Exact Renormalization Group and the O(N) Model

By

Semanti Dutta
PHYS10201604006

The Institute of Mathematical Sciences, Chennai

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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PUBLICATIONS AND PRESENTATIONS

Journals

- *Wilson action for O(N) model*

  Semanti Dutta, B.Sathiapalan, H.Sonoda

  Nucl. Phys. B 956(2020)115022

- *Finite cut-off CFT's and composite operators*

  Semanti Dutta, B.Sathiapalan

  Nucl. Phys. B 973(2021)115524

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- *Bulk Gauge Fields and Holographic RG from Exact RG*

  Pavan Dharanipragada, Semanti Dutta, Bala Sathiapalan

  arXiv: 2201.06240

Talks and Posters

- Conference talk on *" Energy Momentum tensor in Exact Renormlization Group "*

  @ Chennai String Meet, IMSc,2019

- Gong show on *" Energy Momentum tensor in Exact Renormlization Group "*

  @ National String Meet, 2019 ,IISER,Bhopal.

- Poster presentation on *"Energy Momentum tensor in Exact Renormlization Group"*

  @ERG 2020 , Kyoto University, Japan (virtual).

- Poster presentation on *"Composite Operator in ERG"*

  in Strings 2021, @ICTP-SAIFR (virtual).

- Conference talk on *“ Composite operator in ERG ”* @ one day conference on String theory and related areas,2021, HRI, Allahabad(virtual).
- Conference talk on *“Derivation of Bulk Gauge fields from Boundary CFT”* in Indian String Meet 2021 @ IIT Roorkee(virtual).

Contents

Summary

Studying finite cutoff CFT holds much importance in both Condensed Matter Physics and AdS/CFT conjecture. As lattice systems always have an inherent scale, it is important to devise a theory that will be valid for all energy scales. On the other hand, the scheme of Holographic RG requires the study of ERG in boundary CFT.

In this thesis, we apply Exact RG to study a fixed point action and composite operators in finite cutoff O(N) model. In Exact RG the higher energy modes of a theory are integrated out with help of an analytic function resulting in no loss of information as one flows down to the lower energy. This method gives us the most general expression of an action or composite operators which is/are valid at all energy scales.

The main component of our work is Polchinski's ERG equation. In the first part of our work, we have constructed Wilson action at Wilson-Fisher(WF) fixed point in $4-\epsilon$ dimensions for ϕ^4 interaction. This has been done upto the subleading order. At this order, 6-pt vertex appears in the action, which contributes to subleading order terms in 2 and 4-pt vertex. The terms in the action have interpretations in terms of the Feynman diagram which eases the procedure of calculation.

When one has a fixed point action at hand, it is important to find the corresponding irrelevant and relevant operators. Because irrelevant operators define a critical surface, and relevant direction defines directions away from the critical surface. From the AdS/CFT perspective also studying perturbations around the boundary CFT is important, because they give rise to different bulk dual fields.

In the second part of our work we have constructed two important composite operators near the WF fixed point-ϕ^2 and ϕ^4. In continuum theory, the composite operators mix with the same dimension operators. Here expressions are more general, composite operators mixes with all operators which are allowed in the theory. As we are nearby the fixed point, we have

calculated the anomalous dimensions of these operators too. In continuum limit our result matches with results found from the Dimensional Regularization. Also our method of finding composite operators is independent of choice of cutoff functions. This makes it useful for the Holographic RG purpose. However, to find the anomalous dimensions of these operators we have used a specific cutoff function for ease of calculation.

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Chapter 1

Introduction

The Renormalization Group technique was constructed to solve two different problems in physics. One, to understand renormalization which is required to remove the UV divergences in the correlators. Two, in Statistical Mechanics to study critical phenomena. In the first case, why the process of cancelling divergences by adding counterterms in the Lagrangian results in a renormalizable theory was difficult to understand initially. But when one tries to apply RG techniques to the systems of large but finite number of degrees of freedom, a more physical picture emerges. This insight has led to better understanding of renormalization and the problem of UV divergences.

Critical phenomena is observed during second order phase transition where correlation lengths of different order parameters of the system diverges but nearby the phase transition point their behavior is universal. Kadanoff came with his famous block spin formalism to explain the critical point in ferromagnetism [1]. The main idea is to divide the whole system into blocks of a certain size and replace the spins in one block with an effective spin. In next step one has to repeat the same procedure with the new effective spins. This process is continued until the size of block becomes the size of correlation length. In this way one can explain both diverging thermodynamic quantities such as specific heat at the critical point and the universal character of the critical exponents[13, 15].

This opens up a new way of looking at RG- as a process of change of scale of a theory. The bare theory has scale of Λ_0, and when we integrate out the momenta from Λ_0 to Λ, an effective theory of lower energy Λ emerges. If one want the continuum theory one can simply take $\Lambda_0 \to \infty$. In taking $\Lambda_0 \to \infty$, it is important to tune a *finite* number of parameters to special values. This is idea of renormalizability of a theory.

The Exact Renormalization Group is just a smart way of performing the job of integrating out the higher energy modes. Instead of sharply cutting the momentum at higher energies, one can take resort to a smooth analytic function, so that no nonlocal interactions appear in the new effective action in position space [14]. Added benefit to that is the new low energy correlator can always give back the old bare correlator, hence no loss of physics [52]. Using ERG one can show that the theory at the fixed point does not depend on the bare theory chosen, hence the universality of critical exponents arises. In QFT side also this paves a way for proof of perturabtive renormalizability. It was shown for ϕ^4 theory in 4 dimensions, that the irrelavant terms in an action gets supressed in continuum limit, hence resulting in a correlator which can be tuned with finite number of renormalizable parameters [16].

On the other hand, study of Conformal Field Theory has become of much importance in the last few decades. It was argued long back that the theories at the critical point are conformally invariant [3, 58]. The idea of bootstrap was also introduced soon after, which allowed further non perturbative constraints to be placed on the system [4]. Particularly in two dimensions these ideas have been very fruitful [5] and have applications in the world sheet description of string theory. Reviews of later developments and references are given in [6, 7].

The AdS/CFT correpondence [8, 9, 10, 11] or "holography" between a boundary CFT and a bulk gravity theory gives strong motivation for studying CFT's [1]. There is a large amount of literature on this. See, for example, [12] for a review.

ERG allows us to deal with a CFT with finite UV cutoff. In condensed matter systems there is always an underlying lattice structure that provides a natural ultraviolet cutoff. At the critical point the correlation length being much larger than the lattice spacing, one can for many purposes treat it as a continuum theory, much as is done in high energy physics. Nonetheless one shoule be able to construct a theory which is valid at every energy scale . So it is important to understand CFTs with a finite cutoff.[2]

Another motivation for studying finite cutoff CFT using ERG comes from AdS/CFT conjecture. According to this conjecture the radial coordinate of AdS can be interpreted as momentum

[1]It also opens up the amazing possibility of rewriting quantum gravity as a quantum field theory in flat space.

[2]If one speculates as for instance in [61] that space time itself in string theory is dicrete, then that is additional motivation for studying such theories.

scale of the boundary CFT. This scheme comes under the name of 'Holographic RG'. In order to understand this Holographic RG in details it is necesary to study the structure and properties of a fixed point theory in the presence of a finite cutoff, because the cutoff represents the radial coordinate of the (asymptotically) AdS space and one needs to know what the theory looks like as it evolves under RG flow in the radial direction.

Recently it has been shown that one can obtain the Holographic RG equation starting from a ERG equation of the boundary theory[40, 41]. Further evidence for this has been obtained by studying the O(N) model at it's fixed points in [39] where the RG flow of a scalar composite operator was studied. If one starts with a conformally invariant fixed point action in D dimensions and perturbs it, then an ERG describes the evolution of these perturbations. It was shown in [37, 38, 39] that the evolution operator of this ERG can be written as a functional integral of a field theory in AdS_{D+1} space. The boundary values of these fields are typically sources for the perturbing operators, though other interpretations are also possible. It has also been shown recently that gauge fields and metric perturbations in AdS can be obtained from ERG in the boundary[51].

This motivated us to study O(N) model using ERG. O(N) model is a much explored model (see [68, 69] for nice review) in the field theory. From Holographic RG side also, the bulk dual of O(N) model has been found in many forms [63, 64, 65].

In our work, as a first step, we construct a fixed-point Wilson action for this theory to order ϵ^2. It is at this order that the anomalous dimension first shows up. The action is obtained by solving the fixed-point ERG equation perturbatively. The fixed-point equation imposes the constraint of scale invariance. This theory is also conformally invariant. This follows from the tracelessness of the energy momentum tensor [46, 47, 48, 58]. However in our work it has been shown that the EM tensor at zero momentum satisfies traceless condition [43]. Traceless condition of zero momentum EM tensor denotes the theory is scale invariant. In order to prove conformal invaraince one have to find EM tensor at general external momentum which is beyond scope of this thesis.

Next we concentrated on the construction of the "composite" operators in ERG. In continuum field theory these operators have to be renormalized so that Green's functions involving these are finite. This is an interesting problem in its own right. This is described in many textbooks such as [45]. The renormalization of these operators in ϕ^4 theory in four dimensions is described in detail in [48, 49]. Analogous study of ϕ^3 theory in six dimensions has also been done [50]. In presence of interactions, an operator mixes with other operators of the same dimension or less even in continuum field theory. In finite cutoff, one can expect this to mix with higher dimension operators such as $\int_x \phi^4, \int_x \phi^6, \ldots$.

We are interested in those composite operators that maintain their form as they evolve. They should obey the usual properties of operators with definite scaling dimension in a CFT. The eigenvector equation, which is the ERG equation, can be solved perturbatively in powers of λ the coupling constant. This is also related to ϵ since $\lambda \approx O(\epsilon)$. It involves making a fairly general (momentum dependent) ansatz for the eigen operators and solving for the momentum dependence order by order. We do this up to $O(\lambda^2)$. For the simplest case which is the leading order relevant operator, we construct the local operator i.e. $\phi^2(x)$ or in momentum space $\phi^2(q)$ with $q \neq 0$. In all other cases, for reasons of computational simplicity, especially at second order, we have focused on the integrated operators $\int_x \phi^2(x)$ and $\int_x \phi^4(x)$. This amounts to imposing $\sum_i p_i = q = 0$. The unintegrated operator can be extracted from this modulo total derivative terms. The scaling dimensions are also calculated and agree with the literature to this order. It is worthy to note that the expression of action or composite opetaors do not depend on the form of the cutoff function, hence our method of calculation can be accommodated in Holographic RG works mentioned above [37, 38, 39] . But to obtain the anomalous dimension we have used a specific form of cutoff function for ease of calculation.

Construction of the local operators enables us to do other analysis. Important one among them is to find whether these composite operators are primary or not. This has to be done by checking whether the correponding correlation function satisfy the Conformal Ward Identity. However, we did not pursue that in this thesis.

The plan of the thesis as follows:

Chapter 2 gives some background regarding our work. Here we will explain basic procedure of ERG, then state the ERG equation and it's modification in order to find the fixed point action. Then We will talk about the composite operators in continuum theory and ERG. Chapter 3 gives the construction of the Wilson Action for the $O(N)$ model at the Wilson-Fisher fixed point. Chapter 4 gives the construction of two composite operators. Chapter 3 and 4 constitute the work done for the thesis. Finally, Chapter 5 gives some conclusions and outlook of our work.

Chapter 2

Background

In this chapter, we will provide the necessary background for our work. First, we will talk about the basic procedure of ERG. Next, we will state the ERG equation which governs the change of effective action w.r.t scale of the theory. Then we will customize the equation to facilitate the calculation of the IR limit of a critical theory or fixed point theory. This is material for the chapter 3.

To provide background for the chapter 4, next we have elaborated about the composite operators. First we have given details about composite operator in continuum theory. After giving some simple explanation of the meaning of composite operator in Wilsonian RG or ERG, we have stated the definitions and boundary conditions. In the end, we have demonstrated two simple examples of calculating comsposite operators using ERG.

The discussion in this chapter is mainly based on [52, 53].

2.1 Exact Renormalization Group

Renormalization means essentially going from a scale Λ_0 to a lower scale Λ, where the initial scale Λ_0 is typically called a bare scale. One will want to see how physics changes with scale. What do we mean by physics at Λ_0? It means our theory will not be sensitive to momentum $p > \Lambda_0$. This can be done by following the procedure below:

The partition function of the full theory is given by

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

where

$$S = \int_p \frac{1}{2}\phi(p)p^2\phi(-p) + S_I[\phi]$$

To make it a partition function at scale Λ_0 we will try to suppress the kinetic energy term for $\Lambda_0 < p < \infty$. To execute this we will put a smooth cutoff in the kinetic energy term to obtain the bare action

$$S_B[\phi] \equiv \frac{1}{2}\int_p \phi \frac{p^2}{K(p^2/\Lambda_0^2)}\phi + S_{I,B}[\phi] \tag{2.1.1}$$

and the bare partition function

$$Z_B \equiv \int \mathcal{D}\phi \, e^{-S_B[\phi]} \tag{2.1.2}$$

We will choose the cutoff function to follow the condition $K(0) = 1$ and $K(\infty) = 0$. In general cutoff functions satisfy stronger properties , but that will not affect the fixed point values of the couplings [62].

Now we want to go to a lower scale Λ. For that, observe the following identity

$$\begin{aligned}
&\int \mathcal{D}\phi \exp\left[-\frac{1}{2}\int_p \phi(-p)\frac{1}{A(p)+B(p)}\phi(p) - S_{I,B}[\phi]\right] \\
&= \int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\left[-\frac{1}{2}\int_p \frac{1}{A(p)}\phi_1(-p)\phi_1(p) - \frac{1}{2}\int_p \frac{1}{B(p)}\phi_2(-p)\phi_2(p) - S_{I,B}[\phi_1+\phi_2]\right]
\end{aligned}$$

Where a multiplicative constant has been ignored on RHS. Using this we can write

$$\begin{aligned}
Z_B = \int \mathcal{D}\phi_l\mathcal{D}\phi_h \exp\Bigg\{ &-\frac{1}{2}\int_p \frac{p^2}{K(p^2/\Lambda^2)}\phi_l(-p)\phi_l(p) \\
&-\frac{1}{2}\int_p \frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\phi_h(-p)\phi_h(p) - S_{I,B}[\phi_l+\phi_h]\Bigg\}
\end{aligned}$$

We can effectively call $\phi_l(\phi_h)$ as low(high) energy field as it is propagated by low(high) momentum propagator $\Delta_l(\Delta_h)$ defined below

$$\Delta_l = \frac{K(p^2/\Lambda^2)}{p^2}, \quad \Delta_h = \frac{K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2)}{p^2} \tag{2.1.3}$$

So we can write

$$\begin{aligned} Z_B &= \int \mathcal{D}\phi_l \exp\left[-\frac{1}{2}\int_p \phi_l \Delta_l^{-1}\phi_l\right] \int \mathcal{D}\phi_h \exp\left[-\frac{1}{2}\int_p \phi_h \Delta_h^{-1}\phi_h - S_{I,B}[\phi_l + \phi_h]\right] \\ &= \int \mathcal{D}\phi_l \exp\left[-\frac{1}{2}\int_p \phi_l \Delta_l^{-1}\phi_l\right] \exp\{-S_{I,\Lambda}[\phi_l]\} \end{aligned}$$

where

$$\exp\{-S_{I,\Lambda}[\phi_l]\} \equiv \int \mathcal{D}\phi_h \exp\left\{-\frac{1}{2}\int_p \phi_h \Delta_h^{-1}\phi_h - S_{I,B}[\phi_l + \phi_h]\right\} \tag{2.1.4}$$

$S_{I,\Lambda}$ is the interaction part of an effective low energy field theory with a UV cutoff Λ.

Let

$$S_\Lambda[\phi] \equiv \frac{1}{2}\int_p \phi_l \Delta_l^{-1}\phi_l + S_{I,\Lambda}[\phi_l] \tag{2.1.5}$$

be the whole action so that

$$Z_B = \int \mathcal{D}\phi_l \, e^{-S_\Lambda[\phi_l]} \tag{2.1.6}$$

Using (2.1.4), we obtain

$$\begin{aligned} e^{-S_\Lambda[\phi]} = \int \mathcal{D}\varphi \, \exp\Bigg[&-S_B[\varphi] + \frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda_0)}\varphi(p)\varphi(-p) - \frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda)}\phi(p)\phi(-p) \\ &-\frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda_0) - K(p/\Lambda)}\left(\varphi(p) - \phi(p)\right)\left(\varphi(-p) - \phi(-p)\right)\Bigg] \end{aligned} \tag{2.1.7}$$

where we have written ϕ_l as ϕ and ϕ_h as $\varphi - \phi$. This will be useful later. It is to be noted that one can always go back to the bare partition function. For this reason, this scheme is called **"exact"**, i.e. we lose no physical information by varying the scale. It is easy to see this explicitly. Using (2.1.7), we can calculate the generating functional of S_B using S_Λ as

$$\int \mathcal{D}\phi \, \exp\left(-S_B[\phi] - \int_p J(-p)\phi(p)\right)$$
$$= \exp\left[\frac{1}{2}\int_p J(p)J(-p)\frac{1}{p^2}\left\{K(p/\Lambda_0)\left(1-K(p/\Lambda_0)\right) - \left(\frac{K(p/\Lambda_0)}{K(p/\Lambda)}\right)^2 K(p/\Lambda)\left(1-K(p/\Lambda)\right)\right\}\right]$$
$$\times \int \mathcal{D}\phi \, \exp\left(-S_\Lambda[\phi] - \int_p J(-p)\frac{K(p/\Lambda_0)}{K(p/\Lambda)}\,\phi(p)\right) \tag{2.1.8}$$

We observe that the correlation functions of S_B are the same as those of S_Λ up to the trivial (short-distance) contribution to the two-point function and up to the momentum-dependent rescaling of the field by $\frac{K(p/\Lambda_0)}{K(p/\Lambda)}$ [53]. If we ignore the small corrections to the two-point functions (which are disconnected pieces proportional to $\delta(p_i + p_j)$), we can write

$$\prod_{i=1}^{n} \frac{1}{K(p_i/\Lambda)}\ \langle\phi(p_1)\cdots\phi(p_n)\rangle_{S_\Lambda} = \prod_{i=1}^{n} \frac{1}{K(p_i/\Lambda')}\ \langle\phi(p_1)\cdots\phi(p_n)\rangle_{S_{\Lambda'}} \tag{2.1.9}$$

2.2 How to find the fixed point action?

When one flow along the energy scale we may hit a fixed point (more on this in next chapter) i.e. a point where S_Λ does not change with scale. It is scale invariant in the sense that there is only one scale in the theory at that point, that is given by RG scale Λ. So at the fixed point if one writes every quantity in dimensionless variables. Those dimensionless variables will be unchanged even if you change the scale. In addition to this one also needs to adjust the scaling dimension of the field in order to keep the standard form of kinetic energy term.

So to find the fixed point action one needs to find how an action S_Λ changes with scale Λ. This is given by Polchinski's equation (see next subsection). Then one has to modify it as stated above. After all these steps if one puts $-\Lambda\frac{\partial S}{\partial \Lambda} = 0$, one obtains the fixed point action.

2.2.1 Polchinski's ERG equation

We have given an integral formula (2.1.4) for $S_{I,\Lambda}$ and (2.1.7) for S_Λ. It is easy to derive differential equations from these. From (2.1.4), we obtain Polchinski's ERG equation

$$-\Lambda\frac{\partial S_{I,\Lambda}[\phi]}{\partial \Lambda} = \int_p (-)\frac{dK(p/\Lambda)}{dp^2}\left(-\frac{\delta S_{I,\Lambda}[\phi]}{\delta\phi(p)}\frac{\delta S_{I,\Lambda}[\phi]}{\delta\phi(-p)} + \frac{\delta^2 S_{I,\Lambda}[\phi]}{\delta\phi(p)\delta\phi(-p)}\right) \tag{2.2.1}$$

for $S_{I,\Lambda}$. From (2.1.7) we obtain

$$-\Lambda\frac{\partial S_\Lambda[\phi]}{\partial\Lambda} = \int_p \left[-2p^2\frac{d\ln K(p/\Lambda)}{dp^2}\,\phi(p)\frac{\delta S_\Lambda}{\delta\phi(p)} + \frac{dK(p/\Lambda)}{dp^2}\left(-\frac{\delta S_\Lambda}{\delta\phi(p)}\frac{\delta S_\Lambda}{\delta\phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)}\right)\right] \tag{2.2.2}$$

for the entire Wilson action.

The limit $\Lambda \to 0+$

In the limit $\Lambda \to 0+$ we expect $S_\Lambda[\phi]$ approaches something related to the partition function. If we substitute

$$\lim_{\Lambda\to 0+} K(p/\Lambda) = 0 \tag{2.2.3}$$

into (2.1.7), we get

$$\begin{aligned}\lim_{\Lambda\to 0+} e^{-S_\Lambda[\phi]+\frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda)}\phi(p)\phi(-p)} &= \lim_{\Lambda\to 0+} e^{-S_{I,\Lambda}[\phi]}\\ = e^{-\frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda_0)}\phi(p)\phi(-p)} &\int \mathcal{D}\varphi\, \exp\left[-S_B[\varphi] + \int_p \frac{p^2}{K(p/\Lambda_0)}\varphi(p)\phi(-p)\right]\end{aligned} \tag{2.2.4}$$

Hence, rewriting $\phi(p)$ by $\frac{K(p/\Lambda_0)}{p^2}J(p)$, we obtain the generating functional of the bare theory as the $\Lambda \to 0+$ limit of $S_{I,\Lambda}$:

$$\begin{aligned}Z_B[J] &\equiv \int \mathcal{D}\varphi\, \exp\left[-S_B[\varphi] - \int_p \varphi(p)J(-p)\right]\\ &= e^{-\frac{1}{2}\int_p J(p)J(-p)\frac{K(p/\Lambda_0)}{p^2}} \lim_{\Lambda\to 0+} \exp\left(-S_{I,\Lambda}\left[\frac{K(p/\Lambda_0)}{p^2}J(p)\right]\right)\end{aligned} \tag{2.2.5}$$

IR limit of a critical theory

For the bare theory at criticality, we expect that the correlation functions

$$\langle\varphi(p_1)\cdots\varphi(p_n)\rangle_B \equiv \int \mathcal{D}\varphi\, \varphi(p_1)\cdots\varphi(p_n)\, e^{-S_B[\varphi]} \tag{2.2.6}$$

to become scale invariant in the IR limit, i.e., for small momenta compared to Λ_0. To be more precise, we can define the limit

$$\mathcal{C}(p_1,\cdots,p_n) \equiv \lim_{t\to\infty} e^{\frac{n}{2}(-(D+2)+\eta)t}\left\langle\varphi(p_1e^{-t})\cdots\varphi(p_ne^{-t})\right\rangle_B \tag{2.2.7}$$

where $\frac{\eta}{2}$ is the anomalous dimension.

What does this mean for S_Λ in the limit $\Lambda \to 0+$? As we have seen above, the interaction part $S_{I,\Lambda}$ becomes the generating functional of the bare theory in this limit. Since the IR limit of the correlation functions are scale invariant, only the low momentum part of $\lim_{\Lambda \to 0+} S_{I,\Lambda}$ corresponds to the scale invariant theory defined by the IR limit (2.2.7).

To understand the IR limit better, we follow Wilson [13] and reformulate the ERG transformation in two steps:

1. Introduction of an anomalous dimension (section 2.2.1) — the anomalous dimension is an important ingredient of the IR limit. We need to introduce an anomalous dimension of the field within ERG.

2. Introduction of a dimensionless framework (section 2.2.1) — each time we lower the cutoff Λ we have to rescale space-time to restore the same momentum cutoff. This is necessary to realize scale invariance within ERG.

Anomalous dimension in ERG

The cutoff dependent Wilson action $S_\Lambda[\phi]$ has two parts:

$$S_\Lambda[\phi] = \frac{1}{2} \int_p \frac{p^2}{K(p/\Lambda)} \phi(p)\phi(-p) + S_{I,\Lambda}[\phi] \tag{2.2.8}$$

The first term is not the only kinetic term; part of the interaction quadratic in ϕ's also contains the kinetic term. The normalization of ϕ has no physical meaning, and it is natural to normalize the field so that $S_{I,\Lambda}$ contains no kinetic term.

To do this, we modify the ERG differential equation (2.2.2) by adding a number operator [52, 62]:

$$\begin{aligned} -\Lambda\partial_\Lambda S_\Lambda[\phi] = &\int_p \left(-2p^2 \frac{d}{dp^2} \ln K(p/\Lambda)\, \phi(p) \frac{\delta S_\Lambda}{\delta\phi(p)} - \frac{d}{dp^2} K(p/\Lambda) \left\{ \frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)} - \frac{\delta S_\Lambda}{\delta\phi(p)} \frac{\delta S_\Lambda}{\delta\phi(-p)} \right\} \right) \\ &- \frac{\eta_\Lambda}{2} \mathcal{N}_\Lambda[\phi] \end{aligned} \tag{2.2.9}$$

where the number operator $\mathcal{N}_\Lambda[\phi]$ is defined by

$$\mathcal{N}_\Lambda[\phi] \equiv \int_p \left[\phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} + \frac{K(p/\Lambda)\left(1 - K(p/\Lambda)\right)}{p^2} \left\{ \frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)} - \frac{\delta S_\Lambda}{\delta\phi(p)} \frac{\delta S_\Lambda}{\delta\phi(-p)} \right\} \right] \tag{2.2.10}$$

This counts the number of fields:

$$\left\langle \mathcal{N}_\Lambda[\phi]\, \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_\Lambda} = n \left\langle \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_\Lambda} \tag{2.2.11}$$

(Again we are ignoring small corrections to the two-point functions as mentioned before (2.1.9).) Under (2.2.9) the correlation function changes as

$$\prod_{i=1}^n \frac{1}{K(p_i/\Lambda)} \left\langle \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_\Lambda} = \left(\frac{Z_\Lambda}{Z_{\Lambda'}} \right)^{\frac{n}{2}} \prod_{i=1}^n \frac{1}{K(p_i/\Lambda')} \left\langle \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_{\Lambda'}} \tag{2.2.12}$$

where Z_Λ is the solution of

$$-\Lambda \frac{\partial}{\partial \Lambda} Z_\Lambda = \eta_\Lambda\, Z_\Lambda \tag{2.2.13}$$

satisfying the initial condition

$$Z_{\Lambda_0} = 1 \tag{2.2.14}$$

We can choose η_Λ so that S_Λ has the same kinetic term independent of Λ. For (2.2.9), the integral formula (2.1.7) must be changed to [53],

$$e^{S_\Lambda[\phi]} = \int \mathcal{D}\varphi\, e^{S_0[\varphi]} \times \exp\left[-\frac{1}{2} \int_p \frac{p^2}{\frac{1-K(p/\Lambda)}{Z_\Lambda K(p/\Lambda)} - \frac{1-K(p/\Lambda_0)}{K(p/\Lambda_0)}} \left(\frac{\varphi(p)}{K(p/\Lambda_0)} - \frac{\phi(p)}{\sqrt{Z_\Lambda}\, K(p/\Lambda)} \right) \left(\frac{\varphi(-p)}{K(p/\Lambda_0)} - \frac{\phi(-p)}{\sqrt{Z_\Lambda}\, K(p/\Lambda)} \right) \right] \tag{2.2.15}$$

This reduces to (2.1.7) for $Z_\Lambda = 1$.

Dimensionless framework

To reach the IR limit (2.2.7) we must look at smaller and smaller momenta as we lower the cutoff Λ. We can do this by measuring the momenta in units of the cutoff Λ. At the same time, we render all the dimensionful quantities such as $\phi(p)$ dimensionless by using appropriate powers of Λ.

We introduce a dimensionless parameter t by

$$\Lambda = \mu\, e^{-t} \tag{2.2.16}$$

where μ is an arbitary fixed momentum scale. We then define the dimensionless field with dimensionless momentum by

$$\bar{\phi}(p) \equiv \Lambda^{\frac{D+2}{2}} \phi(p\Lambda) \tag{2.2.17}$$

and define a Wilson action parametrized by t:

$$\bar{S}_t[\bar{\phi}] \equiv S_\Lambda[\phi] \tag{2.2.18}$$

We can now rewrite (2.2.9) for $\bar{S}_t$:

$$\begin{aligned} \partial_t \bar{S}_t[\bar{\phi}] = & \int_p \left(-2p^2 \frac{d}{dp^2} \ln K(p) + p \cdot \partial_p + \frac{D+2}{2} \right) \bar{\phi}(p) \cdot \frac{\delta \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}(p)} \\ & + \int_p (-) \frac{d}{dp^2} K(p) \left\{ \frac{\delta^2 \bar{S}_t}{\delta \bar{\phi}(p) \delta \bar{\phi}(-p)} - \frac{\delta \bar{S}_t}{\delta \bar{\phi}(p)} \frac{\delta \bar{S}_t}{\delta \bar{\phi}(-p)} \right\} - \frac{\eta_t}{2} \mathcal{N}_t[\bar{\phi}] \end{aligned} \tag{2.2.19}$$

where we have replaced η_Λ by η_t, and

$$\mathcal{N}_t[\bar{\phi}] \equiv \int_p \bar{\phi}(p) \frac{\delta \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}(p)} + \int_p \frac{K(p)\,(1 - K(p))}{p^2} \left(\frac{\delta^2 \bar{S}_t}{\delta \bar{\phi}(p) \delta \bar{\phi}(-p)} - \frac{\delta \bar{S}_t}{\delta \bar{\phi}(p)} \frac{\delta \bar{S}_t}{\delta \bar{\phi}(-p)} \right) \tag{2.2.20}$$

is the number operator for $\bar{S}_t$.

Rewriting (2.2.12) in terms of dimensionless fields, we obtain

$$\begin{aligned} & \prod_{i=1}^n \frac{1}{K(p_i)} \left\langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \right\rangle_{\bar{S}_t} \\ & = \left(\frac{Z_t}{Z_{t'}} \right)^{\frac{n}{2}} e^{-\frac{n}{2}(D-2)(t-t')} \prod_{i=1}^n \frac{1}{K(p_i e^{-(t-t')})} \left\langle \bar{\phi}(p_1 e^{-(t-t')}) \cdots \bar{\phi}(p_n e^{-(t-t')}) \right\rangle_{\bar{S}_{t'}} \end{aligned} \tag{2.2.21}$$

where Z_t satisfies

$$\partial_t Z_t = \eta_t\, Z_t \tag{2.2.22}$$

(The corrections to the two-point functions are ignored.) Comparing (2.2.21) with (2.2.7), the existence of the IR limit implies that

$$\lim_{t\to\infty} \eta_t = \eta \tag{2.2.23}$$

and

$$\lim_{t\to\infty} \prod_{i=1}^{n} \frac{1}{K(p_i)} \left\langle \bar{\phi}(p_1)\cdots\bar{\phi}\right\rangle_{\bar{S}_t} = \mathcal{C}(p_1,\cdots,p_n) \tag{2.2.24}$$

In other words $\bar{S}_t$ approaches a limit as $t \to +\infty$:

$$\lim_{t\to+\infty} \bar{S}_t = \bar{S}_\infty \tag{2.2.25}$$

We call $\bar{S}_\infty$ a fixed point because the right-hand side of (2.2.19) vanishes for it:

$$\begin{aligned} 0 = &\int_p \left(-2p^2 \frac{d}{dp^2}\ln K(p) + p\cdot\partial_p + \frac{D+2}{2}\right)\bar{\phi}(p)\cdot\frac{\delta \bar{S}_\infty[\bar{\phi}]}{\delta\bar{\phi}(p)} \\ &+ \int_p (-)\frac{d}{dp^2}K(p)\left\{\frac{\delta^2\bar{S}_\infty}{\delta\bar{\phi}(p)\delta\bar{\phi}(-p)} - \frac{\delta\bar{S}_\infty}{\delta\bar{\phi}(p)}\frac{\delta\bar{S}_\infty}{\delta\bar{\phi}(-p)}\right\} - \frac{\eta}{2}\mathcal{N}_\infty[\bar{\phi}] \end{aligned} \tag{2.2.26}$$

Fixed-point equation

Instead of choosing η dependent on t, we may choose η as a constant so that there is a non-trivial fixed-point solution $\bar{S}_\infty$ for which the right-hand side of (2.2.19) vanishes. With a constant anomalous dimension, the dimensionless ERG equation is given by,

$$\begin{aligned} \partial_t\bar{S}_t[\bar{\phi}] = &\int_p \left(-2p^2\frac{d}{dp^2}\ln K(p) + \frac{D+2}{2} - \frac{\eta}{2} + p\cdot\partial_p\right)\bar{\phi}(p)\cdot\frac{\delta\bar{S}_t[\bar{\phi}]}{\delta\bar{\phi}(p)} \\ &+ \int_p\left(-2\frac{d}{dp^2}K(p) - \eta\frac{K(p)\,(1-K(p))}{p^2}\right)\frac{1}{2}\left(\frac{\delta^2\bar{S}_t[\bar{\phi}]}{\delta\bar{\phi}(p)\delta\bar{\phi}(-p)} - \frac{\delta\bar{S}_t[\bar{\phi}]}{\delta\bar{\phi}(p)}\frac{\delta\bar{S}_t[\bar{\phi}]}{\delta\bar{\phi}(-p)}\right) \end{aligned} \tag{2.2.27}$$

For the O(N) model with N fields ϕ^i $(i=1,\cdots,N)$, the ERG equation becomes

$$\begin{aligned} \partial_t\bar{S}_t[\bar{\phi}] = &\int_p \left(-2p^2\frac{d}{dp^2}\ln K(p) + \frac{D+2}{2} - \frac{\eta}{2} + p\cdot\partial_p\right)\bar{\phi}^i(p)\cdot\frac{\delta\bar{S}_t[\bar{\phi}]}{\delta\bar{\phi}^i(p)} \\ &+ \int_p\left(-2\frac{d}{dp^2}K(p) - \eta\frac{K(p)\,(1-K(p))}{p^2}\right)\frac{1}{2}\left(\frac{\delta^2\bar{S}_t[\bar{\phi}]}{\delta\bar{\phi}^i(p)\delta\bar{\phi}^i(-p)} - \frac{\delta\bar{S}_t[\bar{\phi}]}{\delta\bar{\phi}^i(p)}\frac{\delta\bar{S}_t[\bar{\phi}]}{\delta\bar{\phi}^i(-p)}\right) \end{aligned} \tag{2.2.28}$$

where the repeated indices i are summed over.

This is the equation that we will solve in chapter 3.

2.3 Composite Operators

Composite operator in the field theory is simply a product of two or more operators in the same or different space-time points. Let us consider two local operators $O(x_1)$ and $O(x_2)$. In general, this product has singularity as $x \to y$. In free scalar field theory just by subtracting the vacuum expectation value, a well-defined operator product can be constructed i.e.

$$: O^2(x) := \lim_{x \to y} \{O(x)O(y) - \langle O(x)O(y) \rangle\}$$

Things become complex when one adds interaction. Wilson made a hypothesis to deal with this by writing the product in a series of the following form:-

$$O_1(x+\xi)O_2(x-\xi) \equiv \sum_{i=1}^{\infty} E_i(\xi)\mathcal{O}_j(x)$$

Singularity of the product of operators as $\xi \to 0$ is captured by coefficients of $E_i(\xi)$. The above scheme is called Operator Product Expansion. How to define composite operators at the same space-time point? Those operators themselves will be divergent when put inside a Green's function. E.g. consider the following 2-pt Green's function with a composite operator $\phi^2(x)$ in ϕ^4 interaction in $4 - \epsilon$ dimensions.

$$\langle \phi^2(x)\phi(y)\phi(z) \rangle$$

In free theory all that needs to be done is to reproduce ϕ^2 by $: \phi^2(x) :$. But in an interacting theory, one needs to make it finite at every order of perturbation.

At tree level,

$$\langle \phi^2(k)\phi(p)\phi(q) \rangle = 2.\frac{i}{p^2}\frac{i}{q^2}$$

The 1-PI one-loop correction to this is,

$$= \frac{i}{p^2}\frac{i}{q^2}\int \frac{d^d r}{(2\pi)^d}(-i\lambda)\frac{i}{r^2}\frac{i}{(k+r)^2}$$
$$= \frac{i}{p^2}\frac{i}{q^2}\left[-\frac{\lambda}{(4\pi)^2}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}\right] \tag{2.3.1}$$

Where Δ is a function of external momenta. For on-shell condition, one can add a suitable counterterm in the Lagrangian to cancel this divergent 1-loop contribution. That results in an anomalous dimension of the value

$$\gamma_{\phi^2} = \frac{\lambda}{16\pi^2}$$

In general the process of adding counterterms can be encapsulated in the following expression,

$$O_B(p) = Z_O(\lambda, D)[O](p) \tag{2.3.2}$$

Where $O_B(p)$ and $[O](p)$ denote bare and renormalized composite operators respectively.

Define an insertion of a composite operator at momentum k in a renormalized Green's function.

$$G(p_1, p_2, ...p_n; k) = \langle \phi(p_1)\phi(p_2)....\phi(p_n)[O](k)\rangle$$

which is related to Green's function of the bare theory by

$$G(p_1, p_2, ...p_n; k) = Z^{-D/2} Z_O^{-1} \langle \phi(p_1)\phi(p_2)....\phi(p_n)O(k)\rangle_B \tag{2.3.3}$$

Where Z is the usual wavefunction renormalization factor comes with a fundamental field. From the above expression, one finds that Green's functions with a composite operator obey the Callen-Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+D\gamma(\lambda)+\gamma_O(\lambda)\right]G(p_1,p_2,...p_n;k)=0$$

Where $\beta(\lambda)$,$\gamma(\lambda)$,γ_O are the beta function, anomalous dimension of the fundamental field, and dimension of the operator O respectively. μ is the arbitrary scale of the theory. γ and γ_O is defined as

$$\gamma=\frac{1}{2}\mu\frac{\partial}{\partial\mu}\log Z$$

$$\gamma_O=\mu\frac{\partial}{\partial\mu}\log Z_O$$

In a theory where there exists several operators with the same engineering dimension, the relation (2.3.2) becomes

$$O_B^i=Z_O^{ij}O^j \tag{2.3.4}$$

So the anomalous dimension function in the Callen-Symanzik equation also gets generalized to a matrix

$$\gamma_O^{ij}=[Z_O^{-1}]^{ik}\mu\frac{\partial}{\partial\mu}[Z_O]^{kj}$$

Renormalization of composite operators in continuum theory is described in many field theory textbooks (for eg [45]). A careful analysis of the composite operators is described in [48, 49] for ϕ^4 theory in four dimensions using dimensional regularization , in [45] for ϕ^3 theory in six dimensions. In particular, the composite energy- momentum tensor operator is constructed there. A similar analysis has been done recently for the ϕ^3 theory in six dimensions [50].

In contrast, in the Wilsonian RG, one studies the evolution of an operator as longer and longer wavelength modes are integrated out. This is done by requiring that ΔS obey the Wilsonian RG equation linearized about a fixed point. This leads to the definition of a composite operator in ERG given below. In this thesis, we will calculate the composite operators using ERG.

2.4 Composite operator in ERG

2.4.1 Toy example

In free field theory, at $D = 4$, the scalar field ϕ has engineering dimension one. The composite ϕ^n thus has dimension n. Thus we consider a term in the action $\Delta S_2 = \frac{1}{2}\int m^2\phi^2$. Let the UV cutoff be Λ. We write this action in terms of dimensionless fields and coordinates. Define

$$\phi = \Lambda\bar{\phi}, \quad x = \frac{\bar{x}}{\Lambda}$$

Then

$$\Delta S_2 = \frac{1}{2}\int d^4\bar{x}\ \frac{m^2}{\Lambda^2}\bar{\phi}^2 = \frac{1}{2}\int d^4\bar{x}\ r\bar{\phi}^2$$

Here r is dimensionless. On coarse graining, Λ decreases, so for fixed m^2, r increases. Thus if we write $\Lambda = \Lambda_0 e^{-t}$ we see that

$$\frac{d\Delta S_2}{dt} \equiv d_m \Delta S_2 = 2\Delta S_2$$

and we call it relevant. d_m is the overall length scaling dimension of ΔS_2 (not counting the parameter m^2, which is included to make the whole thing have dimension zero).

If we add a term

$$\Delta S_4 = \int d^4x\ u\phi^4$$

one immediately sees that u is already dimensionless and

$$\frac{d\Delta S_4}{dt} \equiv d_m \Delta S_4 = 0$$

and we call it marginal.

But this is not the whole story even in a free theory. The operation d/dt refers not to just changing Λ that was introduced here to make things dimensionless, but it refers to the whole process of *integrating out* modes between Λ and $\Lambda(1-dt)$. This physical coarse graining process fixes the Λ dependence of the action. It introduces an extra Λ dependence over and above what is required for writing everything in terms of dimensionless variables.

We illustrate this with a simple calculation. Write $\phi = \phi_h + \phi_l$. We assume that ϕ_h are modes between Λ, Λ_0 and are integrated out. Thus

$$\phi^4 = \phi_l^4 + 6\phi_l^2\phi_h^2 + \phi_h^4$$

Integrating out ϕ_h in the second term gives

$$\int_x 6\phi_l^2 \frac{1}{(4\pi)^2}\int_{\Lambda^2}^{\Lambda_0^2} dp^2 p^2 \frac{1}{p^2} = \frac{6}{(4\pi)^2}[\Lambda_0^2 - \Lambda^2]\int_x \phi_l^2$$

If we take Λ and $\Lambda(1-dt)$ instead of Λ_0 and Λ we get

$$\frac{d\Delta S_4}{dt} = \frac{6u}{(4\pi)^2}[2\Lambda^2]\int_x \phi_l^2 \tag{2.4.1}$$

Thus we see that $\dot{\Delta} S_4 \neq 0$ even in a free theory. One must add ΔS_2 with $r_0 = -\frac{6u}{(4\pi)^2}$. So in dimensionless variables

$$\Delta S = \frac{1}{4!}\int d^4\bar{x}\ u\bar{\phi}^4 + \frac{1}{2}\int d^4\bar{x}\ r_0\bar{\phi}^2 \tag{2.4.2}$$

satisfies $\dot{\Delta} S = 0$ and has $d_m = 0$. This is the usual "quadratic" divergence in scalar field theory in another guise.

The simple calculation above is in the spirit of the Wilsonian RG and is described further in the next section below. The above simple calculation also indicates the need to renormalize the operators when taking the continuum limit. In the interacting case, the Λ dependence will be more complicated. There will in general be mixing among all operators of a given dimension.

2.4.2 Formal definition

Composite Operators of definite scaling dimension using the ERG were discussed in [13]. A good discussion of composite operators is given in [52] and some of it is summarized in this section below. Many other aspects of composite operators in ϕ^4 field theory are discussed in [54, 55, 56, 57]. In particular, few works on energy-momentum tensor and corresponding correlators have been done [59, 60].

A Composite operator in ERG is defined as the operator obtained by the evolution of a bare operator under ERG flow. Consider an operator O_B in the bare theory. Define the low energy propagator as

$$\Delta_l = \frac{K(p)}{p^2}$$

where $K(p)$ is a smooth momentum cutoff function. For eg.

$$K(p) = e^{-\frac{p^2}{\Lambda^2}}$$

and

$$K_0(p) = e^{-\frac{p^2}{\Lambda_0^2}}$$

We also define

$$\Delta_h(p) = \frac{K_0(p) - K(p)}{p^2}$$

the high energy propagator. It propagates modes mainly between Λ_0, Λ. The full propagator of the bare theory is $\Delta = \Delta_l + \Delta_h$.

Define the Wilson Action S_Λ and the interacting part of the Wilson Action $S_{I,\Lambda}$ by

$$\int \mathcal{D}\phi_h e^{-S_B[\phi_l+\phi_h]} = \int \mathcal{D}\phi_h e^{-\frac{1}{2}\int \phi_l \Delta_l^{-1}\phi_l - \frac{1}{2}\int \phi_h \Delta_h^{-1}\phi_h - S_{I,B}[\phi_l+\phi_h]} = e^{\frac{1}{2}\int \phi_l \Delta_l^{-1}\phi_l - S_{I,\Lambda}[\phi_l]} = e^{-S_\Lambda} \tag{2.4.3}$$

where $S_{I,B}$ is the interacting part of the bare action. The first equality in this can be proved [52]. The rest are definitions. This defines an ERG flow from Λ_0 to Λ.

S_Λ is a theory where Λ is a UV cutoff. It may be obtained as above by integrating out modes in a bare theory defined at a higher scale. From the point of view of this bare theory, Λ is an IR cutoff during the integration process. Nevertheless, a fixed point Wilson action S_Λ defined as a stationary solution of the ERG equation has an existence in its own right without reference to a bare theory from which it is derived. In this viewpoint, Λ is indeed a UV cutoff. We take this viewpoint in this paper.

We give below some equivalent ways of defining a composite operator in ERG:

Definition I

The composite operator of this operator at scale Λ, O_Λ is defined as:

$$\int \mathcal{D}\phi_h O_B[\phi_l + \phi_h] e^{-\frac{1}{2}\int \phi_h \Delta_h^{-1}\phi_h - S_{I,B}[\phi_l+\phi_h]} = O_\Lambda[\phi_l] e^{-S_{I,\Lambda}[\phi_l]}$$

The composite operator defined as above has the useful property: [52]

$$\langle O_B(p)\phi(p_1)\phi(p_2)..\phi(p_n)\rangle_{\Lambda_0} = \prod_{i=1}^{n} \frac{K_0(p_i)}{K(p_i)} \langle [O]_\Lambda(p)\phi(p_1)\phi(p_2)...\phi(p_n)\rangle_\Lambda$$

Definition II

A useful way to think about composite operators in ERG is in terms of evolution operators. Define an ERG evolution operator $U(f,i)$ from theory of scale Λ_i to Λ_f by

$$e^{-S_\Lambda[\phi_f]} = U(f,i) e^{-S_B[\phi_i]}$$

Then

$$O_\Lambda[\phi_f] U(f,i) e^{-S_B[\phi_i]} = U(f,i) O_B[\phi_i] e^{-S_B[\phi_i]}$$

Thus formally one can write this as

$$O_\Lambda[\phi_f] = U(f,i) O_B[\phi_i][U(f,i)]^{-1} \tag{2.4.4}$$

Definition III

We can also think of perturbing S_B with a term of order ϵ and calculate the change in S_Λ to order ϵ:

$$\int \mathcal{D}\phi_h e^{-\frac{1}{2}\int \phi_h \Delta_h^{-1}\phi_h - S_{B,I}[\phi_l+\phi_h] + \epsilon O_B[\phi_l+\phi_h]} = e^{-S_{I,\Lambda}[\phi_l] + \epsilon O_\Lambda[\phi_l]} \tag{2.4.5}$$

This definition leads to a functional differential equation and is also a convenient way of defining O_Λ. In this thesis, we use this approach. This equation is in fact the linearized ERG equation for a perturbation ΔS obtained from (2.2.1).

$$\frac{\partial \Delta S}{\partial t} = \int_p \Big\{ (-K'(p^2))[\underbrace{\frac{\delta^2 \Delta S}{\delta\phi(p)\delta\phi(-p)}}_{1} \underbrace{-2\frac{\delta S}{\delta\phi(p)}\frac{\delta \Delta S}{\delta\phi(-p)}}_{2}] \underbrace{-2\frac{p^2 K'}{K}\phi(p)\frac{\delta \Delta S}{\delta\phi(p)}}_{3} +$$
$$+ \frac{-\eta}{2}\frac{K(p^2)(1-K(p^2))}{p^2}[\frac{\delta^2 \Delta S}{\delta\phi(p)\delta\phi(-p)} - 2\frac{\delta S}{\delta\phi(p)}\frac{\delta \Delta S}{\delta\phi(-p)}] + \frac{-\eta}{2}\phi(p)\frac{\delta \Delta S}{\delta\phi(p)}\Big\}$$
$$+ [(1-\frac{D}{2})N_\phi + D - N_p]\Delta S \qquad (2.4.6)$$

where $t = -\log\frac{\Lambda}{\Lambda_0}$ and all the variables are dimensionless.

This equation defines the Λ or t dependence, given some starting operator at the initial time. Eigen-operators are defined by the property that

$$\frac{\partial \Delta S}{\partial t} = d_m \Delta S + \beta(\lambda)\frac{\partial \Delta S}{\partial \lambda} \qquad (2.4.7)$$

i.e. under RG evolution at a fixed point they just scale as $e^{d_m t}$ where d_m is the (length) scaling dimension.The second term $\beta(\lambda)$ is zero at the fixed point. Actually this is true for operators integrated over all space. In most places in our work ΔS is chosen to be of the form $g_i \int_x O^i(x)$, i.e. integrated over space and thus correspond to some coupling constant in the action. From the integrated form one can determine $O(x)$ up to *total derivatives*. Thus $O(x)$ and $O(x) + \partial_\mu O^\mu(x)$ will give the same ΔS. To determine $O(x)$ unambiguously one would have to make $g_i(x)$ space dependent. This complicates the (already involved) algebra, especially at two loops and is not attempted in our work. [1]

2.4.3 Boundary Conditions on Composite Operators:

In the first two definitions, it is evident that there is a boundary condition for $O_\Lambda[\phi]$, namely that at $\Lambda = \Lambda_0$ it becomes equal to $O_B[\phi]$. Similarly, while solving the eigenvalue equation at the Wisher-Fisher fixed point in our work we put the initial condition that at $\Lambda = \Lambda_0$ $O_\Lambda[\phi]$

[1] Just as an illustration, the leading order result for the relevant unintegrated operator ϕ^2 is given in Section 3.1

reduces to $O_B[\phi]$. We choose O_B as in the Gaussian theory, namely

$$\mathcal{O}_2 = \phi^2 \quad at \quad \Lambda = \Lambda_0 \tag{2.4.8a}$$

$$\mathcal{O}_4 = \phi^4 \quad at \quad \Lambda = \Lambda_0 \tag{2.4.8b}$$

Correction to this will be evaluated in a perturbation series as powers of λ. Thus $\Delta S(\lambda = 0)$ will be equal to corresponding operator in Gaussian theory (which is given in subsection 4.3.1). The corrections will be chosen to be in terms of the high energy propagator,which vanishes when $\Lambda = \Lambda_0$. All the correction terms thus vanish at $\Lambda = \Lambda_0$. This implements the required boundary condition. In continuum limit one may have to add further counterterms in order to keep the operators finite which will modify the corresponding boundary conditions.

Many aspects of these local operators are discussed in [54, 55, 56, 57]. Some scaling properties are described in Appendix B.1.

It is to be noted that the concept of scaling dimension makes sense only if the theory has scale invariance. Thus S must correspond to a fixed point action that obeys

$$\frac{\partial S}{\partial t} = 0$$

But in general, one can solve a more general equation by putting

$$\frac{\partial S}{\partial t} = \sum_i \beta_i(\lambda_i) \frac{\partial S}{\partial \lambda_i}$$

where λ_i and $\beta_i(\lambda_i)$ are the various coupling constants and the beta functions of the theory. In fact, the expression of action which is a solution of this equation is easier to write down [43], hence we will be using this action in our work.

2.4.4 Simple examples

In our work, we will use definition III, so here to give a feeling of how to calculate a composite operator using the other two definitions we will give two instances.

Calculation of $[\phi(p)]$

We start with insertion of $\phi(p)$ in bare theory,

$$\int \mathcal{D}\phi\ \phi\ e^{-S_B[\phi]+J\phi}$$

Then decompose ϕ as $\phi_l + \phi_h$ to obtain,

$$\int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} \int \mathcal{D}\phi_h \quad (\phi_l + \phi_h) e^{-\frac{1}{2}\phi_h \frac{1}{\Delta_h}\phi_h - S_{I,B}[\phi_l+\phi_h] + J\phi_h}$$

Keeping aside $e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l}$ part

$$= \int \mathcal{D}\phi_h \quad (\phi_l + \phi_h) e^{-\frac{1}{2}\phi_h \frac{1}{\Delta_h}\phi_h - S_{I,B}[\phi_l+\phi_h] + J\phi_h}$$

$$= \int \mathcal{D}\phi_h \quad (\phi_l + \frac{\delta}{\delta J}) e^{-\frac{1}{2}\phi_h \frac{1}{\Delta_h}\phi_h - S_{I,B}[\phi_l+\phi_h] + J\phi_h}$$

Redefining $\phi' = \phi_h - \Delta_h J$ we get

$$= \int \mathcal{D}\phi' (\phi_l + \frac{\delta}{\delta J}) e^{-\frac{1}{2}\phi' \frac{1}{\Delta_h}\phi' - S_{I,B}[\phi_l + \phi' + \Delta_h J] + \frac{1}{2} J \Delta_h J}$$

$$= (\phi_l + \frac{\delta}{\delta J}) e^{-S_{I,\Lambda}[\phi_l + \Delta_h J] + \frac{1}{2} J \Delta_h J}$$

So we can write,

$$\int \mathcal{D}\phi \phi e^{-S_B[\phi]+J\phi}\ |_{J=0} = \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} (\phi_l - \Delta_h \frac{\delta S_{I,\Lambda}}{\delta \phi_l}) e^{-S_{I,\Lambda}[\phi_l + \Delta_h J] + \frac{1}{2} J \Delta_h J}\Big|_{J=0}$$

Hence from the definition I we get,

$$[\phi] = \phi_l - \Delta_h \frac{\delta S_{I,\Lambda}}{\delta \phi_l}$$

where $\Delta_h(p^2) = \frac{K_0(p^2) - K(p^2)}{p^2}$.

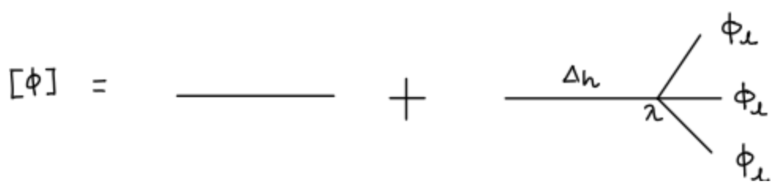
Note that if one considers $\lambda\phi^4$ theory,

$$\phi^4 = (\phi_l + \phi_h)^4 = \phi_l^4 + 4\phi_h \phi_l^3 + \ldots$$

So,

$$\boxed{[\phi] \approx \phi_l + \lambda \langle \phi_h \phi_h \rangle \phi_l^3} \tag{2.4.9}$$

Diagram wise it looks like



Calculation of $\mathcal{N}(p)$

We will use definition II in order to find $\mathcal{N}(p)$, i.e. composite operator corresponding to number operator $\phi \frac{\delta S}{\delta \phi}$. We need the evolution operator $U(f,i)$ of Polchinski's equation. For simplicity let's consider a field theory in zero dimension. The zero-dimensional field is denoted by x. Note that in zero dimensional system both fundamental field x and cut-off function K depends on only scale t as there is no momenta. The Polchinski's equation in zero dimension (from (2.2.2)) stands as

$$\frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2} \dot{K}(t) \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2K^{-1}(t)x \right) \psi(x,t)$$

where $\psi(x,t) = e^{-S[\phi]}$, $t = -\log \frac{\Lambda}{\Lambda_0}$.

Let's change variable $(x,t) \to (y,\tau)$. Where $y(t) = \frac{x(t)}{\sqrt{K(t)}}$ and $\tau = t$. After that define $g(\tau) = -\log K(\tau)$. Finally we get,

$$\frac{\partial \psi}{\partial g} = \frac{1}{2} \left[\frac{\partial^2 \psi}{\partial y^2} + y \frac{\partial \psi}{\partial y} \right]$$

So evolution operator $U(f,i)$ corresponding to the change from initial scale t_i to final scale t_f is,

$$U = e^{\int_{g_i}^{g} dg \frac{1}{2}\left[\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y}\right]} = e^{(g-g_i)\frac{1}{2}\frac{\partial^2}{\partial y^2} + (g-g_i)\frac{1}{2}y\frac{\partial}{\partial y}} \tag{2.4.10}$$

where $g|_{t_i} = g_i = -\log K(\tau_0) = -\log K_0$.

From definition II the composite operator $\mathcal{N}$ is to be found from the following expression

$$e^{a\mathcal{N}} U(f,i) = U(f,i) e^{ax\frac{\partial}{\partial x}}$$

It can be shown

$$\boxed{e^{a\mathcal{N}} = e^{a\left[\frac{K(K_0-K)}{K_0}\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x}\right]}} \tag{2.4.11}$$

Note that in higher dimensional QFT x gets replaced by $\phi(p)$, hence taking the form of (2.2.10) which has been used in Polchinski's equation.

This finishes our discussion about the background of both the next two chapters.

Chapter 3

Wilson action for O(N) model

We will find the fixed-point Wilson action by putting $\frac{\partial \bar{S}_t}{\partial t} = 0$ in (2.2.28). As we will work mostly with dimensionless variables we will remove the bar sign from the dimensionless variables unless otherwise mentioned. Also t dependence of actions and fields being readily implied, the subscript t will be omitted too. We give the fixed point action S in the following form:

$$S = S_2 + S_4 + S_6$$

where S_2 and S_4 are given by

$$S_2 = \int \frac{d^D p}{(2\pi)^D} U_2(p) \frac{1}{2} \phi^I(p) \phi^I(-p) \tag{3.0.1}$$

$$S_4 = \frac{1}{2} \prod_{i=1}^{3} \int \frac{d^D p_i}{(2\pi)^D} U_4(p_1, p_2; p_3, p_4) \frac{1}{2} \phi^I(p_1) \phi^I(p_2) \frac{1}{2} \phi^J(p_3) \phi^J(p_4) \tag{3.0.2}$$

where $p_1 + p_2 + p_3 + p_4 = 0$ is implied. Instead of putting an explicit delta function and integrating over p_4 we will simply impose momentum conservation at every stage. Accordingly S_6 is given by

$$S_6 = \frac{1}{3!} \prod_{i=1}^{5} \int \frac{d^D p_i}{(2\pi)^D} U_6(p_1, p_2; p_3, p_4; p_5, p_6) \frac{1}{2} \phi^I(p_1) \phi^I(p_2) \frac{1}{2} \phi^J(p_3) \phi^J(p_4) \frac{1}{2} \phi^K(p_5) \phi^K(p_6) \tag{3.0.3}$$

3.1 Result

The main result of this section is Wilson action at Wilson-Fisher fixed point which is stated below. (we have put D=4 for $\mathcal{O}(\epsilon^2)$ terms),

$$\boldsymbol{U_2(p) = \frac{p^2}{K(p^2)} - \lambda\frac{N+2}{2}\int\frac{d^Dp}{(2\pi)^D}K'(p^2) + \tilde{U}_2(p)} \tag{3.1.1}$$

The expression for $\tilde{U}_2(p)$ is given as

$$\begin{aligned}\boldsymbol{\tilde{U}_2(p_1)} &= \boldsymbol{-\frac{\lambda^2}{(16\pi^2)^2}\frac{(N+2)^2}{4}h(p_1) - \frac{(N+2)^2}{4}\frac{\lambda^2}{(16\pi^2)^2}} \\ &\quad \boldsymbol{+p_1^2\int_{p^2=0}^{p_1^2} dp^2 \frac{\int\frac{d^Dq}{(2\pi)^D}\Big\{-6\lambda^2(N+2)(-K'(q^2))F(p+q)\Big\} - \eta\, p^2}{2p^4}}\end{aligned} \tag{3.1.2}$$

$$\begin{aligned}\boldsymbol{U_4(p_1,p_2;p_3,p_4)} =& \boldsymbol{(4-D)\frac{16\pi^2}{N+8} + \frac{(N+2)}{2}\frac{\lambda^2}{16\pi^2}\sum_{j=1}^{4}h(p_j)} \\ & \boldsymbol{-\lambda^2\Big[(N+4)F(p_1+p_2) + 2F(p_1+p_3) + 2F(p_1+p_4)\Big]}\end{aligned} \tag{3.1.3}$$

where

$$F(p) = \frac{1}{2}\int\frac{d^Dp}{(2\pi)^D}h(q)\Big[h(p+q) - h(q)\Big]$$

and

$$h(p) = \frac{K(0) - K(p^2)}{p^2}$$

$$U_6(p_1,p_2;p_3,p_4;p_5,p_6) = -\lambda^2 \Big\{ h(p_1+p_2+p_3) + h(p_1+p_2+p_4) + h(p_1+p_2+p_5)$$
$$+h(p_1+p_2+p_6) + h(p_1+p_3+p_4) + h(p_2+p_3+p_4) \Big\} \tag{3.1.4}$$

and the anomalous dimension is given by

$$\frac{\eta}{2} = \lambda^2 \frac{N+2}{4} \frac{1}{(16\pi^2)^2} = \frac{N+2}{(N+8)^2} \frac{\epsilon^2}{4} \tag{3.1.5}$$

To evaluate the integrals we have put $D=4$ and used a specific form of $K(p^2) = e^{-p^2}$.

3.2 Details of the calculation

When one puts the ansatz (3.0.1) and (3.0.3) in the eq.(2.2.28) one gets the equations for the vertices U_2, U_4 and U_6. In this section we will state this procedure. As a corollary, we also got the fixed point value of the coupling constant λ. We have also found the anomalous dimension of the field by considering U_2 vertex equation in the subleading order.

3.2.1 Equations for the vertices

Equation for U_2

$$0 = \int \frac{d^Dp}{(2\pi)^D} \Bigg\{ \left(\frac{-\eta}{2} \frac{K(1-K)}{p^2} - K'(p^2) \right) \frac{1}{8} \Big[4NU_4(p_1,-p_1;p,-p) + 8U_4(p_1,p;-p_1,-p) \Big]$$
$$-\frac{1}{2!} 2U_2(p)U_2(p)\delta^D(p-p_1) \Bigg\} + \left(\frac{-\eta}{2} + 1 - 2\frac{p_1^2}{K(p_1^2)} K'(p_1^2) \right) U_2(p_1) - \frac{1}{2!} p_1 \frac{dU_2(p_1)}{dp_1} \tag{3.2.1}$$

Equation for U_4

$$
\begin{aligned}
0 = &\int \frac{d^Dp}{(2\pi)^D}\left(\frac{-\eta}{2}\frac{K(1-K)}{p^2} - K'(p^2)\right)\frac{1}{48} \\
&\times\Big\{6NU_6(p_1,p_2;p_3,p_4;p,-p) + 12U_6(p_1,p;p_2,-p;p_3,p_4) + 12U_6(p_1,p_2;p_3,p;p_4,-p)\Big\} \\
&-\sum_{j=1}^{4}\left(\frac{-\eta}{2}\frac{K(1-K)}{p_j^2} - K'(p_j^2)\right)U_2(p_j)\ \frac{2}{8}U_4(p_1,p_2;p_3,p_4) \\
&+\sum_{j=1}^{4}\left(\frac{-\eta}{2} - 2\frac{p^2}{K(p_j^2)}K'(p_j^2)\right)\ \frac{1}{8}U_4(p_1,p_2;p_3,p_4) \\
&+\left[4-D-\sum_{i=1}^{4}p_i\frac{d}{dp_i}\right]\frac{1}{8}U_4(p_1,p_2;p_3,p_4)
\end{aligned} \tag{3.2.2}
$$

Here $p = p_a + p_b + p_n = -(p_i + p_j + p_m)$.

Equation for U_6

$$
\begin{aligned}
0 =&\frac{2}{48}\sum_{6\ perm\ of\ (m,n)}\left(\frac{-\eta}{2}\frac{K(1-K)}{(p_i+p_j+p_m)^2)} - K'((p_i+p_j+p_m)^2)\right)U_4(p_i,p_j;p_m,p)U_4(p_a,p_b;p_n,-p) \\
&+\sum_{j=1}^{6}\left(K'(p_j^2) - \frac{-\eta}{2}\frac{K(1-K)}{p_j^2}\right)U_2(p_j)\frac{2}{48}U_6(p_1,p_2;p_3,p_4;p_5,p_6) \\
&+\sum_{j=1}^{6}\left(\frac{-\eta}{2} - 2\frac{p^2}{K(p_j^2)}K'(p_j^2)\right)\frac{1}{48}U_6(p_1,p_2;p_3,p_4;p_5,p_6) + \left[6-2D-\sum_{i=1}^{6}p_i\frac{d}{dp_i}\right]\frac{1}{48}U_6(p_1,p_2;p_3,p_4;p_5,p_6)
\end{aligned} \tag{3.2.3}
$$

3.2.2 Solving the Equations

We know that $U_4 \approx \mathcal{O}(\epsilon)$ and $U_6 \approx \mathcal{O}(\epsilon^2)$ and $\eta \approx \mathcal{O}(\epsilon^2)$, where $\epsilon = 4 - D$.

$\mathcal{O}(1)$: Retrieving Gaussian theory

We start with (3.2.1) for U_2. Neglecting U_4 and η and collecting coefficients of ϕ^2 we get

$$
0 = K'(p^2)U_2(p)U_2(p) + \left(1 - 2\frac{p^2}{K(p^2)}K'(p^2)\right)U_2(p) - p^2\frac{dU_2(p)}{dp^2} \tag{3.2.4}
$$

$U_2(p) = \frac{p^2}{K(p^2)}$ solves this equation. This is expected since the Gaussian theory is expected to be a fixed point - and this ERG was obtained from Polchinski's ERG by adding on the kinetic term $\frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\phi(p)\frac{p^2}{K(p^2)}\phi(-p)$. Thus our solution can be written as

$$\boldsymbol{U_2(p) = \frac{p^2}{K(p^2)} + \underbrace{U_2^{(1)}(p)}_{\mathcal{O}(\epsilon)} + \mathcal{O}(\epsilon^2)} \tag{3.2.5}$$

$\mathcal{O}(\epsilon)$: Correction to mass term

We go back to (3.2.1) and keep U_4 which is $\mathcal{O}(\epsilon)$ but drop η which is $\mathcal{O}(\epsilon^2)$.

$$\begin{aligned}0 = &\int \frac{d^Dp}{(2\pi)^D}\left(\frac{-\eta}{2}\frac{K(1-K)}{p^2} - K'(p^2)\right)\times \\ &\left\{\frac{1}{8}\Big[4NU_4(p_1,p_2;p,-p) + 8U_4(p_1,p;-p,-p_1)\Big] - \frac{1}{2!}2U_2(p)U_2(p)\delta^D(p-p_1)\right\} \\ &+\left(\frac{-\eta}{2} + 1 - 2\frac{p_1^2}{K(p_1^2)}K'(p_1^2)\right)U_2(p_1) - \frac{1}{2!}p_1\frac{dU_2(p_1)}{dp_1}\end{aligned} \tag{3.2.6}$$

We use (3.2.5) in the above equation and look at the terms of order ϵ. To leading order we set $U_4 = \lambda$, which is $\mathcal{O}(\epsilon)$. The equation for $U_2^{(1)}$ is given by

$$0 = -\lambda\frac{4N+8}{8}\int \frac{d^Dp}{(2\pi)^D}K'(p^2) + 2\frac{p_1^2}{K(p_1^2)}U_2^{(1)}\Big(p_1)K'(p_1^2) + (1 - 2\frac{p_1^2}{K(p_1^2)}K'(p_1^2)\Big)U_2^{(1)}(p_1) - p_1^2\frac{dU_2^{(1)}(p_1)}{dp_1^2}$$

To leading order this equation is solved by a constant $U_2^{(1)}$, i.e.

$$0 = -\lambda\frac{4N+8}{8}\int \frac{d^Dp}{(2\pi)^D}K'(p^2) + U_2^{(1)} \tag{3.2.7}$$

Thus

$$U_2^{(1)} = \lambda\frac{N+2}{2}\int \frac{d^Dp}{(2\pi)^D}K'(p^2) \tag{3.2.8}$$

Here

$$\int \frac{d^Dp}{(2\pi)^D} = \frac{1}{2^D\pi^{D/2}\Gamma(D/2)}\int (p^2)^{\frac{D-2}{2}}dp^2$$

To get leading results we can set $D = 4$:

$$U_2^{(1)} = \lambda\frac{4N+8}{8}\frac{1}{(4\pi)^2}\int_0^\infty dp^2p^2K'(p^2) = -\lambda\frac{4N+8}{8}\frac{1}{(4\pi)^2}\int_0^\infty dp^2K(p^2) \tag{3.2.9}$$

We have used $K(0) = 1, K(\infty) = 0$. This gives the fixed point value of the dimensionless mass parameter:

$$U_2^{(1)} = m_\star^2 = -\lambda \frac{N+2}{2} \frac{1}{(4\pi)^2} \int_0^\infty dp^2 K(p^2) \tag{3.2.10}$$

To evaluate the integral explicitly we need a specific form for K. We use $K(p^2) = e^{-p^2}$. Then the integral is equal to 1.

$\mathcal{O}(\epsilon^2)$: Expression for the six-point vertex

Let us turn to (3.2.3) reproduced below:

$$\begin{aligned}
0 = &-\frac{2}{48} \sum_{6\ perm\ of\ (i,j,m)} \left(\frac{-\eta}{2} \frac{K(1-K)}{(p_i+p_j+p_m)^2} - K'((p_i+p_j+p_m)^2) \right) U_4(p_i,p_j;p_m,p) U_4(p_a,p_b;p_n,-p) \\
&+ \sum_{j=1}^{6} \left\{ \left(K'(p_j^2) - \frac{-\eta}{2} \frac{K(1-K)}{p_j^2} \right) 2U_2(p_j) + \left(\frac{-\eta}{2} - 2\frac{p^2}{K(p_j^2)} K'(p_j^2) \right) \right\} \frac{1}{48} U_6(p_1,p_2;p_3,p_4;p_5,p_6) \\
&+ \left[6 - 2D - \sum_{i=1}^{6} p_i \frac{d}{dp_i} \right] \frac{1}{48} U_6(p_1,p_2;p_3,p_4;p_5,p_6)
\end{aligned} \tag{3.2.11}$$

where $p = p_a + p_b + p_n = -(p_i + p_j + p_m)$.

In this equation we keep terms of $\mathcal{O}(\epsilon^2)$. Since η is $\mathcal{O}(\epsilon^2)$, and multiplies terms of $\mathcal{O}(\epsilon^2)$, it contributes only at $\mathcal{O}(\epsilon^4)$ in this equation, so it can be dropped here. Furthermore then, if we use the leading order solution for $U_2 = \frac{p^2}{K(p^2)}$, the second and third terms cancel each other. So we are left with

$$\begin{aligned}
0 = &-\frac{2}{48} \sum_{6\ perm\ (i,j,m)} K'\big((p_i+p_j+p_m)^2\big) U_4(p_i,p_j;p_n,p) U_4(p_a,p_b;p_n,-p) \\
&+ \left[(6 - 2D - \sum_{i=1}^{6} p_i \frac{d}{dp_i}) \right] \frac{1}{48} U_6(p_1,p_2;p_3,p_4;p_5,p_6)
\end{aligned} \tag{3.2.12}$$

Since $U_4 = \lambda$ to this order, we obtain

$$0 = \lambda^2 \frac{2}{48} \sum_{6\ perm\ (i,j,m)} K'((p_i+p_j+p_m)^2) + \left[6 - 2D - \sum_{i=1}^{6} p_i \frac{d}{dp_i} \right] \frac{1}{48} U_6(p_1,p_2;p_3,p_4;p_5,p_6) \tag{3.2.13}$$

The solution for one permutation is

$$U_6(p_1, p_2; p_3, p_4; p_5, p_6) = \lambda^2 \frac{K((p_1 + p_2 + p_3)^2) - K(0)}{(p_1 + p_2 + p_3)^2}$$

The full solution is given by

$$\begin{aligned} U_6(p_1, p_2; p_3, p_4; p_5, p_6) = -\lambda^2 \{ & h(p_1 + p_2 + p_3) + h(p_1 + p_2 + p_4) + h(p_1 + p_2 + p_5) \\ & + h(p_1 + p_2 + p_6) + h(p_1 + p_3 + p_4) + h(p_2 + p_3 + p_4) \} \end{aligned} \tag{3.2.14}$$

where $h(x) = \frac{K(0)-K(x)}{x^2}$.

Fixed Point value of λ: Solution for U_4 at $\mathcal{O}(\epsilon)$

The U_4 equation is given by (3.2.2). In this equation, η can be neglected as $-\eta \approx \mathcal{O}(\epsilon^2)$. Also, we put the value of U_2 up to order of ϵ found above. There is a cancellation between the second and third terms on the R.H.S and we obtain

$$\begin{aligned} & \left[\left(4 - D - \sum_{i=1}^{4} p_i \frac{d}{dp_i} \right) - \sum_{j=1}^{4} 2K'(p_j^2) \frac{\lambda}{16\pi^2} \frac{N+2}{2} \right] \frac{1}{8} U_4(p_1, p_2; p_3, p_4) \\ = & \int \frac{d^D p}{(2\pi)^D} K'(p^2) \frac{1}{48} \left\{ 6N U_6(p_1, p_2; p_3, p_4; p, -p) + 12 U_6(p_1, p; p_2, -p; p_3, p_4) + 12 U_6(p_1, p_2; p_3, p; p_4, -p) \right\} \end{aligned} \tag{3.2.15}$$

The solution is given in the Appendix (A.1.1). The fixed point value λ^* given below solves the above equation:

$$\lambda^* = (4 - D) \frac{16\pi^2}{N + 8} \tag{3.2.16}$$

3.2.3 Determining Anomalous Dimension

U_2 equation at $\mathcal{O}(\epsilon^2)$

$$\begin{aligned} 0 = & \int \left\{ \frac{d^D p}{(2\pi)^D} \left(\frac{-\eta}{2} \frac{K(1-K)}{p^2} - K'(p^2) \right) \left[\frac{\delta^2 S_4}{\delta\phi^I(p) \delta\bar{\phi}^I(-p)} - \frac{\delta S_2}{\delta\phi^I(p)} \frac{\delta S_2}{\delta\phi^I(-p)} \right] \right\} \\ & + \left\{ -\frac{\eta}{2} - 2 \frac{p^2}{K(p^2)} K'(p^2) \right\} \phi(p). \frac{\delta S}{\delta\phi(p)} + \mathcal{G}^c_{dil} S_2 \end{aligned}$$

where we plug in:

$$\begin{aligned} U_4(p_1,p_2;p_3,p_4) &= \lambda + \underbrace{\tilde{U}_4(p_1,p_2;p_3,p_4)}_{O(\epsilon^2)} \\ U_2(p) &= \frac{p^2}{K} - \lambda\frac{N+2}{2}\int\frac{d^Dp}{(2\pi)^D}K'(p^2) + \underbrace{\tilde{U}_2(p)}_{O(\epsilon^2)} \end{aligned} \tag{3.2.17}$$

and keep only $\mathcal{O}(\epsilon^2)$ terms in the above equation to get

$$\begin{aligned} 0 = &\int\frac{d^Dp}{(2\pi)^D}\left(\frac{-\eta}{2}\frac{K(1-K)}{p^2} - K'(p^2)\right)\times \\ &\left\{\frac{1}{8}\left[4N\tilde{U}_4(p_1,-p_1;p,-p) + 8\tilde{U}_4(p_1,p;-p,-p_1)\right] - \frac{1}{2!}2U_2(p)U_2(p)\delta^D(p-p_1)]\right\} \\ &+\left(\frac{-\eta}{2} + 1 - 2\frac{p_1^2}{K(p_1^2)}K'(p_1^2)\right)U_2(p_1) - p_1^2\frac{dU_2(p_1)}{dp_1^2} \end{aligned} \tag{3.2.18}$$

On simplification it gives

$$-\frac{-\eta}{2}\frac{(1-K)}{K}p_1^2 - \int\frac{d^Dp}{(2\pi)^D}K'(p^2)\frac{1}{8}\left[4N\tilde{U}_4(p_1,-p_1;p,-p) + 8\tilde{U}_4(p_1,p_1;-p,-p_1)\right] + K'(p_1^2)U_2(p_1)U_2(p_1)$$

$$+\frac{-\eta}{2}\frac{p_1^2}{K} + \tilde{U}_2(p_1) - p_1^2\frac{d\tilde{U}_2(p_1)}{dp_1^2} = 0 \tag{3.2.19}$$

In the L.H.S the third term will cancel with part of the second term (shown in A.1.3). Also the raison d'etre for introducing η is to ensure that $U_2 = p^2 + \mathcal{O}(p^4)$. So we let $\tilde{U}_2 = \mathcal{O}(p^4)$. So The anomalous dimension is given by

$$\boldsymbol{\frac{\eta}{2} = -\frac{d}{dp_1^2}\int\frac{d^Dp}{(2\pi)^D}K'(p^2)\frac{1}{8}\left[4N\tilde{U}_4^{II}(p_1,-p_1;p,-p) + 8\tilde{U}_4^{II}(p_1,p;-p_1,-p)\right]\Bigg|_{p_1^2=0}} \tag{3.2.20}$$

Here the superscript II is explained in Appendix A and refers to a class of Feynman diagrams.

$\tilde{U}_4$ is determined by solving (3.2.15). So using (3.2.20) and (A.1.19) one can determine η. This is done in the Appendix (A.1.4). The result is of course well known [13]:

$$\frac{\eta}{2} = \lambda^2\frac{N+2}{4}\frac{1}{(16\pi^2)^2} = \frac{N+2}{(N+8)^2}\frac{\epsilon^2}{4} \tag{3.2.21}$$

Chapter 4

Composite Operator

In chapter 2 we have studied the definitions and simple computations of composite operators. We will choose ϕ^4 and ϕ^2 operators in a bare O(1) theory with quartic interaction at 4-ϵ dimensions and calculate the corresponding composite operators at the Wilson-Fisher fixed point. Then we will calculate those operators with their scaling dimension up to $\mathcal{O}(\epsilon^2)$.

4.1 Operators near fixed point

The velocities of the RG trajectories at the critical surface is non-zero everywhere except at the fixed point. So any theory on the critical surface ends up in the fixed point (point P_∞ in Fig. 4.1). Hence it is important to study the basis of the critical surface. This surface is spanned by irrelevant or marginally irrelevant operators. Irrelevant operators are the negative mass dimension operators, hence as expected will get suppressed by negative powers of $\frac{\Lambda}{\Lambda_0}$. A typical example of irrelevant operators in ϕ^4 theory is ϕ^6, ϕ^8,.... etc(curve E in Fig. 4.1). A dimension zero operator in bare theory can become irrelevant near the fixed point for example here it is ϕ^4 operator (section 4.3).

On the other hand, as velocity is zero at the fixed point, the way out of the critical surface to high temperature fixed point (point P_0 in Fig. 4.1) can be only through the fixed point P_∞. These directions are described by the positive mass dimension or relevant operators. Studying these directions is also important because critical exponents bear proof of universality. ϕ^4 theory has only one relevant direction that is ϕ^2 (curve G in Fig. 4.1).

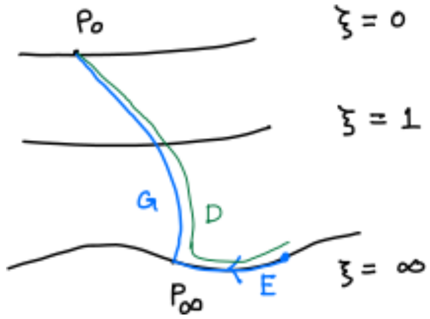


Figure 4.1: Curves nearby the critical surface and fixed point intersecting surfaces of different correlation lengths ξ. Curve G is the only route to be out from the critical surface. Any point E on the critical surface ends up at P_∞. The curve D spends an infinite amount of time near the fixed point. Trajectories with finite ξ end up at P_0.

4.2 Problem

We want to find the anomalous dimension of two important composite operators upto $\mathcal{O}(\epsilon^2)$ at the Wilson-Fisher fixed point of O(1) model with quartic interaction -1) Relevant operator ϕ^2 whose dimension is always positive, 2)Marginally Irrelevant operator ϕ^4 whose dimension is zero at the bare theory but becomes negative near the fixed point. The same dimensions in leading order have been found out in [13]. We find agreement with our result. For subleading order, the expected value of the anomalous dimension using dimensional regularization has been stated. In limit $\Lambda_0 \to \infty$, we recover them from our ERG analysis. So, using (2.4.6), (2.4.7) and the W-F action found in the previous chapter we set to find the anomalous dimension of the above bare operators at the W-F fixed point of the O(1) model. In this process, the respective composite operators automatically come out.

The Action

We give the action S in the form:

$$S = S_2 + S_4 + S_6$$

S_2 and S_4 are given by

$$S_2 = \int \frac{d^D p}{(2\pi)^D} U_2(p) \frac{1}{2} \phi^I(p) \phi^I(-p) \tag{4.2.1}$$

$$S_4 = \frac{1}{2} \prod_{i=1}^{3} \int \frac{d^D p_i}{(2\pi)^D} U_4(p_1, p_2; p_3, p_4) \frac{1}{2} \phi^I(p_1) \phi^I(p_2) \frac{1}{2} \phi^J(p_3) \phi^J(p_4) \tag{4.2.2}$$

where $p_1 + p_2 + p_3 + p_4 = 0$ is implied. Instead of putting an explicit delta function and integrating over p_4 we will simply impose momentum conservation at every stage. Accordingly, we write

$$S_6 = \frac{1}{3!} \prod_{i=1}^{5} \int \frac{d^D p_i}{(2\pi)^D} U_6(p_1, p_2; p_3, p_4; p_5, p_6) \frac{1}{2} \phi^I(p_1) \phi^I(p_2) \frac{1}{2} \phi^J(p_3) \phi^J(p_4) \frac{1}{2} \phi^K(p_5) \phi^K(p_6) \tag{4.2.3}$$

The expression of the action which satisfies (2.2.19) is given in section 3.1 . But note that the value of 2-pt vertex $U_2(p)$ at $\mathcal{O}(\epsilon^2)$ is complicated. Hence we solve a more general equation where the LHS of (2.2.19) is not set to zero but to $\frac{\partial S}{\partial t} = \beta_J \frac{\partial S}{\partial \lambda_J}$. The parameters can be chosen so that the beta functions are zero. This has the effect that the equations are modifed at each order by terms of higher order. The advantage is that the solutions are easier to write down. The vertices in this set-up have been found in[43],

$$\boldsymbol{U_2(p) = -\lambda(N+2)v_2 - \lambda^2 \left(3(N+2)G(p) + (N+2)^2 (v_2)^2 h(p)\right)} \tag{4.2.4a}$$

$$\begin{aligned}\boldsymbol{U_4(p_1, p_2; p_3, p_4)} &= \boldsymbol{-\lambda^2 \Big((N+4)F(p_1+p_2) + 2F(p_1+p_3) + 2F(p_1+p_4)} \\ &\quad \boldsymbol{-(N+2)v_2 \sum_{i=1}^{4} h(p_i) \Big)}\end{aligned} \tag{4.2.4b}$$

$$\begin{aligned}\boldsymbol{U_6(p_1, p_2; p_3, p_4; p_5, p_6)} &= \boldsymbol{-\lambda^2 \,(h(p_1+p_2+p_3) + h(p_1+p_2+p_4) + h(p_1+p_2+p_5)} \\ &\quad \boldsymbol{+h(p_1+p_2+p_6) + h(p_3+p_4+p_1) + h(p_3+p_4+p_2))}\end{aligned} \tag{4.2.4c}$$

where

$$f(p) = -2K'(p^2); \;\; h(p) = \frac{K(0) - K(p^2)}{p^2}$$

and

$$\boldsymbol{v_2 = \frac{1}{2-\epsilon} \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} f(p)} \tag{4.2.5}$$

If we take the limit $\epsilon \to 0$ and $K(p^2) = e^{-p^2}$ we get

$$v_2 = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} e^{-p^2} = \frac{1}{2} \frac{1}{16\pi^2}$$

$$F(p) = \frac{1}{2}\int \frac{d^Dp}{(2\pi)^D} h(q)\Big[h(p+q) - h(q)\Big]$$

The coupling constant λ is given, to order $\epsilon = 4 - D$, as

$$\boldsymbol{\lambda = \frac{\epsilon}{-\beta_N^{(1)}} = \frac{(4\pi)^2}{N+8}}\,\epsilon \tag{4.2.6}$$

The anomalous dimension is given, to order ϵ^2, as

$$\boldsymbol{\eta = \frac{N+2}{2(N+8)^2}\,\epsilon^2} \tag{4.2.7}$$

4.2.1 Anomalous Dimension-what to expect

As we will calculate the anomalous dimension of the composite operators at the Wilson-Fisher fixed point, let us do some simple calculation to understand what to expect as the anomalous dimensions. Consider a bare action at scale Λ_0 and evolve to Λ which is close to Λ_0.

$$S_{\Lambda_0} = \int_x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m_0^2\phi^2 + \lambda_0\frac{\phi^4}{4!}\right] \tag{4.2.8}$$

The operator $\frac{\phi^4}{4!}$ is the marginal operator of the bare theory. Later it will turn out to be irrelavnt operator for $D < 4$ when one will consider higher order contributions. If one see the term ϕ^4 as perturbation, the bare operator ϕ^4 can be interpreted as a composite operator at scale Λ_0 (see (2.4.5)) and can be defined as:-

$$\frac{\partial S_{\Lambda_0}}{\partial\lambda_0} = \int_x \frac{\partial\mathcal{L}_{\Lambda_0}}{\partial\lambda_0} = \int_x \frac{\phi^4}{4!} \tag{4.2.9}$$

Similarly, the relevant operator $\frac{1}{2}\phi^2$ is defined as

$$\frac{\partial S_{\Lambda_0}}{\partial m_0^2} = \int_x \frac{\partial\mathcal{L}_{\lambda_0}}{\partial\lambda_0} = \int_x \frac{\phi^2}{2} \tag{4.2.10}$$

S_Λ is obtained by evolving down from Λ_0 to Λ i.e. by integrating modes $\Lambda < p < \Lambda_0$.

If we apply Definition III ((2.4.5)) of the composite operators given in the previous chapter, $\frac{\partial S_\Lambda}{\partial \lambda_0}$ is a composite operator and defines in fact what we call $[\phi^4]/4!$.

$$\frac{\partial S_\Lambda}{\partial \lambda_0} \equiv \int_x \frac{[\phi^4]_\Lambda}{4!} \tag{4.2.11}$$

We can expect S_Λ to look like the following:

$$S_\Lambda = \int_x [(1-\delta Z(t))\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}(m_0^2 + \delta m_0(t)^2)\phi^2 + (\lambda_0 + \delta\lambda_0(t))\frac{\phi^4}{4!} + O(1/\Lambda)] \tag{4.2.12}$$

Here δZ is the correction to the kinetic term coming from the two loop diagram at $O(\lambda^2)$, $\delta m_0^2 \approx O(\lambda)$ and $\delta\lambda_0 \approx O(\lambda^2)$ are the corrections starting at one loop.

Adding and subtracting terms we can write S_Λ as:

$$= \int_x [\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}(m_0^2 + \delta m_0(t)^2 + \delta Z m_0^2)\phi^2 + (\lambda_0 + \underbrace{\delta\lambda_0(t) + 2\delta Z\lambda_0}_{\bar{\delta}\lambda_0(t)})\frac{\phi^4}{4!} + O(1/\Lambda)]$$

$$-\delta Z[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m_0^2\phi^2 + 2\lambda_0\frac{\phi^4}{4!}]$$

The beta function is defined by

$$\delta\lambda_0 \approx \beta(\lambda_0)t \tag{4.2.13}$$

and

$$\delta Z \approx -\frac{\eta}{2}t$$

with

$$\frac{\eta}{2} = \frac{\lambda_0^2}{(16\pi^2)^2}\frac{1}{12} \tag{4.2.14}$$

The mass anomalous dimension is defined by,

$$\delta m_0(t) \approx \gamma_m t \tag{4.2.15}$$

We write $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi = -\frac{1}{2}\phi\Box\phi$ and then use

$$-\delta Z[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m_0^2\phi^2 + 2\lambda_0\frac{\phi^4}{4!}] = -\delta Z\frac{1}{2}\phi\frac{\delta S}{\delta\phi}$$

$\phi\frac{\delta S}{\delta\phi}$ is called as the equation of motion operator. [1]

$$\mathcal{N} = -\int_p K e^{+S_\Lambda}\frac{\delta}{\delta\phi}([\phi]_\Lambda e^{-S_\Lambda}) \approx \int_p \phi\frac{\delta S}{\delta\phi} \tag{4.2.16}$$

$$\begin{aligned} S_\Lambda = &\int_x [\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}(m_0^2 + m_0^2\gamma_m t)\phi^2 + (\lambda_0 + \beta(\lambda_0)t)\frac{\phi^4}{4!} + O(1/\Lambda)] \\ &+ \frac{\eta}{2}t\int_x \frac{1}{2}\phi\frac{\delta S}{\delta\phi(x)} \end{aligned} \tag{4.2.17}$$

According to (4.2.11)

$$\frac{\partial S_\Lambda}{\partial\lambda_0} \equiv \int_x \frac{[\phi^4]_\Lambda}{4!} = (1 + \frac{\partial\beta(\lambda_0)}{\partial\lambda_0}t)\frac{\phi^4}{4!} + m_0^2\frac{\partial\gamma_m(\lambda_0)}{\partial\lambda_0}t\frac{1}{2}\phi^2 + \frac{1}{2}\frac{\partial\eta(\lambda_0)}{\partial\lambda_0}t\mathcal{N} + O(1/\Lambda)] \tag{4.2.18}$$

So,

$$\boxed{\frac{\partial}{\partial t}\int_x \frac{[\phi^4]_\Lambda}{4!} = (\frac{\partial\beta(\lambda_0)}{\partial\lambda_0})\frac{\phi^4}{4!} + m_0^2\frac{\partial\gamma_m(\lambda_0)}{\partial\lambda_0}\frac{1}{2}\phi^2 + \frac{1}{2}\frac{\partial\eta(\lambda_0)}{\partial\lambda_0}\mathcal{N} + O(1/\Lambda)]} \tag{4.2.19}$$

From chapter 11 of [67, 69][2], we get $\beta(\lambda_0)$ in our convention as,

$$\beta(\lambda_0) = \lambda_0(\epsilon - \frac{1}{16\pi^2}3\lambda_0 + \frac{1}{(16\pi^2)^2}\frac{17}{3}\lambda_0^2) \tag{4.2.20}$$

In the critical theory, we can set $m_0^2 = 0$. So if we collect the coefficient of ϕ^4 we get what we have defined above as d_m in the ERG evolution. We denote anomalous dimension of irrelevant operator as d_4 and that of relevant operator as d_2.

$$\begin{aligned} d_4 &= \epsilon - \frac{1}{16\pi^2}6\lambda_0 + \frac{1}{(16\pi^2)^2}17\lambda_0^2 + \frac{4}{(16\pi^2)^2}\frac{2\lambda_0^2}{12} \\ &= \epsilon - \frac{1}{16\pi^2}6\lambda_0 + \frac{1}{(16\pi^2)^2}\frac{53\lambda_0^2}{3} \end{aligned} \tag{4.2.21}$$

[1]More correctly at higher orders it should include the change in measure and becomes the "number operator". Here $[\phi]$ is the "composite operator" corresponding to ϕ and is defined by [52]

$$[\phi]_\Lambda(p) = \frac{K_0}{K}\phi(p) + \frac{K_0 - K}{p^2}\frac{\delta S_\Lambda}{\delta\phi(-p)}$$

[2]The coupling constants in the relevant equations in these two books differ by a factor of 2

For the relevant operator ϕ^2, analogously one can define it as

$$\frac{\partial S_\Lambda}{\partial m_0^2} = \int_x \frac{[\phi^2]_\Lambda}{2} \tag{4.2.22}$$

So applying this to (4.2.17)

$$\frac{\partial}{\partial t} \int_x \frac{[\phi^2]_\Lambda}{2} = \gamma_m \int_x \frac{[\phi^2]_\Lambda}{2} \tag{4.2.23}$$

From [67, 69][3] we get for the two-loop anomalous dimension

$$\gamma_m = \frac{\lambda_0}{16\pi^2} - \frac{1}{(16\pi^2)^2}\frac{5}{6}\lambda_0^2 \tag{4.2.24}$$

So length scaling dimension d_2 (in our notation) of the relevant operator $\int_x \phi^2$ is given by,

$$d_2 = 2 - \frac{\lambda_0}{16\pi^2} + \frac{1}{(16\pi^2)^2}\frac{5}{6}\lambda_0^2 \tag{4.2.25}$$

Note that the results of [67, 69] are obtained using the mass-independent dimensional renormalization scheme or "minimal subtraction". The scheme used in this paper is also mass independent. In mass-independent schemes the first two orders in the power series expansion of the beta function are well known to be scheme independent. The proof is given below:

Let

$$\beta(\lambda) = \frac{d\lambda}{dt} = b_2\lambda^2 + b_3\lambda^3$$

Let

$$\lambda' = \lambda + a\lambda^2$$

and

$$\beta'(\lambda') = \frac{d\lambda'}{dt} = b_2'\lambda'^2 + b_3'\lambda'^3 = b_2'(\lambda + a\lambda^2)^2 + b_3'(\lambda + a\lambda^2)^3 = b_2'\lambda^2 + (b_3' + 2ab_2')\lambda^3 + ...$$

But also

$$\frac{d\lambda'}{dt} = \beta(\lambda) + a2\lambda\beta(\lambda) = b_2\lambda^2 + b_3\lambda^3 + 2a\lambda(b_2\lambda^2 + b_3\lambda^3) = b_2\lambda^2 + (b_3 + 2ab_2)\lambda^3$$

[3]There is a factor of two in the definition of d_m

Comparing, we see that $b_2 = b'_2$ and $b_3 = b'_3$.

Thus, upto and including $O(\lambda^3)$, the beta functions in the ERG calculation and in dimensional regularization MS scheme are identical. This also means that at the fixed point (given by vanishing of beta function) the expresions relating ϵ and λ are scheme independent to the same order. Now, at the fixed point, the dimensions of operators expressed in terms of ϵ are eigenvalues of the dilatation operator of the CFT and thus universal (to any order in ϵ). These universal expressions in powers of ϵ, when re-expressed in terms of λ, will thus have to match to the lowest two orders in any mass-independent scheme. Thus the expressions obtained for d_2, d_4 in the ERG scheme must agree with the expressions given above. These expectations will be confirmed in this chapter.

4.2.2 Result

Leading Order

The anomalous dimension at the leading order we get as,

$$\boxed{\begin{aligned} d_2 =& 2 - \lambda F \\ d_4 =& \epsilon - 6F\lambda \end{aligned}} \tag{4.2.26a, 4.2.26b}$$

The corresponding eigenvectors are given by,

$$\mathcal{O}_2(q) = \frac{1}{2}\int_{p_1,p_2} \delta(p_1+p_2-q)[1+\lambda\frac{F}{2}(h(p_1-q)+h(p_2-q))+\lambda\mathcal{F}(q)] \quad \phi(p_1)\phi(p_2)$$
$$-\frac{1}{4!}\int_{p_1,p_2,p_3} \delta(p_1+p_2+p_3+p_4-q)\lambda\sum_{i=1}^{4} h(p_i-q)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{4.2.27a}$$

$$\mathcal{O}_4(0) = -\frac{1}{6!}\int_{p_1,p_2,p_3,p_4,p_5} \sum_{10 \; perm \; (i,j,k)} 2\lambda h(p_i+p_j+p_k)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$
$$+\frac{1}{4!}\int_{p_1,p_2,p_3}\left[\sum_{i=1}^{4} F\lambda h(p_i) - \sum_{3 \; perm \; (i,j)} 2\lambda\mathcal{F}(p_i+p_j)\right]\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$
$$+\frac{1}{2}\int_{p_1}\frac{F}{d_4-2}\phi(p_1)\phi(p_2) \tag{4.2.28a}$$

Subleading order

The anomalous dimensions in subleading order are found to be as,

$$\boxed{\begin{aligned} d_4 =& \frac{53}{3}\lambda^2F^2 \\ d_2 =& \frac{5}{6}\lambda^2F^2 \end{aligned}} \tag{4.2.29a, 4.2.29b}$$

The corresponding eigenoperators are given by,

$$
\begin{aligned}
\mathcal{O}_4(0) = & \int_{p_1,..p_7} \frac{\lambda^2}{8!}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)\phi(p_7)\phi(p_8) \\
& \sum_{28\ perm\ (i,j,k)} \sum_{10\ perm\ (m,n)} 3\ h(p_i+p_j+p_k)h(p_i+p_j+p_k+p_m+p_n) \\
& + \int_{p_1,..p_5} \frac{\lambda^2}{6!}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6) \\
& \Big(3 \sum_{10\ perm\ (i,j,k)} \sum_{3\ perm\ (\alpha,\beta)} \int_p \big\{h(p_i+p_j+p_k)[h(p_\alpha+p_\beta+p)h(p)-h(p)h(p)]\big\} \\
& + \frac{-3F}{2} \sum_{10\ perm\ (i,j,k)} h(p_i+p_j+p_k)h(p_i+p_j+p_k) + \frac{-3F}{2}\sum_{l=1}^{6} \sum_{10\ perm\ (i,j,k)} h(p_l)h(p_i+p_j+p_k) \\
& + \frac{1}{2}\int_p \sum_{15\ perm\ (i,j)} \sum_{6\ perm\ (\alpha,\beta)} \Big\{h(p_i+p_j+p)h(p_i+p_j+p_\alpha+p_\beta+p)h(p)\Big\}\Big) \\
& + \int_{p_1,p_2,p_3} \frac{1}{4!}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \\
& \Big(-\frac{6\lambda^2F}{4}\sum_{l=1}^{4}\big\{h(p_l)\big\} \sum_{3\ perm\ (i,j)} \mathcal{F}(p_i+p_j) - \frac{3\lambda^2F}{2}\sum_{3\ perm\ (i,j)} \bar{H}_3(p_i+p_j) \\
& + \frac{3}{4}\lambda^2F^2\sum_{l=1}^{4}\big\{h(p_l)h(p_l)\big\} + \frac{3}{8}\frac{\lambda^2F^2}{4!}\sum_{i\neq j} h(p_i)h(p_j) + \frac{F\lambda\epsilon}{2}\sum_{i=1}^{4}h(p_i) - \frac{3F^2\lambda^2}{2}\sum_{i=1}^{4}h(p_i) \\
& + \frac{3\lambda^2}{4}\sum_{6\ perm\ (i,j)} \{I_4(p_i+p_j;p_i)+I_4(p_i+p_j;p_j)\} \\
& - 12\lambda^2 \sum_{3\ perm\ (i,j)} \int_{p,q} \{h(p_i+p_j+p+q)h(p+q)h(q)h(p) - h(q)h(p)h(p+q)h(p+q)\} \\
& + 6\lambda^2 \sum_{3\ perm\ (i,j)} \int_{p,q} \{h(p_i+p_j+q)h(p+q)h(q)h(p) - h(q)h(p+q)h(q)h(p)\} \\
& + 6\lambda^2 \sum_{3\ perm\ (i,j)} \int_{p,q} \{h(p_i+p_j+p)h(p+q)h(q)h(p) - h(p)h(p+q)h(q)h(p)\} \\
& + \frac{\lambda^2}{2}\sum_{i=1}^{4}h(p_i)F_3(p_i) + 3\lambda^2\sum_{i=1}^{4}h(p_i)\int_q f(q)\mathcal{F}(q) + 3\lambda^2\sum_{3\ perm\ (i,j)} \mathcal{F}(p_i+p_j)\mathcal{F}(p_i+p_j) \\
& + \sum_{i=1}^{4}\frac{\eta}{2\epsilon}p_i^2h(p_i) + 9F\lambda^2\sum_{3\ perm\ (i,j)} \int_\Lambda^\infty \int_{\bar{q}} \frac{d\Lambda'}{\Lambda'}\Big\{h\Big(\frac{p_i}{\Lambda'}+\frac{p_j}{\Lambda'}+\bar{q}\Big)h(\bar{q}) - h(\bar{q})\,h(\bar{q})\Big\}\Big) \\
& + \frac{1}{2}\lambda\int_p\Big(\frac{2F^2}{2-\epsilon} - \frac{2}{3}F_3(p) - \int_q f(q)h(q) - \frac{F^2}{2}h(p)\Big)\phi(p)\phi(-p) \qquad (4.2.30a)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{O}_2(0) = \frac{\lambda^2}{6!}\int_{p_1,..p_5} \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6) \\
&+ \Big(\sum_{10\ perm\ (i,j,k)} h(p_i+p_j+p_k)h(p_i+p_j+p_k) + \sum_{10\ perm\ (i,j,k)} h(p_i+p_j+p_k)\sum_{l=1}^{6} h(p_l) \Big) \\
&+ \frac{\lambda^2}{4!}\int_{p_1,p_2,p_3} \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \\
&\Big(\sum_{3\ perm\ (i,j)} \bar{H}_3(p_i+p_j) + \sum_{l=1}^{4} h(p_l) \sum_{3\ perm\ (i,j)} \mathcal{F}(p_i+p_j) - F\{\frac{1}{2}\sum_{i\neq j} h(p_i)h(p_j) + \sum_{l=1}^{4} h^2(p_l)\} \Big) \\
&+ \frac{1}{2}\int_p \phi(p)\phi(-p) \\
&\Big(-\frac{\lambda^2}{3}\int_{q,k} \{h(p+q+k)h(p)h(q)h(k) - h(q)h(q+k)h(k)\} \\
&- \frac{\lambda^2}{2}\int_{q,k} \{h(p+q+k)h(q)h(q)h(k) - h(q+k)h(q)h(q)h(k)\} \\
&- \lambda^2F^2h(p) + \frac{\epsilon\lambda}{2}h(p) + \frac{3}{4}F^2\lambda^2h^2(p) - \lambda^2h(p)\int_q f(q)\mathcal{F}(q) + \frac{\eta}{\epsilon}p^2h(p) \Big) \qquad (4.2.31a)
\end{aligned}
$$

Where all the functions have been defined in Appendix (B.6). As mentioned in the previous section here also we did not find the 2-pt vertex to second order. The leading order 2-pt vertex is enough to find the next order anomalous diemsnion.

4.3 Details of Calculation

The procedure of calculation is straightforward. We need to solve the eigenvalue equation (2.4.7) for two composite operators ϕ^2 and ϕ^4. We need to use a suitable ansatz consistent with the boundary condition mentioned above in (2.4.8). As a warm-up calculation, we perform this calculation for the Gaussian fixed point action.

4.3.1 Gaussian Theory ERG

As mentioned above, one fixed point action is the free scalar field theory in four (or any other) dimensions. As a warm-up exercise let us solve the eigenvalue equation (2.4.7) for the two bare operators, ϕ^2 and ϕ^4.

As discussed in subsection (2.4.3) the composite operators found here will be the $\lambda \to 0$

limit of the composite operators at the Wilson-Fisher fixed point.

The action we take to be

$$S = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\phi(p)\frac{p^2}{K(p)}\phi(-p) \tag{4.3.1}$$

It obeys Polchinski equation:

$$\frac{\partial S}{\partial t} = \int_p \{-K'(p^2)\}[\frac{\delta^2 S}{\delta\phi(p)\delta\phi(-p)} - \frac{\delta S}{\delta\phi(p)}\frac{\delta S}{\delta\phi(-p)}] - 2\frac{p^2 K'}{K}\phi(p)\frac{\delta S}{\delta\phi(p)} +$$

$$+ [(1-\frac{D}{2})N_\phi + D - N_p]S \tag{4.3.2}$$

and is also a fixed point solution, i.e.

$$\frac{\partial S}{\partial t} = 0 \tag{4.3.3}$$

(Anomalous dimension $\frac{\eta}{2}$, beta function $\beta(\lambda)$ has been set to zero since it is a Gaussian fixed point.)

Let $\Delta S(q)$ be a *local* composite operator of momentum q with definite dimension - added to the action. So as a composite operator it obeys the linearized equation

$$\frac{\partial \Delta S(q)}{\partial t} = \int_p \{-K'(p^2)\}[\frac{\delta^2 \Delta S(q)}{\delta\phi(p)\delta\phi(-p)} - 2\frac{\delta S}{\delta\phi(p)}\frac{\delta \Delta S(q)}{\delta\phi(-p)}] - 2\frac{p^2 K'}{K}\phi(p)\frac{\delta \Delta S(q)}{\delta\phi(p)} +$$

$$+ \underbrace{[(1-\frac{D}{2})N_\phi - N_p]}_{\mathcal{G}^c_{dil}}\Delta S(q) = (d_m + q\frac{d}{dq})\Delta S(q) \tag{4.3.4}$$

Here d_m is the length dimension.

The expression N_p in $\mathcal{G}^c_{dil}$ in (4.3.4) stands for $\sum_i p_i\frac{\partial}{\partial p_i}$.

Take

$$\Delta S(q) = \frac{1}{2}\int_{p_1}\int_{p_2} A(p_1,p_2,q)\phi(p_1)\phi(p_2) \tag{4.3.5}$$

The second and third terms in (4.3.4) cancel (and the first term is field-independent), so we get

(set $D = 4 - \epsilon$)

$$(d_m + q\frac{d}{dq})A(p_1, p_2, q) = (2 - D - \sum_{i=1}^{2} p_i \frac{\partial}{\partial p_i})A(p_1, p_2, q) \tag{4.3.6}$$

1. From (B.1.4) we see that

$$A(p_1, p_2, q) = \delta(p_1 + p_2 - q)$$

satisfies this equation. Note that $d_\phi^x = \frac{D}{2} - 1$ so $d_m = -2d_\phi^x + D = 2$. This is the (length) dimension of $\int_p \phi(p)\phi(q-p)$ as mentioned earlier.

2. Take $A(p_1, p_2, q) = p_1.p_2\ \delta(p_1 + p_2 - q)$. We get

$$(d_m + q\frac{d}{dq})A(p_1, p_2, q) = (2 - D - \sum_{i=1}^{2} p_i \frac{\partial}{\partial p_i})A(p_1, p_2, q) \tag{4.3.7}$$

From (B.1.7) and the subsequent discussion we see that $d_m = 0$.

3. Now we consider higher-dimensional operators:

Take

$$\Delta S(q) = \frac{1}{4!} \int_{p_1,p_2,p_3,p_4} B(p_1, p_2, p_3, p_4, q)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

$$+ \frac{1}{2} \int_{p_1} \int_{p_2} A(p_1, p_2, q)\phi(p_1)\phi(p_2) \tag{4.3.8}$$

Assume once again that this operator has definite momentum q. We get

$$
\begin{aligned}
&(d_m + q\frac{d}{dq}) \\
&\times \left\{ \frac{1}{4!}\int_{p_1,p_2,p_3,p_4} B(p_1,p_2,p_3,p_4,q)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \right. \\
&\left. + \frac{1}{2}\int_{p_1}\int_{p_2} A(p_1,p_2,q)\phi(p_1)\phi(p_2) \right\} \\
&= -\frac{1}{2}\int_p K'(p^2)\int_{p_1,p_2} B(p_1,p_2,p,-p,q)\phi(p_1)\phi(p_2) \\
&+ [(1-\frac{D}{2})2 - \sum_i p_i\frac{\partial}{\partial p_i}]\frac{1}{2}\int_{p_1}\int_{p_2} A(p_1,p_2,q)\phi(p_1)\phi(p_2)) \\
&+ [(1-\frac{D}{2})4 - \sum_i p_i\frac{\partial}{\partial p_i}]\frac{1}{4!}\int_{p_1,p_2,p_3,p_4} B(p_1,p_2,p_3,p_4,q)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)
\end{aligned}
$$

We see that a quartic term cannot be an eigen-operator by itself - need a quadratic piece.

For simplicity, we take

$$B(p_1,p_2,p_3,p_4,q) = \delta(p_1+p_2+p_3+p_4-q)$$

$$A(p_1,p_2,q) = A\delta(p_1+p_2-q)$$

we find ($D = 4-\epsilon$) using (B.1.6) and its generalization:

$$\sum_{i=1}^{4} p_i\frac{\partial}{\partial p_i}\delta(\sum_{j=1}^{4} p_j - q) = -D\delta(\sum_{j=1}^{4} p_j - q) + q\frac{d}{dq}\delta(\sum_{j=1}^{4} p_j - q)$$

from the ϕ^4 term:

$$d_m - (4-2D) + D = 0 \implies d_m = \epsilon \tag{4.3.9}$$

This operator is relevant in the Gaussian theory in $D < 4$ as is also obvious from simple dimensional analysis.

From the quadratic term, we get an equation for A

$$\delta(\sum_{j=1}^{2} p_j - q)[\frac{1}{2}F + (2-D)\frac{A}{2} + D\frac{A}{2}] + q\frac{d}{dq}\frac{A}{2}\delta(\sum_{j=1}^{2} p_j - q) = (\epsilon + q\frac{d}{dq})\frac{A}{2}\delta(\sum_{j=1}^{2} p_j - q) \tag{4.3.10}$$

where

$$F = \int_p (-K'(p^2)) = \frac{1}{16\pi^2}$$

Since $d_m = \epsilon$, $A = -\frac{F}{2-\epsilon}$. Thus our operator is

$$\Delta S = \frac{1}{4!}\int_{p_1,p_2,p_3} \phi(p_1)\phi(p_2)\phi(p_3)\phi(-p_1-p_2-p_3+q) - \frac{F}{2-\epsilon}\frac{1}{2}\int_p \phi(p)\phi(q-p) \quad (4.3.11)$$

which agrees with (2.4.2) if we take $u = \frac{1}{4!}$ for $q = 0$ and $\epsilon = 0$.

4.3.2 Wilson-Fisher fixed point theory at leading order

In this section we will construct, for the Wilson-Fisher fixed point theory for O(1) model, the two lowest dimension composite operators that were studied in the last section for the Gausian fixed point theory namely ϕ^2 and ϕ^4. ϕ^2 is a relevant operator at both fixed points. ϕ^4, which was relevant at the Gaussian fixed point in $D = 4-\epsilon$ (and marginal in $D = 4$) turns out to be irrelevant at the W-F fixed point. We use perturbation theory in λ. In principle one can also do perturbation in ϵ. At the W-F fixed point $\lambda \approx \epsilon$ and there is not much difference. However even in the Gaussian theory in $D = 4-\epsilon$, we have seen that ϵ shows up in the dimension so it is clear that the two expansions are in principle different. The relevant and irrelevant operator for W-F fixed point is denoted by $\mathcal{O}_2(q)$ and $\mathcal{O}_4(q)$. Though for simplicity we have taken external momentum $q = 0$ for all the calculation except while finding $\mathcal{O}_2^{(1)}(q)$. In this calculation, in principle one can put the fixed point condition right in the begining itself to interpret $O(\lambda^n)$ terms as $O(\epsilon^n)$, but there is a subtlety there - ideally all the momentum integrations are to be done in $D = 4-\epsilon$ dimensions. So there are implicit factors of ϵ hidden in there. It therefore makes sense to keep track of ϵ and λ separately and to take the fixed point condition $\lambda = O(\epsilon)$ in the end. Our expressions are in general true for general $D = 4-\epsilon$, but while calculating the anomalous dimension, in order to compare with known results for ϕ^4 in $D = 4$ [67, 69] that have been obtained using dimensional regularization, we have performed the final integrals in four dimensions.

The Ansatz We make the following general ansatz for both $\mathcal{O}_2(q)$ and $\mathcal{O}_4(q)$ as :

$$\Delta S(q) = \frac{1}{2}\int_{p_1}\int_{p_2} A(p_1,p_2)\phi(p_1)\phi(p_2)$$

$$+\frac{1}{4!}\int_{p_1,p_2,p_3,p_4} B(p_1,p_2,p_3,p_4)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

$$+\frac{1}{6!}\int_{p_1,..p_5,p_6} D(p_1,...,p_6)\phi(p_1)...\phi(p_6) \quad + \quad O(\phi^8) \quad + ... \tag{4.3.12}$$

We will assume an ansatz of the form:

$$\begin{aligned}
A(p_1,p_2) &= \delta(p_1+p_2-q)[A^{(0)} + A^{(1)}(p_1,p_2,q) + ...] \\
B(p_1,p_2,p_3,p_4) &= \delta(p_1+p_2+p_3+p_4-q)[B^{(0)} + B^{(1)}(p_1,p_2,p_3,p_4,q) + ...] \\
D(p_1,p_2,p_3,p_4,p_5,p_6) &= \delta(p_1+p_2+p_3+p_4+p_5+p_6-q)[D^{(1)}(p_1,p_2,p_3,p_4,p_5,p_6,q) + ...]
\end{aligned} \tag{4.3.13}$$

Further, we will write each term as a sum of several terms with different momentum structures. For instance, $B^{(1)}$ will turn out to be:

$$B^{(1)}(p_1,p_2,p_3,p_4,q) = \lambda\sum_{i=1}^{4} B_I(p_i,q) + \lambda\frac{1}{2}\underbrace{\sum_{i,j=1,2,3,4}}_{6\ perm} B_{II}(p_i+p_j,q) + ... \tag{4.3.14}$$

For the irrelevant operator, $\mathcal{O}_4(q)$, our starting approximation will be to take $B^{(0)} = 1$. Thus

$$B(p_1,..,p_4) = \delta(p_1+p_2+p_3+p_4-q)[1+O(\lambda)] \tag{4.3.15}$$

Since even in the Gaussian theory this is accompanied by a ϕ^2 term it is clear that $A^{(0)}$ also starts at $O(1)$. Thus

$$A(p_1,p_2) = \delta(p_1+p_2-q)[\frac{F}{\epsilon-2} + O(\lambda)] \tag{4.3.16}$$

Everything else is $O(\lambda)$ or higher.

On the other hand for the relevant operator , $\mathcal{O}_2(q)$ we start with

$$A(p_1, p_2) = \delta(p_1 + p_2 - q)[1 + O(\lambda)] \tag{4.3.17}$$

and everything else is higher order in λ.

The strategy will be to take these as the starting inputs and solve the linearized ERG equation (4.3.18) order by order in λ. Typically, at each order the coefficient of a new higher dimensional irrelevant operator enters the equation.

The Wilson-Fisher action We write the WF fixed point action upto $\mathcal{O}(\epsilon)$ as given in section (4.2).

$$S = \frac{1}{2}\int_p \left(\frac{p^2}{K} + U_2(p)\right)\phi(p)\phi(-p) + \frac{1}{4!}\int_{p_1,p_2,p_3}(\lambda + U_4)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

$$p_4 = -p_1 - p_2 - p_3$$

$$+\frac{1}{6!}\int_{p_1,...p_5} V_6\phi(p_1)...\phi(p_6)$$

$$p_6 = -p_1 - ... - p_5$$

$$U_2 = -\lambda\frac{1}{2-\epsilon}\underbrace{\int_p(-K'(p^2))}_{F} + O(\lambda^2)$$

$$V_6 = -\lambda^2 \sum_{10\ perm\ i,j,k} h(p_i + p_j + p_k)$$

$$h(p) \equiv \frac{1 - K(p^2)}{p^2}$$

$$U_4 \approx O(\lambda^2)$$

We number the Polchinski's equation (the terms with the anomalous dimension is not required at this order since $\eta \approx O(\lambda^2)$) in the following way:

$$\frac{\partial \Delta S(q)}{\partial t} = \int_p -K'(p^2)[\underbrace{\frac{\delta^2 \Delta S}{\delta\phi(p)\delta\phi(-p)}}_{\textbf{(1)}} \underbrace{-2\frac{\delta S}{\phi(p)}\frac{\delta \Delta S}{\phi(-p)}}_{\textbf{(2)}}] \underbrace{-2\frac{p^2 K'}{K}\phi(p)\frac{\delta \Delta S}{\phi(p)}}_{\textbf{(3)}} + \underbrace{[(1-\frac{D}{2})N_\phi + D - N_p]}_{\mathcal{G}^c_{dil}=\textbf{(4a)}} \Delta S$$
$$= (d_m + q\frac{d}{dq})\Delta S(q) \qquad (4.3.18)$$

The second equality is the requirement that $\Delta(q)$ be a scaling operator of length dimension d_m. Note that we donot have to include the term $\beta(\lambda)\frac{\partial \Delta S}{\partial \lambda}$ in this order. We have calculated different parts of (4.3.18) in Appendix B.2.

The Relevant Operator

We start with $A = 1$ and $d_2 \approx 2$.

$$\begin{aligned}
A(p_1,p_2) &= \delta(p_1+p_2-q)[1+A^{(1)}(p_1,p_2,q)+...\\
B(p_1,p_2,p_3,p_4) &= \delta(p_1+p_2+p_3+p_4-q)[\sum_{i=1}^{4} B_I^{(1)}(p_i,q) + \frac{1}{2}\sum_{6\ perm\ (i,j)} B_{II}^{(1)}(p_i+p_j,q)...]\\
D(p_1,p_2,p_3,p_4,p_5,p_6) &= \delta(p_1+p_2+p_3+p_4+p_5+p_6-q)[D^{(1)}(p_1,p_2,p_3,p_4,p_5,p_6,q)+...]\\
d_2 &= 2 + d_2^{(1)} + ...
\end{aligned} \qquad (4.3.19)$$

It turns out that in the leading order we can set $B_{II}^{(1)}(p_I+p_j,q) = D^{(1)}(p_1,p_2,p_3,p_4,p_5,p_6,q) = 0$.

$O(\lambda)$ Equation for ϕ^4

$$-\frac{2\lambda}{4!}\sum_{i=1}^{4} -K'((p_i-q)^2)\delta(\sum p_i - q) + \frac{1}{4!}((4-D)+q\frac{d}{dq})\delta(\sum p_i - q)(\sum_{i=1}^{4} B_I^{(1)}(p_i,q))$$

$$-\frac{1}{4!}\delta(\sum p_i - q)(\sum_i p_i\frac{\partial}{\partial p_i} + q\frac{d}{dq})\sum_{i=1}^{4} B_I^{(1)}(p_i,q) = (d_2 + q\frac{d}{dq})\delta(\sum p_i - q)(\sum_{i=1}^{4} B_I^{(1)}(p_i,q))$$

Canceling terms and dropping $O(\epsilon\lambda)$ or $O(\lambda^2)$ terms we get

$$-2\lambda(-K'((p_i-q)^2)) - (\sum_i p_i\frac{\partial}{\partial p_i} + q\frac{d}{dq})B_I^{(1)}(p_i,q) = 2B_I^{(1)}(p_i,q)$$

This is solved by

$$B_I^{(1)}(p_i,q) = -\lambda h(p_i - q) \tag{4.3.20}$$

The ϕ^2 terms in the equation are:

$$\frac{F}{2}\delta(p_1+p_2-q)(B_I(p_1,q)+B_I(p_2,q)) + \frac{1}{2}\int_p (-K'(p^2))(B_I(p,q)+B_I(-p,q))\delta(p_1+p_2-q)$$

$$+\frac{\lambda F}{2-\epsilon}\sum_i -K'(p_i^2)\delta(p_1+p_2-q)$$

$$+\frac{1}{2}(2-D+D+q\frac{d}{dq})\delta(p_1+p_2-q)(1+A^{(1)}(p_1,p_2,q)) - \frac{1}{2}\delta(p_1+p_2-q)(\sum_i p_i\frac{\partial}{\partial p_i}+q\frac{d}{dq})A^{(1)}(p_1,p_2,q)$$

$$= (d_2+q\frac{d}{dq})\delta(p_1+p_2-q)\frac{1}{2}(1+A^{(1)}(p_1,p_2,q))$$

O(1)

The $O(1)$ part of this equation (after canceling terms) gives

$$d_2^{(0)} = 2 \tag{4.3.21}$$

O(λ)

We substitute (4.3.20) in the $O(\lambda)$ part to get

$$\frac{\lambda F}{2}[-h(p_1-q)-h(p_2-q)] + \lambda\frac{1}{2}\int_p(-K'(p^2))[-h(p-q)-h(p+q)]$$

$$+\frac{\lambda F}{2-\epsilon}[-K'((p_1-q)^2)-K'((p_2-q)^2)]$$

$$+A^{(1)}(p_1,p_2,q) - \frac{1}{2}(\sum_i p_i\frac{\partial}{\partial p_i}+q\frac{d}{dq})A^{(1)}(p_1,p_2,q)$$

$$= d_2^{(1)}\frac{1}{2} + A^{(1)}(p_1,p_2,q)$$

The second term of the first line can be rewritten as

$$\frac{1}{2}\lambda\int_p(-K'(p^2))[(h(p)-h(p-q))+(h(p)-h(p+q))] + \frac{1}{2}\lambda\int_p(-K'(p^2))[-2h(p)] \tag{4.3.22}$$

The q independent term evaluates to $-F$ and we thus get

$$\boxed{d_2^{(1)} = -\lambda F \approx -\frac{\epsilon}{3}} \tag{4.3.23}$$

The first term in (4.3.22) which is independent of p_i can be canceled by choosing

$$A^{(1)}(p_1, p_2, q) = \frac{1}{2}\lambda(\mathcal{F}(q) + \mathcal{F}(-q)) = \lambda\mathcal{F}(q) \tag{4.3.24}$$

$\mathcal{F}(q)$ is defined in Appendix F. Note that $\mathcal{F}(0) = 0$.

The remaining equation is satisfied by setting

$$A^{(1)}(p_1, p_2, q) = \frac{\lambda F}{2}(h(p_1 - q) + h(p_2 - q)) \tag{4.3.25}$$

$A^{(1)}(p)$ looks like first diagram in Fig4.2.

This gives $d_2^{(1)} = -\lambda F = -\frac{\epsilon}{3}$. The value of $d_2^{(1)}$ is coming from the second diagram in Fig. 4.2. As expected the origin of the anomalous dimension is the logarithmically divergent diagram.

Thus the relevant eigen-operator and its dimension are given as:

$$\boxed{d_2 = 2 - \lambda F}$$

$$\boxed{\mathcal{O}_2(q) = \frac{1}{2}\int_{p_1,p_2} \delta(p_1 + p_2 - q)[1 + \lambda\frac{F}{2}(h(p_1 - q) + h(p_2 - q)) + \lambda\mathcal{F}(q)] \;\; \phi(p_1)\phi(p_2)}$$

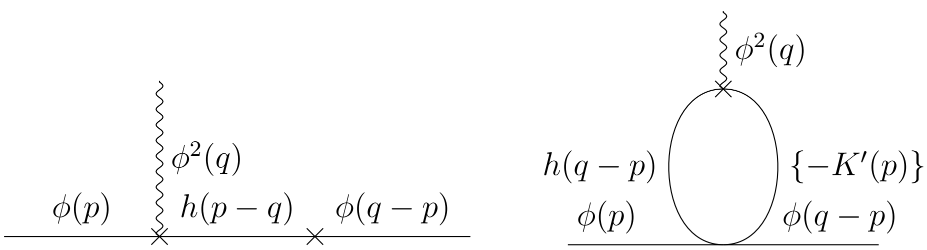


Figure 4.2: The left diagram is for the the relevant operator $A^{(1)}(p)$. The right one is the diagrammatic representation of the term contributing to the anomalous dimension $d_2^{(1)}$. Note that the right diagram is a loagarithmically divergent diagram made finite by replacing the propagator $h(p)$ by $-K'(p^2)$. It is the q independent part that gives $d_2^{(1)}$.

$$-\frac{1}{4!}\int_{p_1,p_2,p_3}\delta(p_1+p_2+p_3+p_4-q)\lambda\sum_{i=1}^{4}h(p_i-q)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

Note that the value of the anomalous dimension agrees with (4.2.25) to this order.

The Irrelevant Operator

For simplicity we set $q=0$. The ansatz simplifies to (momentum conservation is implicit, i.e. $\sum_i p_i = 0$):

$$A(p) = A^{(0)} + A^{(1)}(p) + ...$$

$$B(p_1,p_2,p_3,p_4) = B^{(0)}(p_1,p_2,p_3,p_4) + B_I^{(1)}(p_1,p_2,p_3,p_4) + B_{II}^{(2)}(p_1,p_2,p_3,p_4)$$

$$= \sum_{i=1}^{4} B^{(0)}(p_i) + \sum_{i=1}^{4} B_I^{(1)}(p_i) + \sum_{3\ perm\ (i,j)} B_{II}^{(1)}(p_i+p_j)$$

$$D(p_1,p_2,p_3,p_4,p_5,p_6) = D^{(1)}(p_1,p_2,p_3,p_4,p_5,p_6)$$

$$d_4 = d_4^{(1)} + ...$$

Below we are writing ϕ^2, ϕ^4, and ϕ^6 terms separately to obtain different quantities.

Equation for $\phi^2 : \mathcal{O}(1)$

Different parts of (4.3.18) gives,

(1)

$$\int_q \{-K'(q^2)\}\{B^{(0)}(q) + B_I^{(1)}(q)\} + F\{B^{(0)}(p) + B^{(1)}(p)\} + \frac{1}{2}F\lambda B_{II}(0)$$

$$-\frac{1}{2}\int_q K'(q^2)\lambda[B_{II}(p+q) + B_{II}(p-q)]$$

(2)+(3)

$$-2(-K'(p^2))U_2(p)A^{(0)}(p)$$

(4a)

$$A^{(0)} - p^2 \frac{d}{dp^2} A^{(0)}$$

Collecting terms of $O(1)$:

$$A^{(0)}(p) - p^2 \frac{d}{dp^2} A^{(0)}(p) + \frac{1}{2} F(4B = 1) = d_4 \frac{A^{(0)}(p)}{2}$$

Assuming that $A^{(0)}(p)$ is a constant and $O(1)$ we obtain (neglecting $O(\lambda)$ terms)

$$\boxed{A^{(0)} = \frac{F}{d_4 - 2} \approx -\frac{1}{2} F} \tag{4.3.26}$$

d_4 is expected to be of $O(\epsilon)$ since ϕ^4 is marginal in $D = 4$.

Equation for ϕ^6: $\mathcal{O}(\lambda)$

Now we turn to the ϕ^6 equation:

$$-\frac{4}{6!} \int_{p_1,..p_5} \sum_{10\ perm} -K'((p_i + p_j + p_k)^2) \lambda(4B = 1) \phi(p_1) ... \phi(p_6)$$

$$+\frac{1}{6!} \int_{p_1...p_5} (6 - 2D - 2 \sum_i 2p_i^2 \frac{d}{dp_i^2}) \sum_{10\ perm} D^{(1)}(p_i + p_j + p_k) \phi(p_1) ... \phi(p_6)$$

$$= \frac{d_m}{6!} \int_{p_1,..p_5} \sum_{10\ perm} D^{(1)}(p_i + p_j + p_k) \phi(p_1) ... \phi(p_6)$$

At order λ the equation is

$$(1 + p^2 \frac{d}{dp^2}) D^{(1)}(p) = 2\lambda K'(p^2)$$

considering (B.6.1) we see that

$$\boxed{D^{(1)}(p) = -2\lambda h(p)} \tag{4.3.27}$$

Equation for ϕ^4

Now we turn to the ϕ^4 equation:

(1)

$$\frac{1}{4!}\int_p\{-K'(p^2)\}\int_{p_1,p_2,p_3}\left[D^{(1)}(p_1)+D^{(1)}(p_2)+D^{(1)}(p_3)+D^{(1)}(p_4)\right]\phi(p_1)...\phi(p_4)\quad (\text{ ``}\textit{type}\ 1\text{''})$$

$$+\frac{1}{4!}\int_p\{-K'(p^2)\}\int_{p_1,p_2,p_3}[D^{(1)}(p+p_1+p_2)+D^{(1)}(p+p_1+p_3)+D^{(1)}(p+p_1+p_4)$$
$$+D^{(1)}(p-p_1-p_2)+D^{(1)}(p-p_1-p_3)+D^{(1)}(p-p_1-p_4)]\phi(p_1)...\phi(p_4)\quad (\text{``}\textit{type}\ 2\text{''})$$

We have written the expression in the first line as type 1 because we will see below that quadratically divergent 4-pt vertex will be obtained from these expressions, while from type 2 expressions logarithmically divergent 4-pt vertex will be obtained. We will see the contribution from type 1 diagram will be canceled and those from the type 2 diagram will contribute to the anomalous dimension.

(2)+(3)

$$(-2)\left[\frac{1}{4!}\int_{p_1,p_2,p_3}[\sum_i\{-K'(p_i^2))A(p_i\}]\lambda+\frac{1}{4!}\int_{p_1,p_2,p_3}[\sum_i\{-K'(p_i^2)\}U_2(p_i)][1+...]\right]\phi(p_1)...\phi(p_4)$$

(4a)

$$\frac{1}{4!}\int_{p_1,p_2,p_3}(4-D-2\sum_i p_i^2\frac{d}{dp_i^2})[1+\sum_{i=1}^{4}B_I^{(1)}(p_i)+\sum_{3\ perm(i,j)}B_{II}^{(1)}(p_i+p_j)]\phi(p_1)..\phi(p_4)$$

Collect type 1 terms and (2)+(3) part above, we get,

$$\int_p\{-K'(p^2)\}[D^{(1)}(p_1)+D^{(1)}(p_2)+D^{(1)}(p_3)+D^{(1)}(p_4)]+2[\sum_i(K'(p_i^2))A(p_i)]\lambda+2[\sum_i(K'(p_i^2))U_2(p_i)]$$

$$+(4-D-2\sum_i p_i^2\frac{d}{dp_i^2})[1+\sum_{i=1}^4 B_I^{(1)}(p_i)] = d_4\sum_{i=1}^4 B_I^{(1)}(p_i)$$

Ignoring $\mathcal{O}(\epsilon\lambda)$ or $O(\lambda^2)$ terms from (B.6.1) we get,

$$\boxed{B_I^{(1)}(p) = \lambda F h(p)} \tag{4.3.28}$$

Where $h(p) = \frac{K_0 - K}{p^2}$. It looks like first diagram in Fig.4.3.

The leftover terms on LHS is

$$(4-D) - d_4\sum_{i=1}^4 B_I^{(1)}(p_i)$$

We will keep a record of all leftover terms in LHS as we need those in sub-leading order calculation.

Now we collect type 2 terms and the rest of the equation, we have put $D(p) = -2\lambda h(p)$.

$$4\lambda\int_p K'(p^2)[h(p+p_1+p_2)+h(p+p_1+p_3)+h(p+p_1+p_4)]$$

$$+(4-D-\sum_{l=1}^4 p_l.\frac{d}{dp_l})[B_{II}^{(1)}(p_1+p_2)+B_{II}^{(1)}(p_1+p_3)+B_{II}^{(1)}(p_1+p_4)] = d_4$$

Considering (B.6.2), if we add and subtract $6F\lambda$ as momentum independent term we get,

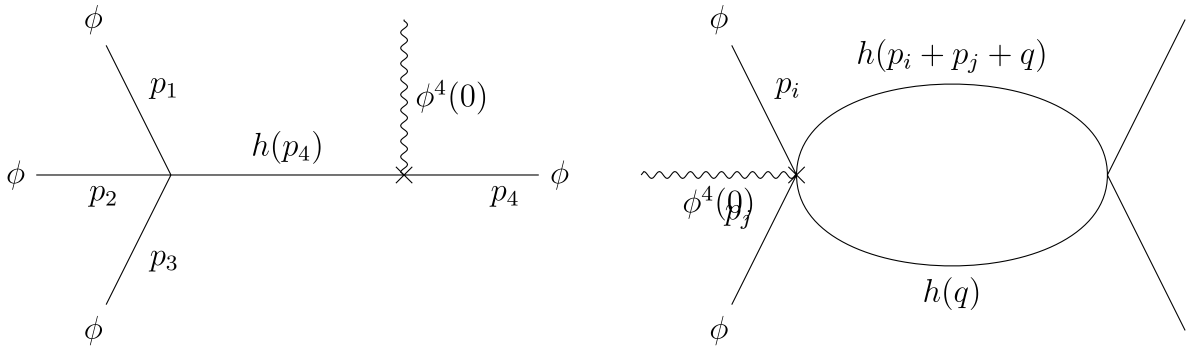


Figure 4.3: The left diagram represents Type-I diagram corresponding to $B_I(p)$, while the right one represents type-II diagram representing $B_{II}(p_i+p_j)$. Anomalous dimension is coming from the process of making the latter diagram zero at zero external momenta. Note that the $B_{II}(p_i+p_j)$ is nothing but the usual logarithmic divergent diagram made finite by adding a counterterm.

$$\boxed{B_{II}^{(1)}(p) = -2\lambda \mathcal{F}(p)} \tag{4.3.29}$$

which looks like second diagram in Fig.4.3. Here $\mathcal{F}(p) = \frac{1}{2}\int_q \{h(p+q)h(q) - h(q)h(q)\}$, it is defined in (B.6.2) in Appendix B.6.

While the Leftover terms in the L.H.S are:

$$(4-D) - d_4 \sum_{i=1}^{4} B^{(1)}(p_i) - 6F\lambda$$

Keeping only λ^1 and ϵ^1 terms and equating with R.H.S we get,

$$4 - D - 6F\lambda = d_4 \sum_{i=1}^{4} B^{(0)}(p_i)$$

so we get the anomalous dimension at the leading order as,

$$d_4^{(1)} = \epsilon - 6F\lambda \tag{4.3.30}$$

in agreement with (4.2.21) at this order.

At $F\lambda = \frac{\epsilon}{3}$ we get,

$$\boxed{d_4^{(1)} = -\epsilon}$$

It is to be noted the origin of the anomalous dimension is the Type-II diagram (second diagram in Fig.4.3). It is expected as anomalous dimension should come from the process of logarithmic divergent digram finite as it happens in the continuum field theory.

So the irrelevant eigen-operator and its anomalous dimension are given as:

$$\boxed{d_4 = \epsilon - 6F\lambda}$$

$$\boxed{\mathcal{O}_4(0) = -\frac{1}{6!}\int_{p_1,p_2,p_3,p_4,p_5} \sum_{10\ perm\ (i,j,k)} 2\lambda h(p_i+p_j+p_k)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)}$$

$$+\frac{1}{4!}\int_{p_1,p_2,p_3}\left[\sum_{i=1}^{4}F\lambda h(p_i)-\sum_{3\ perm\ (i,j)}2\lambda\mathcal{F}(p_i+p_j)\right]\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

$$+\frac{1}{2}\int_{p_1}\frac{F}{d_4-2}\phi(p_1)\phi(p_2)$$

Where $F=\frac{1}{16\pi^2}$. Thus at the fixed point, we get a composite operator with a dimension $-\epsilon$ which is (just a little) irrelevant in contrast with the Gaussian case. [4]

Wilson-Fisher Composite operator at the subleading order

The Polchinski's equation Now we turn to find the irrelevant and relevant operators with their respective anomalous dimensions at the order ϵ^2. We set $q=0$ for simplicity. At this order we have to include anomalous dimension $\frac{\eta}{2}$ in Polchinski's equation i.e.

$$\frac{\partial\Delta S}{\partial t}=\int_p\Big\{(-K'(p^2))[\frac{\delta^2\Delta S}{\delta\phi(p)\delta\phi(-p)}-2\frac{\delta S}{\delta\phi(p)}\frac{\Delta S}{\delta\phi(-p)}]-2\frac{p^2K'}{K}\phi(p)\frac{\delta\Delta S}{\delta\phi(p)}+[(1-\frac{D}{2})N_\phi+D-N_p]\Delta S$$
$$-\frac{\eta}{2}\frac{K(p^2)(1-K(p^2)}{p^2}[\frac{\delta^2\Delta S}{\delta\phi(p)\delta\phi(-p)}-2\frac{\delta S}{\delta\phi(p)}\frac{\delta\Delta S}{\delta\phi(-p)}]+\frac{-\eta}{2}\phi(p)\frac{\delta\Delta S}{\delta\phi(p)}\Big\}=d_m\Delta S+\beta(\lambda)\frac{\partial\Delta S}{\partial\lambda} \tag{4.3.31}$$

The action S up to $\mathcal{O}(\epsilon^2)$ as given in section (4.2),

$$S=\int_p\Big\{\frac{(-F\lambda)}{2-\epsilon}-\frac{1}{2}\lambda^2G(p)+\frac{1}{2}\frac{(-\lambda F^2)}{4}h(p)\Big\}\phi(p)\phi(-p)$$
$$+\frac{1}{4!}\int_{p_1,p_2,p_3}\Big\{\lambda-\lambda^2[\mathcal{F}(p_1+p_2)+\mathcal{F}(p_1+p_3)+\mathcal{F}(p_1+p_4)]+\frac{F\lambda^2}{2}\sum_{i=1}^{4}h(p_i)\Big\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$
$$+\frac{1}{6!}\int_{p_1,p_2,p_3,p_4,p_5,p_6}(-\lambda^2)\sum_{10\ perm\ (i,j,k)}h(p_i+p_j+p_k)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$

Where

[4] This also agrees with Kogut and Wilson (page 109)[13] to this order.

$$G(p) = \frac{1}{3}\int_{q,k}\frac{h(q)}{2}[h(p+q+k)h(k)-h(k)h(k)] - \frac{1}{3}\int_q\frac{h(q)}{2}[h(q+k)h(k)-h(k)h(k)]$$
$$+\frac{\eta p^2}{2\epsilon} - \frac{1}{2-2\epsilon}\left\{\frac{2}{3}\beta^{(1)}v_2^{(1)} + \int_q f(q)\mathcal{F}(q)\right\}$$

Where

$$\beta^{(1)} = -\int_q f(q)h(q) \to_{\epsilon\to 0} -F;\, v_2^{(1)} = -\int_q f(q)h(q) \to_{\epsilon\to 0} -\frac{F}{2}$$

and

$$\mathcal{F}(p_i+p_j) = \frac{1}{2}\int_q\left\{h(p_i+p_j+q)h(q) - h(q)h(q)\right\}$$

$\mathcal{F}(p)$ is defined by eq.(B.6.2).

$$h(p) = \frac{K_0 - K}{p^2};\quad f(q) = -2K'(q)$$

$$\beta(\lambda) = \epsilon\lambda + \beta_1^{(1)}(\lambda);\ \beta_1^{(1)}(\lambda) = -3F\lambda^2$$

The Irrelevant Operator, $\mathcal{O}_4^{(2)}(0)$ The form of the irrelevant operator in the subleading order is given below. Note that at this order, we need to include 8-pt vertex which is of $\mathcal{O}(\epsilon^2)$. We have just given the expressions of $\mathcal{O}_4(0)$ in this section. Equations to find them are given in Appendix B.3.

$$\mathcal{O}_4(0) = \Delta S_2 + \Delta S_4 + \Delta S_6 + \Delta S_8$$
$$= \frac{1}{2!}\int_p\left\{\frac{F}{d_4-2} + A^{(1)}(p)\right\}\phi(p)\phi(-p)$$
$$+\frac{1}{4!}\int_{p_1,p_2,p_3}\left\{1 + F\lambda\sum_{l=1}^{4}h(p_l) - \lambda\int_k\sum_{3\ perm\ (i,j)}[h(p_i+p_j+k)h(k)-h(k)h(k)]\right.$$
$$\left.+B^{(2)}(p_1,p_2,p_3,p_4)\right\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$
$$+\frac{1}{6!}\int_{p_1,p_2,p_3,p_4,p_5}\left\{-2\lambda\sum_{10\ perm\ (i,j,k)}^{6}h(p_i+p_j+p_k) + D^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)\right\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)$$
$$+\frac{1}{8!}\int_{p_1,p_2,p_3,p_4,p_5,p_6,p_7}E^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)\phi(p_7)\phi(p_8)$$

with

$$d_4 = \epsilon - 6F\lambda + d_4^{(2)} + ...$$

ϕ^8 equation-Determination of $\Delta S_8^{(2)}$

The 8-pt vertex is found by solving the ϕ^8 equation at $\mathcal{O}(\lambda^2)$. The ϕ^8 equation is obtained as:

$$-2\int(-K'(p^2))\left\{\frac{\delta}{\delta\phi(p)}\frac{\lambda}{4!}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\right\}\times$$
$$\left\{\frac{\delta}{\delta\phi(-p)}\sum_{10perm(i,j,k)}\frac{D^{(1)}(p_i+p_j+p_k)}{6!}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)\right\} \tag{4.3.32}$$

$$-2\int(-K'(p^2))\left\{\frac{\delta}{\delta\phi(p)}\sum_{i=1}^{4}\frac{1}{4!}B^{(0)}(p_i)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\right\}\times$$
$$\left\{\frac{\delta}{\delta\phi(-p)}\sum_{10perm(i,j,k)}\frac{V^{(2)}(p_i+p_j+p_k)}{6!}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)\right\}$$
$$+\frac{1}{8!}\left\{8-3D-\sum_{i=1}^{8}p_i.\frac{d}{dp_i}\right\}E^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)=0 \tag{4.3.33}$$

The solution is given by:

$$E^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)=\sum_{28\ perm\ (i,j,k)}\sum_{10\ perm\ (m,n)}3\lambda^2\ h(p_i+p_j+p_k)h(p_i+p_j+p_k+p_m+p_n) \tag{4.3.34}$$

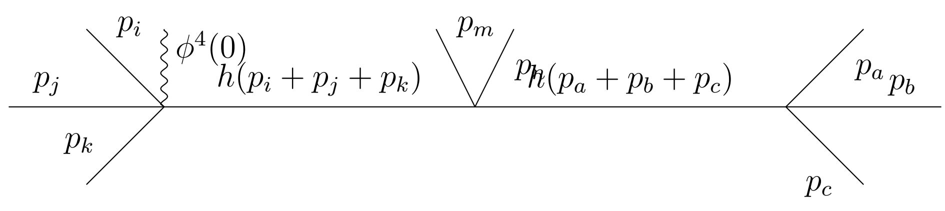


Figure 4.4: The diagram for $E^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)$

ϕ^6 equation - Determination of $\Delta S_6^{(2)}$

Solving ϕ^6 equation we get four kinds of solutions for 6-pt vertex at order $O(\lambda^2)$ based on their tensor structure (see B.3.1 for details)

$$D_I^{(2)}(p_1, p_2..., p_6) = 3\lambda^2 \sum_{10\ perm\ (i,j,k)} \sum_{3\ perm\ (\alpha,\beta)} \int_p \{h(p_i + p_j + p_k)[h(p_\alpha + p_\beta + p)h(p) - h(p)h(p)]\} \tag{4.3.35a}$$

$$D_{II}^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) = \frac{-3\lambda^2 F}{2} \sum_{10\ perm\ (i,j,k)} h(p_i + p_j + p_k)h(p_i + p_j + p_k) \tag{4.3.35b}$$

$$D_{III}^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) = \frac{-3\lambda^2 F}{2} \sum_{l=1}^{6} \sum_{10\ perm\ (i,j,k)} h(p_l)h(p_i + p_j + p_k) \tag{4.3.35c}$$

$$D_{IV}^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) = \frac{\lambda^2}{2} \int_p \sum_{15\ perm\ (i,j)} \sum_{6\ perm\ (\alpha,\beta)} \left\{h(p_i + p_j + p)h(p_i + p_j + p_\alpha + p_\beta + p)h(p)\right\} \tag{4.3.35d}$$

ϕ^2 equation at $\mathcal{O}(\epsilon)$: Determination of $A^{(1)}(p)$

The ϕ^2 equation at order λ^1 is given below (Note that we do not have to consider $\beta(\lambda)\frac{\partial \Delta S}{\partial \lambda}$ part because we want to find $A(p)$ at order ϵ^1 or λ^1 only):

$$\begin{aligned} &\int_q (-K'(q^2))B_I^{(1)}(q) + FB_I^{(1)}(p) + \frac{1}{2}FB_{II}^{(1)}(0) - \frac{1}{2}\int_q K'(q^2)[B_{II}^{(1)}(p+q) + B_{II}^{(1)}(p-q)] \\ &- 2(-K'(q^2))U_2^{(1)}(p)A^{(0)}(p) + A^{(1)}(p) - \frac{1}{2}p.\frac{d}{dp}A^{(1)}(p) = d_m^{(1)}\frac{A^{(1)}(p)}{2} \end{aligned} \tag{4.3.36}$$

Solving the ϕ^2 equation we found the $A^{(0)}(p)$ and three kinds of 2-pt vertices based on their tensor structure.

$$A_I^{(1)}(p) = \frac{2F^2\lambda}{2-\epsilon} \tag{4.3.37a}$$

$$A_{II}^{(1)}(p) = -\frac{2\lambda}{3}F_3(p) - \lambda \int_q f(q)h(q) \tag{4.3.37b}$$

$$A_{III}^{(1)}(p) = -\frac{F^2\lambda}{2}h(p) \tag{4.3.37c}$$

From (4.3.26) we get,

$$A^{(0)} = -\frac{F}{2} - \frac{F\epsilon}{4}$$

Where $\bar{F}_3(p) = \int_{q,k} h(p+q+k)h(q)h(k)$, $F_3(p) = \bar{F}_3(p) - \bar{F}_3(0) = \int_q 2h(q)\left[\mathcal{F}(p+q) - \mathcal{F}(q)\right]$. They are defined by (B.6.3) and (B.6.4).

ϕ^4 **equation-Determination of** $B^{(2)}(p_1, p_2, p_3, p_4)$

Solving the ϕ^4 equation we get total of nine kinds of 4-pt vertices based on their tensor structure (see Appendix B.3.2 for more details).

$$\frac{1}{4!}B_I^{(2)}(p_1,p_2,p_3,p_4) = -\frac{6\lambda^2}{4}\frac{F}{4!}\sum_{l=1}^{4}\{h(p_l)\}\sum_{3\ perm\ (i,j)}\mathcal{F}(p_i+p_j) \tag{4.3.38a}$$

$$\frac{1}{4!}B_{II}^{(2)}(p_1,p_2,p_3,p_4) = -\frac{1}{4!}\frac{3\lambda^2 F}{2}\sum_{3\ perm\ (i,j)}\bar{H}_3(p_i+p_j) \tag{4.3.38b}$$

$$\frac{1}{4!}B_{III}(p_1,p_2,p_3,p_4) = \frac{3}{4}\frac{\lambda^2F^2}{4!}\sum_{l=1}^{4}\{h(p_l)h(p_l)\} + \frac{3}{8}\frac{\lambda^2F^2}{4!}\sum_{i\neq j}h(p_i)h(p_j) \tag{4.3.38c}$$

$$B_{IV}^{(2)}(p_1,p_{2,3},p_4) = \frac{F\lambda\epsilon}{2}\sum_{i=1}^{4}h(p_i) - \frac{3F^2\lambda^2}{2}\sum_{i=1}^{4}h(p_i) \tag{4.3.38d}$$

$$\frac{1}{4!}B_V^{(2)}(p_1,p_2,p_3,p_4) = \frac{1}{4!}\frac{3\lambda^2}{4}\sum_{6\ perm\ (i,j)}\{I_4(p_i+p_j;p_i)+I_4(p_i+p_j;p_j)\} \tag{4.3.38e}$$

$$\begin{aligned}
&\frac{1}{4!}B_{VI}^{(2)}(p_1,p_2,p_3,p_4)\\
&= -\frac{\lambda^2}{2}\sum_{3\ perm\ (i,j)}\int_{p,q}\{h(p_i+p_j+p+q)h(p+q)h(q)h(p) - h(q)h(p)h(p+q)h(p+q)\}
\end{aligned} \tag{4.3.38f}$$

$$+\frac{\lambda^2}{4}\sum_{3\ perm\ (i,j)}\int_{p,q}\{h(p_i+p_j+q)h(p+q)h(q)h(p) - h(q)h(p+q)h(q)h(p)\}$$

$$+\frac{\lambda^2}{4}\sum_{3\ perm\ (i,j)}\int_{p,q}\{h(p_i+p_j+p)h(p+q)h(q)h(p) - h(p)h(p+q)h(q)h(p)\}$$

$$\begin{aligned}
&\frac{1}{4!}B_{VII}(p_1,p_2,p_3,p_4)\\
&= \frac{1}{4!}B_{VII}^{(2)}(p_1,p_2,p_3,p_4)|_1 + \frac{1}{4!}B_{VII}^{(2)}(p_1,p_2,p_3,p_4)|_2\\
&= \frac{1}{4!}\frac{\lambda^2}{2}\sum_{i=1}^{4}h(p_i)F_3(p_i) + \frac{1}{4!}3\lambda^2\sum_{i=1}^{4}h(p_i)\int_q f(q)\mathcal{F}(q)
\end{aligned} \tag{4.3.38g}$$

$$\frac{1}{4!}B_{VIII}^{(2)}(p_1,p_2,p_3,p_4) = \frac{3\lambda^2}{4!}\sum_{3\ perm\ (i,j)}\mathcal{F}(p_i+p_j)\mathcal{F}(p_i+p_j) \tag{4.3.38h}$$

$$\frac{1}{4!}B_{IX}(p_1,p_2,p_3,p_4) = \frac{1}{4!}\sum_{i=1}^{4}\frac{\eta}{2\epsilon}p_i^2h(p_i) \tag{4.3.38i}$$

$$\frac{1}{4!}B_X^{(2)}\left(\frac{p_1}{\Lambda},\frac{p_2}{\Lambda},\frac{p_3}{\Lambda},\frac{p_4}{\Lambda}\right) = \frac{9F\lambda^2}{4!}\sum_{3\ perm\ (i,j)}\int_\Lambda^\infty\int_{\bar{q}}\frac{d\Lambda'}{\Lambda'}\left\{h\left(\frac{p_i}{\Lambda'}+\frac{p_j}{\Lambda'}+\bar{q}\right)h(\bar{q}) - h(\bar{q})h(\bar{q})\right\} \tag{4.3.38j}$$

Where $\bar{F}_3(p) = \int_{q,k} h(p+q+k)h(q)h(k)$, $F_3(p) = \bar{F}_3(p) - \bar{F}_3(0) = \int_q 2h(q)\left[\mathcal{F}(p+q) - \mathcal{F}(q)\right]$. They are defined by (B.6.3) and (B.6.4).

Also

$$\bar{H}_3(p) = \int_q h(p+q)h(q)h(q)$$

and

$$\begin{aligned} I_4(p_i+p_j;p_i) =& \bar{I}_4(p_i+p_j;p_i) - \bar{I}_4(0;0) \\ &= \sum_{6\ perm\ (i,j)} \int_{p,q} \{h(p_i+p_j+q)h(p+q+p_i)h(p)h(q) - h(p+q)h(p)h(q)h(q)\} \end{aligned} \tag{4.3.39}$$

$\bar{H}_3(p)$ and $I_4(p_i+p_j;p_i)$ are defined by (B.6.5) and (B.6.6) respectively.

Equation for $B^{(2)}_{IV}(p_1,p_2,p_3,p_4)$ and $B_V(p_1,p_2,p_3,p_4)$

We will show one sample calculation here to explain how we have used the Feynman diagram as a guide in the calculations.

Taking (B.3.8b), (B.3.11b) and (B.3.11c), we get

$$\begin{aligned} &\frac{3\lambda^2}{4!}\int_{p,q}\{-K'(p^2)\}\sum_{6\ perm\ (i,j)}\{h(p_i+p_j+p)[h(p+q+p_j)+h(p+q+p_i)-2h(q)]h(q)\} \\ &+\frac{2\lambda^2}{4!}\int_p\int_q\{-K'(p^2)\}\sum_{6\ perm\ (i,j)}\{h(p_i+p_j+q)[h(p+q+p_i)+h(p+q+p_j)]h(q)\} \\ &+\frac{\lambda^2}{4!}\int_q\int_p\{-K'(p^2)\}\sum_{l=1}^{4}\sum_{3\ perm\ (i,j)}\{h(p_l+p+q)h(p_l+p_i+p_j+p+q)h(q)\} \\ &+\{4-D-\sum_{i=1}^{4}p_i.\frac{d}{dp_i}\}\frac{1}{4!}B^{(2)}_{IV}(p_1,p_2,p_3,p_4)=0 \end{aligned}$$

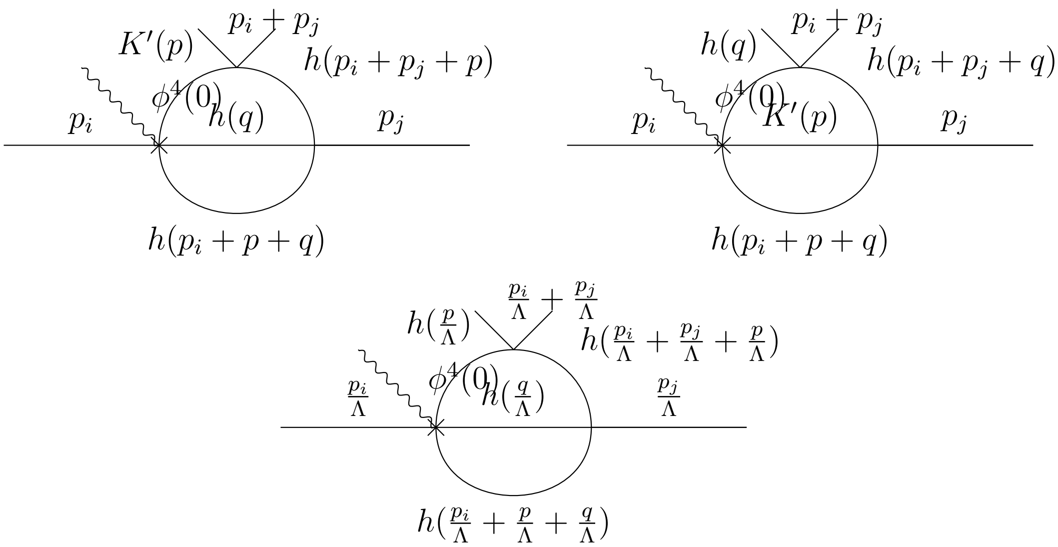


Figure 4.5: Application of $\Lambda\frac{d}{d\Lambda}$ on the diagram at the bottom gives the two diagrams at the top

We aim to solve

$$
\begin{aligned}
&\frac{3\lambda^2}{4!}\int_{p,q}\{-K'(p^2)\}\sum_{6\ perm\ (i,j)} h(p_i+p_j+p)\big\{[h(p+q+p_j)+h(p+q+p_i)]h(q)-2h(p+q)h(q)\big\}\\
&+\frac{2\lambda^2}{4!}\int_p\int_q\big\{-K'(p^2)\big\}\sum_{6\ perm\ (i,j)} h(p_i+p_j+q)\big\{[h(p+q+p_i)+h(p+q+p_j)]h(q)-2h(p+q)h(q)\big\}\\
&+\frac{\lambda^2}{4!}\int_q\int_p\big\{-K'(p^2)\big\}\sum_{l=1}^{4}\sum_{3\ perm\ (i,j)} h(p_l+p+q)h(p_l+p_i+p_j+p+q)h(q)\\
&+\frac{\lambda^2}{6}\int_{p,q}\big\{K'(p^2)\big\}\sum_{3\ perm\ (i,j)} h(p+q)h(p_i+p_j+p+q)h(q)\\
&+\Big\{-2(4-D)-\sum_{i=1}^{4}p_i.\frac{d}{dp_i}\Big\}\frac{1}{4!}B^{(2)}_{IV}(p_1,p_2,p_3,p_4)=0
\end{aligned}
\tag{4.3.40}
$$

To solve this equation first note that the second and third term on the LHS are equal. The first and second term is represented by the first and second diagram respectively on the top of Fig.4.3.40. Now observe we are basically trying to find ΔS such that $-\Lambda\frac{d}{d\Lambda}\Delta S \propto \Delta S$, so if we write Λ explicitly i.e. $p_i \to \frac{p_i}{\Lambda}$ we get,

$$
p_i.\frac{d}{dp_i} = -\Lambda\frac{d}{d\Lambda} \tag{4.3.41}
$$

Now if we consider the third diagram at the bottom of Fig 4.3.40 and apply $\Lambda\frac{d}{d\Lambda}$ we get back terms corresponding to the other two diagrams i.e.

$$
\begin{aligned}
&\Lambda\frac{d}{d\Lambda}\left[\int_{\frac{p}{\Lambda},\frac{q}{\Lambda}} h\left(\frac{p_i+p_j+p}{\Lambda}\right) h\left(\frac{p}{\Lambda}\right) h\left(\frac{p_i+p+q}{\Lambda}\right) h\left(\frac{q}{\Lambda}\right)\right]\\
=&4\int_{\frac{p}{\Lambda},\frac{q}{\Lambda}} K'\left(\frac{p}{\Lambda}\right) h\left(\frac{p_i+p_j+p}{\Lambda}\right) h\left(\frac{q}{\Lambda}\right) h\left(\frac{p_i+p+q}{\Lambda}\right)\\
&+4\int_{\frac{p}{\Lambda},\frac{q}{\Lambda}} K'\left(\frac{q}{\Lambda}\right) h\left(\frac{(p_i+p_j+p)}{\Lambda}\right) h\left(\frac{q}{\Lambda}\right) h\left(\frac{p_i+p+q}{\Lambda}\right)
\end{aligned}
$$

We can expect $B_{IV}(p_1,p_2,p_3,p_4)$ to be of the form $\int_{\frac{p}{\Lambda},\frac{q}{\Lambda}} h\left(\frac{p_i+p_j+p}{\Lambda}\right) h\left(\frac{p}{\Lambda}\right) h\left(\frac{p_i+p+q}{\Lambda}\right) h\left(\frac{q}{\Lambda}\right)$. So we use (B.6.6) and get the solution as:

$$
\begin{aligned}
&\frac{1}{4!}B^{(2)}_{IV}(p_1,p_2,p_3,p_4)\\
&=\frac{1}{4!}\frac{3}{4}\lambda^2\int_{p,q}\sum_{6\ perm\ (i,j)}\Big\{h(p_i+p_j+q)\sum_{a=i,j}h(p+q+p_a)h(p)h(q)-2h(p+q)h(p)h(q)h(q)\Big\}\\
&=\frac{1}{4!}\frac{3\lambda^2}{4}\sum_{6\ perm\ (i,j)}\left[I_4(p_i+p_j;p_i)+I_4(p_i+p_j;p_j)\right]
\end{aligned}
\tag{4.3.42}
$$

In the L.H.S of (B.3.7) we are left with

$$
\begin{aligned}
&\frac{1}{4!}3(4-D)B^{(2)}_{IV}(p_1,p_2,p_3,p_4)\\
&+\frac{\lambda^2}{2}\big\{-K'(p^2)\big\}\sum_{3\ perm\ (i,j)}h(p_i+p_j+p)\Big\{h(p+q)h(q)-h(q)h(q)\Big\}\\
&+\frac{\lambda^2}{3}\big\{-K'(p^2)\big\}\sum_{3\ perm\ (i,j)}h(p_i+p_j+q)h(p+q)h(q)\\
&+\frac{\lambda^2}{6}\big\{-K'(p^2)\big\}\sum_{3\ perm\ (i,j)}h(p_i+p_j+p+q)h(p+q)h(q)
\end{aligned}
\tag{4.3.43}
$$

Ignoring $\mathcal{O}(\epsilon^3)$ term and aiming to solve the following equation from the left over terms:

$$+\frac{\lambda^2}{2}\big\{-K'(p^2)\big\}\sum_{3\ perm\ (i,j)}\big[h(p_i+p_j+p)\big\{h(p+q)h(q)-h(q)h(q)\big\}-h(p)\big\{h(p+q)h(q)-h(q)h(q)\big\}\big]$$
$$+\frac{\lambda^2}{3}\big\{-K'(p^2)\big\}\sum_{3\ perm\ (i,j)}[h(p_i+p_j+q)h(p+q)h(q)-h(q)h(p+q)h(q)]$$
$$+\frac{\lambda^2}{6}\big\{-K'(p^2)\big\}\sum_{3\ perm\ (i,j)}[h(p_i+p_j+p+q)h(p+q)h(q)-h(p+q)h(p+q)h(q)]$$
$$+\left\{-2(4-D)-\sum_{i=1}^{4}p_i.\frac{\partial}{\partial p_i}\right\}\frac{1}{4!}B_V^{(2)}(p_1,p_2,p_3,p_4)=0 \tag{4.3.44}$$

We can write a solution symmetric in variables p and q.

$$\frac{1}{4!}B_V^{(2)}(p_1,p_2,p_3,p_4)=-\frac{\lambda^2}{2}\sum_{3\ perm\ (i,j)}\{h(p_i+p_j+p+q)h(p+q)h(q)h(p)-h(p)h(q)h(p+q)h(p+q)\}$$
$$+\frac{\lambda^2}{4}\sum_{3\ perm\ (i,j)}\{h(p_i+p_j+q)h(p+q)h(q)h(p)-h(q)h(p+q)h(q)h(p)\}$$
$$+\frac{\lambda^2}{4}\sum_{3\ perm\ (i,j)}\{h(p_i+p_j+p)h(p+q)h(q)h(p)-h(p)h(p+q)h(q)h(p)\} \tag{4.3.45}$$

And on LHS of (B.3.7) we are left with

$$\frac{1}{4!}3(4-D)B_V^{(2)}(p_1,p_2,p_3,p_4)$$
$$+\int_{p,q}\left\{\frac{3\lambda^2}{2}\big\{K'(p^2)\big\}h(p)\left[h(p+q)h(q)-h(q)h(q)\right]+\lambda^2\big\{K'(p^2)\big\}h(q)h(p+q)h(q)\right.$$
$$\left.+\frac{\lambda^2}{2}\big\{K'(p^2)\big\}h(p+q)h(p+q)h(q)\right\} \tag{4.3.46}$$

Following this procedure, we can solve all the equations given in B.3.2 to get the 4-point composite operator vertices given above.

Calculation of Anomalous Dimension

To get the anomalous dimension we collect the leftover terms which remain unused- (4.3.46)and (B.3.14) in the LHS. All other left over terms are either cancelled or of $\mathcal{O}(\epsilon\lambda^2)$ or $O(\epsilon^3)$.

$$\int_{p,q}\Big\{\frac{3\lambda^2}{2}\{-K'(p^2)\}h(p)\left[h(p+q)h(q)-h(q)h(q)\right]+\lambda^2\{-K'(p^2)\}h(q)h(p+q)h(q)$$
$$+\frac{\lambda^2}{2}\{-K'(p^2)\}h(p+q)h((p+q)h(q)\Big\}-\frac{4}{4!}\frac{\eta}{2}\sum_{i=1}^{4}B^{(0)}(p_i)=\frac{d_m}{4!}\{\sum_{i=1}^{4}B^{(0)}(p_i)\} \quad (4.3.47)$$

The first three terms on the LHS can be written as:

$$\frac{3\lambda^2}{2}\left[\int_{p,q}\{-K'(p^2)\}h(p)h(p+q)h(q)+\{-K'(p^2)\}h(q)h(p+q)h(q)\right]$$
$$-\frac{3\lambda^2}{2}\int_{p,q}\{-K'(p^2)\}h(p)h(q)h(q)$$
$$=-\frac{1}{4}\frac{3\lambda^2}{2}\Lambda\frac{\partial}{\partial\Lambda}\int_{\frac{p}{\Lambda},\frac{q}{\Lambda}}h\left(\frac{p}{\Lambda}\right)h\left(\frac{p}{\Lambda}\right)h\left(\frac{p+q}{\Lambda}\right)h\left(\frac{q}{\Lambda}\right)+\frac{3\lambda^2}{2}\int_{p,q}\{K'(p^2)\}h(p)h(q)h(q)$$
$$(4.3.48)$$

Where in the second line we have rewritten the integral in terms of dimensionful momenta and written Λ explicitly. This gives a convenient way of doing the integrals. It also reveals the relation with log divergences in Feynman diagrams. While evaluating the integral we have taken $h(p/\Lambda)$ as $\frac{K(p/\Lambda_0)-K(p/\Lambda)}{p^2/\Lambda^2}$ instead of $\frac{1-K(p/\Lambda)}{p^2/\Lambda^2}$. We keep Λ_0 finite initially to make all the integrals finite and well defined and take $\Lambda_0\to\infty$ at the end.

Now we note the Feynman diagrams of the above terms. The first(second)term in the first line of (4.3.48) represent the first(second) diagram at the top of Fig.4.3.40 (if we make all external momenta as zero). Similarly, the first term on the second line represents the diagram at the bottom of the same figure. As written above we will find this integral of the second line of (4.3.48) and then apply $-\frac{1}{4}\Lambda\frac{\partial}{\partial\Lambda}$ to get our desired integral (see Appendix B.4).

The value of the integrals in the limit of $\Lambda_0\to\infty$ is listed below.

$$a.\quad \int_{p,q}\left[\{-K'(p^2)\}h(p)h(p+q)h(q)+\{-K'(p^2)\}h(q)h(p+q)h(q)\right]$$
$$=F^2\left(\frac{1}{2}-\log 2+\frac{1}{2}\log\frac{\Lambda_0^2}{\Lambda^2}\right) \quad (4.3.49)$$

and similarly one can calculate using method shown in Appendix B.4.

$$b. \quad \int_{p,q} \{K'(p^2)\} h(p)h(q)h(q) = -F^2 \left\{ -\log 2 + \frac{1}{2}\log \Lambda_0^2 + \frac{1}{2}\log \Lambda^2 + \log\left(\frac{1}{\Lambda^2} + \frac{1}{\Lambda_0^2}\right)\right\} \tag{4.3.50}$$

So in (4.3.47) we use $B^{(0)}(p_i) = \frac{1}{4}$, combine (4.3.49) and (4.3.50) to get the anomalous dimension. Note that the logarithmic divergences gets exactly cancelled so the (4.3.50) is in fact originated from a counterterm.

$$\boxed{\frac{1}{4!}d_4 = \frac{3}{4}\lambda^2 F^2 - \frac{4}{4!}\frac{\eta}{2} = \frac{1}{4!}\frac{53}{3}\lambda^2 F^2}$$

Where $\frac{\eta}{2} = \frac{\lambda^2 F^2}{12}$ at the fixed point and $F = \frac{1}{16\pi^2}$. This value matches with (4.2.21).

The Relevant Operator, $\mathcal{O}_2^{(2)}(0)$ The form of the relevant composite operator $\mathcal{O}_2(0)$ in the subleading order is assumed as.

$$\begin{aligned}
&\mathcal{O}_2(0) = \Delta S_2 + \Delta S_4 + \Delta S_6 \\
&= \frac{1}{2!}\int_p \left\{1 + A^{(1)}(p)\right\}\phi(p)\phi(-p) \\
&+ \frac{1}{4!}\int_{p_1,p_2,p_3} \left\{1 - \lambda\sum_{l=1}^{4} h(p_l) + B^{(2)}(p_1,p_2,p_3,p_4)\right\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \\
&+ \frac{1}{6!}\int_{p_1,p_2,p_3,p_4,p_5} \left\{D^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)\right\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)
\end{aligned}$$

with

$$d_2 = 2 - F\lambda + d_2^{(2)}$$

In this section, we have written the final expressions of ΔS. The details are given in Appendix B.4.

Determination of $D^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6)$ from ϕ^6 equation

There are two kinds of 6-pt vertices distinguished according to their tensor structure (see B.4.1 for details).

$$D_I^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) = \lambda^2 \sum_{10\ perm\ (i,j,k)} h(p_i + p_j + p_k) h(p_i + p_j + p_k) \tag{4.3.51a}$$

$$D_{II}^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) = \lambda^2 \sum_{10\ perm\ (i,j,k)} h(p_i + p_j + p_k) \sum_{l=1}^{6} h(p_l) \tag{4.3.51b}$$

Determination of $B^{(2)}(p_1, p_2, p_3, p_4)$ from ϕ^4 equation

Similarly there are 3 kinds of 4-pt vertices (see B.4.2 for details).

$$B_I^{(2)}(p_1, p_2, p_3, p_4) = \lambda^2 \sum_{3\ perm\ (i,j)} \bar{H}_3(p_i + p_j) \tag{4.3.52a}$$

$$B_{II}^{(2)}(p_1, p_2, p_3, p_4) = \lambda^2 \sum_{l=1}^{4} h(p_l) \sum_{3\ perm\ (i,j)} \mathcal{F}(p_i + p_j) \tag{4.3.52b}$$

$$B_{III}^{(2)}(p_1, p_2, p_3, p_4) = -F\lambda^2 \{\frac{1}{2} \sum_{i \neq j} h(p_i) h(p_j) + \sum_{l=1}^{4} h^2(p_l)\} \tag{4.3.52c}$$

$\bar{H}_3(p)$ and $\mathcal{F}(p)$ is defined in (B.6.5) and (B.6.2) respectively.

Determination of $A^{(1)}(p)$ from ϕ^2 equation

This ϕ^2 equation is solved by six kinds of $A^{(2)}$s according to different tensor structures (see B.4.3 for details).

$$A_I^{(2)}(p) = -\frac{\lambda^2}{3} \int_{q,k} \{h(p+q+k)h(p)h(q)h(k) - h(q)h(q+k)h(k)\} \tag{4.3.53a}$$

$$A_{II}^{(2)}(p) = -\frac{\lambda^2}{2} \int_{q,k} \{h(p+q+k)h(q)h(q)h(k) - h(q+k)h(q)h(q)h(k)\} \tag{4.3.53b}$$

$$A_{III}^{(2)}(p) = -\lambda^2 F^2 h(p) + \frac{\epsilon\lambda}{2} h(p) \tag{4.3.53c}$$

$$A_{IV}^{(2)}(p) = \frac{3}{4} F^2 \lambda^2 h^2(p) \tag{4.3.53d}$$

$$A_V^{(2)}(p) = -\lambda^2 h(p) \int_q f(q) \mathcal{F}(q) \tag{4.3.53e}$$

$$A_{VI}^{(2)}(p) = \frac{\eta}{\epsilon} p^2 h(p) \tag{4.3.53f}$$

Anomalous dimension

We collect the unused leftover terms as we did in the previous subsection to get the anomalous dimension:

$$\lambda^2 \int_{q,k} \big\{ - K'(q) \big\} h(q+k)h(k)h(k) + \lambda^2 \int_{q,k} \big\{ - K'(q)h(q) \big\} \big\{ h(q+k)h(k) - h(k)h(k) \big\}$$
$$+ \frac{3}{2} \int_q K'(q)h^2(q) + \lambda^2 \frac{1}{2} \int_k h(k)h(k)h(k) - \frac{\eta}{2} A^{(0)}(p) = d_2 \frac{1}{2} A^{(0)}(p) \tag{4.3.54}$$

Like we have seen in the calculation of anomalous dimension of the irrelevant operator here also the anomalous dimension is coming from a diagram as shown in Fig.4.6 which is logarithmically divergent but made finite by adding a counterterm.

Evaluation of Integrals

$$a. \quad \int_q K'(q)h^2(q) = 2\log 2 - \log 3$$

$$b. \quad \int_k h(k)h(k)h(k) = 3\log 3 - 6\log 2$$

So the third and fourth terms on LHS of (4.3.54) cancels among each other. The rest of the integrals in the LHS we know from the previous subsection. So in the limit of $\Lambda_0 \to \infty$ we obtain the anomalous dimension as,

$$\boxed{d_2 = 2\left(\frac{\lambda^2 F^2}{2} - \frac{\eta}{2}\right) = \frac{5}{6}\lambda^2 F^2}$$

Where $F = \frac{1}{16\pi^2}$ and $\frac{\eta}{2} = \frac{\lambda^2 F^2}{12}$. This agrees with (4.2.25).

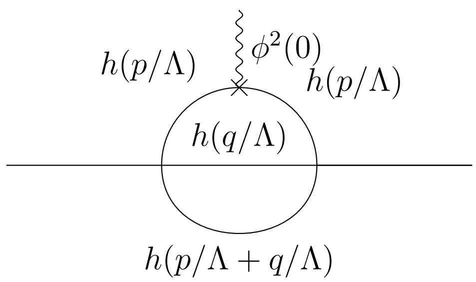


Figure 4.6: Diagram contributing to $d_2^{(2)}$ for the relevant operator

Chapter 5

Conclusion

In this thesis, we have studied some aspects of the $O(N)$ model using the Exact RG formalism. We have done two things:

1) We have constructed the Wilson action for the $O(N)$ model at the Wilson-Fisher fixed point in $4-\epsilon$ dimensions up to order ϵ^2. This is done by solving the fixed point equation, order by order in ϵ.

2) Using the above action we have also constructed two composite operators in the ϕ^4 scalar field theory at the Wilson-Fisher fixed point in $D = 4-\epsilon$ dimension. The composite operators and their anomalous dimensions are listed in (4.2.26),(4.2), (4.3), (4.2.29), (4.2.30) and (4.6).

Dimension of an operator is a well-defined concept only if the underlying theory is scale-invariant (at least in some approximation). The fixed point condition of the ERG equation is a condition for scale invariance of the action. This was solved to $O(\epsilon^2)$ in [43]. The energy-momentum tensor was also shown to be traceless, thus verifying that this theory is also conformal invariant - as expected on general grounds. Thus the operators constructed in this theory should correspond to primary operators of this CFT. However this needs to be verified by checking the Conformal Ward Identities, which requires a *local* operator, i.e. $\mathcal{O}_2(q), \mathcal{O}_4(q)$ with $q \neq 0$. We leave this for the future.

As mentioned in the introduction, one of the motivations for this construction is to use the ideas in [40, 41] and construct the AdS action corresponding to this CFT. A related problem is to construct the AdS action for sources for composite operators such as $\phi^i \phi^i$. Even more interesting would be to study the massless spin 2 field that would be the source for the energy-momentum tensor. This would give dynamical gravity in the bulk as a consequence of Exact RG in the boundary by a direct change of variables similar to what was done for the scalar field in [40, 41].

The main point of our work is that the UV cutoff is kept finite throughout. Thus both the fixed point action and the composite operators constructed here are valid at all length scales. In particular, scale and conformal invariance of the action is not an approximate statement valid at energies $p << \Lambda$ but is valid for all p. In the same way, the expressions for the composite operators in terms of fundamental fields are valid even when the internal momentum circulating is arbitrarily large. (Note that because of the analytic form of the cutoff function, loop momenta are not restricted to be less than Λ.)

CFTs and more generally field theories with a finite UV cutoff are conceptually interesting and generalize the notion of scale invariance in the presence of a UV cutoff. These could have applications in condensed matter physics and critical phenomena because these systems always have an underlying short distance cutoff.

The results of our work are also relevant for a better understanding of holography. The bulk AdS dual of the O(N) model has been studied. The connection between ERG and Holographic RG has also been studied recently and in these approaches, a finite cutoff plays a crucial role [40, 41, 42]. In fact, as we have used a very general cut-off in our work, the cutoff functions used in [40, 41, 42] can be accommodated.

There are several other open questions. One is to understand the precise role of the irrelevant terms in the Wilson Action when constructing the bulk AdS-dual. It would also be interesting to have more examples of such constructions in other CFTs and in other dimensions where a Lagrangian description is available, for eg., Wess-Zumino-Witten models and $O(N)$ models in 3 dimensions.

Finally and perhaps most important is the inclusion of gravity in these theories and the connection with string theory. If one were to speculate (as for instance in [61]) that underlying space-time in string theory is not a continuum then it may also be necessary to understand properties of theories with finite cutoff where the underlying "lattice" is dynamical.

Appendix A

Wilson Action

A.1 Fixed point action

A.1.1 Evaluation of U_4

We need to solve

$$\left[\left(4-D-\sum_{i=1}^{4}p_i\frac{d}{dp_i}\right)+\sum_{j=1}^{4}2K'(p_j^2)U_2^{(1)}(p_j)\right]\frac{1}{8}U_4(p_1,p_2;p_3,p_4)$$
$$=\int\frac{d^Dp}{(2\pi)^D}K'(p^2))\frac{1}{48}\Big\{6NU_6(p_1,p_2;p_3,p_4;p,-p)+12U_6(p_1,p;p_2,-p;p_3,p_4)+12U_6(p_1,p_2;p_3,p;p_4,-p)\Big\}$$
$$=\int\frac{d^Dp}{(2\pi)^D}K'(p^2))\Big\{-\frac{(N+2)}{8}\Big(h(p_1)+h(p_2)+h(p_3)+h(p_4)\Big)$$
$$-\frac{(N+4)}{4}\Big(h(p+p_1+p_2)+2h(p+p_1+p_3)+2h(p+p_1+p_4)\Big)\Big\} \tag{A.1.1}$$

where

$$\int\frac{d^Dp}{(2\pi)^D}K'(p^2))\Big\{-\frac{(N+2)}{8}\Big(h(p_1)+h(p_2)+h(p_3)+h(p_4)\Big)\Big\} \tag{A.1.2}$$

corresponds to the kind of diagrams shown in A.1. Here the external loop does not involve momenta p_i+p_j. We will call it type I diagrams. Considering only leading order terms in p_j^2 the contribution from type I diagram in (A.1.1) is

$$=-\frac{N+2}{8}\frac{\lambda^2}{16\pi^2}\;4K'(p_j^2)\Big|_{p_j=0} \tag{A.1.3}$$

Now consider the second term in L.H.S of (A.1.1). In the limit of small external momenta after putting the value of $U_2^{(1)}(p) = -\frac{N+2}{2}\frac{\lambda}{16\pi^2}$(as we are considering terms of $\mathcal{O}(\epsilon^2)$ we have put D=4 to find $U_2^{(1)}$) we get

$$
\begin{aligned}
&-\sum_{j=1}^{4} 2K'(p_j^2)\Big|_{p_j\to 0} \frac{\lambda}{16\pi^2}\frac{N+2}{2}\frac{1}{8}V_4(p_1,p_2;p_3,p_4) \\
&= -\,4K'(p_j^2)\Big|_{p_j\to 0}\frac{\lambda^2}{16\pi^2}\frac{N+2}{8}
\end{aligned}
\tag{A.1.4}
$$

This cancels exactly with (A.1.3).

Similarly in (A.1.1) the term

$$
\int \frac{d^Dp}{(2\pi)^D}K'(p^2)\Big\{-\frac{(N+4)}{4}\Big(h(p+p_1+p_2)+2h(p+p_1+p_3)+2h(p+p_1+p_4)\Big)\Big\} \tag{A.1.5}
$$

corresponds to the kind of diagram shown in A.2. We will call it Type II diagram. In the limit $p_i \to 0$ the above term becomes

$$
\begin{aligned}
&\lambda^2\,\frac{(N+8)}{4}\frac{1}{16\pi^2}\int_0^\infty dp^2K'(p^2)\Big(K(p^2)-K(0)\Big) \\
=&\lambda^2\frac{(N+8)}{4}\frac{1}{16\pi^2}\int_0^\infty dp^2\Big\{\frac{1}{2}\frac{d(K^2)}{dp^2}-K(0)K'(p^2)\Big\}
\end{aligned}
$$

Using $K(\infty)=0$ and $K(0)=1$, this integral gives $\frac{1}{2}$. Equating this contribution with $\epsilon\frac{\lambda}{4!}$ from L.H.S of (A.1.1) we obtain

$$
\frac{1}{8}(4-D)\lambda = \frac{N+8}{8}\frac{\lambda^2}{(4\pi)^2}
$$

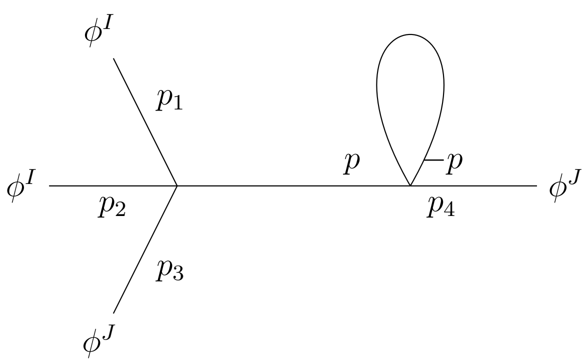


Figure A.1: Type I diagram

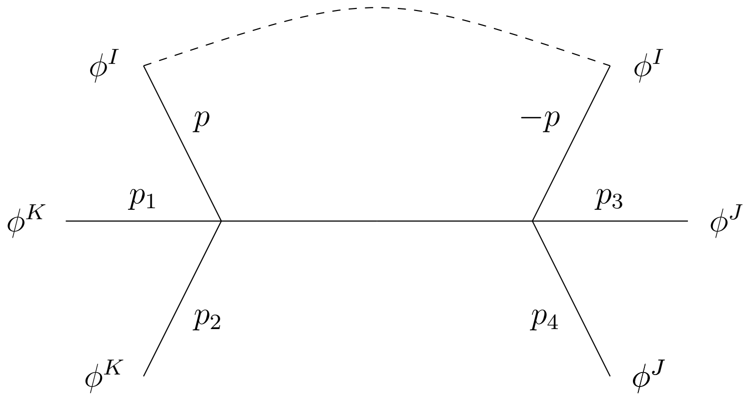


Figure A.2: Type II diagram

Thus in addition to the trivial fixed point $\lambda = 0$, we have a non trivial fixed point:

$$\lambda = (4 - D)\frac{16\pi^2}{N + 8} \tag{A.1.6}$$

A.1.2 Solving for $\tilde{U}_4$

$\tilde{U}_4$ will have contribution from both type I and II diagram explained above. We write

$$\tilde{U}_4 = \tilde{U}_4^I + \tilde{U}_4^{II}$$

according to contributions from type I(II) diagrams.

(We shall set $D = 4$ while evaluating integrations in those terms that are already of $\mathcal{O}(\epsilon^2)$.)

Type I diagram In (A.1.1) the first term on the LHS and the first terms on the RHS (Type I) cancel only in leading order. In general, their difference is

$$\lambda^2 \frac{N+2}{8} \times \frac{1}{(4\pi)^2} \int_0^\infty dp^2 K'(p^2) \left[\sum_j \frac{K(p_j^2) - K(0)}{p_j^2} - K'(p_j^2) \right]$$

Taylor expanding we find

$$\lambda^2 \frac{N+2}{8} \times \frac{1}{(4\pi)^2} \int dp^2 K'(p^2) K''(0) \frac{1}{2} \sum_j p_j^2 \equiv c \sum_j p_j^2$$

This is a contribution to $\tilde{U}_4(p_1, p_2; p_3, p_4)$ that we can call $\Delta U_4^I(p_1, p_2; p_3, p_4)$. Consider a type I graph where the line at one end has p_1 and lines with momenta p_2, p_3, p_4 are at the other end.

This corresponds to the term

$$\lambda^2\frac{N+2}{8}\times\frac{1}{(4\pi)^2}\int dp^2K'(p^2)K''(0)\frac{1}{2}p_1^2\equiv cp_1^2$$

when contracted in a loop in order to contribute to $\tilde{U}_2$, so that say $p_3=-p_4$, we have $p_2=-p_1$. It contributes to $\tilde{U}_2(p_1^2)$ an amount

$$\int dp^2K'(p^2)\frac{1}{2}\Delta U_4^I(p_1,-p_1,p,-p)=\int dp^2K'(p^2)\frac{1}{2}c(p_1^2)=\Big[c\int dp^2K'(p^2)\Big]p_1^2\equiv Ap_1^2$$

This is just a simple wave function renormalization that does not depend on p_1. There is no contribution to the mass. The same argument applies to all the other permutations of the type I terms. A simple wave function renormalization $\phi'^2=(1+A)\phi^2$ can ensure the normalization of the kinetic term. They do not affect the physics or contribute to η. However, type I term contributes to sub-leading order term of m^2 or U_2.

$\tilde{U}_4^I$ satisfies the following equation:

$$-\sum_{i=1}^4 p_i\frac{d}{dp_i}\frac{1}{8}\tilde{U}_4^I(p_1,p_2;p_3,p_4)=\lambda^2\frac{N+2}{8}\times\frac{1}{(4\pi)^2}\int_0^\infty dp^2K'(p^2)\Bigg[\sum_j\frac{K(p_j^2)-K(0)}{p_j^2}-K'(p_j^2)\Bigg]$$

The solution is

$$\tilde{U}_4^I(p_1,p_2;p_3,p_4)=-\lambda^2\frac{(N+2)}{2}\frac{1}{16\pi^2}\sum_{j=1}^4\frac{K(p_j^2)-K(0)}{p_j^2} \tag{A.1.7a}$$

$$=\lambda^2\frac{(N+2)}{2}\frac{1}{16\pi^2}\sum_{j=1}^4 h(p_j) \tag{A.1.7b}$$

where $K(p)=e^{-p^2}$ is assumed.

Type II Diagram In (A.1.1) if we keep terms upto $\mathcal{O}(\epsilon^2)$,

$$\frac{1}{8}\Bigg[\sum_{j=1}^4 p_j\frac{d}{dp_j}\Bigg]\tilde{U}_4^{II}(p_1,p_2;p_3,p_4)$$
$$=\frac{\lambda^2}{4}\int\frac{d^Dp}{(2\pi)^D}K'(p^2)\Big\{(N+4)h(p+p_1+p_2)+2h(p+p_1+p_3)+2h(p+p_1+p_4)-(N+8)h(p)\Big\}$$
$$\tag{A.1.8}$$

where $h(p) = \frac{K(0)-K(p)}{p^2}$. It is to be noted in the momentum independent part $-\epsilon\frac{\lambda}{4!}$ we have written ϵ in terms of λ using the fixed-point value of λ.

The solution at $\mathcal{O}(\epsilon^2)$, analytic at zero external momenta, is given by

$$\begin{aligned}
&\tilde{U}_4^{II}(p_1, p_2; p_3, p_4) \\
&= -\frac{\lambda^2}{2}\int \frac{d^D p}{(2\pi)^D} h(p)\Big[(N+4)h(p_1+p_2+p) + 2h(p+p_1+p_3) + 2h(p+p_1+p_4) - (N+8)h(p)\Big] \quad \text{(A.1.9a)} \\
&= -\lambda^2\Big[(N+4)F(p_1+p_2) + 2F(p_1+p_3) + 2F(p_1+p_4)\Big] \quad \text{(A.1.9b)}
\end{aligned}$$

where $F(q) = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D} h(p)\Big(h(p+q) - h(p)\Big)$.

A.1.3 Equation for $\tilde{U}_2$

From (3.2.19) we get

$$\begin{aligned}
0 = &\int \frac{d^D p}{(2\pi)^D}\Big(-K'(p^2)\Big)\times \\
&\Big\{\frac{1}{8}\Big[4N\tilde{U}_4^I(p_1,-p_1;p,-p) + 4N\tilde{U}_4^{II}(p_1,-p_1;p,-p) + 8\tilde{U}_4^I(p_1,p;-p_1,-p) + 8\tilde{U}_4^{II}(p_1,p;-p_1,-p)\Big] \\
&-v_2^{(1)}(p)v_2^{(1)}(p)\delta^D(p-p_1)\Big\} - \frac{\eta}{2}p_1^2 + \tilde{U}_2(p_1) - p_1^2\frac{d\tilde{U}_2(p_1)}{dp_1^2} \quad \text{(A.1.10)}
\end{aligned}$$

From(A.1.7a)

$$\begin{aligned}
&\frac{1}{8}\Big\{4N\tilde{U}_4^I(p_1,-p_1;p,-p) + 8\tilde{U}_4^I(p_1,p;-p,-p_1)\Big\} \\
&= \frac{1}{2}(N+2)^2\frac{\lambda^2}{16\pi^2}\Big\{h(p) + h(p_1)\Big\} \quad \text{(A.1.11)}
\end{aligned}$$

and from (A.1.9a)

$$\begin{aligned}
&\frac{1}{8}\Big\{4N\tilde{U}_4^{II}(p_1,-p_1;p,-p) + 8\tilde{U}_4^{II}(p_1,p;-p,-p_1)\Big\} \\
&= -\frac{3\lambda^2}{2}(N+2)\int_r\Big\{h(r)\Big[h(r+p_1+p) - h(r)\Big]\Big\} \quad \text{(A.1.12)}
\end{aligned}$$

If we decompose $\tilde{U}_2$ in two parts namely $\tilde{U}_2^I$ and $\tilde{U}_2^{II}$ respectively, in the following way, 1.

$$\tilde{U}_2^I(p_1) - p_1^2\frac{d\tilde{U}_2^I(p_1)}{dp_1^2} = \int\frac{d^Dp}{(2\pi)^D}K'(p^2)\frac{1}{2}(N+2)^2\frac{\lambda^2}{16\pi^2}h(p_1) - \big(U_2^{(1)}\big)^2K'(p_1^2) \qquad \text{(A.1.13)}$$

which gives

$$\tilde{U}_2^I(p_1) = -\frac{\lambda^2}{(16\pi^2)^2}\frac{(N+2)^2}{4}h(p_1) \qquad \text{(A.1.14)}$$

2.

$$\begin{aligned}&-2\tilde{U}_2^{II}(p_1) + 2p_1^2\frac{d\tilde{U}_2^{II}(p_1)}{dp_1^2}\\ =&-6\lambda^2(N+2)\int\frac{d^Dp}{(2\pi)^D}\Big(-K'(p^2)\Big)F(p_1+p) + (N+2)^2\frac{\lambda^2}{16\pi^2}\int\frac{d^Dp}{(2\pi)^D}\Big(-K'(p^2)\Big)h(p) - \eta\, p_1^2\end{aligned} \qquad \text{(A.1.15)}$$

which gives

$$\tilde{U}_2^{II}(p_1) = \ p_1^2\int_{p^2=0}^{p_1^2}dp^2\frac{\int\frac{d^Dq}{(2\pi)^D}\Big\{-6\lambda^2(N+2)(-K'(q^2))F(p+q)\Big\} - \eta\, p^2}{2p^4} - \frac{(N+2)^2}{4}\frac{\lambda^2}{(16\pi^2)^2} \qquad \text{(A.1.16)}$$

The second term in the expression of $\tilde{U}_4^{II}$ is evaluated using $K(p) = e^{-p^2}$.

Hence The full expression of $\tilde{U}_2(p_1)$ is given by

$$\begin{aligned}\tilde{U}_2(p_1) =& -\frac{\lambda^2}{(16\pi^2)^2}\frac{(N+2)^2}{4}h(p_1)\\ &+p_1^2\int_{p^2=0}^{p_1^2}dp^2\frac{\int\frac{d^Dq}{(2\pi)^D}\Big\{-6\lambda^2(N+2)(-K'(q^2))F(p+q)\Big\} - \eta\, p^2}{2p^4} - \frac{(N+2)^2}{4}\frac{\lambda^2}{(16\pi^2)^2}\end{aligned} \qquad \text{(A.1.17)}$$

A.1.4 Expression for η

Only Type II diagrams contribute to η. Because we need the external momentum to flow through the loop - to get a momentum dependence in U_2. This can happen only in Type II terms and that too for **certain contractions**. (Calculation of this section requires us to go back to bar denoted variable as a dimensionless variable. So p's from the last section are replaced with $\bar{p}$). From (3.2.20) we have

$$\frac{\eta}{2} = -\frac{1}{8}\frac{d}{d\bar{r}^2}\int_{\bar{q}} K'(\bar{q}^2)\Big\{4N\tilde{U}_4^{II}(\bar{q},-\bar{q};\bar{r},-\bar{r}) + 8\tilde{U}_4^{II}(\bar{q},\bar{r};-\bar{r},-\bar{q})\Big\}\Big|_{\bar{r}^2=0} \tag{A.1.18}$$

We can convert differentiation w.r.t p_j into that w.r.t Λ , i.e.

$$-\sum_{j=1}^{4}\bar{p}_j\frac{d}{d\bar{p}_j} = \Lambda\frac{d}{d\Lambda}$$

So (A.1.8) gives following expression for $\tilde{U}_4^{II}$:

$$\frac{1}{8}\tilde{U}_4^{II}(\frac{p_1}{\Lambda},\frac{p_2}{\Lambda};\frac{p_3}{\Lambda},\frac{p_4}{\Lambda})$$
$$=\frac{\lambda^2}{4}\int_0^{\ln\Lambda} d\ln\Lambda' \int_{\bar{p}} K'(\bar{p}^2)\Big[(N+4)h(\bar{p}+\frac{p_1}{\Lambda'}+\frac{p_2}{\Lambda'}) + 2h(\bar{p}+\frac{p_1}{\Lambda'}+\frac{p_3}{\Lambda'}) + 2h(\bar{p}+\frac{p_1}{\Lambda'}+\frac{p_4}{\Lambda'}) - (N+8)h(\bar{p})\Big] \tag{A.1.19}$$

Hence

$$\frac{1}{8}\Big\{4N\tilde{U}_4^{II}(\bar{q},-\bar{q};\bar{r},-\bar{r}) + 8\tilde{U}_4^{II}(\bar{q},\bar{r};-\bar{r},-\bar{q})\Big\}$$
$$=\frac{\lambda^2}{4}\int_0^{\ln\Lambda} d\ln\Lambda' \int_{\bar{p},\bar{r}} K'(\bar{p}^2)\Big\{(12N+48)h(\bar{p}+\frac{q}{\Lambda'}+\frac{r}{\Lambda'}) + (12N+48)h(\bar{p}+\frac{q}{\Lambda'}-\frac{r}{\Lambda'}) - 24(N+2)h(\bar{p})\Big\} \tag{A.1.20}$$

So we need to find the coefficient of $\bar{r}^2$ in $\Big[h(\bar{p}+\frac{q}{\Lambda'}+\frac{r'}{\Lambda'}) + h(\bar{p}+\frac{q}{\Lambda'}-\frac{r'}{\Lambda'})\Big]$ which is calculated as

$$\frac{1}{2}\frac{r^\mu r^\nu}{\Lambda'^2}\frac{d^2}{dr'^\mu dr'^\nu}\Big[h(\bar{p}+\frac{q}{\Lambda'}+\frac{r'}{\Lambda'}) + h(\bar{p}+\frac{q}{\Lambda'}-\frac{r'}{\Lambda'})\Big]\Big|_{r'=0}$$
$$=-\frac{\bar{r}^2}{4}\frac{d^2}{d\bar{r}^\mu d\bar{r}_\mu}\frac{K(\bar{r}^2)-1}{\bar{r}^2}\Big|_{\bar{r}=\bar{p}+\frac{q}{\Lambda'}}$$
$$=\bar{r}^2K''((\bar{p}+\frac{q}{\Lambda'})^2) \tag{A.1.21}$$

where we have used the facts: in 4 dimensions $(\frac{d}{dp_\mu}\frac{1}{p^2}) = \delta^4(p)$ and $K(0)=1$.

From (A.1.18),(A.1.20) and (A.1.21) we get

$$\frac{\eta}{2} = 3\lambda^2(N+2)\int_{\bar q} K'(\bar q^2)\int_0^{\ln\Lambda} d\ln\Lambda'\,(\frac{\Lambda}{\Lambda'})^2\int_{\bar p} K'(\bar p^2)K''((\bar p+\frac{q}{\Lambda'})^2) \tag{A.1.22}$$

Evaluation of integral: Let us use $\bar q' = \frac{q}{\Lambda'}$ and Λ' as variables of integration, rather than $\bar q = \frac{q}{\Lambda}$ and Λ'. So change variables:

$$\bar q = \bar q'\frac{\Lambda'}{\Lambda}\ ; \quad \bar q^2 = \bar q'^2\Big(\frac{\Lambda'}{\Lambda}\Big)^2\ ; \quad \int d^4\bar q = \int d^4\bar q\Big(\frac{\Lambda'}{\Lambda}\Big)^4$$

to get

$$\frac{\eta}{2} = -3\lambda^2(N+2)\int_0^{\ln\Lambda} d\ln\Lambda'\int_{\bar q'}\Big(\frac{\Lambda'}{\Lambda}\Big)^{-2}K'(\bar q'^2)\Big(\frac{\Lambda'}{\Lambda}\Big)^2\int_{\bar p}K'(\bar p^2)K''((\bar p+\frac{q}{\Lambda'})^2)$$

Using $K'(\bar q'^2) = \frac{dK}{d\Lambda'}\frac{d\Lambda'}{d\bar q'^2} = -\frac{\Lambda'}{2\bar q'^2}\frac{dK}{d\Lambda'}$ we get

$$\frac{\eta}{2} = -3\lambda^2(N+2)\int_0^{\Lambda} d\Lambda'\,\frac{dK}{d\Lambda'}\int_{\bar q'}\frac{1}{2\bar q'^2}\int_{\bar p}K'(\bar p^2)K''((\bar p+\bar q')^2)$$

Since $\bar q'$ is an independent variable we can write this as

$$\frac{\eta}{2} = -3\lambda^2(N+2)\int_{\bar q'}\int_0^{\Lambda} d\Lambda'\,\frac{dK}{d\Lambda'}\frac{1}{2\bar q'^2}\int_{\bar p}K'(\bar p^2)K''((\bar p+\bar q')^2)$$

The integral over $\bar p$ is a function of $\bar q'$ and not Λ'. So we can do the Λ' integral easily. Using $K(\infty) = 0$ we get

$$\frac{\eta}{2} = -\frac{3\lambda^2}{2}(N+2)\underbrace{\int_{\bar q'}K(\bar q'^2)\frac{1}{\bar q'^2}\int_{\bar p}K'(\bar p^2)K''((\bar p+\bar q')^2)}_{-\frac{\pi^4}{6(2\pi)^8}} = \frac{1}{4}\lambda^2(N+2)\frac{1}{(16\pi^2)^2}$$

The integral underbraced above is calculated to give $-\frac{\pi^4}{6(2\pi)^8}$ for $K(x) = e^{-x}$. But it can be shown to give identical result for any smooth $K(x)$ [66]. Using $\lambda = \frac{16\pi^2}{N+8}\epsilon$ we can write the anomalous dimension as:

$$\frac{\eta}{2} = \frac{1}{4}\lambda^2(N+2)\frac{1}{(16\pi^2)^2} = \frac{N+2}{(N+8)^2}\frac{\epsilon^2}{4} \tag{A.1.23}$$

A.2 Asymptotic behaviors of $F(p)$ and $G(p)$

The function $F(p)$ is defined by

$$(p \cdot \partial_p + \epsilon)\, F(p) = \int_q f(q)\Big(h(q+p) - h(q)\Big) \tag{A.2.1}$$

For large p, we obtain an equation satisfied by the asymptotic form $F_{\mathrm{asymp}}(p)$:

$$(p \cdot \partial_p + \epsilon)\, F_{\mathrm{asymp}}(p) = - \int_q f(q)h(q) = -\frac{1}{(4\pi)^2} + \mathrm{O}(\epsilon) \tag{A.2.2}$$

This implies

$$F_{\mathrm{asymp}}(p) = -\frac{1}{\epsilon} \int_q f(q)h(q) + C_F(\epsilon) p^{-\epsilon} \tag{A.2.3}$$

where $C_F(\epsilon)$ is independent of p. Since $F(p)$ is finite in the limit $\epsilon \to 0+$, we must find

$$C_F(\epsilon) = \frac{1}{\epsilon} \frac{1}{(4\pi)^2} + \cdots \tag{A.2.4}$$

Hence, expanding in ϵ, we obtain

$$F_{\mathrm{asymp}}(p) = -\frac{1}{(4\pi)^2} \ln p + \mathrm{const} + \mathrm{O}(\epsilon) \tag{A.2.5}$$

We next consider $G(p)$ satisfying

$$(p \cdot \partial_p - 2 + 2\epsilon)\, G(p) = \int_q f(q)F(q+p) + 2v_2 \int_q f(q)h(q) + \eta^{(2)} p^2 \tag{A.2.6}$$

where

$$\eta^{(2)} = -\frac{d}{dp^2} \int_q f(q)F(q+p)\Big|_{p=0} = \frac{1}{6(4\pi)^4} + \mathrm{O}(\epsilon) \tag{A.2.7}$$

The asymptotic form $G_{\mathrm{asymp}}(p)$ satisfies

$$(p \cdot \partial_p - 2 + 2\epsilon)\, G_{\mathrm{asymp}}(p) = \eta^{(2)} p^2 \tag{A.2.8}$$

This gives

$$G_{\mathrm{asymp}}(p) = \frac{1}{2\epsilon} \eta^{(2)} p^2 + C_G(\epsilon) p^{2-2\epsilon} \tag{A.2.9}$$

Since $G(p)$ is finite as $\epsilon \to 0+$, we obtain

$$C_G(\epsilon) = -\frac{1}{\epsilon}\frac{1}{12(4\pi)^4} + \cdots \tag{A.2.10}$$

Hence,

$$G_{\mathrm{asymp}}(p) = p^2\left(\frac{1}{6(4\pi)^4}\ln p + \mathrm{const}\right) + \mathrm{O}(\epsilon) \tag{A.2.11}$$

Appendix B

Composite Operators

B.1 Local Operators

Under a scale transformation

$$\bar{x} = \lambda x \quad , \quad \bar{p} = \frac{p}{\lambda} \tag{B.1.1}$$

$$\bar{\phi}(\bar{p}) = \lambda^{-d^p_\phi}\phi(p)$$

Here d^x_O is the scaling dimension of any operator $O(x)$ and $d^p_O = d^x_O - D$ is the scaling dimension of $O(p)$. Let $\lambda = e^{-t}$ and $\bar{p} = pe^t$.

$$\bar{\phi}(pe^t) = e^{d^p_\phi t}\phi(p)$$

$$e^{-d^p_\phi t}\bar{\phi}(pe^t) = \phi(p)$$

We hold p fixed and change t:

$$\frac{\partial \phi(p)}{\partial t} = (-d^p_\phi + p\frac{d}{dp})\phi(p)$$

and more generally for any operator with mass scaling dimension d^p_O:

$$\frac{\partial O(p)}{\partial t} = (-d^p_O + p\frac{d}{dp})O(p) \tag{B.1.2}$$

One can also call $-d^p_O$ the length scaling dimension.

Let us consider operators of the form

$$\Delta S = \int_q B(q) O(q) \tag{B.1.3}$$

Then the change under scaling can be written as

$$\frac{\partial \Delta S}{\partial t} = \int_q B(q)(-d_O^q + q\frac{d}{dq})O(q)$$

$$= \int_q [(-d_O^p - D - q\frac{d}{dq})B(q)]O(q) = \int_q [(-d_O^x - q\frac{d}{dq})B(q)]O(q)$$

This gives the action on the coefficient functions in the composite operator.

Thus if we have

$$O = \int_{p_1}\int_{p_2} A(p_1, p_2)\phi(p_1)\phi(p_2)$$

Then

$$\frac{\partial O}{\partial t} = \int_{p_1}\int_{p_2} [(-p_1\frac{d}{dp_1} - p_2\frac{d}{dp_2} - 2d_\phi^x)A(p_1, p_2)]\phi(p_1)\phi(p_2) \tag{B.1.4}$$

The operator acting on the coefficient functions A has been called $\mathcal{G}_{dil}^c$ in the literature. The superscript c denotes that it is the contribution to scaling due to the classical or engineering dimensions. (see for eg.[18, 19]).

Let us consider some simple examples that will be used.

1.

$$A(p_1, p_2) = \delta(p_1 + p_2 - q) \tag{B.1.5}$$

Then using

$$(p_1\frac{d}{dp_1} + p_2\frac{d}{dp_2} + q\frac{d}{dq})\delta(p_1 + p_2 - q) = -D\delta(p_1 + p_2 - q) \tag{B.1.6}$$

we obtain

$$\frac{\partial O}{\partial t} = \int_{p_1}\int_{p_2} (-2d_\phi^x + D + q\frac{d}{dq})\delta(p_1 + p_2 - q)\phi(p_1)\phi(p_2)$$

$$\frac{\partial O}{\partial t} = (-d_O^p + q\frac{d}{dq})\int_{p_1}\int_{p_2} \delta(p_1 + p_2 - q)\phi(p_1)\phi(p_2)$$

as required.

2. More generally

$$A(p_1,p_2)=\delta(p_1+p_2-q)B(p_1,p_2,q) \tag{B.1.7}$$

Then going through the same steps one obtains

$$\frac{\partial O}{\partial t}=\Big((-2d_\phi^x+D+q\frac{d}{dq})\int_{p_1}\int_{p_2}\delta(p_1+p_2-q)B(p_1,p_2,q)$$

$$+\int_{p_1}\int_{p_2}\delta(p_1+p_2-q)(-p_1\frac{d}{dp_1}-p_2\frac{d}{dp_2}-q\frac{d}{dq})B(p_1,p_2,q)\Big)\phi(p_1)\phi(p_2)$$

If $B(p_1,p_2,q)$ has a well defined scaling dimension it adds to d_ϕ^p. For eg if $B(p_1,p_2,q)=p_1.p_2$ the operator is just the kinetic term and we get $-2d_\phi^x+D-2=0$, which is the dimension of $\int_p(p.(q-p))\phi(p)\phi(q-p)$.

B.2 Composite operators at the leading order

In this appendix we have calculated different parts of (4.3.18) upto λ^1. Note that we have marked different parts as **(1)**, **(2)**,**(3)** and **(4a)** respectively. As we have considered only the leading order terms we remove the superscript (1) from 4 and 6-pt vertices B_I , B_{II} and D.

(1)

$$\int_p\{-K'(p^2)\}\frac{\delta^2\Delta S}{\delta\phi(p)\delta\phi(-p)}$$

$$=\int_p\{-K'(p^2)\}\frac{1}{2}\int_{p_1,p_2}\delta(p_1+p_2-q)\Big(B^{(0)}+(B_I(p_1,q)+B_I(p_2,q)+B_I(p,q)+B_I(-p,q))+$$

$$+\frac{1}{2}[B_{II}(p_1{+}p_2,q){+}B_{II}(p_1{+}p,q){+}B_{II}(p_1{-}p,q){+}B_{II}(p_2{+}p,q){+}B_{II}(p_2{-}p,q){+}B_{II}(0,q)]\Big)\phi(p_1)\phi(p_2)$$

$$+\frac{1}{4!}\int_p\{-K'(p^2)\}\frac{1}{2}\int_{p_1,p_2,p_3,p_4}\delta(p_1+p_2+p_3+p_4-q)[(D(p_1,q)+D(p_2,q)+D(p_3,q)+D(p_4,q)+$$

$$+D(p_1+p_2+p_3,q)+D(p_1+p_2+p_4,q)+D(p_1+p_3+p_4,q)+D(p_2+p_3+p_4,q))+$$

$$+(D(p_1{+}p_2{+}p,q){+}D(p_1{+}p_3{+}p,q){+}D(p_1{+}p_4{+}p,q){+}D(p_3{+}p_2{+}p,q){+}D(p_4{+}p_2{+}p,q){+}D(p_3{+}p_4{+}p,q))+$$

$$+(D(p_1{+}p_2{-}p,q){+}D(p_1{+}p_3{-}p,q){+}D(p_1{+}p_4{-}p,q){+}D(p_3{+}p_2{-}p,q){+}D(p_4{+}p_2{-}p,q){+}D(p_3{+}p_4{-}p,q))]$$

$$\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

If we set $q = 0$ in the above things simplify considerably:

$$\int_p \{-K'(p^2)\} \frac{\delta^2 \Delta S}{\delta\phi(p)\delta\phi(-p)} = \int_p \{-K'(p^2)\} B_I(p) \int_q \phi(q)\phi(-q) + F \int_q B_I(q)\phi(q)\phi(-q) +$$

$$\frac{1}{2} F B_{II}(0) \int_q \phi(q)\phi(-q) + \frac{1}{2} \int_p \{-K'(p^2)\} \int_q [B_{II}(p+q) + B_{II}(p-q)]\phi(q)\phi(-q)$$

$$+\frac{1}{4!} \int_p \{-K'(p^2)\} \int_{p_1,p_2,p_3} [D(p_1) + D(p_2) + D(p_3) + D(p_4)$$

$$+D(p+p_3+p_4) + D(p+p_3+p_2) + D(p+p_3+p_1)$$

$$+D(p-p_3-p_4) + D(p-p_3-p_2) + D(p-p_3-p_1)]$$

$$\phi(p_1)...\phi(p_4) \qquad p_4 = -p_3 - p_2 - p_1$$

(2)+(3)

$$-2 \int_p \{-K'(p^2)\} \frac{\delta S}{\delta\phi(p)} \frac{\delta \Delta S}{\delta\phi(-p)} - \int_p 2\frac{p^2 K'}{K}\phi(p) \frac{\delta \Delta S}{\delta\phi(p)} =$$

$$-\int_{p_1,p_2} \sum_i \{-K'(p_i^2)\} U_2(p_i) \delta(p_1+p_2-q)[A^{(0)} + A^{(1)}(p_1,p_2,q)]\phi(p_1)\phi(p_2) +$$

$$-\frac{2}{4!}\lambda \int_{p_1,p_2,p_3,p_4} \delta(p_1+p_2+p_3+p_4-q) \sum_{i=1}^{4} \{-K'((p_i-q)^2)\}[A^{(0)}+A^{(1)}(p_i, p_j+p_k+p_l)]\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

$$-\frac{2}{4!} \int_{p_1,p_2,p_3,p_4} \delta(p_1+p_2+p_3+p_4-q) \sum_i \{-K'(p_i^2)\} U_2(p_i) \Big(B^{(0)} + (B_I(p_1)+B_I(p_2)+B_I(p_3)+B_I(p_4)$$

$$+B_{II}(p_1+p_2,q) + B_{II}(p_1+p_3,q) + B_{II}(p_1+p_4,q) +$$

$$+B_{II}(p_2+p_3,q) + B_{II}(p_2+p_4,q) + B_{II}(p_3+p_4,q)) \Big) \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

Once again if we set $q = 0$ the result is simpler:

$$-2 \int_p \{-K'(p^2)\} \frac{\delta S}{\delta\phi(p)} \frac{\delta \Delta S}{\delta\phi(-p)} - \int_p 2\frac{p^2 K'}{K}\phi(p) \frac{\delta \Delta S}{\delta\phi(p)} = -2\Big[\int_p \{-K'(p^2)\} U_2(p) A(p)\phi(p)\phi(-p)$$

$$+\frac{1}{3!} \int_p \{-K'(p^2)\} A(p) \int_{p_2,p_3} (\lambda + U_4(p,p_2,p_3,p_4))\phi(-p)\phi(p_2)\phi(p_3)\phi(p_4) \ \ ; \ p = p_2 + p_3 + p_4$$

$$+\frac{1}{3!}\int_p\{-K'(p^2)\}U_2(p)\int_{q_2,q_3}[B_I(p)+B_I(q_2)+B_I(q_3)+B_I(q_4)+B_{II}(p+q_2)+B_{II}(p+q_3)+B_{II}(p+q_4)]$$

$$\phi(p)\phi(q_2)\phi(q_3)\phi(q_4)\Big] \quad ; -p = q_2+q_3+q_4$$

Rename $p->p_1$ and then symmetrize:

$$= -2\Big[\int_p\{-K'(p^2)\}U_2(p)A(p)\phi(p)\phi(-p)$$

$$+\frac{1}{4!}\int_{p_1,p_2,p_3}(\sum_{i=1}^{4}\{-K'(p_i^2)\}A(p_i))(\lambda+U_4(p_1,p_2,p_3,p_4))\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

$$+\frac{1}{4!}\int_{p_1,p_2,p_3}(\sum_{i=1}^{4}\{-K'(p_i^2)\}U_2(p_i))[B_I(p)+B_I(q_2)+B_I(q_3)+B_I(q_4)+B_{II}(p+q_2)+B_{II}(p+q_3)+B_{II}(p+q_4)]$$

$$\phi(p)\phi(q_2)\phi(q_3)\phi(q_4)\Big] \quad ; p_4 = -(p_1+p_2+p_3)$$

We write the ϕ^6 terms separately (we set $q=0$ here since these terms are not required for the relevant operator at leading order):

$$-\frac{4}{6!}\int_{p_1,...p_5}\sum_{10\ perm\ i,j,k}\{-K'((p_i+p_j+p_k)^2)\}[\lambda+U_4(p,p_i,p_j,p_k)]$$

$$[B_I(p)+B_I(p_a)+B_I(p_b)+B_I(p_c)+B_{II}(p+p_a)+B_{II}(p+p_b)+B_{II}(p+p_c)]\phi(p_1)....\phi(p_6)$$

$$+\frac{2}{6!}\int_{p_1,...p_5}[\sum_i\{\{-K'(p_i^2)\}\}U_2(p_i)][\sum_{10\ perm\ i,j,k}D(p_i+p_j+p_k)]\phi(p_1)....\phi(p_6)$$

$$+\frac{2}{6!}\int_{p_1,...p_5}[\sum_i -K'(p_i^2)A(p_i)][V_6(p_1,...p_6)]\phi(p_1)....\phi(p_6)$$

$$p = p_i+p_j+p_k = -(p_a+p_b+p_c)$$

(4a)

The general form of the action of $\mathcal{G}^c_{dil}$ is given by

$$\mathcal{G}^c_{dil}\delta(\sum p_i - q)X(p_1,..,p_N) = ((1-\frac{D}{2})N - \sum p_i\frac{\partial}{\partial p_i})\delta(\sum p_i - q)X(p_1,..,p_N)$$

$$= ((1-\frac{D}{2})N + D + q\frac{d}{dq})\delta(\sum p_i - q)X(p_1,..,p_N) -$$

$$\delta(\sum p_i - q)(\sum p_i\frac{\partial}{\partial p_i} + q\frac{d}{dq})X(p_1,..,p_N) \qquad \text{(B.2.1)}$$

When $q = 0$ we get:

$$\mathcal{G}^c_{dil}\frac{1}{2}\int_p A(p)\phi(p)\phi(-p) = \int_p (A(p) - p^2\frac{d}{dp^2}A(p))\phi(p)\phi(-p)$$

$$\mathcal{G}^c_{dil}\frac{1}{4!}\int_{p_1,p_2,p_3}[\sum_i\{B^{(0)}(p_i)+B_I(p_i)\}+(B_{II}(p_1+p_2)+B_{II}(p_1+p_3)+B_{II}(p_1+p_4))]\phi(p_1)...\phi(p_4) =$$

$$\frac{1}{4!}\int_{p_1,p_2,p_3}[(4-D-\sum_i p_i\frac{d}{dp_i})[\sum_i\{B^{(0)}(p_i)+B_I^{(1)}(p_i)\}+(B_{II}(p_1+p_2)+B_{II}(p_1+p_3)+B_{II}(p_1+p_4))]\phi(p_1)...\phi(p_4)$$

$$\mathcal{G}^c_{dil}\frac{1}{6!}\int_{p_1,...p_5}\sum_{10\ perm\ i,j,k} D(p_i+p_j+p_k)\phi(p_1)...\phi(p_6)$$

$$= \frac{1}{6!}\int_{p_1,...p_5}(6-2D-\sum_i p_i\frac{d}{dp_i})\sum_{10\ perm\ i,j,k} D(p_i+p_j+p_k)\phi(p_1)...\phi(p_6)$$

$$p_6 = -p_1... - p_5$$

B.3 Irrelevant Operator at subleading order

B.3.1 The ϕ^6 equation

$$\int_p\{-K'(p^2)\}\frac{\delta^2\Delta S_8^{(2)}(0)}{\delta\phi(p)\delta\phi(-p)} - \frac{4}{6!}\sum_{10\ perm\ (i,j,k)}\{-K'(p_i+p_j+p_k)\}\{\lambda + U_4^{(2)}(p,p_i,p_j,p_k)\}$$
$$\{B^{(0)}(p)+B^{(0)}(p_a)+B^{(0)}(p_b)+B^{(0)}(p_c)+B_I^{(1)}(p)+B_I^{(1)}(p_a)+B_I^{(1)}(p_b)+B_I^{(1)}(p_c)$$
$$+B_{II}^{(1)}(p+p_a)+B_{II}^{(1)}(p+p_b)+B_{II}^{(1)}(p+p_c)\}$$
$$-\frac{2}{6!}\{\sum_{l=1}^{6}(-K(p_l^2))U_2^{(1)}(p_l)D^{(1)}(p_1,p_2,p_3,p_4,p_5,p_6)\} - \frac{2}{6!}\{\sum_{l=1}^{6}(-K(p_l^2))A^{(0)}(p_l)V_6^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)\}$$
$$+\frac{1}{6!}\{6-2D-\sum_{i=1}^{6}p_i.\frac{d}{dp_i}\}D^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6) + \frac{1}{6!}(2\epsilon)\{D^{(1)}(p_1,p_2,p_3,p_4,p_5,p_6)$$
$$=\frac{1}{6!}\{\epsilon - 6F\lambda\}D^{(1)}(p_1,p_2,p_3,p_4,p_5,p_6) + \frac{1}{6!}\{\epsilon\lambda + \beta_1^{(1)}(\lambda)\}\frac{\partial}{\partial\lambda}D^{(1)}(p_1,p_2,p_3,p_4,p_5,p_6) \quad (B.3.1)$$

The last term on LHS comes from putting $D = 4-\epsilon$ in $(6-2D)D^{(1)}$ term. Where

$$\beta_1^{(1)}(\lambda) = -3F\lambda^2$$

So the first and 3rd term combined in RHS cancels the last term in LHS.

$$
\begin{aligned}
&U_4^{(2)}(p, p_i, p_j, p_k) \\
&= \underbrace{-\lambda^2 \big\{ \mathcal{F}(p+p_i) + \mathcal{F}(p+p_j) + \mathcal{F}(p+p_k) \big\}}_{U_4^I} + \underbrace{\frac{F\lambda^2}{2} \sum_{i=1}^{4} h(p_i)}_{U_4^{II}}
\end{aligned}
\tag{B.3.2}
$$

Where $\mathcal{F}(p) = \frac{1}{2} \int_k \big\{ h(p+k)h(k) - h(k)h(k) \big\}$.

$\frac{\delta^2 \Delta S_8^{(2)}(0)}{\delta\phi(p)\delta\phi(-p)}$ in ϕ^6 equation

$$
\begin{aligned}
&\frac{\delta^2 \Delta S_8^{(2)}(0)}{\delta\phi(p)\delta\phi(-p)} \\
=&\frac{28 \times 3\lambda^2}{8!} \Bigg\{ \sum_{10\ perm\ (1,j,k)} \sum_{l=1}^{6} h(p_l) h(p_i+p_j+p_k) + 4 \sum_{10\ perm\ (i,j,k)} \sum_{3\ perm\ (\alpha,\beta)} h(p_i+p_j+p_k) h(p_\alpha+p_\beta+p) \\
&+56 \sum_{10\ perm\ (i,j,k)} h(p_i+p_j+p_k) h(p_i+p_j+p_k) + 28 \sum_{l=1}^{6} \sum_{10\ perm\ (i,j,k)} h(p_l) h(p_i+p_j+p_k) \\
&+112 \sum_{10\ perm\ (i,j,k)} h(p_i+p_j+p_k) \sum_{3\ perm\ (\alpha,\beta)} h(p_\alpha+p_\beta+p) \\
&+56 \sum_{15\ perm\ (i,j)} \sum_{6\ perm\ (\alpha,\beta)} h(p_i+p_j+p) h(p_i+p_j+p_\alpha+p_\beta+p) \Bigg\} \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)
\end{aligned}
\tag{B.3.3}
$$

Equation for $D_I^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)$

We take 2nd and 5th term of R.H.S of (B.3.3), Note that the coefficients $\epsilon D^{(1)}$ terms cancel, now considering all terms in RHS we get:

$$\begin{aligned}&\Big\{6-2D-2\sum_{l=1}^{6}p_i.\frac{d}{dp_i}\Big\}D_I^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)\\&+\frac{12\lambda^2}{6!}\int_p\big(-K'(p^2)\big)\sum_{10\ perm\ (i,j,k)}\ \sum_{3\ perm\ (\alpha,\beta)}h(p_i+p_j+p_k)\Big\{h(p_\alpha+p_\beta+p)-h(p)\Big\}\\&+\frac{4}{6!}\sum_{10\ perm\ (i,j,k)}K'(p_i+p_j+p_k)U_4^I(p,p_i,p_j,p_k)\sum_{l=1}^{4}B^{(0)}(p_l)\\&+\frac{4}{6!}\sum_{10\ perm\ (i,j,k)}\lambda K'(p_i+p_j+p_k)\Big\{B_{II}^{(1)}(p+p_a)+B_{II}^{(1)}(p+p_b)+B_{II}^{(1)}(p+p_c)\Big\}=0\end{aligned}\tag{B.3.4}$$

Equation for $D_{II}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)$ and $D_{III}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)$

We take 1st, 3rd and 4th term from (B.3.3) and remaining all terms in (B.3.1):

$$\begin{aligned}&\frac{4}{6!}\sum_{10\ perm\ (i,j,k)}\Big(\{K'(p_i+p_j+p_k)\}\{\lambda\}\{\underbrace{B_I^{(1)}(p)}_{1}+B_I^{(1)}(p_a)+B_I^{(1)}(p_b)+B_I^{(1)}(p_c)\}\\&+\{K'(p_i+p_j+p_k)\}\{B^{(0)}(p)+B^{(0)}(p_a)+B^{(0)}(p_b)+B^{(0)}(p_c)\}\underbrace{\frac{F\lambda^2}{2}\Big\{\underbrace{h(p)}_{1}+h(p_i)+h(p_j)+h(p_k)\Big\}}_{U_4^{II}}\Big)\\&-\frac{2}{6!}\sum_{l=1}^{6}\Big\{-K'(p_l^2)U_2^{(1)}(p_l)\Big\}\Big\{D^{(1)}(p_1,p_2,p_3,p_4,p_5,p_6)\Big\}\\&-\frac{2}{6!}\sum_{l=1}^{6}\Big\{-K'(p_l^2)A^{(0)}(p_l)\Big\}\Big\{V_6^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)\Big\}\\&+\frac{3\lambda^2}{6!}\sum_{l=1}^{6}\sum_{10\ perm\ (i,j,k)}Fh(p_l)h(p_i+p_j+p_k)\\&+\int_p\big(-K'(p^2)\big)\frac{3\lambda^2}{6!}\sum_{10\ perm\ (i,j,k)}\Big\{h(p_i+p_j+p_k)h(p_i+p_j+p_k)\Big\}\\&+\frac{1}{6!}\Big(6-2D-\sum_i p_i.\frac{d}{dp_i}\Big)\Big\{\underbrace{D_{II}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)}_{1}+D_{III}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)\Big\}=0\end{aligned}\tag{B.3.5}$$

Let's take collect all terms marked with "1" marked and the 6 th term on LHS,

$$\frac{4}{6!}\sum_{10\ perm\ (i,j,k)}\{K'(p_i+p_j+p_k)\}\{\lambda\}B_I^{(1)}(p)$$
$$+\frac{4}{6!}\sum_{10\ perm\ (i,j,k)}\{K'(p_i+p_j+p_k)\}\{B^{(0)}(p)+B^{(0)}(p_a)+B^{(0)}(p_b)+B^{(0)}(p_c)\}\frac{F\lambda^2}{2}h(p)$$
$$+\int_p\big(-K'(p^2)\big)\frac{3\lambda^2}{6!}\sum_{10\ perm\ (i,j,k)}\Big\{h(p_i+p_j+p_k)h(p_i+p_j+p_k)\Big\}$$
$$+\frac{1}{6!}\Big(6-2D-\sum_i p_i.\frac{d}{dp_i}\Big)D_{II}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)=0$$

Collecting other terms in (B.3.5) we get equation to solve $D_{III}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)$.

Equation for $D_{IV}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)$

At last, only term remains in (B.3.3) is the 6 th term. So the equation for $D_{IV}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)$

$$\frac{3\lambda^2}{6!}\sum_{15\ perm\ (i,j)}\sum_{6\ perm\ (\alpha,\beta)}\int_p\{-K'(p^2)\}h(p_i+p_j+p)h(p_i+p_j+p_\alpha+p_\beta+p)$$
$$+\big(6-2D-p_i.\frac{d}{dp_i}\big)\frac{1}{6!}D_{IV}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)=0 \tag{B.3.6}$$

B.3.2 The ϕ^4 equation to determine $B^{(2)}(p_1,p_2,p_3,p_4)$

Now we will write ϕ^4 contribution in (4.3.31). We recall that while calculating 4-pt vertex of leading order there were two left over terms $(4-D)B_I^{(1)}(p_1,p_2,p_3,p_4)$ and $2(4-D)B_{II}^{(1)}(p_1,p_2,p_3,p_4)$. We have added those terms in LHS of the equation below.

$$\overbrace{\int_p K'(p^2)\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)}D^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)}^{A}$$

$$-\overbrace{\frac{2}{4!}\sum_{i=1}^{4}\{-K'(p_i^2)\}\{A^{(0)}(p_i)+A^{(1)}(p_i)\}\{\lambda+U_4^{(2)}(p_1,p_2,p_3,p_4)\}}^{B}$$

$$-\overbrace{\frac{2}{4!}\sum_{i=1}^{4}\{-K'(p_i^2)\}\{U_2^{(1)}(p_i)+U_2^{(2)}(p_i)\}\{\sum_{i=1}^{4}B^{(0)}(p_i)+\sum_{i=1}^{4}B_I^{(1)}(p_i)+B_{II}^{(1)}(p_1,p_2,p_3,p_4)\}}^{C}$$

$$\overbrace{-\frac{\eta}{2}\Bigg\{\sum_{i=1}^{4}\frac{4}{4!}B^{(0)}(p_i)\Bigg\}+\frac{1}{4!}\eta\sum_{i=1}^{4}p_i^2h(p_i)}^{D}$$

$$+\overbrace{\frac{1}{4!}\{(4-D)-p_i.\frac{d}{dp_i}\}\{B^{(2)}(p_1,p_2,p_3,p_4)\}+2(4-D)B_{II}^{(1)}(p_1,p_2,p_3,p_4)+(4-D)B_I^{(1)}(p_1,p_2,p_3,p_4)}^{E}$$

$$=\frac{\epsilon-6F\lambda}{4!}\{B_I^{(1)}(p_1,p_2,p_3,p_4)+B_{II}^{(1)}(p_1,p_2,p_3,p_4)+B^{(2)}(p_1,p_2,p_3,p_4)\}+\frac{d_4^{(2)}}{4!}\{\sum_{i=1}^{4}B^{(0)}(p_i)\}$$

$$+\frac{1}{4!}\{\epsilon\lambda+\beta_1^{(1)}(\lambda)\}\frac{\partial}{\partial\lambda}\{B_I^{(1)}(p_1,p_2,p_3,p_4)+B_{II}^{(1)}(p_1,p_2,p_3,p_4)\} \qquad \text{(B.3.7)}$$

Where $B_I^{(1)}(p_1,p_2,p_3,p_4)=\lambda\sum_{i=1}^{4}h(p_i)$ and $B_{II}^{(1)}(p_1,p_2,p_3,p_4)=-2\lambda\sum_{3\ perm\ (i,j)}\mathcal{F}(p_i+p_j)$.

$$\beta_1^{(1)}(\lambda)=-3F\lambda^2$$

$$U_2^1(p)=-\frac{\lambda F}{2-\epsilon};\quad U_2^{(2)}(p)=-\lambda^2G(p)-\frac{\lambda^2F^2}{4}h(p)$$

Where

$$\begin{aligned}G(p)=&\frac{1}{3}\int_{q,k}\frac{h(q)}{2}[h(p+q+k)h(k)-h(k)h(k)]-\frac{1}{3}\int_q\frac{h(q)}{2}[h(q+k)h(k)-h(k)h(k)]\\&+\frac{\eta p^2}{2\epsilon}-\frac{1}{2-2\epsilon}\Bigg\{\frac{2}{3}\beta^{(1)}v_2^{(1)}+\int_q f(q)\mathcal{F}(q)\Bigg\}\end{aligned}$$

$$\beta^{(1)}=-\int_q f(q)h(q)\rightarrow_{\epsilon\rightarrow 0}-F;v_2^{(1)}=-\int_q f(q)h(q)\rightarrow_{\epsilon\rightarrow 0}-\frac{F}{2}$$

Different parts of (B.3.7)

In the LHS, A. Calculation of $\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}D^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)$

1.

$$\int_p \{-K'(p^2)\}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}D_I^{(2)}(p_1,p_2,p_3,p_4)\phi(p_1)\phi(p_2\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$
$$=\int_p \{-K'(p^2)\}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}$$
$$\left\{\frac{3\lambda^2}{6!}\sum_{10\ perm\ (i,j,k)}\sum_{3\ perm\ (\alpha,\beta)}\int_q h(p_i+p_j+p_k)[h(p_\alpha+p_\beta+q)h(q)-h(q)h(q)]\right\}$$
$$\times\,\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$

$$=\int_q\frac{1}{4!}\frac{3\lambda^2F}{2}\left\{\sum_{3\ perm\ (i,j)}\sum_{l=1}^{4}h(p_l)[h(p_i+p_j+q)h(q)-h(q)h(q)]\right\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.8a}$$

$$+\frac{3\lambda^2}{4!}\int_{p,q}\{-K'(p^2)\}\left\{\sum_{6\ perm\ (i,j)}h(p_i+p_j+p)[\sum_{a=i,j}h(p+q+p_a)-2h(q)h(q)]\right\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.8b}$$

$$+\int_{p,q}\frac{6\lambda^2}{4!}\{-K'(p^2)\}\sum_{3\ perm\ (i,j)}h(p_i+p_j+p)\{h(p_i+p_j+q)h(q)-h(q)h(q)\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.8c}$$

$$+\int_{p,q}\{-K'(p^2)\}\frac{3\lambda^2}{4!}\sum_{i=1}^{4}h(p_i)\{h(p_i+p+q)h(q)-h(q)h(q)\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.8d}$$

2.

$$\int_p \{-K'(p^2\}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}D_{II}^{(2)}(p_1,p_2,p_3,p_4)\phi(p_1)\phi(p_2\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$
$$=\int_p \{-K'(p^2\}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\frac{1}{6!}\frac{-3\lambda^2F}{2}\sum_{l=1}^{6}\sum_{10\ perm\ (i,j,k)}h(p_l)h(p_i+p_j+p_k)\phi(p_1)\phi(p_2)...\phi(p_5)\phi(p_6)$$

gives

$$+\frac{-3}{2}\frac{\lambda^2 F}{4!}\int_p \big\{-K'(p^2)\big\}\big\{2h(p)\big\}\sum_{l=1}^{4}\big\{h(p_l)\big\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.9a}$$

$$+\frac{-3}{2}\frac{\lambda^2 F}{4!}\int_p \big\{-K'(p^2)\big\}\big\{\sum_{l=1}^{4}h(p_l)h(p_l)+\sum_{i\neq j}h(p_i)h(p_j)\big\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.9b}$$

$$+\frac{-3\lambda^2 F}{4!}\int_p \big\{-K'(p^2)\big\}\big\{2h(p)\big\}\sum_{3\ perm\ (i,j)}\big\{h(p_i+p_j+p)\big\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.9c}$$

$$+\frac{-3\lambda^2 F}{4!}\int_p \big\{-K'(p^2)\big\}\big\{\sum_{l=1}^{4}h(p_l)\big\}\sum_{3\ perm\ (i,j)}\big\{h(p_i+p_j+p)\big\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.9d}$$

3.

$$\int_p \big\{-K'(p^2)\big\}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}D^{(2)}_{III}(p_1,p_2,p_3,p_4)\phi(p_1)\phi(p_2\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$

$$=\int_p \big\{-K'(p^2\big\}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\frac{1}{6!}\frac{-3\lambda^2 F}{2}\sum_{10\ perm\ (i,j,k)}h(p_i+p_j+p_k)h(p_i+p_j+p_k)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p$$

gives

$$\frac{1}{4!}\frac{-3\lambda^2 F^2}{2}\sum_{l=1}^{4}h(p_l)h(p_l)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.10a}$$

$$+\int_p \big\{-K'(p^2)\big\}\frac{-3\lambda^2 F}{4!}\sum_{3\ perm\ (i,j)}h(p_i+p_j+p)h(p_i+p_j+p)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.10b}$$

4.

$$\int_p \big\{-K'(p^2)\big\}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}D^{(2)}_{IV}(p_1,p_2,p_3,p_4)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$

$$=\int_p \big\{-K'(p^2\big\}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}$$

$$\Big\{\frac{1}{6!}\frac{\lambda^2}{2}\int_q\sum_{15\ perm\ (i,j)}\sum_{6\ perm\ (\alpha,\beta)}[h(p_i+p_j+q)h(p_i+p_j+p_\alpha+p_\beta+q)h(q)]\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)\Big\}$$

gives

$$= \frac{3\lambda^2 F}{4!} \sum_{3\ perm\ (i,j)} \int_q \big\{ h(p_i+p_j+q)h(q)h(q) \big\} \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.11a}$$

$$+ \frac{2\lambda^2}{4!} \int_p \int_q \{ -K'(p^2\} \sum_{6\ perm\ (i,j)} \big\{ h(p_i+p_j+q)[h(p+q+p_i)+h(p+q+p_j)]h(q) \big\} \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.11b}$$

$$+ \frac{\lambda^2}{4!} \int_q \int_p \{ -K'(p^2\} \sum_{l=1}^{4} \sum_{3\ perm\ (i,j)} \big\{ h(p_l+p+q)h(p_l+p_i+p_j+p+q)h(q) \big\} \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.3.11c}$$

B

$$-\frac{2}{4!} \sum_{i=1}^{4} \big\{ -K'(p_i^2) \big\} \big\{ A^{(0)}(p_i) + A^{(1)}(p_i) \big\} \big\{ \lambda + U_4^{(2)}(p_1,p_2,p_3,p_4) \big\}$$

$$= -\frac{2}{4!} \sum_{i=1}^{4} \{ -K'(p_i^2) \} \left(\left\{ -\frac{F}{2} \right\} \left\{ -\lambda^2 \sum_{3\ perm\ (i,j)} \mathcal{F}(p_i+p_j) + \frac{F\lambda^2}{2} \sum_{i=1}^{4} h(p_i) \right\} + [A^{(0)} + A^{(1)}(p_i)]\lambda \right)$$

Where $\mathcal{F}(p_1+p_2) = \frac{1}{2} \int_q \big\{ h(p_1+p_2+q)h(q) - h(q)h(q) \big\}$

$$= \frac{1}{4!} \frac{F\lambda^2}{2} \sum_{l=1}^{4} \big\{ K'(p_l^2) \big\} \sum_{3\ perm\ (i,j)} \int_q \big\{ h(p_i+p_j+q)h(q) - h(q)h(q) \big\} \tag{B.3.12a}$$

$$- \frac{1}{4!} \frac{F^2\lambda^2}{2} \sum_{i=1}^{4} \big\{ K'(p_i^2) \big\} \sum_{l=1}^{4} h(p_l) \tag{B.3.12b}$$

$$+ \frac{2}{4!} \sum_{i=1}^{4} \big\{ K'(p_i^2) \big\} \lambda \big\{ A^{(0)} + A^{(1)}(p_i) \big\} \tag{B.3.12c}$$

C

$$-\frac{2}{4!} \sum_{i=1}^{4} \big\{ -K'(p_i^2) \big\} \big\{ U_2^{(1)}(p_i) + U_2^{(2)}(p_i) \big\} \big\{ \sum_{i=1}^{4} B^{(0)}(p_i) + B_I^{(1)}(p_1,p_2,p_3,p_4) + B_{II}^{(1)}(p_1,p_2,p_3,p_4) \big\}$$

$$= \frac{2}{4!}\frac{-F\lambda\epsilon}{4}\sum_{i=1}^{4}K'(p_i^2) \tag{B.3.13a}$$

$$= \frac{\lambda^2 F}{4!}\sum_{l=1}^{4}\big\{K'(p_l^2)\big\}\big\{\int_q\sum_{3\ perm\ (i,j)}\big(h(p_i+p_j+q)h(q)-h(q)h(q)\big)\big\} \tag{B.3.13b}$$

$$-\frac{F^2\lambda^2}{4!}\sum_{i=1}^{4}\big\{K'(p_i^2)\big\}\big\{\sum_{i=1}^{4}h(p_i)\big\} \tag{B.3.13c}$$

$$-\frac{1}{3}\frac{2\lambda^2}{4!}\sum_{i=1}^{4}\big\{K'(p_i^2)\big\}\big\{\int_{q,k}h(q)\left[h(p_i+q+k)h(k)-h(k)h(k)\right]-\int_{q,k}h(q)\left[h(q+k)h(k)-h(k)h(k)\right]\big\} \tag{B.3.13d}$$

$$-\frac{2\lambda^2}{4!}\sum_{i=1}^{4}\big\{K'(p_i^2)\big\}\big\{\frac{\eta p_i^2}{2\epsilon}\big\} \tag{B.3.13e}$$

$$-\frac{\lambda^2}{4!}\sum_{i=1}^{4}K'(p_i^2)\int_q K'(q^2)\big\{h(q+k)h(k)-h(k)h(k)\big\} \tag{B.3.13f}$$

$$+\frac{2\lambda^2}{4!}\sum_{i=1}^{4}\big\{K'(p_i^2)\big\}\big\{\frac{1}{2-2\epsilon}\left(\frac{2}{3}\beta^{(1)}v_2^{(1)}\right)\big\} \tag{B.3.13g}$$

$$-\frac{1}{4!}\frac{F^2\lambda^2}{2}\sum_{i=1}^{4}\big\{K'(p_i^2)h(p_i)\big\} \tag{B.3.13h}$$

D.

$$-\frac{4}{4!}\frac{\eta}{2}\sum_{i=1}^{4}B^{(0)}(p_i) \tag{B.3.14}$$

$$+\frac{1}{4!}\eta\sum_{i=1}^{4}p_i^2 h(p_i) \tag{B.3.15}$$

Where at the fixed point $F\lambda = \frac{\epsilon}{3}$, $\frac{\eta}{2} \longrightarrow \frac{\epsilon^2}{108}$ **E.**

$$\frac{1}{4!}(4-D)B_I^{(1)}(p_1,p_2,p_3,p_4) \tag{B.3.16a}$$

$$+\frac{1}{4!}2(4-D)B_{II}^{(1)}(p_1,p_2,p_3,p_4) \tag{B.3.16b}$$

$$+\frac{1}{4!}(4-D-\sum_{i=1}^{4}p_i.\frac{d}{dp_i})B^{(2)}(p_1,p_2,p_3,p_4) \tag{B.3.16c}$$

In the RHS

$$= \frac{\epsilon - 6F\lambda}{4!}\{\sum_{i=1}^{4} B_I^{(1)}(p_i)\} \tag{B.3.17a}$$

$$+ \frac{\epsilon - 6F\lambda}{4!} B_{II}^{(1)}(p_1, p_2, p_3, p_4) \tag{B.3.17b}$$

$$+ \frac{\epsilon - 6F\lambda}{4!}\{B^{(2)}(p_1, p_2, p_3, p_4)\} \tag{B.3.17c}$$

$$+ \frac{d_4^{(2)}}{4!}\{\sum_{i=1}^{4} B^{(0)}(p_i)\} \tag{B.3.17d}$$

$$+ \epsilon\frac{1}{4!} B_I^{(1)}(p_1, p_2, p_3, p_4) \tag{B.3.17e}$$

$$+ \epsilon\frac{1}{4!} B_{II}^{(2)}(p_1, p_2, p_3, p_4) \tag{B.3.17f}$$

$$+ (-3F\lambda)\frac{1}{4!} B_I^{(1)}(p_1, p_2, p_3, p_4) \tag{B.3.17g}$$

$$+ (-3F\lambda)\frac{1}{4!} B_{II}^{(1)}(p_1, p_2, p_3, p_4) \tag{B.3.17h}$$

We know all necessary terms to find $B^{(2)}(p_1, p_2, p_3, p_4)$. We will reorganize the terms and will make suitable ansatz about $B^{(2)}(p_1, p_2, p_3, p_4)$ so that (B.3.7) is satisfied and at the end we get some number proportional to $\sum_{i=1}^{4} B^{(0)}(p_i)$ in the LHS of (B.3.7) so that we can equate that with $\frac{d_m^{(2)}}{4!}\{\sum_{i=1}^{4} B^{(0)}(p_i)\}$ in RHS to get the anomalous dimension.

Equation for $B_I^{(2)}(p_1, p_2, p_3, p_4)$

Taking (B.3.8a),(B.3.9d),(B.3.12a) and (B.3.13b) and adding suitable couterterm,

$$\int_q \frac{1}{4!}\frac{3\lambda^2 F}{2} \sum_{3\ perm\ (i,j)} \sum_{l=1}^{4} \{h(p_l)\}\{h(p_i + p_j + q)h(q) - h(q)h(q)\}$$
$$+ \frac{-3\lambda^2 F}{4!}\int_p \{-K'(p^2)\}\sum_{l=1}^{4}\{h(p_l)\}\sum_{3\ perm\ (i,j)}\{h(p_i + p_j + p)\} + \frac{9}{2}\frac{\lambda^2 F^2}{4!}\sum_{l=1}^{4} h(p_l)$$
$$+ \frac{1}{4!}\frac{\lambda^2 F}{2}\sum_{l=1}^{4}\{K'(p_l^2)\}\{\int_q \sum_{3\ perm\ (i,j)} \big(h(p_i + p_j + q)h(q) - h(q)h(q)\big)\}$$
$$+ \frac{\lambda^2 F}{4!}\sum_{l=1}^{4}\{K'(p_l^2)\}\{\int_q \sum_{3\ perm\ (i,j)} \big(h(p_i + p_j + q)h(q) - h(q)h(q)\big)\}$$
$$+ \{-(4 - D) - \sum_{i=1}^{4} p_i.\frac{d}{dp_i}\frac{1}{4!}\} B_I^{(2)}(p_1, p_2, p_3, p_4) = 0 \tag{B.3.18}$$

On LHS of (B.3.7) we are left with,

$$2(4-D)\frac{1}{4!}B_I^{(2)}(p_1,p_2,p_3,p_4) - \frac{1}{4!}\frac{9\lambda^2 F}{2}\sum_{l=1}^{4} h(p_l) \tag{B.3.19}$$

Equation for $B_{II}^{(2)}(p_1,p_2,p_3,p_4)$

Taking (B.3.9c),(B.3.10b) and (B.3.11a) we get,

$$\begin{aligned}
&\frac{-3\lambda^2 F}{4!}\int_p \{-K'(p^2)\}\{2h(p)\}\sum_{3\ perm\ (i,j)}\{h(p_i+p_j+p)\} \\
&+\int_p \{-K'(p^2)\}\frac{-3\lambda^2 F}{4!}\sum_{3\ perm\ (i,j)} h(p_i+p_j+p)h(p_i+p_j+p) \\
&+\frac{3\lambda^2 F}{4!}\sum_{3\ perm\ (i,j)}\int_q h(p_i+p_j+q)h(q)h(q) \\
&+\{-(4-D)-\sum_{i=1}^{4} p_i.\frac{d}{dp_i}\}\frac{1}{4!}B_{II}^{(2)}(p_1,p_2,p_3,p_4) = 0
\end{aligned} \tag{B.3.20}$$

On LHS of (B.3.7) we are left with,

$$+2(4-D)\frac{1}{4!}B_{II}^{(2)}(p_1,p_2,p_3,p_4)$$

Equation for $B^{(2)}_{III}(p_1, p_2, p_3, p_4)$

Taking (B.3.9b),(B.3.10a),(B.3.12b),(B.3.13c), (B.3.13h) we get,(Note that we need to $A^{(1)}_{III}$ for the equation to be satisfied.

$$\begin{aligned}
&\frac{-3}{2}\frac{\lambda^2 F}{4!}\int_p \{-K'(p^2)\}\{\sum_{l=1}^{4} h(p_l)h(p_l) + \sum_{i\neq j} h(p_i)h(p_j)\} \\
&+ \frac{1}{4!}\frac{-3\lambda^2 F^2}{2}\sum_{l=1}^{4} h(p_l)h(p_l) \\
&- \frac{1}{4!}\frac{F^2\lambda^2}{2}\{\sum_{i=1}^{4} K'(p_i^2)h(p_i) + \sum_{i\neq j} K'(p_i^2)h(p_j)\} \\
&- \frac{F^2\lambda^2}{4!}\sum_{i=1}^{4}\{\sum_{i=1}^{4} K'(p_i^2)h(p_i) + \sum_{i\neq j} K'(p_i^2)h(p_j)\} \\
&- \frac{1}{4!}\frac{F^2\lambda^2}{2}\sum_{i=1}^{4}\{K'(p_i^2)h(p_i)\} \\
&+ \frac{2}{4!}\sum_{i=1}^{4}\{K'(p_i^2)\}\lambda A^{(1)}_{III}(p_i) - \sum_{i=1}^{4} p_i.\frac{d}{dp_i}\frac{1}{4!}B^{(2)}_{III}(p_1, p_2, p_3, p_4) = 0
\end{aligned} \tag{B.3.21}$$

where $A^{(1)}_{III}(p) = \frac{-\lambda F^2}{2}h(p)$.

On LHS of (B.3.7) we are left with

$$(4-D)\frac{1}{4!}B^{(2)}_{III}(p_1, p_2, p_3, p_4) \tag{B.3.22}$$

Equation for $B_{IV}^{(2)}(p_1,p_2,p_3,p_4)$

Collecting (B.3.9a),(B.3.13a),(B.3.13g),(B.3.16a), (B.3.17a), (B.3.17e), (B.3.17g) and the second term of (B.3.19) we get,

$$\frac{-3}{2}\frac{\lambda^2F}{4!}\int_p\{-K'(p^2)\}\{2h(p)\}\sum_{l=1}^{4}\{h(p_l)\}$$
$$+\frac{2}{4!}\frac{-F\lambda\epsilon}{4}\sum_{i=1}^{4}K'(p_i^2)$$
$$+\frac{2\lambda^2}{4!}\sum_{i=1}^{4}\{K'(p_i^2)\}\{\frac{1}{2-2\epsilon}\left(\frac{2}{3}\beta^{(1)}v_2^{(1)}\right)\}$$
$$+\frac{2}{4!}\sum_{i=1}^{4}\{K'(p_i^2)\}\lambda\{A^{(0)}+A_I^{(1)}(p_i)\}+\left\{(4-D)-\sum_{i=1}^{4}p_i\cdot\frac{d}{dp_i}\right\}\frac{1}{4!}B_{IV}^{(2)}(p_1,p_2,p_3,p_4)+\frac{4-D}{4!}\sum_{i=1}^{4}\lambda Fh(p_i)$$
$$-\frac{1}{4!}\frac{9}{2}F^2\lambda^2\sum_{l=1}^{4}h(p_l)$$
$$=\frac{\epsilon-6F\lambda}{4!}\sum_{i=1}^{4}\lambda Fh(p_i)+(\epsilon\lambda-3F\lambda^2)F\sum_{i=1}^{4}h(p_i)$$

Where $\frac{1}{3}\beta^{(1)} = -\int_q f(q)h(q) \rightarrow_{\epsilon\rightarrow 0} -F$, $v_2^{(1)} = -\frac{1}{2-\epsilon}\int_p -K'(p^2) \rightarrow_{\epsilon\rightarrow 0} \frac{-F}{2}$, $A^{(0)} = -\frac{F\epsilon}{4}$,$A_I^{(1)}(p)=F^2\lambda$.

Equation for $B^{(2)}_{VI}(p_1,p_2,p_3,p_4)$

Taking (B.3.8d), (B.3.13d) and (B.3.13f), we get

$$\int_{p,q}\Big\{-K'(p^2)\Big\}\frac{3\lambda^2}{4!}\sum_{4\ perm\ (i,j,k)}\{h(p_i+p_j+p_k)h(p_\alpha+p+q)h(q)-h(p+q)h(q)\}$$
$$\frac{1}{2}\frac{\lambda^2}{4!}\int_{p,q}K'(p)h(p+q)h(q)$$
$$-\frac{1}{3}\frac{\lambda^2}{4!}\sum_{i=1}^{4}\{K'(p_i^2)\}\{\int_{q,k}h(q)\,[h(p_i+q+k)h(k)-h(q+k)h(k)]\,\}$$
$$+\frac{\lambda^2}{4!}\sum_{i=1}^{4}K'(p_i^2)\int_{p,q}\{-K'(q)\}\{h(q+k)h(k)-h(k)h(k)\}$$
$$+\frac{3\lambda^2}{4!}\sum_{i=1}^{4}h(p_i)\int_{p,q}\{-K'(q)\}\{h(q+k)h(k)-h(k)h(k)\}$$
$$+\frac{2}{4!}\sum_{i=1}^{4}K'(p_i^2)A_{II}(p_i)\lambda$$
$$+\{-2(4-D)-\sum_{i=1}^{4}p_i.\frac{d}{dp_i}\}\frac{1}{4!}\Big\{B^{(2)}_{VI}(p_1,p_2,p_3,p_4)|_1\Big\}+\{-\sum_{i=1}^{4}p_i.\frac{d}{dp_i}\}\frac{1}{4!}\Big\{B^{(2)}_{VI}(p_1,p_2,p_3,p_4)|_2\Big\}=0 \quad \text{(B.3.23)}$$

$A^{(1)}_{II}(p_i)=-\frac{\lambda}{3}\int_{p,q}[h(p_i+p+q)h(p)h(q)-h(p+q)h(p)h(q)]-\lambda\int_p K'(p)\{h(p+q)h(q)-h(q)h(q)\}$.

On LHS of (B.3.7) we are left with

$$3(4-D)\frac{1}{4!}B^{(2)}_{VI}(p_1,p_2,p_3,p_4)|_1+(4-D)\frac{1}{4!}B^{(2)}_{VI}|_2(p_1,p_2,p_3,p_4)|_2 \quad \text{(B.3.24)}$$

Equation for $B^{(2)}_{VII}(p_1,p_2,p_3,p_4)$

Considering (B.3.8c),

$$\int_{p,q}\frac{6\lambda^2}{4!}\{-K'(p)\}\sum_{3\ perm\ (i,j)}h(p_i+p_j+p)\{h(p_i+p_j+q)h(q)-h(q)h(q)\}$$
$$+\{-2(4-D)-p_i.\frac{d}{dp_i}\}\frac{1}{4!}B^{(2)}_{VII}(p_1,p_2,p_3,p_4)=0 \quad \text{(B.3.25)}$$

On LHS of (B.3.7) we are left with

$$3(4-D)\frac{1}{4!}B^{(2)}_{VII}(p_1,p_2,p_3,p_4) \tag{B.3.26}$$

Equation for $B^{(2)}_{VIII}(p_1,p_2,p_3,p_4)$

Collecting (B.3.13e) and (B.3.15) we get (because of the expected structure of $B^{(2)}_{VIII}$ as $\frac{\eta}{\epsilon}$ we consider the term $\epsilon\lambda\frac{\partial B_{IX}}{\partial\lambda}$ from RHS of (B.3.7)),

$$\begin{aligned}&-\frac{2}{4!}\sum_{i=1}^{4}\{K'(p_i^2)\}\{\frac{\eta p_i^2}{2\epsilon}\}+\frac{1}{4!}\eta\sum_{i=1}^{4}p_i^2h(p_i)+\left\{(4-D)-\sum_{i=1}^{4}p_i.\frac{\partial}{\partial p_i}\right\}\frac{1}{4!}B_{IX}(p_1,p_2,p_3,p_4)\\&=\{\epsilon-6F\lambda\}\frac{1}{4!}B_{IX}(p_1,p_2,p_3,p_4)+\frac{1}{4!}\epsilon\lambda\frac{\partial B_{IX}(p_1,p_2,p_3,p_4)}{\partial\lambda}\end{aligned}$$

We ignore $\lambda B^{(2)}_{IX}$ or $\epsilon B^{(2)}_{IX}$ terms being higher-order and get

$$-\frac{2}{4!}\sum_{i=1}^{4}\{K'(p_i^2)\}\{\frac{\eta p_i^2}{2\epsilon}\}+\frac{1}{4!}\eta\sum_{i=1}^{4}p_i^2h(p_i)+\left\{-\sum_{i=1}^{4}p_i.\frac{\partial}{\partial p_i}\right\}\frac{1}{4!}B_{IX}=\frac{1}{4!}\epsilon\lambda\frac{\partial B_{IX}}{\partial\lambda} \tag{B.3.27}$$

And on LHS of (B.3.7) we are left with

$$(\epsilon-6\lambda F)\frac{1}{4!}B_{IX}(p_1,p_2,p_3,p_4) \tag{B.3.28}$$

Equation for $B^{(2)}_{IX}(p_1,p_2,p_3,p_4)$

At last we collect the terms (B.3.16b), (B.3.17b), (B.3.17f) and (B.3.17h) to get,

$$\begin{aligned}&\{4-D-\sum_{i=1}^{4}p_i.\frac{d}{dp_i}\}\frac{1}{4!}B^{(2)}_{X}(p_1,p_2,p_3,p_4)+2(4-D)\frac{1}{4!}B_{II}(p_1,p_2,p_3,p_4)\\&=\left\{\epsilon-6F\lambda+\epsilon-3F\lambda^2\right\}\frac{1}{4!}B^{(1)}_{II}(p_1,p_2,p_3,p_4)\end{aligned} \tag{B.3.29}$$

We ignore the term $\epsilon\frac{1}{4!}B^{(2)}_{X}(p_1,p_2,p_3,p_4)$ and get the following euqations:

$$-\sum_{i=1}^{4}p_i.\frac{d}{dp_i}B^{(2)}_{X}(p_1,p_2,p_3,p_4)-9F\lambda^2\sum_{3\ perm\ (i,j)}\int_q\left\{h(p_i+p_j+q)h(q)-h(q)h(q)\right\}=0 \tag{B.3.30}$$

To solve this, we use $\bar{p} = \frac{p}{\Lambda}$. In this noatation $-\sum_{i=1}^{4} \bar{p}_i . \frac{d}{dp_i} B_X^{(2)}(\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4)$ can be written as,

$$-\sum_{i=1}^{4} \bar{p}_i . \frac{d}{d\bar{p}_i} B_X^{(2)} \left(\frac{p_1}{\Lambda}, \frac{p_2}{\Lambda}, \frac{p_3}{\Lambda}, \frac{p_4}{\Lambda} \right) = \Lambda . \frac{d}{d\Lambda} B_X^{(2)} \left(\frac{p_1}{\Lambda}, \frac{p_2}{\Lambda}, \frac{p_3}{\Lambda}, \frac{p_4}{\Lambda} \right)$$

So the solution is given by,

$$\frac{1}{4!} B_X^{(2)}(\frac{p_1}{\Lambda}, \frac{p_2}{\Lambda}, \frac{p_3}{\Lambda}, \frac{p_4}{\Lambda}) = \frac{9F\lambda^2}{4!} \sum_{3\ perm\ (i,j)} \int_{\Lambda}^{\infty} \int_{\bar{q}} \frac{d\Lambda'}{\Lambda'} \left\{ h\left(\frac{p_i}{\Lambda'} + \frac{p_j}{\Lambda'} + \bar{q} \right) h\left(\bar{q}\right) - h\left(\bar{q}\right) h\left(\bar{q}\right) \right\} \tag{B.3.31}$$

In LHS we are left with

$$\frac{1}{4!}(4 - D) B_X^{(2)}(p_1, p_2, p_3, p_4) \tag{B.3.32}$$

B.4 Relevant operator at sub-leading operator

B.4.1 The ϕ^6 equation to find $D^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6)$

ϕ^6 equation is given by (we donot have to consider $\beta(\lambda)\frac{\partial \Delta S}{\partial \lambda}$ part because there is no $D^{(1)}(p_1, p_2, p_3, p_4)$ in this case).

$$\begin{aligned}
&-\frac{4}{6!} \sum_{10\ perm\ (i,j,k)} \{ -K'(p_i + p_j + p_k) \} \{\lambda\} \{ \underbrace{B(p_i) + B(p_j) + B(p_k)}_{1} + \underbrace{B(p_i + p_j + p_k)}_{2} \} \\
&-\frac{2}{6!} \sum_{i=1}^{4} \{ -K'(p_i^2) \} \underbrace{\{A(p_i)\} V_6^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6)}_{3} \\
&+\frac{1}{6!}\left(6 - 2D - \sum_{i=1}^{4} p_i . \frac{\partial}{\partial p_i} \right) D^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) \\
&= \frac{d_2^{(0)}}{6!} D^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6)
\end{aligned} \tag{B.4.1}$$

$$d_2^{(0)} = 2$$

We collect the terms marked '2' to find first kind of $D^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)$.

$$\begin{aligned}
&-\frac{4}{6!}\sum_{10\ perm\ (i,j,k)}\big\{-K'(p_i+p_j+p_k)\big\}\big\{\lambda\big\}\big\{B(p_i+p_j+p_k)\big\}\\
&+\frac{1}{6!}\left(6-2D-\sum_{i=1}^{4}p_i.\frac{\partial}{\partial p_i}\right)D^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)\\
&=\frac{d_2^{(0)}}{6!}D_I^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)
\end{aligned}\tag{B.4.2}$$

Similarly collecting the terms marked as '1' and '3' we get the following equation,

$$\begin{aligned}
&-\frac{4}{6!}\sum_{10\ perm\ (i,j,k)}\big\{-K'(p_i+p_j+p_k)\big\}\big\{\lambda\big\}\big\{B(p_i)+B(p_j)+B(p_k)\big\}\\
&-\frac{2}{6!}\sum_{i=1}^{4}\big\{-K'(p_i^2)\big\}\big\{A(p_i)\big\}V_6^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)\\
&=\frac{2}{6!}D_{II}^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)
\end{aligned}\tag{B.4.3}$$

B.4.2 The ϕ^4 equation to determine $B^{(2)}(p_1,p_2,p_3,p_4)$

The ϕ^4 equation is given by,

$$\begin{aligned}
&\frac{1}{6!}\int_p\big\{-K'(p)\big\}\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)}D^{(2)}(p_1,p_2,p_3,p_4,p_5,p_6)\\
&-\frac{2}{4!}\sum_{i=1}^{4}\big\{-K'(p_i)\big\}\big\{A^{(0)}(p)+A^{(1)}(p)\big\}\big\{\lambda+U_4^{(2)}(p_1,p_2,p_3,p_4)\big\}\\
&-\frac{2}{4!}\sum_{i=1}^{4}\big\{-K'(p_i)\big\}\big\{U_2^{(1)}(p_i)+U_2^{(2)}(p_i)\big\}\big\{\sum_{i=1}^{4}B^{(1)}(p_i)\big\}+\frac{1}{4!}\left(4-D-\sum_{i=1}^{4}p_i.\frac{\partial}{\partial p_i}\right)B^{(2)}(p_1,p_2,p_3,p_4)\\
&+(4-D)B^{(1)}(p_1,p_2,p_3,p_4)\\
&=\frac{d_2^{(1)}}{4!}B^{(1)}(p_1,p_2,p_3,p_4)+\frac{d_2^{(0)}}{4!}B^{(2)}(p_1,p_2,p_3,p_4)+\frac{1}{4!}\{\epsilon\lambda+\beta_1^{(1)}(\lambda)\}\frac{\partial}{\partial\lambda}B^{(1)}(p_1,p_2,p_3,p_4)
\end{aligned}\tag{B.4.4}$$

Where

$$U_4^{(2)}(p_1,p_2,p_3,p_4)=\underbrace{-\lambda^2\sum_{3\ perm\ (i,j)}\mathcal{F}(p_i+p_j)}_{U_4^I}+\underbrace{\frac{F\lambda^2}{2}\sum_{i=1}^{4}h(p_i)}_{U_4^{II}}$$

$$\mathcal{F}(p_i + p_j) = \frac{1}{2}\int_k \{h(p_i + p_j + k)h(k) - h(k)h(k)\}$$

$$d_2^{(1)} = -F\lambda$$

$$B^{(1)}(p_1, p_2, p_3, p_4) = -\lambda \sum_{i=1}^{4} h(p_i)$$

Calculation of $\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)}D^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6)$

a.

$$\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} D_I^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$

$$= \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\{\sum_{10\ perm\ (i,j,k)} h(p_i + p_j + p_k)\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$

$$= 30 \times \{\sum_{l=1}^{4} h(p_l)h(p_l)\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.4.5a}$$

$$+ 60 \times \sum_{3\ perm\ (i,j)} \{h(p_i + p_j + p)h(p_i + p_j + p)\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.4.5b}$$

b.

$$\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} D_{II}^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$

$$\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \sum_{10\ perm\ (i,j,k)} h(p_i + p_j + p_k)\sum_{l=1}^{6} h(p_l)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi(p_5)\phi(p_6)$$

$$= 30 \times \sum_{i=1}^{4} h(p_i)2h(p)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.4.6a}$$

$$+ 30 \times \sum_{i=1}^{4} h(p_i)\sum_{j=1}^{4} h(p_j)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.4.6b}$$

$$+ 60 \times \sum_{3\ perm\ (i,j)} h(p_i + p_j + p)\{2h(p)\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.4.6c}$$

$$+ 60 \times \sum_{3\ perm\ (i,j)} h(p_i + p_j + p)\sum_{k=1}^{4} h(p_k)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \tag{B.4.6d}$$

Equation for $B_I^{(2)}(p_1, p_2, p_3, p_4)$

Collecting (B.4.5b) and (B.4.6c), we get the following equations:

$$\lambda^2 \int_p \{-K'(p)\}[2 \sum_{3\ perm\ (i,j)} h(p_i + p_j + p)h(p_i + p_j + p) + 4 \sum_{3\ perm\ (i,j)} h(p_i + p_j + p)h(p)]$$
$$\left\{(4 - D) - p_i.\frac{d}{dp_i}\right\} B_I^{(2)}(p_1, p_2, p_3, p_4) = d_2^{(0)} B_I^{(2)}(p_1, p_2, p_3, p_4) \tag{B.4.7}$$

Equation for $B_{II}^{(2)}(p_1, p_2, p_3, p_4)$

We take $\beta_1^{(1)}(\lambda)\frac{\partial}{\partial\lambda}B^{(1)}$ term from RHS. Collecting (B.4.6d) and the term with U_4^I in the second line of (B.4.4) we get the following equation,

$$2\lambda^2 \sum_{3\ perm\ (i,j)} h(p_i + p_j + p)\{-K'(p)\} \sum_{l=1}^{4} h(p_l) - 3\lambda^2 F \sum_{l=1}^{4} h(p_l)$$
$$+ 2\sum_{i=1}^{4} K'(p_i)A^{(0)}(p_i)\left\{-\frac{\lambda^2}{2}\int_k \sum_{3\ perm\ (i,j)} [h(p_i + p_j + k)h(k) - h(k)h(k)]\right\}$$
$$+ \left\{(4 - D) - p_i.\frac{d}{dp_i}\right\} B_{II}^{(2)}(p_1, p_2, p_3, p_4) = d_2^{(0)} B_{II}^{(2)}(p_1, p_2, p_3, p_4)$$

Equation for $B_{III}^{(2)}(p_1, p_2, p_3, p_4)$

Collecting (B.4.5a), (B.4.6b) and the term containing U_4^{II} we get

$$\int_p \{-K'(p)\}\{\sum_{l=1}^{4} h(p_l)h(p_l) + \sum_{i=1}^{4} h(p_i) \sum_{j=1}^{4} h(p_j)\}$$
$$+ 2\sum_{i=1}^{4} K'(p_i^2)\{A^{(0)}(p_i)U_4^{(2)}|_2 + \lambda A^{(1)}(p_i)\}$$
$$+ 2\sum_{i=1}^{4} K'(p_i)U_2^{(1)}(p_i) \sum_{j=1}^{4} B^{(1)}(p_i)$$
$$+ (4 - D - p_i.\frac{d}{dp_i})B_{III}^{(2)}(p_1, p_2, p_3, p_4) = d_2^{(0)} B_{III}^{(0)}(p_1, p_2, p_3, p_4)$$

We have,

$$A^{(0)}(p) = 1 \; ; A^{(1)}(p) = F\lambda h(p) \; ; U_2^{(1)}(p) = -\frac{\lambda F}{2-\epsilon} \; ; B^{(1)}(p) = -\lambda h(p)$$

So the equation for $B_{III}^{(2)}(p_1, p_2, p_3, p_4)$ becomes,

$$F\Big\{\sum_{l=1}^{4} h(p_l)h(p_l) + \sum_{i\neq j} h(p_i)h(p_j)\Big\} + F\sum_{i\neq j} K'(p_i^2)h(p_j)$$
$$2F\sum_{i\neq j} K'(p_i)h(p_j) + 4F\sum_{i=1}^{4} K'(p_i)h(p_i) + \{4 - D - p_i.\frac{d}{dp_i}\}B_{III}^{(2)}(p_1,p_2,p_3,p_4) = d_2^{(0)}B_{III}^{(2)}(p_1,p_2,p_3,p_4) \tag{B.4.8}$$

Cancellation

Note that the last term in LHS and the third term in RHS of (B.4.4) cancels. Also the term (B.4.6a) cancels with the term $d_2^{(1)}B^{(1)}(p_1, p_2, p_3, p_4)$.

B.4.3 The ϕ^2 equation to determine $A^{(2)}(p)$

ϕ^2 equation is given by,

$$\int (-K'(q))\frac{\delta^2}{\delta\phi(q)\phi(-q)}\{B^{(2)}(p_1,p_2,p_3,p_4)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\}$$
$$+ \left[A(p) - p^2A'(p) - 2\{-K'(p^2)\}U_2(p)A(p)\right]\phi(p)\phi(-p)$$
$$+ \eta\frac{K(p^2)(1-K(p^2))}{p^2}\frac{p^2}{K(p^2)}\phi(p)\phi(-p) - \frac{\eta}{2}A^{(0)}(p)\phi(p)\phi(-p) = \frac{d_2}{2}A(p) + (\epsilon\lambda + \beta(\lambda))\frac{\partial}{\partial\lambda}\frac{A^{(2)}(p)}{2} \tag{B.4.9}$$

$$U_2^1(p) = -\frac{\lambda F}{2-\epsilon}; \quad U_2^{(2)}(p) = -\lambda^2 G(p) - \frac{\lambda^2 F^2}{4}h(p)$$

Where

$$G(p) = \frac{1}{3}\int_{q,k}\frac{h(q)}{2}[h(p+q+k)h(k)-h(k)h(k)] - \frac{1}{3}\int_q \frac{h(q)}{2}[h(q+k)h(k)-h(k)h(k)]$$
$$+\frac{\eta p^2}{2\epsilon} - \frac{1}{2-2\epsilon}\left\{\frac{2}{3}\beta^{(1)}v_2^{(1)} + \int_q f(q)\mathcal{F}(q)\right\}$$

$$\beta^{(1)} = -\int_q f(q)h(q) \to_{\epsilon\to 0} -F; v_2^{(1)} = -\int_q f(q)h(q) \to_{\epsilon\to 0} -\frac{F}{2}$$

A.

1. $$\frac{\delta^2}{\delta\phi(q)\delta\phi(-q)}B_I^{(2)}(p_1,p_2,p_3,p_4)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$
$$= \frac{\delta^2}{\delta\phi(q)\delta\phi(-q)}\frac{1}{4!}\sum_{3\ perm\ (i,j)}\int_k \left\{h(p_i+p_j+k)h(k)h(k)\right\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$
$$= \int_{q,k}\left\{h(p+q+k)h(k)h(k)\right. \tag{B.4.10}$$
$$+\frac{1}{2}h(k)h(k)h(k) \tag{B.4.11}$$

2. $$\frac{\delta^2}{\delta\phi(q)\delta\phi(-q)}\left\{B_{II}^{(2)}(p_1,p_2,p_3,p_4)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\right\}$$
$$=\frac{\delta^2}{\delta\phi(q)\delta\phi(-q)}\frac{1}{4!}\frac{1}{2}\int_k \sum_{3\ perm\ (i,j)}\left\{h(p_i+p_j+k)h(k)-h(k)h(k)\right\}\sum_{l=1}^{4}h(p_l)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$
$$=\int_{q,k}\left\{h(p+q+k)h(k)-h(k)h(k)\right\}\left\{h(p)\right\} \tag{B.4.12}$$
$$+\int_{q,k}\left\{h(p+q+k)h(k)-h(k)h(k)\right\}\left\{h(q)\right\} \tag{B.4.13}$$

3. $$\int_q \{-K'(q)\} \frac{\delta^2}{\delta\phi(q)\delta\phi(-q)} B^{(2)}_{III}(p_1,p_2,p_3,p_4)\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

$$= \int_q \{-K'(q)\} \frac{\delta^2}{\delta\phi(q)\delta\phi(-q)} \frac{1}{4!}(-F)\{\sum_{i\neq j}\frac{1}{2}h(p_i)h(p_j) + \sum_{l=1}^{4} h^2(p_l)\}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

$$= \int_q \{-K'(q)\} \frac{-F}{4!}\left[6\times 2\left\{\frac{2h^2(p)+2h^2(q)+8h(p)h(q)}{2}\right\} + 6\times 2\{2h^2(p)+2h^2(q)\}\right]$$

$$= \frac{3\lambda^2 F}{2}\int K'(q)h^2(q) \tag{B.4.14}$$

$$-\frac{3}{2}\lambda^2F^2h^2(p) \tag{B.4.15}$$

$$-\lambda^2F^2h(p) \tag{B.4.16}$$

B. $$2K'(p^2)\{U_2^{(2)}(p)A^{(0)}(p) + U_2^{(1)}A^{(1)}(p)\}$$

$$= 2K'(p^2)\frac{-\lambda F}{2-\epsilon} \tag{B.4.17}$$

$$= 2K'(p^2)\frac{-\lambda^2F^2}{4}h(p) \tag{B.4.18}$$

$$= -2K'(p^2)A^{(0)}(p)\frac{1}{3}\left[\int_{q,k}\frac{h(q)}{2}\{h(p+q+k)h(k) - h(q+k)h(k)\}\right] \tag{B.4.19}$$

$$-2K'(p^2)A^{(0)}(p)\frac{\eta p^2}{2\epsilon} \tag{B.4.20}$$

$$+2K'(p^2)A^{(0)}(p)\frac{1}{2-2\epsilon}\{\frac{2}{3}\beta^{(1)}v_2^{(1)}\} \tag{B.4.21}$$

$$+2K'(p^2)A^{(0)}(p)\frac{1}{2-2\epsilon}\{\int_q f(q)\mathcal{F}(q)\} \tag{B.4.22}$$

$$+2K'(p^2)\{-\frac{\lambda F}{2-\epsilon}\}\{\lambda F h(p)\} \tag{B.4.23}$$

C. $$-\eta\frac{K(p^2)(1-K(p^2))}{p^2}\frac{p^2}{K(p^2)} - \frac{\eta}{2}A^{(0)}(p)$$

$$= -\eta\, p^2 h(p)\phi(p)\phi(-p) \tag{B.4.24}$$

$$-\frac{\eta}{2}A^{(0)}(p) \tag{B.4.25}$$

Equation for $A_I^{(2)}(p)$

We collect (B.4.12) and (B.4.19) to write the following equation,

$$\int_{q,k} \big\{-K'(q)\big\}\big\{h(p+q+k)h(k) - h(q+k)h(k)\big\}\big\{h(p)\big\}$$
$$- 2K'(p^2)A^{(0)}(p)\frac{1}{3}\left[\int_{q,k} \frac{h(q)}{2}\big\{h(p+q+k)h(k) - h(q+k)h(k)\big\}\right]$$
$$+ A_I^{(2)}(p) - p^2\frac{\partial A_I^{(2)}(p)}{\partial p^2} = d_2\frac{A_I^{(2)}(p)}{2} \tag{B.4.26}$$

On LHS of (B.4.9) we are left with,

$$\int_{q,k} \big\{-K'(q)\big\}h(p)\big\{h(q+k)h(k) - h(k)h(k)\big\} \tag{B.4.27}$$

Equation for $A_{II}^{(2)}(p)$

We collect (B.4.10) and (B.4.13) to write

$$\int_{q,k} \big\{-K'(q)\big\}\big\{h(p+q+k)h(k)h(k) - h(q+k)h(k)h(k)\big\}$$
$$+ \int_{q,k} \big\{-K'(q)\big\}\big\{h(p+q+k)h(k) - h(q+k)h(k)\big\}\big\{h(q)\big\}$$
$$+ A_{II}^{(2)}(p) - p^2\frac{\partial A_{II}^{(2)}(p)}{\partial p^2} = d_2^{(0)}\frac{A_{II}^{(2)}(p)}{2} \tag{B.4.28}$$

On LHS of (B.4.9) we are left with

$$\int_{q,k} \big\{-K'(q)h(q+k)h(k)h(k)\big\} + \int_{q,k} \big\{-K'(q)h(q)\big\}\big\{h(q+k)h(k) - h(k)h(k)\big\} \tag{B.4.29}$$

Equation for $A_{III}^{(2)}(p)$

We collect (B.4.17), (B.4.21) and (B.4.16) to get,

$$-\lambda^2F^2h(p) + 2K'(p^2)\big\{-\frac{\lambda F\epsilon}{4}\big\} + 2K'(p^2)A^{(0)}(p)\frac{1}{2-2\epsilon}\big\{\frac{2}{3}\beta^{(1)}v_2^{(1)}\big\} + A_{III}^{(2)}(p) - p^2\frac{\partial}{\partial p^2}A_{III}^{(2)}(p)$$
$$= d_2^{(0)}\frac{A_{III}^{(2)}(p)}{2} + (d_2^{(1)})\frac{A^{(1)}(p)}{2} + \{\epsilon\lambda + \beta(\lambda)\}\frac{1}{2}\frac{\partial A}{\partial \lambda} \tag{B.4.30}$$

Where $d_2^{(1)} = -\lambda F$ and $A^{(1)}(p) = \lambda F h(p)$.

Equation for $A^{(2)}_{IV}(p)$

Collecting (B.4.18), (B.4.23) and (B.4.15) we get,

$$-\frac{3}{2}\lambda^2F^2h^2(p)+2K'(p^2)\frac{-\lambda^2F^2}{4}h(p)+2K'(p^2)\{-\frac{\lambda F}{2-\epsilon}\}\{\lambda F h(p)\}+A^{(2)}_{IV}(p)-p^2\frac{\partial}{\partial p^2}A^{(2)}_{IV}(p)$$
$$=d_2^{(0)}\frac{A^{(2)}_{IV}(p)}{2} \tag{B.4.31}$$

Equation for $A^{(2)}_{V}(p)$

We collect (B.4.22) and (B.4.11) to get the following equation,

$$2K'(p^2)A^{(0)}(p)\frac{1}{2-2\epsilon}\{\int_q f(q)\mathcal{F}(q)\}+\int_{q,k}\{-K'(q)\}h(p)\{h(q+k)h(k)-h(k)h(k)\}$$
$$+A^{(2)}_{V}(p)-p^2\frac{\partial}{\partial p^2}A^{(2)}_{V}(p)=d_2^{(0)}\frac{A^{(2)}_{V}(p)}{2} \tag{B.4.32}$$

Equation for $A^{(2)}_{VI}(p)$

We collect 5th term of (4.3.36), (B.4.20) to get the following equation

$$\eta\frac{K(p^2)(1-K(p^2))}{p^2}\frac{p^2}{K(p^2)}-2K'(p^2)A^{(0)}(p)\frac{\eta p^2}{2\epsilon}+A^{(2)}_{VI}(p)-\frac{1}{2}p.\frac{\partial}{\partial p}A^{(2)}_{VI}(p)$$
$$=d_2^{(0)}\frac{A_{VI}(p)}{2}+d_2^{(1)}\frac{A^{(2)}_{VI}(p)}{2}+\epsilon\lambda\frac{\partial}{\partial\lambda}\frac{A^{(2)}_{VI}(p)}{2} \tag{B.4.33}$$

B.5 Evaluation of Integrals

$$\int_{p,q}h\left(\frac{p+q}{\Lambda}\right)h\left(\frac{q}{\Lambda}\right)h\left(\frac{p}{\Lambda}\right)h\left(\frac{p}{\Lambda}\right)$$
$$=\int_{p,q}\frac{K\left(\frac{p+q}{\Lambda_0}\right)-K\left(\frac{p+q}{\Lambda}\right)}{(p+q)^2}\;\frac{K\left(\frac{q}{\Lambda_0}\right)-K\left(\frac{q}{\Lambda}\right)}{(q)^2}\;\frac{K\left(\frac{p}{\Lambda_0}\right)-K\left(\frac{p}{\Lambda}\right)}{(p)^2}\;\frac{K\left(\frac{p}{\Lambda_0}\right)-K\left(\frac{p}{\Lambda}\right)}{(p)^2}$$

We evaluate the integral for $K(x)=e^{-x^2}$.

$$=\int_{p,q}\frac{e^{-\frac{(p+q)^2}{\Lambda_0^2}}-e^{-\frac{(p+q)^2}{\Lambda^2}}}{(p+q)^2}\;\frac{e^{-\frac{q^2}{\Lambda_0^2}}-e^{-\frac{q^2}{\Lambda^2}}}{(q)^2}\;\frac{e^{\frac{p^2}{\Lambda_0^2}}-e^{\frac{p^2}{\Lambda^2}}}{(p)^2}\;\frac{e^{\frac{p^2}{\Lambda_0^2}}-e^{\frac{p^2}{\Lambda^2}}}{(p)^2}$$

Now we apply Schwinger parametrization.

$$\int_{p,q}\int_{x,y,u,v=\frac{1}{\Lambda^2}}^{\frac{1}{\Lambda_0^2}} e^{-(p+q)^2x}e^{-q^2y}e^{-p^2u}e^{-p^2v}$$

Now we do q inetgral first. We Complete the square on q and change integration varibale q. After that we do p inetegral. Also we change x, y as $x \to \frac{1}{x},\ y \to \frac{1}{y}$. At the end we take $\Lambda_0 \to \infty$.

$$\begin{aligned}
&\int_{x,y,u,v}\int_{p,q}\frac{1}{(x+y)^2}e^{-q^2}e^{-p^2\left(\frac{xy}{x+y}+u+v\right)}\\
&= F^2\int_{x,y,u,v=\frac{1}{\Lambda^2}}^{\frac{1}{\Lambda_0^2}}\frac{1}{\{1+(p+q)(u+v)\}^2}\\
&= F^2\Big(\frac{1}{2}\{\log 2\}^2-\frac{1}{2}\{\log\frac{2\Lambda_0^2}{\Lambda^2}\}^2+\{\log\frac{\Lambda_0^2}{\Lambda^2}\}^2+2\log 2-6\log 3+2\log\frac{\Lambda_0^2}{\Lambda^2}\\
&+\frac{1}{4}\{\log 4\}^2-\frac{1}{2}\{\log\frac{2\Lambda_0^2}{\Lambda^2}\}^2+\frac{1}{4}\{\log\frac{4\Lambda_0^2}{\Lambda^2}\}^2-8\log 2+5\log 5-\log\frac{\Lambda_0^2}{\Lambda^2}\\
&+\frac{1}{4}\{\log\frac{4\Lambda^2}{\Lambda_0^2}\}^2-\frac{1}{2}\{\log 2\}^2+\frac{1}{4}\{\log 4\}^2+4\log 2-6\log 3+5\log 5\Big)
\end{aligned}$$

So,

$$\begin{aligned}
&\int_{p,q}\big[\{-K'(p^2)\}h(p)h(p+q)h(q)+\{-K'(p^2)\}h(q)h(p+q)h(q)\big]\\
&= F^2\left(\frac{1}{2}-\log 2+\frac{1}{2}\log\frac{\Lambda_0^2}{\Lambda^2}\right) \qquad \text{(B.5.1)}
\end{aligned}$$

Using same procedure we can find all other integrals of used in this thesis.

B.6 Useful Mathematical identities

In this section, we give various mathematical identities about the functions $h(p)$, $\mathcal{F}(p)$, $F_3(p)$, etc which were used in the thesis to find the composite operators.

$$h(p)=\frac{1-K(p^2)}{p^2}\quad f(p)=-2K'(p)$$

$$-p.\frac{\partial}{\partial p}h(p)=-f(p)+2h(p) \qquad \text{(B.6.1)}$$

$$\mathcal{F}(p) = \frac{1}{2}\int_q \{h(p+q)h(q) - h(q)h(q)\}$$

$$\left(p.\frac{d}{dp} + \epsilon\right)\mathcal{F}(p) = \int_q f(q)\{h(p+q) - h(q)\} \tag{B.6.2}$$

$$\bar{F}_3(p) = \int_{q,k} h(p+q+k)h(q)h(k)$$

$$, \quad F_3(p) = \bar{F}_3(p) - \bar{F}_3(0) = \int_q 2h(q)\left[F(p+q) - F(q)\right]$$

$$\left(-\frac{p}{2}.\frac{d}{dp} + 1\right)\bar{F}_3(p) = -6\int_{q,k} f(q)h(p+q+k)h(k) \tag{B.6.3}$$

$$\left(p.\frac{d}{dp} - 2 + 2\epsilon\right)F_3(p) = 3\int_{q,k} f(k)h(q)\left[h(q+k+p) - h(q+k)\right] \tag{B.6.4}$$

$$\bar{H}_3(p) = \int_q h(p+q)h(q)h(q)$$

$$\left(p.\frac{\partial}{\partial p} + 2 + \epsilon\right)\bar{H}_3(p) = \int_q f(p)h(p+q)^2 + 2\int_q f(q)h(q)h(p+q) \tag{B.6.5}$$

$$I_4(p_i+p_j;p_i) = \bar{I}_4(p_i+p_j;p_i) - \bar{I}_4(0;0)$$

$$= \sum_{6\ perm\ (i,j)} \int_{p,q} \big\{h(p_i+p_j+q)h(p+q+p_i)h(p)h(q) - h(p+q)h(p)h(q)h(q)\big\}$$

$$\left(-\sum_{l=1}^{4} p_l.\frac{d}{dp_l} - 2\epsilon\right)\bar{I}_4(p_i+p_j;p_i) = -2\int_{p,q} f(p)\left[h(p_i+p_j+p) + h(p_i+p_j+q)\right]h(p_i+p+q)h(q) \tag{B.6.6}$$

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