COHOMOLOGY OF A MODULI SPACE OF VECTOR BUNDLES

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A thesis submitted to the University of Madras for the degree of Doctor of Philosophy

By

V. BALAJI

THE INSTITUTE OF MATHEMATICAL SCIENCES
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1989

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CERTIFICATE

This is to certify that the Ph.D. thesis submitted by Mr.V.Balaji, to the University of Madras, entitled : Cohomology of a moduli space of vector bundles , is a record of bonafide research work done by him under my supervision. The research work presented in this thesis has not been presented in part or full for any other Degree, Diploma, Associateship or other similar titles. It is further certified that the thesis represents independent work on the part of the candidate.

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Supervisor.

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PREFACE

This thesis is concerned with a study of the cohomological properties of certain moduli spaces of vector bundles over a compact Riemann surface. A detailed discussion is given in the Introduction. This research was funded by the National Board for Higher Mathematics.

Let me take this opportunity to express my thanks to various people who have helped me in the course of my mathematical career.

First and foremost, my grateful thanks to my teacher Prof C.S.Seshadri. He taught me with infinite patience and has encouraged me at every step. It was a great privilege to learn this subject from him.

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Numerous colleagues have assisted me at various times. Adequate thanksgiving would require space which this page is too small to provide.

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But for my parents, my brother and my sister I would never have been able to set sail towards these far off shores of research mathematics. They have constantly encouraged me and have showered on me their love. To them my eternal gratitude.

CONTENTS

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INTRODUCTION

He listens for Inspirations' postman knock
And takes delivery of the priceless gift
A little spoilt by the receiver mind
Or mixed with the manufacture of his brain;
When least defaced, then is it most divine.

Sri Aurobindo, Savitri, Book seven

The Betti numbers of the moduli spaces of semi-stable vector bundles of rank r and degree d, with $(r,d) = 1$ have been the subject of study beginning with Newstead [PN-1](in 1967 for $r=2, d=1$), Harder-Narasimhan [H-N](in 1972, $\forall$ $r,d,(r,d)=1$) and in recent years by Atiyah-Bott [A-B](in 1982 $\forall$ r,d, $(r,d)=1$ .More recently, F.Kirwan [K] (in 1986) has paid attention to the non-coprime case and has computed the intersection Betti numbers of these moduli spaces. N.Nitsure [NN-2] (in 1987) has computed the Betti numbers of the moduli space of parabolic vector bundles.

Apart from throwing light on the topology of these moduli varieties the cohomology groups provide us with subtle geometric information especially regarding the rationality of these varieties.

We make a brief digression here to discuss the Lüroth problem in the present context.

In 1876, Lüroth proved that every unirational curve is rational. In 1894, Castelnuovo proved that every unirational surface over $\mathbb{C}$ is rational. (This is not true for non-algebraically closed base fields;(for example cf Manin (Cubic forms ,'Arithmetic, Algebra, Geometry',(North Holland 1985). The problem of Lüroth is ''*Are all unirational varieties of dim* ≥ 3 *over* $\mathbb{C}$ *rational ?*''

Subtle invariants have since been defined to answer this question. One such is the Brauer-Grothendieck group for schemes, introduced and investigated by Grothendieck [AG].

If V is a smooth proper unirational variety over $\mathbb{C}$ then Grothendieck observed that Br(V) (the Brauer-Grothendieck group) is a birational invariant of V and is actually isomorphic to $H^3(V,\mathbb{Z})_{tor}$. Hence V cannot be rational unless $H^3(V,\mathbb{Z})_{tor} = (0)$. But, in reality this provides us with only a negative criterion as has been recently shown by J.Colliot-Thélène and Ojanguren.

In 1972, M.Artin and D.Mumford in [A-M], constructed unirational conic bundles V over a surface, for which $\mathbb{Z}_2 \subset H^3(V,\mathbb{Z})$. These give us examples of unirational varieties which are not rational, thereby answering Lüroth's problem in the negative.

One is thus led naturally to pose this restricted question:

"*What is* $H^3(V,\mathbb{Z})_{tor}$ *for these moduli varieties ?*"(*)

In [A-B] Atiyah and Bott show that all the cohomology groups of M(n,d), (n,d)=1, are in fact torsion-free.

It is known that for (n,d)≠1 the varieties M(n,d) are singular.(cf [N-1]) Smooth models of M(2,0) (the moduli space of vector bundles of rank 2 and degree 0 with trivial determinant) have been constructed by Narasimhan-Ramanan [N-2] and Seshadri [S-1]. N.Nitsure [NN-1] was led to pose this question (*) for the smooth compactification of $M(2,0)^s$ (the stable bundles) constructed by Narasimhan-Ramanan [N-2], which we denote by N'. Nitsure shows that $H^3(N',\mathbb{Z})_{tor} = (0)$.

This was our starting point. Nitsure's proof of this fact was somewhat lengthy and since $H^3(V,\mathbb{Z})_{tor}$ is a birational invariant we were led to seeing if a simpler proof could be obtained using the canonical desingularisation model of M(2,0) (which we denote by N) of [S-1]. (This is canonical in the sense of representing a natural moduli functor). We give a considerably shorter proof of Nitsure's theorem in Chapter I using the variety N.(We denote this variety by N_o in Chapter I for technical reasons)

The proof led us to define a natural stratification of N as given below:

We have a canonical family of quadratic forms $\{Q_x\}_{x\in N}$ on a 3-dimensional vector space parametrised by N. We define closed subschemes $\{N_i\}$, i=1,2,3, by the condition

$$N_i = \left\{ x \in N \mid \text{rank of } Q_x \le 3 - i \right\}$$

One observes that $N_i - N_{i+1}$, i=1,2,3 ($N_4 = \emptyset$) are all smooth subschemes of N.

We use this stratification to compute some low cohomology groups of N. In particular the main theorems of Chapter I are

Theorem (A). *The third cohomology group of* N *is torsion-free,* $g \geq 3$.

Theorem (B). *Let* B_i *denote the Betti numbers of* N. *Then we have:*

$$B_3 = 2g, \quad B_4 = {}^{2g}C_2 + 4, \quad g \geq 4.$$

The basic principle in these computations is the following:

One explicitly determines the strata and their normal bundles and examine the Thom-Gysin sequence. (see also [A-B] pp 537). This principle has been exploited to the full in [B-2] to give a complete description of the strata of N and to compute about $\frac{2}{3}$rd's of the Betti numbers of N.

This concludes Chapter I of the thesis.

Chapter II is devoted to giving an application of the computations of Chapter I by studying the third intermediate jacobian of N.

For non-singular projective varieties V with the plurigenera $h^{3,0}(V) = h^{0,3}(V) = 0$, an interesting invariant is the intermediate Jacobian attached to $H^3(V)$. In this situation(e.g. if V is unirational) the Weil intermediate Jacobian coincides with the Griffiths construction and we have the Weil-Griffiths Jacobian $J^2(V)$,an abelian variety which is by definition

$$J^2(V) \approx H^3(V,\mathbb{R})/\text{Image } H^3(V,\mathbb{Z})$$

where $H^3(V,\mathbb{R})$ is given a complex structure via the decomposition

$$H^3(V,\mathbb{R}) \otimes \mathbb{C} \approx H^{1,2} \oplus H^{2,1}$$

(cf [G]).

A polarisation on V canonically induces one on $J^2(V)$ as follows:

Let ω denote the Kähler class defined by this polarisation. Then ω defines a bilinear pairing on $H^3(V,\mathbb{C})$ as follows:

$$H^3(V,\mathbb{C}) \times H^3(V,\mathbb{C}) \longrightarrow \mathbb{C}$$

$$(\alpha\ ,\ \beta) \longrightarrow \int_V \omega^{n-3} \wedge \alpha \wedge \beta \qquad (n = \dim_{\mathbb{C}} V)$$

(Where we have tacitly assumed that all classes in $H^3(V,\mathbb{C})$ are primitive since $h^{0,3} = h^{3,0} = 0$, $h^{0,1} = h^{1,0} = 0$ say when V is unirational)

This pairing induces a polarisation on the torus $J^2(V)$ making it a polarised abelian variety which depends holomorphically on V (cf [G]).

For the moduli space $M(n,d)_L$ of semi-stable vector bundles of rank n and degree d with $\det E = L$, Mumford-Newstead [M-N] and later Narasimhan-Ramanan [N-3] have shown that if $(n,d) = 1$ $J^2(M(n,d)_L)$ is <u>canonically isomorphic</u> to the principally polarised Jacobian J(X) of X.

If $(n,d) \neq 1$, $M(n,d)_L$ is no longer smooth. For the case $n=2, d=0$ $L = \mathcal{O}_X$ we have the desingularisation model N, which solves a natural moduli problem. Moreover, from the computations in Chapter I, we have $H^3(N,\mathbb{Z}) = \mathbb{Z}^{2g}$, and since N provides us with a smooth compactification of $M(2,0)^*_{\mathcal{O}_X}$ it is natural to pose an analogous question for the intermediate jacobian of this variety N. Then the main theorem of Chapter II reads as follows:

Theorem (C) *There is a canonical polarisation* Θ' *on* $J^2(N(X))$ *and an isogeny* ϕ *of degree* 2^{2g}

$$\phi : J(X) \longrightarrow J^2(N(X))$$

such that $\phi^*(\Theta') \sim \Theta$.

In fact

$$\operatorname{Ker} \phi = \{ \text{ points of order 2 of } J(X) \}$$

Thus by Torelli's theorem we have,

Corollary. *If* $N(X_1)$ *is isomorphic to* $N(X_2)$ *for two curves* X_1 *and* X_2 *, then* X_1 *is isomorphic to* X_2.

CHAPTER I

COHOMOLOGY OF A MODULI SPACE

Our tasks are given, we are but instruments:

Sri.Aurobindo, Savitri, Book seven.

§1 Preliminaries

In this section we shall outline very briefly the definitions and terminologies of [S-1]. The proof of most of the statements made in this section can be found in [S-1] or [S-2]. We state at the very outset that the ground field of all our varieties is the field $\mathbb{C}$ of complex numbers.

(*i*) X is a smooth irreducible projective curve of genus $g \geq 3$.

(*ii*) Let V be a vector bundle on X. By a *parabolic structure* at a point $P \in X$ we mean

(*a*) a *quasi parabolic structure* i.e a flag

$$V_P = F^1V_P \supseteq F^2V_P \supseteq \ldots \supseteq F^rV_P.$$

(*b*) weights $\alpha_1, \ldots, \alpha_r$ attached to $F^1V_P, \ldots, F^1V_P$ such that

$$0 \leq \alpha_1 < \alpha_2 < \ldots \alpha_r < 1.$$

Call $k_1 = \dim F^1V_P - \dim F^2V_P, \ldots\ k_r = \dim\ F^rV_P$ the multiplicities of $\alpha_1, \ldots, \alpha_r$.

The *parabolic degree* of V is defined by

$$\text{par deg } V = \deg V + \sum_i k_i \alpha_i$$

and write par $\mu(V)$ = par deg V/rk V.

If W is a subbundle of V, it acquires, in an obvious way, a quasi-parabolic structure. To make it a parabolic subbundle, we attach weights as follows:

Given i_o, $F^{i_o}W \subset F^jV$ for some j; let j_o be such that $F^{i_o}W \subset F^{j_o}V$ and $F^{i_o}W \not\subset F^{j_o+1}V$; then the weight of $F^{j_o}V$ = wt $F^{i_o}W$. Define V to be parabolic stable (resp semi stable) if for every proper subbundle W of V, one has par $\mu(W)$ <par $\mu(V)$ (resp $\leq$)

If V_n is the category of semistable vector bundles on X of rank n and degree 0, then we denote by PV_n the category of parabolic semistable vector bundles at a fixed point $P \in X$ and fixed parabolic structure. (cf [S-1] for this notation).Recall that, one can choose the weights (α) small enough so as to have the condition 'parabolic semistable' equivalent to'parabolic stable'.

(*iii*) N is the isomorphism classes of $(V,\Delta) \in PV_4$ (Δ a parabolic structure), such that End V is a specialisation of $\mathcal{M}_2$-the (2 × 2) matrix algebra.

In fact, if (V,Δ_1) and (V,Δ_2) belong to N, they represent the same element of N (i.e isomorphic in PV_4) iff the underlying vector bundles V_1 and V_2 are isomorphic (cf [S-1]). Hence we often simply write $V \in N$.

(iv) $\mathcal{A}$ is the variety of all algebra structures on a fixed 4-dimensional vector space which are specializations of $\mathcal{M}_2$ and admit a fixed identity element. We have a *canonical group of automorphisms* acing on $\mathcal{A}$, namely the subgroup of $GL(4)$, fixing this identity element.

(v) M denotes the normal projective variety of equivalence classes of vector bundles of rank 2 and degree 0 under the equivalence relation $V \sim V'$ if and only if $gr\ V = gr\ V'$.

(vi) M^s will be the open subset of M consisting of the stable bundles.

It is known that $M - M^s$ is precisely the singular locus of M (cf [N-1]). The main theorem of [S-1] is stated below.

Theorem 1. (*Seshadri*) *There is a natural structure of a smooth projective variety on* N *and there exists a canonical morphism* $p : N \longrightarrow M$, *which is an isomorphism over* M^s. *More precisely, if* $V \in N$, *then* $gr\ V = D \oplus D$, *with* $rk\ D = 2$, $gr\ D$ *is a direct sum of stable line bundles of degree* 0 *and the morphism* $p : N \longrightarrow M$ *is given by* $V \longrightarrow D$. *Further* $V \in p^{-1}(M^s)$ iff $End\ V \cong \mathcal{M}_2$ *or equivalently (which is easily seen)* $V = W \oplus W$, *where* W *is stable.*

In the course of proving the smoothness of N, Seshadri defines a morphism from neighbourhoods U of a given point of N into $\mathcal{A}$ which we shall denote by

$$\phi^{U} : U \longrightarrow \mathcal{A}$$

We shall briefly indicate the construction of ϕ^{U}: The functor defining the moduli space N being representable, we have a defining vector bundle E on $X \times N$ of rank 4. Let $f : X \times N \longrightarrow N$ be the canonical projection and End E the vector bundle associated to the sheaf of endomorphism of E. Set

$$\mathcal{B} = f_{*}(\text{End } E)$$

$\mathcal{B}$ is the canonical family of specialisation of $\mathcal{M}_2$, parametrized by N (cf Prop.5 [S-1] for details). Consider any given point $u \in N$; then choosing a neighbourhood U of u, which trivialises B, we get a natural morphism

$$\phi^{U} : U \longrightarrow \mathcal{A} \quad \text{given by} \quad V \longrightarrow \text{End } V, \quad V \in U.$$

This morphism exists by the so-called versal property of $\mathcal{A}$. Further, let A_o = End V_u, V_u the vector bundle corresponding to the point $u \in U$, i.e $A_o = \phi^{U}(u)$. Then, if A_u is the mini-versal deformation space of A_o, the morphism

$$\phi^{U}_{1} : U \longrightarrow A_u$$

induced by the versality of A_u, is in fact *smooth*.

Note 1.1 By an abuse of notation, in the course of this chapter, we shall suppress U and the mini-versal deformation space corresponding to each point, and simply denote by $\phi : N \longrightarrow \mathcal{A}$ the *smooth local morphism* defined above. In fact, we will be using it only in this form throughout this chapter.

Note further that these ϕ^{U} are uniquely determined modulo automorphisms coming from the canonical group of automorphisms acting on $\mathcal{A}$.

Definition 1.1. Let M_o (resp N_o) be the subvarieties of M (resp N) consisting of bundles with trivial determinant. Then it is easy to see that p maps N_o to M_o.

Proposition 1.1 *The restriction of the local morphism* ϕ *to the sub-variety* N_o *remains smooth.*

Proof. Let J denote the Jacobian variety of line bundles of degree zero on X. Then we have a natural morphism

$$\psi : N_o \times J \longrightarrow N$$

$$(E,L) \longrightarrow E \otimes L$$

(that this map is a morphism follows from the universal property of N and the fact that $E \otimes L$ gives a family on X parametrized by $N_o \times J$).

We claim that ψ is *smooth*. In fact ψ is *etale*. For, let $\Gamma \subset J$ be the finite subgroup of J consisting of elements of order 2. Then there is a natural diagonal action of Γ on $N_o \times J$ which is obviously fixed point free. It is not difficult to see that N is actually the quotient of $N_o \times J$ by Γ and $\psi : N_o \times J \longrightarrow N$ the quotient morphism (note that our ground field is $\mathbb{C}$ and if A and B are smooth complex manifolds and G is a finite group acting on A such that B is the set theoretic quotient of A by G, then B is A /G).

This Γ-action being fixed point free, ψ is *etale*.

For $b \in N_o \times J$, choosing a neighbourhood U of $\psi(b) = u$ in N, we get the following diagram

$$\begin{tikzcd}
\psi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr] & & U \arrow[dl, "\phi_1^U"] \\
& A_u &
\end{tikzcd}$$

where A_u is the mini-versal deformation space of the algebra $A_o = \phi^U(u)$ in $\mathscr{A}$. Since ϕ_1^U, ψ are smooth, so is $\phi_1^U \circ \psi$. In other words the *local morphism* (again by abuse of notation)

$$\phi \circ \psi : N_o \times J \longrightarrow \mathscr{A}$$

is smooth. If $L \in J$, then $\mathrm{End}(E \otimes J) = \mathrm{End}\ E$ and hence $\phi \circ \psi$

clearly factors through N_o to give the smoothness of the restriction of ϕ from N_o to $\mathscr{A}$.

Remark 1.1. Because of Prop 1.1, by the same arguments as in [S-1], we see that N_o is a smooth projective variety. We then get an obvious generalization of Theorem.1.1 namely that $p : N_o \longrightarrow M_o$ is a desingularisation of M_o, and that it is an isomorphism over M_o^s etc.

§2.Conic bundles

Definition 1.2 Let S be a variety. A *generalized conic bundle* $\mathscr{C}$ on S is giving

(*a*) a vector bundle V on S of rank 3

(*b*) a closed subscheme $\mathscr{C}$ of $\mathbb{P}(V)$ over S, such that, given $s \in S$, $\exists$ a neighbourhood U of s, where $\mathscr{C} \cap p^{-1}(U)$ is defined by $q = 0$, $q \in \Gamma(p^{-1}(U), H^2)$, H being the tautological line bundle for $\mathbb{P}(V) \xrightarrow{p} S$; i.e $p_*(H) \cong V^*$ and therefore $p_*(H^2) = S^2(V^*)$, etc.

By definition, $\mathscr{C}$ is a Cartier divisor and is therefore defined by a section of a line bundle θ on $\mathbb{P}(V)$. Now locally over S, θ and H^2 coincide and therefore by the *see-saw theorem* (cf Mumford's *Abelian varieties*) there exists a line bundle L on S such that $\theta = H^2 \otimes L = S^2(V^*) \otimes L$, *the condition* (*b*) *above is equivalent* to giving an element q of $\Gamma(S^2(V^*) \otimes L)$ or a *quadratic form*

$$q : V \longrightarrow L.$$

The *discriminant* Δ of q can be defined as a section of $L^3 \otimes (\Lambda^3(V^*))^2$. The equation $\Delta = 0$ gives locally the *degeneracy locus* of $\mathscr{C}$.

We now introduce *subschemes* on S, namely for i = 1,2,3, set

$$S_i = \{s \in S \mid q \text{ restricted to } V_s, \text{ the fibre at } s, \text{ has rank} \leq 3\ \}$$

Thus we have a stratification

$$S_3 \subset S_2 \subset S_1 \subset S = S_0$$

If $g : \mathscr{C} \longrightarrow S$ be the projection, let $\mathscr{C}_i = g^{-1}(S_i)$, $i = 1,2,3$. Then we have S_1 to be the degeneracy locus of $\mathscr{C}$, i.e given by $\Delta = 0$, and $S_2 \subset S_1$ is the singular locus of S_1. The space $\mathscr{C}$ can be described as follows:

$\mathscr{C} - \mathscr{C}_1$ consists of non-degenerate conics; $\mathscr{C}_1 - \mathscr{C}_2$ of pairs of lines intersecting transversely; $\mathscr{C}_2 - \mathscr{C}_3$ of repeated lines and $\mathscr{C}_3$ of the whole plane. We call S_i the *canonical subschemes associated to the conic bundle $\mathscr{C}$* on S. Accordingly we make the following

Definition 1.3 A generalized conic bundle $\mathscr{C}$ is of type I if $\mathscr{C}_1 = \emptyset$, of type II if $\mathscr{C}_2 = \emptyset$ and of type III if $\mathscr{C}_3 = \emptyset$.

Definition 1.4 (cf p.164 [S-1])Let T be an algebraic scheme and $\{G_t\}_{t\in T}$ a family of algebras parametrized by T and defined by a locally free $\mathcal{O}_T$-module G of rank 4. We say that this is a *family of specialisations of* $\mathcal{M}_2$ if, given $t \in T$, there is a neighbourhood T_1 of t and a morphism $f: T_1 \longrightarrow \mathcal{A}$, such that $\{G_t\}_{t\in T_1}$ is the base change of $\{A_y\}_{y\in\mathcal{A}}$ by f, where A_y is the algebra structure corresponding to $y \in \mathcal{A}$.

Remark 1..2. By Remark 3 [S-1], the above definition has an equivalent formulation as follows: Let T = Spec R, and G be an R-algebra with identity e_0 such that the underlying R-module is free of rank 4. Let $J = G/Re_0$. Consider the canonical structure of a Lie algebra on J induced by the associative algebra structure on G. This gives a canonical skew-symmetric bilinear map $J \times J \longrightarrow J$ or equivalently (in our case)an element of $J \otimes J$. Then we say the algebra gives a *family of specialisations of* $\mathcal{M}_2$ parametrized by T, if this Lie algebra structure is defined by a *symmetric element of* $J \otimes J$. Further, the algebra G is isomorphic to C_q^+, q being the corresponding quadratic form. This definition generalizes, in an obvious way, when T is any scheme, and G is a vector bundle of rank 4 on T; however, the quadratic form q on J takes values in a line bundle on T.

Note 2. We shall use this reformulation in the course of this chapter.

Remark 1.3(*i*) Denote the canonical family of specialisation of $\mathcal{M}_2$ parametrized by N_o by $\mathcal{B}$. .

(*ii*) For $y \in \mathcal{A}$, let $\mathcal{A}_y$ be the corresponding algebra structure; then $\{\mathcal{A}_y\}_{y\in\mathcal{A}}$ gives an obvious family of specialisation of $\mathcal{M}_2$.

(*iii*) Let T = Spec R and G an R-algebra giving a family of specialisations of $\mathcal{M}_2$. Then by Remark 1.2, we get a symmetric element of $J \otimes J = G/Re_o$. This symmetric element naturally gives rise to a symmetric bilinear form on J^*(the R-dual of J) and therefore a quadratic form on J^*. Now J^* being a projective R-module of rank 3, it defines a vector bundle of rank 3 on T. More generally, if we are given an algebraic scheme T, a family $\{G_t\}$of specialisations of $\mathcal{M}_2$, then we have a canonical vector bundle V of rank 3 on T together with an $\mathcal{O}_T$-valued quadratic form $q : V \longrightarrow \mathcal{O}_T$, and thus a conic bundle on T.

(*iv*) The families $\mathcal{B}$ on N and $\{A_y\}_{y\in\mathcal{A}}$ on $\mathcal{A}$ give generalized conic bundles on N_o and $\mathcal{A}$ respectively.

Notation 2. Denote these conic bundles by P on N_o and Q on $\mathcal{A}$.

Proposition 1.2 *The conic bundle* P *on* N_o *is locally the base change of* Q *on* $\mathcal{A}$ *by the local morphism* $\phi : N_o \longrightarrow \mathcal{A}$ *of* §1.

Proof. This is an immediate consequence of the definitions of ϕ, $\mathcal{B}$, and $\{A_y\}_{y\in\mathcal{A}}$.

Remark 1.4. Following §2, we introduce the *canonical subschemes*

$$\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1 \subset \mathcal{A}$$

and

$$N_3 \subset N_2 \subset N_1 \subset N_0$$

associated to the degeneracy locus of P and Q respectively. Then, by Prop 1 2. ϕ locally maps $N_0 - N_2$ into $\mathcal{A} - \mathcal{A}_2$ in such a way that $N_1 - N_2 \longrightarrow \mathcal{A}_1 - \mathcal{A}_2$, $N_0 - N_1 \longrightarrow \mathcal{A} - \mathcal{A}_1$.

Remark 1.5 By Theorem 1 [S-1], we know that $\mathcal{A} \cong \Phi \times \Lambda$, where Λ is the 3-dimensional affine space and Φ the 6-dimensional affine space whose points are identified with the set of quadratic forms on a fixed 3-dimensional vector space (or algebras of the form C_q^+ -the even degree elements of the Clifford algebra associated to the quadratic form q). Therefore we have for $i = 1,2,3$

$$\mathcal{A}_i = \{ q \in \Phi \mid \text{rank } q \leq 3 - i \}$$

Note that

$$\mathcal{A}_{\_} - \mathcal{A}_1 = \{ q \mid q \in \Phi,\ C_q^+ \cong \mathcal{M}_2 \} \times \mathbb{A}^3 \quad \text{or equivalently}$$

$$\mathcal{A} - \mathcal{A}_1 = \{ y \mid A_y \cong \mathcal{M}_2 \}$$

Notation 3. We denote the subsets $N_o - N_2$ and $N_1 - N_2$ of N_o by Z and Y respectively.

Let $K = M_o - M_o^s$, be the singular locus of M_o. The bundles here are of the form $L \oplus L^{-1}$, where L is a line bundle of degree 0. Let K_o be the *nodes* of K (i.e consisting of bundles of the type $L \oplus L$ with L^2 trivial). Then

$$K - K_o = \{ L \oplus L^{-1} \mid L \in J - \Gamma \}$$

J and Γ as in §1. It may be noted that K *is the Kummer variety of dim* g (cf [N-1]).

Proposition 1.3 *The subsets Z and Y of N_o are precisely $N_o - \rho^{-1}(K_o)$ and $\rho^{-1}(K - K_o)$ respectively, where $\rho : N_o \longrightarrow M_o$ is the desingularisation morphism. In particular, $Z - Y = \rho^{-1}(M_o^s)$.*

Proof. By Remark 1.3, we know that $V \in p^{-1}(M_o^s)$ iff End $V \cong \mathcal{M}_2$.

Therefore it is enough to show that, for $E \in p^{-1}(K - K_o)$, End E is isomorphic to the algebra C_q^+, for a quadratic form q of rank 2 on a 3-dimensional vector space and conversely.

We consider a point E in $p^{-1}(L \oplus L^{-1})$ where $E = V \oplus W$, $V \in Ext(L, L^{-1})$, $W \in Ext(L^{-1}, L)$, $L \in J - \Gamma$. i.e

$$0 \longrightarrow L \longrightarrow V \longrightarrow L^{-1} \longrightarrow 0$$

$$0 \longrightarrow L^{-1} \longrightarrow W \longrightarrow L \longrightarrow 0 \tag{1}$$

It is clear that points of this type are actually in $p^{-1}(K - K_o)$. Using (1), it is easy to see that End($V \oplus W$) has four generators, which in terms of block matrices can be described as

$$x = \begin{pmatrix} 0 & 0 \\ \gamma_2 & 0 \end{pmatrix} \quad w = \begin{pmatrix} 0 & \gamma_1 \\ 0 & 0 \end{pmatrix} \quad u = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad v = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

where I is 2×2 identity matrix, and γ_1 and γ_2 coming from identification of the line bundles in the exact sequence (1). The defining relations can be given as

$$\left.\begin{array}{c} u^2 = u, \quad v^2 = v, \quad uv = 0, \quad u+v = I, \\ w^2 = x^2 = wx = 0, \quad uw = w, \quad wu = 0, \\ ux = 0, \quad xu = x, \quad vw = 0, \quad wv = w, \\ vx = x, \quad xv = 0. \end{array}\right] \tag{2}$$

If q is a quadratic form of rank 2 on a 3-dimensional vector space over an algebraically closed field k then it is easily seen that C_q^+ is a 4-dimensional k-algebra with

$$C_q^+ = k + k\alpha + k\beta + k\gamma \quad \text{such that}$$

$$\alpha^2 = -1, \quad \alpha\beta = -\gamma, \quad \alpha\gamma = \beta,$$

$$\beta\alpha = \gamma, \quad \gamma\alpha = -\beta.$$

Now put $a = \frac{1}{2}(1 + \iota\alpha)$, $b = \frac{1}{2}(1 - \iota\alpha)$, $c = (\iota\beta + \gamma)$, $d = (\iota\beta - \gamma)$, where $\iota = \sqrt{-1} \in k$. Then a, b, c, d are new generators of C_q^+ with the following defining relations

$$\left.\begin{array}{c} a^2 = a, \quad b^2 = b, \quad ab = 0, \quad a + b = 1, \\ c^2 = d^2 = cd = 0, \quad ac = c, \quad ca = 0 \\ ad = 0, \quad da = d, \quad bc = 0, \quad cb = c, \\ bd = d, \quad db = 0, \end{array}\right] \qquad (3)$$

A glance at (2) and (3) proves that End E $\cong C_q^+$.

Conversely, let $E \in p^{-1}(K - K_o)$; then, End W has four generators x, w, u, v with the relations (2) as above. Consider $u \in$ End E, and let V = ker u. Then V is a subbundle of E and we have an exact sequence

$$0 \longrightarrow V \longrightarrow E \longrightarrow W \longrightarrow 0.$$

It is clear then that W is in fact ker v, $v \in$ End E and therefore we get a splitting of the exact sequence, implying $E = V \oplus W$.

Now using Prop 1. of [S-1], V and W cannot be of the type

$L \oplus L$ or $L^{-1} \oplus L^{-1}$. For the same reason, since $E \in PV_4$, we rule out $V = L \oplus L^{-1}$, $W = L^{-1} \oplus L$. Hence we are left with $V \in \mathbb{P}(\mathrm{Ext}(L,L^{-1}))$, $W \in \mathbb{P}(\mathrm{Ext}(L^{-1},L))$ or vice versa.

Note that for $L \in K - K_o$, $\mathrm{Ext}(L,L^{-1}) = H^1(X,L^{-2})$ has dimension $g - 1$ and therefore Y is a $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$-fibration over $K - K_o$. The vector bundle to which this is associated has fibre at any $L \in K - K_o$ to be $\mathrm{Ext}(L,L^{-1}) \oplus \mathrm{Ext}(L^{-1},L)$.

Corollary 1.1 *The map*

$$p : Y \longrightarrow K - K_o$$

is a $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ *fibration associated to a vector bundle on* $K - K_o$.

Corollary 1.2 *The fibration* $Y \xrightarrow{p} K - K_o$ *is locally trivial in the Zariski topology.*

Proof. This follows from Cor 1.1 and Serre (cf[JPS-1]).

Proposition 1.4 *Let* $P - P_2$ *be the restriction of the conic bundle* P *over points of* $N_o - N_2$(i.e Z). *Then the total space of* $P - P_2$ *is smooth.*

Proof. By Prop 1.2., $P - P_2$ is locally the base change of $Q - Q_2$ (the restriction of Q over the points of $\mathcal{A} - \mathcal{A}_2$). Since $\phi : N_o \longrightarrow \mathcal{A}$

is a smooth local morphism, the total space of $P - P_2$ is smooth iff the total space of $Q - Q_2$ is so.
Consider any point $(a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{A}^6$. This defines a quadratic form

$$q = a_1X^2 + a_2XY + a_3Y^2 + a_4XZ + a_5YZ + a_6Z^2.$$

We therefore have a conic bundle C over $\mathbb{A}^6$ by considering the conics defined by the quadratic forms. By Remark 4. it is clear that the conic bundle Q on $\mathscr{A}$ is *essentially* the conic bundle C. Thus we would have proved our claim if we show that the total space of $C \longrightarrow \mathbb{A}^6 - S'$ is smooth, where S is the degeneracy locus of C and $S' \subset S$ its singular locus. We have in fact more:

Lemma 1.1 *Let* $\theta : C \longrightarrow \mathbb{A}^6$ *be the canonical morphism. Then* $\theta^{-1}(\mathbb{A}^6 - (0))$ *is smooth.*

Proof. Let $P \in C$ be any point. Then P can be given by

$$(a_1, a_2, a_3, a_4, a_5, a_6, X, Y, Z)$$

where not all a_i are zero and not all $X, Y, Z = 0$, P lying on the conic defined by $q = a_1X^2 + \ldots\ldots + a_6Z^2$. Taking partial derivatives of q with respect to a_i, $i = 1, \ldots, 6$, we have

$$\partial q/\partial a_i = 0, \quad i = 1, \ldots, 6 \quad \Rightarrow X = Y = Z = 0.$$

§3 Cohomology computations

Let W be a conic bundle of the type I (cf Def 1.3.) on the variety S. This gives rise to a topological Brauer class b_W in $H^3(S,\mathbb{Z})_{tor}$ (i.e the torsion subgroup of $H^3(S,\mathbb{Z})$).

Let W be a conic bundle of type II (cf Def 1.3). Then if W degenerates to a pair of lines over an irreducible divisor $S_1 \subset S$, the restriction W_1 of W over S gives rise in a natural way to a double cover of S_1(cf Lemma on p 29 of [PN2])and $W - W_1$ is a conic bundle of type I over $S - S_1$. We shall denote by 'α' the element in $H^2(S_1,\mathbb{Z})$ coming from this double cover. Consider the part of the Gysin sequence for $S_1 \subset S$ which involves $H^3(S,\mathbb{Z})$, i.e.

$$H^1(S_1,\mathbb{Z}) \longrightarrow H^3(S,\mathbb{Z}) \longrightarrow H^3(S - S_1,\mathbb{Z}) \xrightarrow{g} H^2(S_1,\mathbb{Z}).$$

Then we have here the

Theorem 1.2 *(Nitsure) Let* W *be a conic bundle of type II on* S*. If the total space of* W *is smooth, then the image of* $b_{W - W_1} \in H^3(S - S_1,\mathbb{Z})_{tor}$ *under the Gysin map g, is precisely* $\alpha \in H^2(S_1,\mathbb{Z})$*. In particular if* $\alpha \neq 0$*, then* $b_{W - W_1} \neq 0$.

Proof. For the proof cf [NN-1] and [NN-2].

Proposition 1.5 *Let* W *be a conic bundle of type* I *over* S *where* $H^1(S,\mathbb{Z}) = 0$ *and with* $b_W \neq 0$ *in* $H^3(S,\mathbb{Z})_{tor}$*. Suppose that there exists another topological* $\mathbb{P}^1$*-bundle* $\bar{U} \longrightarrow S$ *with the property* *that* $H^3(U,\mathbb{Z})_{tor} = (0)$. *Then* $b_W = \pm b_U$ *and* $H^3(S,\mathbb{Z})_{tor}$ *is generated by* b_W.

Proof. To prove this proposition, we shall appeal to the following well known (cf [NN-1])

Lemma 1.2 *Let* $U \longrightarrow S$ *be a* $\mathbb{P}^1$*-bundle over a path connected space* S *with* $H^1(S) = 0$. *Then the kernel of the induced homomorphism* $H^3(S,\mathbb{Z}) \longrightarrow H^3(U,\mathbb{Z})$ *is generated by* b_U.

We now apply the lemma to the bundle $U \longrightarrow S$. Since we have $H^3(U,\mathbb{Z})_{tor} = (0)$, we get $H^3(S,\mathbb{Z})_{tor}$ to be generated by b_U, which is a 2-torsion element. Also b_W lies in $H^3(S,\mathbb{Z})_{tor}$, and $b_W \neq 0$ which implies $b_W = \pm b_U$. This proves Prop 1.5.

The next step is to construct explicitly a $\mathbb{P}^1$-bundle on the subspace $Z - Y$ which satisfy the property of Prop 1.5. For this purpose, we elaborate in some detail, what is called the *Hecke correspondence* of [N-2] in terms of parabolic bundles as remarked in [M-S].

Let V be a vector bundle of rank 2 and degree 0. Suppose we are given a parabolic structure at a point $x \in X$, defined by a 1-dimensional subspace

$$F^2V_x \subset F^1V_x = V_x \quad \text{and weights } (\alpha_1, \alpha_2) \text{ such that}$$

(i) parabolic stable = parabolic semi-stable

(*ii*) parabolic stable $\Rightarrow$ underlying bundle is semi-stable, and
(*iii*) underlying bundle is stable $\Rightarrow$ any parabolic structure is stable.
Let T be the torsion $\mathcal{O}_x$-module given by

$$T_x = V_x/F^2V_x, \quad T_y = 0, \; x \neq y$$

Then we have a homomorphism of V onto T (as $\mathcal{O}_X$-modules). If W is the kernel of this map, we have $0 \longrightarrow W \longrightarrow V \longrightarrow T \longrightarrow 0$ and W is locally free of rank 2 and degree -1.

Let H be the moduli space of parabolic stable bundles of rank 2, degree 0 on X and M_{-1} the moduli space of stable bundles of rank 2, degree -1, $f : H \longrightarrow M$, the canonical morphism ; let $H_0 = f^{-1}(M_0)$.

Proposition 1.6 *If* $V \in H$ *then* W *defined above, is in* M_{-1} *and the map* $\psi : H \longrightarrow M_{-1}$, $V \longrightarrow W$ *is a* $\mathbb{P}^1$*-bundle, locally trivial in the Zariski topology In fact it is the dual projective Poincaré bundle on* M_{-1}.

Proof. We first claim that if V is parabolic stable then W is stable. To see this, let $F \subset W$ be a line subbundle. We need to show that deg F < 0. Suppose this is not the case, i.e. deg F $\geq$ 0.

Let G be the line subbundle of V generated by the image of F in V. Then deg F $\leq$ deg G. Since the underlying bundle of V is certainly semi-stable, we have $\deg G \leq 0$. By our assumption $\deg F \geq 0$ and hence we have $\deg F = \deg G = 0$. This implies that the canonical homomorphism $F \longrightarrow G$ is an isomorphism. We also see that by the definition of T

$$G_x \subset F^2 V_x,$$

but V being parabolic stable with weights $0 < \alpha_1 < \alpha_2$, we get

$$\text{par deg } G = \alpha_2 < \tfrac{1}{2}(\alpha_1 + \alpha_2) = \text{par deg } V/\text{rk } V$$

which leads to a contradiction. Hence W is stable. Conversely, we claim that H is isomorphic to the dual projective Poincaré bundle of M_{-1} restricted to M_{-1}. To see this, we start with a $W \in M_{-1}$. Then, given a point in $\mathbb{P}(W_x^*)$, $x \in X$, one can easily obtain a vector bundle V of rank 2 and degree 0 and an injection $W \longrightarrow V$ as $\mathcal{O}_X$-modules. The cokernel then gives a 1-dimensional subspace F^2V_x of V_x and therefore a *quasi parabolic structure*. The stability of W together with an argument as above, makes V parabolic stable. That this map is an *isomorphism* is a consequence of the universal property of the moduli space of parabolic stable bundles.

That the map $H \longrightarrow M_{-1}$ is locally trivial in the Zariski topology, now follows from Serre [JPS-1].

Proposition 1.7 *Consider the canonical morphism* $f : H_0 \longrightarrow M_0$. *Then* f *is a* $\mathbb{P}^1$*-fibration over* M_0^s *and* $f^{-1}(K)$ *has codimension* $g - 1$ *in* H.

Proof. That f is a $\mathbb{P}^1$-fibration over M_0^s is immediate by the property (3) mentioned before Prop 1.6. Let $L \oplus L^{-1} \in K - K_0$. Then the points of H lying over $L \oplus L^{-1}$ are of the following form :

Case 1. V *is a non-trivial extension of* L^{-1} *by* L *(or* L *by* L^{-1}*)*

We claim that a parabolic structure on V which is equivalent to giving a subspace F^2V_p of V_p of dimension 1, is stable iff $L_p \subset F^2V_p$. This is necessary to ensure parabolic stability, for otherwise if $L_p \subset F^2V_p$, then par deg L = deg L + $\alpha_2 = \alpha_2$ and $\alpha_2 <$ par deg V/rk V $- \frac{1}{2}(\alpha_1 + \alpha_2)$, since $\alpha_1 < \alpha_2$.

Case 2. $V = L \oplus L^{-1}$

We claim that a parabolic structure F^2V_p such that $F^2V_p \neq L_p$ or L_p^{-1} is stable. This is easily checked as above. In fact we see by an argument as in Prop 1. of [S-1] all the parabolic structures of Case 2 are isomorphic and give one point of M. Hence the total dimension of the fibre at $L \oplus L^{-1}$ = dim $\mathrm{Ext}(L, L^{-1}) + 1 = g - 1$. Therefore, dim $f^{-1}(K - K_0) = 2g - 1$.

In fact, it is not difficult to see that for $x \in K - K_0$, $f^{-1}(x)$ is the union of two projective spaces of dim $g - 1$ meeting at a point.

Finally, let $V \in M_0$ be such that $grV = L \oplus L$, (L of order 2). Then the following can be easily checked.

(*i*) Vhas a parabolic structure rendering it parabolic stable iff V is a non-trivial extension of L by L.

(*ii*) A parabolic structure given by F^2V_p is stable iff $F^2V_p \neq L_p$ (where L is the unique line subbundle of V).

Once again by an argument as in Prop 1. [S-1] we see that all the parabolic structures on a non-trivial extension V of L by L are isomorphic. Hence the fibre of f over $L \oplus L$ is isomorphic to $\mathbb{P}(H^1(X,\mathcal{O}_X))$ which has dimension $g - 1$, implying $\operatorname{codim}(f^{-1}(K), H) = g - 1$.

Remark 1.6. Thus we have the following diagram

$$\begin{tikzcd} & H \arrow[dl, "\psi"'] \arrow[dr, "f"] & \\ M_{-1} & & M \end{tikzcd}$$

which gives a correspondence between M_{-1} and M.

Proposition 1.8 *The fibration* $Y \xrightarrow{p} K - K_o$ *with fibre* $F = \mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ *satisfies the conditions of the Leray-Hirsch theorem and consequently we have*

$$H^*(Y,\mathbb{R}) \cong H^*(K - K_o,\mathbb{R}) \otimes H^*(F,\mathbb{R}).$$

Proof The following form of the Leray-Hirsch theorem will suit our purposes. (cf R.Bott and L.Tu -Differential forms in Algebraic topology.)

Leray-Hirsch. Let E be a fibre bundle over B and compact fibre F. Suppose that B has a finite *good cover*. If there are global cohomology classes $e_1, \quad , e_r$ on E which, when restricted to the fibre freely generate the cohomology of the fibre, then $H^*(E,\mathbb{R})$ is a free module over $H^*(B,\mathbb{R})$ with basis $e_1, \ldots , e_r$; or more precisely, if the canonical map $j: H^*(E,\mathbb{R}) \longrightarrow H^*(F,\mathbb{R})$, is surjective, then for any subspace W of $H^*(E,\mathbb{R})$ such that $j|W : W \longrightarrow H^*(F,\mathbb{R})$ is an isomorphism, one has

$$H^*(E,\mathbb{R}) = H^*(B,\mathbb{R}) \otimes W$$

Since F in our case is $\mathbb{P}^{g-z} \times \mathbb{P}^{g-z}$, $H^*(F,\mathbb{R})$ is generated by line bundles on F. Therefore it is enough that any line bundle on F can be extended to a line bundle on Y.

By Cor 1.2., $Y \longrightarrow K - K_o$ is locally trivial in the Zariski topology. Let L be a line bundle on F, and $U \subset K - K_o$ be a trivialising Zariski open subset. Then L can obviously be extended to a line bundle on $U \times F$, which we continue to denote by L. Since Y is *smooth*, the bundle L on the open subset $U \times F$ of Y can be extended to a line bundle on Y.

Proposition 1.9 *The element* $\alpha \in H^2(Y,\mathbb{Z})$, *associated to the double on* Y *arising from the conic bundle* P *is non-zero.*

Proof. By Prop 1.8. and Spanier [Sp], $H^1(Y,\mathbb{Z}) = 0$. Hence if we consider the cohomology exact sequence for

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0$$

we get

$$H^1(Y,\mathbb{Z}/(2)) \subset H^2(Y,\mathbb{Z})$$

$\alpha \in H^2(Y,\mathbb{Z})$ is the image of the covering element in $H^1(Y,\mathbb{Z}/(2))$, and is non-zero if the covering is non-split.

We claim that this double on Y is in fact the pull back of the covering

$$J - \Gamma \longrightarrow K - K_o \qquad (*)$$

J being the Jacobian of X (line bundles of deg 0) (for notations cf §1).

Since this covering is non-split, and since $H^1(K-K_o,\mathbb{Z}) = 0$ it follows that the covering element corresponding to (*) is a non-zero element β in $H^2(K - K_o,\mathbb{Z})$. Thus by Cor 1.1., and the Serre sequence for the fibration $Y \longrightarrow K - K_o$ (cf [Sp-2]) we have

$$H^2(K - K_o,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$$

maps β to α <u>which is therefore non-zero.</u>
Thus to complete the proof of Prop 1.9, it is enough to prove the claim.

Fix $t_o \in X$. Then if $E \in N_o$, one can easily see that E_{t_o} can be identified with the right regular representation of $A = \text{End } E$ (see for e.g Prop 5 [S-1]).

Let $E = V \oplus W$ be an element of Y as in Prop 1.3. It is easy to see that the scalar in A do not meet V_{t_o} and W_{t_o} under the above identification. So if we consider the projective space $\mathbb{P}(A')$, A' being A/(scalars), then V_{t_o} and W_{t_o} give a pair of lines in $\mathbb{P}(A')$. As in the proof of Prop 1.3., identifying the algebra A with a C_q^+ corresponding to a quadratic form q in Φ, it is clear that this pair of lines are the ones in the conic bundle over Y.

Then the one dimensional subspaces L_{t_o} and $L_{t_o}^{-1}$ give pair of points $\bar{L}_{t_o}$ and $\bar{L}_{t_o}^{-1}$ in $\mathbb{P}(A')$. Then the correspondence

$$E \longrightarrow (\bar{L}_{t_o}, \bar{L}_{t_o}^{-1}).$$

gives a double covering on Y since we have a definig family of vector bundles $E_y = \{V_y \oplus W_y\}_{y \in Y}$. Obviously, this is the canonical double cover associated to the conic bundle on Y.

Note that $\{L_y \oplus L_y^{-1}\}_{y \in Y}$ gives a family on Y which is clearly the pull back $p^*\{L_u \oplus L_u^{-1}\}_{u \in K - K_o}$, under the map $p: Y \longrightarrow K - K_o$.

The double cover of Y given above is therefore the pull back of the double cover of $K - K_o$ given by $J - \Gamma \longrightarrow K - K_o$.

Proposition 1.10 *(a) Let Z and Y be as in §2. Then there exists a* *topological $\mathbb{P}^1$-bundle D on Z − Y with $H^*(D,\mathbb{Z})$ torsion free. In fact* $D = f^{-1}(M_o^s)$.

(b) The topological Brauer class $b_D \neq 0$.

Proof. (a) By Prop 1.7. $f^{-1}(K)$ has codim $g - 1$ in H_o and $D = H_o - f^{-1}(K)$. Consider $\psi : H_o \longrightarrow M_{-1,x}$, $M_{-1,x}$ being the set of bundles in M_{-1} with det L_x. Since the $\mathbb{P}^1$-fibration ψ is locally trivial in the Zariski topology a line bundle on the fibre $\mathbb{P}^1$ can be extended obviously to $\mathbb{P}^1 \times U$, where U is a Zariski open subset of $M_{-1,x}$. Since H_o is smooth, the closure of L in H_o gives a line bundle on H_o. Now the cohomology on $\mathbb{P}^1$ is generated by line bundles and we can apply the Leray-hirsch theorem to conclude that the cohomology groups of H_o are those of $M_{-1,x} \times \mathbb{P}^1$.

By Atiyah-Bott [A-B], all the cohomology groups of $M_{-1,x}$ are torsion free and therefore all the cohomology groups of H_o are torsion free.

Since $g \geq 3$, the complex codim of $f^{-1}(K)$ in H_o is $g - 1$ which is ≥ 2. This implies $\text{codim}_{\mathbb{R}} f^{-1}(K)$ in $H_o \geq 4 = g - 1 \geq 2$. Consider the homology exact sequence of the pair (H_o, D)

$$H_k(H_o, D, \mathbb{Z}) \longrightarrow H_{k-1}(D,\mathbb{Z}) \longrightarrow H_{k-1}(H_o,\mathbb{Z}) \longrightarrow H_{k-1}(H_o, D, \mathbb{Z})$$

Now H_o is a compact complex manifold and therefore we can apply the Alexander duality theorem to the pair (H_o, D) to get

$$H_k(H_o,D,\mathbb{Z}) \cong H^{n-k}(H_o-D,\mathbb{Z})$$
$$= H^{n-k}(f^{-1}(K),\mathbb{Z})$$

$n = \dim_{\mathbb{R}} H_o$.

Since $\dim_{\mathbb{R}} f^{-1}(K) \leq n - 4$, we therefore get

$$H_2(H_o,D,\mathbb{Z}) = H^{n-2}(f^{-1}(K),\mathbb{Z}) = 0$$

and similarly $H_3(H_o,D,\mathbb{Z}) = 0$.
Thus we have

$$H_2(D,\mathbb{Z}) = H_2(H_o,\mathbb{Z})$$

By the *universal coefficient theorem* one has torsion subgroup of $H_k(T,\mathbb{Z})$ to be that of $H^k(T,\mathbb{Z})$, T any topological space, and therefore we conclude that

$$H^3(D,\mathbb{Z})_{tor} = H^3(H_o,\mathbb{Z})_{tor} = (0).$$

Note that $Z - Y = H_o^s$ and this completes the proof.
The claim (*b*) is due to Ramanan (p.52 [R])

Theorem 1.3. *$H^3(Z,\mathbb{Z})$ is torsion free.*

Proof. Consider the Gysin sequence for $(Z, Z - Y)$,

$$H^1(Y,\mathbb{Z}) \longrightarrow H^3(Z,\mathbb{Z}) \longrightarrow H^3(Z - Y,\mathbb{Z}) \xrightarrow{g} H^2(Y,\mathbb{Z})$$

Now by Cor.1.2., Y is $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$-fibration over $K - K_o$ and by ([Sp] p.159) $H^1(K - K_o,\mathbb{Z}) = 0$ implying by standard arguments $H^1(Y,\mathbb{Z}) = 0$ (note that $H^1(Y,\mathbb{Z})$ is torsion free by the universal coefficient theorem).

Thus we have from the Gysin sequence an injection

$$H^3(Z,\mathbb{Z}) \subset H^3(Z - Y,\mathbb{Z})) \qquad (*)$$

Now note that $H^1(Z - Y,\mathbb{Z}) = 0$. This follows for example from the Gysin sequence. For, note that $H^1(Z - Y,\mathbb{Z}) \cong H^1(Z,\mathbb{Z})$. Also we will be seeing in §4 that the codimension of $N_o - Z$ in N_o is actually 6. But N_o is unirational and is therefore simply connected, being smooth projective (cf Serre [JPS-2]). Hence $H^1(N_o,\mathbb{Z}) = 0$ implying $H^1(Z,\mathbb{Z}) = 0 = H^1(Z - Y,\mathbb{Z})$.

Thus we can apply Prop 1.5., and Prop 1.10., to see that $H^3(Z - Y,\mathbb{Z})_{tor}$ is generated by b_{P-P_1}, the Brauer element coming from the conic bundle $P - P_1$ over $N_o - N_1$ which $Z - Y$. By Prop 1.4. the total space of $P - P_2$ is smooth and the theorem due to Nitsure mentioned in §3.1 is applicable. Thus we have

$$g(b_{P-P_1}) = \alpha \neq 0 \quad (\alpha \neq 0 \text{ by Prop } 1.9.)$$

This together with (*) and the exactness of the Gysin sequence gives $H^3(Z,\mathbb{Z})_{tor} = (0)$.

Lemma 1.3. *Pic Z is generated by* Pic(Z - Y) *and the element* [Y] *coming from the irreducible divisor* $Y \subset Z$.

Proof. This follows from the following general fact:

If X is a smooth variety, $U \subset X$ open with $Y = X - U$ an irreducible divisor, then

$$\mathrm{Pic}\, X \longrightarrow \mathrm{Pic}\, U$$

is a surjection and the kernel of this homomorphism is generated by [Y].

Lemma 1.4. *Let* $N_1 \subset N_0$ *be as in* §3. *Then* Pic N_0 *is generated by* Pic M_0 *and* $[N_1]$ *over* $\mathbb{Q}$. (in fact over $\mathbb{Z}$ (cf Remark in Appendix 2, [B-1])

Proof. Firstly, we remark that N_1 is precisely $\bar{Y}$ in N_0. Actually, we will be showing in §4 that $Y \subset N_1$ is precisely the set of non-singular points of N_1. Let us assume this. Suppose N_1 is not irreducible and let A, B be subvarieties such that $N_1 = A \cup B$. Then $A \cap B \subset N_1 - Y$ and hence $A \cap Y$ and $B \cap Y$ will disconnect Y which is false since Y is connected. Thus N_1 is irreducible. Also since Y is irreducible it follows that $\bar{Y} = N_1$.

An application of Lemma 1.3. and the result of Appendix 2 ([B-1]) yields the result.

Remark 1.7. Thus by the above lemma, any $L \in \mathrm{Pic}\ N_0$ can be expressed as $L = aL_1 + bL_2$, $L_1 = [N_1]$ and $L_2 \in \mathrm{Pic}\ M_0$, $a,b \in \mathbb{Q}$.

In particular, let L be chosen ample. Then if F is the fibre of $Y \longrightarrow K - K_0$, L when restricted to F is $(aL_1 + bL_2)|F$. But, since $L_2 \in \mathrm{Pic}M_0$, which is trivial on F, we have

$$L|F = (aL_1)|F$$

Now F is isomorphic to $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ and L is ample, therefore we have the restriction of L_1 to each $\mathbb{P}^{g-2}$ to be either ample or negatively ample.

Let $e \in H^2(Y,\mathbb{R})$ be the Euler class of the irreducible divisor Y in Z. Then by the *adjunction formula*, we have

$$e = [Y]|_{[Y]}$$

where [Y] is the class of $Y \subset Z$. Now $L_1 = [N_1]$ and $[N_1] = \bar{Y}$, hence it follows from the above reasoning that the *Euler class* e *when restricted to the factors of* F*is ample or negatively ample.*

Proposition 1.11 *Let* E *be the normal bundle of* Y *in* Z *and* E_0 *be* *the complement of the zero section. Consider the Gysin sequence for the 2-plane bundle* (E, E_o)

$$H^k(Y,\mathbb{R}) \longrightarrow H^{k+2}(Y,\mathbb{R}) \longrightarrow H^{k+2}(E_o,\mathbb{R}) \longrightarrow H^{k+1}(Y,\mathbb{R}) \longrightarrow H^{k+3}(Y,\mathbb{R})$$

Then the Gysin homomorphism

$$h: H^k(Y,\mathbb{R}) \longrightarrow H^{k+2}(Y,\mathbb{R})$$

given by wedging *with the Euler class* $e \in H^2(Y,\mathbb{R})$ *is an injection for* $k \le \dim_{\mathbb{R}}\mathbb{P}^{g-2} - 2 = 2g - 6$.

Proof. By Prop 1.8., we have

$$H^k(Y) \cong \sum_{l+m=k} H^l(K - K_o)\otimes H^m(F)$$

or using the subspace W of $H^*(Y)$ as in Prop 1.8., we have, any $v \in H^k(Y)$ $v \neq 0$ for $k \le \dim_{\mathbb{R}}F$, to be expressible as

$$v = \sum_i u_i \otimes w_i, \quad u_i \in H^*(K - K_o), \; w_i \in W,$$

where not all w_i are zero (this is so since $k \le \dim_{\mathbb{R}}F$). Without loss of generality, the u_i's can be chosen linearly independent.

Now consider $u \otimes e$, e the Euler class in $H^2(Y,\mathbb{R})$

$$u \otimes e = \sum_i u_i \otimes (\omega_i \otimes e).$$

Consider the class $\omega_i \otimes e$. This when restricted to the fibre F is non-zero, since by Remark 1.7., the class e restricted to the factors of F is ample or negatively ample and ω_i by the definition lies in W and so $\omega_i \wedge e$ is non-zero on F for $\omega_i \in H^k(F,\mathbb{R})$, $k \le \dim_{\mathbb{R}} \mathbb{P}^{g-2} - 2$. Hence by the linear independence of the u_i's we get

$$u \otimes e = \sum_i u_i \otimes (\omega_i \otimes e) \neq 0$$

$$\Rightarrow \qquad h: H^k(Y,\mathbb{R}) \longrightarrow H^{k+2}(Y,\mathbb{R})$$

is an injection for $k \le \dim_{\mathbb{R}} \mathbb{P}^{g-2} - 2 = 2g - 6$.

Corollary 1.3 *The Gysin map considered in Theorem 3. i.e.*

$$h: H^k(Y,\mathbb{R}) \longrightarrow H^{k+2}(Z,\mathbb{R})$$

is also an injection for $k \le 2g - 6$.

Proof In fact, the Gysin sequences for (E,E_o) and $(Z, Z - Y)$ are related as follows

$$\begin{tikzcd} H^k(Y,\mathbb{R}) \arrow[rr, "h"] \arrow[dr, "h'"'] & & H^{k+2}(Y,\mathbb{R}) \arrow[dl, "\mathrm{Res}"] \\ & H^{k+2}(Z,\mathbb{R}) & \end{tikzcd}$$

and therefore, since h is an injection by Prop 1.11., so is h'.

Corollary 1.4 $H^k(Z,\mathbb{R}) = H^{k-2}(Y,\mathbb{R}) \oplus H^k(Z-Y,\mathbb{R})$, $k \leq 2g - 4$.

Proof. Consider the Gysin sequence for $(Z, Z-Y)$.

$$\longrightarrow H^{k-2}(Y,\mathbb{R}) \longrightarrow H^k(Z,\mathbb{R}) \longrightarrow H^k(Z-Y,\mathbb{R}) \longrightarrow H^{k-1}(Y,\mathbb{R}) \longrightarrow H^{k+1}(Z,\mathbb{R}) \longrightarrow$$

Since h' is an injection for $k \leq 2g - 6$, we get

$$0 \longrightarrow H^{k-2}(Y,\mathbb{R}) \longrightarrow H^k(Z,\mathbb{R}) \longrightarrow H^k(Z-Y,\mathbb{R}) \longrightarrow 0$$

for $k \leq 2g - 4$ and this proves the corollary.

Remark 1.8. In Balaji [B-2], the Betti numbers of M_0^s are computed using the *Hecke correspondence* if the genus g of the curve X is ≥ 4, for $i \leq 2g - 3$. This has also been obtained by Kirwan [K]. This together with Prop 1.8., Cor 1.4., and Spanier [Sp], yields the Betti numbers of Z for $i < 2g - 3$.

For the sake of completeness we shall give the above computation in full.

For any pair (X,Y) in the complex projective space $H^*(X,Y)$ will denote cohomology with coefficients in $\mathbb{R}$, in the usual topology. We shall mainly be dealing with cohomology (or homology) groups of the type

(i) $H^r(X)$, X a projective variety.

(ii) $H^r(X,Y)$, X a projective variety, Y a closed sub-variety.

(iii) $H^r(X,X - Y)$ under same conditions as in (ii).

Hence by Spanier [Sp2] the singular cohomology groups coincide with the $\bar{H}$ groups of 6.1 of [Sp2].

Lemma 1.5 *Let* $D = f^{-1}(K) \subset H$. *Then we have an isomorphism*

$$H_k(H - D) \longrightarrow H_k(H) \quad \textit{for } k < 2g - 3$$

Proof Consider the pair (H,H - D) which falls under the type (iii) above. Writing the homology exact sequence for this pair, we have

$$\longrightarrow H_{k+1}(H,H - D) \longrightarrow H_k(H - D) \longrightarrow H_k(H) \longrightarrow H_k(H,H - D) \longrightarrow$$

Now, H is a smooth projective variety, therefore by the Alexander duality theorem (6.2.16 of [Sp2]) we get

$$H_k(H,H - D) \longrightarrow H^{\lambda-k}(D), \qquad \lambda = \dim_{\mathbb{R}}(H) \qquad (*)$$

If $\lambda - k > \dim_{\mathbb{R}}(D)$, then $k < \operatorname{codim}_{\mathbb{R}}(D)$, i.e by Prop 1.7 implies that $k < 2(g - 1)$. Also if $\lambda - k > \dim_{\mathbb{R}}(D)$, then $H^{\lambda-k}(D) = 0$, hence we get using (*)

$$H_k(H, H - D) = 0 \quad \text{for } k < 2g - 2.$$

Hence the exact sequence of the pair gives

$$H_k(H - D) \cong H_k(H) \quad \text{for } k < 2g - 3.$$

Lemma 1.6 *The Leray-Hirsch theorem for real cohomology groups holds for the* $\mathbb{P}^1$*-fibration* $f : H - D \longrightarrow M_o^s$ *and hence*

$$H^*(H - D) \cong H^*(M_o^s) \otimes H^*(\mathbb{P}^1).$$

Proof The Leray-Hirsch theorem as stated in Prop 1.8 will suit our purposes.

Since f is a projective morphism, we can consider the relatively ample line bundle on H - D, (or in this case, we could just restrict the ample line bundle on the projective variety H to H - D). This when restricted to the fibres will give a power of the hyperplane bundle. Since our cohomology groups have coefficients in $\mathbb{R}$ and since $H^*(\mathbb{P}^1)$ is generated by the hyperplane bundle, it is clear that the conditions for the Leray-Hirsch theorem are satisfied by the map f and the lemma follows.

Remark 1.9 From the above Lemma 1.6 we obtain the following relation between the Betti numbers of M_o^s in terms of those of $H - D$:

$$B_k(H - D) = B_k(M_o^s) + B_{k-2}(M_o^s).$$

Also, Lemma 1.5 gives

$$B_k(H) = B_k(H - D), \quad k < 2g - 3.$$

By Prop 1.6 the Betti numbers of H are the same as those of $\mathbb{P}^1 \times M_{-1}$ and can therefore be obtained using [A-B].

By [A-B], pp.593, the Poincaré polynomial of M_{-1} is given by

$$P_t(M_{-1}) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)\,(1-t^4)}$$

and therefore

$$P_t(H) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)^2}$$

Using this we can compute recursively, as much as $2g - 2$ of the Betti numbers of M_o^s.

Theorem 1.4. $H^3(Z,\mathbb{Z}) = \mathbb{Z}^{2g}$, *when* $g \geq 4$.

Proof. By Theorem 1.3., $H^3(Z,\mathbb{Z})$ is torsion free. By Cor 1.4.,

$$H^3(Z,\mathbb{R}) = H^1(Y,\mathbb{R}) \oplus H^3(Z-Y,\mathbb{R})$$

Since $H^1(Y,\mathbb{R}) = 0$, and since $Z-Y \cong M_o^3$, using Remark 1.9 we conclude that $H^3(Z,\mathbb{Z}) = \mathbb{Z}^{2g}$.

§4. The main theorem.

Consider the stratification of N_0 in terms of the degeneracy locus as in §3, i.e $N_2 \subset N_2 \subset N_1 \subset N_0$.

Proposition 1.12 *The subvariety* N_2 *has codimension* 3 *in* N_0.

Proof. Consider the local morphism

$$\phi : N_0 \longrightarrow \mathscr{A}$$

of §2. We have already seen that ϕ maps N_1 into $\mathscr{A}_1$ and N_2 into $\mathscr{A}_2$. Moreover, ϕ being a smooth local morphism, its fibres are equidimensional. Hence the codimension of N_2 in N_0 equals the codimension of $\mathscr{A}_2$ in $\mathscr{A}$. We have also seen that $\mathscr{A}_1 \subset \mathscr{A}$ is a hyper-surface given by $\Delta = 0$ and $\mathscr{A}_2 \subset \mathscr{A}$ is precisely the singular locus of $\mathscr{A}_1$. So we would like to show that

$$\text{codim of } \mathscr{A}_2 \text{ in } \mathscr{A}_1 = 2.$$

Consider the natural conic bundle C on $\mathbb{A}^6$ as in Lemma 1. Let S be the hypersurface of $\mathbb{A}^6$ given by $\Delta = 0$ and let $S' \subset S$ be its singular locus . Then by Remark 1.5, it is enough to show that

$$\text{codim of } S' \text{ in } S = 2$$

By definition, if

$$q = aX^2 + bY^2 + cZ^2 + fYZ + gXZ + hXY,$$

then Δ is given by

$$\Delta = \det \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

Thus, if $\mathrm{Sym}(\mathcal{M}_3)$ is all (3×3)-symmetric matrices

$$S = \{A \in \mathrm{Sym}(\mathcal{M}_3) \mid \text{rank } A \le 2\}$$

The conditions $\partial\Delta/\partial a = \partial\Delta/\partial b = \partial\Delta/\partial c = \partial\Delta/\partial f = \partial\Delta/\partial g = \partial\Delta/\partial h = 0$, gives

$$bc = f^2, \quad ac = g^2, \quad ab = h^2, \quad af = hg, \quad fh = bg, \quad ch = fg.$$

$$a/h = h/b = g/f \quad \text{and} \quad a/g = h/f = g/c$$

$$S' = \{A \in \mathrm{Sym}(\mathcal{M}_3) \mid \text{rank} \le 1\}$$

From which we obtain the codim of S' in S.

Corollary 1.5 $H_k(N_o,\mathbb{Z}) = H_k(Z,\mathbb{Z})$, $k \le 4$. (*and therefore by the Universal coefficients theorem* $H^k(N_o,\mathbb{Z}) = H^k(Z,\mathbb{Z})$, $k \le 4$.

Proof. Consider the homology sequence of the pair (N_o, Z)

$$H_{k+1}(N_o, Z, \mathbb{Z}) \longrightarrow H_k(Z, \mathbb{Z}) \longrightarrow H_k(N_o, \mathbb{Z}) \longrightarrow H_k(N_o, Z, \mathbb{Z}).$$

Since N_o is a compact complex manifold, the Alexander duality as in Theorem 1.3., gives

$$H_k(N_o, Z, \mathbb{Z}) \cong H^{n-k}(N_o - Z, \mathbb{Z}) = H^{n-k}(N_2, \mathbb{Z}), \qquad n = \dim_{\mathbb{R}} N_o.$$

By Prop 1.12., $\dim_{\mathbb{R}} N_2 = n - 6$ since $\operatorname{codim}_{\mathbb{C}}(N_2, N_o) = 3$. Hence

$$H^{n-k}(N_2, \mathbb{Z}) = 0 \text{ for } k < 6.$$

$$\Rightarrow \quad H_k(N_o, Z, \mathbb{Z}) = 0 \qquad k < 6$$

$$\Rightarrow \quad H_k(N_o, \mathbb{Z}) = H_k(Z, \mathbb{Z}), \qquad k < 4.$$

Theorem 1.5. $H^3(N_o, \mathbb{Z}) = \mathbb{Z}^{2g}$.

Proof. Firstly, $H^3(N_o, \mathbb{Z})$ is torsion free. For, by Cor 1.5., $H_2(N_o, \mathbb{Z}) = H_2(Z, \mathbb{Z})$ and since

$$H^3(N_o, \mathbb{Z})_{tor} = H_2(N_o, \mathbb{Z})_{tor}$$

we have (by the universal coefficient theorem,)

$$H^3(N_o, \mathbb{Z})_{tor} = H^3(Z, \mathbb{Z})_{tor} = (0)$$

by Theorem 1.3.

Now using Theorem 1.4.,and Cor 1.5.,we get

$$H^3(N_o,\mathbb{Z}) = \mathbb{Z}^{2g}.$$

Theorem 1.6. *The Betti number* B_4 *of* N_o *is given by*

$$B_4(N_o) = {}^{2g}C_2 + 4.$$

Proof. We use Remark 1.8 to get the Betti numbers of M_o^s to be
$B_0(M_o^s) = 1$, $B_1(M_o^s) = 0$, $B_2(M_o^s) = 1$, $B_3(M_o^s) = 2g$, $B_4(M_o^s) = 2$ etc
By Cor 1.4.,

$$B_4(Z) = B_2(Y) + B_4(Z-Y) \qquad (*)$$

Now, by Prop 1.8., $B_2(Y) = B_2(K - K_o) + B_2(\mathbb{P}^{g-2}\times \mathbb{P}^{g-2})$. Hence, by Spanier [Sp]

$$B_2(Y) = {}^{2g}C_2 + 2.$$

Also, $B_3(Y) = 0$, since the odd Betti numbers of $K - K_o$ and $\mathbb{P}^{g-2}\times \mathbb{P}^{g-2}$ are zero (cf [Sp] again). Combining this with (*), we get

$$B_4(Z) = {}^{2g}C_2 + 4.$$

Hence by Cor 1.5., we get

$$B_4(N_o) = {}^{2g}C_2 + 4.$$

CHAPTER II

THE INTERMEDIATE JACOBIAN

Nothing is our own that we create

Sri Aurobindo Savitri Book seven

§1 Preliminaries

Let us recall the definition of the Weil map relating the intermediate jacobian $J^2(V)$ to codimension 2 cycles on V where V is a non-singular projective variety.(cf [G],[M-N],[AW])

Let A be an algebraic cycle on $V\times T$ of codimension 2, T some parameter space. Then we have an element $\alpha \in H^4(V\times T,\mathbb{Z})$, the cohomology class defined by A. Assume that $H^3(V,\mathbb{Z})_{tor}= 0$, and consider the (3,1) component of the Künneth decomposition of α given by $\alpha_{3,1}\in H^3(V,\mathbb{Z}) \otimes H^1(T,\mathbb{Z})$, i.e a homomorphism

$$\alpha_{3,1}:H_1(T,\mathbb{Z}) \longrightarrow H^3(V,\mathbb{Z})$$

which defines map(the Weil map)

$$\phi_A:H_1(T,\mathbb{R})/_{H_1(T,\mathbb{Z})} \longrightarrow H^3(V,\mathbb{R})/_{H^3(V,\mathbb{Z})}$$

If V is as above and T a smooth projective curve ,then ϕ_A defines a morphism

$$\phi_A : J(T) \longrightarrow J^2(V)$$

Recall that $H^3(V,\mathbb{R})/_{H^3(V,\mathbb{Z})}$ has a complex structure etc as in the introduction.

Remark 2.1. If we assume that V is a unirational variety then we have noted that $J^2(V)$ is an abelian variety; and ϕ then is an abelian variety morphism.

§2. The universal bundle on $N \times X$.

By a universal bundle E on $X \times N$, we mean a vector bundle on $X \times N$ whose restriction to $X \times \{n\}$ for any $n \in N$ is exactly the vector bundle E_n on X corresponding to the point $n \in N$.

Remark 2.2. Let $V = N$. Since the functor defining N is actually shown in [S-1] to be representable $\exists$ a universal bundle E on $X \times N$. This gives us a canonical algebraic cycle $c_2(E) \in H^4(X \times N,\mathbb{Z})$ and the induced map $\phi_A : J(X) \longrightarrow J^2(N)$

Remark 2.3 Recall the Hecke correspondence of Chapter 1, Remark 1.6

Then there exist universal bundles F on $X \times H$ and G on $X \times M_{-1}$ (cf [M-N],[S-2]). Moreover, it is easy to see from the definition of $\psi : H \longrightarrow M_{-1}$ that the sequence

$$0 \longrightarrow (1 \times \psi)^{*}G \longrightarrow F \longrightarrow \mathcal{L} \longrightarrow 0$$

is exact on $X \times H$, where $\mathcal{L}$ is a coherent sheaf on $X \times H$ defined by the torsion sheaves T on X (i.e $\forall\ h \in H$, $\mathcal{L}_h \cong T$ as $\mathcal{O}_X$-modules, and the torsion sheaves T on X are defined as in the discussion after Lemma 1.2).

Remark 2.4 . Before proceeding to the main proposition, we shall recall a few definitions from [S-1].

(1) Let T be any parameter variety and $V = \{V_t\}_{t \in T}$ a family of rank 4 vector bundles on $X \times T$. Fix a point $P \in X$ and let V_P be the restriction of V to T, T being identified as a subscheme of $X \times T$ by $t \longrightarrow t \times P$. Then to give a *family of parabolic structures* (for the moduli problem associated to the variety N) $(V,\Delta) = \{(V_t,\Delta_t)\}_{t \in T}$ is to give a section

$$\Delta : T \longrightarrow \mathbb{P}(V_P^{*})$$

where V_P^{*} is the dual of V_P (cf Ch I §1 also)

To give a *rigidified family of parabolic structures* $(V,\Delta) = \{(V_t,\Delta_t)\}_{t \in T}$ is to give a nowhere vanishing section

$$\tilde{\Delta} : T \longrightarrow V_P^{*}$$

such that the associated morphism $\Delta : T \longrightarrow \mathbb{P}(V_p^*)$ is a family of parabolic structures.

(2) Define the functor

$$\mathcal{G} : (\text{Schemes}) \longrightarrow (\text{sets})$$

$\mathcal{G}(T) := \{$Isomorphism classes of rigidified families in PV_4 (i.e $(V_t, \Delta_t) \in PV_4 \ \forall \ t \in T$) parametrised by T $\}$

Define the functor

$$\mathcal{F} : (\text{Schemes}) \longrightarrow (\text{sets})$$

$\mathcal{F}(T) := \{(V, \tilde{\Delta}) \in \mathcal{G}(T) \mid \text{End } V_t \text{ is a specialisation of } \mathcal{M}_2 \ \forall \ t \in T\}$

Then one of the main theorems of [S-1] is that $\mathcal{F}$ <u>is representable and the scheme N of Chapter 1. represents it.</u>

Note that if $(V, \tilde{\Delta}) \in \mathcal{F}(T)$, then $\text{Aut}(V_t, \tilde{\Delta}_t) = \text{identity}$. (cf [S-1], p 172)

Proposition 2.1. *Let* E' *be the restriction of* E *to* $X \times M^s$ *and* F' *the restriction of* F *to* $X \times H^s$. *Then there exists a vector bundle* Q *of rank* 2 *on* H^s *such that*

$$(1 \times f)^*(E') \approx F' \otimes q^*(Q)$$

as bundles on $X \times H^s$ *(q:* $X \times H^s \longrightarrow H^s$ *is the projection).*

Proof . Consider the families $(1 \times f)^*(E')$ and F' on $X \times H^s$. From the definitions of $\pi : Z - Y \longrightarrow M^s$ and $f : H^s \longrightarrow M^s$, it is clear that if $t \in H^s$,

$$(1 \times f)^*(E'_t) \cong F'_t \oplus F'_t \qquad (*)$$

as bundles on X.

Also by Remark 2.4 we know that since E' is the restriction of the *universal bundle* E on $X \times N$, there is a natural family of rigidified parabolic structures on E', which we denote by $\tilde{\Delta}$.

Denote the bundle $F' \oplus F'$ by $F' \otimes \mathfrak{I}$, where $\mathfrak{I}$ is the *trivial* rank 2 bundle on $X \times H^s$. Note that since F' is stable, $Aut(F' \otimes \mathfrak{I}) \cong GL(\mathfrak{I}) \cong GL(2)$. Fix a $t \in H^s$. Then by (*) above, the bundle $(F' \otimes \mathfrak{I})_t$ acquires a rigidified parabolic structure from that of $(1 \times f)^*(E'_t)$, which is defined by a point ξ_t in $(F' \otimes \mathfrak{I})^*_{P,t}$. Let $((F' \otimes \mathfrak{I})^*_{P,t})^s \subset (F' \otimes \mathfrak{I})^*_{P,t}$ be the open subset consisting of parabolic structures which make $(F' \otimes \mathfrak{I})_t$ parabolic stable; observe that $GL(\mathfrak{I})$ acts transitively on this open subset. Further these open subsets patch up to define an open subset $((F' \otimes \mathfrak{I})^*_P)^s$ of $(F' \otimes \mathfrak{I})^*_P$. Clearly $\xi_t \in ((F' \otimes \mathfrak{I})^*_{P,t})^s$. Trivialise $(F' \otimes \mathfrak{I})^*_P$ in a neighbourhood U of t in H^s. Then we get a nowhere vanishing section (going to a smaller open if need be)

$$\xi_U : U \longrightarrow (F' \otimes \mathfrak{I})^*_P |_{U \times X}$$

such that $\xi_U(t) \in ((F' \otimes \mathfrak{I})^*_{P,t})^s$ $\forall\, t \in U$. Thus, we have an open covering $\{U_i\}$ of H^s such that $\forall$ i,$\exists$ a rigidified family of parabolic structures on $(F' \otimes \mathfrak{I})|_{U \times X}$, coming from

$$\xi_{U_i} = \xi_i : U_i \longrightarrow (F' \otimes \mathfrak{I})^*_P.$$

Hence by the representability of the functor $\mathcal{F}$ of Remark 2.4 ,we have $\forall$ i, an *isomorphism of rigidified families*

$$(1 \times f)^*(E')|_{U_i \times X} \cong (F' \otimes \mathfrak{I})|_{U_i \times X} \qquad (**)$$

Since the ξ_i are rigidifications, via the isomorphisms (**) we get canonical isomorphisms

$$(F' \otimes \mathfrak{I}, \tilde{\Delta}_i)|_{U_{ij} \times X} \cong (F' \otimes \mathfrak{I}, \tilde{\Delta}_j)|_{U_{ij} \times X}$$

where $U_{ij} = U_i \cap U_j$.

These give us functions

$$s_{ij} : U_i \cap U_j \longrightarrow GL(\mathfrak{I}) \cong GL(2)$$

with $s_{ij}.s_{jk}.s_{ki} = 1$ on $U_{ijk} = U_i \cap U_j \cap U_k$, (because the ξ_i's are rigidifications). These transition functions define a rank 2 vector bundle Q on H^s, and the isomorphisms of (**) patch up to define a (rigidified) isomorphism

$$(1 \times f)^*(E') \cong F' \otimes q^*(Q)$$

on $X \times H^s$.

§3. Computation of $c_{3,1}(E)$.

Before proceeding to the computation, we shall prove some trivial facts on Chern classes.

Lemma 2.1. *Let* A, B *and* C *be three vector bundles on* $X \times T$, T *some parameter variety with* $H^1(T,\mathbb{Z}) = 0$. *Suppose that the sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact. Then

$$c_{3,1}(B) = c_{3,1}(A) + c_{3,1}(C).$$

Proof . Consider $c_2(B) \in H^4(X \times T,\mathbb{Z})$. Then clearly $c_2(B) = c_2(A) + c_1(A).c_1(C) + c_2(C)$. Now $c_1(A)$, $c_1(C)$ are in $H^2(X \times T,\mathbb{Z})$. Let the Künneth decompositions of

$$c_1(A) = \alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2}$$

and

$$c_1(C) = \beta_{2,0} + \beta_{1,1} + \beta_{0,2}$$

where $\alpha_{i,j}$ and $\beta_{i,j} \in H^i(X) \otimes H^j(T)$, $\forall$ i, j. Now since $H^1(T) = 0$ we have

$$\alpha_{1,1} = \beta_{1,1} = 0$$

Hence

$$[c_1(A).c_1(C)]_{3,1} = 0$$

$$\Rightarrow \qquad c_{3,1}(B) = c_{3,1}(A) + c_{3,1}(C)$$

Lemma 2.2. *Let* W *be a bundle on* $X \times T$ *with* $H^1(T) = 0$ *and let* $V = W \otimes L$, *where* L *is a line bundle. Then*

$$c_{3,1}(V) = c_{3,1}(W)$$

Proof. $c_2(V) = c_2(W \otimes L) = c_1(W).c_1(L) + c_2(W) + c_1(L)^2$. Again, as in Lemma 2.1.

$$[c_1(W).c_1(L)]_{3,1} = 0$$

and similarly $[\, c_1(L)^2]_{3,1} = 0$.

Therefore $c_{3,1}(V) = c_{3,1}(W)$.

Lemma 2.3. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of vector bundles on $X \times T$, *with* $H^1(T) = 0$. *Suppose that* $c_2(C) = 0$. *Then*

$$c_{3,1}(B) = c_{3,1}(A)$$

Proof. Trivial from Lemma 2.1

Lemma 2.4. *Consider the exact sequence of* Remark 2.3,

$$0 \longrightarrow (1 \times \psi)^{*}G \longrightarrow F \longrightarrow \mathcal{L} \longrightarrow 0$$

on $X \times H$.

Then

$$c_{3,1}((1 \times \psi)^{*}G) = c_{3,1}(F).$$

Proof. We claim that $c_2(\mathcal{L}) = 0$. For, by definition of the sheaf $\mathcal{L}$, we have an exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_{X\times H} \longrightarrow \mathcal{L} \longrightarrow 0$$

where I is the pull-back of the ideal sheaf at $P \in X$. Thus we have the relation

$$c_t(I).c_t(\mathcal{L}) = 1 \qquad (\dagger)$$

where the c_t denote Chern polynomials. Since I is a line bundle $c_t(I) = 1 + t\, c_1(I)$ and $(\dagger)$ implies that

$$c_t(\mathcal{L}) = 1 - t\, c_1(I) + (t\, c_1(I))^2 - \ldots.$$

But since I is the pull-back of a line bundle on the curve X, $(c_1(I).t)^k = 0$, $k \geq 2$. Hence $c_2(\mathcal{L}) = 0$. Now, since the variety H is unirational and projective, $H^1(H,\mathbb{Z}) = 0$ (cf Chapter 1), therefore by Lemma 2.3.,

$$c_{3,1}((1 \times \psi)^* G) = c_{3,1}(F)$$

This proves Lemma 2.4.

Consider the $\mathbb{P}^1$-bundle $f : H^5 \longrightarrow M^3$. Then it has been shown in Prop 1.5 that we have an exact sequence

$$0 \longrightarrow \{\gamma\,'\} \longrightarrow H^3(M^3,\mathbb{Z}) \longrightarrow H^3(H^5,\mathbb{Z}) \longrightarrow 0$$

where γ' is a 2-torsion element, lying in the topological Brauer group $Br_{top}(M^s)$ (identified with $H^3(M^s,\mathbb{Z})_{tor}$), coming from the $\mathbb{P}^1$-bundle $f : H^s \longrightarrow M^s$. Moreover, we have the following diagram

$$\begin{tikzcd}
 & & 0 & & \\
 & & \{\gamma'\} \arrow[u] & & \\
0 \arrow[r] & \{\gamma'\} \arrow[r] \arrow[ur] & H^3(M^s,\mathbb{Z}) \arrow[r, "f^*"] \arrow[u] & H^3(H^s,\mathbb{Z}) \arrow[r] \arrow[dl, "\delta"] & 0 \\
 & & H^3(Z,\mathbb{Z}) \arrow[u] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

and δ is an isomorphism. The vertical exact sequence is as in Theorem 1.3., above and the fact that γ' maps to γ is precisely the contents of Theorem 1.3.

Consider $Z \xrightarrow{i} N$ and let $(1 \times i)^*(E) = E_1$ be the restriction of E to $X \times Z$. By Cor 1.5, since $H^k(N,\mathbb{Z}) \cong H^k(Z,\mathbb{Z})$, $k \leq 4$, it follows that $H^4(X \times N,\mathbb{Z}) \cong H^4(X \times Z,\mathbb{Z})$; therefore by naturality of Chern classes, it is clear that we need to compute only $c_2(E_1) \in H^4(X \times Z,\mathbb{Z})$. Since there is no ambiguity, we shall drop the subscript and call $(1 \times i)^*(E) = E_1$ as E itself on $X \times Z$.

Lemma 2.5 *The following diagram is commutative*

$$\begin{tikzcd} c_{3,1}(E) : H_1(X,\mathbb{Z}) \arrow[r] \arrow[d, "\parallel"] & H^3(Z,\mathbb{Z}) \arrow[d] \\ c_{3,1}((1 \times f)^{*}E) : H_1(X,\mathbb{Z}) \arrow[r] & H^3(H^3,\mathbb{Z}) \end{tikzcd}$$

Proof . E' is the restriction of E to $X \times M^3$ $(Z - Y \cong M^3)$ and the lemma follows trivially from the naturality of Chern classes.

Lemma 2.6 *The following diagram is commutative*

$$\begin{tikzcd} c_{3,1}(F) : H_1(X,\mathbb{Z}) \arrow[r] & H^3(H,\mathbb{Z}) \\ c_{3,1}(G) : H_1(X,\mathbb{Z}) \arrow[r] \arrow[u, "\parallel"'] & H^3(M_{-1},\mathbb{Z}) \arrow[u] \end{tikzcd}$$

Proof . Since $\psi : H \longrightarrow M_{-1}$ is by Prop 1.7, a $\mathbb{P}^1$-bundle locally trivial in the Zariski topology, an application of the Leray-Hirsch theorem implies that $\psi^{*}: H^3(M_{-1},\mathbb{Z}) \longrightarrow H^3(H,\mathbb{Z})$ is an isomorphism (cf Prop 1.10.)

The naturality of Chern classes therefore implies that $c_{3,1}(G) = c_{3,1}((1 \times \psi)^{*}(G))$. Now using Lemma 2.4, we get $c_{3,1}(G) = c_{3,1}(F)$ and the lemma follows.

Theorem 2.1 *The map*

$$
\begin{array}{ccc}
c_{3,1}(E) : H_1(X,\mathbb{Z}) & \longrightarrow & H^3(N,\mathbb{Z}) \\
\| & & \| \\
\mathbb{Z}^{2g} & & \mathbb{Z}^{2g}
\end{array}
$$

is given by multiplication by '2'.

Proof . Firstly, by Prop 2.1, $(1 \times f)^*(E') \cong F' \otimes q^*(Q)$, where Q is a rank 2 bundle on H^5. By the *splitting principle*, we can assume, for the computation of Chern classes, that $q^*(Q) = L_1 \oplus L_2, L_1, L_2$ two line bundles. Therefore, it is enough to compute $c_{3,1}(F' \otimes (L_1 \oplus L_2))$.

Using Lemma 2.1 and Lemma 2.2,

$$c_{3,1}(F' \otimes (L_1 \oplus L_2)) = 2.c_{3,1}(F')$$

Thus,

$$c_{3,1}((1 \times f)^*(E')) = 2.c_{3,1}(F').$$

By Remark 1.9 $H^3(H,\mathbb{Z}) \cong H^3(H^5,\mathbb{Z})$, and so by the naturality of Chern classes $c_{3,1}(F) = c_{3,1}(F')$. By ([M-N],[R]) ,

$$c_{3,1}(G) : H_1(X,\mathbb{Z}) \longrightarrow H^3(M_{-1},\mathbb{Z})$$

is an isomorphism. Hence by Lemma 2.5 and Lemma 2.6, we get

$c_{3,1}(E)$ to be multiplication by '2'. (Loosely put $c_{3,1}(E) = 2.c_{3,1}(F') = 2.c_{3,1}(F) = 2.c_{3,1}(G)$).

Corollary 2.1 The induced Weil map (cf §2.2)

$$\phi_E : J(X) \longrightarrow J^2(N)$$

is an *isogeny* of degree 2^{2g}; in fact,

$$\text{Ker } \phi_E = \{\text{points of order 2 of } J(X)\}$$

Remark 2.5 From the proof of Theorem 2.1, it follows that the induced Weil map

$$\phi_F : J(X) \longrightarrow J^2(H)$$

is an isomorphism. Here H is nothing but P(2,0)the moduli space of parabolic stable bundles of rank 2 and degree 0. The same proof together with a general Hecke correspondence for rank n bundles, together with [N-3] would show that

$$\phi_F : J(X) \longrightarrow J^2(P(n,0))$$

is an *isomorphism* ∀ n.

§4 The Polarisation on $J^2(N)$.

As we have noted in the Introduction, in the case when V is unirational, a polarisation on V induces canonically one on $J^2(V)$. This a priori depends on the choice of the Kähler class on V.

Remark 2.6 The varieties N and $M(n,d)_L, (n,d) = 1$, are smooth projective *unrational* varieties (cf [S-2]) and so satisfy the conditions on the plurigenera, viz, $h^{0,3} = h^{3,0} = 0$.

Remark 2.7 Consider the case $V = M(n,d)_L$, $(n,d) = 1$ dealt with in [M-N], [N-3] and [R]. In [M-N], it has been remarked that since $\operatorname{Pic} M(n,d)_L = \mathbb{Z}$, there is a canonical polarisation induced on $J^2(M(n,d)_L)$ and that under the Weil map , which in this case is an isomorphism, this is equivalent to the principal polarisation on $J(X)$. (Since in these cases , numerical equivalence is equivalent to algebraic equivalence, we shall denote it using '$\equiv$ '). *(A proof due to S. Ramanan of this fact is given below)*

For convenience, denote $M(n,d)_L$ by V. Let $G \longrightarrow X \times V$ be the universal vector bundle (cf Remark 2.3) and $\phi_G : J(X) \longrightarrow J^2(V)$, be the induced Weil map . Then we know that ϕ_G is an isomorphism of abelian varieties (cf [M-N], [N-3], [R]).

Let $\{X_t\}_{t\in T}$ be a family of curves parametrised by T and let $t_o \in T$ be any fixed point. Let $\{V_t\}_{t\in T}$ be the corresponding moduli spaces of vector bundles (of type $M(n,d)_L, (n,d) \neq 1$). Then since

Pic $V_t = \mathbb{Z}$, $\forall\, t \in T$, it is clear that given a polarisation L_{t_o} of V_{t_o}, it can be lifted to a family of polarisations $\{L_t\}_{t\in T}$ parametrised by T. (Since we have a projective morphism $\mathcal{V} \longrightarrow$ T with fibres V_t etc)

Now consider M_g the moduli space of curves of genus g. Then it is <u>known</u> *that for a generic curve* $X \in M_g$. *the Neron-Severi group* $NS(J(X)) \cong \mathbb{Z}$, and we have a non-empty *open* subset $\{X \in M_g \mid NS(J(X)) \cong \mathbb{Z}\}$(cf [SM])

We are interested in proving that $\forall\, X \in M_g$, the isomorphism

$$\phi_a : J(X) \longrightarrow J^2(V)$$

is polarisation preserving . So fix an X in M_g and let $\{X_t\}_{t\in T}$ be a family of curves parametrised by T *such that* $X_{t_o} = X$ *and there exists a* $t_1 \in T$ *such that* $NS(J(X_{t_1})) = \mathbb{Z}$.

By the discussion above, we have a family of polarisations $\{L_t\}_{t\in T}$ on $\{V_t\}_{t\in T}$, parametrised by T which induce polarisation Θ_t' on $J^2(V_t)$. Thus by Griffiths [G] we have a family of polarised abelian varieties $\{J^2(V_t),\Theta_t'\}_{t\in T}$ parametrised by T. Therefore, we have a family of Jacobians $\{J(X_t)\}_{t\in T}$ with two families of polarisations $\{\Theta_t\}$ and $\{\Theta_t'\}$, Θ_t being the canonical principal polarisation and Θ_t', the one induced from $J^2(V_t)$ (via the Weil map).

Now, since $t_1 \in T$ is such that $NS(J(X_{t_1})) = \mathbb{Z}$, it implies that

$$\Theta_{t_1} \equiv \Theta_{t_1}' .$$

Therefore, since we have a parametrised family of polarisations, and since the Neron-Severi group is *discrete*, we have

$$\Theta_t \equiv \Theta_t' \quad \forall\ t \in T$$

In particular we have

$$\Theta_{t_o} \equiv \Theta_{t_o}'$$

Therefore since $X_{t_o} = X$, we have

$$\phi_\alpha : J(X) \longrightarrow J^2(V)$$

to be polarisation preserving.

Remark 2.8 Note that the above argument cannot be directly applied to the case of the moduli space N, since Pic N = $\mathbb{Z} \oplus \mathbb{Z}$ (cf Chapter I, Lemmas 1.3 and 1.4)

Consider a family of curves $\{X_t\}_{t\in T}$ as above and let $\{N_t\}_{t\in T}$

be the corresponding family of moduli spaces. Thus we have a projective morphism $\mathcal{N} \longrightarrow T$.

Lemma 2.7 *Let* $h \in T$ *and* L_h *any polarisation on* N_h. *Then, there exists an ample bundle* $\mathcal{L}$ *on* $\mathcal{N}$ *such that* $\mathcal{L}_h = L_h$ *(in other words* L_h *lifts to a family of polarisations* $\{L_t\}_{t\in T}$ *on* $\{N_t\}_{t\in T}$ *)*

Proof. By Lemma 1.4, Pic N is generated by Pic M^s and the class of the irreducible divisor $N_1 \subset N$. Let L_1 generate Pic M^s (Pic $M^s = \mathbb{Z}$) and $L_2 = [N_1]$, the class of N_1 in Pic N.

Then we *claim that there exist line bundles* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *on* $\mathcal{N}$ *such that , if* $t \in T$ *be any point* $(\mathcal{L}_1)_t \cong (L_1)_t$ *and* $(\mathcal{L}_2)_t \cong (L_2)_t$.

Firstly note that Pic $M^s \cong$ Pic M (cf [D-N]) (in fact, this isomorphism over $\mathbb{Q}$ is all that we need and that follows from [B-1]). Now the moduli space construction generalizes to a family of curves. Therefore, L_1 lifts to a family of line bundles $\{(L_1)_t\}_{t\in T}$.

Also the construction of the desingularisation model N generalizes to a family of curves. From these considerations and the fact that the line bundle L_2 coming from the divisor N_1 of the degeneracy locus (cf Remark 1.4) is *canonical* , we deduce that there exists $\mathcal{L}_2$ on $\mathcal{N}$ such that $(\mathcal{L}_2)_t \cong (L_2)_t$ $\forall$ $t \in T$, thereby proving the claim.

Since by Lemma 1.4 Pic N = Pic $M^s + \mathbb{Z}.[N_1]$, where $[N_1]$ is the class of the divisor N_1, the Lemma follows from the above claim.

INSTITUTE OF MATHEMATICAL SCIENCES
MDS-600 113

Theorem 2.2 *Let* Θ_1' *and* Θ_2' *be two polarisations on* $J^2(N)$ *induced from polarisations* L_1 *and* L_2 *on* N. *Then*

$$\Theta_1' \equiv \Theta_2'$$

so that we have a canonical polarisation Θ' *on* $J^2(N)$. *Further* Θ' *is equivalent to the canonical theta divisor* Θ *on* $J(X)$ *via the Weil map* ϕ.

Proof, Let $\{X_t\}_{t\in T}$ be a family of curves as in Remark 2.7, viz $X_{t_o} = X$ and there exists $t_1 \in T$ such that $NS(J(X_t)) = \mathbb{Z}$. Let $\{N_t\}_{t\in T}$ be the corresponding family of desingularisations. Then by Lemma 2.7 ,*any* polarisation L_t on N_t can be lifted to a family of polarisations $\{L_t\}_{t\in T}$. Let $\{\Theta_t'\}$ be a family of polarisations on $\{J^2(N_t)\}$ induced from $\{L_t\}$. Then we have by Griffiths [G] a family $\{J^2(N_t), \Theta_t'\}$ of polarised abelian varieties varying analytically in 't'.

Consider the family of jacobians $\{J(X_t)\}$ and the two families of polarisations $\{\Theta_t\}$ and $\{\phi^*\Theta_t'\}$. Proceeding as in Remark 2.7, we conclude that

$$\Theta_t \equiv \phi^*\Theta_t' \quad \forall\ t \in T$$

In particular we have

$$\Theta \equiv \phi^{*}\Theta \text{ ,}$$

(where $\Theta = \Theta_{t_0}$).

By the canonical nature of the isogeny ϕ, we see that the polarisation induced on $J^2(N)$ is independent of the choice of the polarisation on N and this polarisation is equivalent to the *theta divisor* on $J(X)$ via ϕ.

Corollary 2.2. *Let* X_1 *and* X_2 *be two curves such that the varieties* $N(X_1)$ *and* $N(X_2)$ *are isomorphic. Then*

$$X_1 \cong X_2$$

Proof. An isomorphism $N(X_1) \longrightarrow N(X_2)$ maps a polarisation on $N(X_1)$ to one on $N(X_2)$. Then we get an isomorphism of $J^2(N(X_1))$ onto $J^2(N(X_2))$ which by Thm 2.2 is polarisation preserving for the canonical polarisations. Thus using the *canonically defined isogenies*

$$\phi_{E_i} : J(X_i) \longrightarrow J^2(N(X_i)), \quad i = 1,2$$

we have

$$(J(X_1),\Theta_1) \cong (J(X_2),\Theta_2)$$

implying by Torelli's theorem, that

$$X_1 \cong X_2$$

Remark 2.9. Using Remark 2.5 and arguments similar to the one above, we have similar statements for P(n,0), the moduli space of parabolic stable bundle of rank n and degree 0

References:

[A-M] M.Artin and D.Mumford,*Some elementary examples of unirational varieties which are not rational*,Proc.Lond.Math.Soc.25 (1972) 75-95.

[A-B] M.F.Atiyah and R.Bott,*The Yang-Mills equation on a Riemann surface*,Philos.Trans.R.Soc.London A 308 (1982) 523-621.

[B-1] V.Balaji,*Cohomology of certain moduli spaces of vector bundles*,Proc.Ind.Acad.Sci.(Math.Sci) Vol 98 (1988) 1-24

[B-2] V.Balaji and C.S.Seshadri,*Cohomology of a moduli space of vector bundles*,(to appear in a volume dedicated to Prof A.Grothendieck on his sixtieth birthday)

[B-3] V.Balaji,*Intermediate jacobian of some moduli spaces of vector bundles on curves*, Amer.J.Math.,(to appear).

[D-N] J.M.Drezet and M.S.Narasimhan,*Groupe de Picard des variétiés de modules de fibrés semistables sur les courbes algébriques*, Invent.Math.Vol 97 (1989) 53-95.

[G] Ph.Griffiths,*Periods of integrals on algebraic manifolds-1*,Amer.J.Math.90 (1968) 568-626.

[AG] A.Grothendieck,*Le groupe de Brauer I,II,III* ,Dix exposes sur la cohomologie des schemas. (Amsterdam, North Holland , 1968)

[H-N] G.Harder and M.S.Narasimhan,*On the cohomology groups of moduli spaces vector bundles over curves*,Math.Annalen. 212 (1975) 215-248.

[K] F.Kirwan,*On the homology of the compactifications of the moduli spaces of vector bundles over a Riemann surface*,Proc.Lond.Math.Soc.53 (1986) 237-267.

[M-S] V.B.Mehta and C.S.Seshadri,*Moduli space of parabolic vector bundles on curves*,Math.Annalen.248 (1980) 205-239.

[M-N] D.Mumford and P.E.Newstead,*Periods of a moduli space of vector bundles on curves*,Amer.J.Math.90 (1968) 1201-1208.

[N-1] M.S.Narasimhan and S.Ramanan,*Moduli of vector bundles on a compact Riemann surface*,Ann.Math.89 (1969) 14-51.

[N-2] M.S.Narasimhan and S.Ramanan,*Geometry of Hecke cycles-1*, C.P.Ramanujam - A Tribute (Bombay T.I.F.R)(1978).

[N-3] M.S.Narasimhan and S.Ramanan,*Deformations of moduli space of vector bundles over an algebraic curve*,Ann. Math. 101 (1975) 391-417.

[NN-1] N.Nitsure, *Cohomology of desingularisation of moduli space of vector bundles*,Comp.Math (to appear)

[NN-2] N.Nitsure,*Cohomology of the moduli of parabolic vector* *bundles*, Proc.Ind.Acad.Sci.(Math.Sci) 95 (1986) 61-77.

[PN-1] P.Newstead, *Topological properties of some spaces of stable bundles*, Topology 6 (1967) 540-567.

[PN-2] P.Newstead, *Comparison theorem for conic bundles*, Math.Proc.Cambridge.Philos.Soc.(1981) 9-21.

[R] S.Ramanan, *The moduli spaces of vector bundles over an algebraic curve*, Math.Annalen 200 (1973) 69-84.

[JPS-1] J.P.Serre, *Espaces fibrés algébriques* (Seminaire Chevalley, 1958)

[JPS-2] J.P.Serre, *On the fundamental group of a unirational variety*, J.Lond.Math.Soc. 34 (1959) 481-484.

[S-1] C.S.Seshadri, *Desingularisations of moduli varieties of vector bundles on curves*, Int.Symp.on Algebraic Geometry ,Ed M.Nagata, (Kyoto)(1977) 155-184.

[S-2] C.S.Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque 96 (1982).

[Sp] E.H.Spanier, *The homology of Kummer manifolds*, Proc. Amer.Math.Soc 7 (1956), 155-160.

[Sp2] E.H.Spanier, *Algebraic Topology*, (Springer Verlag, 1987)

[SM] S.Mori, *The endomorphism ring of abelian varieties, I, II*, Japanese.J.Math.2 (1976) 109-130, 3 (1977), 105-109.

[AW] A.Weil, *On Picard varieties*, Amer.J.Math. Vol 74, (1962), 865-895.