Infrared Structure Of Gauge Theory : A Comparison Between $\mathcal{N} = 4$ SYM and QCD

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Amlan Chakraborty. 28.6.2021

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2. **Infrared structure of $\mathcal{N} = 4$ SYM and leading transcendentality principle in gauge theory.**

   Taushif Ahmed, Pulak Banerjee, Amlan Chakraborty, Prasanna K. Dhani, V. Ravindran and Satyajit Seth *PoS RADCOR2019 (2019) 059*

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- **Resummed prediction for Higgs boson production through $b\bar{b}$ annihilation at N^3LL**

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- **Higgs pair production from bottom quark annihilation to NNLO in QCD**

  A.H. Ajjath, Pulak Banerjee, Amlan Chakraborty, Prasanna K. Dhani, Pooja Mukherjee, Narayan Rana and V. Ravindran

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- **The Curious Case of Leading Transcendentality: Three Point Form Factors.**

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My Parents, Teachers and Friends

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SYNOPSIS

Scattering amplitudes and Cross Sections are most prominent observable in the context of collider experiments. Almost everything we know experimentally about gauge theory is based on scattering processes with asymptotic, on-shell and partially off-shell (Form factor) states, evaluated in perturbation theory. So far the Standard Model (SM) of Particle Physics is the most successful theory to understand the quantum nature and dynamics of the elementary constituents of matter excluding Gravity. The quest for searching the new physics beyond SM has intensified after the discovery of Higgs Boson at Large Hadron Collider (LHC)in CERN, Geneva. For discovering the new physics , we will need to understand the SM with unprecedented accuracy. Till date the only reliable way to predict SM cross sections in general lies under the framework of perturbative Quantum Field Theory (QFT). The matrix elements ($\mathcal{M}$) as well as the observable in perturbative QFT are expanded in powers of coupling constants, c, present in the theory which is needed to be small enough in order to have a asymptotic series expansion :

$$\mathcal{M} = \sum_{n=0}^{\infty} c^n \mathcal{M}\,. \tag{0.0.1}$$

The theoretical uncertainties for different processes are due to our lack of understanding higher order terms in perturbative QFT. For achieving the desirable accuracy one has to give enormous effort for computing the multiloop and multileg scattering amplitudes and cross sections. For devising new techniques to accomplish this task one has to understand the universal structure of scattering amplitudes and cross sections in different gauge theories. We can exploit the understanding on the universal features of scattering cross section to predict some of the missing higher order terms in the perturbation series and this will reduce the theoretical uncertainties substantially

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The structure of infrared (IR) singularities of the scattering amplitudes are universal (process independent) quantities of a gauge theory. Understanding of such singularities consisting of soft and collinear ones is ubiquitous for the application of Quantum Chromodynamics (QCD)[1] to predict cross section at LHC. Calculations of cross sections beyond tree level involve delicate cancellations of such singularities in the sum over the external states. A detailed understanding of the singularities is therefore a preliminary condition for making precise predictions. Apart from phenomenological significance, infrared singularities pave a window into the all-order structure of perturbation theory. They acknowledge a simple, iterative structure, which is common to all gauge theories. Investigating this structure is an important step towards the understanding scattering amplitudes in a wide class of non abelian gauge theories, in general.

Being a cousin of QCD, scattering amplitudes in the maximally supersymmetric Yang-Mills theory ($\mathcal{N}$ = 4 SYM) admits many surprising and remarkable similarities with the former one . In addition to having all the symmetries of QCD, $\mathcal{N}$ = 4 SYM theory enjoys supersymmetry and UV conformal symmetry, which makes it interesting to study. As physicists, our obvious goal is to solve the problem related to QCD and one can ask naturally whether it is worth to consider $\mathcal{N}$ = 4 SYM problems. From the calculations $\mathcal{N}$ = 4 SYM one can gain many insights about the choice and transcendental[2] nature of the basis Integrals and it helps to device cutting edge tools to perform complicated real life QCD calculations. Also being a UV finite theory $\mathcal{N}$ = 4 SYM is a perfect playground for understanding the Infrared singularity structure. Besides that one can test and refine new concepts like generalised unitarity, BCFW recursion relations, symbol of functions, scattering amplitude/wilson loop duality, dual conformal and Yangian symmetries and its connection with Integrability. It's also very interesting to consider the amplitudes at different physically singular limit (regge and collinear) which we studied rigorously for different four point double massive amplitude in $\mathcal{N}$ = 4 SYM and compared with corresponding QCD results to demonstrate the universality. Sometimes its very difficult to understand the hidden symmetries of pure Yang-Mills theory from a geometrical point of view (Amplituhedron [1]) and rigorous investigation of

[1]Theory describing the storng interactions

[2]The transcendentality weight, τ, of a function, f, is defined as the number of iterated integrals required to define it, e.g. $\tau(\log) = 1$, $\tau(\mathrm{Li}_n) = n$, $\tau(\zeta_n) = n$ and moreover, we define $\tau(f_1 f_2) = \tau(f_1) + \tau(f_2)$. Algebraic factors are assigned to weight zero and dimensional regularisation parameter ϵ to -1.

scattering amplitude data in $\mathcal{N} = 4$ SYM can help to reveal new hidden symmetries.

Before starting the work for this thesis we asked following questions in the direction of IR and transcendental structure of different quantities in $\mathcal{N} = 4$ SYM :

- Splitting functions are the universal (process independent) quantities of a generic massless non abelian gauge theory and it controls the evolutions of some non perturbative objects (Parton Distribution Functions namely). Computation of this quantities in $\mathcal{N} = 4$ SYM can reveal the transcendental structure of IR sensitive quantities and particularly universal features at singular limit. Also we can gain a idea to decompose it with respect to anomolous dimentions of the theory and therefore a clear understanding about the factorisation properties of different observable in a theory independent manner.
- Whether we can use $\mathcal{N} = 4$ SYM calculations to predict some of features of QCD.
- Transcendental structure of multileg Form-Factors (FF) involving several UV protected and non-protected composite operator insertion in $\mathcal{N} = 4$ SYM and comparison with similar kind of FFs in QCD in special singular limits.

0.1 Second order splitting functions and infrared safe cross sections in $\mathcal{N} = 4$ SYM theory

Field theoretic results from Quantum Chromodynamics (QCD) play an important role in understanding the physics of strong interactions. Inclusive and differential cross sections computed using perturbative QCD not only helped to discover several of elementary particles of the Standard Model (SM) but also provided a laboratory to understand the field theoretical structure of non-abelian gauge theories. For example, both theoretical and experimental results from high energetic collision processes, such as the deep-inelastic scattering and the Drell-Yan production provides the complete knowledge of the internal structure as well as the dynamics of hadrons in terms of their constituents such as quarks and gluons. Scattering cross sections computed in high energetic collision processes such as the Drell-Yan and the deep-inelastic scattering processes can be expressed in terms of perturbatively computed partonic cross section, convoluted with the parton distribution functions (PDFs). The partons refer to quarks and gluons and the PDFs describe the probabilities of finding the partons in a bound state.These scattering cross sections at high energies can be expressed in terms of the perturbatively calculable scatterings involving constituents of hadrons properly convoluted with parton distribution functions.These constituents at high energies are light quarks and gluons often called partons and the corresponding PDFs describe their probabilities to exist in the hadron. Such a description of hadronic cross section goes by the name parton model. $\mathcal{N} = 4$ SYM is an UV renormalizable quantum field theory in four dimensional space. In addition to having SU(N) gauge symmetry, $\mathcal{N} = 4$ SYM theory enjoys supersymmetry and conformal symmetry that make it fascinating to study. Although the study of cross sections in such a theory has no phenomenological implications, yet they can help us to understand the factorization properties of the IR singularities, the latter being useful to extract the Altaresi Parisi (AP) [2] kernels at each order in the perturbation theory. Undoubtedly, higher order computation of the FFs and the amplitudes unravel the IR structure of the $\mathcal{N} = 4$ SYM theory in an elegant way. However purely real emission processes, which appear in cross sections, can also give important informations about the nature of soft and collinear emissions. In QCD, the gluons in a virtual loop can become soft and contribute to poles in ϵ in a dimensionally regulated theory, similar situation also happens when gluons in a real emission process carry a small fraction of the momentum of the incoming particles. More precisely, when we perform the phase space integrations for such real emission processes, we encounter poles in ϵ, at every order in perturbation series. These soft contributions from real and virtual diagrams cancel order by order when they are added together, thanks to the Kinoshita-Lee-Nauenberg (KLN) theorem [3, 4]. In addition, the real emissions of gluons and quarks are sensitive to collinear singularities; while the final state divergences are taken care by the KLN theorem, the initial state counterparts are removed by mass factorization. Similar scattering of massless gluons, quarks, scalars and pseudo-scalars in $\mathcal{N} = 4$ SYM theory can be studied within a supersymmetric preserving $\overline{DR}$ regularisation scheme. The cancellation of soft singularities and factorisation of collinear singularities in the scattering cross sections will also provide wealth of information on the IR structure of $\mathcal{N} = 4$ gauge theory. One can investigate the soft plus virtual part of these finite cross sections after mass factorisation in terms of universal cusp and collinear anomalous dimensions.

In [5] we focused on computing AP Spitting functions and try to understand the universal fac-

torisation properties of Soft-Virtual(SV) cross-sections for certain composite operators (BPS and Stress tensor) upto NNLO and then to predict the NNNLO SV cross-sctions using the known three loop anomalous dimensions $\mathcal{N}$=4 SYM. We studied the universal decomposition of the Spitting functions with respect to the anomalous dimensions of the theory at SV limit. Also we investigated the transcendental structure of the soft gluon and scalar radiation in $\mathcal{N} = 4$. Our findings are consistent with the previous results available in the literature related to the topic.

0.2 Form factors with two-operator insertion and the principle of transcendentality principles

Understanding the analytical structures of on-shell amplitudes and FFs in $\mathcal{N} = 4$ SYM has been an active area of investigation, not only to uncover the hidden structures of these quantities but also to establish the connections with other gauge theories, such as QCD. One of the most intriguing facts is the appearance of uniform transcendental (UT) weight terms in certain class of quantities in $\mathcal{N} = 4$ SYM. The two-point or Sudakov FFs of primary half-BPS operator belonging to the stress-energy supermultiplet is observed [6–8] to exhibit the UT property to three-loops, more specifically, they are composed of only highest transcendental (HT) terms with weight 2L at loop order L. This is a consequence of the existence of an integral representation of the FFs with every Feynman integral as UT [8]. Knowing the existence of such a basis has profound implications in choosing the basis of integrals while evaluating Feynman integrals using differential equations method [9]. Also one can gain a clear understanding on the transcendental weight contributions to the scattering amplitude from the different type of singularity (UV and IR). The three-point FFs of half-BPS operator is also found to respect this wonderful UT property [10]. On the contrary, this property fails for the two- [11, 12] and three-point [13] FFs of the unprotected Konishi operator which are investigated up to three- and two-loops, respectively. Three-point FFs of a Konishi descendant operator is also found not to exhibit the UT property [14]. Having seen the beautiful property of UT in FFs of one-operator insertion, the question arises whether it is respected for the two-point FFs with SUSY protected two-operator insertion and whether this property can be extrapolated to generalised FFs with n-number of operators insertion. Also the connection between quantities in $\mathcal{N} = 4$ SYM and that of QCD is of fundamental importance. In addition to deepening our theoretical understanding, it is motivated from the fact that computing a quantity in QCD is much more difficult, and in the absence of our ability to calculate a quantity in QCD, if it is possible to obtain the result, at least partially, from that of simpler theory, such as $\mathcal{N} = 4$ SYM. In refs. [5, 15], it is found that the anomalous dimensions of leading twist-two-operator in $\mathcal{N} = 4$ SYM are identical to the HT counterparts in QCD [16], and consequently, the principle of maximal transcendentality (PMT) is conjectured. The PMT says that the algebraically most complex part of certain quantities in $\mathcal{N} = 4$ SYM and QCD are identical.

In [17], for the first time, we address this question in the context of general class of FF in $\mathcal{N} = 4$ and found that the UT property does not hold true at two-loop for the FF of double half-BPS insertion. Also we found the wonderful conjecture of PMT is not respected in the context of double half-BPS insertions in $\mathcal{N} = 4$ and its QCD counterpart. Nevertheless the UT and PMT property found to be restored at regge and collinear limit in the phase space region of the FF. We have discussed superiority of the Feynman diagrammatic approach over unitarity method in the context of the computation of double operator insertion.

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1 Introduction

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[1] Within the framework of Lattice Gauge Theory one can predict some of the not perturbative features.

enough in order to have a well define asymptotic series expansion :

$$\mathcal{M} = \sum_{n=0}^{\infty} c^n \mathcal{M} . \tag{1.0.1}$$

The theoretical uncertainties for different processes are due to our lack of understanding of higher order terms in perturbative QFT. For achieving the desirable accuracy comparable to current experimental uncertainties one has to give enormous effort for computing the multiloop and multileg scattering amplitudes and cross sections. For devising new techniques to accomplish this task one has to understand the universal structure of scattering amplitudes and cross sections in different gauge theories. One can exploit the understanding on the universal features of scattering cross section to predict some of the missing higher order terms in the perturbation series and this will reduce the theoretical uncertanties substantially .

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Being a cousin of QCD, scattering amplitudes in the maximally supersymmetric Yang-Mills theory ($\mathcal{N}$ = 4 SYM) admits many surprising and remarkable similarities with the former one. In addition to having all the symmetries of QCD , $\mathcal{N}$ = 4 SYM theory enjoys supersymmetry and UV conformal symmetry makes it interesting to study to many researchers. Many fantastic progresses made over the last two decades to compute multi loop/leg scattering amplitudes in $\mathcal{N}$ = 4 SYM to understand the IR singularity structure of it.

[2]Theory describing the strong interactions

As physicists our obvious goal is to solve the problem related to QCD and one can ask naturally whether it is worth to consider $\mathcal{N} = 4$ SYM problems. From the calculations $\mathcal{N} = 4$ SYM one can gain many insights about the choice and transcendental[3] nature of the basis Integrals and it helps to device cutting edge tools to perform complicated real life QCD calculations. Also being a UV finite theory $\mathcal{N} = 4$ SYM is a perfect playground for understanding the singularity structure. Besides that one can test and refine new concepts like generalised unitarity, BCFW recursion relations, symbol of functions, scattering amplitude/Wilson loop duality, dual conformal and Yangian symmetries and its connection with Integrability. It's also very interesting to consider the amplitudes at different physically singular limit (regge and collinear) which will be a important topic of discussion later part of this thesis. Sometimes its very difficult to understand the hidden symmetries of pure Yang-Mills theory from a geometrical point of view (Amplituhedron [1]) and rigorous investigation of scattering amplitude data in $\mathcal{N} = 4$ SYM can help to reveal new hidden symmetries.

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- Validity of the principle of maximal transcendentality in the context of stress tensor operator in $\mathcal{N} = 4$ SYM and QCD.

Let us conclude this part citing one statement of Mikhail Shifman [18, 19] about the importance of considering the SUSY gauge theory :

> *“ Although the ultimate goal of the B-theory is calculating QCD amplitudes, the concept design of various ideas and methods is carried out in supersymmetric theories, which provide an excellent testing ground. Looking at super Yang Mills theory offers a lot of insight into how one can deal with the problems in QCD. Of all supersymmetric theories probably the most remarkable one is N = 4 Yang Mills. Its special status is due to the fact that (a) it is conformal, and (b) in the planar strong coupling limit it is dual to string theory on AdS5 × S5. ”*

1.1 Scattering Amplitudes

In Quantum Mechanics, probability amplitude is a complex quantity that encapsulates all of the information in describing the behavior of microscopic systems. The modulus squared of the quantity represents a probability or probability density. QFT, a relativistic extension of standard quantum mechanics also have similar quantity called scattering amplitude or scattering matrix (S matrix). S-matrix relates the initial state and the final state of an undergoing a scattering process. S-matrix is constructed by an unitary matrix connecting asymptotically free in and out-states in the Fock-space of physical states. For any initial state connecting to a final state one can define the S matrix following way :

$$\mathcal{M}_{i\to f} = \langle f|S|i\rangle = \langle f|\sum_{n=0}^{\infty}(-i)^n/n! \int_{-\infty}^{\infty} \mathrm{d}x_1^4 ... \int_{-\infty}^{\infty} \mathrm{d}x_n^4 \, T\left[\boldsymbol{H}(t_1)...\boldsymbol{H}(t_n)\right]|i\rangle \tag{1.1.1}$$

where $\boldsymbol{H}$ is the Hamiltonian density, Legendre transform of Lagrangian density. Lagrangian density can be built as a functional of the fields present in the theory and thus one can define the action of the system by integrating over the coordinate space -

$$S = \int d^4x \, \mathcal{L}[\phi_i(x)] \; . \tag{1.1.2}$$

S-matrix of any physically meaningful theory in flat space should satisfy the following criterion :

- It has to be *Lorentz* invariant.
- It has to be *gauge* invariant if one is considering a QFT respecting local symmetry.
- It should be *crossing* invariant: the antiparticle scattering to be described by the analytic continuation of the particle scattering.
- It has to be *unitary*.
- It should follow the Landau property : physical singularities of scattering matrix are due to the poles and cuts corresponding to exchange of real particles on shell.

There exist several prescriptions to compute S matrix in QFT . There are two popular approach worth mentioning and used for computing various multiloop and multileg amplitudes in QCD and $\mathcal{N}$ =4 SYM :

1. *Diagrammatic approach:* Generate relevant Feynman diagrams and correspondingly compute the associated integrals appearing at particular perturbative order for the multiloop/leg scattering amplitude under consideration.
2. *Unitary-based approach:* Exploit the generalised unitary properties of the amplitude and compute quantities without considering the diagrams.

Although in the community most of the computation in $\mathcal{N}$ =4 SYM performed through Unitary-based approach , we preferred to choose the diagrammatic approach to compute. Particularly a large portion of the thesis dealing with the real radiation diagrams with the integral over final state phase space suits the diagrammatic approach with respect to the setup we have.

In this thesis, we will be dealing with the perturbative aspects of $\mathcal{N}$ =4 SYM amplitudes and cross sections. We will start our discussion of $\mathcal{N}$ =4 SYM by introducing the basic aspects of it which will be followed by the writing down the Lagrangian and corresponding Feynman rules. Using these tools in hand, we will discuss how to compute multiloop/leg observable in $\mathcal{N}$ =4 SYM and eventually get analytical expression of different IR sensitive physical quantities and compare directly with the similar quantities in QCD exists in literature.

1.2 Brief introduction to $\mathcal{N}$ =4 SYM

Scattering amplitudes in generic gauge theories are shown to have a very simpler structure due to the underlying local symmetries than it is expected with respect to the Feynman diagrams point of view from scratch. Often a huge cancellations takes place between the individual diagrams and finally one gets a simple analytical results. Particularly a very simple analytical behavior of generic gauge theory amplitudes at multiloop level motivates the field theorists to investigate thoroughly questing after hidden symmetries behind it and also to develop new tools to avoid directly computing huge number of Feynman diagrams. $\mathcal{N}$ = 4 SYM is an UV renormalizable quantum field theory in four dimensional space. In addition to having SU(N) gauge symmetry, $\mathcal{N}$ = 4 SYM theory enjoys supersymmetry and conformal symmetry that make it fascinating to study Specifically the unbroken UV conformal symmetry at the quantum level can be attributed to the UV finiteness of the theory. $\mathcal{N}$ = 4 SYM is suggested to be an integrable gauge theory in planar limit in 4 space time dimensionsThe theory has many special properties. In addition to unbroken superconformal symmetry at the quantum level it is conjectured via the AdS/CFT correspondence that $\mathcal{N}$ = 4 SYM to be dual to String theory in AdS space relating a conformal field theory at strong coupling with a supergravity theory at weak coupling. As a consequence the all order scattering amplitude in $\mathcal{N}$ = 4 SYM must have simple and compact structures. Besides that one can test and refine new concepts like generalised unitarity, BCFW recursion relations, symbol of functions, scattering amplitude/wilson loop duality, dual conformal and Yangian symmetries and its connection with integrability in $\mathcal{N}$ = 4 SYM in a more efficient way due to its inherent simplicity.

1.2.1 Lagrangian and Feynman Rules

Let us briefly describe the Lagrangian density and its particular form we have choosen according to our convenience. The effective Lagrangian contains color singlet composite operators such as a half-BPS, Konishi and energy momentum tensor of $\mathcal{N} = 4$ SYM. These are denoted by $\mathcal{O}^I$ with I = BPS, $\mathcal{K}$ and T being the three singlet operators respectively. The Lagrangian density including the effective interactions reads as follows [20–23],

$$\mathcal{L} = \mathcal{L}_{\text{SYM}}^{\mathcal{N}=4} + \mathcal{L}_{\text{int}} \tag{1.2.1}$$

where $\mathcal{L}_{\text{SYM}}^{\mathcal{N}=4}$ [20–23], is given by

$$\begin{aligned}\mathcal{L}_{\text{SYM}}^{\mathcal{N}=4} =& -\frac{1}{4}G_{\mu\nu}^{a}G^{\mu\nu a} - \frac{1}{2\xi}(\partial_{\mu}A^{a\mu})^2 + \partial_{\mu}\bar{\eta}^{a}D^{\mu}\eta_{a} + \frac{i}{2}\bar{\lambda}_{m}^{a}\gamma^{\mu}D_{\mu}\lambda_{m}^{a} + \frac{1}{2}(D_{\mu}\phi_{i}^{a})^2 \\ &+ \frac{1}{2}(D_{\mu}\chi_{i}^{a})^2 - \frac{g}{2}f^{abc}\bar{\lambda}_{m}^{a}[\alpha_{m,n}^{i}\phi_{i}^{b} + \gamma_5\beta_{m,n}^{i}\chi_{i}^{b}]\lambda_{n}^{c} - \frac{g^2}{4}\Big[(f^{abc}\phi_{i}^{b}\phi_{j}^{c})^2 \\ &+ (f^{abc}\chi_{i}^{b}\chi_{j}^{c})^2 + 2(f^{abc}\phi_{i}^{b}\chi_{j}^{c})^2\Big].\end{aligned} \tag{1.2.2}$$

η^a and $A^{a,\mu}$ represent ghost and the gauge fields respectively, the fermion(Majorana) are denoted by λ_m^a, with $m = 1, ..., 4$ denoting their generation, and ϕ_i^a and χ_j^a (indices $i,\ j = 1, 2, 3$) represent scalars and pseudoscalars fields of the theory, all in four space-time dimensions. All the fields transform in adjoint representation of SU(N) gauge group. g is the coupling constant and ξ is the gauge fixing parameter. The strength of the gluon field is given by $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$, while covariant derivative is $D_\mu = \partial_\mu - igT^a A_\mu^a$. The matrices T^a follows $[T^a, T^b]_- = if^{abc}T^c$ algbra, where f^{abc} is the totally antisymmetric structure constant of the group. The generators are normalized as $\text{Tr}(T^a\,T^b) = 1/2\,\delta^{a\,b}$. The antisymmetric matrices α and β satisfy the relations

$$[\alpha^i, \alpha^j]_+ = [\beta^i, \beta^j]_+ = -2\delta^{ij}\mathbb{I}, \quad [\alpha^i, \beta^j]_- = 0. \tag{1.2.3}$$

and in addition,

$$\text{tr}(\alpha^i) = \text{tr}(\beta^i) = \text{tr}(\alpha^i\beta^j) = 0, \quad \text{tr}(\alpha^i\alpha^j) = \text{tr}(\beta^i\beta^j) = -4\delta^{ij}. \tag{1.2.4}$$

Since we work with the dimensionally regulated version of the Lagrangian density in the space time dimensions $d = 4 + \epsilon$, and use supersymmetry preserving dimensional reduction scheme ($\overline{DR}$) [24, 25], the number of α and β matrices is dependent on d. Hence care is needed when we perform the contraction of indices i in d dimensions, for example

$$\alpha^i\alpha^i = \beta^i\beta^i = \Big(-3 + \frac{\epsilon}{2}\Big)\mathbb{I} \,, \alpha^i\alpha^j\alpha^i = \alpha^j\Big(1 - \frac{\epsilon}{2}\Big)\mathbb{I} \,, \quad \beta^i\beta^j\beta^i = \beta^j\Big(1 - \frac{\epsilon}{2}\Big)\mathbb{I}. \tag{1.2.5}$$

The interaction part of the Lagrangian density in eq. 1.2.1 is given by

$$\mathcal{L}_{\rm int} = \mathcal{L}^{\rm BPS} + \mathcal{L}^{\mathcal{K}} + \mathcal{L}^{\rm T}. \tag{1.2.6}$$

where

$$\mathcal{L}^{\rm BPS} = J^{\rm BPS}_{rt} O^{\rm BPS}_{rt}, \quad \mathcal{L}^{\mathcal{K}} = J^{\mathcal{K}} O^{\mathcal{K}}, \quad \mathcal{L}^{\rm T} = J^{T}_{\mu\nu} T^{\mu\nu}. \tag{1.2.7}$$

In above, the different singlet states are denoted by external currents J's (namely $J^{\rm BPS}_{rt}$, $J^{\mathcal{K}}$ and $J^{T}_{\mu\nu}$) which couple to a half-BPS ($O^{\rm BPS}_{rt}$), a Konishi ($O^{\mathcal{K}}$) and a tensorial operator,T ($T^{\mu\nu}$).

namely $J^{\rm BPS}_{rt}$, $J^{\mathcal{K}}$ and $J^{T}_{\mu\nu}$) which couple to a half-BPS, a Konishi and a tensorial operators respectively.

The half-BPS operator that we use is given by [6, 26]

$$O^{\rm BPS}_{rt} = \phi^a_r\phi^a_t - \frac{1}{3}\delta_{rt}\phi^a_s\phi^a_s, \tag{1.2.8}$$

The factor 1/3 has been used to ensure the tracelessness property in 4 dimension. The primary operator of the Konishi supermultiplet, the Konishi, has the following form

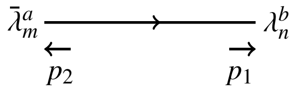
$$O^{\mathcal{K}} = \phi^a_r\phi^a_r + \chi^a_r\chi^a_r. \tag{1.2.9}$$

The tensor operator, T, is the energy-momentum tensor of $\mathcal{N} = 4\,\mathrm{SYM}$ theory. In terms of the Majorana, gauge, scalar and pseudoscalar fields, we find

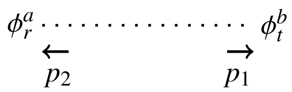
$$\begin{aligned} T^{\mathcal{N}=4\,\mathrm{SYM}}_{\mu\nu} = {} & G^a_{\mu\lambda} G^\lambda_{a\nu} + \frac{1}{4}\eta_{\mu\nu} G^a_{\rho\lambda} G^{\rho\lambda}_a - \frac{1}{\xi}\,\partial_\lambda A^\lambda \left[\partial_\mu A_\nu + \partial_\nu A_\mu\right] + \frac{1}{2\xi}\,\eta_{\mu\nu}(\partial_\rho A^\rho{}_a)^2 \\ & + (\partial_\mu \bar{\eta}^a)(D_\nu \eta_a) + (\partial_\nu \bar{\eta}^a)(D_\mu \eta_a) - \eta_{\mu\nu}(\partial_\rho \bar{\eta}^a)(D^\rho \eta_a) + \frac{i}{4}\Big[\bar{\lambda}^a_m \gamma_\mu D_\nu \lambda^a_m \\ & - \frac{1}{2}\partial_\mu(\bar{\lambda}^a_m \gamma_\nu \lambda^a_m) + \bar{\lambda}^a_m \gamma_\nu D_\mu \lambda^a_m - \frac{1}{2}\partial_\nu(\bar{\lambda}^a_m \gamma_\mu \lambda^a_m)\Big] - \frac{i}{2}\Big[\eta_{\mu\nu}\bar{\lambda}^a_m \gamma^\rho D_\rho \lambda^a_m \\ & - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\left(\bar{\lambda}^a_m \gamma^\rho \lambda^a_m\right)\Big] + (D_\mu \phi^a_i)(D_\nu \phi^a_i) - \frac{1}{2}\eta_{\mu\nu}(D_\rho \phi^a_i)^2 + (D_\mu \chi^a_i)(D_\nu \chi^a_i) \\ & - \frac{1}{2}\eta_{\mu\nu}(D_\rho \chi^a_i)^2 + \frac{g}{2}\eta_{\mu\nu} f^{abc}\,\bar{\lambda}^a\left[\alpha^i_{m,n}\phi^b_i + \gamma_5 \beta^i_{m,n}\chi^b_i\right]\lambda^c_n + \frac{g^2}{4}\eta_{\mu\nu}\Big[(f^{abc}\phi^b_i \phi^c_j)^2 \\ & + (f^{abc}\chi^b_i \chi^c_j)^2 + 2(f^{abc}\phi^b_i \chi^c_j)^2\Big]. \end{aligned} \tag{1.2.10}$$

We denote Majorana fermions through straight lines, scalar through dotted lines and pseudo-scalars through dash dot lines.

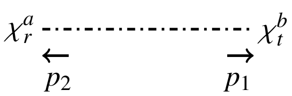
- The propagators for Majorana fermions, scalars, pseudo-scalars, gluon and ghosts are given by:



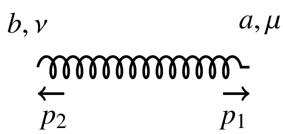
$$i\,(2\pi)^4\,\delta^{(4)}\,(p_1 + p_2)\,\frac{\not{p}_1}{p_1^2 + i\varepsilon}\delta^{ab}\delta_{mn}$$



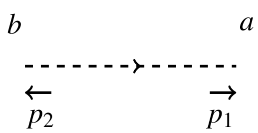
$$i\,(2\pi)^4\,\delta^{(4)}\,(p_1 + p_2)\,\frac{1}{p_1^2 + i\varepsilon}\delta^{ab}\delta_{rt}$$



$$i\,(2\pi)^4\,\delta^{(4)}\,(p_1 + p_2)\,\frac{1}{p_1^2 + i\varepsilon}\delta^{ab}\delta_{rt}$$

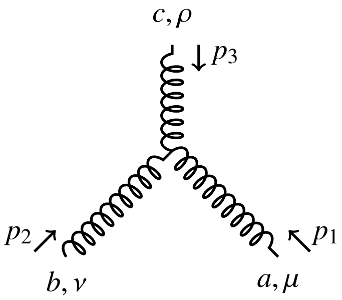


$$i\,(2\pi)^4\,\delta^{(4)}\,(p_1 + p_2)\,\delta_{ab}\frac{1}{p_1^2}\left[-g_{\mu\nu} + (1-\xi)\frac{p_{1\mu}p_{1\nu}}{p_1^2}\right]$$

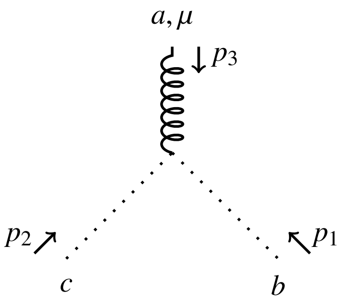


$$i\,(2\pi)^4\,\delta^{(4)}\,(p_1 + p_2)\,\delta_{ab}\frac{1}{p_1^2}$$

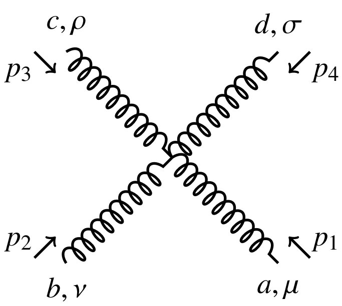
- The interacting vertices are given by:



$$\frac{g_s}{3!}\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,f^{abc}$$
$$\times\left[g^{\mu\nu}(p_1-p_2)^\rho+g^{\nu\rho}(p_2-p_3)^\mu+g^{\rho\mu}(p_3-p_1)^\nu\right]$$



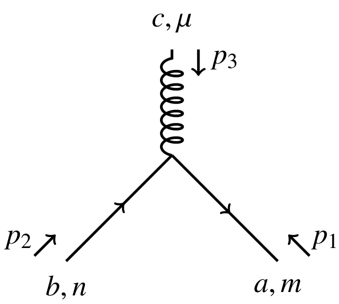
$$-g_s\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,f^{abc}p_1^\mu$$



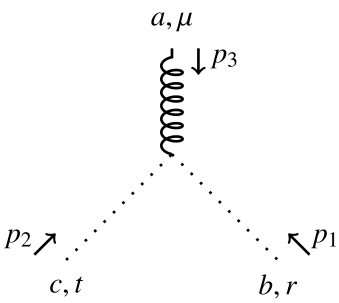
$$-\frac{g_s^2}{4!}\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4)$$
$$\left\{\left(f^{ac,bd}-f^{ad,cb}\right)g^{\mu\nu}g^{\rho\sigma}+\left(f^{ab,cd}-f^{ad,bc}\right)g^{\mu\rho}g^{\nu\sigma}\right.$$
$$\left.+\left(f^{ac,db}-f^{ab,cd}\right)g^{\mu\sigma}g^{\nu\rho}\right\}$$

with

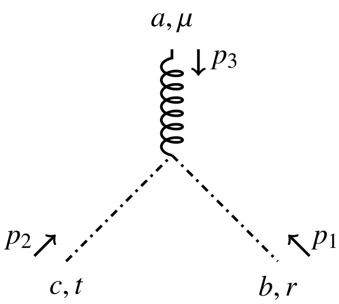
$$f^{ab,cd}\equiv f^{abx}f^{cdx}$$



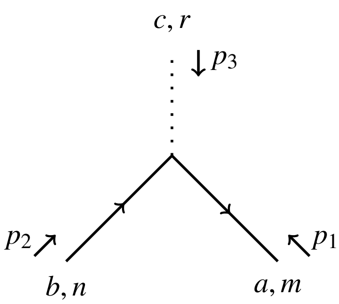
$$g\gamma^{\mu}\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,f^{abc}\,\delta_{mn}$$



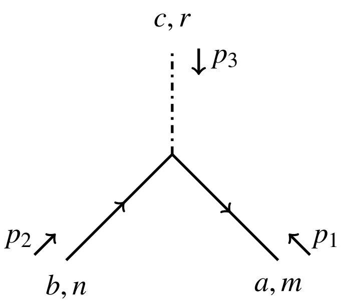
$$-g\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,(p_1-p_2)^{\mu}f^{abc}\delta_{rt}$$



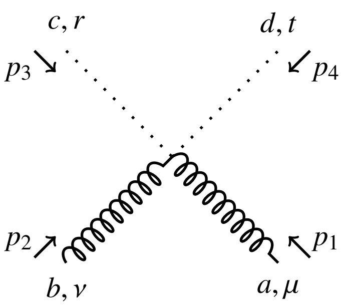
$$-g\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,(p_1-p_2)^{\mu}f^{abc}\delta_{rt}$$



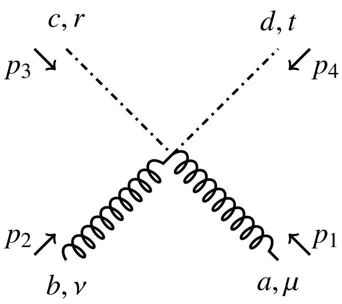
$$ig\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,\alpha^{r}_{mn}f^{abc}$$



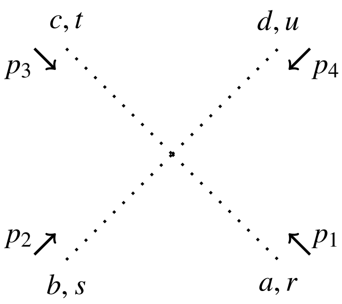
$$-g\gamma_5\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,\beta^r_{mn}f^{abc}$$



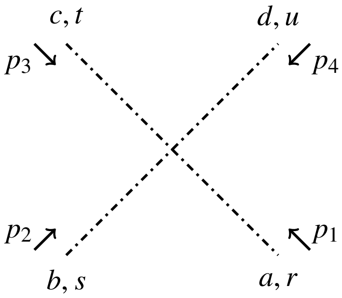
$$ig^2\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4)$$
$$\times\left(f^{ad,bc}+f^{ac,bd}\right)g^{\mu\nu}\delta_{rt}$$



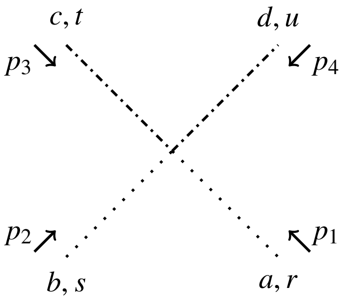
$$ig^2\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4)$$
$$\times\left(f^{ad,bc}+f^{ac,bd}\right)g^{\mu\nu}\delta_{rt}$$



$$ig^2\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4)$$
$$\times\left\{(\delta_{ru}\delta_{st}-\delta_{rt}\delta_{su})\,f^{ab,cd}+(\delta_{ru}\delta_{st}-\delta_{rs}\delta_{tu})\,f^{ac,bd}\right.$$
$$\left.+\,(\delta_{rt}\delta_{su}-\delta_{rs}\delta_{tu})\,f^{ad,bc}\right\}$$

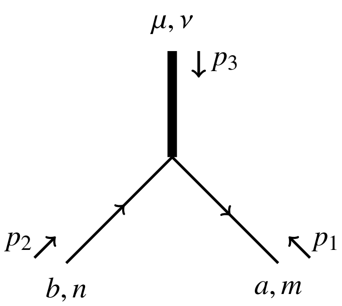


$$
\begin{aligned}
& ig^2\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4) \\
& \times \Big\{ (\delta_{ru}\delta_{st}-\delta_{rt}\delta_{su})\, f^{ab,cd} + (\delta_{ru}\delta_{st}-\delta_{rs}\delta_{tu})\, f^{ac,bd} \\
& + (\delta_{rt}\delta_{su}-\delta_{rs}\delta_{tu})\, f^{ad,bc} \Big\}
\end{aligned}
$$

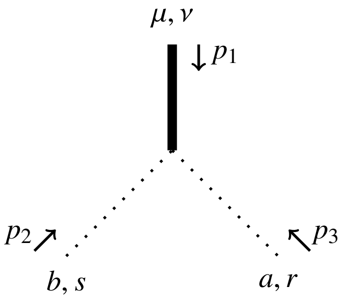


$$
\begin{aligned}
& -ig^2\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4) \\
& \times \left(f^{ad,bc} + f^{ac,bd} \right) \delta_{rs}\delta_{tu}
\end{aligned}
$$

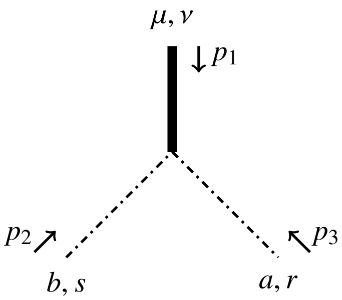
- Stress tensor interaction feynman rule:



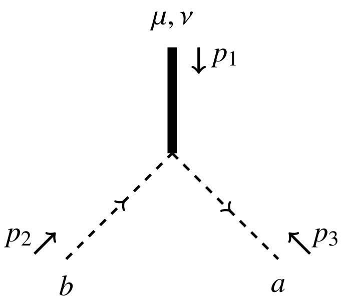
$$
\begin{aligned}
& \frac{1}{8} i\kappa\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,\delta_{ab}\delta_{mn} \\
& \times \left\{ 2\eta^{\mu\nu}(\not{p}_2-\not{p}_1) + (p_1^\nu-p_2^\nu)\gamma^\mu + (p_1^\mu-p_2^\mu)\gamma^\nu \right\}
\end{aligned}
$$



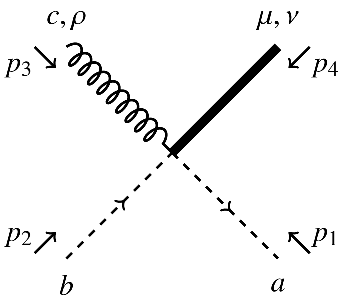
$$\frac{1}{2} i\kappa \,(2\pi)^4\, \delta^{(4)}\,(p_1+p_2+p_3)\,\delta_{ab}\delta_{rs}$$
$$\times\Big\{-\eta^{\mu\nu}(p_2\cdot p_3)+p_2^{\nu}p_3^{\mu}+p_2^{\mu}p_3^{\nu}\Big\}$$



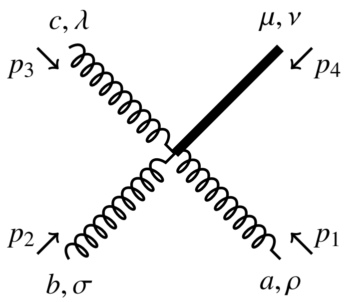
$$\frac{1}{2} i\kappa \,(2\pi)^4\, \delta^{(4)}\,(p_1+p_2+p_3)\,\delta_{ab}\delta_{rs}$$
$$\times\Big\{-\eta^{\mu\nu}(p_2\cdot p_3)+p_2^{\nu}p_3^{\mu}+p_2^{\mu}p_3^{\nu}\Big\}$$



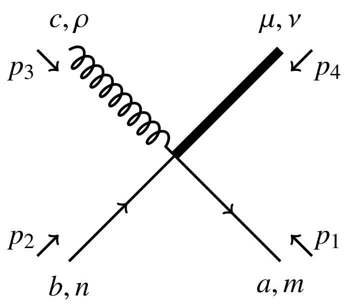
$$\frac{1}{2} i\kappa \,(2\pi)^4\, \delta^{(4)}\,(p_1+p_2+p_3)\,\delta_{ab}$$
$$\times\Big\{-\eta^{\mu\nu}(p_2\cdot p_3)+p_2^{\nu}p_3^{\mu}+p_2^{\mu}p_3^{\nu}\Big\}$$



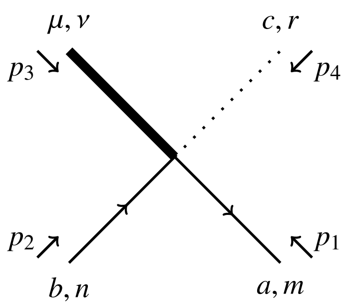
$$\frac{1}{2} g\,\kappa\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4)$$
$$\times f_{c,a,b}\Big\{\eta^{\rho\mu}p_1^{\nu}+\eta^{\rho\nu}p_1^{\mu}-\eta^{\mu\nu}p_1^{\rho}\Big\}$$



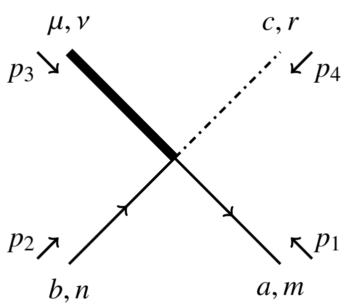
$$
\begin{aligned}
&\frac{1}{2} g\,\kappa\,(2\pi)^4\,\delta^{(4)}(p_1+p_2+p_3+p_4)\\
&\times f_{c,a,b}\Big\{-\eta^{\rho\nu}\eta^{\sigma\lambda}p_2^{\mu}+\eta^{\rho\nu}\eta^{\sigma\lambda}p_3^{\mu}-\eta^{\rho\nu}\eta^{\sigma\mu}p_1^{\lambda}+\eta^{\rho\nu}\eta^{\sigma\mu}p_2^{\lambda}\\
&\quad+(\eta^{\rho\mu}\eta^{\sigma\lambda}-\eta^{\rho\lambda}\eta^{\sigma\mu})p_3^{\nu}-\eta^{\rho\lambda}\eta^{\sigma\nu}p_3^{\mu}-\eta^{\rho\mu}\eta^{\sigma\nu}p_1^{\lambda}\\
&\quad+\eta^{\rho\mu}\eta^{\sigma\nu}p_2^{\lambda}+\eta^{\rho\nu}\eta^{\lambda\mu}p_1^{\sigma}-\eta^{\rho\nu}\eta^{\lambda\mu}p_3^{\sigma}-\eta^{\lambda\mu}\eta^{\sigma\nu}p_2^{\rho}\\
&\quad+\eta^{\lambda\mu}\eta^{\sigma\nu}p_3^{\rho}+(\eta^{\lambda\rho}\eta^{\sigma\mu}-\eta^{\lambda\mu}\eta^{\sigma\rho})p_1^{\nu}\\
&\quad+(\eta^{\lambda\mu}\eta^{\sigma\rho}-\eta^{\rho\mu}\eta^{\sigma\lambda})p_2^{\nu}+\eta^{\lambda\nu}\eta^{\sigma\rho}p_2^{\mu}\\
&\quad+\eta^{\rho\mu}\eta^{\lambda\nu}p_1^{\sigma}-\eta^{\rho\mu}\eta^{\lambda\nu}p_3^{\sigma}-\eta^{\lambda\nu}\eta^{\sigma\mu}p_2^{\rho}\\
&\quad+\eta^{\lambda\nu}\eta^{\sigma\mu}p_3^{\rho}+(\eta^{\lambda\rho}\eta^{\sigma\nu}-\eta^{\lambda\nu}\eta^{\sigma\rho})p_1^{\mu}\\
&\quad+\eta^{\mu\nu}\big[\eta^{\sigma\rho}(p_1^{\lambda}-p_2^{\lambda})+\eta^{\lambda\rho}(p_3^{\sigma}-p_1^{\sigma})+\eta^{\sigma\lambda}(p_2^{\rho}-p_3^{\rho})\big]\Big\}
\end{aligned}
$$



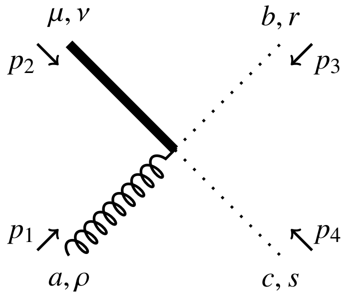
$$
\begin{aligned}
&-\frac{1}{4} g\,\kappa\,(2\pi)^4\,\delta^{(4)}(p_1+p_2+p_3+p_4)\\
&\times f_{c,a,b}\delta_{mn}\big[\eta^{\rho\mu}\gamma^{\nu}+\eta^{\rho\nu}\gamma^{\mu}-2\eta^{\mu\nu}\gamma^{\rho}\big]
\end{aligned}
$$



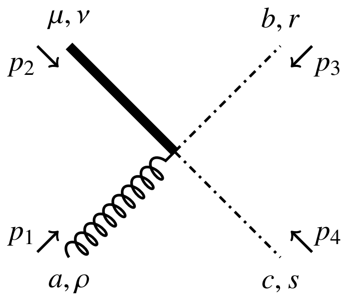
$$
\frac{i}{2}\, g\,\kappa\,(2\pi)^4\,\delta^{(4)}(p_1+p_2+p_3+p_4)\, f_{c,a,b}\eta^{\mu\nu}\alpha^{r}_{mn}
$$



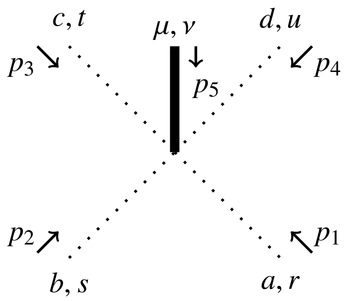
$$
-\frac{i}{2}\, g\,\kappa\,(2\pi)^4\,\delta^{(4)}(p_1+p_2+p_3+p_4)\, f_{c,a,b}\eta^{\mu\nu}\beta^{r}_{mn}\gamma_5
$$



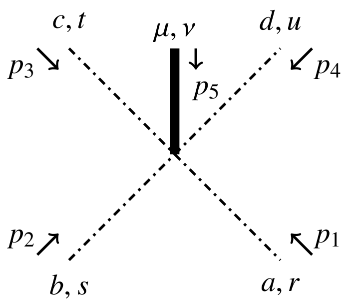
$$\frac{1}{2}\, g\, \kappa\, (2\pi)^4\, \delta^{(4)}\,(p_1 + p_2 + p_3 + p_4)\, f_{b,a,c}\delta_{rs}$$
$$\times \{\eta^{\rho\mu}(p_4^\nu - p_3^\nu) + \eta^{\rho\nu}(p_4^\mu - p_3^\mu) + \eta^{\mu\nu}(p_3^\rho - p_4^\rho)\}$$



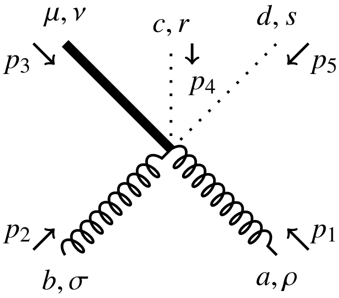
$$\frac{1}{2}\, g\, \kappa\, (2\pi)^4\, \delta^{(4)}\,(p_1 + p_2 + p_3 + p_4)\, f_{b,a,c}\delta_{rs}$$
$$\times \{\eta^{\rho\mu}(p_4^\nu - p_3^\nu) + \eta^{\rho\nu}(p_4^\mu - p_3^\mu) + \eta^{\mu\nu}(p_3^\rho - p_4^\rho)\}$$



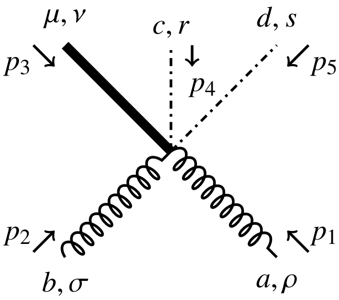
$$\frac{i}{2} g^2\, \kappa\, (2\pi)^4\, \delta^{(4)}\,(p_1 + p_2 + p_3 + p_4 + p_5)$$
$$\times\, \eta^{\mu\nu}\Big\{ (\delta_{ru}\delta_{st} - \delta_{rt}\delta_{su})\, f^{ab,cd} + (\delta_{ru}\delta_{st} - \delta_{rs}\delta_{tu})\, f^{ac,bd}$$
$$+ (\delta_{rt}\delta_{su} - \delta_{rs}\delta_{tu})\, f^{ad,bc}\Big\}$$



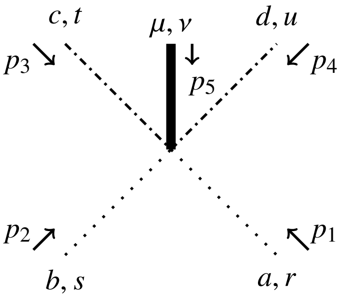
$$\frac{i}{2} g^2\, \kappa\, (2\pi)^4\, \delta^{(4)}\,(p_1 + p_2 + p_3 + p_4 + p_5)$$
$$\times\, \eta^{\mu\nu}\Big\{ (\delta_{ru}\delta_{st} - \delta_{rt}\delta_{su})\, f^{ab,cd} + (\delta_{ru}\delta_{st} - \delta_{rs}\delta_{tu})\, f^{ac,bd}$$
$$+ (\delta_{rt}\delta_{su} - \delta_{rs}\delta_{tu})\, f^{ad,bc}\Big\}$$



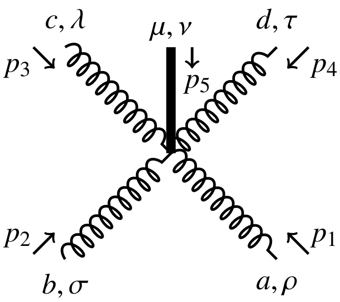
$$-\frac{i}{2}g^2\,\kappa\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4+p_5)$$
$$\times\,\delta_{rs}(f^{ad,bc}+f^{ac,bd})\Big\{\eta^{\rho\nu}\eta^{\sigma\mu}+\eta^{\rho\mu}\eta^{\sigma\nu}-\eta^{\rho\sigma}\eta^{\mu\nu}\Big\}$$



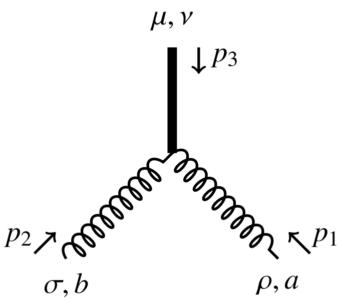
$$-\frac{i}{2}g^2\,\kappa\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4+p_5)$$
$$\times\,\delta_{rs}(f^{ad,bc}+f^{ac,bd})\Big\{\eta^{\rho\nu}\eta^{\sigma\mu}+\eta^{\rho\mu}\eta^{\sigma\nu}-\eta^{\rho\sigma}\eta^{\mu\nu}\Big\}$$



$$-\frac{i}{2}g^2\,\kappa\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4+p_5)$$
$$\times\,\eta^{\mu\nu}\delta_{rs}\delta_{tu}\{f^{ac,bd}+f^{ad,bc}\}$$



$$\frac{i}{2}g^2\,\kappa\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4+p_5)$$
$$\times\,\{f^{ac,bd}[-\eta^{\rho\tau}\eta^{\sigma\nu}\eta^{\lambda\mu}+\eta^{\rho\sigma}\eta^{\tau\nu}\eta^{\lambda\mu}-\eta^{\rho\tau}\eta^{\sigma\mu}\eta^{\lambda\nu}$$
$$+\,\eta^{\rho\sigma}\eta^{\lambda\nu}\eta^{\tau\nu}+\eta^{\rho\nu}(\eta^{\sigma\mu}\eta^{\lambda\tau}-\eta^{\sigma\lambda}\eta^{\tau\mu})$$
$$+\,\eta^{\rho\mu}(\eta^{\sigma\nu}\eta^{\lambda\tau}-\eta^{\sigma\lambda}\eta^{\tau\nu})+\eta^{\mu\nu}(\eta^{\rho\tau}\eta^{\sigma\lambda}-\eta^{\rho\sigma}\eta^{\lambda\tau})]$$
$$+\,c\Leftrightarrow d,\lambda\Leftrightarrow\tau+c\Leftrightarrow b,\lambda\Leftrightarrow\sigma\}$$



$$
\begin{aligned}
& -\frac{i}{2}\,\kappa\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3) \\
& \times\,\delta_{ab}\times\{[\eta^{\mu\sigma}\eta^{\nu\rho}+\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\nu}\eta^{\sigma\rho}]p_2.p_1+\eta^{\mu\nu}\,p_1^{\rho}p_2^{\sigma}- \\
& \qquad(\eta^{\mu\rho}\,p_1^{\nu}p_2^{\sigma}+\eta^{\mu\sigma}\,p_1^{\rho}p_2^{\nu}-\eta^{\rho\sigma}\,p_1^{\mu}p_2^{\nu}+\eta^{\nu\rho}\,p_1^{\mu}p_2^{\sigma}+\eta^{\nu\sigma}\,p_1^{\rho}p_2^{\mu} \\
& \qquad-\eta^{\rho\sigma}\,p_1^{\nu}p_2^{\mu})-\eta^{\mu\nu}(p_1^{\sigma}\,p_1^{\rho}+p_2^{\sigma}\,p_2^{\rho}+p_1^{\sigma}\,p_2^{\rho}) \\
& \qquad-(\eta^{\nu\rho}\,p_1^{\mu}p_2^{\sigma}+\eta^{\nu\sigma}\,p_1^{\mu}p_2^{\rho}+\eta^{\mu\rho}\,p_1^{\nu}p_2^{\sigma}+\eta^{\mu\sigma}\,p_1^{\nu}p_2^{\rho})\}
\end{aligned}
$$

In addition to these rules, we have to keep in mind the following points:

- For any Feynman diagram, the symmetry factor needs to be multiplied appropriately. The symmetry factor is defined as the number of ways one can obtain the topological configuration of the Feynman diagram under consideration.
- For each loop momenta, the integration over the loop momenta, k, needs to be performed with the integration measure $d^d k/(2\pi)^d$ in d-dimensions (in dimensional regularisation).
- For each quark/Majorana fermion/ghost loop, one has to multiply a factor of (-1) which we implemented via a private code written by a member of our group.

1.3 Perturbative S-matrix

In a general QFT in principle the theoretical predictions of any physical observables can be made through the computations of various S-matrix elements order by order in perturbation theory. As mentioned earlier we are following the feynman diagrammatic method to evaluate the S-matrix elements and this expansion is represented through the direct evaluation of Feynman diagrams. Using the Feynman rules presented in Sec. 1.2.1, we can evaluate all the Feynman diagrams order by order in the perturbation theory.

S-matrix calculations beyond leading order: Beyond the leading order (LO) the massless gauge theory amplitudes are plagued by short distance or ultraviolet (UV) and long distance and collinear singularities ie. Infrared (IR). The singularities can arise from virtual and real emission diagrams contributing beyond LO .To accomplish the corresponding cross sections finite, the determination of these singularities are essential.

The UV divergences arise from very high momentum region of the Feynman integrals and through a systematic renormalisation procedure one can get rid of these divergences. For being UV conformal (beta functions zero for all order in perturbation theory) the onshell scattering amplitudes in $\mathcal{N} = 4$ SYM are free of UV divergences. But nevertheless the Form Factors consisting of non-protected[4] composite operator are UV divergent and required a renormalisation to get a sensible result. Before performing the UV renormalisation, we need to regulate the Feynman integrals which is required to understand the nature of divergences. There are several popular ways to regulate the integrals. The most consistent and widely used prescription for our current interest is the framework of dimensional regularisation introduced in [27–29]. Within the framework, we need to regulate the integrals in general d-dimensions which is taken as $4 + \epsilon$ in this thesis. Upon performing the integrals, all the UV singularities appear as poles in ϵ. The UV renormalisation, which is performed through the redefinition of all of the physical quantities present in the Lagrangian and adding counterterms to absorb these poles to get a UV finite result. The UV renormalisation is done at an energy scale, known as renormalisation scale, μ_R. On the other hand, the soft divergences arise from the very low momentum region of the loop and phase space integrals and the collinear ones arise when any loop or phase space momentum becomes collinear to any of the external massless particles. The collinear divergence appears only in the QFT with massless particle vertices . Hence, even after performing the UV renormalisation, the resulting expressions obtained through the evaluation the loop and phase space integrals are not finite and requires additional technicalities to deal with this.

Removing the IR singularities through Mass factorisation : For removing the IR divergences one has to combine the contributions arising from the real emission diagrams at same perturbative order. The IR contains soft as well as collinear divergences coming from the phase space integral. Once we add the virtual and real emission diagrams and evaluate the phase space integrals, the resulting expressions become free from soft and final state collinear singularities by Kinoshita-Lee-Nauenberg (KLN) [3, 4] theorem. An analogous result for quantum electrodynamics with massive fermion is known as Bloch-Nordsieck [30, 31] cancellation. However, the collinear singularities arising from the initial state configurations remain uncancelled and removed at the hadronic level through the mass factorisation techniques. Where the uncancelled singularities are absorbed into

[4]Protected operators follow the Noether current conservaton law and satisfy a corresponding ward indentity. They do not require overall UV renormalisation.

the bare parton distribution functions (PDF). So, the observables at the hadronic level are finite when it is compared with the experimental findings. Mass factorisation is also done at some energy scale, called factorisation scale, μ_F. The dependence of the fixed order results on the unphysical scales like μ_R and μ_F is an artifact of the truncation of the perturbative series to a finite order. The philosophy of the *mass factorisation* technique is quite similar with the UV one and sometimes it is called as collinear counterterm in the context of differential cross sections.

Although $\mathcal{N} = 4$ SYM is an UV conformal theory, nevertheless it suffers from the divergences coming from both soft and collinear origin and the prescriptions for removing these divergences at the cross-section level are similar with QCD. However the presence of explicit IR divergences in scattering amplitudes in $\mathcal{N} = 4$ introduce a regularisation scale and break the conformal invariance of the theory.

The core part of this thesis deals with the computation of IR sensitive observables in $\mathcal{N} = 4$ SYM and comparison of the former with QCD observables. Also there is rigorous dicussion about the transcendental structure of soft and collinear radiations and general class of FF in $\mathcal{N} = 4$.

More specifically, the thesis contains

- Second order splitting functions and infrared safe cross sections in $\mathcal{N} = 4$ SYM theory [5],
- Form factors with two-operator insertion and the transcendentality principles [17].
- Violations of maximal trancendentality principle in the context of stress tensor operator [32].

In the subsequent subsections, we will discuss the above things in brief.

1.3.1 Second order splitting functions and infrared safe cross sections in $\mathcal{N} = 4$ SYM theory

Field theoretic results from Quantum Chromodynamics (QCD) play an important role in understanding the physics of strong interactions. Inclusive and differential cross sections computed using perturbative QCD not only helped to discover several of elementary particles of the Standard Model (SM) but also provided a laboratory to understand the field theoretical structure of non-abelian gauge theories. For example, both theoretical and experimental results from high energetic collision processes, such as the deep-inelastic scattering and the Drell-Yan production provides the complete knowledge of the internal structure as well as the dynamics of hadrons in terms of their constituents such as quarks and gluons. Scattering cross sections computed in high energetic collision processes such as the Drell-Yan [33] and the deep-inelastic scattering processes can be expressed in terms of perturbatively computed partonic cross section, convoluted with the parton distribution functions (PDFs). The partons refer to quarks and gluons and the PDFs describe the non perturbative probabilities of finding the partons in a bound state. These scattering cross sections at high energies can be expressed in terms of the perturbatively calculable scatterings involving constituents of hadrons properly convoluted with parton distribution functions. These constituents at high energies are light quarks and gluons often called partons and the corresponding PDFs describe their probabilities to exist in the hadron. Such a description of hadronic cross section goes by the name parton model. $\mathcal{N} = 4$ SYM is an UV renormalizable quantum field theory in four dimensional space. In addition to having SU(N) gauge symmetry, $\mathcal{N} = 4$ SYM theory enjoys supersymmetry and conformal symmetry that make it fascinating to study. Although the study of cross sections in such a theory has no phenomenological implications, yet they can help us to understand the factorization properties of the IR singularities, the latter being useful to extract the Alatrelli-Parisi(AP) [2] splitting kernels at each order in the perturbation theory. Undoubtedly, higher order computation of the FFs and the amplitudes unravel the IR structure of the $\mathcal{N} = 4$ SYM theory in an elegant way. However purely real emission processes, which appear in cross sections, can also give important informations about the nature of soft and collinear emissions. In QCD, the gluons in a virtual loop can become soft and contribute to poles in ϵ in a dimensionally regulated theory, similar situation also happens when gluons in a real emission process carry a small fraction of the momentum of the incoming particles. More precisely, when we perform the phase space integrations for such real emission processes, we encounter poles in ϵ, at every order in perturbation series. These soft contributions from real and virtual diagrams cancel order by order when they are added together, thanks to the Kinoshita-Lee-Nauenberg (KLN) theorem [3,4]. In addition, the real emissions of gluons and quarks are sensitive to collinear singularities; while the final state divergences are taken care by the KLN theorem, the initial state counterparts are removed by mass factorization. Similar scattering of massless gluons, quarks, scalars and pseudo-scalars in $\mathcal{N} = 4$ SYM theory can be studied within a supersymmetric preserving $\overline{DR}$ [5] regularisation scheme which protects the supersymmetry throughout. The cancellation of soft singularities and factorisation of collinear singularities in the scattering cross sections will also provide wealth of information on the IR structure of $\mathcal{N} = 4$ gauge theory. One can investigate the soft plus virtual part of these finite cross sections after mass factorisation in terms of universal cusp and collinear anomalous dimensions. Also, the factorisation of initial state collinear singularities provides valuable information about the AP splitting functions in $\mathcal{N}$=4 SYM theory. Understanding such cross sections in the light of well known results in QCD will help us to investigate the resummation of soft gluon contributions to all orders in perturbation theory in a process independent manner.

We focus on computing Spitting functions and try to understand the universal factorisation properties of Soft-Virtual(SV) cross-sections for certain composite operators (BPS and Stress tensor) upto NNLO and then to predict the NNNLO SV cross-sctions using the known three loop anamolous dimentions $\mathcal{N}$= 4 SYM. We studied the universal decomposition of the Spitting functions with respect to the anomolous dimentions of the theory at SV limit. Also we investigated the transcendental stucture of the soft gluon and scalar radiation in $\mathcal{N} = 4$. Our findings are consistent with the previous results available in the literature related to the topic.

1.3.2 Form factors with two-operator insertion and the transcendentality principles

Understanding the analytical structures of on-shell amplitudes and FFs in $\mathcal{N} = 4$ SYM has been an active area of investigation, not only to uncover the hidden structures of these quantities but also to establish the connections with other gauge theories, such as QCD. One of the most intriguing facts is the appearance of uniform transcendental (UT) weight terms in certain class of quantities in $\mathcal{N} = 4$ SYM. The two-point or Sudakov FFs of primary half-BPS operator belonging to the stress-energy supermultiplet is observed [6–8] to exhibit the UT property to three-loops, more

[5] In this scheme, the number of generations of scalar and pseudoscalar fields are such that the resulting bosonic degrees of freedom is same as that of fermions, preserving the supersymmetry. Since the gauge fields have $2 + \epsilon$ degrees of freedom, there are $3 - \epsilon/2$ scalars and $3 - \epsilon/2$ pseudoscalars in the regularised version of the theory so that the total number of bosonic degrees of freedom in d dimensions is same as in four dimensions, namely 8. Further details can be found in [24, 25]

specifically, they are composed of only highest transcendental (HT) terms with weight 2L at loop order L. This is a consequence of the existence of an integral representation of the FFs with every Feynman integral as UT [8]. Knowing the existence of such a basis has profound implications in choosing the basis of integrals while evaluating Feynman integrals using differential equations method [9]. Also one can gain a clear understanding on the transcendental weight contributions to the scattering amplitude from the different type of singularity (UV and IR). The three-point FFs of half-BPS operator is also found to respect this wonderful UT property [10]. On the contrary, this property fails for the two- [11, 12] and three-point [13] FFs of the unprotected Konishi operator which are investigated up to three- and two-loops, respectively. Three-point FFs of a Konishi descendant operator is also found not to exhibit the UT property [14]. Having seen the beautiful property of UT in FFs of one-operator insertion, the question arises whether it is respected for the two-point FFs with SUSY protected two-operator insertion and whether this property can be extrapolated to generalised FFs with n-number of operators insertion. Also the connection between quantities in $\mathcal{N} = 4$ SYM and that of QCD is of fundamental importance. In addition to deepening our theoretical understanding, it is motivated from the fact that computing a quantity in QCD is much more difficult, and in the absence of our ability to calculate a quantity in QCD, if it is possible to obtain the result, at least partially, from that of simpler theory, such as $\mathcal{N} = 4$ SYM. In refs. [5, 15], it is found that the anomalous dimensions of leading twist-two-operator in $\mathcal{N} = 4$ SYM are identical to the HT counterparts in QCD [16], and consequently, the principle of maximal transcendentality (PMT) is conjectured. The PMT says that the algebraically most complex part of certain quantities in $\mathcal{N} = 4$ SYM and QCD are identical.

In [17], for the first time, we address this question in the context of general class of FF in $\mathcal{N} = 4$ and found that the UT property does not hold true at two-loop for the FF of double half-BPS insertion. Also we found the wonderful conjecture of PMT is not respected in the context of double half-BPS insertions in $\mathcal{N} = 4$ and its QCD counterpart. Nevertheless the UT and PMT property found to be restored at regge and collinear limit in the phase space region of the FF. We have discussed superiority of the Feynman diagrammatic approach over unitarity method in the context of the computation of double operator insertion.

1.3.3 Three point form factor and violation of trancendentality principle

On-shell scattering amplitudes and off-shell correlation functions are quantities of fundamental importance in any gauge theory. A generic quantum field theory is completely specified by the knowledge of these quantities. Form factors (FFs), the overlap of an n-particle on-shell state with a state created by the action of a local gauge invariant operator on the vacuum, are a fascinating bridge between aforementioned two quantities which have been studied extensively in quantum chromodynamics (QCD) and $\mathcal{N} = 4$ super Yang-Mills (SYM) theory:

$$\mathcal{F}_{O}(1, \ldots, n; q) \equiv \langle 1, \ldots, n|O(0)|0\rangle \,. \tag{1.3.1}$$

The numbers $1, \ldots, n$ denote the on-shell particles and $q^2 \neq 0$ is the off-shell momentum associated with the operator. The central objects which are considered in this article are two- and three-point FFs. Very recently, the first calculations of generalisations of such FFs to the case of two operator insertions with one non-protected operator was performed [34] by some of us.

It is conjectured in ref. [35] that at two-loop level the HT weight parts of every two-point minimal FF (number of fields present in the operator is same as number of external on-shell states) are identical and those are same as that of half-BPS operator belonging to the stress-energy supermultiplet [6]. In ref. [12], this conjecture is verified to three-loops level for the FF of unprotected Konishi operator. In ref. [36], the minimal FFs of long BPS operators (more than two fields) are computed to two-loops, and the corresponding HT piece is found to be a universal in all FFs with long, unprotected operators [35, 37–41].

It is also conjectured in [35], that the HT terms of two-loop remainder function of the three-point FF of every length-two operator should agree with the corresponding half-BPS remainder found in ref. [10]. The latter conjecture is falsified in ref. [13] where, for the first time, it is shown that for three-point FF of the Konishi operator (length-two), the HT part depends on the nature of external on-shell states; it fails to match with that of half-BPS if the external on-shell states are $\phi\lambda\lambda$.

In past few decades, people have been investigating the connection among quantities computed in different gauge theories. In particular, the connection between on-shell amplitudes or FFs of $\mathcal{N} = 4$ SYM and that of QCD are of fundamental importance. Besides theoretical understanding, this is motivated from the fact that calculating any quantity in QCD is much more complex and in absence of our ability to calculate a quantity in QCD, whether it is possible to get the result, at least partially, from some other simpler theory. In refs. [15, 42, 43], the connection between anomalous dimensions of leading twist-two operators of these theories is found and it is shown that the results in $\mathcal{N} = 4$ SYM is related to the HT part of the QCD results and consequently, the principle of maximal transcendentality (PMT) is conjectured. Same conclusion is obtained by some of us in [5] through a different procedure based on momentum fraction space. At this level, this connection involves only pure number. This property is later examined in the context of two-point FFs in [8] to three-loops level and surprisingly, found to hold true: the HT pieces of quark (vector interaction) and gluon (scale interaction) FFs in QCD [44] are identical to scalar FFs of half-BPS operator in $\mathcal{N} = 4$ SYM upon changing the representation of fermions in QCD from fundamental to adjoint. The same behaviour is also found for the quark and gluon two-point FFs [45, 46] associated to tensorial interaction through energy-momentum tensor. In refs. [10, 38–40, 47], the same behaviour is found to replicate for three-point scalar and pseudo-scalar FFs. This is the first scenario where non-trivial kinematics is involved and the validity of this principle implies this not only holds true for pure numbers but also for functions containing non-trivial kinematic dependence. Even after including the dimension seven operators in the effective theory of Higgs boson in the Standard model, PMT is also found to hold true [39, 48, 49] for three-point FFs. Using this principle, the four-loop collinear anomalous dimension in planar $\mathcal{N} = 4$ SYM is determined in ref. [50]. The complete domain of validity of this principle is still not fully clear and it is under active investigation. For on-shell scattering amplitudes, it breaks down even at one loop [51] for cases with four or five external gluons.

In the article [32], we consider Energy-momentum tensor (T) operator eq. 1.2.10 in $\mathcal{N} = 4$ is considered in this article, has been studied extensively in the context of three-point FFs in QCD [52, 53], and is found to behave like the half-BPS at one- as well as two-loop order and it is independent of the associated external on-shell states. This is in accordance with our classical expectation: since the stress-tensor is protected and lies in the same multiplet as the half-BPS, an exact SUSY Ward identity would relate these two FFs. Quite surprisingly, the process with three external partons violates the PMT even at one-loop while comparing the corresponding quantities in QCD! This is observed for the first time at the level of three-point FF.

2 Second order splitting functions and infrared safe cross sections in N = 4 SYM theory

The materials presented in this chapter are the result of an original research done in collaboration with Pulak Banerjee, Prasanna K. Dhani, V. Ravindran and Satyajit Seth, an these are based on the published article [5].

Contents

2.1 Prologue

The state-of-art results from Quantum Chromodynamics (QCD) play a pivotal role in understanding the physics of strong interactions. Inclusive and differential cross sections computed using perturbative QCD not only helped to discover several of elementary particles of the Standard Model

(SM) but also provided a laboratory to understand the field theoretical structure of non-abelian gauge theories. For example, both theoretical and experimental results from high energetic collision processes, such as the deep-inelastic scattering and the Drell-Yan [33] production provides the complete knowledge of the internal structure as well as the dynamics of hadrons in terms of their constituents such as quarks and gluons. These scattering cross sections at high energies can be expressed in terms of the perturbatively calculable scatterings involving constituents of hadrons properly convoluted with parton distribution functions (PDF).These constituents at high energies are light quarks and gluons often called partons and the corresponding PDFs describe their probabilities to exist in the hadron. Such a description of hadronic cross section goes by the name parton model. While the scattering of partons are calculable order by order in perturbative QCD (pQCD), the non-perturbative PDFs depend on the long distance part of the hadronic cross section and hence are process independent and can be computed only by non-perturbative techniques. The PDFs can be in principle computable using non-perturbative techniques. However, due to the complexity involved in the computations, they are fitted from the data from various high energy scattering experiments. However, the evolution of PDFs as functions of energy scale is controlled by pQCD through Altarelli-Parisi (AP) [2] splitting functions. The study of AP splitting functions at different orders give a wealth of informations about the structure infrared (IR) divergences.

UV and IR singularities appear in the Feynman diagrams through loop and phase space integrals at the intermediate stages of computations of hadronic observables such as cross sections and decay rates involving hadrons. In particular, they show up when the parton level cross sections are computed beyond the leading order in perturbation theory. Computation of partonic cross sections beyond the leading order (LO) in pQCD introduces these divergences and the origin of these singularities is due to virtual loop and external state phase space integrations. The UV divergences arise due to the high energy modes of virtual particles in the loop while the IR divergences such as soft and collinear ones, come from gluons and light quarks respectively. In the calulations of inclusive and differential cross sections the soft divergences cancel among themselves between real emission and virtual diagrams at every order in perturbation theory, the collinear divergences from degenerate final states again cancel among themselves when they are appropriately summed. Hence, for scatterings or decays where quarks and/or gluons are absent in the initial state the resultant observables are infrared safe. If the incoming states contain quarks and/or gluons, there will be uncancelled initial state collinear singularities. Thanks to the existence of bound states and the factorisation properties of the initial state collinear singularities, one can remove these singularities by appropriately redefining the PDFs. In other words, collinear unsafe parton level cross sections resulting from scatterings of initial light partonic states can be factorised into process independent kernels and collinear finite coefficient functions order by order in pQCD. The kernels satisfy renormalisation group equations controlled by AP splitting functions [2], which are known exactly up to third order in perturbation series [2, 16, 54–64]; the four loop counterparts in planar and large n_f (number of flavours) limit were calculated in [65, 66]. Thus in QCD, the nature of UV and IR divergences and their cancellation at cross section level have been studied in details and is quite well understood.

It is very natural to generalise this concept in a general non abelian gauge theory context, particularly in *supersymmetric* gauge theories and the knowledge of UV and IR singularities in QCD can guide us to investigate the divergence structure arising in different quantum field theoretic context. $\mathcal{N} = 4$ SYM, the most interesting candidate among the supersymmetric theories has a number of beautiful properties, such as UV conformal invaiance at quantuam level, internal cancellation of quadratic divergencies arising from scalar particle interaction, several non-renormalization theorems and many more. Also motivated from QCD it is very interesting to study the factorisation [67] properties of Soft-Virtual (SV) cross-sections in the context of $\mathcal{N} = 4$ SYM. Although the study of cross sections in such a theory has no phenomenological implications, yet they can help us to understand the factorization properties of the IR singularities, the latter being useful to extract the AP kernels at each order in the perturbation theory.

One of the goals in this article is to compute the AP splitting functions up to two-loop order in the perturbation series from explicit calculation of certain inclusive cross sections in $\mathcal{N} = 4$ SYM theory.

The most widely studied quantities in $\mathcal{N} = 4$ SYM theory are the on-shell amplitudes. Owing to the supersymmetric Ward identities [68], the tree level on-shell identical-helicity amplitudes vanish [69]. In addition to unbroken superconformal symmetry at the quantum level it is conjectured via the AdS/CFT [70] correspondence that $\mathcal{N} = 4$ SYM to be dual to String theory in AdS space relating a conformal field theory at strong coupling with a supergravity theory at weak coupling. As a consequence the all order scattering amplitude in N = 4 SYM must have simple and compact structures. In addition these on-shell amplitudes satisfy the Anti-de-Sitter/conformal field theory (AdS/CFT) conjecture [70] which relates the $\mathcal{N}$ =4 SYM theory in four dimensions and gravity theory in five-dimensional anti-de Sitter(AdS) space. The property of supersymmetric amplitudes has been extensively studied in the works [71–74]. The factorization property of the finite terms for n-point m-loop amplitudes in terms of one-loop counterparts was shown in the article [75]. However this factorization property fails beyond two-loop five-point maximally helicity violating (MHV) amplitudes [76, 77].

Like on-shell amplitudes, form factors (FFs) of composite operators also contribute to the scattering cross sections and provide important information about the IR structure of the gauge theories. The FFs are defined as the matrix elements of the composite operator between an off-shell initial state and on-shell final states. The most widely studied composite operator in $\mathcal{N} = 4$ SYM theory is the half-BPS operator, whose UV anomalous dimensions vanish to all orders in perturbation theory [78–82]. As a result the FFs of this composite operator look relatively simple. The first computation of a two-point FF up to two-loop order for the half-BPS operator was done by van Neerven [6]. The three loop computation was done in [8] where the authors have shown an interesting connection between their results and the corresponding ones in non-supersymmetric SU(N) gauge theory containing n_f fermions, with the following replacement of the color factors: $C_A = C_F = n_f = N$, where C_A, C_F are the Casimir for the adjoint, fundamental representations respectively. Study of FFs of composite operators also shed light on the ADS/CFT correspondence, see [7, 8, 10, 83–85] for details. Over the past few years calculation of FFs for non-BPS type composite operators, such as the Konishi [86] have also gained interest. However this operator is non-protected and hence develop UV anomalous dimensions at each order in perturbation theory. In this regard, study of the FFs of the Konishi operator in $\mathcal{N} = 4$ SYM theory helps to understand the IR structure in a more general way. For computation of one-loop two-point, two-loop two-point and one-loop three-point FFs see [11]. In [12], some of the authors of the present paper have presented the three-loop two-point FF for the Konishi operator and also predicted up to $1/\epsilon$ pole at four loops in d $(= 4 + \epsilon)$ dimensions. The two-loop three-point FF and their finite remainders for the half-BPS [10] and the Konishi operator were recently calculated in [13]. Several other results on n-point FFs of the Konishi operator are now available, see [35, 37, 87, 88] for details.

The FFs of composite operators as well as the on-shell amplitudes offer a wide scope to investigate the IR structure of quantum field theory. In QCD, the two-point FFs satisfy the K+G equation [89–92] and the IR structure of these quantities are already well understood. The universal nature of IR singularities for a n-point QCD amplitude up to two-loop order was predicted by Catani in [93]. It was then realized in [94] that the above predictions are a consequence of the underlying factorization and resummation properties of the QCD amplitudes. Later on the generalisation of the results in [93] and [94] and [95, 96] in SU(N) gauge theory, at any loop order, having n_f light flavours in terms of cusp, collinear and soft anomalous dimensions was conjectured in [97] and independently in [98]. All these studies have helped to understand the iterative structure of IR divergences which subsequently lead to the program of resummation of observables, the latter being an important area of study at the energies of the hadron colliders.

Undoubtedly, higher order computation of the FFs and the amplitudes unravel the IR structure of the $\mathcal{N} = 4$ SYM theory in an elegant way. However purely real emission processes, which appear in cross sections, can also give important informations about the nature of soft and collinear emissions. In QCD, the gluons in a virtual loop can become soft and contribute to poles in ϵ in a dimensionally regulated theory, similar situation also happens when gluons in a real emission process carry a small fraction of the momentum of the incoming particles. More precisely, when we perform the phase space integrations for such real emission processes, we encounter poles in ϵ, at every order in perturbation series. These soft contributions from real and virtual diagrams cancel order by order when they are added together, thanks to the Kinoshita-Lee-Nauenberg (KLN) theorem [3, 4]. In addition, the real emissions of gluons and quarks are sensitive to collinear singularities; while the final state divergences are taken care by the KLN theorem, the initial state counterparts are removed by mass factorization. Similar scattering of massless gluons, quarks, scalars and pseudo-scalars in $\mathcal{N} = 4$ SYM theory can be studied within a supersymmetric preserving regularised scheme. The cancellation of soft singularities and factorisation of collinear singularities in the scattering cross sections will also provide wealth of information on the IR structure of $\mathcal{N} = 4$ gauge theory. One can investigate the soft plus virtual part of these finite cross sections after mass factorisation in terms of universal cusp and collinear anomalous dimensions. Also, the factorisation of initial state collinear singularities provides valuable information about the AP splitting functions in $\mathcal{N}$=4 SYM theory. Understanding such cross sections in the light of well known results in QCD will help us to investigate the resummation of soft gluon contributions to all orders in perturbation theory in a process independent manner. In other words, $\mathcal{N}$=4 SYM theory offers an easier framework to appreciate IR structure of not only on-shell amplitudes but also scattering cross sections. Such an exercise helps us to appreciate better the underlying principles of quantum field theory. In this article, we make such an attempt to compute inclusive cross sections for the production of various singlet states through effective interactions of certain composite operators, namely the half-BPS, the Konishi and energy momentum (EM) tensor with fields of $\mathcal{N}$=4 SYM theory. In contrast to the half-BPS and the Konishi operator, the EM tensor couples universally to all the fields; thus the number of processes contributing becomes overwhelmingly large. We compute all the subprocesses contributing up to two-loop order in the perturbation theory and use them to extract the AP kernels up to the same order in perturbative expansion. We notice interesting aspects of the splitting functions, namely, presence of transcendental terms ranging $2l$ (l = loop order) to 0. We also compare the cross sections calculated in $\mathcal{N} = 4$ SYM theory to the standard model counterparts, namely Drell-Yan and Higgs boson productions and find interesting similarities and differences, which we shall elucidate in the later part of the paper in detail.

Sec. 2.2 contains the methodology to compute scattering cross sections using the regularised version of the Lagrangian. In Sec. 2.3, we present the results for the splitting functions and the coefficient functions up to two-loop level and discuss our findings in detail. Finally Sec. 2.4 is devoted to conclusions. Appendix contains the Mellin-j space results of AP splitting functions in a compact form.

2.1.1 Computation of splitting functions and IR finite cross sections

In this section, we describe how the inclusive cross sections for the production of singlet states corresponding to the operators O^I (the half-BPS or the Konishi or the stress tensor (T) operators introduced in subsec. 1.2.1, through the scattering of particles in $\mathcal{N} = 4$ SYM theory, can be used to obtain various splitting functions and infrared safe coefficient functions. The generic scattering process in $\mathcal{N} = 4$ SYM theory is given by

$$a(p_1) + b(p_2) \to I(q) + \sum_{i=1}^{m} X(l_i), \tag{2.1.1}$$

where $a, b \in \{\lambda, g, \phi, \chi\}$ can be a Majorana or gauge or scalar or pseudoscalar particles.

I represents a color singlet state denoted by half-BPS or Konishi or T with invariant mass given by $Q^2 = q^2$, where q is its four momentum.

X denotes the final inclusive state comprising of $\{\lambda, g, \phi, \chi\}$. In the above equation, the momenta of the corresponding particles are given inside their parenthesis with the invariant mass of the singlet state denoted by $Q^2 = q^2$. Except the singlet state all other particles are massless.

The cross section, $\hat{\sigma}^I_{ab}(\hat{s}, Q^2, \epsilon)$, for the scattering process in Eq. (3.2.2) in $4 + \epsilon$ dimensions is given by

$$\hat{\sigma}^I_{ab}(\hat{s}, Q^2, \epsilon) = \frac{1}{2\hat{s}} \int [dPS_{m+1}] \overline{\sum} |\mathcal{M}_{ab}|^2 \,{}^{1}, \tag{2.1.2}$$

where $\hat{s} = (p_1+p_2)^2$ is the partonic center of mass energy. The phase space integration, $\int [dPS_{m+1}]$, is given by

$$\int [dPS_{m+1}] = \int \prod_{i=1}^{m+1} \frac{d^n l_i}{(2\pi)^n} 2\pi\delta_+(l_i^2 - q_i^2)(2\pi)^n \delta^n\Big(\sum_{j=1}^{m+1} l_j - p_1 - p_2\Big), \tag{2.1.3}$$

with $l_{m+1} = q$, $q_i^2 = 0$ for $i = 1, \cdots m$ and $q_{m+1}^2 = Q^2$. The symbol $\overline{\sum}$ indicates sum of all the spin/polarization/generation and color of the final state particles X and the averaging over them for the initial state scattering particles a, b. $\mathcal{M}_{ab}$ is the amplitude for the scattering reaction depicted in Eq. (3.2.2). We follow the Feynman diagrammatic approach to compute these amplitudes.

[1] The number of the particle (m) at the external state phase space depend on the perterbative order under consideration.

The cross sections $\hat{\sigma}^I_{ab}$ can be expanded in powers of t'Hooft coupling constant 'a' defined by

$$a \equiv \frac{g^2 N}{16\pi^2} \exp[\frac{\epsilon}{2}(\gamma_E - \ln 4\pi)], \tag{2.1.4}$$

where N is the number of colors in SU(N) gauge theory and $\gamma_E = 0.5772 \cdots$, is the Euler-Mascheroni constant. Note that the spherical factor that appears at every order in the perturbation theory resulting from the loop and phase space integrals, is absorbed into the coupling constant. We compute the inclusive cross section order by order in perturbation theory as

$$\hat{\sigma}^I_{ab}(z, Q^2, \epsilon) = \sum_{i=0}^{\infty} a^i \hat{\sigma}^{I,(i)}_{ab}(z, Q^2, \epsilon), \tag{2.1.5}$$

where the scaling variable is defined by $z = Q^2/\hat{s}$. For the half-BPS and Konishi, at LO, only scalar and pseudoscalars contribute, but for the T, at LO, all the particles namely Majoranas, gluons, scalars and pseudoscalars contribute. At next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) level, there will be plethora of processes that will be available for study. At NLO, we need to evaluate the amplitudes involving purely virtual diagrams, called FFs and single real emission processes to the LO processes. For the NNLO, we need in addition the interference of processes with single real emission and one virtual loop with an emission.

Beyond LO, that is starting from order a, we encounter both virtual and real emission processes. The virtual diagrams involve loop integrals and real emission ones, the phase space integrals.

Beyond LO, evaluation of the Feynman diagrams involves performing the loop integrals for the FFs and the phase space integrals arising in the real emission processes. Both the loop and the phase space integrals are often divergent in four space-time dimensions due to the presence of UV and IR divergences, hence they need to be regulated. Dimensional regularization (DR) has been tremendously successful in regulating both UV as well as IR singularities on an equal footing, where all the singularities show up as poles in ϵ.

If we regulate these integrals and express all the matrix elements in d dimensions, the divergences show up as poles in ϵ.

There are several schemes of DR that exist. In the scheme proposed by 't Hooft and Veltman [99], called DREG scheme, the gauge bosons in the loops are treated in $4 + \epsilon$ dimensions with $2 + \epsilon$

helicity states but the external physical ones in 4 dimensions having 2 helicity states. In the CDR scheme proposed by Ellis and Sexton [100] one treats both the physical and unphysical gauge fields in $4 + \epsilon$ dimensions. There is yet another scheme, namely the four dimensional helicity (FDH) scheme by [101, 102] wherein both the physical and unphysical gauge fields are treated in 4 dimensions. In all these schemes the loop integrals are performed in $4 + \epsilon$ dimensions. FDH scheme has been the most popular one in supersymmetric theories.

In this paper, we choose to work with the modified dimensional reduction ($\overline{\mathrm{DR}}$) scheme [24, 25] which keeps the supersymmetry intact throughout of the computation. In this scheme, the number of generations of scalar and pseudoscalar fields are such that the resulting bosonic degrees of freedom is same as that of fermions, preserving the supersymmetry. Since the gauge fields have $2 + \epsilon$ degrees of freedom, there are $3 - \epsilon/2$ scalars and $3 - \epsilon/2$ pseudoscalars in the regularised version of the theory so that the total number of bosonic degrees of freedom in d dimensions is same as in four dimensions, namely 8. It was shown in [12] that this scheme has advantage over the others as it can be used even for operators that depend on space-time dimensions. An example of such an operator is the Konishi operator (see Eq. (1.2.9)). In [12], three-loop FFs of the Konishi operator was computed in $\overline{\mathrm{DR}}$ scheme which correctly reproduces its anomalous dimensions up to the same level.

In the $\overline{\mathrm{DR}}$ scheme, in addition to analytically continuing the loop integrals of virtual amplitudes and phase space integrals of real emission processes to d space-time dimensions, all the traces of Dirac gamma matrices, flavour matrices α and β, and various flavour sums/averages for the Majorana, scalar, pseudoscalar particles and polarisation sums/averages for the gauge fields are done in d dimensions.

The renormalisation of the fields and couplings are done with the help of renormalisation constants. Due to supersymmetry, the coupling constant g does not require any renormalization, the beta function of the coupling is zero to all orders in the perturbation theory [22,23]. Hence $\frac{\hat{a}}{\mu^\epsilon} = \frac{a}{\mu_R^\epsilon}$, where renormalization scale is denoted by μ_R and an arbitrary scale μ is introduced to keep the coupling dimensionless in d dimensions. In addition, the amplitudes involving protected operators such as the half-BPS and the space-time conserved operator like T do not require overall renormalisation constant. Since the Konishi operator is not protected by supersymmetric ward identity, we need to perform an overall renormalisation order by order in perturbation theory. The corresponding renormalization constant $Z^{\mathcal{K}}(a(\mu_R), \epsilon)$, satisfies the following renormalization group equation (RGE):

$$\frac{d \ln Z^{\mathcal{K}}}{d \ln \mu_R^2} = \gamma^{\mathcal{K}} = \sum_{i=1}^{\infty} a^i \gamma_i^{\mathcal{K}} . \tag{2.1.6}$$

The solution to the above equation is

$$Z^{\mathcal{K}} = \exp\left(\sum_{n=1}^{\infty} a^n \frac{2\gamma_n^{\mathcal{K}}}{n\epsilon} \right). \tag{2.1.7}$$

Here γ^K is the anomalous dimension whose value up to two-loop was computed in [73, 103, 104] while the three-loop results are available in [12, 43, 105]

The real emission processes start contributing from NLO, where any one of the particles $\in \{\lambda, g, \phi, \chi\}$ can be emitted ($m = 1$ in Eq. (3.2.2)). Note that at NNLO level, there will be two classes of real emission processes, namely amplitudes with double real emissions ($m = 2$ in Eq. (3.2.2)) and those with the interference of one real and one virtual associated with a radiation. The UV finite virtual amplitudes involving half-BPS, T and Konishi are sensitive to IR singularities. The massless gluons and scalar can give soft singularities and the massless states in virtual loops can become parallel to one another, giving rise to collinear singularities. The soft singularities from the virtual diagrams cancel against the those from the real emission processes, thanks to the Kinoshita-Lee-Nauenberg (KLN) theorem [3, 4]. Similarly, the final state collinear singularities cancel among themselves in these inclusive cross sections leaving only initial state collinear singularities. The soft and collinear singularities from the virtual diagrams cancel against the soft and final state collinear divergences from the real emission processes, thanks to the KLN theorem [3, 4]. Since the initial degenerate states are not summed in the scattering cross sections, collinear divergences originating from incoming states remain as poles in ϵ. Hence, like in QCD, the inclusive cross sections in $\mathcal{N} = 4$ SYM theory, are singular in four dimensions. Following perturbative QCD [106], these singular cross sections can be shown to factorize at the factorization scale μ_F:

$$\hat{\Delta}^I_{ab}\left(z, Q^2, \frac{1}{\epsilon}\right) = \left(\prod_{i=1}^{3}\int_0^1 dx_i\right)\delta\left(z - \prod_{i=1}^{3} x_i\right) \sum_{c,d} \Gamma_{ca}\left(x_1, \mu_F^2, \frac{1}{\epsilon}\right)$$
$$\times\Gamma_{db}\left(x_2, \mu_F^2, \frac{1}{\epsilon}\right) \Delta^I_{cd}\left(x_3, Q^2, \mu_F^2, \epsilon\right), \tag{2.1.8}$$

where the sum extends over the particle content $\{\lambda, g, \phi, \chi\}$. In the above expression $\hat{\Delta}^I_{ab}(z, Q^2, 1/\epsilon) = \hat{\sigma}^I_{ab}(z, Q^2, \epsilon)/z$; the corresponding one after factorisation is denoted by Δ^I_{ab}. If this is indeed the case, then we should be able to obtain Γ_{ab} order by order in perturbation theory from the collinear singular $\hat{\Delta}^I_{ab}$ by demanding Δ^I_{ab} is finite as $\epsilon \to 0$. The fact that the $\hat{\Delta}^I_{ab}$ are independent of the scale μ_F leads the following RGE:

$$\mu_F^2 \frac{d}{d\mu_F^2}\Gamma(x, \mu_F^2, \epsilon) = \frac{1}{2}P(x) \otimes \Gamma\left(x, \mu_F^2, \epsilon\right), \tag{2.1.9}$$

where the function $P(x)$ is matrix valued and their elements $P_{ab}(x)$ are finite as $\epsilon \to 0$ and they are called splitting functions. This is similar to Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation [2, 107–109] in QCD for the parton distribution functions. In the $\overline{\mathrm{DR}}$ scheme, the solution to the RGE in terms of the splitting functions, the latter expanded in a as,

$$P_{ca}(x) = \sum_{i=1}^{\infty} a^i P^{(i-1)}_{ca}(x), \tag{2.1.10}$$

can be found to be

$$\Gamma_{ca}\left(x, \mu_F^2, \frac{1}{\epsilon}\right) = \sum_{k=0}^{\infty} a^k \Gamma^{(k)}_{ca}\left(x, \mu_F^2, \frac{1}{\epsilon}\right),$$
$$\Gamma^{(0)}_{ca} = \delta_{ca}\delta(1-x),$$
$$\Gamma^{(1)}_{ca} = \frac{1}{\epsilon}P^{(0)}_{ca}(x),$$
$$\Gamma^{(2)}_{ca} = \frac{1}{\epsilon^2}\left(\frac{1}{2}P^{(0)}_{ce} \otimes P^{(0)}_{ea}\right) + \frac{1}{\epsilon}\left(\frac{1}{2}P^{(1)}_{ca}\right).$$

Knowledge of $\hat{\Delta}^I_{cd}$ up to sufficient order both in a as well as in ϵ, combined with the solution of Eq. (2.1.9) will give us the desired $P^{(i)}_{ab}(z)$, order by order in perturbation theory. Note that in the $\overline{\mathrm{DR}}$ scheme, the AP kernels contain only $\frac{1}{\epsilon^n}$ where n is positive definite.

The splitting functions $P^{(i)}_{ab}(z)$ can extracted from the collinear singular cross sections $\hat{\Delta}^I_{cd}(z, Q^2, 1/\epsilon)$ by demanding that $\Delta^I_{cd}(z, Q^2, \mu_F^2, \epsilon)$ is finite order by order in perturbation theory. Hence, the splitting functions appearing in these kernels can be determined uniquely provided $\hat{\Delta}^I_{cd}$ are known to sufficient order both in a as well as in ϵ.

Splitting function from collinear factorisation :

Scattering amplitudes devlops singularity in phase space, when the momenta of two external particles become collinear.

In the collinear limit tree level n+1 point amplitude factorises :

$$\lim_{p_i \parallel p_j} \mathcal{A}^{(0)}_{n+1}(..., p_i, p_j) = \sum_{\lambda} \text{Split}^{(0)}(p_i, p_j)\, \mathcal{A}^{(0)}_n(..., p). \tag{2.1.11}$$

where p_i and p_j are the momenta of two collinear legs and the sum is over all polarisations. Squaring the splitting amplitudes

$$\begin{aligned} P^{(0)}_{\text{quark}}(\lambda_1, \lambda_2) &= \sum_{\lambda_i, \lambda_j} u(p, \lambda_1)\ \ \text{Split}^{(0)}(p_i, p_j)^* \,\text{Split}^{(0)}(p_i, p_j)\, \bar{u}(p, \lambda_2), \\ P^{(0)}_{\text{gluon}}(\lambda_1, \lambda_2) &= \sum_{\lambda_i, \lambda_j} \epsilon^{\mu}(p, \lambda_1)^*\ \ \text{Split}^{(0)}(p_i, p_j)^* \,\text{Split}^{(0)}(p_i, p_j)\, \epsilon^{\nu}(p, \lambda_2). \end{aligned} \tag{2.1.12}$$

AP splitting functions can be obtained by averaging Eq. (2.1.12) over the spin indices

$$\begin{aligned} P^{(0)}_{\text{quark}}\Big|_{spin\,avg.} &= \frac{1}{2} \sum_{\lambda} P^{(0)}_{\text{quark}}(\lambda, \lambda), \\ P^{(0)}_{\text{gluon}}\Big|_{spin\,avg.} &= \frac{1}{(2+\epsilon)} \sum_{\lambda} P^{(0)}_{\text{gluon}}(\lambda, \lambda). \end{aligned}$$

The generalisation of the factorisation formula for higher loop order can be found in [110–117].

Computation detalils :

In Eq. (2.1.10) $c, a \in \{\lambda, g, \phi, \chi\}$ thus we have 16 splitting functions P_{ab} at every order in perturbation theory. To determine LO $P^{(0)}_{ab}$ and NLO $P^{(1)}_{ab}$, we need to evaluate the scattering cross sections $\hat{\sigma}^I_{ab}$ for various choices of initial states 'ab' up to second order in the coupling constant a. Since these are inclusive cross sections, sum over all the allowed final states need to be done. We find more than one splitting functions $P^{(i)}_{ab}$ appear in single $\hat{\Delta}^{I,(i)}_{ab}$ which makes it difficult to determine them separately. For example the non-diagonal terms such as $P_{\lambda\phi}$ and $P_{\phi\lambda}$ would appear together with some numerical coefficients in $\hat{\Delta}^{I,(i)}_{\lambda\phi}$, at every order. We can disentangle them if we compute the contributions from more than one partonic cross sections, *i.e.* I = half-BPS and T. In addition we have observed that $\hat{\sigma}^{I}_{\lambda\phi} = \hat{\sigma}^{I}_{\lambda\chi}$, which is valid up to second order in a for any I. Hence, the number of P_{ab} that we need to determine reduces to 10. They are given by $P_{gg}, P_{\lambda\lambda}, P_{\phi\phi}, P_{g\lambda}, P_{\lambda g}, P_{g\phi}, P_{\phi g}, P_{\lambda\phi}, P_{\phi\lambda}$ and $P_{\phi\chi}$.

The LO diagonal splitting functions $P^{(0)}_{cc}$ requires cross sections $\hat{\sigma}^{\mathrm{T},(i)}_{cc}$ with $i = 0, 1$ and the relevant processes are

$$
\begin{aligned}
&c + c \rightarrow \mathrm{T}, \qquad \mathrm{c} + \mathrm{c} \rightarrow \mathrm{T} + \text{one loop}, \\
&c + c \rightarrow \mathrm{T} + X, \qquad\qquad\qquad\qquad (2.1.13)
\end{aligned}
$$

where $X = g$ for $c \in \{\phi, g\}$ and $X \in \{g, \phi, \chi\}$ for $c = \lambda$. Each of the above processes at $O(a)$ contains only one $P^{(0)}_{cc}$, hence it is straightforward to obtain each of them independently. If we use the half-BPS operator, we can compute only $P^{(0)}_{\phi\phi}$ which we find agrees with that obtained using the T operator. The non-diagonal LO splitting functions $P^{(0)}_{cb}$ requires the computation of $\hat{\sigma}^{\mathrm{T},(i)}_{cb}$ with $i = 0, 1$. At one loop the processes that contribute are given by

$$
c + b \rightarrow \mathrm{T} + c, \qquad (2.1.14)
$$

where we have chosen: $c \neq b$ with $(c, b) \in \{(\lambda, \phi), (\lambda, g), (\phi, g)\}$. It is interesting to note that in each of the above subprocesses only the following combination of splitting functions appears: $\hat{\sigma}^{\mathrm{T},(0)}_{cc} P^{(0)}_{cb} + \hat{\sigma}^{\mathrm{T},(0)}_{bb} P^{(0)}_{bc}$. We can disentangle $P^{(0)}_{cb}$ and $P^{(0)}_{bc}$ separately by comparing the coefficients of $\hat{\sigma}^{\mathrm{T},(0)}_{cc}$ and $\hat{\sigma}^{\mathrm{T},(0)}_{bb}$. The remaining LO splitting function $P^{(0)}_{\phi\chi} = P^{(0)}_{\chi\phi}$ is found to be identically zero as they start at $O(a^2)$.

At NLO level, the diagonal splitting function $P^{(1)}_{cc}$ requires the computation of $\hat{\sigma}^{\mathrm{T},(2)}_{cc}$ and $\hat{\sigma}^{\mathrm{T},(i)}_{cb}$ with $i = 0, 1$, for different combinations of c and b. $\hat{\sigma}^{\mathrm{T},(2)}_{cc}$ gets contribution from two-loop virtual processes

$$
c + c \rightarrow \mathrm{T} + \text{two loops}\,, \qquad (2.1.15)
$$

one-loop with a single real emission processes

$$c + c \to \mathrm{T} + X + \text{one loop}, \tag{2.1.16}$$

where $X = g$ for $c \in \{\phi, g\}$, $X \in \{g, \phi, \chi\}$ for $c = \lambda$ and pure double emission processes

$$c + c \to \mathrm{T} + b + b, \tag{2.1.17}$$

where for every pair of initial states made up of a pair of c's with $c = \lambda, g, \phi$, the allowed final states contain a pair of b's where $b = \lambda, g, \phi$. Since the half-BPS operator couples to only ϕ's at LO, we can compute $P^{(1)}_{\phi\phi}$ from $\hat{\sigma}^{\mathrm{BPS},(2)}_{\phi\phi}$ as well. This provides an independent check on our results.

Unlike the diagonal splitting functions, the non-diagonal ones can not be determined from $\hat{\sigma}^{\mathrm{T},(2)}_{cb}$ alone. The cross sections $\hat{\sigma}^{\mathrm{T},(2)}_{cb}$ where $c \neq b$ always contain the combinations of $P^{(1)}_{cb}$ and $P^{(1)}_{bc}$. Hence determining them from single cross section is not possible. Therefore we resort to $\hat{\sigma}^{\mathrm{BPS},(2)}_{cb}$ which can give $P^{(1)}_{cb}$ unambiguously. Knowing $P^{(1)}_{cb}$ and using $\hat{\sigma}^{\mathrm{T},(2)}_{cb}$, we determine $P^{(1)}_{bc}$. The relevant processes to determine $P^{(1)}_{c\lambda}$ and $P^{(1)}_{\lambda c}$ where $c = g, \phi$ are given by

$$\begin{aligned} \lambda + c &\to I + \lambda + \text{one loop}, \\ \lambda + c &\to I + \lambda + b, \end{aligned} \tag{2.1.18}$$

where $b \in \{\phi, \chi, g\}$ and I = T, half-BPS. The cross sections, $\hat{\sigma}^{I}_{g\phi}$ where I =T, half-BPS that contribute to $P^{(1)}_{\phi g}$ and $P^{(1)}_{g\phi}$ can be obtained . The relevant processes are

$$\begin{aligned} g + \phi &\to I + \phi + \text{one loop}, \\ g + \phi &\to I + g + \phi, \\ g + \phi &\to I + \lambda + \lambda. \end{aligned} \tag{2.1.19}$$

Finally, the splitting function $P^{(1)}_{\phi\chi} = P^{(1)}_{\chi\phi}$ is obtained from the cross sections $\hat{\sigma}^{I}_{\phi\chi}$ with I =T, half-BPS which get contributions from the subprocesses

$$\phi + \chi \to I + \phi + \chi,$$

$$\phi + \chi \to I + \lambda + \lambda. \tag{2.1.20}$$

In QCD, the kernel Γ_{ab} contains 9 different splitting functions because $a, b \in \{q, \overline{q}, g\}$ for a given flavour quark. The Mellin moments of them namely

$$\int_0^1 dz z^{j-1} P_{ab}(z) = \gamma_{ab,j}, \tag{2.1.21}$$

are anomalous dimensions of gauge invariant local operators made up of quark, anti-quark and gluon fields, see [55,118–122]. Following QCD, we can relate the Mellin moments of P_{ab} obtained in $\mathcal{N} = 4$ SYM theory with the anomalous dimensions of composite operators given by

$$O^{\lambda}_{\mu_1 \cdots \mu_j} = S\left\{\overline{\lambda}^a_m \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_j} \lambda^a_m\right\}, \tag{2.1.22}$$

$$O^{g}_{\mu_1 \cdots \mu_j} = S\left\{G^a_{\mu\mu_1} D_{\mu_2} \cdots D_{\mu_{j-1}} G^{a\mu}_{\mu_j}\right\}, \tag{2.1.23}$$

$$O^{\phi}_{\mu_1 \cdots \mu_j} = S\left\{\phi^a_i D_{\mu_1} \cdots D_{\mu_j} \phi^a_i\right\}, \tag{2.1.24}$$

$$O^{\chi}_{\mu_1 \cdots \mu_j} = S\left\{\chi^a_i D_{\mu_1} \cdots D_{\mu_j} \chi^a_i\right\}. \tag{2.1.25}$$

The symbol S indicates symmetrisation of indices $\mu_1 \cdots \mu_j$. Note that these operators mix under renormalisation and the corresponding anomalous dimensions are given by $\gamma_{ab,j}$. In addition, when $j = 2$, the sum reproduces the gauge invariant part of energy momentum tensor which does not require any overall renormalisation. In other words, the sum $\sum_a O^a_{\mu_1\mu_2}$ is UV finite, hence

$$\mu_R^2 \frac{d}{d\mu_R^2}\left(\sum_a O^a_{\mu_1\mu_2}\right) = 0, \qquad a \in \{\lambda, g, \phi, \chi\}. \tag{2.1.26}$$

This implies

$$\sum_a \gamma_{ab,2} = 0 \qquad a, b \in \{\lambda, g, \phi, \chi\}. \tag{2.1.27}$$

We will show that splitting functions computed in the present paper satisfy the above relation up to NLO level, namely at each perturbative order i

$$\sum_a \int_0^1 dz\, z P^{(i)}_{ab}(z) = 0\,, \text{ where } \quad i = 0, 1, \tag{2.1.28}$$

with a, b given in Eq. (2.1.27). In the next section, we shall discuss the methodology that we have adopted to compute the individual partonic cross sections $\hat{\sigma}^I_{bc}$.

2.2 Methodology

The computation of $\hat{\Delta}^I_{ab}(z, Q^2, \epsilon)$ *i.e.* $\hat{\sigma}^I_{ab}(z, Q^2, \epsilon)/z$ beyond the LO involves evaluating processes with real emissions and virtual loops. We generate relevant Feynman diagrams by using the package QGRAF [123]. The QGRAF output is processed to a suitable format for further manipulation by using our in-house codes written in FORM [124]. We then compute the square of the diagrams by summing over the spins of Majoranas, polarization vectors of gluons and generation indices of Majoranas, scalars and pseudoscalars. In addition, we sum the colors of all the external states. The resulting expression contains large number of Feynman integrals and phase space integrals. Using a Mathematica based package LiteRed [125,126] we reduce all the Feynman integrals to few Master Integrals (MIs). While there were brisk developments in evaluating the loop diagrams, progress in computing the phase space integrals for real emission processes took place slowly. It is worthwhile to mention that the NNLO QCD corrections to DY pair production [127] was achieved by choosing Lorentz frames in such a way that the integrals can be achieved. An alternate approach was proposed in [128] to obtain the inclusive production of Higgs boson. In this approach, the phase space integrals were done after expanding the matrix elements around the scaling variable $z = 1$. These approaches pose the problem of dealing with large number of integrals. An elegant formalism was developed by Anastasiou and Melnikov [129] which helps to reduce these large number of phase space integrals to a set of few master integrals. In this formalism, the phase space integrals are first converted to loop integrals by employing the method of reverse unitarity. One replaces the δ_+ functions, coming from phase space integrals (see Eq. (2.1.3)), by the difference of propagators,

$$\delta_+(q^2 - m^2) \sim \left(\frac{1}{q^2 - m^2 + i\varepsilon} - \frac{1}{q^2 - m^2 - i\varepsilon}\right) \theta(q^0) \tag{2.2.1}$$

This replacement results in loop integrals which can be simplified to fewer set of MIs with the help of integration by parts (IBP) identities. Care is needed while using IBP identities because the shifts of momenta are not allowed between cut and uncut propagators. The MIs that remain after the application of IBP identities are transformed back to phase space integrals by appropriately replacing those propagators that were introduced in place of δ_+ functions. Since the number of integrals at this stage is much smaller, the problem reduces to evaluation of fewer integrals using standard techniques. The phase space integrals relevant up to NNLO level can be found in [130]. We used this approach to obtain $\hat{\sigma}^I_{ab}$ up to $O(a^2)$ in perturbation theory. For more details on the implementation, see [129, 131, 132].

2.3 Analytical results and discussion

The splitting functions $P^{(i)}_{ab}(z)$ for $i = 0, 1$ are extracted from the collinear singular cross sections $\hat{\Delta}^I_{cd}(z, Q^2, 1/\epsilon)$ by demanding that $\Delta^I_{cd}(z, Q^2, \mu_F^2, \epsilon)$ are finite order by order in perturbation theory. In the $\overline{\text{DR}}$ scheme, at LO level, the diagonal ones are found to be

$$
\begin{aligned}
P^{(0)}_{\lambda\lambda}(z) &= 8\left[\mathcal{T}(z) + 3 - 2z\right], \\
P^{(0)}_{gg}(z) &= 8\left[1 - \mathcal{V}(z) + \mathcal{T}(z) + z(1-z)\right], \\
P^{(0)}_{\phi\phi}(z) &= 8\left[\mathcal{T}(z) + 1\right],
\end{aligned}
\tag{2.3.1}
$$

and the non-diagonal ones are

$$
\begin{aligned}
P^{(0)}_{g\lambda}(z) &= 4\left[z - 2\mathcal{V}(z)\right], & P^{(0)}_{\lambda g}(z) &= 16\left[1 - 2\,z\,(1-z)\right], \\
P^{(0)}_{\phi\lambda}(z) &= 6\,z, & P^{(0)}_{\lambda\phi}(z) &= 16, \\
P^{(0)}_{\phi g}(z) &= 12\,z\,(1-z), & P^{(0)}_{g\phi}(z) &= -8\,\mathcal{V}(z).
\end{aligned}
\tag{2.3.2}
$$

The LO splitting functions involving χ are obtained using

$$
\begin{aligned}
P^{(0)}_{\chi\chi}(z) &= P^{(0)}_{\phi\phi}(z), \\
P^{(0)}_{\chi\phi}(z) &= P^{(0)}_{\phi\chi}(z) = 0, \\
P^{(0)}_{b\chi}(z) &= P^{(0)}_{b\phi}(z), \\
P^{(0)}_{\chi b}(z) &= P^{(0)}_{\phi b}(z) \quad \text{where } b \in \{\lambda, g\}.
\end{aligned}
\tag{2.3.3}
$$

The extraction of the splitting functions at NNLO level involves use of both the half-BPS as well as T operators because of the presence of more than one splitting functions in a single cross section. By appropriately choosing the singlet final states and the corresponding pair of particles in the initial states, as described in the previous section, we obtain

$$
\begin{aligned}
P^{(1)}_{\lambda\lambda}(z) &= 24\,\zeta_3\,\delta(1-z) + 8\left[\log^2(z) - 2\,\zeta_2\right][\mathcal{T}(z) + \mathcal{T}(-z) + 6] - 32\,\log(z)\log(1-z) \\
&\times [\mathcal{T}(z) + 3 - 2z] - 32\,[\mathrm{Li}_2(-z) + \log(z)\log(1+z)]\,[\mathcal{T}(-z) + 3 + 2z] \\
&+64\log(z)\left[3 + z + \frac{4}{3}z^2\right] + \frac{640}{9}\frac{1}{z} + 128z - \frac{1792}{9}z^2, \\
P^{(1)}_{g\lambda}(z) &= 32\zeta_2 + 16\,[\mathrm{Li}_2(-z) + \log(z)\log(1+z)]\,[2\mathcal{V}(-z) + z] - 16\log^2(z) \\
&+16\,\log(z)\log(1-z)\,[2\mathcal{V}(z) - z] - 16\log(z)\left[9 + 2z + \frac{4}{3}z^2\right] \\
&+80 - \frac{1072}{9}\frac{1}{z} + \frac{352}{9}z^2, \\
P^{(1)}_{\phi\lambda}(z) &= 24z\,[\mathrm{Li}_2(-z) + \log(z)\log(1+z) - \log(z)\log(1-z)] - 8\log(z)\left[3 + 2z + 4z^2\right] \\
&+16 + 24\mathcal{V}(-z) - 64z + 80z^2, \\
P^{(1)}_{gg}(z) &= 24\zeta_3\delta(1-z) + \left[2\zeta_2 - \log^2(z)\right]\left[64 - 8\mathcal{T}(-z) - 8\mathcal{T}(z) + 16z^2\right] \\
&+32\,[\mathrm{Li}_2(-z) + \log(z)\log(1+z)]\left[\mathcal{V}(-z) - \mathcal{T}(-z) - 1 + z + z^2\right] \\
&+32\,[\log(z)\log(1-z)]\left[\mathcal{V}(z) - \mathcal{T}(z) - 1 - z + z^2\right] \\
&- \log(z)\left[144 + 112z - \frac{352}{3}z^2\right] + 80 - \frac{1072}{9}\frac{1}{z} - 208z + \frac{2224}{9}z^2, \\
P^{(1)}_{\lambda g}(z) &= \left[\log^2(z) - 2\zeta_2\right]\left[32 + 64z^2\right] - 64\,[\mathrm{Li}_2(-z) + \log(z)\log(1+z)]\left[1 + 2z + 2z^2\right] \\
&-64\,[\log(z)\log(1-z)]\left[1 - 2z + 2z^2\right] + \log(z)\left[192 + 320z + \frac{1792}{3}z^2\right] \\
&+\frac{640}{9}\frac{1}{z} + 896z - \frac{8704}{9}z^2, \\
P^{(1)}_{\phi g}(z) &= 24z^2\left[2\zeta_2 - \log^2(z)\right] + 48z(1+z)\,[\mathrm{Li}_2(-z) + \log(z)\log(1+z)] \\
&-48z(1-z)\log(z)\log(1-z) - \log(z)\left[24 + 104z + 240z^2\right] \\
&-64 + 24\mathcal{V}(-z) - 344z + 360z^2, \\
P^{(1)}_{\phi\phi}(z) &= 24\zeta_3\delta(1-z) + \left[8\log^2(z) - 16\zeta_2\right][\mathcal{T}(z) + \mathcal{T}(-z) + 2] \\
&-32\,[\mathrm{Li}_2(-z) + \log(z)\log(1+z)]\,[\mathcal{T}(-z) + 1] - 32\,[\mathcal{T}(z) + 1]\log(z)\log(1-z) \\
&+ \log(z)\left[-24 + 24z + 16z^2\right] - 64 + 24\mathcal{V}(-z) + 40z - 24z^2, \\
P^{(1)}_{\lambda\phi}(z) &= 32\left[\log^2(z) - 2\zeta_2 - 2\mathrm{Li}_2(-z) - 2\log(z)\log(1+z) - 2\log(z)\log(1-z)\right]
\end{aligned}
$$

$$
\begin{aligned}
& + \log(z)\left[192 - 64z - \frac{128}{3}z^2\right] + \frac{640}{9}\frac{1}{z} - 128z + \frac{512}{9}z^2, \\
P^{(1)}_{g\phi}(z) &= 16\left[2\zeta_2 - \log^2(z)\right] + 32\left[\mathrm{Li}_2(-z) + \log(z)\log(1+z)\right]\mathcal{V}(-z) \\
& +32\mathcal{V}(z)\log(z)\log(1-z) + \log(z)\left[-144 + 16z + \frac{32}{3}z^2\right] \\
& +80 - \frac{1072}{9}\frac{1}{z} + 48z - \frac{80}{9}z^2, \\
P^{(1)}_{\phi\chi}(z) &= 8\log(z)\left[-3 + 3z + 2z^2\right] + 24\,\mathcal{V}(-z) - 64 + 40z - 24z^2\,, \qquad (2.3.4)
\end{aligned}
$$

and the splitting functions involving χ are obtained using

$$
\begin{aligned}
P^{(1)}_{\chi\chi}(z) &= P^{(1)}_{\phi\phi}(z)\,, \qquad P^{(1)}_{\chi\phi}(z) = P^{(1)}_{\phi\chi}(z), \\
P^{(1)}_{b\chi}(z) &= P^{(1)}_{b\phi}(z)\,, \qquad P^{(1)}_{\chi b}(z) = P^{(1)}_{\phi b}(z) \quad \text{where } b \in \{\lambda, g\}. \qquad (2.3.5)
\end{aligned}
$$

In above $\mathcal{T}(z) = 1/(1-z)_+ - 2$ and $\mathcal{V}(z) = 1 - 1/z$. The action of "+ distribution" on a dummy function $f(z)$ is defined by

$$
\int_0^1 dz f(z)\left[\frac{\log^n(1-z)}{1-z}\right]_+ = \int_0^1 dz\left[f(z) - f(1)\right]\frac{\log^n(1-z)}{1-z}. \qquad (2.3.6)
$$

Consistency check :

We find that the both LO and NLO splitting functions satisfy the following relations:

$$
\sum_{a=\lambda,g,\phi,\chi} P^{(i)}_{a\lambda} = \sum_{a=\lambda,g,\phi,\chi} P^{(i)}_{ag} = \sum_{a=\lambda,g,\phi,\chi} P^{(i)}_{a\phi} = \sum_{a=\lambda,g,\phi,\chi} P^{(i)}_{a\chi} = I^{(i)}(z), \qquad i = 0, 1, \qquad (2.3.7)
$$

where

$$
\begin{aligned}
I^{(0)}(z) &= 8\left[\frac{1}{(1-z)_+} + \frac{1}{z}\right], \\
I^{(1)}(z) &= 24\zeta_3\delta(1-z) + 32\frac{1}{z}\left[\mathrm{Li}_2(-z) + \log(z)\log(1+z) - \log(z)\log(1-z)\right] \\
& + \frac{1}{(1-z)_+}\left[-32\log(z)\log(1-z) + 8\log^2(z) - 16\zeta_2\right] \\
& + \frac{1}{1+z}\left[-32\mathrm{Li}_2(-z) - 32\log(z)\log(1+z) + 8\log^2(z) - 16\zeta_2\right]. \qquad (2.3.8)
\end{aligned}
$$

- All of the expressions in eq. 2.3.10 are *uniform transcendental* at each perturbative order!

Using the above relations, we confirm the identity given in Eq. (2.1.28) *i.e.*

$$\sum_{a=\lambda,g,\phi,\chi} \int_0^1 dz\, z P^{(i)}_{ab} = \int_0^1 dz\, z I^{(i)}(z) = 0\,, \quad i = 0, 1 \text{ and } b = \{\lambda, g, \phi, \chi\}. \tag{2.3.9}$$

- The identities in eq. 2.3.9 are satisfied at each order of perturbation theory and served as a crucial cross check of our results.

Universal properties of the diagonal splitting functions at SV limit

We find that both at NLO and NNLO, only the diagonal splitting functions contain "+" distributions. In addition, at NNLO level, terms proportional to $\delta(1-z)$ start contributing to diagonal splitting functions as the collinear/virtual anomalous dimension is zero [12] at NLO for $\mathcal{N} = 4$. Hence, in the limit z → 1, the diagonal splitting functions can be decomposed as

$$P^{(i)}_{aa}(z) = 2A_{i+1}\frac{1}{(1-z)_+} + 2B_{i+1}\delta(1-z) + R^{(i)}_{aa}(z), \tag{2.3.10}$$

where A_{i+1} and B_{i+1} are the cusp [12, 133–135] and collinear [12] anomalous dimensions respectively. $R^{(i)}_{aa}(z)$ is the regular function as $z \to 1$. We find that

$$A_1 \quad = \quad 4, A_2 = -8\zeta_2\,, \qquad \text{and} \qquad B_1 = 0, B_2 = 12\zeta_3\,, \tag{2.3.11}$$

which are in agreement with the result obtained from the FFs of the half-BPS operator [12, 133–135].

Using the supersymmetric extensions of Balitskii-Fadin-Kuraev-Lipatov (BFKL) [136–138] and DGLAP [2, 107–109] evolution equations, Kotikov and Lipatov [15, 42, 43, 139–142] conjectured leading transcendentality (LT) principle which states that the eigenvalues of anomalous dimension [143] matrix of twist two composite operators made out of λ, g and complex ϕ fields in $\mathcal{N} = 4$ SYM theory contain uniform transcendental terms at every order in perturbation theory. Interestingly they are related to the corresponding quantities in QCD [16, 64]. In [43] it has been shown that the eigenvalues of the anomalous dimension matrix are related to the universal anomalous dimension by shifts in spin-j up to three-loop level. Unlike [141], we distinguish scalar and pseudo-scalar fields and compute their anomalous dimensions and their mixing in Mellin-j space. We find two of the eigenvalues of the resulting anomalous dimension matrix coincide with the universal eigenvalues obtained in [141] after finite shifts and the remaining two coincide with the universal ones only in the large j limit (i.e. $z \to 1$). For reference, we explicitly present the eigenvalues computed in this paper in appendices 2.4. One can associate the transcendentality weight n to terms such as $\zeta(n)$, ϵ^{-n} and also to the weight of the harmonic polylogarithms that appear in the perturbative calculations. Similar relations were found in certain scattering amplitudes [144,145], FFs of BPS type operators [8,10,146,147], light-like Wilson loops [148,149] and correlation functions [147,149–153] computed in $\mathcal{N} = 4$ SYM theory. It is shown that in [10], the two-loop three-point MHV FFs of the half-BPS operator have uniform transcendental terms in the finite reminder functions. Several bosonic FFs in QCD after putting $C_A = C_F = n_f = N$ coincide with simialr kind of FFs in $\mathcal{N} = 4$ SYM theory, and the LT terms of the amplitude for Higgs boson decaying to three on-shell gluons in QCD [44, 154] are related to the two-loop three-point MHV FFs of the half-BPS operator [10]. Similar correspondence was shown between two-loop three-point FF [13] of the half-BPS operator and the pseudoscalar Higgs boson plus three-gluon amplitudes [47] in minimal supersymmetric SM. Two-point FFs of quark current operator [155], pseudoscalar [156] operators, energy momentum tensor [45,46] of the QCD up to three loops also show the same behaviour. It was shown in [12, 13], unlike BPS operators, the Konishi operators do not have uniform transcendental terms but their LT terms in FFs between $\phi\phi$ and in remainder function computed between $g\phi\phi$ external state coincide with the corresponding ones of the half-BPS.

As can be seen from the results of splitting functions (see Eq. (2.3.4)), at each order n, the splitting functions consist of terms which have trancendentality ranging from $2n$ to 0. It is worth comparing the splitting functions in $\mathcal{N} = 4$ SYM theory, P_{ab} with the ones obtained in QCD, $P^{\rm QCD}_{ab}$. We apply the following color transformation on the QCD ones for comparison: $C_A = C_F = n_f = N$. We find that the one loop splitting functions $P^{\rm QCD,(0)}_{gq}$ and $P^{(0)}_{gl}$ are identical; $P^{\rm QCD,(0)}_{qg}$ and $P^{(0)}_{lg}$ are also identical up to an overall factor. For $P^{\rm QCD,(1)}_{qg}$ and $P^{(1)}_{lg}$, apart from an overall factor, we find that only terms proportional to $\log^2(z)$ are different. We also observe that LT parts of $P^{\rm QCD,(1)}_{gq}$ and $P^{(1)}_{gl}$ differ only in their $\log^2(z)$ terms.

IR safe results

Let us study the IR finite cross sections Δ^{I}_{ab} up to NNLO level. These cross sections are computed in power series of the coupling constant a as

$$\Delta^{I}_{ab} = \delta(1-z)\delta_{ab} + a\,\Delta^{I,(1)}_{ab} + a^2\,\Delta^{I,(2)}_{ab} + \cdots \qquad (2.3.12)$$

These $\Delta^{I,(i)}_{ab}$ contain both regular functions as well as distributions in the scaling variable z. The former are made up of polynomials and multiple polylogarithms of z that are finite as $z \to 1$ and they are from hard particles. The distributions are from soft and collinear particles, which show up at every order in the perturbation theory in the form of $\delta(1-z)$ and $\mathcal{D}_i(z)$ where

$$\mathcal{D}_i(z) = \left(\frac{\log^i(1-z)}{1-z}\right)_+ , \qquad (2.3.13)$$

and its action on a regular function is shown in Eq. (2.3.6). Real emission diagrams contribute to these distributions through the expansion

$$(1-z)^{-1+\epsilon} = \frac{1}{\epsilon}\delta(1-z) + \sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\mathcal{D}_k. \qquad (2.3.14)$$

These distributions constitute what is called the threshold or soft plus virtual (SV) part of the cross section, denoted by $\Delta^{\rm SV}_{ab}$. One can decompose the total cross section as,

$$\Delta^{I,(i)}_{ab} = \Delta^{I,(i),\rm SV}_{ab} + \Delta^{I,(i),\rm Reg}_{ab}, \qquad (2.3.15)$$

where,

$$\Delta^{I,(i),\rm SV}_{ab} = \delta_{ab}\left(c^I_i\delta(1-z) + \sum_{j=0}^{2i-1} d^I_{ij}\mathcal{D}_j(z)\right). \qquad (2.3.16)$$

The constants c^I_i and d^I_{ij} are absent when $a \neq b$. For the diagonal ones ($a = b$), they depend on the final singlet state I and are in general functions of rational terms and irrational ζ. For the diagonal ones, $\Delta^{I,(i),\rm SV}_{aa}$ are found identical to each other for I = BPS, T. Up to NNLO level, they are found to be

$$\begin{aligned}
\Delta_{aa}^{I,(0),\mathrm{SV}} &= \delta(1-z)\,, \\
\Delta_{aa}^{I,(1),\mathrm{SV}} &= 8\zeta_2\delta(1-z) + 16\mathcal{D}_1(z)\,, \\
\Delta_{aa}^{I,(2),\mathrm{SV}} &= -\frac{4}{5}\zeta_2^2\delta(1-z) + 312\zeta_3\mathcal{D}_0(z) - 160\zeta_2\mathcal{D}_1(z) + 128\mathcal{D}_3(z). \qquad (2.3.17)
\end{aligned}$$

We observe that at every order, the above terms demonstrate **uniform transcendentality** which is 1 at NLO and 3 at NNLO. Note that $\delta(1-z)$ has -1 transcendental weight which can be understood from Eq. (2.3.14) by noting that the term ϵ^{-n} has transcendentality n. We also notice that the highest distribution at every order determines the transcendental weight at that order. It is interesting to note that the above coefficient functions are exactly identical to the LT parts of the corresponding result in the SM for the Higgs boson production through gluon fusion computed in the effective theory, upon proper replacement of the color factors in the following way *i.e.* $C_A = C_f = n_f = N$. On the other hand for $I = \mathcal{K}$, we find up to NNLO level,

$$\begin{aligned}
\Delta_{aa}^{\mathcal{K},(0),\mathrm{SV}} &= \delta(1-z)\,, \\
\Delta_{aa}^{\mathcal{K},(1),\mathrm{SV}} &= \left[-28 + 8\zeta_2\right]\delta(1-z) + 16\mathcal{D}_1(z)\,, \\
\Delta_{aa}^{\mathcal{K},(2),\mathrm{SV}} &= \left[604 - 272\zeta_2 - \frac{4}{5}\zeta_2^2\right]\delta(1-z) + 312\zeta_3\mathcal{D}_0(z) \\
&\quad - \left[160\zeta_2 + 448\right]\mathcal{D}_1(z) + 128\mathcal{D}_3(z). \qquad (2.3.18)
\end{aligned}$$

Unlike BPS and T type, for Konishi, $\Delta_{aa}^{\mathcal{K},(i),\mathrm{SV}}$ does not have uniform transcendentality but its LT terms coincide with those of BPS/T.

General factorisation equation of SV cross section

In perturbative QCD, the fixed order predictions for the observables become often unreliable in certain regions of phase space due to the presence of large logarithms. For example, at the hadron colliders, the inclusive observables like total cross section or invariant mass distribution of finial state colorless state and some differential distributions contain large logarithms which can spoil the reliability of fixed order results. Also fixed order results are not trustworthy due the large uncertainity coming from mising higher order terms. For example, at the partonic threshold i.e.

when the initial partons have just enough energy to produce the final state colorless particle and soft gluons, the phase space available for the gluons become severely constrained giving large logarithms. In the resummation approaches these large logarithms are systematically resummed to all orders in perturbation theory leading to reliable predictions.

The SV part of the inclusive observables in QCD is well understood to all orders in perturbation theory. For example, the SV part of the inclusive cross section gets contribution from virtual part, namely the form factor and the soft, collinear configurations of the real emission processes. In these observables, the soft singularities cancel between virtual and real emission processes, while the initial collinear ones are removed by mass factorisation, thus giving IR finite results. Interestingly, the factorisation property of these cross sections can be used to identify the process independent soft distribution function which depends only the incoming states. In addition, they satisfy certain differential equation similar to K+G equation of FFs. The solution gives all order prediction for the soft part of the observable in terms of soft anomalous dimensions f_a with $a = q, g$. Following [67] and noting that only $\Delta^{I,\mathrm{SV}}_{aa}$ contains threshold logarithms, its all order structure can be expressed as

$$\begin{aligned}\Delta^{I,\mathrm{SV}}_{aa} &= \left(Z^I(a,\epsilon)\right)^2 |\hat{F}^I_{aa}(Q^2,\epsilon)|^2 \delta(1-z) \otimes \mathcal{C}\exp\left(2\Phi^I_{aa}(z,Q^2,\epsilon)\right) \\ &\quad \otimes \Gamma^{-1}_{aa}(z,\mu_F^2,\epsilon) \otimes \Gamma^{-1}_{aa}(z,\mu_F^2,\epsilon). \end{aligned} \tag{2.3.19}$$

In above I can be any one of the three operators considered in our current work. $Z^I(a,\epsilon)$ is the overall operator renormalization constant, which is unity for I = half-BPS and T operators; however, for $I = \mathcal{K}$, up to three loop, the pertubative coefficients of $Z^{\mathcal{K}}$ are available [12, 43, 73, 103–105]. $\hat{F}^I_{aa}(Q^2)$ is the FF contribution, *i.e.*, the matrix elements of the half-BPS or T or $\mathcal{K}$ between the on-shell state aa where $a = \{\lambda, g, \phi, \chi\}$ and vacuum, normalised by the Born contribution, which reads as

$$\hat{F}^I_{aa}(Q^2) = \frac{\langle a(p_1), a(p_2)|\tilde{O}^I|0\rangle}{\langle a(p_1), a(p_2)|\tilde{O}^I|0\rangle^{(0)}}, \qquad Q^2 = (p_1+p_2)^2\,. \tag{2.3.20}$$

$\tilde{O}^I$ is the Fourier transform of O^I and the superscript 0 indicates that it is the Born contribution. $\Phi^I_{aa}(z,Q^2)$ is the soft distribution function resulting from the soft radiation and Γ_{aa} are the AP kernels that can be written in terms diagonal splitting functions as given in Eq. (2.3.10).

Γ_{aa} and they contain only the distributions $\delta(1-z)$ and $\mathcal{D}_i(z)$ from the diagonal splitting functions $P^{(i)}_{aa}(z)$.

The symbol $\otimes$ denotes convolution and the $C\exp(f(z))$ is defined by

$$Ce^{f(z)} = \delta(1-z) + \frac{1}{1!}f(z) + \frac{1}{2!}f(z)\otimes f(z) + \frac{1}{3!}f(z)\otimes f(z)\otimes f(z) + \cdots \tag{2.3.21}$$

In the above, we drop all the regular terms resulting from the convolutions and keep only distributions.

IR structure of soft distribution function

In [12], the FFs are shown to satisfy the K+G equation [89–92] and its solution at each order can be expressed in terms of the universal cusp (A^I), soft (f^I) and collinear anomalous (B^I) dimensions along with some operator dependent contributions [67, 157]. $\Delta^{I,\mathrm{SV}}_{aa}$ is finite in the limit $\epsilon \to 0$, thus the pole structure of soft distribution function should be similar to that of $\hat{F}^I_{aa}$ and Γ_{aa}. One can show that the *soft distribution function* Φ^I_{aa} also satisfies a Sudakov type differential equation [67] namely $\bar{K} + \bar{G}$ whose solution is straightforward to obtain:

$$\Phi^I_{aa} = \sum_{i=1}^{\infty} a^i \left(\frac{q^2(1-z)^2}{\mu_F^2}\right)^{i\epsilon/2} \left(\frac{1}{1-z}\right)\left[\frac{2A_i}{i\epsilon} - f_i + \overline{\mathcal{G}}^I_{ia}(\epsilon)\right], \tag{2.3.22}$$

where, A_i and f_i are the cusp and soft anomalous dimenssions in $\mathcal{N}=4$ SYM.

$$A_1 = 4\,, \qquad A_2 = -8\zeta_2 \qquad A_3 = \frac{176}{5}\zeta_2^2. \tag{2.3.23}$$

$$f_1 = 0\,, \qquad f_2 = -28\zeta_3 \qquad f_3 = \frac{176}{3}\zeta_2\zeta_3 + 192\zeta_5. \tag{2.3.24}$$

Detailed discussion on the structure and solutions of Φ^I_{aa} is presented in the appendix.

We find that Φ^I_{aa} does not depend on I and in addition they are identical for $a = \lambda, g, \phi$ and χ. Hence, $\overline{\mathcal{G}}^I_{ia} = \overline{\mathcal{G}}_i$. From the known coefficient functions, $\Delta^{I,(i),\mathrm{SV}}$, up to two loops we can determine $\overline{\mathcal{G}}_i$ and they are found to be

$$\begin{aligned}
\overline{\mathcal{G}}_1(\epsilon) &= -3\zeta_2\epsilon + \frac{7}{3}\zeta_3\epsilon^2 - \frac{3}{16}\zeta_2^2\epsilon^3 + \left[\frac{31}{20}\zeta_5 - \frac{7}{8}\zeta_2\zeta_3\right]\epsilon^4 \\
&\quad + \left[\frac{49}{144}\zeta_3^2 - \frac{57}{640}\zeta_2^3\right]\epsilon^5 + O(\epsilon^6), \\
\overline{\mathcal{G}}_2(\epsilon) &= 4\zeta_2^2\epsilon + 43\zeta_5\epsilon^2 + \left[\frac{413}{6}\zeta_3^2 + \frac{715}{84}\zeta_2^3\right]\epsilon^3 \\
&\quad + \left[\frac{9}{2}\zeta_7 - \frac{2527}{20}\zeta_2\zeta_5 + \frac{559}{120}\zeta_2^2\zeta_3\right]\epsilon^4 + O(\epsilon^5).
\end{aligned} \tag{2.3.25}$$

The above result is found to be exactly identical to Φ^q and Φ^g that appear in the inclusive cross sections of the Drell-Yan and the Higgs productions respectively up to two loops, after setting the Casimirs of SU(N) as $C_F = n_f = C_A$ and retaining only the LT terms. Our explicit computation demonstrates that the soft distribution function Φ contains uniform transcendental terms and in addition it obeys leading transcendentality principle. In [158], third order contribution to Φ^I for $I = q, g$ were obtained from [159] which we use here to predict the corresponding result for Φ of $\mathcal{N} = 4$ SYM theory after suitably adjusting the color factors and retaining the leading transcendental terms. That is, we find

$$\begin{aligned}
f_3 &= \frac{176}{3}\zeta_2\zeta_3 + 192\zeta_5. \\
\overline{\mathcal{G}}_3(\epsilon) &= -4006\zeta_6 + \frac{536}{3}\zeta_3^2 + \frac{289192}{315}\zeta_2^3 + O(\epsilon).
\end{aligned} \tag{2.3.26}$$

The three-loop results for the FFs, $\hat{F}^I$ are already known [12], up to the same order the distribution parts of Γ_{aa} (see Eq. (2.3.10)) can be obtained by using A_3 [12, 135] and B_3 [12]. Using f_3 and $\overline{\mathcal{G}}_3(\epsilon)$ from Eq. (2.3.26) we determine Φ^I up to three loops. Having known the form factors, soft distribution function and the AP kernels to third order, it is now straight forward to predict the SV part cross section at third order using Eq. (2.3.19). For $I = \mathcal{K}$, we find

$$\begin{aligned}
\Delta^{\mathcal{K},(3),\mathrm{SV}}_{\phi\phi} &= \left[-\frac{8012}{3}\zeta_6 + \frac{13216}{3}\zeta_3^2 + 480\zeta_5 - \frac{992}{5}\zeta_2^2 - 432\zeta_3 + 6512\zeta_2 - 11552\right]\delta(1-z) \\
&\quad + \left[11904\zeta_5 - \frac{23200}{3}\zeta_2\zeta_3 - 8736\zeta_3\right]\mathcal{D}_0 + \left[-\frac{9856}{5}\zeta_2^2 + 3712\zeta_2 + 9664\right]\mathcal{D}_1 \\
&\quad + 11584\zeta_3\mathcal{D}_2 + \left[-3584\zeta_2 - 3584\right]\mathcal{D}_3 + 512\mathcal{D}_5.
\end{aligned} \tag{2.3.27}$$

and for the I = half-BPS and T, we find

$$\Delta_{aa}^{I,(3),\mathrm{SV}} = \Delta_{\phi\phi}^{\mathcal{K},(3),\mathrm{SV}}\Big|_{\mathrm{LT}}\,, \tag{2.3.28}$$

where for I = half-BPS, $a = \phi$ and for I = T, $a = \{\lambda, g, \phi, \chi\}$. In addition we find that for I = half-BPS, our third order prediction, Eq. (2.3.28), agrees with the result [160] obtained by explicit computation.

2.4 Conclusion

In this paper, we have studied the perturbative structure of $\mathcal{N} = 4$ SYM gauge theory in the infrared sector and report our findings. We achieved this by computing various inclusive scattering cross sections of on-shell particles belonging to this theory. There are already many important perturbative results in $\mathcal{N} = 4$ SYM theory and most of them are obtained by studying on-shell scattering amplitudes. These amplitudes are computed in perturbation theory at leading as well as beyond the leading order in t'Hooft coupling, a. Computation of multi-loop FFs of the half-BPS operators in dimensionally regulated version of the theory gives perturbative coefficients such as cusp and collinear anomalous dimensions. Unprotected operators like Konishi also demonstrate universal structure in the infrared sector of $\mathcal{N} = 4$ SYM theory. Resummed results also exist for the amplitudes and they play an important role in the context of AdS/CFT correspondence.

Number of computations in perturbative QCD exists, motivated to understand the physics of strong interaction from the high energy colliders. For example, scattering cross sections in QCD for many observables are known very precisely and they are compared against the results from the experiments. In addition, these computations provide theoretical laboratory to unravel the rich infrared structure of not only QCD but also a wide class of non-abelian gauge theories. Factorisation of IR sensitive contributions and their universal structure in QCD amplitudes and in scattering cross sections provide unique opportunity to understand the infrared structure of the theory.

Motivated by these computations in QCD, we have calculated inclusive cross sections for producing a singlet state through the half-BPS, the energy-momentum tensor and the Konishi operators to understand the soft and the collinear properties of $\mathcal{N} = 4$ SYM theory. By defining infrared safe observables in $\mathcal{N} = 4$ SYM theory, we obtain collinear splitting functions up to second order in perturbation theory. This is possible because of the factorisation of collinear singularities in the inclusive observables, the property that infrared safe observables in QCD enjoy. In addition, we establish the cancellation of soft divergences between virtual and real emission processes order by order in perturbation theory leaving only factorizable collinear singularities. The former is in accordance with the KLN theorem. The systematic factorisation of collinear singularities and ambiguity associated with the collinear finite terms lead to RGE in the collinear sector of the theory. The latter is governed by universal collinear splitting functions, analogue of AP splitting functions in perturbative QCD. These splitting functions show several remarkable similarities with those of QCD. In particular, only the diagonal ones contain distributions $\mathcal{D}_0$ and $\delta(1-z)$ with cusp and collinear anomalous dimensions as their coefficients, like in QCD. In addition, several of the regular terms in z are in close resemblance with those in QCD when the color factors of QCD are taken as $C_F = C_A = n_f = N$. We find that the Mellin moments of the diagonal splitting functions in the large j limit agree with the universal anomalous dimensions of twist-2 operators when the spin j becomes large. In particular, unlike [141] we distinguish the scalar and the pseudo-scalar fields and compute the eigenvalues of the anomalous dimension matrix. We find that two of the eigenvalues coincide with the universal eigenvalues obtained in [141] after finite shifts and the remaining two coincide with the universal ones only in the large j limit. Here we wish to point out few checks on the validity of our calculation of splitting functions:

- Many of the splitting functions are calculated by considering completely different set of processes and they are found to be identical.
- Our splitting function results satisfy the identity in Eq. (2.3.9).
- The LT terms of SV cross sections calculated in this paper matches exactly with the SM counterparts which provides third but not last non-trivial check on our computation. We elaborate further on this point below.

We have investigated the structure of infrared safe cross sections resulting after collinear factorisation. We find that the LT terms of SV part of the cross sections agree with that of Drell-Yan or Higgs production cross sections in QCD when we set $C_A = C_F = n_f = N$ in the latter. This corresponds to leading transcendentality principle advocated in [15, 43, 139–141] between the anomalous dimensions of twist-2 Wilson operators in $\mathcal{N} = 4$ SYM theory and those of splitting functions in QCD. In addition, we find that the soft parts of the cross sections for the half-BPS, T and Konishi are all identical and are independent of incoming states. We extract the soft distribution functions from inclusive cross sections and found that they are process independent, namely they do not depend on the incoming states and also on the nature of singlet final state. This distribution up to second order in a coincides with that of Drell-Yan or Higgs production when $C_A = C_F = n_f = N$ in QCD. This is again an example for the leading transcendentality principle in the context of soft distribution functions in inclusive scattering cross sections. Extending this principle to third order in a and using the three loop FFs of the half-BPS,T and Konishi and the third order soft distribution function obtained from Drell-Yan or Higgs production cross sections, we have predicted third order inclusive cross section $\Delta^{I,(3),\mathrm{SV}}$ for I = half-BPS,T and Konishi. Our prediction for the half-BPS agrees with the result obtained by explicit computation in [160]. $\Delta^{\mathrm{T},(3),\mathrm{SV}}$ coincides identically with the half-BPS because because their three loop FFs are also identical to each other. For the Konishi, the SV part of the cross section contains sub-leading transcendental terms unlike the case of the half-BPS or T but the leading ones coincide with those of the half-BPS and T. In summary, collinear finite inclusive cross sections in $\mathcal{N} = 4$ SYM theory provide several valuable informations on the perturbative IR structure of the theory.

The Mellin j-space results for two-loop splitting functions

In the following, we list the results of two-loop splitting functions after transforming them into Mellin j-space. Using Eq. (2.1.21) order by order in perturbation theory and splitting function results in Eq. (2.3.4), we obtain

$$
\begin{aligned}
\gamma^{(1)}_{\phi\phi,j} &= \frac{24}{j-1} + \frac{24}{j^2} - \frac{112}{3j} - \frac{24}{(j+1)^2} + \frac{40}{j+1} - \frac{16}{(j+2)^2} - \frac{24}{(j+2)} \\
&\quad + 2\hat{Q}(j) + \frac{8}{3} S_1(j-1)\,, \\
\gamma^{(1)}_{gg,j} &= -\frac{1072}{9(j-1)} - \frac{32}{j^3} + \frac{144}{j^2} + \frac{248}{3j} + \frac{112}{(j+1)^2} - \frac{208}{j+1} - \frac{32}{(j+2)^3} + \frac{352}{3(j+2)^2} \\
&\quad + \frac{2224}{9(j+2)} - 32K(j-1) + 32K(j) - 32K(j+1) + 32K(j+2) + 2\hat{Q}(j)
\end{aligned}
$$

$$+\frac{8}{3}S_1(j-1)\,,$$

$$\gamma^{(1)}_{\lambda\lambda,j} = \frac{640}{9(j-1)} + \frac{64}{j^3} - \frac{192}{j^2} + \frac{8}{3j} - \frac{64}{(j+1)^2} + \frac{128}{j+1} - \frac{256}{3(j+2)^2} - \frac{1792}{9(j+2)}$$
$$-64K(j) + 64K(j+1) + 2\hat{Q}(j) + \frac{8}{3}S_1(j-1)\,,$$

$$\gamma^{(1)}_{\lambda g,j} = \frac{640}{9(j-1)} + \frac{64}{j^3} - \frac{192}{j^2} - \frac{320}{(j+1)^2} + \frac{896}{(j+1)} + \frac{128}{(j+2)^3} - \frac{1792}{3(j+2)^2}$$
$$-\frac{8704}{9(j+2)} - 64K(j) + 128K(j+1) - 128K(j+2)\,,$$

$$\gamma^{(1)}_{\phi g,j} = \frac{24}{j-1} + \frac{24}{j^2} - \frac{40}{j} + \frac{104}{(j+1)^2} - \frac{344}{(j+1)} - \frac{48}{(j+2)^3} + \frac{240}{(j+2)^2} + \frac{360}{(j+2)}$$
$$-48K(j+1) + 48K(j+2)\,,$$

$$\gamma^{(1)}_{g\lambda,j} = -\frac{1072}{9(j-1)} - \frac{32}{j^3} + \frac{144}{j^2} + \frac{80}{j} + \frac{64}{3(j+2)^2} + \frac{352}{9(j+2)} + \frac{32}{(j+1)^2}$$
$$-32K(j-1) + 32K(j) - 16K(j+1)\,,$$

$$\gamma^{(1)}_{\phi\lambda,j} = \frac{24}{j-1} + \frac{24}{j^2} - \frac{40}{j} + \frac{16}{(j+1)^2} - \frac{64}{(j+1)} + \frac{32}{(j+2)^2} + \frac{80}{(j+2)} - 24K(j+1)\,,$$

$$\gamma^{(1)}_{g\phi,j} = -\frac{1072}{9(j-1)} - \frac{32}{j^3} + \frac{144}{j^2} + \frac{240}{3j} - \frac{16}{(j+1)^2} + \frac{48}{(j+1)} - \frac{32}{3(j+2)^2} - \frac{80}{9(j+2)}$$
$$-32K(j-1) + 32K(j)\,,$$

$$\gamma^{(1)}_{\lambda\phi,j} = \frac{640}{9(j-1)} + \frac{64}{j^3} - \frac{192}{j^2} + \frac{64}{(j+1)^2} - \frac{128}{(j+1)} + \frac{128}{3(j+2)^2} + \frac{512}{9(j+2)}$$
$$-64K(j)\,,$$

$$\gamma^{(1)}_{\chi\phi,j} = \frac{24}{j-1} - \frac{24}{(j+1)^2} + \frac{40}{(j+1)} - \frac{16}{(j+2)^2} - \frac{24}{(j+2)} + \frac{24}{j^2} - \frac{40}{j}, \tag{2.4.1}$$

where

$$K(j) = \frac{S_1(j)}{j^2} + \frac{S_2(j)}{j} + \frac{\hat{S}_2(j)}{j}\,,$$
$$\hat{Q}(j) = -\frac{4}{3}S_1(j) + 16S_1(j)S_2(j) + 8S_3(j) - 88\hat{S}_3(j) + 16\hat{S}_{1,2}(j)\,,$$
$$S_k(j) = \sum_{i=1}^{j}\frac{1}{i^k}\,, \quad \hat{S}_k(j) = \sum_{i=1}^{j}\frac{(-1)^i}{i^k}\,, \quad \hat{S}_{k,l}(j) = \sum_{i=1}^{j}\frac{\hat{S}_l(i)}{i^k}\,. \tag{2.4.2}$$

Eigenvalues of the anomalous dimension matrix

In the following, we list the expressions for the eigenvalues of the anomalous dimension matrix in

Mellin j-space and they are found to be

$$
\begin{aligned}
\lambda_1 &= \frac{8}{3j} + \frac{32}{j^3} - 32K(j) - 32K(j-1) + 2\hat{Q}(j) + \frac{8}{3}S_1(j-1), \\
\lambda_2 &= \frac{8}{3j} + 2\hat{Q}(j) + \frac{8}{3}S_1(j-1), \\
\lambda_3 &= \mathcal{E}_1 + 16\sqrt{\mathcal{E}_2}, \\
\lambda_4 &= \mathcal{E}_1 - 16\sqrt{\mathcal{E}_2},
\end{aligned}
\tag{2.4.3}
$$

with

$$
\begin{aligned}
\mathcal{E}_1 =& \frac{8}{3j} - \frac{16}{(j+2)^3} + 16K(j+1) + 16K(j+2) + 2\hat{Q}(j) + \frac{8}{3}S_1(j-1), \\
\mathcal{E}_2 =& \frac{64}{(j+1)^2} + \frac{8}{(j+1)^3} - \frac{8}{(j+1)^4} + \frac{64}{(j+2)^2} + \frac{8}{(j+2)^3} - \frac{24}{(j+2)^4} + \frac{1}{(j+2)^6} \\
& - \frac{128}{(j+1)(j+2)} - \frac{8}{(j+1)^2(j+2)} - \frac{8}{(j+1)(j+2)^2} - \frac{40}{(j+1)^2(j+2)^2} \\
& + \frac{8}{(j+1)(j+2)^3} + \frac{8}{(j+1)^2(j+2)^3} + K(j+1)\left[\frac{8}{j+1} - \frac{8}{j+2} + \frac{8}{(j+2)^2}\right. \\
& \left. - \frac{2}{(j+2)^3}\right] - K(j+2)\left[\frac{8}{j+1} + \frac{8}{(j+1)^2} - \frac{8}{j+2} + \frac{2}{(j+2)^3}\right] \\
& + \Big(K(j+1) + K(j+2)\Big)^2,
\end{aligned}
\tag{2.4.4}
$$

where $K(j)$, $\hat{Q}(j)$ and $S_1(j)$ can be found in Eq. (2.4.2).

3 Form factors with two-operator insertion and principle of maximal transcendentality

The materials presented in this chapter are the result of an original research done in collaboration with Taushif Ahmed, Pulak Banerjee, Prasanna K. Dhani and V. Ravindran, an these are based on the published article [17].

Contents

3.1 Prologue

A generic quantum field theory is entirely specified by the knowledge of on-shell scattering amplitudes and off-shell correlation functions. There exists another class of fascinating objects, called form factors (FFs), which interpolate between amplitudes and correlators. This object is defined through the overlap between a state created by the action of composite gauge invariant operators on the vacuum and a state consisting of only on-shell particles. The FFs in maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ sYM) are expected to inherit much of the remarkable simplicity of the on-shell amplitudes, and at the same time to reflect some of the non-trivial behaviour of the off-shell correlators. In past few decades, FFs have been studied extensively starting from the seminal works in refs. [6, 7, 84, 85, 146, 161]. Very recently, the first step is taken to go beyond the horizon of FFs with single operator insertion and the scenario with two-operator insertion is addressed in ref. [34]. In this article, we take this step forward by performing a state-of-the-art computation to explore the nature of two-loop two-point FFs with insertions of two identical operators. Consequently, for the first time, we examine the validation of several conjectures in view of generalised FFs.

In this work, we consider two local gauge invariant operators:

$$\mathcal{O}_{rt}^{\rm BPS} = \phi_r^b \phi_t^b - \frac{1}{3}\delta_{rt}\phi_s^b\phi_s^b\,, \qquad\qquad \mathcal{O}^{\mathcal{K}} = \phi_r^b\phi_r^b + \chi_r^b\chi_r^b, \tag{3.1.1}$$

where $\mathcal{O}_{rt}^{\rm BPS}$ and $\mathcal{O}^{\mathcal{K}}$ are the SUSY protected half-BPS primary belonging to the stress-energy supermultiplet and unprotected Konishi operators, respectively. The scalar and pseudo-scalar fields are denoted by ϕ_r^b and χ_s^b, respectively, where their number of generations is represented through $r, s, t \in [1, n_g]$ with $n_g = 3$ in 4-dimensions. All the fields in $\mathcal{N} = 4$ sYM theory transform under adjoint representation which is represented through the SU(N) colour index b.

Understanding the analytical structures of on-shell amplitudes and FFs in $\mathcal{N} = 4$ sYM has been an active area of investigation, not only to uncover the hidden structures of these quantities but also to establish the connections with other gauge theories, such as QCD. One of the most intriguing facts is the appearance of uniform transcendental[1] (UT) weight terms in certain class of quantities in $\mathcal{N} = 4$ sYM. This is indeed an observational [5, 8, 10, 13, 43, 140, 144, 146, 147, 149, 150], albeit unproven fact. The two-point or Sudakov FFs of primary half-BPS operator belonging to the stress-energy supermultiplet is observed [6–8] to exhibit the UT property to three-loops, more

[1]The transcendentality weight, τ, of a function, f, is defined as the number of iterated integrals required to define it, e.g. $\tau(\log) = 1\,, \tau(\mathrm{Li}_n) = n\,, \tau(\zeta_n) = n$ and moreover, we define $\tau(f_1 f_2) = \tau(f_1) + \tau(f_2)$. Algebraic factors are assigned to weight zero and dimensional regularisation parameter ϵ to -1.

specifically, they are composed of only highest transcendental (HT) terms with weight 2L at loop order L. This is a consequence of the existence of an integral representation of the FFs with every Feynman integral as UT [8]. Knowing the existence of such a basis has profound implications in choosing the basis of integrals while evaluating Feynman integrals using differential equations method [9]. The three-point FFs of half-BPS operator is also found to respect this wonderful UT property [10]. On the contrary, this property fails for the two- [11, 12] and three-point [13] FFs of the unprotected Konishi operator which are investigated up to three- and two-loops, respectively. Three-point FFs of a Konishi descendant operator is also found not to exhibit the UT property [14]. All the aforementioned results are in accordance with the general belief that the FFs of a supersymmetry (SUSY) protected operator, such as half-BPS primary, exhibit UT behaviour. Having seen the beautiful property of UT in FFs of one-operator insertion, the question arises whether it is respected for the two-point FFs with SUSY protected two-operator insertion and whether this property can be extrapolated to generalised FFs with n-number of operators insertion. In this article, for the first time, we address this question and, we find that the UT property does not hold true at two-loop for the FF of double half-BPS insertion.

It is conjectured in ref. [35] that the HT weight parts of every two-point minimal FFs (presence of equal number of fields in the operator and external on-shell state) are identical and those are equal to that of half-BPS, $\mathcal{O}_{rt}^{\text{BPS}}$. This conjecture is verified to four-loops order in ref. [12] for the Sudakov FFs of operator $\mathcal{O}^{\mathcal{K}}$. Naturally, it is curious to see if this conjecture holds true for the generalised two-point FFs with two-operator insertion. In particular, we address whether the HT parts of the two-point FFs with double $\mathcal{O}_{rt}^{\text{BPS}}$ and double $\mathcal{O}^{\mathcal{K}}$ insertion match. It turns out they are different both at one- and two-loop.

The connection between quantities in $\mathcal{N} = 4$ sYM and that of QCD is of fundamental importance. In addition to deepening our theoretical understanding, it is motivated from the fact that computing a quantity in QCD is much more difficult, and in the absence of our ability to calculate a quantity in QCD, if it is possible to obtain the result, at least partially, from that of simpler theory, such as $\mathcal{N} = 4$ sYM. In refs. [5, 15, 42, 43], it is found that the anomalous dimensions of leading twist-two-operator in $\mathcal{N} = 4$ sYM are identical to the HT counterparts in QCD [16], and consequently, the principle of maximal transcendentality (PMT) is conjectured. The PMT says that the algebraically most complex part of certain quantities in $\mathcal{N} = 4$ sYM and QCD are identical. The conjecture is found to hold true for two-point FFs to three-loops level [8], more specifically, the HT pieces of quark and gluon FFs in QCD [44] are identical, up to a normalization factor of 2^L, to scalar FFs of the operator $\mathcal{O}^{\text{BPS}}_{rt}$ in $\mathcal{N} = 4$ sYM upon changing the representation of fermions in QCD from fundamental to adjoint. The diagonal elements of the two-point pseudo-scalar [156] and tensorial FFs [45,46] also obey the conjecture. The three-point scalar and pseudo-scalar FFs are also found to respect the PMT [10,35,38–40,47–49,88]. Employing this conjecture, the four-loop collinear anomalous dimension in the planar $\mathcal{N} = 4$ sYM is computed [50]. In ref. [162], the asymptotic limit of energy-energy correlator and in ref. [5], the soft function are also observed to be consistent with PMT. However, the complete domain of validity of this principle is still unknown. For on-shell amplitudes, it fails even at one loop [51] with four or five external gluons. In this article, we investigate whether the wonderful conjecture of PMT holds true for two-point FFs of two-operator insertion with $\mathcal{O}^{\text{BPS}}_{rt}$. Surprisingly, we find that the PMT also doesn't hold true.

3.2 Computation of Two-Loop Form Factor

The Lagrangian density [20–23] encapsulating the dynamics of $\mathcal{N} = 4$ sYM and describing the interactions with the gauge invariant local operators, $\mathcal{O}^{\text{BPS}}_{rt}$ and $\mathcal{O}^{\mathcal{K}}$, is given by

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \mathcal{J}^{\text{BPS}}_{rt}\mathcal{O}^{\text{BPS}}_{rt} + \mathcal{J}^{\mathcal{K}}\mathcal{O}^{\mathcal{K}}. \tag{3.2.1}$$

The quantity $\mathcal{J}$ represents the off-shell state described by the corresponding operator. We are interested in investigating the two-point FFs with two-operator insertion of the following scattering processes

$$\phi(p_1) + \phi(p_2) \to \begin{cases} \mathcal{J}^{\text{BPS}}(p_3) + \mathcal{J}^{\text{BPS}}(p_4)\,, \\ \mathcal{J}^{\mathcal{K}}(p_3) + \mathcal{J}^{\mathcal{K}}(p_4)\,, \end{cases} \tag{3.2.2}$$

where p_i are the corresponding four-momentum with $p_1^2 = p_2^2 = 0$ and $p_3^2 = p_4^2 = m_\lambda^2 \neq 0$. The m_λ^2 is the invariant mass square of the colour singlet state described by the operators in (3.1.1) i.e. $\lambda \in \{\text{BPS}, \mathcal{K}\}$. The underlying Mandelstam variables are defined as $s \equiv (p_1 + p_2)^2$, $t \equiv (p_1 - p_3)^2$ and $u \equiv (p_2 - p_3)^2$ satisfying $s + t + u = 2m_\lambda^2$. For convenience, we introduce the dimensionless variables x, y, z through

$$s = m_\lambda^2 \frac{(1+x)^2}{x}, \quad t = -m_\lambda^2 y, \quad u = -m_\lambda^2 z. \tag{3.2.3}$$

The quantity x is called the Landau variable. The aforementioned choice of variables are necessary to rationalise the square root, $\sqrt{s(s-4m_\lambda^2)}$, which appears in the differential equation. The square root is associated to the threshold of two massive particles production.

In perturbation theory, the scattering amplitude of the processes (3.2.2) can be expanded in powers of the 't Hooft coupling constant, $a \equiv g^2 N (4\pi e^{-\gamma_E})^{-\frac{\epsilon}{2}}/(4\pi)^2$, as $|\mathcal{M}\rangle_\lambda = \sum_{n=0}^{\infty} a^n |\mathcal{M}^{(n)}\rangle_\lambda$ where the quantity $|\mathcal{M}^{(n)}\rangle_\lambda$ represents the n-th loop amplitude of the process involving $\mathcal{J}^\lambda$. The quadratic Casimir in the adjoint representation of SU(N) group is given by N. The dimensional regulator, ϵ, is defined through $d = 4+\epsilon$ with the space-time dimension d. We regulate the theory by adopting a SUSY preserving modified dimensional reduction ($\overline{\text{DR}}$) scheme [24, 25] which keeps the number of bosonic and fermionic degrees of freedom equal. This is attained by representing the number of scalar and pseudo-scalar generations from $n = 3$ to $n_\epsilon = 3 - \epsilon/2$ in d-dimensions. The FFs are constructed out of the transition matrix elements through

$$\mathcal{F}_\lambda = 1 + \sum_{n=1}^{\infty} a^n \mathcal{F}_\lambda^{(n)} \equiv 1 + \sum_{n=1}^{\infty} a^n \frac{\langle \mathcal{M}^{(0)} | \mathcal{M}^{(n)} \rangle}{\langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle}\bigg|_\lambda . \tag{3.2.4}$$

The primary objective of this article is to calculate the FFs to two-loops i.e. $\mathcal{F}_\lambda^{(1)}$ and $\mathcal{F}_\lambda^{(2)}$. In figure 3.1, we show some examples of two-loop Feynman diagrams.

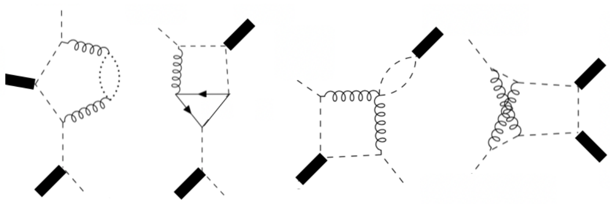


Figure 3.1. Sample of two-loop Feynman diagrams. The thick solid, thin solid, dashed, dotted, curly lines represent $\mathcal{J}^\lambda$, Majorana fermion, scalar, ghost, gluon, respectively.

In contrast to the widely used method of unitarity to evaluate the on-shell amplitudes and FFs in $\mathcal{N} = 4$ sYM, we employ the methodology based on Feynman diagrammatic approach. The relevant Feynman diagrams are generated using QGRAF [123]. Because of the presence of Majorana fermions in the theory, the generated diagrams are plagued with the wrong flow of fermionic currents which is rectified by an in-house algorithm based on PYTHON. There are 440 and 606 number of Feynman diagrams at two-loop for the production of double $\mathcal{J}_{rt}^{\mathrm{BPS}}$ and $\mathcal{J}^{\mathcal{K}}$, respectively. The diagrams are passed through a series of in-house codes based on symbolic manipulating program FORM [124] in order to apply the Feynman rules, perform spinor, Lorentz and colour algebra. To ensure the inclusion of only physical degrees of freedom of gauge bosons, we include the ghosts in the loop. The resulting expressions of the matrix elements are consist of plenty of the scalar Feynman integrals which are reduced to a much smaller set of integrals (called master integral or MI) employing integration-by-parts (IBP) identities [163, 164] with the help of LiteRed [125, 165]. The integrals belong to the class of four-point families with two off-shell legs of same loop order and these are computed in refs. [166, 167] as Laurent series expansion in dimensional regulator ϵ. Employing the results of the MIs, we obtain the FFs (3.2.4) to two-loops.

The on-shell amplitudes in $\mathcal{N} = 4$ sYM are ultraviolet (UV) finite in 4-dimensions due to vanishing β-function. However, the FFs can exhibit UV divergences if the underlying operator is not SUSY protected which, in the present context, gets reflected by the presence of UV poles in the FFs, $\mathcal{F}_{\mathcal{K}}$, arising from the unprotected Konishi operator $O^{\mathcal{K}}$. Being a property inherent to the operator, $O^{\mathcal{K}}$ needs to be renormalised through multiplication of an operator renormalisation constant, $Z_{\mathcal{K}}$, which reads $[O_{\mathcal{K}}]_R = Z_{\mathcal{K}} O_{\mathcal{K}}$. $[O_{\mathcal{K}}]_R$ represents the corresponding renormalised operator. The $Z_{\mathcal{K}}$ can be determined [12, 43, 73, 103–105] by solving the underlying renormalisation group equation and analysing its Sudakov FFs. The result in terms of its anomalous dimensions, $\gamma_{\mathcal{K}}$, is given by $Z_{\mathcal{K}} = \exp\left(\sum_{n=1}^{\infty} a^n 2\gamma_{\mathcal{K},n}/n\epsilon\right)$, $\gamma_{\mathcal{K},1} = -6$ and $\gamma_{\mathcal{K},2} = 24$. For the half-BPS operator, all the anomalous dimensions are identically zero, as guaranteed by the SUSY protection. The UV renormalised FFs are obtained through $[\mathcal{F}_{\lambda}]_R = Z_{\lambda}^2 \mathcal{F}_{\lambda}$.

The UV finite FFs contain soft and collinear (IR) divergences resulting from the low momentum and/or vanishing angle configurations of the loop momenta. The IR divergences are universal [93, 94, 97, 98] for an SU(N) gauge theory which can be expressed as exponentiation of a quantity containing universal light-like cusp and collinear anomalous dimensions. The UV renormalised FFs are found to exhibit the expected universal structures of the IR divergences which serve as the most stringent check of our calculation. We find that there is no additional UV divergence from the contact term of two operators, unlike the di-Higgs production through gluon fusion in heavy quark effective theory in QCD [168, 169]. From the perspective of operator product expansion (OPE), in principle, we could expect to encounter additional divergences in the form factors of two composite operators insertion. The contact terms between two operators are the source of these divergences. Through explicit computation of the FFs, we discover that there is no such contact divergence in the case of double half-BPS or Konishi operators and therefore, we do not need any additional UV counterterm. The absence of contact divergences for two-operator insertion is also found for the production of di-pseudo-scalar in heavy quark (infinite top mass limit) effective theory [170, 171] and di-Higgs boson in bottom quark annihilation [172].

The BDS/ABDK ansatz [71, 75], which says the maximally helicity violating (MHV) amplitude in planar $\mathcal{N} = 4$ sYM is exponentiated in terms of one-loop result along with the universal anomalous dimensions, gets violated for two-loop six-point amplitudes [77, 173]. In order to capture the deviation from the ansatz, a quantity called finite remainder is introduced [77, 173]. For the Sudakov FFs of half-BPS operator at two-loop [6], both the IR divergence and finite part are found to be exponentiated, however, the finite part stops exhibiting this nature at three-loop [8]. In order to capture the deviation, following the line of thought for the MHV amplitudes, a finite remainder function (FR) for the FFs at two-loop is introduced in ref. [7] which reads

$$\mathcal{R}^{(2)}_{\lambda} = \mathcal{F}^{(2)}_{\lambda}(\epsilon) - \frac{1}{2}\left(\mathcal{F}^{(1)}_{\lambda}(\epsilon)\right)^2 - f^{(2)}(\epsilon)\mathcal{F}^{(1)}_{\lambda}(2\epsilon) - C^{(2)},$$

with $f^{(2)}(\epsilon) = -2\zeta_2 + \epsilon\zeta_3 - \epsilon^2\zeta_2^2/5$ and $C^{(2)} = 8\zeta_2^2/5$. The quantities $f^{(2)}(\epsilon)$ and $C^{(2)}$ are independent of the number of operators and external states. Representing a two-loop FF in terms of a quantity dictated by BDS/ABDK ansatz plus an extra part provides a nice way of representing the deviation from the exponentiation - the ansatz part captures the universal IR divergences that exponentiates whereas the extra part encapsulates the finite part in 4-dimensions. We compute the FR for both the FFs at two-loops and conclude that the finite parts of the two-point FFs with two-operator insertion do not exponentiate, unlike the case of single half-BPS operator insertion at two-loop [6]. The results of the form factors and finite remainders are provided as Supplemental Material [174].

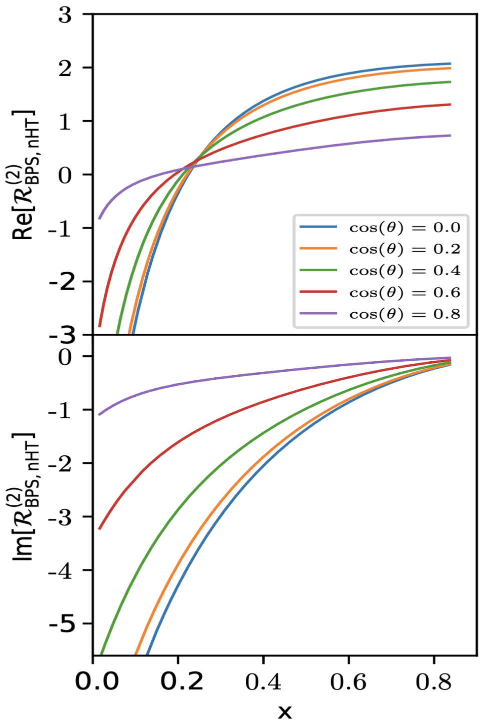


Figure 3.2. Next-to-highest transcendental terms in the two-loop finite remainder of double half-BPS operator

3.3 Principle of Uniform Transcendentality

It is a general belief, albeit based on observations, that the FFs and FRs of a SUSY protected operator, such as half-BPS, exhibit the behaviour of UT i.e. they contain only HT weight terms - commonly known as principle of uniform transcendentality (PUT). No deviation from this conjecture has ever been found. In this article, for the first time, we report that the property of UT does not extrapolate to the FFs of double insertions of SUSY protected operator. We find that though the FF of half-BPS primary is UT at one-loop, it no longer holds true at two-loop:

$$\mathcal{F}^{(1)}_{\text{BPS,nHT}} = 0\,, \qquad \mathcal{F}^{(2)}_{\text{BPS,nHT}} \neq 0\,, \tag{3.3.1}$$

where nHT represents the next-to-highest-transcendental terms. Remainder functions also obey same property. Therefore, the property of UT for SUSY protected operator can not be generalised to more general class of FFs with more than one-operator insertion. To be more specific, among the nHT terms at two-loop, only the transcendental 3 term is non-zero, the remaining lower ones identically vanish:

$$\mathcal{F}^{(2)}_{\mathrm{BPS,nHT}} = \mathcal{F}^{(2),\tau(3)}_{\mathrm{BPS}} \,, \qquad \mathcal{F}^{(2),\tau(<3)}_{\mathrm{BPS}} = 0 \,, \tag{3.3.2}$$

where the FFs are written as $\mathcal{F}^{(n)}_{\lambda} = \sum_{l=0}^{2n} \mathcal{F}^{(n),\tau(l)}_{\lambda}$. The $\tau(l)$ represents the terms with transcendentality weight l. Since the result is too big to be presented here, we provide a graphical presentation of the nHT terms of the FR, $\mathcal{R}^{(2)}_{\mathrm{BPS,nHT}}$, in figure 3.2 to demonstrate the dependence on scaling variables x and y. We plot the real (Re) and imaginary (Im) parts as a function of Landau variable x for the choices of cosθ, where θ is the angle between one of the particles corresponding to half-BPS operator and one of the initial state scalars in their center of mass frame, defined through $2m_\lambda^2 y = (s - 2m_\lambda^2 - \cos\theta\sqrt{s(s-4m_\lambda^2)})$. The FR is also seen to be invariant under cos` ↔ −cos`, as expected for a purely bosonic scattering. Since this symmetry is not used in the setup of the calculation, this serves as a strong check on the finite part of the results. For numerical evaluation of the polylogarithms, we make use of the package from ref. [175].

In contrast to the form factor of half-BPS, the UT is not a property for the two-point FFs with single insertion of unprotected Konishi operator which is verified to three-loops [11, 12]. The FFs with double insertions exhibit the behaviour consistent with this expectation:

$$\mathcal{F}^{(1)}_{\mathcal{K},\,\mathrm{nHT}} \neq 0 \,, \qquad \mathcal{F}^{(2)}_{\mathcal{K},\,\mathrm{nHT}} \neq 0 \,. \tag{3.3.3}$$

In ref. [35], in the context of FFs with one-operator insertion, it is conjectured that the HT weight parts of every two-point minimal FF, including that of Konishi, are identical to that of half-BPS. Through our computation, for the first time, we report the deviation from this conjecture, in particular, this property fails to hold true for the double insertions of operators. Our findings show that

$$\mathcal{F}^{(1)}_{\mathcal{K},\,\mathrm{HT}} \neq \mathcal{F}^{(1)}_{\mathrm{BPS,\,HT}} \,, \qquad \mathcal{F}^{(2)}_{\mathcal{K},\,\mathrm{HT}} \neq \mathcal{F}^{(2)}_{\mathrm{BPS,\,HT}} \,. \tag{3.3.4}$$

Hence, the conjecture fails, in general, to be extrapolated for the case involving two-operator insertion. In addition to the analytical check, we verify it numerically which is displayed graphically through figure 3.3.

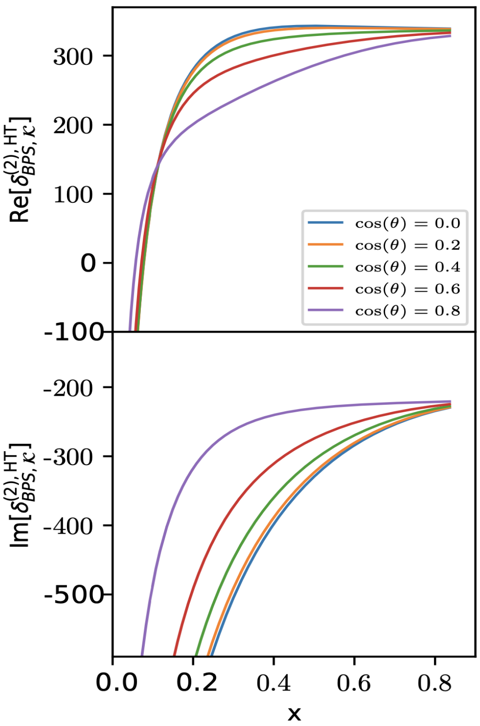


Figure 3.3. Difference (δ) between the highest transcendental terms of two-loop finite remainders of double half-BPS and Konishi

3.4 Principle of Maximal Transcendentality

The conjecture of PMT establishes a bridge between $\mathcal{N} = 4$ sYM and QCD. It states that the HT terms of certain quantities in $\mathcal{N} = 4$ sYM and QCD are identical upon converting the fermions in QCD from fundamental to adjoint representation through $C_A = C_F = 2n_f T_F = N$, where C_A and C_F are the Casimirs in adjoint and fundamental representations, respectively, n_f is the number of light quark flavours and T_F is the normalisation factor in fundamental representation. The conjecture is found to hold true to three loops for two-point FFs with one-operator insertion while comparing the quark/gluon FFs in QCD and that of half-BPS primary. The question arises if the PMT carries over to more general class of FFs and correlation functions. Through our computations, for the first time, we find that the PMT doesn't hold true for the FFs with two-operator insertion. The HT terms of two-point FFs of double half-BPS operator do not match with that of the di-Higgs produced either through gluon fusion [169] or through bottom quark annihilation [172]:

$$\mathcal{F}^{(n)}_{\text{BPS,HT}} \neq \mathcal{F}^{(n)}_{gg\to HH,\text{HT}} \neq \mathcal{F}^{(n)}_{b\bar{b}\to HH,\text{HT}}\,. \tag{3.4.1}$$

Consequently, the conjecture of PMT can not be extrapolated to general class of FFs and correlation functions.

3.5 Regge and collinear limit and universality of the leading transcendental term

Though the PUT and PMT do not hold true for the double half-BPS operator, we intend to see whether it can be restored in some kinematic limit. Keeping this in mind, we investigate the behaviour of the nHT terms analytically with the help of `PolyLogTools` [176] in the kinematic regime captured by x→ 0 and $\cos\theta \to 1$. The former one corresponds to the Regge limit whereas the latter makes one of the color and colorless particles collinear, and thereby effectively converting it into a three point scattering. Upon taking the limits simultaneously, the entire nHT term is found to vanish for the double-BPS form factor at two-loop, which implies the restoration of the PUT:

$$\lim_{\substack{x\to 0\\ \cos\theta\to 1}} \mathcal{F}^{(2)}_{\text{BPS,nHT}} = 0\,. \tag{3.5.1}$$

Moreover, the PMT also gets restored to two-loops only for the di-Higgs boson production through gluon fusion:

$$\lim_{\substack{x\to 0\\ \cos\theta\to 1}} \left[\mathcal{F}^{(n)}_{gg\to HH,\text{HT}} = \mathcal{F}^{(n)}_{\text{BPS,HT}} \neq \mathcal{F}^{(n)}_{b\bar{b}\to HH,\text{HT}}\right]. \tag{3.5.2}$$

We also find that in this limit the difference between the HT terms of half-BPS and Konishi FFs disappears, and thereby the conjecture of having identical HT weight parts of every two-point minimal FF is also reinstated.

The underlying reason behind the transcendentality principles is not well-understood till date. From the perspective of OPE, the generic expansion of the correlator of double operator insertion should contain not only SUSY protected but also unprotected operators, and therefore, a priori there is no reason to believe the existence of transcendentality principles for FFs involving multiple operator insertion. Either the OPE or an explicit computation of the FFs can only reveal the true nature. Our experience also shows that the transcendentality principles for the minimal FFs can be correlated to the existence of a complete factorisation of the leading order amplitude at any loop order under consideration. For the form factor of double operator insertion, the leading order amplitude does not factorise from the complete result either at one- or two-loop. Consequently, the transcendentality principles, in general, fail to hold true for the double operator insertion, which however gets reinstated in the simultaneous Regge and collinear limit. It leads us to the understanding that essentially the process dependence is embedded into the leading order result and therefore, upon a successful factorisation of the leading order amplitude at certain loop order, we end up with a quantity exhibiting universal nature from the perspective of transcendentality.

3.6 Conclusions

For the first time, we present the form factors with insertions of two identical local gauge invariant operators to two-loops in $\mathcal{N} = 4$ sYM theory by performing a state-of-the-art computation. In particular, we compute the scalar FFs with double insertions of half-BPS primary and Konishi operators. Through this calculation, we take a step forward to go beyond the FFs of single operator insertion and enter into the domain of more general class of FFs. To validate our computations, we check the infrared poles which agree with the predictions. Moreover, the appearance of expected kinematic symmetry inherent to the bosonic FFs provides a strong check on the finite parts of our results.

The findings enable us to reach a number of important conclusions. For the first time, the conjecture that the FFs of SUSY protected operators are always UT is found not to hold true at two-loop for the FFs of double-operator insertion, in sharp contrast to the Sudakov FFs. In particular, though the FF of double half-BPS primary is UT at one-loop, it fails to exhibit this property at two-loop. In accordance with our expectation, we find the FFs of SUSY unprotected operator, Konishi, to be not UT. From our experience, we find that the factorisation of the leading order amplitude at any loop order can be accounted for the existence of UT property.

The conjecture that the highest transcendentality weight terms of every two-point minimal FF are identical to that of half-BPS is also found not to hold true for two-operator insertion. In other words, the HT weight terms of unprotected Konishi are not identical to that of half-BPS both at one- and two-loop.

We find that the principle of maximal transcendentality, which says the highest weight terms of quark/gluon FFs in QCD are identical to that of scalar FFs of half-BPS primary, can not be extrapolated to the case of two-operator insertion. The HT weight terms of double half-BPS FFs do not match with that of di-Higgs production through gluon fusion or bottom quark annihilation in QCD, observed for the first time in this context. In the simultaneous Regge and collinear limits, the transcendentality principles are found to be restored.

By computing the finite remainder function at two-loop, we confirm that the finite parts of the FFs of double half-BPS do not exponentiate, in contrast to the corresponding FFs of single half-BPS at two-loop [6]. Moreover, we discover that there is no contact divergence in the case of double half-BPS or Konishi operators and therefore, we do not need any additional UV counterterm. Our work, in addition to providing a better understanding of the nature of generalised FFs, opens the door for further analytic calculations of general class of FFs.

Also, our investigation on the FF of double operator insertion will serve the very important data towards the understanding Wilson-loop (WL) / FF duality, specially on the WL dual of these 3.1.1 two-operator FFs. Right now the only FF for which the WL dual is well established is the single 1/2-BPS: the dual is an infinite periodic Wilson line [7]. The assumption is that other operators like Konishi are also dual to infinite Wilson lines, but with more complicated periodicity conditions. It very well might be that two 1/2-BPS operator FF has a simpler WL double than the single Konishi FF.

4 Three point form factor and violation of trancendentality principle

The materials presented in this chapter are the result of an original research done in collaboration with Taushif Ahmed, Pulak Banerjee, Prasanna K. Dhani and V. Ravindran, an these are based on the article [32].

Contents

4.1 Prologue

On-shell scattering amplitudes and off-shell correlation functions are quantities of fundamental importance in any gauge theory. A generic quantum field theory is completely specified by the knowledge of these quantities. Form factors (FFs), the overlap of an n-particle on-shell state with a state created by the action of a local gauge invariant operator on the vacuum, are a fascinating bridge between aforementioned two quantities which have been studied extensively in quantum chromodynamics (QCD) and $\mathcal{N} = 4$ super Yang-Mills (SYM) theory:

$$\mathcal{F}_{\mathcal{O}}(1, \ldots, n; q) \equiv \langle 1, \ldots, n|\mathcal{O}(0)|0\rangle \,. \tag{4.1.1}$$

The numbers $1, \ldots, n$ denote the on-shell particles and $q^2 \neq 0$ is the off-shell momentum associated with the operator. The central objects which are considered in this article are two- and three-point FFs. Very recently, the first calculations of generalisations of such FFs to the case of two operator insertions with one non-protected operator was performed [34] by some of us.

It is conjectured in ref. [35] that at two-loop level the HT weight parts of every two-point minimal FF (number of fields present in the operator is same as number of external on-shell states) are identical and those are same as that of half-BPS operator belonging to the stress-tensor supermultiplet [6]. In ref. [12], this conjecture is verified to three-loops level for the FF of unprotected Konishi operator. In ref. [36], the minimal FFs of long BPS operators (more than two fields) are computed to two-loops, and the corresponding HT piece is found to be a universal in all FFs with long, unprotected operators [35, 37–41].

It is also conjectured in [35], that the HT terms of two-loop remainder function of the three-point FF of every length-two operator should agree with the corresponding half-BPS remainder found in ref. [10]. The latter conjecture is falsified in ref. [13] where, for the first time, it is shown that for three-point FF of the Konishi operator (length-two), the HT part depends on the nature of external on-shell states; it fails to match with that of half-BPS if the external on-shell states are $\phi\lambda\lambda$.

In past few decades, people have been investigating the connection among quantities computed in different gauge theories. In particular, the connection between on-shell amplitudes or FFs of $\mathcal{N} = 4$ SYM and that of QCD are of fundamental importance. Besides theoretical understanding, this is motivated from the fact that calculating any quantity in QCD is much more complex and in absence of our ability to calculate a quantity in QCD, whether it is possible to get the result, at least partially, from some other simpler theory. In refs. [15, 42, 43], the connection between anomalous dimensions of leading twist-two operators of these theories is found and it is shown that the results in $\mathcal{N}$ = 4 SYM is related to the HT part of the QCD results and consequently, the principle of maximal transcendentality (PMT) is conjectured. Same conclusion is obtained by some of us in [5] through a different procedure based on momentum fraction space. At this level, this connection involves only pure number. This property is later examined in the context of two-point FFs in [8] to three-loops level and surprisingly, found to hold true: the HT pieces of quark (vector interaction) and gluon (scalar interaction) FFs in QCD [44] are identical to scalar FFs of half-BPS operator in $\mathcal{N}$ = 4 SYM upon changing the representation of fermions in QCD from fundamental to adjoint. The same behaviour is also found for the quark and gluon two-point FFs [45, 46] associated to tensorial interaction through energy-momentum tensor. In refs. [10, 38–40, 47], the same behaviour is found to replicate for three-point scalar and pseudo-scalar FFs. This is the first scenario where non-trivial kinematics is involved and the validity of this principle implies this not only holds true for pure numbers but also for functions containing non-trivial kinematic dependence. Even after including the dimension seven operators in the effective theory of Higgs boson in the Standard model, PMT is also found to hold true [39, 48, 49] for three-point FFs. Using this principle, the four-loop collinear anomalous dimension in planar $\mathcal{N}$ = 4 SYM is determined in ref. [50]. The complete domain of validity of this principle is still not fully clear and it is under active investigation. For on-shell scattering amplitudes, it breaks down even at one loop [51] for cases with four or five external gluons. We consider Energy-momentum tensor (T) operator eq. 1.2.10 in $\mathcal{N}$ = 4 is considered in this article, has been studied extensively in the context of three-point FFs in QCD [52, 53], and is found to behave like the half-BPS at one- as well as two-loop order and it is independent of the associated external on-shell states. This is in accordance with our classical expectation: since the stress-tensor is protected and lies in the same multiplet as the half-BPS, an exact SUSY Ward identity would relate these two FFs. Quite surprisingly, the process with three external partons violates the PMT even at one-loop while comparing the corresponding quantities in QCD! This is observed for the first time at the level of three-point FF.

4.2 Theoretical framework

The underlying Lagrangian density encapsulating the interaction of the off-shell states described by gauge invariant local operators, defined in eq. 1.2.10, to the fundamental fields of $\mathcal{N} = 4$ SYM is given by

$$\mathcal{L}^{i}_{\text{int}} = J^{i} O^{i} . \tag{4.2.1}$$

So, the full underlying Lagrangian for the scattering processes under consideration reads

$$\mathcal{L}^{i} = \mathcal{L}_{\mathcal{N}=4} + \mathcal{L}^{i}_{\text{int}} . \tag{4.2.2}$$

We define the kinematic invariants of the processes through

$$s \equiv (p_1 + p_2)^2 , \quad t \equiv (p_2 + p_3)^2 \quad \text{and} \quad u \equiv (p_3 + p_1)^2 , \tag{4.2.3}$$

satisfying $s + t + u = q^2$. We also introduce dimensionless invariants out of these as

$$x \equiv \frac{s}{q^2} , \quad y \equiv \frac{u}{q^2} \quad \text{and} \quad z \equiv \frac{t}{q^2} \tag{4.2.4}$$

which satisfies the constraint $x + y + z = 1$. All the results are presented in terms of Multiple Polylogarithms (MPLs) [177] containing these parameters. In the next section, we define the form factors in terms of matrix elements.

$$T(q) \quad \rightarrow \begin{cases} g(p_1) + g(p_2) + g(p_3) \\ g(p_1) + \phi(p_2) + \phi(p_3) \\ g(p_1) + \lambda(p_2) + \bar{\lambda}(p_3) \\ \phi(p_1) + \lambda(p_2) + \bar{\lambda}(p_3) . \end{cases} \tag{4.2.5}$$

4.3 Three-point form factors

In perturbation theory, any scattering amplitude can be expressed as expansion in powers of coupling constant

$$|\mathcal{M}\rangle^i_{\{C\}} = \quad a^\kappa \sum_{n=0}^{\infty} a^n |\mathcal{M}^{(n)}\rangle^i_{\{C\}} \tag{4.3.1}$$

where the quantity $|\mathcal{M}^{(n)}\rangle^i_{\{C\}}$ represents the n-th loop amplitude of the scattering process depicted in (**??**). The expansion parameter, a, is the 't Hooft coupling [75] given by

$$a \equiv \frac{g^2 N}{(4\pi)^2}(4\pi e^{-\gamma_E})^\epsilon \tag{4.3.2}$$

with the Euler constant $\gamma_E \approx 0.5772$. N is the quadratic Casimir of SU(N) group in adjoint representation. κ in (4.3.1) corresponds to the power of coupling constant of the leading order amplitude. Form factors are constructed out of the transition matrix elements through

$$\mathcal{F}^i_{\{C\}} = 1 + \sum_{n=1}^{\infty} a^n \mathcal{F}^{(n),i}_{\{C\}} \equiv 1 + \sum_{n=1}^{\infty} a^n \left. \frac{\langle \mathcal{M}^{(0)} | \mathcal{M}^{(n)} \rangle}{\langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle} \right|^i_{\{C\}} . \tag{4.3.3}$$

The goal of this article is to calculate the three-point FFs of the scattering process listed in (**??**) to two-loops

4.4 Infrared factorisation

Being a massless theory, the UV renormalised FFs exhibit infrared divergences (IR) from soft and collinear configurations. These appear as poles in dimensional regularisation parameter ϵ, whose universal nature was first demonstrated in [93]. In [94], a detailed derivation was presented by exploiting the factorisation and resummation properties of scattering amplitudes which was later generalised to all orders in [97,98]. The IR divergences can be factored out from UV renormalised quantities through the subtraction operators $\mathbf{I}^{(n)}_{\{C\}}$ which up to two-loops read

$$|\mathcal{M}^{(1)}\rangle^i_{\{C\}} = 2\mathbf{I}^{(1)}_{\{C\}}(\epsilon)|\mathcal{M}^{(0)}\rangle^i_{\{C\}} + |\mathcal{M}^{(1)}_{\text{fin}}\rangle^i_{\{C\}} ,$$

$$|\mathcal{M}^{(2)}\rangle^{i}_{\{C\}} = 4\mathbf{I}^{(2)}_{\{C\}}(\epsilon)|\mathcal{M}^{(0)}\rangle^{i}_{\{C\}} + 2\mathbf{I}^{(1)}_{\{C\}}(\epsilon)|\mathcal{M}^{(1)}\rangle^{i}_{\{C\}} + |\mathcal{M}^{(2)}_{\text{fin}}\rangle^{i}_{\{C\}}\,. \tag{4.4.1}$$

The subtraction operators are independent of the nature of operator insertion, it solely depends on the SU(N) coloured particles. In case of $\mathcal{N} = 4$ SYM where all the fundamental particles transform under the adjoint representation of SU(N) gauge group and coefficients of β-function vanishes to all orders, these take the form

$$\begin{aligned}
\mathbf{I}^{(1)}_{C_1C_2C_3}(\epsilon) &= -\frac{e^{-\frac{\epsilon}{2}\gamma_E}}{\Gamma(1+\frac{\epsilon}{2})}\left(\frac{2}{\epsilon^2}\right)\left\{\left(-\frac{s}{\mu^2}\right)^{\frac{\epsilon}{2}} + \left(-\frac{t}{\mu^2}\right)^{\frac{\epsilon}{2}} + \left(-\frac{u}{\mu^2}\right)^{\frac{\epsilon}{2}}\right\},\\
\mathbf{I}^{(2)}_{C_1C_2C_3}(\epsilon) &= -\frac{1}{2}\left(\mathbf{I}^{(1)}_{C_1C_2C_3}(\epsilon)\right)^2 - e^{\frac{\epsilon}{2}\gamma_E}\frac{\Gamma(1+\epsilon)}{\Gamma(1+\frac{\epsilon}{2})}\zeta_2\, \mathbf{I^{(1)}}_{C_1C_2C_3}(2\epsilon) + \frac{1}{\epsilon}\mathbf{H}^{(2)}_{C_1C_2C_3}(\epsilon)
\end{aligned} \tag{4.4.2}$$

with [97]

$$\mathbf{H}^{(2)}_{C_1C_2C_3}(\epsilon) = -\frac{3}{4}\zeta_3. \tag{4.4.3}$$

The arbitrariness of the finite parts of the subtraction operators define various schemes in which finite part of amplitude is computed. Upon translating the infrared factorisation from amplitudes to the UV renormalised FFs depicted in (4.3.3), we get exactly same relations as (4.4.1):

$$\begin{aligned}
\mathcal{F}^{(1),i}_{\{C\}}(\epsilon) &= 2\mathbf{I}^{(1)}_{\{C\}}(\epsilon) + \mathcal{F}^{(1),i}_{\{C\},\text{fin}}\,,\\
\mathcal{F}^{(2),i}_{\{C\}}(\epsilon) &= 4\mathbf{I}^{(2)}_{\{C\}}(\epsilon) + 2\mathbf{I}^{(1)}_{\{C\}}(\epsilon)\mathcal{F}^{(1),i}_{\{C\}}(\epsilon) + \mathcal{F}^{(2),i}_{\{C\},\text{fin}}\,.
\end{aligned} \tag{4.4.4}$$

4.5 Finite remainders at two-loop

For maximally helicity violating (MHV) amplitude in planar $\mathcal{N} = 4$ SYM, the helicity blind quantity which is obtained by factoring out the tree level amplitude was conjectured to exponentiate in terms of one-loop counter part along with the universal cusp and collinear anomalous dimensions. This is known as BDS/ABDK ansatz [71, 75]. However, this ansatz is found to break down for two-loop six-point amplitudes [77, 173].

Though the IR divergent parts were found to be consistent with the prediction of the ansatz, the finite part was not. Hence, to capture the amount of deviation from the prediction of BDS/ABDK ansatz, a quantity named finite remainder [77, 173] was introduced which is a finite function of dual conformal cross ratios.

Following this line of thought and inspired by the exponentiation of the IR divergences of FFs, the finite remainder function for FFs at two-loop was introduced in ref. [10] as

$$\mathcal{R}^{(2),i}_{\{C\}} \equiv \mathcal{F}^{(2),i}_{\{C\}}(\epsilon) - \frac{1}{2}\left(\mathcal{F}^{(1),i}_{\{C\}}(\epsilon)\right)^2 - f^{(2)}(\epsilon)\mathcal{F}^{(1),i}_{\{C\}}(2\epsilon) - C^{(2)} + O(\epsilon) \tag{4.5.1}$$

where $f^{(2)}(\epsilon) \equiv \sum_{j=0}^{2} \epsilon^j f^{(2)}_j$. Writing a two-loop FF in terms a quantity dictated by BDS/ABDK ansatz plus a finite remainder provides an efficient way of organising the result - the ansatz part captures the IR divergences whereas the remainder encapsulates the finite part in 4-dimensions. Due to the nature of IR divergences of FFs, the $1/\epsilon^4$ and $1/\epsilon^3$ poles cancel between first two terms in eq. 4.6.1, By demanding the finiteness of remainder i.e. the vanishing of the remaining $1/\epsilon^2$ and $1/\epsilon$ poles, one gets

$$f^{(2)}_0 = -2\zeta_2\,, \qquad f^{(2)}_1 = \zeta_3. \tag{4.5.2}$$

The other coefficient $f^{(2)}_2$ and $C^{(2)}$ can not be fixed uniquely. To maintain a close resemblance with the corresponding quantity of MHV amplitudes, the $f^{(2)}_2$ is assigned a value according to

$$f^{(2)}_2 = -\frac{1}{2}\zeta_4\,. \tag{4.5.3}$$

With the help of Sudakov FF, the remaining constant is found to be [10]

$$C^{(2)} = 4\zeta_4\,. \tag{4.5.4}$$

$f^{(2)}(\epsilon)$ and $C^{(2)}$ are independent of the nature of operator and number of external legs. The remainder $\mathcal{R}^{(2),i}_{\{C\}}$ and finite part of the $\mathcal{F}^{(2),i}_{\{C\},\text{fin}}$ can be related to each other which can be found in [13]. In this article, we compute the finite remainders for all the processes and operators which are presented explicitly in the appendix **??** after setting $\mu^2 = -q^2$ and organising those according to their transcendental weights as

$$\mathcal{R}^{(2),i}_{\{C\}} = \sum_{k=0}^{4} \mathcal{R}^{(2),i,\tau(k)}_{\{C\}}\,. \tag{4.5.5}$$

4.6 Finite remainders at two-loop

For maximally helicity violating (MHV) amplitude in planar $\mathcal{N} = 4$ SYM, the helicity blind quantity which is obtained by factoring out the tree level amplitude was conjectured to exponentiate in terms of one-loop counter part along with the universal cusp and collinear anomalous dimensions. This is known as BDS/ABDK ansatz [71, 75]. However, this ansatz is found to break down for two-loop six-point amplitudes [77, 173].

Though the IR divergent parts were found to be consistent with the prediction of the ansatz, the finite part was not. Hence, to capture the amount of deviation from the prediction of BDS/ABDK ansatz, a quantity named finite remainder [77, 173] was introduced which is a finite function of dual conformal cross ratios.

Following this line of thought and inspired by the exponentiation of the IR divergences of FFs, the finite remainder function for FFs at two-loop was introduced in ref. [10] as

$$\mathcal{R}^{(2),i}_{\{C\}} \equiv \mathcal{F}^{(2),i}_{\{C\}}(\epsilon) - \frac{1}{2}\left(\mathcal{F}^{(1),i}_{\{C\}}(\epsilon)\right)^2 - f^{(2)}(\epsilon)\mathcal{F}^{(1),i}_{\{C\}}(2\epsilon) - C^{(2)} + O(\epsilon) \tag{4.6.1}$$

where $f^{(2)}(\epsilon) \equiv \sum_{j=0}^{2} \epsilon^j f^{(2)}_j$. Writing a two-loop FF in terms a quantity dictated by BDS/ABDK ansatz plus a finite remainder provides an efficient way of organising the result - the ansatz part captures the IR divergences whereas the remainder encapsulates the finite part in 4-dimensions. Due to the nature of IR divergences of FFs, the $1/\epsilon^4$ and $1/\epsilon^3$ poles cancel between first two terms in (4.6.1), By demanding the finiteness of remainder i.e. the vanishing of the remaining $1/\epsilon^2$ and $1/\epsilon$ poles, one gets

$$f^{(2)}_0 = -2\zeta_2\,, \qquad f^{(2)}_1 = \zeta_3. \tag{4.6.2}$$

The other coefficient $f^{(2)}_2$ and $C^{(2)}$ can not be fixed uniquely. To maintain a close resemblance with the corresponding quantity of MHV amplitudes, the $f^{(2)}_2$ is assigned a value according to

$$f^{(2)}_2 = -\frac{1}{2}\zeta_4\,. \tag{4.6.3}$$

With the help of Sudakov FF, the remaining constant is found to be [10]

$$C^{(2)} = 4\zeta_4 \,. \tag{4.6.4}$$

$f^{(2)}(\epsilon)$ and $C^{(2)}$ are independent of the nature of operator and number of external legs. The remainder $\mathcal{R}^{(2),i}_{\{C\}}$ and finite part of the $\mathcal{F}^{(2),i}_{\{C\},\text{fin}}$ can be related to each other which can be found in [13]. In this article, we compute the finite remainders for all the processes and operators which are presented explicitly at the end of this chapter 4.10.1 after setting $\mu^2 = -q^2$ and organising those according to their transcendental weights as

$$\mathcal{R}^{(2),i}_{\{C\}} = \sum_{k=0}^{4} \mathcal{R}^{(2),i,\tau(k)}_{\{C\}} \,. \tag{4.6.5}$$

4.7 Principle of uniform transcendentality

The principle of uniform transcendentality (PUT) is a remarkable feature based on numerous observations [5, 8, 10, 43, 140, 144–151], albeit unproven, which states that certain kinds of scattering amplitudes and form factors in $\mathcal{N} = 4$ can be expressed purely in terms of functions having UT. In other words, the L-th loop result is found to contain only polylogarithmic functions of degree 2L. It is conjectured in ref. [35] that at two-loop level the HT weight parts of every two-point minimal FF (number of fields present in the operator is same as number of external on-shell states) are identical and those are same as that of half-BPS operator belonging to the stress-energy supermultiplet [6]. In ref. [36], the minimal FFs of long BPS operators (more than two fields) are computed to two-loops, and the corresponding HT piece is found to be a universal in all FFs with long, unprotected operators [35, 37–41]. It is also conjectured in [35], that the HT terms of two-loop remainder of the three-point FF of every length-two operator should agree with the corresponding half-BPS remainder found in ref. [10] which is falsified in ref. [13] where, for the first time, it is shown that for three-point FF of the Konishi operator (length-two), the HT part depends on the nature of external on-shell states; it fails to match with that of half-BPS if the external on-shell states are $\phi\lambda\lambda$.

In this section, we dissect our new results in light of these principles and conjectures, and discuss how far the PUT holds true in present context.

In case of tensorial interaction through operator $O^3_{\mu\nu}$, for all the processes depicted in **??**, the one-loop and two-loop finite parts of the three-point FFs i.e. $\mathcal{F}^{(1),3}_{C_1C_2C_3,\text{fin}}$, $\mathcal{F}^{(2),3}_{C_1C_2C_3,\text{fin}}$ in (4.4.4), are UT. These contain only the HT terms which involve exclusively polylogarithmic functions of transcendentality 2 and 4 at one- and two-loop, respectively. No lower transcendental terms is present. Moreover, the corresponding finite remainder $\mathcal{R}^{(2),3}_{C_1C_2C_3}$ in (4.6.1) replicates this behaviour which is exactly similar to the corresponding quantity of the half-BPS operator. More specifically, the results in terms of MPLs are obtained as

$$\mathcal{F}^{(n),3}_{ggg,\text{fin}} = \mathcal{F}^{(n),3}_{g\phi\phi,\text{fin}} = \mathcal{F}^{(n),3}_{g\lambda\bar{\lambda},\text{fin}} = \mathcal{F}^{(n),3}_{\phi\lambda\bar{\lambda},\text{fin}} \equiv \mathcal{F}^{(n),3,\tau(2n)}_{C_1C_2C_3,\text{fin}} = \mathcal{F}^{(n),\text{half-BPS}}_{C_1C_2C_3,\text{fin}}\,,$$
$$\mathcal{R}^{(2),3}_{ggg} = \mathcal{R}^{(2),3}_{g\phi\phi} = \mathcal{R}^{(2),3}_{g\lambda\bar{\lambda}} = \mathcal{R}^{(2),3}_{\phi\lambda\bar{\lambda}} \equiv \mathcal{R}^{(2),3,\tau(4)}_{C_1C_2C_3} = \mathcal{R}^{(2),\text{half-BPS}}_{C_1C_2C_3}\,, \tag{4.7.1}$$

where $\tau(i)$ represents terms with transcendentality weight i. The above eq. (4.7.1) implies that the finite FFs as well as the remainders are independent of the on-shell states. Moreover, the results are exactly same as that of the half-BPS operator [10, 13] which is presented explicitly in terms of MPLs in appendices 4.10.1. In summary, the energy-momentum tensor behaves exactly like the half-BPS operator at three-point FFs level. This is in accordance with our classical expectation: since the stress-tensor is protected and lies in the same multiplet as the half-BPS, an exact SUSY Ward identity would relate these two FFs.

4.8 Principle of maximal transcendentality and its violation

The principle of maximal transcendentality (PMT) [15, 42, 43] establishes a bridge between the results in QCD and those of $\mathcal{N} = 4$ SYM which are comparatively simpler. It states that for certain quantities, the results in $\mathcal{N} = 4$ SYM can be obtained from that in QCD by converting the fermions from fundamental to adjoint representation and then retaining only the HT terms. The complete domain of applicability of this principle is still not clear and under active investigation. In this article, we check the applicability of PMT with respect to the energy momentam tensor (T) we considered.

Now let us ask the question, instead of considering the QCD FFs of scalar and vector interactions, if we take the tensorial interaction, does the PMT hold true? Quite remarkably, the answer turns out to be 'No': the HT terms of the three-point QCD FFs of EM tensor [52, 53] do not match with the corresponding quantities of $\mathcal{N} = 4$ SYM upon converting the quarks from fundamental to adjoint representation. To demonstrate this, we consider the processes depicted in 4.2.5 in $\mathcal{N} = 4$ SYM and the corresponding two processes in massless QCD that (a) involves three gluons [52] and (b) one gluon along with Dirac's fermion-antifermion pair [178].

Let's begin with the process (a). We find none of the HT terms of the corresponding FFs of the underlying process in QCD [52] match with that of $\mathcal{N} = 4$ SYM in (4.7.1) upon converting the fermions from fundamental to adjoint representation through $\{C_F \to C_A, 2n_f T_F \to C_A\}$

$$\mathcal{F}^{(1),3,\tau(2)}_{ggg,\text{fin}} \neq \mathcal{F}^{(1),3,\tau(2)}_{ggg,\text{fin,QCD}} \,. \tag{4.8.1}$$

The Casimirs in fundamental and adjoint representations are denoted as $C_F = (N^2-1)/(2N)$ and $C_A = N$, respectively, $T_F = 1/2$ is the conventional normalisation factor and n_f is the number of massless fermionic flavors in QCD. This indicates a clear violation of the PMT. This non-matching may be attributed to the incomplete factorisation of the leading order amplitude from the one loop form factor of EM tensor in QCD.

In $\mathcal{N} = 4$ SYM, it has been observed [10, 13], that the coefficients of the MPLs in HT term at any order, both in the poles as well as the finite part become independent of the kinematics only after dividing the loop matrix elements by their leading order contribution. This independence of kinematics can also be seen for HT terms in QCD. For the poles, one can understand this factorisation of kinematics in terms of their universal nature, but it is not trivial to reason out why it happens for the finite part too. For example, the half-BPS operator in $\mathcal{N} = 4$ SYM which has been seen to contain only HT terms up to two loop order [10, 13] in the perturbation theory, matches with the HT contribution of the Konishi operator [13] (with $g\phi\phi$ as external particles) because of the exact factorisation of the leading order contributions from both sides, in the poles as well as finite parts. For the current process in QCD, the leading order does not factorise from the HT part of one- aand two-loop, see Eq. (4.3) in [52], which is in sharp contrast to the counterpart process in $\mathcal{N} = 4$ SYM. Consequently, the mismatch of the HT terms occur.

In addition, it is also observed that in the finite term of the FF if we take the double collinear limit

$y \to 0$ and $z \to 0$ (which is equivalent to single soft limit), the residual kinematics factorises. Moreover, it turns out to be identical to the $\mathcal{N} = 4$ SYM counterparts upon taking the same limit on latter part. The same things get repeated for the process (b) i.e. violation of PMT. However, in contrast to the process (a), the HT terms for the finite part matches partially with the $\mathcal{N} = 4$ SYM counterpart, while the rest of them do match only in the double collinear limit. More precisely, at one-loop the terms proportional to quantities like $\zeta_2, \log(z)\log(y)$ and $\log(z)\log(1-z)$ match with the $\mathcal{N} = 4$ SYM result. But for the remaining terms, we need to take the double collinear or single soft limit appropriately to get the agreement.

In conclusion, for the first time the PMT is found to break down for three-point FFs arising from the EM tensor.

4.9 Checks on results

We perform a number of consistency checks in order to ensure the reliability of our results.

- All the three-point FFs at one- and two-loop and two-point FFs up to three-loops exhibit the correct universal infrared behaviour depicted in section 4.4 which serves as the most stringent check on our results.

- Appearance of correct IR poles, though, is a powerful check, it does not fully ensure the correctness of the finite term. Often one needs to perform some additional checks on finite parts. For the processes under consideration, we check the kinematics symmetry in finite FF as well as remainder functions. For the processes with three gluons, the result should be symmetric under mutual exchange of momentum of any of the two gluons: $p_i \leftrightarrow p_j$ i.e. $x_i \leftrightarrow x_j$ where $x_1 = x, x_2 = z, x_3 = y$. Similarly, $x_1 \leftrightarrow x_3$ symmetry must be there for the processes with final state particles containing $\phi(p_2)\phi(p_3)$ or $\lambda(p_2)\bar{\lambda}(p_3)$. For all the processes, we have checked the kinematics symmetry in finite FF as well as remainders numerically with the help of `GiNaC` [179, 180] through `PolyLogTools` [176]. Indeed, they exhibit the expected symmetry which in turn raises the reliability of the final results.

4.10 Conclusions

In this article, we have computed several three-point form factors (FFs) of T to two-loops, respectively, in maximally supersymmetric Yang Mills theory by following Feynman diagrammatic approach.

We have analysed the results of the FFs from the perspective of principle of uniform and maximal transcendentality. The energy-momentum (EM) tensor behaves exactly like the half-BPS operator. These are uniform transcendental (UT), and the highest transcendental (HT) weight terms of the FFs and finite remainders are same as that of half-BPS operator. This is in accordance with our classical expectation: since the stress-tensor is protected and lies in the same multiplet as the half-BPS, an exact SUSY Ward identity would relate these two FFs. The most interesting finding of our analysis is that, the three-point FFs of the EM tensor violate the principle of maximal transcendentality (PMT). If we take the three-point quark and gluon FFs of EM tensor in QCD [52,53] and change the fermions from fundamental to adjoint representation, the HT weight terms do not match with the corresponding results in $\mathcal{N}=4$ SYM. This is a clear violation of the PMT and it is observed for the first time for three-point FF.

4.10.1 Three-point form factors of half-BPS operator

$$\mathcal{F}^{(1),\text{half-BPS}}_{C_1C_2C_3,\text{fin}} = -\frac{\pi^2}{6} - GPL(0,y)GPL(0,z) - GPL(0,y)GPL(1,z) - GPL(0,z)GPL(1-z,y) + 2GPL(1,z)GPL(-z,y) + GPL(0,1,z) - GPL(0,1-z,y) + 2GPL(1,0,y) + GPL(1,0,z) - GPL(1-z,0,y) + 2GPL(-z,1-z,y)$$

$$\mathcal{F}^{(2),\text{half-BPS}}_{C_1C_2C_3,\text{fin}} = \frac{49\pi^4}{720} + 2GPL(0,0,y)GPL(0,0,z) + 2GPL(0,0,y)GPL(0,1,z) + 2GPL(0,0,z)GPL(0,1-z,y) - 2GPL(0,1,z)GPL(0,-z,y) + 2GPL(1,0,y)GPL(0,1,z) + 2GPL(0,0,y)GPL(1,0,z) - 2GPL(1,0,z)GPL(0,-z,y) + 2GPL(1,0,y)GPL(1,0,z) + 2GPL(0,0,y)GPL(1,1,z) - 4GPL(1,1,z)GPL(0,-z,y) + 2GPL(0,0,z)GPL(1-z,0,y) - 4GPL(0,1,z)GPL(1-z,0,y) - 4GPL(1,0,z)GPL(1-z,0,y) + 2GPL(0,0,z)GPL(1-z,1-z,y) - 4GPL(0,1,z)GPL(1-z,1-z,y) + 4GPL(1,0,z)GPL(1-z,1-z,y) + 6GPL(0,1,z)GPL(1-z,-z,y) - 2GPL(1,0,z)GPL(1-z,-z,y) + 2GPL(0,1,z)GPL(-z,0,y) + 2GPL(1,0,z)GPL(-z,0,y) - 4GPL(1,1,z)GPL(-z,0,y) + \pi^2\Big(\frac{1}{2}GPL(0,y)GPL(0,z) + \frac{1}{2}GPL(0,z)GPL(1-z,y) + \frac{1}{2}GPL(0,y)GPL(1,z) - \frac{4}{3}GPL(1,z)GPL(1-z,y) + \frac{1}{3}GPL(1,z)GPL(-z,y) + \frac{1}{2}GPL(0,1-z,y) + \frac{1}{2}GPL(1-z,0,y) - \frac{4}{3}GPL(1-$$

$z, 1-z, y) + \frac{1}{3}GPL(-z, 1-z, y) - \frac{4}{3}GPL(0, 1, y) - GPL(1, 0, y) + \frac{4}{3}GPL(1, 1, y) - \frac{1}{2}GPL(0, 1, z) - \frac{1}{2}GPL(1, 0, z)\Big) + 4GPL(0, 1, z)GPL(-z, 1-z, y) - 4GPL(1, 0, z)GPL(-z, 1-z, y) - 8GPL(0, 1, z)GPL(-z, -z, y) + 8GPL(1, 1, z)GPL(-z, -z, y) - 2GPL(0, y)GPL(0, 0, 1, z) + 6GPL(0, 0, 1, z)GPL(1-z, y) - 8GPL(0, 0, 1, z)GPL(-z, y) - 2GPL(0, z)GPL(0, 0, 1-z, y) + 2GPL(1, z)GPL(0, 0, 1-z, y) + 4GPL(1, z)GPL(0, 0, -z, y) - 2GPL(0, 1, 0, y)GPL(0, z) - 2GPL(0, 1, 0, y)GPL(1, z) - 2GPL(0, y)GPL(0, 1, 0, z) - 2GPL(0, 1, 0, z)GPL(1-z, y) - 2GPL(0, y)GPL(0, 1, 1, z) + 4GPL(0, 1, 1, z)GPL(-z, y) - 2GPL(0, z)GPL(0, 1-z, 0, y) + 2GPL(1, z)GPL(0, 1-z, 0, y) + 2GPL(0, z)GPL(0, 1-z, 1-z, y) - 2GPL(1, z)GPL(0, 1-z, -z, y) + 4GPL(1, z)GPL(0, -z, 0, y) - 2GPL(0, z)GPL(0, -z, 1-z, y) - 4GPL(1, z)GPL(0, -z, 1-z, y) - 4GPL(1, 0, 0, y)GPL(0, z) - 4GPL(1, 0, 0, y)GPL(1, z) - 2GPL(0, y)GPL(1, 0, 0, z) - 2GPL(1, 0, 0, z)GPL(1-z, y) - 2GPL(0, y)GPL(1, 0, 1, z) + 4GPL(1, 0, 1, z)GPL(-z, y) + 2GPL(0, z)GPL(1, 0, 1-z, y) - 2GPL(0, y)GPL(1, 1, 0, z) + 8GPL(1, 1, 0, z)GPL(1-z, y) - 4GPL(1, 1, 0, z)GPL(-z, y) + 2GPL(0, z)GPL(1, 1-z, 0, y) + 2GPL(0, z)GPL(1-z, 0, 0, y) + 2GPL(1, z)GPL(1-z, 0, 0, y) - 2GPL(0, z)GPL(1-z, 0, 1-z, y) - 2GPL(1, z)GPL(1-z, 0, -z, y) - 2GPL(0, z)GPL(1-z, 1, 0, y) + 4GPL(1, z)GPL(1-z, 1, 0, y) - 2GPL(0, z)GPL(1-z, 1-z, 0, y) - 4GPL(1, z)GPL(1-z, 1-z, -z, y) - 2GPL(1, z)GPL(1-z, -z, 0, y) - 2GPL(0, z)GPL(1-z, -z, 1-z, y) + 8GPL(1, z)GPL(1-z, -z, -z, y) + 2GPL(0, z)GPL(-z, 0, 1-z, y) - 4GPL(1, z)GPL(-z, 0, 1-z, y) + 2GPL(0, z)GPL(-z, 1-z, 0, y) - 4GPL(1, z)GPL(-z, 1-z, 0, y) - 4GPL(0, z)GPL(-z, 1-z, 1-z, y) + 8GPL(1, z)GPL(-z, 1-z, -z, y) + 8GPL(1, z)GPL(-z, -z, 1-z, y) - 8GPL(1, z)GPL(-z, -z, -z, y) - 4GPL(0, 0, 1, 0, y) + 2GPL(0, 0, 1, 1, z) + 2GPL(0, 0, 1-z, 1-z, y) + 4GPL(0, 0, -z, 1-z, y) + 2GPL(0, 1, 0, 1, z) - 2GPL(0, 1, 0, 1-z, y) + 8GPL(0, 1, 1, 0, y) + 2GPL(0, 1, 1, 0, z) - 2GPL(0, 1, 1-z, 0, y) + 2GPL(0, 1-z, 0, 1-z, y) - 2GPL(0, 1-z, 1, 0, y) + 2GPL(0, 1-z, 1-z, 0, y) - 2GPL(0, 1-z, -z, 1-z, y) + 4GPL(0, -z, 0, 1-z, y) + 4GPL(0, -z, 1-z, 0, y) - 4GPL(0, -z, 1-z, 1-z, y) + 2GPL(1, 0, 0, 1, z) - 4GPL(1, 0, 0, 1-z, y) + 8GPL(1, 0, 1, 0, y) + 2GPL(1, 0, 1, 0, z) - 4GPL(1, 0, 1-z, 0, y) + 8GPL(1, 1, 0, 0, y) + 2GPL(1, 1, 0, 0, z) - 8GPL(1, 1, 1, 0, y) - 4GPL(1, 1-z, 0, 0, y) + 2GPL(1-z, 0, 0, 1-z, y) - 2GPL(1-z, 0, 1, 0, y) + 2GPL(1-z, 0, 1-z, 0, y) - 2GPL(1-z, 0, -z, 1-z, y) - 4GPL(1-z, 1, 0, 0, y) + 4GPL(1-z, 1, 0, 1-z, y) + 4GPL(1-z, 1, 1-z, 0, y) + 2GPL(1-z, 1-z, 0, 0, y) + 4GPL(1-z, 1-z, 1, 0, y) - 4GPL(1-z, 1-z, -z, 1-z, y) - 2GPL(1-z, -z, 0, 1-z, y) - 2GPL(1-z, -z, 1-z, 0, y) + 8GPL(1-z, -z, -z, 1-z, y) - 4GPL(-z, 0, 1-z, 1-z, y) - 4GPL(-z, 1-z, 0, 1-z, y) - 4GPL(-z, 1-z, 1-z, 0, y) + 8GPL(-z, 1-z, -z, 1-z, y) + 8GPL(-z, -z, 1-z, 1-z, y) - 8GPL(-z, -z, -z, 1-z, y) + \zeta_3\Big(-GPL(1-z, y) - GPL(0, y) - GPL(0, z) - GPL(1, z)\Big).$

$\mathcal{R}^{(2),\text{half-BPS}}_{C_1C_2C_3} = -4GPL(0, 1, z)GPL(1-z, 0, y) - 4GPL(1, 0, z)GPL(1-z, 0, y) - 4GPL(0, 1, z)GPL(1-$

$$z,1-z,y)+4GPL(1,0,z)GPL(1-z,1-z,y)+8GPL(0,1,z)GPL(1-z,-z,y)+4GPL(0,1,z)GPL(-z,0,y)+$$
$$4GPL(1,0,z)GPL(-z,0,y)+4GPL(0,1,z)GPL(-z,1-z,y)-4GPL(1,0,z)GPL(-z,1-z,y)+$$
$$\pi^2\Big(-\tfrac{4}{3}GPL(1,z)GPL(1-z,y)+\tfrac{4}{3}GPL(1,z)GPL(-z,y)-\tfrac{4}{3}GPL(1-z,1-z,y)+\tfrac{4}{3}GPL(-z,1-$$
$$z,y)-\tfrac{4}{3}GPL(0,1,y)+\tfrac{4}{3}GPL(1,1,y)\Big)-8GPL(0,1,z)GPL(-z,-z,y)+8GPL(0,0,1,z)GPL(1-$$
$$z,y)-8GPL(0,0,1,z)GPL(-z,y)-4GPL(0,z)GPL(0,0,1-z,y)+4GPL(1,z)GPL(0,0,-z,y)-$$
$$4GPL(0,z)GPL(0,1-z,0,y)+4GPL(1,z)GPL(0,-z,0,y)+4GPL(0,z)GPL(1,0,1-z,y)-4GPL(1,z)GPL(1,0,-z,y)+$$
$$8GPL(1,1,0,z)GPL(1-z,y)-8GPL(1,1,0,z)GPL(-z,y)+4GPL(0,z)GPL(1,1-z,0,y)-4GPL(1,z)GPL(1,-z,0,y)-$$
$$4GPL(0,z)GPL(1-z,0,1-z,y)+4GPL(1,z)GPL(1-z,1,0,y)-4GPL(0,z)GPL(1-z,1-z,0,y)-$$
$$4GPL(1,z)GPL(1-z,1-z,-z,y)+8GPL(1,z)GPL(1-z,-z,-z,y)+4GPL(0,z)GPL(-z,0,1-z,y)-$$
$$4GPL(1,z)GPL(-z,1,0,y)+4GPL(0,z)GPL(-z,1-z,0,y)+4GPL(1,z)GPL(-z,1-z,-z,y)-$$
$$8GPL(1,z)GPL(-z,-z,-z,y)-4GPL(0,0,1,0,y)+4GPL(0,0,-z,1-z,y)+8GPL(0,1,1,0,y)+$$
$$4GPL(0,-z,0,1-z,y)+4GPL(0,-z,1-z,0,y)+4GPL(1,0,1,0,y)-4GPL(1,0,-z,1-z,y)-$$
$$8GPL(1,1,1,0,y)-4GPL(1,-z,0,1-z,y)-4GPL(1,-z,1-z,0,y)+4GPL(1-z,1,0,1-z,y)+$$
$$4GPL(1-z,1,1-z,0,y)+4GPL(1-z,1-z,1,0,y)-4GPL(1-z,1-z,-z,1-z,y)+8GPL(1-$$
$$z,-z,-z,1-z,y)-4GPL(-z,1,0,1-z,y)-4GPL(-z,1,1-z,0,y)-4GPL(-z,1-z,1,0,y)+$$
$$4GPL(-z,1-z,-z,1-z,y)-8GPL(-z,-z,-z,1-z,y)\,.$$

5 Multiloop/leg Scattering Amplitude Computation

Contents

5.1 Prologue

Scattering amplitude and cross sections plays a pivotal role in the context of high energy physics. Computations of multiparticle amplitude and cross sections very precisely not only help us to discover new features but also to unravel the rich field theoretic structure associated with this. However, in perturbative QFT, the theoretical predictions based on the leading order (LO) calculation happens to be unreliable due to our lack of knowledge of higher order term beyond LO. One must go beyond the leading order to make the predictions more accurate and reliable. Unfortunately the complexity of the computations grows enormously with the increasing number of loop and legs. In this Chapter, we will discuss the techniques of higher order virtual or loop corrections and phase space integral computation we followed to deal with real radiation diagrams. There exist several precriptions to compute S-matrix elements in QFT . There are several popular approach worth mentioning and used for computing various multiloop and multileg amplitudes in QCD and $\mathcal{N} = 4$ SYM :

1. *Diagrammatic approach:* Generate relevant Feynman diagrams and correspondingly compute the associated integrals appearing at particular perturbative order for the multiloop/leg scattering amplitude under consideration.

2. *Unitary-based approach:* Exploit the generalised unitary properties of the amplitude and compute quantities without considering the diagrams.

3. Understanding the scattering amplitudes in the planar limit from a geometric [1] point of view. Locality and unitarity emerge as a consequence of the geometrical structure.

Although in the community most of the computation in $\mathcal{N} = 4$ SYM performed through Unitary-based approach , we peferred to choose the diagrammatic approach to compute. Particularly a large portion of the thesis dealing with the real radiation diagrams with the integral over final state phase space suits the diagrammatic approach with respect to the setup we have.

5.2 A Diagrammatic way to compute the higher order IR safe Cross Sections

The matrix elements as well as the observables in a general perturbative QFT are expanded in powers of coupling constants, c, present in the Lagrangian which is needed to be small enough in. One can expand a generic scattering cross section involving n external particle momentum with $p_1, p_2, \cdots, p_n$ in powers of coupling constant following way :

$$\mathcal{O}(p_1, p_2, \cdots, p_n) = \sum_{i=0}^{\infty} c^i \mathcal{O}^{(i)}(p_1, p_2, \cdots, p_n) \,. \tag{5.2.1}$$

The order of the expansion is given by i. The cross section for $i = 0$ is called the leading order (LO) or Born level, $i = 1$ is next-to-leading order (NLO) and so forth. $\mathcal{O}$ can be written in terms of the scattering matrix elements at each order in perturbation theory upto $NNLO$ by

$$\mathcal{O}(0) = 1/\mathcal{K}(p_1, \cdots p_n) \int |\, |\mathcal{A}_n{}^{(0)}\rangle\, |^2 \, d\phi_n \,,$$

$$\mathcal{O}(1) = 1/\mathcal{K}(p_1, \cdots, p_n) (\int 2\,\mathrm{Re}\,(\langle \mathcal{A}_n^{(0)} | \mathcal{A}_n^{(1)} \rangle)\, d\phi_n + \int |\, |\mathcal{A}_{n+1}^{(0)}\rangle\, |^2 \, d\phi_{n+1}),$$

$$O(2) = 1/\mathcal{K}(p_1, \cdots, p_n) \Big(\int 2\,\mathrm{Re}\,(\langle \mathcal{A}_n^{(0)} | \mathcal{A}_n^{(2)} \rangle) d\phi_n + \int (\langle \mathcal{A}_n^{(1)} | \mathcal{A}_n^{(1)} \rangle) d\phi_n + \int 2\,\mathrm{Re}\,(\langle \mathcal{A}_{n+1}^{(0)} | \mathcal{A}_{n+1}^{(1)} \rangle)\, d\phi_{n+1}$$

$$+ \int | \, | \mathcal{A}_{n+2}^{(0)} \rangle \, |^2 \, d\phi_{n+2} \Big) \tag{5.2.2}$$

and the generalisation is quite straitforward at N^3LO or higher order. In previous equations, $|\mathcal{A}_n^{(i)}\rangle$ is the matrix element at i^{th} perturbative order with n external momentum. $d\phi_n$ is the phase space kernel. "Re" denotes the real part of the matrix element and $\mathcal{K}(p_1, \cdots, p_n)$ is an overall factor depending on the external scales.

Beyond the leading order (LO) the massless gauge theory amplitudes are plagued by short distance or ultraviolet (UV) and long distance and collinear singularities ie. Infrared (IR). The singularities can arise from virtual and real emission diagrams contributing beyond LO. To accomplish the corresponding cross sections finite, the determination of these singularities are essential. For removing the IR divergences one has to combine the contributions arising from the real emission diagrams at same perturbative order. The IR contains soft as well as collinear divergences coming from the phase space integral. Once we add the virtual and real emission diagrams and evaluate the phase space integrals, the resulting expressions become free from soft and final state collinear singularities by Kinoshita- Lee-Nauenberg (KLN) theorem. An analogous result for quantum electrodynamics with massive fermion is known as Bloch-Nordsieck cancellation. However, the collinear singularities arising from the initial state configurations remain uncancelled and removed at the hadronic level through the mass factorisation techniques. Where the uncancelled singularities are absorbed into the bare parton distribution functions (PDF). So, the observables at the hadronic level are finite when it is compared with the experimental findings.

We followed the following procedure to accomplish the computations of higher order virtual diagrams.

Automated Feynman Diagram Generation: QGRAF [123]

$\Downarrow$

Conversion of propagators to basis Propagator : REDUZE2 [181, 182]

$\Downarrow$

Symbolic Manipulation of gamma matrices, color factors, Dirac and Lorentz indices and supersymmetric indices. indices : FORM [124]

$\Downarrow$

Integral by parts (IBP) reduction [163,164] to MI : LiteRed [125,126] / FIRE [183] / REDUZE2 [181, 182]

$\Downarrow$

$C_i I_i$

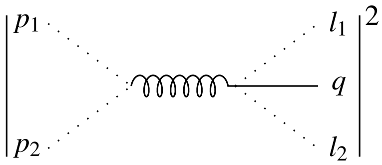
- I_i are the Master Integral (MI) set.

- C_i are the Coefficients of I_i .

Reverse unitarity method to compute real radiation phase space integral

As we are interested on the computation of IR safe cross sections, evalutaion of the phase space interal arising from different real radiations diagrams become essential for accomplishing the goal. We followed an unique technique introduced by [129, 184].

Lets consider a sample double real emission diagram for the illustration to compute phase space integrals: $\phi\ +\ \phi \rightarrow Half - BPS\ + \phi + \phi$



$$\propto \int \frac{d^n l_1}{(2\pi)^{n-1}} \frac{d^n l_2}{(2\pi)^{n-1}} \delta_+(l_1^2)\delta_+(l_2^2)\delta_+(l^2 - m_h^2)[\mathcal{A}]$$

Figure 5.1. Interference of double-real emissions

where, $\delta_+(P^2 - m^2) \equiv \delta(P^2 - m^2)\theta(P^0)$. According to Cutkosky rules [185], the δ_+ functions can be replaced by the difference between two propagators with opposite Feynman ϵ prescriptions for their imaginary parts as a consequence of the casuality:

$$\delta(P^2 - m^2) \sim \frac{1}{P^2 - m^2 + i\varepsilon} - \frac{1}{P^2 - m^2 - i\varepsilon} \tag{5.2.3}$$

Following this replacement, the modulus square of the matrix element, portrayed through Fig. 5.1 can be fused to a virtual like scattering amplitude, presented in Fig. 5.2, where, the blue dotted line denotes the cut propagators which will be be substitued by the RHS of Eq. (5.2.3).

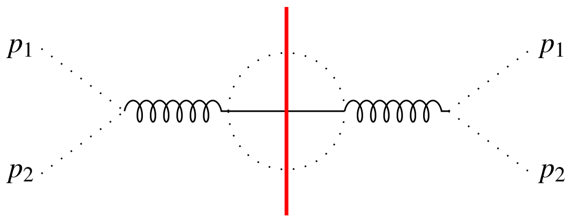


Figure 5.2. Two loop virtual like diagram after the fusion of three cut propagators

We extensively followed the above mentioned techniques to compute the real radiation corrections. As the phase space integral of the square of the modulas of the matrix element can be expressed in terms of a virtual like amplitude, one now can handle in the same way as the multiloop integrals. We applied the IBP identities to reduce this two loop diagram into a set of MIs just like multiloop virtual processes and after the proper replacement of the master intgral results we arrived at the final outcomes.

6 Conclusions and Outlooks

Undoubtedly, $\mathcal{N} = 4$ SYM calculations helps to get clear understanding on the IR structure of gauge theory from a model independent point of view and to develop cutting edge tools to compute most difficult QCD computations. Also, the computations in $\mathcal{N} = 4$ serves as a beautiful playground to unveil many novel structures in QFT like generalised unitarity, BCFW recursion relations, symbol of functions, scattering amplitude/wilson loop duality, dual conformal and Yangian symmetries and its connection with Integrability. On the other hand, similarity of results between perturbative $\mathcal{N} = 4$ and QCD may be the indications of hidden symmetries which is common to both. Universality of the Leading Trancendental (LT) term of the observables at special singular such as soft, collinear or high-energy limits also very interesting from the generic gauge theory point of view. The tools developed for generic QFT calculations, such as efficient Feynman integral calculations, on-shell methods have an enormous impact on QCD computations already.

The state-of-the-art techniques, which mostly use our in-house codes, have been employed extensively to carry out all the computations presented in this thesis.

In chapter 2, we aimed at computing Splitting functions and tried to demonstrate the universal factorisation properties of Soft-Virtual(SV) cross-sections for certain composite operators (BPS and Stress tensor) upto NNLO and then to predict the NNNLO SV cross-sctions using the known three loop anamolous dimentions $\mathcal{N}$=4 SYM. We also demonstrated the operator independent nature of the factorisation properties of SV cross-sections in $\mathcal{N}$=4. We studied the universal decomposition of the Splitting functions with respect to the anomalous dimentions of the theory in the SV limit. Also we investigated the transcendental stucture of the soft gluon and scalar radiation in $\mathcal{N} = 4$. Our findings are consistent with the previous results available in the literature related to the topic.

In chapter 3, for the first time we presented the form factors with insertions of two identical local gauge invariant operators to two-loops in $\mathcal{N} = 4$ SYM theory by performing a state-of-the-art computation. In particular, we compute the scalar FFs with double insertions of half-BPS primary and Konishi operators. Through this calculation, we take a step forward to go beyond the FFs of single operator insertion and enter into the domain of more general class of FFs. To validate our computations, we check the infrared poles which agree with the predictions. Moreover, the appearance of expected kinematic symmetry inherent to the bosonic FFs provides a strong check on the finite parts of our results.

The findings enable us to reach a number of important conclusions. For the first time, the conjecture that the FFs of SUSY protected operators are always UT is found not to hold true at two-loop for the FFs of double-operator insertion, in sharp contrast to the Sudakov FFs. In particular, though the FF of double half-BPS primary is UT at one-loop, it fails to exhibit this property at two-loop. In accordance with our expectation, we find the FFs of SUSY unprotected operator, Konishi, to be not UT. From our experience, we find that the factorisation of the leading order amplitude at any loop order can be accounted for the existence of UT property.

The conjecture that the highest transcendentality weight terms of every two-point minimal FF are identical to that of half-BPS is also found not to hold true for two-operator insertion. In other words, the HT weight terms of unprotected Konishi are not identical to that of half-BPS both at one- and two-loop.

We find that the principle of maximal transcendentality, which says the highest weight terms of quark/gluon FFs in QCD are identical to that of scalar FFs of half-BPS primary, can not be extrapolated to the case of two-operator insertion. The HT weight terms of double half-BPS FFs do not match with that of di-Higgs production through gluon fusion or bottom quark annihilation in QCD, observed for the first time in this context. In the simultaneous Regge and collinear limits, the transcendentality principles are found to be restored.

By computing the finite remainder function at two-loop, we confirm that the finite parts of the FFs of double half-BPS do not exponentiate, in contrast to the corresponding FFs of single half-BPS at two-loop [6]. Moreover, we discover that there is no contact divergence in the case of double half-BPS or Konishi operators and therefore, we do not need any additional UV counterterm. Our work, in addition to providing a better understanding of the nature of generalised FFs, opens the door for further analytic calculations of general class of FFs. Also, our results will serve the important data towards the further understanding on FF/Wilson loop duality.

In Chapter 4, we have computed several three-point form factors (FFs) of energy momentum tensor to two-loops, respectively, in maximally supersymmetric Yang Mills theory by following Feynman diagrammatic approach.
We have analysed the results of the FFs from the perspective of principle of uniform and maximal transcendentality. The energy-momentum (EM) tensor behaves exactly like the half-BPS operator. These are uniform transcendental (UT), and the highest transcendental (HT) weight terms of the FFs and finite remainders are same as that of half-BPS operator. This is in accordance with our classical expectation: since the stress-tensor is protected and lies in the same multiplet as the half-BPS, an exact SUSY Ward identity would relate these two FFs. The most interesting finding of our analysis is that, the three-point FFs of the EM tensor violate the principle of maximal transcendentality (PMT). If we take the three-point quark and gluon FFs of EM tensor in QCD [52, 53] and change the fermions from fundamental to adjoint representation, the HT weight terms do not match with the corresponding results in $\mathcal{N} = 4$ SYM. This is a clear violation of the PMT and it is observed for the first time for a three-point FF.

In conclusion, as an extention of the work presented in the thesis one can compute the observables we presented here at higher loop level in $\mathcal{N} = 4$ and can study the contributions coming from different particle content separately to predict the QCD results after making appropriate transformations. It is worth mentioning that the the validity and the generalisation of Casimir scaling principle [67] for the soft distribution function at N^4LO level in the context of QCD. Also its very interesting to study the factorisation and universality properties of the cross sections $\mathcal{N} = 4$ SYM at Next-to-leading power level (ongoing). Along with the development of cutting edge tools for the computations, hopefully theoreticians like us will continue to unravel new fascinating structure in gauge theories.

Appendices

A Inclusive Production Cross Section

Although the result main results we presented in this thesis are based on $\mathcal{N} = 4$ calculations, nevertheless the technical details of the compuation resembles with QCD. Below we present a short note on the computation of parton model based inclusive cross section.

In QCD improved parton model, the inclusive cross-section for the production of a final state colorless particle with two parton in the initial states can be formulated using

$$\sigma^I(\tau, q^2) = \sum_{a,b=q,\bar{q},g} \int_0^1 dx_1 \int_0^1 dx_2 f_a(x_1, \mu_F^2) f_b(x_2, \mu_F^2) \sigma_{ab}^I \left(z, q^2, \mu_R^2, \mu_F^2\right) \tag{A.0.1}$$

where, f's are the non perterbative partonic distribution functions factorised at the mass scale μ_F. σ_{ab}^I is the partonic cross section for the production of colorless particle I from the partons a and b. This is UV renormalised at renormalisation scale μ_R and mass factorised at μ_F. where,

$$\begin{aligned} q^2 &= m_I^2 \,, \\ \tau &= \frac{q^2}{S} \,, \\ z &= \frac{q^2}{\hat{s}} \,. \end{aligned} \tag{A.0.2}$$

In the above expression, S and $\hat{s}$ are square of the hadronic and partonic center of mass energies, respectively, and they can be related by

$$\hat{s} = x_1 x_2 S \,. \tag{A.0.3}$$

and

$$\int dz \delta(\tau - x_1 x_2 z) = \frac{1}{x_1 x_2} = \frac{S}{\hat{s}} \tag{A.0.4}$$

in Eq. (A.0.1), we can rewrite the Eq. (A.0.1) as

$$\sigma^I(\tau, q^2) = \sigma^{I,(0)}(\mu_R^2) \sum_{ab=q,\bar{q},g} \int_\tau^1 dx\, \chi_{ab}(x, \mu_F^2)\, \Delta^I_{ab}\left(\frac{\tau}{x}, q^2, \mu_R^2, \mu_F^2\right). \tag{A.0.5}$$

The flux χ_{ab} density is defined through

$$\chi_{ab}(x, \mu_F^2) = \int_x^1 \frac{dy}{y} f_a(y, \mu_F^2)\, f_b\left(\frac{x}{y}, \mu_F^2\right) \tag{A.0.6}$$

and the dimensionless quantity Δ^I_{ab} is called the coefficient function of the partonic level cross section. Upon normalising the partonic level cross section by the born one, we obtain Δ^I_{ab} i.e.

$$\Delta^I_{ab} \equiv \frac{\sigma^I_{ab}}{\sigma^{I,(0)}}. \tag{A.0.7}$$

B Anomalous Dimensions

Due to supersymmetry the Cusp (A), Collinear/virtual (B) and Soft (f) anomalous dimentions are independent of the particle content in $\mathcal{N} = 4$. Here we present A [12, 133–135], f [12], and B [12] up to three loop level in $\mathcal{N} = 4$. The A's are given by

$$
\begin{aligned}
A_1 &= C_A \Big\{4\Big\} \,, \\
A_2 &= C_A^2 \left\{-8\zeta_2\right\} \,, \\
A_3 &= C_A^3 \left\{\frac{176\zeta_2^2}{5}\right\} .
\end{aligned} \tag{B.0.1}
$$

The f's are obtained as

$$
\begin{aligned}
f_1 &= 0 \,, \\
f_2 &= C_A^2 \left\{-28\zeta_3\right\} \,, \\
f_3 &= {C_A}^3 \left\{\frac{176}{3}\zeta_2\zeta_3 + 192\zeta_5\right\} .
\end{aligned} \tag{B.0.2}
$$

Similarly the B's are given by

$$
\begin{aligned}
B_1 &= 0 \,, \\
B_2 &= C_A^2 \left\{12\zeta_3\right\} \,, \\
B_3 &= -\,{C_A}^3\, 16 \left\{\zeta_2\zeta_3 + 5\,\zeta_5\right\} .
\end{aligned} \tag{B.0.3}
$$

Interestingly at every order, the above terms demonstrate uniform transcendentality which is 1 at NLO, 3 at NNLO and 5 at NNNLO and one can retain the above anomalous dimentions in $\mathcal{N}$ =

4 from gluon anomalous dimentions in QCD upon proper replacement of the color factors in the following way i.e. $C_A = C_f = n_f = N$ *and taking the LT term of it.*

C Renormalisation Group Evolution Equation

Let us present a short note on the solving of a generic Renormalisation Group Evolution Equation (RGE) at the renormalisation/factorisation scale μ:

$$\mu^2 \frac{d}{d\mu^2} \ln A = C \tag{C.0.1}$$

We want to solve it order by order in perturbation theory. We start by expanding the functions in powers of $c = c(\mu^2)$

$$\begin{aligned} A &= 1 + \sum_{k=1}^{\infty} c^k A^{(k)} , \\ C &= \sum_{k=1}^{\infty} c^k A^{(k)} . \end{aligned} \tag{C.0.2}$$

$\ln A$ can be represented by the expanded form as

$$\ln A = \sum_{k=1}^{\infty} c^k C_k \tag{C.0.3}$$

with

$$\begin{aligned} A_1 &= A^{(1)} , \\ A_2 &= -\frac{1}{2}(A^{(1)})^2 + A^{(2)} , \\ A_3 &= \frac{1}{3}(A^{(1)})^3 - A^{(1)}A^{(2)} + A^{(3)} , \end{aligned}$$

$$A_4 = -\frac{1}{4}(A^{(1)})^4 + (A^{(1)})^2 A^{(2)} - \frac{1}{2}(A^{(2)})^2 - A^{(1)} a^{(3)} + A^{(4)} \,. \tag{C.0.4}$$

upto fouth in the expansion. after using the above expansions in RGE. (C.0.1) one can write

$$\mu^2 \frac{d}{d\mu^2} a_s = \frac{\epsilon}{2} c - \sum_{k=0}^{\infty} \beta_k \, c^{k+2} \tag{C.0.5}$$

and

$$\begin{aligned}
A_1 &= \frac{2}{\epsilon} C^{(1)} \,, \\
A_2 &= \frac{2}{\epsilon^2} \beta_0 C^{(1)} + \frac{1}{\epsilon} C^{(2)} \,, \\
A_3 &= \frac{8}{3\epsilon^3} \beta_0^2 C^{(1)} + \frac{1}{\epsilon^2} \left\{ \frac{4}{3} \beta_1 C^{(1)} + \frac{4}{3} \beta_0 C^{(2)} \right\} + \frac{2}{3\epsilon} C^{(3)} \,, \\
A_4 &= \frac{4}{\epsilon^4} \beta_0^3 C^{(1)} + \frac{1}{\epsilon^3} \left\{ 4\beta_0 \beta_1 C^{(1)} + 2\beta_0^2 C^{(2)} \right\} + \frac{1}{\epsilon^2} \left\{ \beta_2 C^{(1)} + \beta_1 C^{(2)} + \beta_0 C^{(3)} \right\} + \frac{1}{2\epsilon} C^{(4)} \,.
\end{aligned} \tag{C.0.6}$$

using the Eq. (C.0.3) and (C.0.6) we obtain,

$$\begin{aligned}
A^{(1)} &= \frac{2}{\epsilon} C^{(1)} \,, \\
A^{(2)} &= \frac{1}{\epsilon^2} \left\{ 2\beta_0 C^{(1)} + 2(C^{(1)})^2 \right\} + \frac{1}{\epsilon} C^{(2)} \,, \\
A^{(3)} &= \frac{1}{\epsilon^3} \left\{ \frac{8}{3} \beta_0^2 C^{(1)} + 4\beta_0 (C^{(1)})^2 + \frac{4}{3} (C^{(1)})^3 \right\} + \frac{1}{\epsilon^2} \left\{ \frac{4}{3} \beta_1 C^{(1)} + \frac{4}{3} \beta_0 C^{(2)} + 2C^{(1)} C^{(2)} \right\} \\
&\quad + \frac{2}{3\epsilon} C^{(3)} \,, \\
A^{(4)} &= \frac{1}{\epsilon^4} \left\{ 4\beta_0^3 C^{(1)} + \frac{22}{3} \beta_0^2 (C^{(1)})^2 + 4\beta_0 (C^{(1)})^3 + \frac{2}{3} (C^{(1)})^4 \right\} + \frac{1}{\epsilon^3} \left\{ 4\beta_0 \beta_1 C^{(1)} + \frac{8}{3} \beta_1 (C^{(1)})^2 \right. \\
&\quad \left. + 2\beta_0^2 C^{(2)} + \frac{14}{3} \beta_0 C^{(1)} C^{(2)} + 2(C^{(1)})^2 C^{(2)} \right\} + \frac{1}{\epsilon^2} \left\{ \beta_2 C^{(1)} + \beta_1 C^{(2)} + \frac{1}{2} (C^{(2)})^2 + \beta_0 C^{(3)} \right. \\
&\quad \left. + \frac{4}{3} C^{(1)} C^{(3)} \right\} + \frac{1}{2\epsilon} C^{(4)} \,.
\end{aligned} \tag{C.0.7}$$

Example 2: Solution of the splitting function kernel

We used the following solutions of the RGE of mass factorisation kernel

$$\Gamma^{I,(1)}(z,\epsilon) = \frac{1}{\epsilon}\Bigg\{P^{I,(0)}(z)\Bigg\},$$

$$\Gamma^{I,(2)}(z,\epsilon) = \frac{1}{\epsilon^2}\Bigg\{-\beta_0 P^{I,(0)}(z) + \frac{1}{2}P^{I,(0)}(z)\otimes P^{I,(0)}(z)\Bigg\} + \frac{1}{\epsilon}\Bigg\{\frac{1}{2}P^{I,(1)}(z)\Bigg\},$$

$$\begin{aligned}
\Gamma^{I,(3)}(z,\epsilon) = &\frac{1}{\epsilon^3}\Bigg\{\frac{4}{3}\beta_0^2 P^{I,(0)} - \beta_0 P^{I,(0)}\otimes P^{I,(0)} + \frac{1}{6}P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(0)}\Bigg\} \\
&+ \frac{1}{\epsilon^2}\Bigg\{-\frac{1}{3}\beta_1 P^{I,(0)} + \frac{1}{6}P^{I,(0)}\otimes P^{I,(1)} - \frac{4}{3}\beta_0 P^{I,(1)} + \frac{1}{3}P^{I,(1)}\otimes P^{I,(0)}\Bigg\} \\
&+ \frac{1}{\epsilon}\Bigg\{\frac{1}{3}P^{I,(2)}\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\Gamma^{I,(4)}(z,\epsilon) = &\frac{1}{\epsilon^4}\Bigg\{-2\beta_0^3 P^{I,(0)} + \frac{11}{6}\beta_0^2 P^{I,(0)}\otimes P^{I,(0)} - \frac{1}{2}\beta_0 P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(0)} \\
&+ \frac{1}{24}P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(0)}\Bigg\} + \frac{1}{\epsilon^3}\Bigg\{\frac{4}{3}\beta_0\beta_1 P^{I,(0)} - \frac{1}{3}\beta_1 P^{I,(0)}\otimes P^{I,(0)} \\
&+ \frac{1}{24}P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(1)} - \frac{7}{12}\beta_0 P^{I,(0)}\otimes P^{I,(1)} + \frac{1}{12}P^{I,(0)}\otimes P^{I,(1)}\otimes P^{I,(0)} \\
&+ 3\beta_0^2 P^{I,(1)} - \frac{5}{4}\beta_0 P^{I,(1)}\otimes P^{I,(0)} + \frac{1}{8}P^{I,(1)}\otimes P^{I,(0)}\otimes P^{I,(0)}\Bigg\} \\
&+ \frac{1}{\epsilon^2}\Bigg\{-\frac{1}{6}\beta_2 P^{I,(0)} + \frac{1}{12}P^{I,(0)}\otimes P^{I,(2)} - \frac{1}{2}\beta_1 P^{I,(1)} + \frac{1}{8}P^{I,(1)}\otimes P^{I,(1)} \\
&- \frac{3}{2}\beta_0 P^{I,(2)} + \frac{1}{4}P^{I,(2)}\otimes P^{I,(0)}\Bigg\} + \frac{1}{\epsilon}\Bigg\{\frac{1}{4}P^{I,(3)}\Bigg\}. \qquad \text{(C.0.8)}
\end{aligned}$$

Although for $\mathcal{N}=4$ one has to set β_i to zero, but for the sake of generality we tried perform the analysis in a model independent manner.

In the soft/eikonal-virtual limit, only the diagonal parts of the kernels has non vanishing contribution. Our findings are consistent with the existing diagonal solutions which can be found in the article [67].

D Solving K + G Equation

The Sdakov form factor satisfies the K + G differential equation :

$$Q^2 \frac{d}{dQ^2} \ln \mathcal{F}^I_{ij}(\hat{c}, Q^2, \mu^2, \epsilon) = \frac{1}{2}\left[K^I_{ij}\left(\hat{c}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) + G^I_{ij}\left(\hat{c}, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right)\right]. \tag{D.0.1}$$

In this appendix we demonstrate the procedure to solve the K + G equation. RGE invariance of the $\mathcal{F}$ with respect to the renormalisation scale μ_R implies

$$\mu_R^2 \frac{d}{d\mu_R^2} K^I_{ij}\left(\hat{c}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = -\mu_R^2 \frac{d}{d\mu_R^2} G^I_{ij}\left(\hat{c}, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = -A^I_{ij}\left(c, (\mu_R^2)\right) \tag{D.0.2}$$

where, A^I_{ij}'s are the cusp anomalous dimensions. Unlike the previous cases, we expand K^I_{ij} in powers of unrenormalised $\hat{c}$ as

$$K^I_{ij}\left(\hat{c}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = \sum_{k=1}^{\infty} \hat{a}^k S_\epsilon^k \left(\frac{\mu_R^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{K}^I_{ij,k}(\epsilon) \tag{D.0.3}$$

whereas we define the components $A^I_{ij,k}$ through

$$A^I_{ij} = \sum_{k=1}^{\infty} c^k \left(\mu_R^2\right) A^I_{ij,k}. \tag{D.0.4}$$

Following the methodology discussed in Appendix C, we can solve for $\hat{K}^I_{ij,k}(\epsilon)$

$$\hat{K}^I_{ij,1}(\epsilon) = \frac{1}{\epsilon}\left\{-2A^I_{ij,1}\right\},$$
$$\hat{K}^I_{ij,2}(\epsilon) = \frac{1}{\epsilon^2}\left\{2\beta_0 A^I_{ij,1}\right\} + \frac{1}{\epsilon}\left\{-A^I_{ij,2}\right\},$$
$$\hat{K}^I_{ij,3}(\epsilon) = \frac{1}{\epsilon^3}\left\{-\frac{8}{3}\beta_0^2 A^I_{ij,1}\right\} + \frac{1}{\epsilon^2}\left\{\frac{2}{3}\beta_1 A^I_{ij,1} + \frac{8}{3}\beta_0 A^I_{ij,2}\right\} + \frac{1}{\epsilon}\left\{-\frac{2}{3}A^I_{ij,3}\right\},$$

$$\hat{K}^{I}_{ij,4}(\epsilon) = \frac{1}{\epsilon^4}\Big\{4\beta_0^3 A^I_{ij,1}\Big\} + \frac{1}{\epsilon^3}\Big\{-\frac{8}{3}\beta_0\beta_1 A^I_{ij,1} - 6\beta_0^2 A^I_{ij,2}\Big\} + \frac{1}{\epsilon^2}\Big\{\frac{1}{3}\beta_2 A^I_{ij,1} + \beta_1 A^I_{ij,2} + 3\beta_0 A^I_{ij,3}\Big\} + \frac{1}{\epsilon}\Big\{-\frac{1}{2}A^I_{ij,4}\Big\} \tag{D.0.5}$$

After the Integration of the RGE of G^I_{ij}, (D.0.2), we get

$$G^I_{ij}\left(\hat{c}, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) - G^I_{ij}\left(\hat{a}, 1, \frac{Q^2}{\mu^2}, \epsilon\right) = \int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A^I_{ij}$$

$$\Rightarrow \quad G^I_{ij}\left(c(\mu_R^2), \frac{Q^2}{\mu_R^2}, \epsilon\right) = G^I_{ij}\left(c(Q^2), 1, \epsilon\right) + \int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A^I_{ij} \tag{D.0.6}$$

Consider the second part of the above Eq. (D.0.6)

$$\int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A^I_{ij} = \int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} \sum_{k=1}^{\infty} c^k A^I_{ij,k}$$

$$= \sum_{k=1}^{\infty} \int_{\frac{Q^2}{\mu^2}}^{\frac{\mu_R^2}{\mu^2}} \frac{dX^2}{X^2} \hat{c}^k S_\epsilon^k \left(X^2\right)^{k\frac{\epsilon}{2}} \left(Z_c^{-1}(X^2)\right)^k A^I_{ij,k} \tag{D.0.7}$$

where, the integration variable was changed from μ_R to X by $\mu_R^2 = X^2\mu^2$. By evaluating the integral we obtain

$$\int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A^I_{ij} = \sum_{k=1}^{\infty} \hat{c}^k S_\epsilon^k \left(\frac{\mu_R^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \left[\left(\frac{Q^2}{\mu_R^2}\right)^{k\frac{\epsilon}{2}} - 1\right] \hat{K}^I_{ij,k}(\epsilon)\,. \tag{D.0.8}$$

The first part of G^I_{ij} in Eq. (D.0.6) can be expanded in powers of $c(Q^2)$ as

$$G^I_{ij}\left(c(Q^2), 1, \epsilon\right) = \sum_{k=1}^{\infty} c^k(Q^2) G^I_{ij}(\epsilon)\,. \tag{D.0.9}$$

By putting back the Eq. (D.0.3), (D.0.7) and (D.0.8) in the original KG equation (D.0.1), we solve for $\ln \mathcal{F}^I_{ij}(\hat{c}, Q^2, \mu^2, \epsilon)$:

$$\ln \mathcal{F}^I_{ij}(\hat{c}, Q^2, \mu^2, \epsilon) = \sum_{k=1}^{\infty} \hat{c}^k S_\epsilon^k \left(\frac{Q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\mathcal{L}}^I_{ij,k}(\epsilon) \tag{D.0.10}$$

with[1]

$$
\begin{aligned}
\hat{\mathcal{L}}^{I}_{ij,1}(\epsilon) &= \frac{1}{\epsilon^2}\Big\{-2A^{I}_{ij,1}\Big\} + \frac{1}{\epsilon}\Big\{G^{I}_{ij,1}(\epsilon)\Big\},\\
\hat{\mathcal{L}}^{I}_{ij,2}(\epsilon) &= \frac{1}{\epsilon^3}\Big\{\beta_0 A^{I}_{ij,1}\Big\} + \frac{1}{\epsilon^2}\Big\{-\frac{1}{2}A^{I}_{ij,2} - \beta_0 G^{I}_{ij,1}(\epsilon)\Big\} + \frac{1}{\epsilon}\Big\{\frac{1}{2}G^{I}_{ij,2}(\epsilon)\Big\},\\
\hat{\mathcal{L}}^{I}_{ij,3}(\epsilon) &= \frac{1}{\epsilon^4}\Big\{-\frac{8}{9}\beta_0^2 A^{I}_{ij,1}\Big\} + \frac{1}{\epsilon^3}\Big\{\frac{2}{9}\beta_1 A^{I}_{ij,1} + \frac{8}{9}\beta_0 A^{I}_{ij,2} + \frac{4}{3}\beta_0^2 G^{I}_{ij,1}(\epsilon)\Big\}\\
&\quad + \frac{1}{\epsilon^2}\Big\{-\frac{2}{9}A^{I}_{ij,3} - \frac{1}{3}\beta_1 G^{I}_{ij,1}(\epsilon) - \frac{4}{3}\beta_0 G^{I}_{ij,2}(\epsilon)\Big\} + \frac{1}{\epsilon}\Big\{\frac{1}{3}G^{I}_{ij,3}(\epsilon)\Big\},\\
\hat{\mathcal{L}}^{I}_{ij,4}(\epsilon) &= \frac{1}{\epsilon^5}\Big\{A^{I}_{ij,1}\beta_0^3\Big\} + \frac{1}{\epsilon^4}\Big\{-\frac{3}{2}A^{I}_{ij,2}\beta_0^2 - \frac{2}{3}A^{I}_{ij,1}\beta_0\beta_1 - 2\beta_0^3 G^{I}_{ij,1}(\epsilon))\Big\}\\
&\quad + \frac{1}{\epsilon^3}\Big\{\frac{3}{4}A^{I}_{ij,3}\beta_0 + \frac{1}{4}A^{I}_{ij,2}\beta_1 + \frac{1}{12}A^{I}_{ij,1}\beta_2 + \frac{4}{3}\beta_0\beta_1 G^{I}_{ij,1}(\epsilon) + 3\beta_0^2 G^{I}_{ij,2}(\epsilon))\Big\}\\
&\quad + \frac{1}{\epsilon^2}\Big\{-\frac{1}{8}A^{I}_{ij,4} - \frac{1}{6}\beta_2 G^{I}_{ij,1}(\epsilon) - \frac{1}{2}\beta_1 G^{I}_{ij,2}(\epsilon) - \frac{3}{2}\beta_0 G^{I}_{ij,3}(\epsilon)\Big\}\\
&\quad + \frac{1}{\epsilon}\Big\{\frac{1}{4}G^{I}_{ij,4}(\epsilon)\Big\}. \qquad \text{(D.0.11)}
\end{aligned}
$$

where $G^{I}_{li,j}$ has the following decomposition, first conjectured by [?] and then confirmed by [157] at three loops. Recently [?] showed that the decompsition is true even at 4 loops.

$$
G^{I}_{li,j}(\epsilon) = 2\left(B_{li,j} - \gamma^{I}_{li,j}\right) + f_{li,j} + \sum_{k=1}^{\infty}\epsilon^k g^{I,k}_{li,j}
$$

where, $B = \sum_{j=1}^{\infty} a^j B_j$ and $f = \sum_{j=1}^{\infty} a^j f_j$ are the collinear and soft anamolous dimensions in $\mathcal{N} = 4$ SYM and γ^{I}_{j} is overall UV anamolous dimention for effective operator taken.

[1]For $\mathcal{N} = 4$ one has to set β_i to zero.

E Soft Distribution Function

In Sec. 2.3, we introduced the soft distribution Φ^I_{aa} in the context of the factoristion of SV cross section. A generic SV cross section in x momenta fraction space have following form

$$\Delta^{I,\mathrm{SV}}(z,q^2,\mu_R^2,\mu_F^2) = \mathcal{C}\exp\left(\psi^I\left(z,q^2,\mu_R^2,\mu_F^2,\epsilon\right)\right)\Big|_{\epsilon=0} \tag{E.0.1}$$

where, $\psi^I\left(z,q^2,\mu_R^2,\mu_F^2,\epsilon\right)$ is finite in d $= 4+\epsilon$ and $\mathcal{C}$ is the standard convolution. The ψ^I is given by, Eq. (2.3.22)

$$\begin{aligned}\psi^I\left(z,q^2,\mu_R^2,\mu_F^2,\epsilon\right) = &\left(\ln\left[Z^I(\hat{a}_s,\mu_R^2,\mu^2,\epsilon)\right]^2 + \ln\left|\mathcal{F}^I(\hat{c},Q^2,\mu^2,\epsilon)\right|^2\right)\delta(1-z)\\ &+ 2\Phi^I(\hat{c},q^2,\mu^2,z,\epsilon) - 2\mathcal{C}\ln\Gamma^I(\hat{c},\mu^2,\mu_F^2,z,\epsilon)\,.\end{aligned} \tag{E.0.2}$$

where, Z^I, Φ^I, Γ^I is the overall renormalisation constant, soft distribution function[1] and splitting functions kernel respectively.

Putting $\mu_R = \mu_F$ we get,

$$\begin{aligned}\ln\left(Z^{I,(1)}\right)^2 &= c(\mu_F^2)\frac{4\gamma_1^I}{\epsilon}\,,\\ \ln|\mathcal{F}^{I,(1)}|^2 &= c(\mu_F^2)\left(\frac{q^2}{\mu_F^2}\right)^{\frac{\epsilon}{2}}\left[-\frac{4A_1^I}{\epsilon^2} + \frac{1}{\epsilon}\left(2f_1^I + 4B_1^I - 4\gamma_1^I\right)\right],\\ 2\mathcal{C}\ln\Gamma^{I,(1)} &= 2c(\mu_F^2)\left[\frac{2B_1^I}{\epsilon}\delta(1-z) + \frac{2A_1^I}{\epsilon}\mathcal{D}_0\right]\end{aligned} \tag{E.0.3}$$

[1] Soft distribution functions contains informations about the real radiations only.

$$
\begin{aligned}
\psi^I &= \sum_{k=1}^{\infty} c^k \left(\mu_F^2\right) \psi^{I,(k)} \,, \\
\ln(Z^I)^2 &= \sum_{k=1}^{\infty} c^k \left(\mu_F^2\right) Z^{I,(k)} \,, \\
\ln|\mathcal{F}^I|^2 &= \sum_{k=1}^{\infty} c^k \left(\mu_F^2\right) \left(\frac{q^2}{\mu_F^2}\right)^{k\frac{\epsilon}{2}} \ln|\mathcal{F}^{I,(k)}|^2 \,, \\
\Phi^I &= \sum_{k=1}^{\infty} c^k(\mu_F^2) \Phi_k^I \,, \\
\ln\Gamma^I &= \sum_{k=1}^{\infty} c^k(\mu_F^2) \ln\Gamma^{I,(k)}
\end{aligned}
\tag{E.0.4}
$$

and the "+" distribution kernel arising from real radiations

$$
\mathcal{D}_i \equiv \left[\frac{\ln^i(1-z)}{1-z}\right]_+ \,. \tag{E.0.5}
$$

Collecting the first order coefficients in $c(\mu_F^2)$ of the singular part of ψ^I we get,

$$
\psi^{I,(1)}|_{\text{singular}} = \left[\left\{-\frac{4A_1^I}{\epsilon^2} + \frac{2f_1^I}{\epsilon}\right\}\delta(1-z) - \frac{4A_1^I}{\epsilon}\mathcal{D}_0\right] + 2\Phi_1^I \tag{E.0.6}
$$

For getting a finite ψ^I in the above expression Φ^I must have the opposite pole structure.

$$
2\Phi_1^I|_{\text{singular}} = -\left[\left\{-\frac{4A_1^I}{\epsilon^2} + \frac{2f_1^I}{\epsilon}\right\}\delta(1-z) - \frac{4A_1^I}{\epsilon}\mathcal{D}_0\right] \tag{E.0.7}
$$

Also, as Φ^I contains information about the real radiations only, it should be RG invariant with respect to μ_R:

$$
\mu_R^2 \frac{d}{d\mu_R^2}\Phi^I = 0 \,. \tag{E.0.8}
$$

Following the *ansatz* [67] and considering the above equations one can show that Φ^I satisfies a the K+G alike integro-differential equation : namely $\overline{K+G}$:

$$
q^2 \frac{d}{dq^2}\Phi^I\left(\hat{c}, q^2, \mu^2, z, \epsilon\right) = \frac{1}{2}\left[\overline{K}^I\left(\hat{c}, \frac{\mu_R^2}{\mu^2}, z, \epsilon\right) + \overline{G}^I\left(\hat{c}, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon\right)\right] \,. \tag{E.0.9}
$$

where, $\overline{K}^I$ accommodates all the singular terms whereas $\overline{G}^I$ contains of the finite terms in ϵ. From the RGE invariance of of Φ^I one gets

$$\mu_R^2 \frac{d}{d\mu_R^2} \overline{K}^I = -\mu_R^2 \frac{d}{d\mu_R^2} \overline{G}^I = \gamma^I \tag{E.0.10}$$

where, we define a quantity γ^I. The solution of Φ^I can be written as

$$\Phi^I(\hat{c}, q^2, \mu^2, z, \epsilon) = \sum_{k=1}^{\infty} \hat{c}^k S_\epsilon^k \left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Phi}_k^I(z, \epsilon) \tag{E.0.11}$$

with

$$\hat{\Phi}_k^I(z, \epsilon) = \hat{\mathcal{L}}_k^I \left(A_i^I \to \gamma_i^I, G_i^I \to \overline{G}_i^I(z, \epsilon)\right). \tag{E.0.12}$$

where the perturbative expansions of the quantities are

$$\begin{aligned} \gamma^I &= \sum_{k=1}^{\infty} c^k(\mu_F^2) Y_k^I, \\ \overline{G}^I(z, \epsilon) &= \sum_{k=1}^{\infty} c^k(\mu_F^2) \overline{G}_k^I(z, \epsilon). \end{aligned} \tag{E.0.13}$$

This solution directly follows from the Eq. (D.0.10). Hence we get

$$2\hat{\Phi}_1^I(z, \epsilon) = \frac{1}{\epsilon^2}\Big\{-4\gamma_1^I\Big\} + \frac{2}{\epsilon}\Big\{\overline{G}_1^I(z, \epsilon)\Big\}. \tag{E.0.14}$$

Expanding the components of Φ^I in powers of c, we obtain

$$\begin{aligned} \Phi^I(\hat{c}, q^2, \mu^2, z, \epsilon) &= \sum_{k=1}^{\infty} \hat{c}^k S_\epsilon^k \left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Phi}_k^I(z, \epsilon) \\ &= \sum_{k=1}^{\infty} c^k(\mu_F^2) \left(\frac{q^2}{\mu_F^2}\right)^{k\frac{\epsilon}{2}} Z_{a_s}^k \hat{\Phi}_k^I(z, \epsilon) \\ &\equiv \sum_{k=1}^{\infty} c^k(\mu_F^2) \left(\frac{q^2}{\mu_F^2}\right)^{k\frac{\epsilon}{2}} \Phi_k^I(z, \epsilon) \end{aligned} \tag{E.0.15}$$

and considering the $c(\mu_F^2)$ order term , $\hat{\Phi}_1^I(x, \epsilon) = \Phi_1^I(x, \epsilon)$ upon discarding the terms like $\log(q^2/\mu_F^2)$.

Hence, by comparing the Eq. (E.0.7) and (E.0.14), we get

$$\gamma_1^I = -A_1^I \delta(1-z)\,, \overline{G}_1^I(z,\epsilon) = -f_1^I \delta(1-z) + 2A_1^I \mathcal{D}_0 + \sum_{k=1}^{\infty} \epsilon^k \overline{g}_1^{I,k}(z)\,.$$

$\overline{g}_1^{I,k}(z)$ determined through explicit computations of the soft functions or can be extracted from the inclusive SV cross section using E.0.2.This uniquely fixes the structure of the soft distribution Φ^I at one loop level. The generalisation of the prescription is very straight forward for higher order in perturbation theory. It is worth mentioning that the generalisation of the modified casimir scaling for the soft distribution function at N^4LO level in the context of QCD is going to be a very interesting topic to consider in future. Let us propose a solution of $\overline{K+G}$ equation :

$$\begin{aligned}\hat{\Phi}_k^I(z,\epsilon) &\equiv \left\{ k\epsilon \frac{1}{1-z} \left[(1-z)^2\right]^{k\frac{\epsilon}{2}} \right\} \hat{\Phi}_k^I(\epsilon) \\ &= \left\{ \delta(1-z) + \sum_{j=0}^{\infty} \frac{(k\epsilon)^{j+1}}{j!} \mathcal{D}_j \right\} \hat{\Phi}_k^I(\epsilon)\,.\end{aligned} \tag{E.0.16}$$

The RG invariance of Φ^I, Eq. (E.0.8), implies

$$\mu_R^2 \frac{d}{d\mu_R^2} \overline{K}^I = -\mu_R^2 \frac{d}{d\mu_R^2} \overline{G}^I \equiv \gamma'^I \tag{E.0.17}$$

where, we introduce a quantity γ', analogous to γ. Hence, the solution can be obtained as

$$\hat{\Phi}_k^I(\epsilon) = \hat{\mathcal{L}}_k^I \left(A_i^I \to \gamma_i'^I, G_i^I \to \overline{\mathcal{G}}_i^I(\epsilon) \right)\,. \tag{E.0.18}$$

Hence, according to the Eq. (D.0.11), for $k = 1$ we get

$$2\Phi_1^I(z,\epsilon) = \left\{ \frac{1}{\epsilon^2} \left(-4\gamma_1'^I\right) + \frac{2}{\epsilon} \overline{\mathcal{G}}_1^I(\epsilon)) \right\} \delta(1-z) + \left\{ -\frac{4\gamma_1'^I}{\epsilon^2} + \frac{2}{\epsilon} \overline{\mathcal{G}}_1^I(\epsilon) \right\} \sum_{j=0}^{\infty} \frac{\epsilon^{j+1}}{j!} \mathcal{D}_j \tag{E.0.19}$$

where, γ'^I and $\overline{\mathcal{G}}^I$ are expanded similar to Eq. (E.0.13). Comparing between the two solutions Eq. (E.0.7) and (E.0.19), we find

$$\gamma_1'^I = -A_1^I$$

$$\overline{\mathcal{G}}_1^I(\epsilon) = -f_1^I + \sum_{k=1}^{\infty} \epsilon^k \overline{\mathcal{G}}_1^{I,k}. \qquad \text{(E.0.20)}$$

Explicit computation is required to determine the coefficients of ϵ^k, $\overline{\mathcal{G}}_1^{I,k}$.

F Harmonic Polylogarithms

The transcendental functions like logarithms, polylogarithms ($\mathrm{Li}_n(x)$) and generalised Nielsen's polylogarithm ($\mathrm{S}_{n,p}(x)$) appear in the in the feynman integral computations in pQCD which are defined as follows

$$
\begin{aligned}
&\ln(x) = \int_1^x \frac{dt}{t}\,,\\
&\mathrm{Li}_n(x) \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^n} = \int_0^x \frac{dt}{t}\mathrm{Li}_{n-1}(t)\,, \qquad \text{e.g.} \quad \mathrm{Li}_1(x) = -\ln(1-x)\,,\\
&S_{n,p}(x) \equiv \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{dt}{t}[\ln(t)]^{n-1}[\ln(1-xt)]^p\,,\\
&\quad \text{e.g.} \quad S_{n-1,1}(x) = \mathrm{Li}_n(x)\,.
\end{aligned} \tag{F.0.1}
$$

For the evaluation of multiloop/leg feynman integral or phase phase integral the above mentioned functions are not adequate. To overcome this a new set of functions which are called **Harmonic Polylogarithms (HPLs)** was introduced. Here, we briefly describe the definition and properties of HPL [186] and 2dHPL. HPL is denoted by $HPL(\vec{m}_w;y)$ or sometimes by $H(\vec{m}_w;y)$ with a w-dimensional vector $\vec{m}_w$ of parameters and its argument y. w is called the weight of the HPL. The elements of $\vec{m}_w$ belong to $\{1,0,-1\}$ through which the following rational functions are defined

$$
f(1;y) \equiv \frac{1}{1-y}, \qquad f(0;y) \equiv \frac{1}{y}, \qquad f(-1;y) \equiv \frac{1}{1+y}\,. \tag{F.0.2}
$$

The weight 1 ($w=1$) HPLs are given by

$$
HPL(1,y) \equiv -\ln(1-y), \qquad HPL(0,y) \equiv \ln y, \qquad HPL(-1,y) \equiv \ln(1+y)\,. \tag{F.0.3}
$$

For $w > 1$, $HPL(m, \vec{m}_w; y)$ can be defined as

$$HPL(m, \vec{m}_w; y) \equiv \int_0^y dx\, f(m, x)\, HPL(\vec{m}_w; x), \qquad m \in 0, \pm 1\,. \tag{F.0.4}$$

The 2dHPLs are defined in the same way as Eq. (F.0.4) with the new elements $\{2, 3\}$ in $\vec{m}_w$ defining a new class of functions

$$f(2; y) \equiv f(1 - z; y) \equiv \frac{1}{1 - y - z}, \qquad f(3; y) \equiv f(z; y) \equiv \frac{1}{y + z} \tag{F.0.5}$$

and correspondingly with the weight 1 ($w = 1$) 2dHPLs

$$HPL(2, y) \equiv -\ln\Big(1 - \frac{y}{1 - z}\Big), \qquad HPL(3, y) \equiv \ln\Big(\frac{y + z}{z}\Big). \tag{F.0.6}$$

F.0.1 Properties

Shuffle algebra : A product of two HPL with weights w_1 and w_2 of the same argument y is a combination of HPLs with weight $(w_1 + w_2)$ and argument y, such that all possible permutations of the elements of $\vec{m}_{w_1}$ and $\vec{m}_{w_2}$ are considered preserving the relative orders of the elements of $\vec{m}_{w_1}$ and $\vec{m}_{w_2}$,

$$HPL(\vec{m}_{w_1}; y) HPL(\vec{m}_{w_2}; y) = \sum_{\vec{m}_w = \vec{m}_{w_1} \uplus \vec{m}_{w_2}} HPL(\vec{m}_w; y). \tag{F.0.7}$$

Integration-by-parts identities : The ordering of the elements of $\vec{m}_w$ in an HPL with weight w and argument y can be reversed using integration-by-parts and in the process, some products of two HPLs are generated in the following way

$$\begin{aligned} HPL(\vec{m}_w; y) \equiv HPL(m_1, m_2, ..., m_w; y) &= HPL(m_1, y) HPL(m_2, ..., m_w; y) \\ &- HPL(m_2, m_1, y) HPL(m_3, ..., m_w; y) \\ &+ ... + (-1)^{w+1} HPL(m_w, ..., m_2, m_1; y)\,. \end{aligned} \tag{F.0.8}$$

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