited the dual invariance of the Maxwell's theory. It is basically the Q.E.D. action in the presence of spin zero electric charges ϕ with the correspondence

$$\varphi \leftrightarrow \phi \;,\; e_i(n) \leftrightarrow b_i(n) \;\&\; \theta_i(n) \leftrightarrow \tilde{\theta}_i(n) \;.$$

Here $\vec{e}_i$ & $\vec{b}_i$ are the electric and magnetic fields. $\theta_i(n)$, $\tilde{\theta}_i(n)$ are the electric and magnetic vector potentials respectively.

The above Hamiltonian has the local U(1) gauge invariance :

$$\varphi^*(n) \longrightarrow (\exp\, i\, g\, \Lambda(n))\, \varphi^*(n) \tag{3.31a}$$

$$\varphi(n) \longrightarrow (\exp\, i\, g\, \Lambda(n))\, \varphi(n) \tag{3.31b}$$

and

$$U_i(n) \longrightarrow \left(\exp -ig\Lambda(n)\right)\, U_i(n)\, \left(\exp\ ig\ \Lambda(n+i)\right) \tag{3.31c}$$

The Hamiltonian (3.30) corresponds to the path integral

(2.13).

We characterize all the states in the Hilbert space by the eigenvalues of $\vec{b}_i(n)$, N_a and N_b : $|N_a, N_b, \vec{b}_i(n)\rangle$. Out of these, the physical states $|N_a, N_b, \vec{b}_i\rangle_{Physical}$ are those which satisfy the Gauss Law constraint in the presence of matter

$$\sum_i \left(\vec{\Delta}_i \vec{b}_i(n) \right) | N_a , N_b , \vec{b}_i(n) \rangle_{Physical}$$

$$= \left(N_a(n) - N_b(n) \right) | N_a , N_b , \vec{b}_i(n) \rangle_{Physical}$$

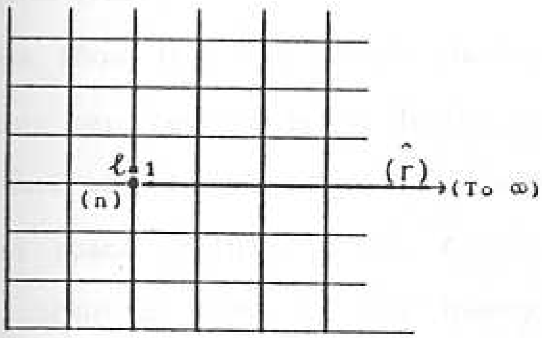
$$\equiv \ell(n) | N_a , N_b , \vec{b}_i(n) \rangle_{Physical} \tag{3.32}$$

at every lattice site (n). Here $N_a - N_b$ is the magnetic charge operator in the units of g and we have used the notations of section (III.1,2).

This Gauss law can be solved non-locally in terms of the electric vector potential $\vec{\ell}_i(n)$,

$$\vec{b}_i(n) = \epsilon_{ijk} \Delta_j \vec{\ell}_k(n) + \hat{r}_i \sum_{m=0}^{\infty} \ell(n+\hat{r}m) \tag{3.33}$$

Here the summation is done along a string starting from the lattice site (n), the location of the magnetic charge, and going to ∞ along any arbitrary direction $\hat{r}$. This is the Dirac string in the Hamiltonian formulation [Fig.9].



[Fig.9]: The Dirac String in the lattice Hamiltonian formulation.

The theory in terms of the electric vector potential $\vec{\ell}_k(n)$ again has local U(1) gauge invariance.

$$\varphi(n) \longrightarrow (\exp\ i\ g\ \lambda(n))\ \varphi(n) \qquad (3.34a)$$

$$\tilde{U}_i(n) \equiv \exp\ i\ (g\ \ell_k(n)) \longrightarrow \Big(\exp\ -ig\ \lambda(n)\Big)\tilde{U}_i(n)\Big((\exp\ +\ ig\ \lambda(n)\Big)$$

(3.34b)

The equation (3.32) clearly implies that the representation $|N_a, N_b, \vec{\ell}_k(n)\rangle$, in the presence of monopoles will lead to a non-local dynamics involving the Dirac string (3.30b,3.33). This was the origin of the numerous problems associated with the previous attempts to construct a theory of monopoles dyons.

In our formulation, instead of the standard $|N_a(n),N_b(n)\rangle$ basis, we chose the dual $|h_i(n),n_c(n)\rangle$ representation for the spin zero monopoles (III.1).

Now we show that the matrix elements of the Hamiltonian (3.30) in this new representation $|h_i(n),n_c(n), \tilde{\ell}_i(n)\rangle$ completely evade the need of the Dirac string and hence the dynamics in this dual Hilbert space $|h_i(n)\ ,\ n_c(n), \vec{\ell}_i(n)\rangle$ is local. Moreover, the naive continuum limit of this theory corresponds to the theory described by (2.18) and hence is manifestly Lorentz covariant.

We redefine the magnetic field (3.33) in terms of the dual variable $\vec{h}_i(n)$ and the electric vector potential $\vec{\ell}_k(n)$ as :

$$\vec{b}_i(n) = \epsilon_{ijk}\, \Delta_j\, \vec{\ell}_k(n) + \vec{h}_i(n) \qquad (3.35)$$

This is the Hamiltonian version of the corresponding path integral equation (2.17) with the identification (3.13)

$$\vec{h}_i(n) \equiv \tilde{H}_{oi} \qquad (3.13)$$

The defining relations (3.35) and (3.11) for the electric vector potential $\vec{\ell}_i(n)$ and the spin zero matter in it's dual form

$h_i(n)$ are invariant under the combined transformations

$$\vec{\ell}_i(n) \longrightarrow \vec{\ell}_i(n) + \vec{\Lambda}_i(n) \tag{3.36a}$$

and

$$\vec{h}_i(n) \longrightarrow \vec{h}_i(n) - \epsilon_{ijk}\, \Delta_j\, \vec{\Lambda}_k(n) . \tag{3.36b}$$

which leave the values of the magnetic field $\vec{b}_i(n)$ and the charge operator $\ell(n)$ invariant.

This is the Hamiltonian version of the "m-type" gauge invariance (2.19a) and (2.19b) associated with the magnetic matter. Therefore all the states $|\vec{h}_i(n), n_a(n), \vec{\ell}_i(n)\rangle$ related by (3.36-a,b) are physically equivalent. The Gauss law constraint for the dual theory is

$$\exp i \left(\sum_i [(\vec{\nabla}\times\vec{\chi}(n))_i + \vec{\theta}_i(n)] \right) = 1 \tag{3.37}$$

As already discussed in the sections [II-A.2,A.3], [III.1,2] and appendix B, the matter and the gauge parts of the Hamiltonian H_m, H_g (3.30-a,b) have local matrix elements in this dual basis[5].

Now we show that the effect of the interaction part of the Hamiltonian (3.30c) is also independent of the Dirac string and hence local in the dual basis $|h_i(n), n_a(n), \vec{\ell}_i(n)\rangle$.

The interaction Hamiltonian in the (N_a,N_b) number basis can be written as

$$H_{int} = x \sum_{n,l} \left\{ a^+(n)\ U_l(n)\ a(n+l) + a^+(n)\ U_l(n)\ b^+(n+l) + \right.$$

$$\left. b(n)\ U_l(n)\ a(n+l) + b(n)\ U_l(n)\ b^+(n+l) \right\} + h.c \qquad (3.38)$$

This interaction Hamiltonian acting on the basis $|N_a(n),N_b(n),b_l(n)\rangle$ basis has the following effects:

Each of the first four terms in (3.38)

a] Increases the magnetic field on the l^{th} link at the lattice site (n) by one unit.

$$\delta\ \vec{b}_l(n) = +1 \qquad (3.39a)$$

b] Increases the magnetic charge $\tilde{\ell}(n)$ at the site (n), by one unit

$$\delta\ \tilde{\ell}(n) = +1\ , \qquad (3.39b)$$

c] decreases the magnetic charge at site (n+l), $\tilde{\ell}(n+l)$, by one unit.

$$\delta\, \tilde{\ell}(n+i) = -1\,, \tag{3.39c}$$

In $|N_a, N_b, \ell_{ia}(n)\rangle$ representation, these local changes are consistent with the Gauss Law constraint (3.32) only if the electric vector potential $\tilde{\ell}_i(n)$ changes non-locally resulting in non-local dynamics (Fig.10).

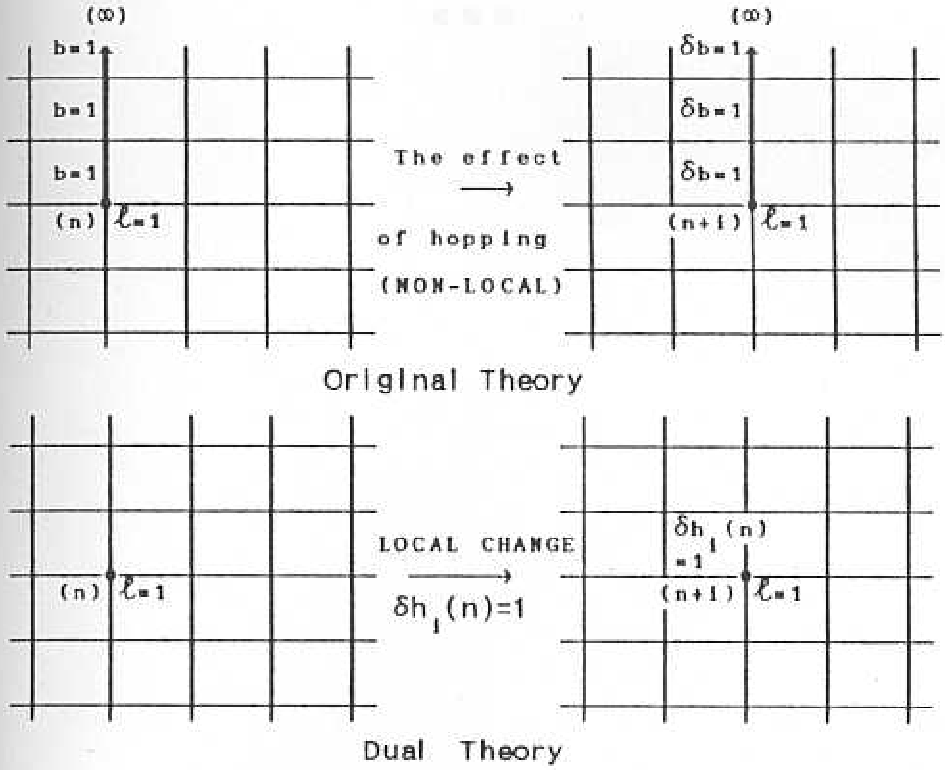


Fig.10] The changes due to the hopping terms in the original and the dual theory.

In the dual representation $|\vec{h}_i(n),\ n_c(n),\ \vec{\ell}_i(n)\rangle$ all the above changes (3.39-a,b,c) can be taken care of by a local change in the dual field $h_i(n)$ on the i^{th} link at the lattice site (n) (Fig.10)

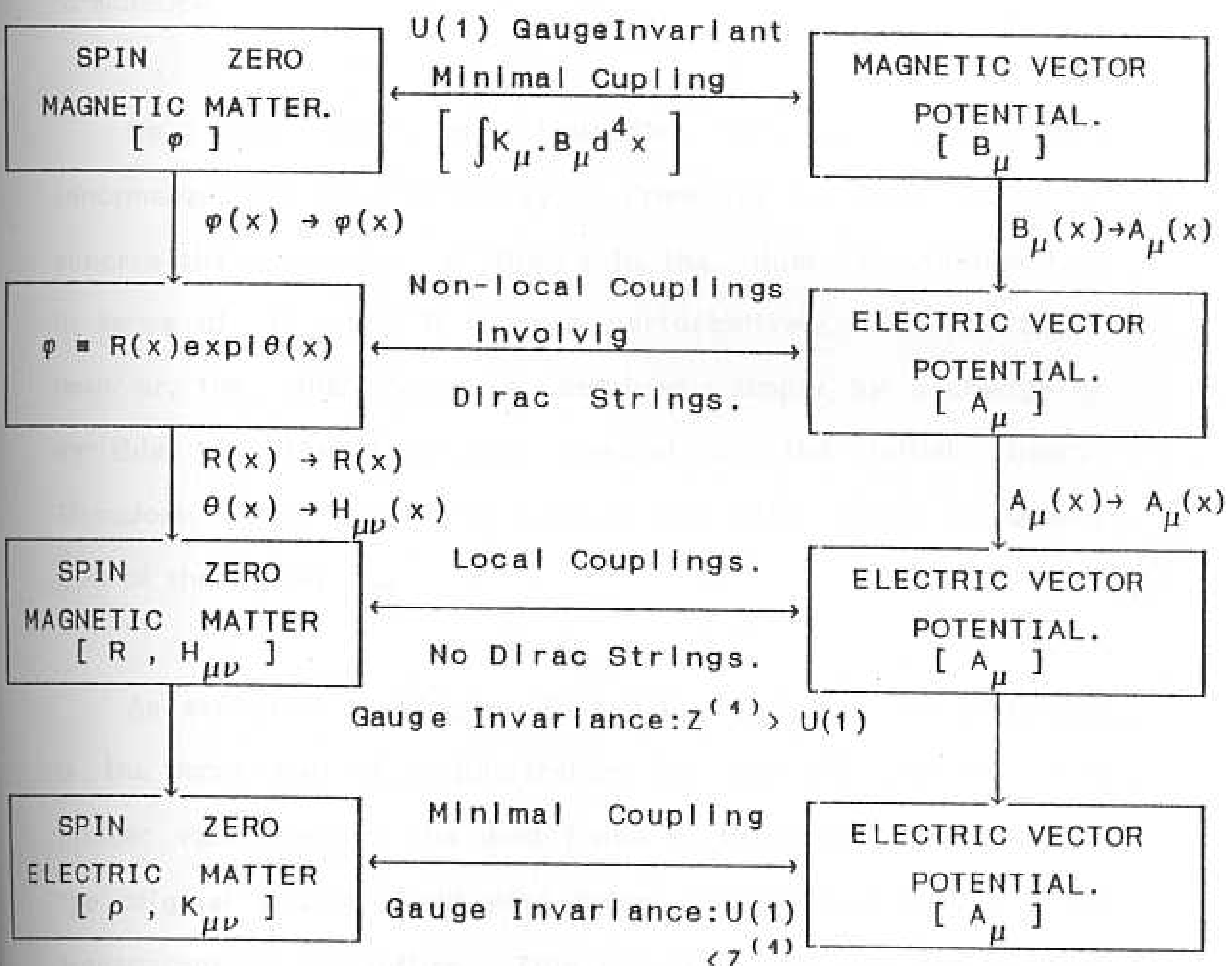
$$\delta\, h_i(n) \;=\; +\,1 \tag{3.40}$$

This change is also compatible with the Gauss law.

■ ■ ■

CHAPTER IV

IV.1] THE BASIC IDEA ON A FLOW CHART :



Invariant Under $Z^{(4)}$ iff

$$e\, g = 2\pi \, (\text{Integer})$$

DISCUSSION

In this section we analyze the various subtle features of our formulation.

We start with the question of the perturbative renormalizability of the theory. From the equation (2.18), it appears that even the φ^4 theory in the dual formulation i.e., in terms of $H_{\mu\nu}$ and R is not perturbatively renormalizable. However, the dual theory is obtained simply by a change of variables in the functional integral of the initial theory. Therefore if the continuum limit of the latter exists so should that of the former ■

As explained in Chapter II & III, the lattice regularization is the most natural regularization for our formulation. The integer valuedness of the dual fields $H_{\mu\nu}(x)$ is a consequence of the original angular field $\theta(x)$ being compact and this is most transparent on the lattice. This aspect is also reflected by the nature of the quantization of charges. In the initial theory ,as a consequence of quantization, the charge operator,

$$Q = e\int d^3x\ (\pi^*\varphi - \pi\ \varphi^*)$$

has only integer eigenvalues in units of e. In the dual theory, the corresponding integer valuedness of the dual (topological) charge operator,

$$\tilde{Q} = e \int d^3x \, \partial_i \tilde{H}_{0i}$$

is seen on the lattice as a consequence of the integer valuedness of $H_{\mu\nu}$ themselves.

Integer valuedness of $H_{\mu\nu}$ lattice fields need not be regarded as unacceptable in field theory. The continuum φ^4 theory, for instance, can be obtained(15) by the Ising model, where the variables take only ± 1 values. The continuum field is rather related to the average value of the lattice field in a block.(15) ■

Non-perturbative studies(15) suggest that QED might not exist as an interacting continuum theory. From this point of view it is important to study this question when magnetic charges are also present. This can drastically alter the critical behavior of the theory[12] ■

It is well known that when both electric and magnetic charge are present, perturbation theory is not of any use. This is a consequence of the Dirac quantization due to which couplings e and

$2\pi/e$ both appear in the action. Our formulation on lattice is amenable to non-perturbatively analysis, for e.g. by Monte-Carlo methods ■

The interaction of the antisymmetric tensor $H_{\mu\nu}$ in our formulation are quite similar to interactions in certain supergravity[6] models. The experience gained here might throw some light on the questions regarding the spectrum, renormalizability, phase structure etc. of these models ■

IV-3] PROBLEMS YET TO BE ADDRESSED :

a] A New Representation For Fermions !

A relatively more interesting problem is to find out the "natural variables" for for spin half particles with magnetic charges. For spin zero particles the results regarding the antisymmetric representation were partly known[9]. For fermions, to our knowledge such representation does not exist in the literature. This new representation might turn out to be useful even outside the context of monopoles and dyons.

b] "A Self-Dual Theory Of Dyons" !

The present theory involves only the electric vector potential A_μ. Therefore, in a sense, we have not treated the electric and magnetic charges on an equal footing. It will be interesting to construct a manifestly Lorentz covariant formulation of Zwanziger's theory[3-d,e] involving electric as well as magnetic vector potentials. Such a theory, unlike the present one, will be manifestly invariant and hence self dual under the duality transformations [section (II-A.1)] which leave the Maxwell's equations invariant. This formulation will have the following two additional advantages :

a] One may expect that it suffices to use φ and φ^* fields, which are directly related to the spectrum of the theory, for describing both electric and magnetic charges.

b] It will be possible to have a manifestly Lorentz covariant and local formulation of particle mechanics involving both electric and magnetic charges without the Dirac string at the level of dynamics.

Moreover, in such a formulation, it will be extremely interesting to analyze the origin of the Dirac quantization condition.

IV.4] CONCLUSION WITH A SIMPLE ANALOGY

We conclude that the natural variables to describe the dynamics of spin zero monopoles are the antisymmetric tensor field $H_{\mu\nu}(x)$ along with the radial field R(x) and not the conventional fields ϕ, ϕ^* describing the physical degrees of freedom of monopoles and anti-monopoles. Thus by introducing some redundant degrees of freedom in the dynamics, we are able to formulate a manifestly Lorentz covariant, local gauge theory of monopoles / dyons.

This situation is exactly analogous to a theory of (say) free photons written just in terms two physical transverse polarizations. In such a theory , one is bound to encounter precisely the same problems which were present in the earlier theories of monopoles. By giving two extra components to photons we not only recover manifestly Lorentz covariant & local interactions, but also get a U(1) gauge invariant theory of photons. An important difference is that in the case of monopoles the $Z^{(4)}$ gauge invariance is truly a quantum invariance . It is the invariance of the partition function and not of the action, thus implying Dirac quantization

APPENDIX - A

U(1) GAUGE THEORY ON LATTICE :

In this appendix we give a brief introduction of lattice formulation of U(1) gauge theory[8]. We start with a discussion on the non-compact version of the lattice Q.E.D. and then go over to describe the corresponding compact formulation.

The most direct procedure for going from continuum to lattice formulation is to replace derivative in the action by difference operators as was done in the section II ,i.e, $\partial_\mu \to \Delta_\mu$. Here Δ_μ is the lattice difference operator defined in the section *(III.1)* The range of the photon field is $[-\infty,+\infty]$ as in the continuum fomulation. The term describing the interaction of the photon with the (spin zero) matter field $\varphi(n)$ is

$$\left[\ \varphi^*(n)\left(\exp\ i\ e\ a_\mu(n)\right)\varphi(n+\mu) + \text{h.c}\ \right] \tag{A-1}$$

In the naive continuum limit this coupling gives the standard minimal coupling $(D_\mu\varphi)^*(\ D_\mu\varphi)$. Here D_μ $(= \partial_\mu - ieA_\mu)$ is the covariant derivative with the identification $A_\mu \equiv a_\mu(n)/a$ and a is the lattice spacing. In this non-compact formulation, unlike the kinetic energy term for the photon,the matter coupling (A.1) is sensitive to only eA_μ modulo 2π. This formulation has non-compact gauge invariance,

$$a_\mu(n) \longrightarrow a_\mu(n) + \Delta_\mu \Lambda(n) \tag{A-2a}$$

$$\varphi(n) \longrightarrow \exp i[e \Lambda (n)] \varphi(n) \tag{A-2b}$$

where $\Lambda(n)$ is the gauge parameter defined on the lattice sites (n) with the range $[-\infty,+\infty]$ and Δ_μ is the lattice difference operator in the $(\mu)^{th}$ direction. This version of the lattice Q.E.D. has been exploited in our formulation.

However in the non-abelian gauge theories, because of the modified Liebnitz rule on the lattice, this procedure gives rise to a Lagrangian density which is not gauge invariant. The lattice action which respects the local gauge invariance and has the correct continuum limit was constructed by Wilson[8-b]. This construction for the U(1) case is described below.

We define link variables.

$$U_\mu(n) \equiv \exp i (\theta_\mu(n) \tag{A-3}$$

where $U_\mu(n)$ and $\theta_\mu(n)$ are defined on lattice links of unit length with the identification $\theta_\mu(n) \equiv - \theta_{-\mu}(n+\mu)$ and $\theta_\mu(n)$ being the angular variable $\theta_\mu(n) \equiv \theta_\mu(n) + 2\pi$. The action is defined by

$$S = \frac{1}{e^2} \sum_{(n,\mu,\nu)} \mathrm{Re}\left[1 - U_\mu(n)\, U_\nu(n+\mu) \left(U_\mu(n+\nu) \right)^{-1} \left(U_\nu(n) \right)^{-1}\right] \tag{A-4}$$

The product of four $U_\mu(n)$ link variables in [A-2] can be thought of as circulating in a counter - clockwise sense around a plaquette [Fig.11].

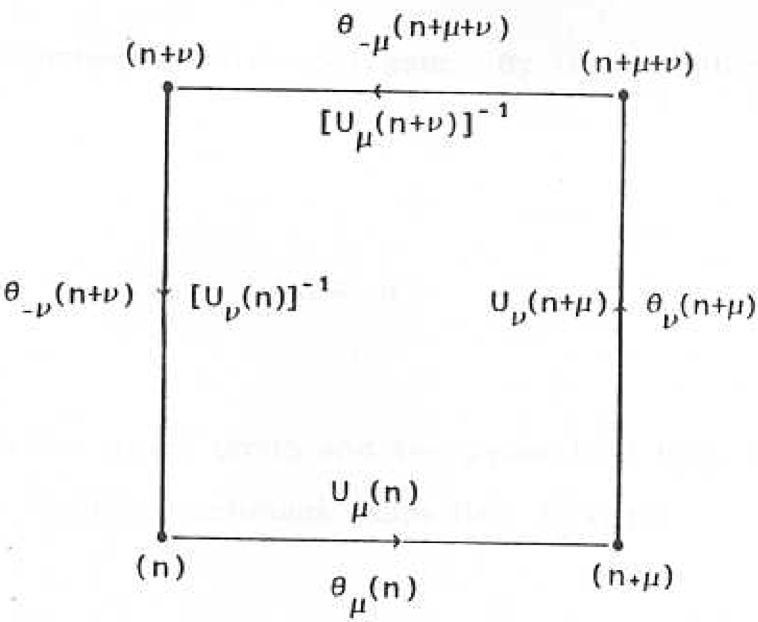


Fig.11]: An oriented plaquette on the lattice and the Q.E.D. plaquette action.

The gauge transformations can be directly written in terms of the link variables,

$$U_\mu(n) \rightarrow V(n)\, U_\mu(n)\, (V(n+\mu))^{-1} \tag{A-5}$$

The gauge transformation functions V(n) are defined at the lattice sites and they are elements of the group U(1) i.e.

$$\left(V(n)\right)^{-1} = V^{+}(n) \tag{A-6}$$

Here + denotes Hermitian conjugate. By it's definition it is clear that

$$-\pi \leq \theta_\mu(n) \leq \pi \tag{A-7}$$

Now both the gauge group and the gauge field $\theta_\mu(n)$ have become compact. Defining continuum gauge field $A_\mu(x)$ by

$$A_\mu(x) = \frac{\theta_\mu(n)}{a} \tag{A-8}$$

where x = na. The range of $A_\mu(x)$ is

$$- \pi / a \leq A_\mu(x) \leq \pi / a \qquad \text{(A-9)}$$

Hence in the continuum limit $a \longrightarrow 0$ it's range becomes infinite and the action simplifies to

$$S = 1 / 4 e^2 \int \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right)^2 d^4x$$

corresponding to standard quantum electrodynamics action in the continuum.The lattice Q.E.D action (A-2) can also be written as

$$S = (1 / e^2) \sum_{\substack{(n,\mu,\nu) \\ \mu<\nu}} \left[1 - \cos \theta_{\mu\nu}(n) \right] \qquad \text{(A-10)}$$

Here $\theta_{\mu\nu}(n) = \Delta_\mu \theta_\nu(n) - \Delta_\nu \theta_\mu(n)$ is defined over a plaqutte $(\mu\nu)$. We will come back to this form of the Q.E.D action in Appendix C.

So, in the compact formulation of the lattice Q.E.D.,the photon and the gauge fields both being compact, all the charges get quantized with respect to each other. In contrast in the non-compact formulation, any arbitrary values of the electric charges are allowed.

APPENDIX - B

THE DUAL REPRESENTATIONS OF SPIN ZERO MATTER

In this appendix we summarize the work of Deser & Witten[9] on the dynamical content of the antisymmetric representation of a real massless scalar field φ. Their motivation to study this equivalence was quite different from ours. It was regarding the gauge hierarchy problem in the grand unified theories. In the end of this appendix we briefly discuss their motivation.

A] MASSLESS CASE

We start with the action of a free massless antisymmetric tensor field $H_{\mu\nu}$.

$$S = \int d^4x \left(\partial_\nu \tilde{H}_{\mu\nu}\right)^2 . \qquad \text{(B-1)}$$

This action has gauge invariance.

$$H_{\mu\nu} \rightarrow H_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \qquad \text{(B-2)}$$

Defining "electric fields" and "magnetic fields" by $\vec{\mathcal{B}}_i = \tilde{H}_{oi}$ & $\vec{\mathcal{E}}_i = H_{oi}$ respectively and after some algebra we can rewrite (B-1) as

$$S = \int d^4x \left[\left(\vec{\nabla} . \vec{\mathcal{B}}\right)^2 + \left(\dot{\vec{\mathcal{B}}} - \vec{\nabla} \times \vec{\mathcal{E}}\right)^2\right] \quad . \tag{B-3}$$

We decompose $\vec{\mathcal{E}}$ & $\vec{\mathcal{B}}$ into longitudinal and transverse parts.

$$\vec{\mathcal{E}} = \vec{\mathcal{E}}_L + \vec{\mathcal{E}}_T \ , \quad \vec{\mathcal{B}} = \vec{\mathcal{B}}_L + \vec{\mathcal{B}}_T \tag{B-4}$$

with

$$\vec{\mathcal{E}}_L = \vec{\nabla}\left[(\vec{\nabla}^2)^{-1}\ \vec{\nabla}.\vec{\mathcal{E}}\right] \qquad \vec{\mathcal{E}}_T = \vec{\nabla} \times \left[(\vec{\nabla}^2)^{-1}\ (\vec{\nabla} \times \vec{\mathcal{E}})\right]$$

$$\tag{B-5}$$

$$\vec{\mathcal{B}}_L = \vec{\nabla}\left[(\vec{\nabla}^2)^{-1}\ \vec{\nabla}.\vec{\mathcal{B}}\right] \qquad \vec{\mathcal{B}}_T = \vec{\nabla} \times \left[(\vec{\nabla}^2)^{-1}\ (\vec{\nabla} \times \vec{\mathcal{B}})\right]$$

Using the fact that $\int d^4x\ \vec{\mathcal{B}}_L . \vec{\mathcal{B}}_T = 0$, we can rewrite (B-2) as

$$S = \int d^4x \left[\left\{-\left(\vec{\nabla} . \vec{\mathcal{B}}_L\right)^2 + \left(\dot{\vec{\mathcal{B}}}_L\right)^2\right\} + \left\{\left(\dot{\vec{\mathcal{B}}}_T\right) - \left(\vec{\nabla} \times \vec{\mathcal{E}}_T\right)\right\}^2\right] \tag{B-6}$$

Hence the longitudinal part of electric field $\vec{\mathcal{E}}_L$ decouples from the dynamics completely. (In the presence of gauge couplings it can be gauged away using one degree of freedom of Λ_μ). From (B-6) it is clear that the transverse part of electric field $\vec{\mathcal{E}}_T$ does not have any dynamical content and can be eliminated in terms of $\vec{\mathcal{B}}_T$ and $\vec{\mathcal{B}}_T$ in turn can be gauged away using the transverse part of Λ_μ. This way we are just left with only one longitudinal degree of freedom of the magnetic field $\vec{\mathcal{B}}$ and the gauge parameter Λ_μ. The later is used to eliminate the longitudinal polarization of photon. Identifying

$$\vec{\mathcal{B}}_L = \vec{\nabla} (\vec{\nabla}^2)^{-1} \varphi \quad . \tag{B-7}$$

We recover the free scalar field action.

$$S = \int \partial_\mu \varphi \, \partial_\mu \varphi \, d^4x \; . \tag{B-8}$$

The motivation of Deser & Witten to study this equivalence was a search for a formalism for spin zero field in which bare mass would be impossible leading to a possible solution of gauge hierarchy problem. It is clear that the gauge invariance of the dual theory [B-2] prohibits such mass terms for real spin zero matter. For complex spin zero matter in the dual theory the mass term for scalar field is $m^2 R^2$ where R is the radial degree of freedom for the scalar field ϕ. This mass term is not protected by any symmetry of the theory.Therefore by the dual description, one can not protect the masses of the complex spin zero matter.

B] MASSIVE CASE

In this section we explicitly demonstrate the equivalence of massive antisymmetric theory with massive spin one theory. This coresspondence was used in our formulation (chapter-2) to show the equivalence of the spectrums of the initial and the final (dual) theories in the broken Higg's phase.

In the presence of a mass term for the antisymmetric tensor field $H_{\mu\nu}$ we have

$$S = \int d^4x \left\{ \left(\partial_\nu \tilde{H}_{\mu\nu} \right)^2 + \beta H^2_{\mu\nu} \right\} \qquad \text{(B-9)}$$

Here β is the mass parameter. Again using B(5) it can be rewritten as

$$S = \int d^4x \left\{ \left[\left\{ -\left(\vec{\nabla} \cdot \vec{\mathcal{B}}_L \right)^2 + \left(\dot{\vec{\mathcal{B}}}_L \right)^2 + \beta \left(\vec{\mathcal{B}}_L \right)^2 \right\} - \beta \left(\vec{\mathcal{E}}_L \right)^2 \right] + \left[\left\{ \left(\dot{\vec{\mathcal{B}}}_T \right) - \left(\vec{\nabla} \times \vec{\mathcal{E}}_T \right) \right\}^2 + \beta \left(\vec{\mathcal{B}}_T \right)^2 - \beta \left(\vec{\mathcal{E}}_T \right)^2 \right] \right\}$$

(B-10)

Here $\vec{\mathcal{E}}_L$ and $\vec{\mathcal{E}}_T$ being auxiliary fields, can be eliminated in terms of the magnetic field $\vec{\mathcal{B}}$.

$$\vec{\mathcal{E}}_L = 0 \tag{B-11}$$

$$\vec{\mathcal{E}}_T = -\left(\nabla^2 + \beta^2 \right)^{-1} \left(\vec{\nabla} \times \dot{\vec{\mathcal{B}}}_T \right) \tag{B-12}$$

Using the identity

$$\int d^4x \; \vec{\mathcal{B}}_T \cdot \left(\vec{\nabla} \times \dot{\vec{\mathcal{B}}}_T \right) = 0 \tag{B-13}$$

and after some algebra

$$S = \int d^4x \left\{ \left[\left(\dot{\vec{\mathcal{B}}}_L \right)^2 - \left(\vec{\nabla}.\vec{\mathcal{B}}_L \right)^2 + \beta \left(\vec{\mathcal{B}}_L \right)^2 \right] \right.$$

$$\left. + \quad \beta \left[\vec{\mathcal{B}}_T^2 + \dot{\vec{\mathcal{B}}}_T \left(\vec{\nabla}^2 + \beta \right)^{-1} \dot{\vec{\mathcal{B}}}_T \right] \right\} \tag{B-14}$$

This corresponds in general to three massive degrees of freedom (ε_L, $\vec{\varepsilon}_T$)defined according to

$$\vec{\mathcal{B}}_L \equiv \vec{\nabla}\left(- \vec{\nabla}^2 \right)^{-1/2} \varepsilon_L \tag{B-15}$$

$$\vec{\mathcal{B}}_T \equiv \left(1 + \frac{\nabla^2}{\beta} \right)^{1/2} \vec{\varepsilon}_T \tag{B-16}$$

with

$$\left(\text{Mass} \right)^2 \equiv m^2 = \beta \tag{B-17}$$

$$S = \int d^4x \left[\dot{\varepsilon}_L^2 + \varepsilon_L \left(\nabla^2 - m^2 \right) \varepsilon_L \right]$$

$$+ \left[\dot{\vec{\varepsilon}}_T^2 + \vec{\varepsilon}_T \left(\nabla^2 - m^2 \right) \vec{\varepsilon}_T \right] \tag{B-18}$$

This is the action for massive spin one particle.

APPENDIX - C

ON LATTICE MONOPOLES

Magnetic monopoles also arise in the context of pure compact U(1) lattice gauge theories. They represent certain topological degrees of freedom, which show up because of the compact nature of the dynamical variables. These degrees of freedom are known to play a crucial role for the confinement mechanism in U(1) gauge theories and can be extracted out by making certain "duality transformations".[8,10,11] In pure U(1) lattice gauge theories they lead to a confining phase above a critical coupling[10,11]. In this appendix we briefly discuss these duality transformations in the context of pure lattice Q.E.D and show that the kinematical aspects of these lattice monopoles are quite similar to those of spin zero monopoles we have talked about in the thesis.

As mentioned earlier lattice monopoles arise due to periodicity of dynamical variables. Hence they do not have any dynamical content. To see the origin of these topological degrees of freedom, we start with the partition function of pure compact U(1) lattice gauge theory (A-10),

$$Z = \int_{0}^{2\pi} \prod_{n,\mu} d\theta_{\mu}(n) \exp - S_{q.e.d} \tag{C-1}$$

with

$$S_{q.e.d} = 1/e^2 \sum_{\substack{n,\mu,\nu \\ \mu < \nu}} (1-\cos\theta_{\mu\nu}(n)) \tag{C-2}$$

$\theta_{\mu\nu}(n)$ is defined over a plaquette and e is the electric charge and $\theta_{\mu\nu}(n)$ is the plaquette term in the action defined in the Appendix (A).

The interactions described by (C-1) are non-polynomial in the nature and therefore extremely difficult to handle. The exponential in (C-1) being a periodic function of the plaquette variable $\theta_{\mu\nu}(n)$, can be approximated by a Gaussian functional integral,

$$Z = \int \prod_{n,\mu} d\theta_{\mu}(n) \sum_{h_{\mu\nu}(n)=-\infty}^{+\infty} \exp -1/2\, e^2 \sum_{\substack{n,\mu,\nu \\ \mu < \nu}} \left[\theta_{\mu\nu}(n) - 2\pi\, h_{\mu\nu}(n)\right]^2$$

(C-3)

Here $h_{\mu\nu}(n)$ are the integer valued antisymmetric tensor fields.

The partition function (C-3) retains the periodicity property $\left[\theta_{\mu}(n) \equiv \theta_{\mu}(n) + 2\pi \ \ Z, Z \in \text{Integers}\right]$ and invariances of compact Q.E.D. described by (C-1,2) and we can take the range of $\theta_{\mu}(n)$ in (C-3) to be non-compact $[-\infty,+\infty]$. Moreover both the models have similar Gaussian behaviour around their Minima $\theta_{\mu\nu}(n) = 2\pi\, Z$. Therefore we may expect this approximated Gaussian model is in the same universality class as the original one and to have the same physical content near the critical points. This defines the Villain approximation for lattice Q.E.D..

The partition function in this Villain / Gaussian form (C-3) clearly shows the origin and the significance of the $h_{\mu\nu}$ degrees of freedom. Their role is to retain the periodicity properties of the dynamical variable $\theta_{\mu}(n)$ in the corresponding Gaussian model (C-3)which is much easier to solve.

These degrees of freedom also show that the pure compact Q.E.D described by (C-1,2) is an interacting theory and not a free theory of photons as indicated by its naive continuum limit. In fact the partition function (C-3) is quite similar to the action of our theory (2.26). This shows that $h_{\mu\nu}$ fields are associated with monopole degrees of freedom with magnetic charge (1/e). Hence compact Q.E.D. described by (C-1,2) is a theory of monopoles interacting via photons. As mentioned earlier, it is these degrees of freedom which become relevant above a critical coupling e_c leading to a confining phase in compact Q.E.D.[10,11]

These lattice monopoles differ from the ones described in this thesis in the sense that they do not have the kinetic energy term in the action.

The Maxwell equations from (C-3):

$$\Delta_\nu F_{\mu\nu}(n) = 0$$

$$\Delta_\nu \tilde{F}_{\mu\nu}(n) = 2\pi/e\ \Delta_\nu \tilde{h}_{\mu\nu}(n) \equiv (2\pi/e)\ K_\mu(n)\ .$$

imply that the Dirac quantization condition for lattice monopoles automatically follows.

APPENDIX - D

THE MONOPOLE - ELECTRIC CHARGE INTERACTIONS

AND

GEOMETRICAL INTERPRETATION

In this section we demonstrate that in our theory the magnetic monopole electric charge interactions have an elegant geometrical interpretation, which again leads to Dirac quantization condition.

In the absence of θ term integration over photon in (2.18) gives

$$Z = \sum_{K_\mu J_\mu} \exp \Bigg[1/2\ g^2 \sum_{n,n} K_\mu(R)\ G(R-R')\ K_\mu(R')$$

$$- \frac{1}{2} \left(\frac{2\Pi . s}{g_o} \right) \sum_{RR'} J_\mu(R)\ G(R-R')\ J_\mu(R')$$

$$+ \ Is \sum_{R,R'} K_\mu(R)\ \theta_{\mu\nu}(R-R')\ J_\nu(R') \Bigg] \tag{D-1}$$

Here we have ignored all self interactions of electric and magnetic matter and considered only terms involving their couplings to photon. (s) is a scale in the theory which fixes the magnitude of minimum electric charge (e_o) in terms of minimum magnetic charge (g_o) i.e.

$$e_o = \left(\frac{2\pi}{g_o}\right)s \quad , \quad e_q = n_q \; e_o \; ,$$

$$g_p = m_p \; g_0 \qquad (n_q, \; m_p, s \in \text{Integers})$$

(D-2)

$G(|R'-R'|)$ is the Green function of the Euclidean Laplacian operator in four dimensions and is equal to $\left(\frac{1}{|R-R'|^2}\right)$

$$K_\mu(R) = \sum_{p=1}^{M} m_p \; \partial_\nu \; \tilde{H}^p_{\mu\nu}(R) \qquad \text{(D-3)}$$

$$J_\mu(R) = \sum_{q=1}^{M} n_p \; \partial_\nu \; \tilde{K}^q_{\mu\nu}(R) \qquad \text{(D-4)}$$

$$\theta_{\mu\nu} \; (R-R') = 2\pi \; \epsilon_{\mu\nu\lambda\sigma} \; u_\lambda (u.\partial)^{-1} \; \partial_\sigma \; G\;(|R-R'|) \qquad \text{(D-5)}$$

Here u_μ is an arbitrary four vector to be identified with Dirac string. $(u.\partial)^{-1}$ is the kernel of $(u.\partial)$ operator and we choose it to be

$$(u.\partial)^{-1} = \frac{1}{2}\int_0^\infty ds\left[\delta^4(R-us)-\delta^4(R+us)\right] \qquad (D-6)$$

The first two terms in (D-1) are the standard coulomb interactions of electric and magnetic charges among themselves. The last term describes the non-local interaction between electric and magnetic charges. The geometrical interpretation of this term is transparent if we replace the conserved currents K_μ & J_μ by integrals over closed particle loops

$$K_\mu(R) = \sum_p m_p \oint_p dx^\mu \, \delta^4(x-R) \qquad (D-7)$$

$$J_\mu(R) = \sum_q n^q \oint_q dy^\mu \, \delta^4(y-R) \qquad (D-8)$$

With this (D-1) can be written as

$$Z = \sum_{\text{loops}} \exp\left[-\frac{1}{2} g_o^2 m_p m_{\bar{p}} \oint_p dx^\mu \oint_{\bar{p}} d\bar{x}^\mu G(x-\bar{x}) - \frac{1}{2}\left(\frac{2\pi}{g_0}s\right)^2 n_q n_{\bar{q}} \oint_q dy^\mu \oint_{\bar{q}} d\bar{y}^\mu G(y-\bar{y})\right]$$

$$\times\left(\exp 2\pi i \sum_{p,q} m_p n_q \Omega_{pq}\right) \qquad (D-9)$$

Here

$$\Omega_{pq} = \oint_p dx^\mu \oint_q dy^\nu \, \epsilon_{\mu\nu\lambda\sigma} \, u_\lambda (u.\partial)^{-1} \; \partial_\sigma \, G(x-y) \qquad \text{(D-10)}$$

Using Stoke's theorem and identities of the ϵ symbols

$$\Omega_{pq} = \int_p d\sigma^{\mu\rho} \oint dy^\mu \left[u_\rho (u.\partial)^{-1} \, \delta^4(x-y) + \partial_\rho \, G(x-y) \right] \qquad \text{(D-11)}$$

Here $\sigma^{\mu\rho}$ is the surface enclosed by the loop (p) and we have used

$$\partial^2 G(x-y) = \delta^4(x-y).$$

This effective form of our theory after integration over photon and replacing field currents by loop currents, exactly matches with the effective action of Zwanziger formulation. This gives the indication that the physical content of these two, a priori completely different looking theories, may not be different. So from here onwards for the sake of completeness we will be essentially repeating the work of Zwanziger[3-d,e] which led him to Dirac quantization condition.

From (D-11) we find that all the string dependence is hidden in the first term. Using the kernel (D-6) for $(u.\partial)^{-1}$ operator, this term can be explicitly written as

$$\frac{1}{2}\int u^{\nu}ds \int_{p} d\tilde{\sigma}^{(x)}_{\mu\nu} \oint_{q} dy^{\mu} \left[\delta^4(x-y-us) - \delta^4(x-y+us)\right]$$

This term has a value $Z(u)/2$ where $Z(u)$ is a topological winding number that counts the number of times the cylinder along u erected on the loop and intersects the surface bounded by the loop p. Using the fact that $\Omega_{pq} = -\Omega_{qp}$, we get the charge quantization condition

$$\left[e_q g_p - e_p g_q = 4\pi \times \text{integer}\right] \qquad \text{(D-12)}$$

Instead of the symmetrical string (D-6), if we had chosen to work with an asymmetrical string, we would have got (2π) in the above quantization condition. This arbitrariness is uniquely fixed in our formulation as a consequence of the gauge invariance.

As mentioned in Chapter (2), the presence of θ term modifies the electric current J_μ to $J_\mu + \theta/2\pi\, K_\mu$. So the θ term does not affect the last term in (D-1) and hence, unlike our formulation, this procedure for obtaining the quantization condition is insensitive to the θ term. In fact, the gauge invariance (2.19) and (2.25), in the absence of the θ term, puts more severe constraint on the electric and magnetic charges in our theory (2.27) compared to Zwanziger's formulation (D-12).

APPENDIX - E

YET ANOTHER DUALITY IN OUR DUAL THEORY

In this appendix we show that in the Higg's phase where the radial degrees of freedom of electric and magnetic matter are frozen and large, our theory reduces to Z(s) gauge theory . Here (s) is the scale which fixes the magnitude of the minimum electric charge in the unit of ($2\pi/g_o$) (D-2). The phase structure of Z(s) lattice gauge theories has been extensively studied by Cardy et al[12]. They show that the invariances of this theory (including self-duality) completely fix it's phase structure.For the sake of completeness, we will briefly summarize their results in this appendix.

In the Higg's phase mentioned above, ignoring the radial degrees of freedoms of the electric and magnetic matter(2.26)

$$Z = \int da_\mu(n) \sum_{[h_{\mu\nu}, k_{\mu\nu}]} \exp - \left\{ \sum_{n,\mu,\nu} \frac{1}{4} \Big(\Delta_\mu a_\nu(n) - \Delta_\nu a_\mu(n) + g_o h_{\mu\nu}(n)\Big) \right.$$

$$\left. + \, i \sum_{n,\mu,\nu} \left(\frac{2\pi}{g_o}\right)(s) \; a_\mu(n)\Big(\Delta_\nu \; \tilde{k}_{\mu\nu}(n)\Big)\right\} \qquad \text{(E-1)}$$

with

$$h_{\mu\nu}(n) = \sum_{p} m^{p}\ h^{p}_{\mu\nu}(n)$$

$$k_{\mu\nu}(n) = \sum_{q} n^{q}\ k^{q}_{\mu\nu}(n) \qquad \text{(E-2)}$$

Here m_p and n_q are integers denoting sites on the electric and magnetic charge lattice [Fig -5]. Rescaling a_μ by $2\pi/g_o$, we can rewrite [E-1] as

$$Z = \int da_\mu(n) \sum_{\substack{h_{\mu\nu} \\ k_{\mu\nu}}} \exp - \left\{ \sum_{n,\mu,\nu} \frac{g_o^2}{16\pi^2} \left(\Delta_\mu a_\nu(n) - \Delta_\nu a_\mu(n) + 2\pi\ h_{\mu\nu}(n)\right)^2 \right.$$

$$\left. + i\, s \sum_{n,\mu,\nu} a_\mu(n) \left(\Delta_\nu\ \tilde{k}_{\mu\nu}(n)\right) \right\} \qquad \text{(E-3)}$$

In this functional integral the integer valued electric current $\Delta_\nu \tilde{k}_{\mu\nu}$ being traced over, the effect of the last term is to discretize $a_\mu(n)$ in the units of $2\pi/s$. Hence in this particular Higg's phase our theory reduces to Z(s) gauge theory.

The effective theory (in the presence of θ term) after integration over photon can be written in a compact notation.

$$Z = \sum_{(J_\mu, K_\mu)} \exp\left[-\frac{2\pi\ S}{(\xi + \xi^*)} \left\{ \xi.\xi^* \ K_\mu(n)\ G(|n-n|)\ K_\mu(n') \right.\right.$$

$$\left. + J_\mu(n)\ G(|n-n'|)\ J_\mu(n') + i(\xi^* - \xi)\ J_\mu(n)\ G(|n-n'|)\ K_\mu(n') \right\}$$

$$\left. + i\ S\ K_\mu(n)\ \theta_{\mu\nu}(|n-n'|)\ J_\nu(n') \right] \qquad \text{(E-4)}$$

Here $G(|n-n'|)$is the Green function of the Euclidean Laplacian operator.J_μ and K_μ are the electric and magnetic currents respectively.

$$\xi = \frac{g_0^2}{2\pi S} + \frac{i\theta}{2\pi}$$

$$\xi^* = \frac{g_0^2}{2\pi S} + \frac{i\theta}{2\pi} \qquad \text{(E-5)}$$

or inverting these relations

$$g_o^2 = n\pi\ (\xi + \xi^*)\ \&\ \theta = l\pi\ (\xi^* - \xi\ ) \qquad \text{(E-6)}$$

The effective action (E-4) has the following invariances

A] <u>Duality [D]</u> :

$$\begin{aligned} \xi &\rightarrow \xi^{-1} \\ K_\mu &\rightarrow J_\mu \\ J_\mu &\rightarrow -K_\mu \end{aligned} \qquad \text{(E-7)}$$

B] <u>Periodicity [Δ]</u> :

$$\begin{aligned} \xi &\rightarrow \xi + l \\ K_\mu &\rightarrow K_\mu \\ J_\mu &\rightarrow J_\mu - K_\mu \end{aligned} \qquad \text{(E-8)}$$

C] <u>Time Reversal [T]</u> :

$$\begin{aligned} \xi &\rightarrow \xi^* \\ K_\mu &\rightarrow K_\mu \\ J_\mu &\rightarrow -J_\mu \end{aligned} \qquad \text{(E-9)}$$

Under all these symmetry transformations the theory gets mapped into itself. Hence the theory is self dual under [E-7], [E-8] & [E-9].These invariances completely fix the phase structure of the theory[12-b].

E.b] <u>The Phase Diagram</u>

As shown by Cardy Z(s) theories have a very rich phase structure depending on the values of (s), (g_o) & θ. This structure can be completely determined using the above symmetry transformations. Instead we will briefly give Cardy's free energy arguments[12-a] here to give a qualitative picture of the phase structure. This latter approach although approximate gives an intuitive understanding of the various phases of Z(s) gauge theory and will be useful even when some of the symmetries mentioned in the previous section are not exact.

We consider a large loop of length L, carrying electric and magnetic charge $\left(N + \frac{\theta}{2\pi} M, M\right)$. Approximating it's energy by self energy in (E-4) i.e. $n = n'$.

$$\text{Energy} = \left[g_o^2 M^2 + \frac{1}{2} s^2 \left(\frac{2\pi}{g_o}\right)^2 \left(N + \frac{\theta}{2\pi} M\right)^2\right] G(o)L \qquad \text{(E-10)}$$

On the other hand, it's entropy is roughly L $\ell n(7)$, since at each step the loop can choose seven different directions. The loop will condense in the vacuum provided it's free energy is negative i.e.

$$\left[\frac{M^2}{T} + \left(N + \frac{\theta}{2\pi} M\right)^2 T\right] s < C \qquad \text{(E-11)}$$

Here

$$T = \frac{2\pi s}{g_0^2} \quad \& \quad C = \frac{\ell n 7}{\pi G(o)}$$

The criterion [E-11] defines the interior of an ellipse in the plane of electric charge and magnetic charge coupling Fig[12].

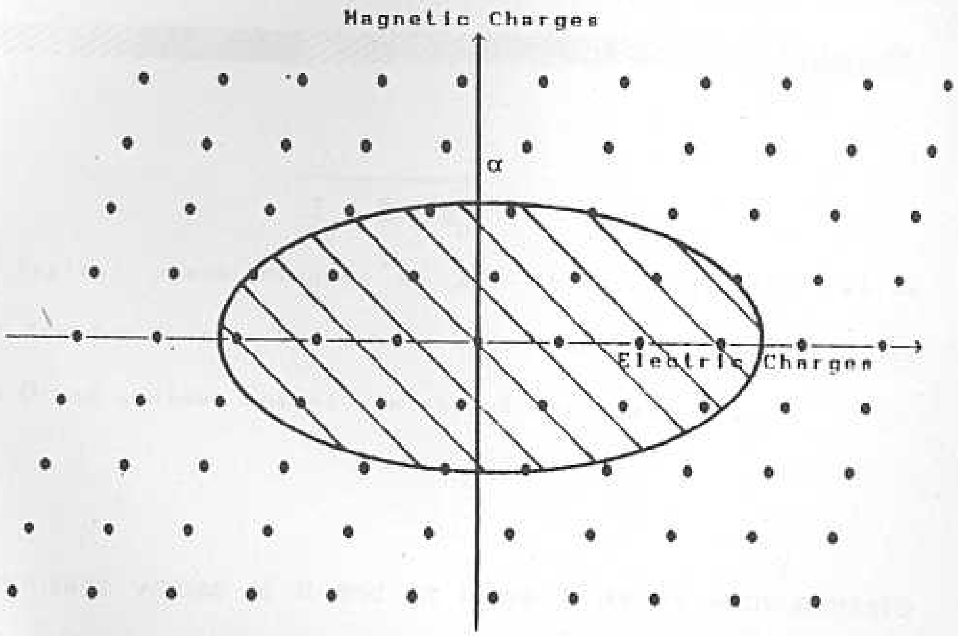


Fig.12]: The allowed values of the electric and magnetic charges. An ellipse for particular values of g_0, s and θ is shown. Charges within the ellipse can condense.

When $\theta = 0$ & $s < C$, for large T magnetic charges will condense and for small T electric charges will condense in the vacuum. This is also intuitively obvious. These are the well known confining (i.e. monopole condensation) and the Higg's phases (i.e. electric charge condensation) of U(1) lattice gauge theory. As the value of (s) increases (s>C) there are intermediate values of T where [E-(11)] is not satisfied and no charges condense. This corresponds to the Coulomb phase [Fig.13].

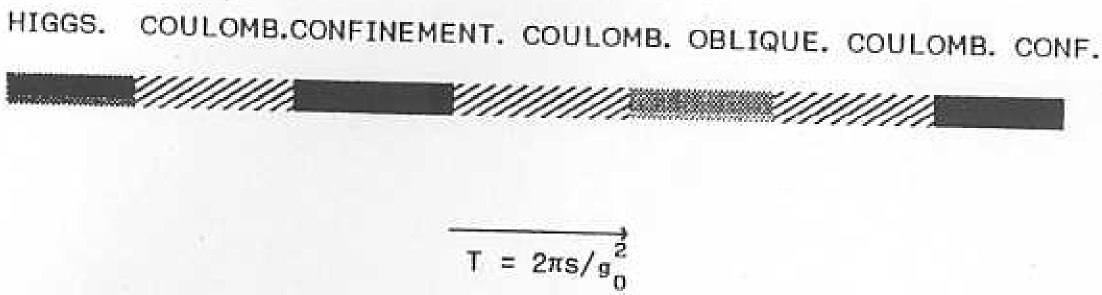


Fig.13] : Typical phase diagram of our theory with the radial degrees of freedom frozen..The number of oblique confinement phases depends on θ and coulomb phases are absent for small s.

For nonzero values of θ and at large T dyons with nonzero values of electric and magnetic charges condense (oblique confinement). For small T it is the electric charges which minimize the energy of the vacuum implying Higg's phase. For intermediate values of T, depending on the relative values of C and s, there are Coulomb and Higg's phases (Fig.13). These features of Z(s) gauge theories are analyzed in detail in the excellent papers of Cardy et al.

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