SOME STUDIES IN COHERENT STATES AND SQUEEZED COHERENT STATES

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PREFACE

This thesis is based on the research work done by the author during August 1981 to July 1986 at the Institute of Mathematical Sciences(I.M.Sc.) Madras-600113 under the guidance of Professor T.S.Santhanam.

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CONTENTS

AN OVERVIEW

This thesis deals with certain aspects of coherent states and squeezed coherent states and their applications.

Coherent states for the harmonic oscillator were originally discovered by Schrodinger[1] when he was looking for the states whose wave packets had classical motion. These were rediscovered by Sudarshan[2] and Glauber[3] in the study of the quantum theory of optical coherence. The coherent states provide a good description for the various optical fields especially for a laser light. The coherent states and their generalizations are also extensively employed in other areas of physics. A veritable source of information in the form of a collection of reprints has been published by Klauder and Skagerstam.[4]

The main results contained in this thesis are briefly discussed below:

1) Schrodinger showed that for the harmonic oscillator given by the hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(\frac{p^2}{M} + M\omega^2 x^2\right) \tag{1}$$

(p = momentum and x = position)
the states whose probability density wave packets have classical motion viz.,

$$\langle x \rangle = x_{cl}(t) \equiv A \cos(\omega t + \phi) \tag{2a}$$

$$\langle p \rangle = p_{cl}(t) \equiv M \dot{x} \tag{2b}$$

where

$$A = |\alpha| \left(\frac{2\hbar}{M\omega} \right)^{1/2} \tag{3}$$

are given by

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{4}$$

where the states $|n\rangle$ are the eigenstates of $\mathcal{H}$.

Equ.(4) defines a coherent state (c.s).

Equ.(4) could also be obtained as:

$$|\alpha\rangle \equiv D(\alpha)\,|0\rangle \tag{5}$$

where
$$D(\alpha) \equiv \mathrm{Exp}(\alpha a^{\dagger} - \alpha^{*} a) \tag{6}$$

is known as the displacement operator, and a and $a^{\dagger}$ are the annihilation and creation operators.

Chapter 1 contains a brief introduction to the coherent states of the harmonic oscillator and their use in the formulation of the quantum theory of coherence.

Roy and Virendra Singh(5) and Boiteux and Levelut(6) obtained generalized coherent states (g.c.s) for the harmonic oscillator as given below:

$$|n,\alpha\rangle \equiv D(\alpha)\,|n\rangle \tag{7}$$

A detailed discussion of the g.c.s is given in Chapter 2. It is also shown that $|n,\alpha\rangle$ could be obtained from the c.s $|\alpha\rangle$. To be precise the states $|n,\alpha\rangle$ are proved to be the excited states of the displaced oscillator vacuum $|\alpha\rangle$. Further, it is also shown how a c.s. $|\alpha\rangle$ could be expanded in the basis of $|n,\alpha\rangle$. These results are discussed in Chapter 2.

2) In the language of Lie algebra, we can say that the harmonic oscillator algebra (Heisenbery-Weyl algebra) can be used to define the coherent state $|\alpha\rangle$, given by equ.(4).

Coherent states for the angular momentum i.e., for SU(2) algebra were introduced by Radcliffe[7] and Arecchi[8] et al. Owing to apparent similarities in the treatment of these two coherent states, it was shown, using group contraction procedure,[9] that the harmonic oscillator coherent state could be obtained as a certain limit of angular momentum coherent state. Not much has been known about the meaning of this contraction procedure except that the two different coherent states are related as a consequence of a limit. We show that the group contraction procedure merely means a well known concept of limiting distributions in probability theory. We make use of these concepts to associate a probability distribution with an arbitrary Lie algebra and further the contraction of Lie algebras is shown to be 'contraction of probability distributions'.

Some remarks are made regarding the coherent states of phase operator in quantum mechanics in finite dimensions. These results are discussed in Chapter 3.

3) Chapter 4 gives a detailed review of squeezed coherent states (s.c.s). They are defined as[10]

$$|z,\alpha\rangle \equiv D(\alpha)\, S(z)\, |0\rangle \tag{8}$$

where

$$S(z) \equiv \mathrm{Exp}\left[\frac{z}{2} a^{\dagger}a^{\dagger} - \frac{z^{*}}{2} aa\right] \tag{9}$$

is known as squeezing operator.

Unlike $D(\alpha)$ which just shifts a and $a^{\dagger}$, $S(z)$ mixes a and $a^{\dagger}$ into another equivalent boson system as given below:

$$\left.\begin{aligned} S a S^{\dagger} &= a \cosh r + e^{i\theta} a^{\dagger} \sinh r = b \\ S a^{\dagger} S^{\dagger} &= a e^{-i\theta} \sinh r + a^{\dagger} \cosh r = b^{\dagger} \end{aligned}\right\} \tag{10}$$

and $z = r e^{i\theta}$.

The squeezed states correspond to Gaussian wave packets with widths distorted from that of vacuum state and those states follow classical motion; <u>but</u> the uncertainties oscillate.[11]

The nonclassical nature of squeezed states comes from the nonexistence of Glauber-Sudarshan function $P(\alpha)$ as a well behaved positive definite function.

In Sec.4.2 a new class of squeezed states known as logarithmic states have been introduced and their squeezing properties have been studied.

4) Motivated by an interesting result of Fisher et al[12] who proved that 'squeezing could not be naively generalized', it is

shown that the para-Bose oscillator does not admit even squeezing of order two (See Sec.5.1).

5) Though for the harmonic oscillator one cannot naively generalize squeezed states, one can obtain generalized squeezed coherent states in a more useful way as

$$|n, z, \alpha\rangle \equiv D(\alpha)\, S(z)\, |n\rangle . \tag{11}$$

The possibility of squeezing the state $|n\rangle$ and the states $|n, z, \alpha\rangle$ are discussed in Sec.5.2.

6) In Chapter 6, the formalism of s.c.s is extended to the hydrogen atom and it is shown that the dynamical symmetry group $O(4)$ of the hydrogen atom already possesses such squeezed states.

7) The other nonclassical effect namely 'antibunching' is discussed in Chapter 7. The bunching and antibunching properties of various coherent states and squeezed coherent states are also studied in this chapter.

8) Many counting distributions arising in quantum optics possess the property known as scaling property. In the Addendum it is shown that the pure squeezed states have the scaling form.

The study of the squeezed states is useful for various reasons: i) for their nonclassical nature, ii) their use in nonlinear optical processes and optical communications ; iii) the detection of gravitational waves. Recently the squeezed states have been observed in four-wave mixing.[13]

The results in this thesis not only generalize the concept of squeezed coherent states also discuss about their bunching and antibunching properties. The results also throw some more light on the squeezing operator. Extension of s.c.s to other quantum mechanical systems like hydrogen atom are also studied.

The results in this thesis are based on the following papers and preprints:

1) 'Impossibility of squeezed coherent states for a para-Bose oscillator' - T.S. Santhanam and M. Venkata Satyanarayana, Phys. Rev. 30 D, 2251 (1984).
2) 'Generalized coherent states and generalized squeezed coherent states' - M. Venkata Satyanarayana, Phys. Rev. 32 D, 400 (1985).
3) 'Bunching and Antibunching properties of various coherent states of the radiation field,' - M.H. Mahran and M. Venkata Satyanarayana, Phys. Rev. A, 1986 (In Press).
4) 'Squeezed coherent states of Hydrogen atom' - M. Venkata Satyanarayana, J. Phys. A, 1986 (In Press).
5) 'A note on contraction of Lie algebras' - M. Venkata Satyanarayana, J. Phys. A, 1986 (In Press).
6) 'Logarithmic States of the radiation field' - R. Simon and M. Venkata Satyanarayana, Preprint 1986 (To be submitted for publication).
7) ' A remark on Angular Momentum Coherent States' - M. Venkata Satyanarayana, Preprint 1984 (Unpublished)

8) 'Scaling property of squeezed states'

- B.A. Bambah and M. Venkata Satyanarayana,

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PART I

COHERENT STATES

CHAPTER 1

INTRODUCTION

The mathematical description of the radiation field from the point of view of quantum mechanics with the objective of interpreting the results in the sense of classical mechanics has resulted in developing a powerful machinery called the 'coherent states' of the harmonic oscillator. The fact that the light is essentially quantum mechanical in nature was brought to light by Planck.(1)

It is very well known now, that one of the consequences of the second quantization(2) of the radiation field is to think of the radiation field itself as an assemblage of harmonic oscillators(3) and therefore the Fock space, constituted by $|n\rangle$ $(n = 0, 1, 2, \ldots)$ provides a natural setting to describe every mode of the field. Depending on the situation, alternative states like 'coherent states' have been found to be more effectively useful to represent the field, and these states have now occupied a high status as the 'fundamental tools of the art' in the study of quantum optics. In this chapter the coherent states of the harmonic oscillator are introduced in order to facilitate necessary background for the rest of the thesis enabling us to get to other 'alternative states' like 'squeezed coherent states'.

1.1 Definition

We briefly review the different ways of defining the harmonic oscillator coherent states:

A) Minimum Uncertainty Coherent States

The hamiltonian of the harmonic oscillator is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \tag{1.1}$$

(m : mass and ω : circular frequency)

and the solution to the classical problem is given by the equations

$$\left.\begin{aligned} x_{cl}(t) &= \left(\frac{2E}{m\omega^2}\right)^{1/2} \sin(\omega t + \phi) \\ p_{cl}(t) &= (2mE)^{1/2} \cos(\omega t + \phi) \end{aligned}\right\} \tag{1.2}$$

The coherent states of the quantum harmonic oscillator can be defined as the states which result in the minimum uncertainty[4,5] of the measurements in the canonical variables

$$(\Delta x)^2 (\Delta p)^2 = \hbar^2/4 \tag{1.3}$$

with the restriction[6]

$$(\Delta x)^2/(\Delta p)^2 = 1/(m\omega)^2 \tag{1.4}$$

where the self-adjoint operators x and p satisfy the Heisenberg relation

$$[x, p] = i\hbar \tag{1.5}$$

and

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2.$$

From the measurement point of view these states are the closest analogs of the classical states which restrict the Heisenberg uncertainty product

$$(\Delta x)^2 (\Delta p)^2 \geq \hbar^2/4$$

to the minimum. An example of such a state is the ground state of the harmonic oscillator.

In deriving equ.(1.3) from equ.(1.5) the Schwarz inequality is made use of which gives equality for the states which satisfy

$$(\Delta x)\psi = -i\lambda(\Delta p)\psi, \ (\lambda \text{ real}) \tag{1.6}$$

or

$$(x + i\lambda p)\psi = \alpha\psi \tag{1.7}$$

with

$$\alpha = \langle x + i\lambda p \rangle . \tag{1.8}$$

For

$$\alpha = \frac{1}{2}\left[\frac{\langle x \rangle}{\Delta x} + i\frac{\langle p \rangle}{\Delta p}\right],$$

the normalised state which satisfies equ.(1.3) is

$$\psi(x) = \left[2\pi(\Delta x)^2\right]^{-1/4} \operatorname{Exp}\left\{-\left[\frac{x - \langle x \rangle}{2(\Delta x)}\right]^2 + \frac{i}{\hbar}\langle p \rangle x\right\}, \tag{1.9}$$

which is Gaussian.

Now consider the state $|\alpha\rangle$, α (complex)

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{1.10}$$

With the use of the generating function of the Hermite polynomials, it could be shown that $\psi(x)$ could be obtained from $|\alpha\rangle$ (up to a phase). Therefore the states $|\alpha\rangle$ are coherent states.

It is easy to check

$$\left.\begin{array}{l} \langle\alpha|x(t)|\alpha\rangle = x_{cl}(t) \equiv A\cos(\omega t+\phi) \\ \text{and} \\ \langle\alpha|p(t)|\alpha\rangle = p_{cl}(t) \equiv m\dot{x} \end{array}\right\} \tag{1.11}$$

where

$$A = |\alpha|\left(2\hbar/m\omega\right)^{1/2}$$

The above equations are the same as the equs.(1.2)

Further,

$$\left(\Delta x(t)\right)^2 = \hbar/2m\omega \; ; \; \left(\Delta p(t)\right)^2 = m\omega\hbar/2 \tag{1.12}$$

So, we observe that for the coherent state $|\alpha\rangle$ whose energy is E, $\langle\alpha|x(t)|\alpha\rangle$ will have the same motion as $x(t)$ for a classical particle of energy $E-E_0$ where $E_0=\frac{\hbar\omega}{2}$. Therefore, the shape and the minimum-uncertainty property of the wave packet are preserved in time.[7] So the oscillator in a coherent state will have classical energy.

B) <u>Annihilation Operator Eigen States</u>

Quantum mechanics of the harmonic oscillator could be formulated by non-hermitian operators a and $a^\dagger$ (defined below) whose spectra are complex unlike the hermitian operators x and p which have real spectra.[9]

$$\left.\begin{aligned} a &= \left(\frac{1}{2m\hbar\omega}\right)^{1/2}(m\omega x + ip) \\ a^{\dagger} &= \left(\frac{1}{2m\hbar\omega}\right)^{1/2}(m\omega x - ip) \end{aligned}\right\} \tag{1.13}$$

Equ.(1.5) implies $[a, a^{\dagger}] = 1$

Therefore $x, p \longrightarrow a, a^{\dagger}$ is non-canonical.

Considering the eigen states of a,

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad , \quad \alpha\,(\text{Complex}) \tag{1.14}$$

$|\alpha\rangle$ could be shown to be that given by equ.(1.10)

C) Displacement Operator Coherent States

The state $|\alpha\rangle$ could also be obtained by the action of the unitary displacement operator[10] $D(\alpha)$ on the vacuum state $|0\rangle$.

$$|\alpha\rangle \equiv D(\alpha)|0\rangle \quad , \alpha\,(\text{Complex}) \tag{1.15}$$

and

$$D(\alpha) \equiv \mathrm{Exp}(\alpha a^{\dagger} - \alpha^{*} a) \tag{1.16}$$

Using the Baker-Campbell-Hausdorff formula[11] Equ. (1.10) could be easily derived from equ.(1.15).

The above results could be viewed from a group theoretic approach. The coherent states of the oscillator could be obtained as the displaced ground state of the oscillator, the unitary displacement operator $D(\alpha)$ representing an element of the Heisenberg-Weyl algebra.

$$|\alpha\rangle = D(\alpha)|0\rangle \quad , \quad \alpha \text{(complex)}$$

where $|0\rangle$ satisfies

$$a|0\rangle = 0 \quad , \quad N|0\rangle = 0 \; ; \tag{1.17}$$

and

$$N = a^{\dagger}a \tag{1.18}$$

Also

$$[a, N] = a \quad ; \quad [N, a^{\dagger}] = a^{\dagger} \tag{1.19}$$

In order that equ.(1.14) to hold, we have as a consequence of equs.(1.17) and (1.18)

$$(\operatorname{adj} a) D(\alpha) \equiv [a, D(\alpha)] = \pm \alpha D(\alpha) \tag{1.20}$$

i.e., $D(\alpha)$ furnish a basis for $(\operatorname{adj} a)$. Since $a \sim \frac{\partial}{\partial a^{\dagger}}$ in view of $[a, a^{\dagger}] = 1$, equ. (1.20) gives the solution

$$D(\alpha) = \operatorname{Exp}(\alpha a^{\dagger} - \alpha^{*} a)$$

where the second term has been added to make $D(\alpha)$ unitary.[12]

The operators $a, a^{\dagger}$ and $\mathbb{1}$ form the Weyl group $\mathcal{W}$, whose general element is the unitary operator

$$D(t, \alpha) = e^{it\mathbb{1}} e^{\alpha a^{\dagger} - \alpha^{*} a} = e^{it} D(\alpha) \tag{1.21}$$

and t real.

Now

$$D(t_1, \alpha_1)\, D(t_2, \alpha_2) = e^{i\phi} D(\alpha_1 + \alpha_2) \tag{1.22}$$

where

$$\phi = (t_1 + t_2 + \mathcal{I}m\,\alpha_1\alpha_2^*). \tag{1.23}$$

The discussion in sections (B) and (C) could be summarised as follows:

The operators $D(\alpha)\in\mathcal{W}$ generate the coherent states $|\alpha\rangle$ from the ground state $|0\rangle$ which is also a coherent state.

The displacement operators provide a complete and orthonormal basis for the adjoint group of the Weyl group with a scalar product

$$\left(D(\alpha), D(\alpha')\right) = \mathcal{T}r\left(D(\alpha)\,D^{\dagger}(\alpha')\right) = \pi\,\delta(\alpha-\alpha'). \tag{1.24}$$

All the three definitions (A), (B) and (C) are equivalent in the simple harmonic oscillator system.

1.2 Properties

(a) The states $|\alpha\rangle$ are not orthogonal.

$$\langle\alpha|\beta\rangle = \mathrm{Exp}\left[\alpha^*\beta - \frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)\right] \tag{1.25}$$

is known as Bargmann's delta function $\mathcal{B}(\alpha,\beta)$. (The reason for calling it a delta function would be clear by observing equ.(1.33).

If $(\alpha-\beta)$ is large then the states are 'almost' orthogonal since

$$|\langle\alpha|\beta\rangle|^2 = \mathrm{Exp}\left(-|\alpha-\beta|^2\right) \tag{1.26}$$

(b) The expectation value of a normally ordered product of a and $a^\dagger$ is

$$\langle\alpha|(a^\dagger)^n a^m|\alpha\rangle = (\alpha^*)^n\alpha^m \tag{1.27}$$

The above equation is of great use in the application of coherent states. If $F(a,a^\dagger)$ is a normally ordered function of a and $a^\dagger$, then

$$\langle\alpha|F(a,a^\dagger)|\alpha\rangle = \langle 0|F(a+\alpha, a^\dagger+\alpha^*)|0\rangle. \tag{1.28}$$

(c) The completeness relation is given by

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \mathbb{1} \tag{1.29}$$

where $d^2\alpha \equiv d\alpha_1\, d\alpha_2$ and $\alpha = \alpha_1 + i\alpha_2$

(d) They are overcomplete. A sub set of $|\alpha\rangle$ is already complete. In fact, coherent states on a lattice with

$$\alpha = \sqrt{\pi}(m+in) = \alpha_{mn}\,;\; m,n = 0,\pm 1,\pm 2,\ldots\ldots \tag{1.30}$$

are already complete[13]

(e) An arbitrary state $|\Psi\rangle$ can be expanded as

$$|\Psi\rangle = \int \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha|\Psi\rangle. \tag{1.31}$$

For $|\Psi\rangle = |\beta\rangle$ a coherent state ,

$$|\beta\rangle = \int \frac{d^2\alpha}{\pi}\, \langle\alpha|\beta\rangle\, |\alpha\rangle \tag{1.32}$$

$$= \int \frac{d^2\alpha}{\pi}\, B(\alpha,\beta)\,|\alpha\rangle \tag{1.33}$$

(f) Diagonal Representation

Overcompleteness property enables that an operator $F(a,a^{\dagger})$ could be expanded as

$$F(a,a^{\dagger}) = \int \frac{d^2\alpha}{\pi} P(\alpha,\alpha^{*}) |\alpha\rangle\langle\alpha| . \tag{1.34}$$

The possibility of expressing an arbitrary density operator of a quantum-mechanical system in the form (1.34) known as the 'diagonal coherent state representation' was first discovered by Sudarshan.[14] A representation of this type for a more restricted class of operators was discussed by Glauber,[15] under the name P-representation.

For $\rho(a,a^{\dagger})$ a density operator, its diagonal representation satisfies

$$\int P(\alpha,\alpha^{*}) \frac{d^2\alpha}{\pi} = 1 \tag{1.35}$$

since

$$\mathrm{Tr}[\rho(a,a^{\dagger})] = 1 . \tag{1.36}$$

If the dynamical variables are in normal ordered form, then their expectation values could be written as averaging a classical function over a probability distribution as

$$\mathrm{Tr}[\rho\, a^{\dagger m} a^{n}] = \int P(\alpha,\alpha^{*}) \alpha^{*m} \alpha^{n} \frac{d^2\alpha}{\pi} \tag{1.37}$$

where ρ is the density operator.

For the existence of $P(\alpha,\alpha^{*})$ to represent a density operator, it must possess the following properties:

1. $P(\alpha,\alpha^*)$ should be regular, real and positive definite.

2. Projections $|\alpha\rangle\langle\alpha|$ are linearly independent.

3. $P(\alpha,\alpha^*)$ should be normalised to unity.

If the third condition is realised then the second is never of it. Also, in certain regions of the complex plane, $P(\alpha,\alpha^*)$ could become negative. But the diagonal representation holds under very general conditions provided that it is interpreted in the sense of generalized function theory. That is why $P(\alpha,\alpha^*)$ is also called by the name 'quasi-probability'.

(g) Differential operator representation for a and $a^\dagger$:

$$\langle\alpha|a|\alpha'\rangle = \alpha'\langle\alpha|\alpha'\rangle = \left(\frac{\alpha}{2} + \frac{\partial}{\partial\alpha^*}\right)\langle\alpha|\alpha'\rangle . \tag{1.38}$$

Multiplying by $\langle\alpha'|\Psi\rangle$ and integrating over α' gives

$$\langle\alpha|a|\Psi\rangle = \left(\frac{\alpha}{2} + \frac{\partial}{\partial\alpha^*}\right)\langle\alpha|\Psi\rangle , \tag{1.39}$$

which means

$$a = \left(\frac{\alpha}{2} + \frac{\partial}{\partial\alpha^*}\right) \tag{1.40}$$

and

$$a^\dagger = \left(\frac{\alpha^*}{2} + \frac{\partial}{\partial\alpha}\right) \tag{1.41}$$

1.3 Coherent States in Optics

After Schrodinger's discovery(7) of the coherent states in 1926, the interest in them was revived and the first fruitful use of these states were made in 1963 in the formulation of the quantum theory of coherence by Sudarshan(14) and Glauber.(15, 16) Such an interest was necessitated due to a class of experiments(17) which are considered to be initiators of a new discipline of optics called 'Quantum Optics'.

The observable quantities of the electromagnetic field are the electric field $\underset{\sim}{E}(\underset{\sim}{r},t)$ and the magnetic field $\underset{\sim}{B}(\underset{\sim}{r},t)$ and they satisfy Maxwell's equations.

$\underset{\sim}{E}(\underset{\sim}{r},t)$ could be decomposed as

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t) + \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r},t) \tag{1.42}$$

$\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)$ is the positive frequency part (associated with the photon annihilation operator) and $\overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r},t)$ is the negative frequency part (associated with the photon creation operator) and they are not hermitian. In addition, we define vacuum state $|0\rangle$

$$\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\,|0\rangle = 0 \tag{1.43}$$

Supposing the field makes a transition from the initial state $|i\rangle$ to the final state $|f\rangle$ in which one photon has been absorbed, the relevant matrix element is

$$\langle f|\,\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\,|i\rangle \tag{1.44}$$

Now, the probability per unit time that a photon is absorbed by an ideal detector is proportional to

$$\begin{aligned}\sum_f \left|\langle f|\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)|i\rangle\right|^2 &= \sum_f \langle i|\,\overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\,|f\rangle\langle f|\,\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\,|i\rangle \\ &= \langle i|\,\overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\,\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\,|i\rangle\end{aligned} \tag{1.45}$$

One can also keep detectors at $\underset{\sim}{r}$ and $\underset{\sim}{r}'$ to detect delayed coincidences (as done by Hanbury Brown and Twiss[18]) and in such cases the matrix element is

$$\langle f|\,\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}',t')\,\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\,|i\rangle \tag{1.46}$$

and the total rate at which the transitions occur is proportional to

$$\langle i | \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r},t) \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r}',t') \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}',t') \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t) | i \rangle . \tag{1.47}$$

If ρ is the density operator of the state of the radiation field, then (1.45) and (1.47) could be written as

$$\mathrm{Tr}\left[\rho \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r},t) \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\right] \tag{1.48}$$

and

$$\mathrm{Tr}\left[\rho \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r},t) \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r}',t') \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}',t') \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\right] \tag{1.49}$$

respectively.

The correlation functions could be defined as

$$\overset{(1)}{G}(\underset{\sim}{r}t;\underset{\sim}{r}'t') = \mathrm{Tr}\left[\rho \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r},t) \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}',t')\right] \tag{1.50}$$

and

$$\overset{(2)}{G}(\underset{\sim}{r}_1t_1;\underset{\sim}{r}_2t_2;\underset{\sim}{r}_3t_3;\underset{\sim}{r}_4t_4) = \mathrm{Tr}\left[\rho \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r}_1,t_1) \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r}_2,t_2) \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}_3,t_3) \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}_4,t_4)\right] \tag{1.51}$$

and more generally

$$\begin{aligned}\overset{(n)}{G}(\underset{\sim}{r}_1t_1;\underset{\sim}{r}_2t_2;\ldots;\underset{\sim}{r}_{2n}t_{2n}) = \mathrm{Tr}\Big[\rho\, &\overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r}_1,t_1) \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r}_2,t_2) \ldots\ldots \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r}_n,t_n) \\ &\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}_{n+1},t_{n+1}) \ldots\ldots \overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}_{2n},t_{2n})\Big].\end{aligned} \tag{1.52}$$

According to the classical theory of coherence the first order correlation function factorizes as [19]

$$\overset{(1)}{G}(\underset{\sim}{r}_1t_1;\underset{\sim}{r}_2t_2) = \overset{*}{E}(\underset{\sim}{r}_1,t_1)\, E(\underset{\sim}{r}_2,t_2) . \tag{1.53}$$

Similarly

$$\overset{(2)}{G}(\underset{\sim}{r}_1t_1;\underset{\sim}{r}_2t_2;\underset{\sim}{r}_3t_3;\underset{\sim}{r}_4t_4) = \overset{*}{E}(\underset{\sim}{r}_1,t_1) \overset{*}{E}(\underset{\sim}{r}_2,t_2) E(\underset{\sim}{r}_3,t_3) E(\underset{\sim}{r}_4,t_4) \tag{1.54}$$

and generally

$$\left.\begin{aligned}\overset{(n)}{G}(\underset{\sim}{r}_1t_1;\underset{\sim}{r}_2t_2;\ldots;\underset{\sim}{r}_{2n}t_{2n}) = \overset{*}{E}(\underset{\sim}{r}_1,t_1) \overset{*}{E}(\underset{\sim}{r}_2,t_2) \ldots\ldots \overset{*}{E}(\underset{\sim}{r}_n,t_n) \\ E(\underset{\sim}{r}_{n+1}t_{n+1}) \ldots\ldots E(\underset{\sim}{r}_{2n}t_{2n})\end{aligned}\right\}. \tag{1.55}$$

Such a factorization of correlation functions is possible only for the states $|\,\rangle$ satisfying

$$\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)\,|\,\rangle = E(\underset{\sim}{r},t)\,|\,\rangle \tag{1.56}$$

and we identify these states as coherent states.

Therefore coherent states are the eigenstates of the $\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t)$ and in these states the quantum correlation functions 'mimic' the classical correlation functions and they could be factorized satisfying the classical description of coherence. This is the reason why coherent states are named so.

The second-order correlation function is of great importance both in semi-classical and quantum theories of coherence.[20] The normalised second-order correlation function (for a single mode) is given by

$$g^{(2)}(\underset{\sim}{r}_1 t_1;\underset{\sim}{r}_2 t_2) = \frac{G^{(2)}(\underset{\sim}{r}_1 t_1;\underset{\sim}{r}_2 t_2)}{\langle \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r}_1,t_1)\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}_1,t_1)\rangle\langle \overset{(-)}{\underset{\sim}{E}}(\underset{\sim}{r}_2,t_2)\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r}_2,t_2)\rangle} \tag{1.57}$$

which could be rewritten as (for $\underset{\sim}{r}_1=\underset{\sim}{r}_2$ and $t_1=t_2$)

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} \tag{1.58}$$

using the Fourier expansion

$$\overset{(+)}{\underset{\sim}{E}}(\underset{\sim}{r},t) = i\sum_k (\hbar\omega_k/2)^{1/2} a_k u_k(\underset{\sim}{r})\, e^{-ik\omega_k t}. \tag{1.59}$$

Since $[a,a^\dagger]=1$,

$$g^{(2)}(0) = \frac{\langle a^\dagger a a^\dagger a\rangle - \langle a^\dagger a\rangle}{\langle a^\dagger a\rangle^2} \tag{1.60}$$

$$= 1 + \frac{(\sigma^2 - \langle n\rangle)}{\langle n\rangle^2} \tag{1.61}$$

where $\sigma^2 = \langle n^2\rangle - \langle n\rangle^2$.

$g^{(2)}(0)$ is the quantity which determines the classical and non-classical properties of various states of the radiation field. These will be discussed in Chapter 7.

So a coherent state $|\alpha\rangle$ of the radiation field (single mode) is the one given in equ.(1.10).

The probability that there are n photons in a coherent state $|\alpha\rangle$ is given by

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{(|\alpha|^2)^n}{n!} \tag{1.62}$$

which is a Poisson distribution.

A typical state of the radiation field of a laser operating well above the threshold is closer to a coherent state and the photon counting probabilities have been experimentally verified to be closer to a Poisson distribution.[21]

Thus the use of coherent states in quantum optics is clear from their role in the formulation of the theory of coherence.

There are certain non-classical aspects in the quantum theory of radiation like 'squeezing' and 'antibunching' and the states possessing such effects are discussed in Part II. To understand these effects the understanding of the diagonal representation is essential.

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1) M. Planck, Verh. dt. phys. Ges 2, 202 (1900). Reprinted in The Old Quantum Theory: D. ter Haar, 1967, Pergamon, Oxford.

2) Though the classical and semi-classical theories of light have proved to be effective and successful, a fully quantized version of light is a must at various instances. For example, the quantized version of the theory of laser is needed since, the semi-classical theory does not account for the spontaneous emission (noise source).

3) The vector potential associated with a cavity mode gives the harmonic oscillators.

4) An interesting lemma concerning the minimum value of $\Delta x . \Delta p$ for Bose-like oscillators is given by Ohnuki and Kamefuchi as

$$\Delta x . \Delta p \geqslant E_{min}$$

where E_{min} is the minimum energy available for the Bose-like oscillator. Ref: Y. Ohnuki and S. Kamefuchi, Z. Phys. C2, 367(1979).

5) Coherent states of the harmonic oscillator are also characterized by the minimum uncertainty sum $(\Delta x)^2 + (\Delta p)^2$.
For $[A, B] = iC$, the minimum uncertainty sum states are necessarily the minimum uncertainty product states and for such states

$$(\Delta A)^2 = (\Delta B)^2 = |\langle C \rangle| / 2$$

Ref: C.L. Mehta and Sunil Kumar, Pramana 10, 75(1978).

6) In the words of Nieto and Simmons this restriction is 'commonly fluffed over'. Ref: M.M. Nieto and L.M. Simmons Jr. 'Analytic coherent states for general potentials' in 'Foundations of Radiation Theory and Quantum Electrodynamics', edited by A.O. Barut (Plenum Publishing Corporation, 1980) pp.203.

7) Historically speaking coherent states were originally introduced by Schrodinger in 1926 when he was looking for the states with probability-density wave packet remaining unchanged in shape as time progresses and have the classical motion as in equ.(1.2). Thus coherent states have a 'long and proud history'[8]
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10) $\mathcal{D}(\alpha)$ is known as displacement operator because

$$\mathcal{D}^{\dagger}(\alpha)\, a\, \mathcal{D}(\alpha) = a+\alpha$$
$$\mathcal{D}^{\dagger}(\alpha)\, a^{\dagger}\, \mathcal{D}(\alpha) = a^{\dagger}+\alpha^{*}$$

In fact for an arbitrary operator function $F(a,a^{\dagger})$

$$\mathcal{D}^{\dagger}(\alpha)\, F(a,a^{\dagger})\,\mathcal{D}(\alpha) = F(a+\alpha,\, a^{\dagger}+\alpha^{*}).$$

11) If $[A,B]=1$, then $\mathrm{Exp}(\alpha A+\beta B) = \mathrm{Exp}(\alpha A)\,\mathrm{Exp}(\beta B)\,\mathrm{Exp}(-\frac{\alpha\beta}{2})$.

12) The fact that the equation $\{A,\mathcal{D}\}_{+} = \alpha\mathcal{D}$ would also give equ.(1.15) was first noted by Santhanam. Ref: T.S. Santhanam 'Generalized Coherent States' in Symmetries in Science, edited by B. Gruber and R.S. Millman (Plenum Publishing Company, New York 1980).

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17) Broadly speaking such experiments are named as 'photon counting experiments' and these involve the 'direct detection' of photons which could be understood only by the quantized version of the electromagnetic field and clearly lie outside the domain of semi-classical theories. In the photon counting experiments a system of detectors records the random arrival times of a beam of photons.

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Note:- The factorization property is the necessary and sufficient condition for coherence.

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CHAPTER 2

GENERALIZED COHERENT STATES*

In Sec 1.1 various definitions of the coherent states of the harmonic oscillator have been given and for the harmonic oscillator all these definitions give the unique state $|\alpha\rangle$ (equ.(1.10)). But for general potentials criteria (A), (B) and (C) lead to different states.[1] In a series of papers Nieto et al[1] have developed a formalism keeping Schrodinger's criterion in mind and successfully applied it to obtain coherent states for various potentials like Morse potential. The essential idea is to transform a system with an arbitrary potential into an <u>equivalent</u> harmonic oscillator system

$$U^{+}\left[\frac{p^2}{2m}+V(x)\right]U = \frac{p_c^2}{2m}+\frac{m\omega_c^2 x_c^2}{2} \tag{2.1}$$

and then construct the coherent states of this harmonic oscillator by the techniques described in Sec 1.1. It is to be pointed out that the method is analogous to the Bogoliubov transformation which maps (<u>approximately</u>) an interacting system into a free particle system. Also the states obtained can obey some of the equations of motion only <u>approximately</u>.

Roy and Virendra Singh[2] showed, by adopting Schrodinger's criterion, i.e., to define coherent states as those with <u>undistored normalizable wave packets with classical motion</u>, that the harmonic oscillator possesses an infinite string of coherent states

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*Based on M. Venkata Satyanarayana, Phys. Rev. <u>D32</u>, 400 (1985).

'hitherto not thought of'. Originally they were known as 'semi-coherent states'[3] introduced earlier by Boiteux and Levelut.[4] Following Roy and Virendra Singh we shall call these states as 'generalized coherent states'.[5]

2.1 Definition and Details

The generalized coherent states (g.c.s) of the harmonic oscillator are

$$|n,\alpha\rangle \equiv D(\alpha)|n\rangle, \; n=0,1,2,\ldots \tag{2.2}$$

where $|n\rangle$ is the nth state of the harmonic oscillator. Then $|0,\alpha\rangle$ is the usual coherent state $|\alpha\rangle$ introduced in equ.(1.10).

The wave function associated with the state $|n,\alpha\rangle$ could be obtained as

$$\Psi_{n,\alpha}(t) = \Phi_n[x - x_{cl}(t)] \,\mathrm{Exp}\left[\frac{i p_{cl}}{\hbar}\left(x - \frac{x_{cl}}{2}\right) - \frac{i E_n t}{\hbar}\right] \tag{2.3}$$

where Φ_n and E_n are the wave functions and the energies of the states $|n\rangle$ respectively. Therefore the probability density packet $|\Phi_n(x - x_{cl}(t))|^2$ retains its shape and moves classically. In fact this particle-like or 'lump' property is common even to the nonlinear Schrodinger equation[6]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\psi + \lambda\psi g(|\psi|) \tag{2.4}$$

where $g(|\psi|)$ is an arbitrary real function.

Now using the Baker-Campbell-Hausdorff formula, the Fock space representation of $|n,\alpha\rangle$ could be obtained as

$$\left.\begin{aligned}|n,\alpha\rangle = \mathrm{Exp}\left[-\tfrac{1}{2}|\alpha(0)|^2\right]\sum_{m=0}^{\infty}\left(\frac{n!}{m!}\right)^{1/2} L_n^{(m-n)}(|\alpha(0)|^2)\,[\alpha(0)]^{m-n}|m\rangle \\ \times\,\mathrm{Exp}\left[-i\omega t\left(m+\tfrac{1}{2}\right)\right]\end{aligned}\right\} \tag{2.5}$$

where $L_n^{(\alpha)}(x)$ are Laguerre polynomials defined as

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!} \tag{2.6}$$

Also , the uncertainty in the state $|n,\alpha\rangle$ is

$$\Delta x . \Delta p = (n+\tfrac{1}{2})\hbar\omega, \tag{2.7}$$

which means that the minimum uncertainty (i.e., $\hbar\omega/2$) is not necessary for the classical motion of a wave packet. This fact has been also noted by Ohnuki and Kamefuchi.[7]

Also $|n,\alpha\rangle$ satisfy the criterion in equ. (1.4) since

$$m\omega(\Delta x)^2 = \frac{(\Delta p)^2}{m\omega} = \Delta x . \Delta p = (n+1/2)\hbar\omega . \tag{2.8}$$

The expectation value of energy is a sum of a purely classical term and a quantum term:

$$\langle n,\alpha|(a^{\dagger}a + 1/2)|n,\alpha\rangle = E_{cl} + (n+1/2)\hbar\omega \tag{2.9}$$

where $E_{cl} = (m\omega^2 A^2/2)$ is the classical energy for oscillation amplitude A.

Further the $|n,\alpha\rangle$ have all the characteristic properties of the coherent states as described in Sec 1.2. For example the identity operator could be resolved as

$$\int \frac{d^2\alpha}{\pi} |n,\alpha\rangle\langle n,\alpha| = 1 . \tag{2.10}$$

Also, $|n,\alpha\rangle$ provide a diagonal representation for general operators as in equ. (1.34).

We give a brief review of the literature concerning the places were $|n,\alpha\rangle$ could be spotted.

The states $|n,\alpha\rangle$ could be obtained when one considers the hamiltonian[8]

$$\mathcal{H} = \frac{p^2}{2} + \frac{x^2}{2} + x f(t) \tag{2.11}$$

where $f(t)$ is an external force. If the driving term is taken to be linear in a and $a^{\dagger}$ then one obtains $|n,\alpha\rangle$.

Considering $\langle m|n,\alpha\rangle$,

$$|\langle m|n,\alpha\rangle|^2 = e^{-|\alpha|^2}\left[\frac{n!}{m!}\right]|\alpha|^{2(m-n)}\left[L_n^{(m-n)}(|\alpha|^2)\right]^2 . \tag{2.12}$$

It has been shown by Koonin[9] that equ. (2.12) is related to the S-matrix element S_{mn} by

$$|\langle m|n,\alpha\rangle|^2 = |S_{mn}|^2 . \tag{2.13}$$

S_{mn} gives the amplitude for excitation from the initial oscilator state $|n\rangle$ to the final $|m\rangle$.

Also, Hollenhorst[10] has proved equ.(2.12) gives the matrix element for a transition from the state $|n\rangle$ to the state $|m\rangle$ under the influence of a gravitational wave.

Equ. (2.12) is known as Schwinger's formula[11] and is also given by Feynman.[12] There are other places where this formula occurs (Refrs. 13-16).

In fact equ. (2.2) is explicitly given by Ring and Schuck.[17]

2.2 Relationship between q.c.s and c.s

We are interested in obtaining $|n,\alpha\rangle$ from $|\alpha\rangle$. Let

$$|n,\alpha\rangle = A(a,a^{\dagger},n)|\alpha\rangle \tag{2.14}$$

$$= A(a,a^{\dagger},n)D(\alpha)|0\rangle \tag{2.15}$$

where $A(a,a^{\dagger},n)$ is the operator to be determined.

Also

$$|n,\alpha\rangle = D(\alpha)|n\rangle \tag{2.16}$$

$$= D(\alpha)\frac{(a^{\dagger})^n}{\sqrt{n!}}|0\rangle \tag{2.17}$$

Since $D(\alpha)$ translates a and $a^{\dagger}$,

$$A(a,a^{\dagger},n) = \frac{(a^{\dagger}-\alpha^{*})^{n}}{\sqrt{n!}} . \tag{2.18}$$

Therefore

$$|n,\alpha\rangle = \frac{(a^{\dagger}-\alpha^{*})^{n}}{\sqrt{n!}}|\alpha\rangle . \tag{2.19}$$

The meaning of the state $|n,\alpha\rangle$ is very clear as the nth state of the oscillator whose ground state is $|\alpha\rangle$, a coherent state, not $|0\rangle$ as in the case of the usual oscillator. In other words, the g.c.s are the excited states of the displaced oscillator. The above result is clearly depicted in the following diagram

$$\begin{tikzcd} |\alpha\rangle \arrow[d, "A"'] & & |0\rangle \arrow[ll, "D(\alpha)"'] \arrow[d, "\frac{(a^{\dagger})^{n}}{\sqrt{n!}}"] \\ |n,\alpha\rangle & & |n\rangle \arrow[ll, "D(\alpha)"] \end{tikzcd}$$

$|0\rangle$: ground state of the harmonic oscillator

$|\alpha\rangle$: ground state of the displaced harmonic oscillator

The above way of obtaining g.c.s using equ. (2.19) not only establishes the relationship between g.c.s and c.s; it also gives a simple algebraic way to obtain the g.c.s.

Therefore the g.c.s $|n,\alpha\rangle$ is related to the c.s $|\alpha\rangle$ just as the number state $|n\rangle$ is related to the vacuum state $|0\rangle$.

Also, the g.c.s $|n,\alpha\rangle$ is a 'generalised coherent state' in the sense of Perelemov[5] for whom the reference state could be an arbitrary vector in the Fock space.

With the use of equ.(1.41), equ.(2.19) could be given a differential operator representation as

$$|n,\alpha\rangle = \frac{1}{\sqrt{n!}}\sum_{m=0}^{n}\binom{n}{m}(-\alpha^*)^{n-m}\left[\frac{\partial}{\partial\alpha}+\frac{\alpha^*}{2}\right]^m|\alpha\rangle \tag{2.20}$$

The coherent states of the displaced oscillator could be defined as

$$|\beta,\alpha\rangle \equiv \mathrm{Exp}\left[\beta(a^\dagger-\alpha^*)-\beta^*(a-\alpha)\right]|\alpha\rangle \tag{2.21}$$

$$= \mathrm{Exp}(\beta^*\alpha-\beta\alpha^*)\,D(\beta)\,|\alpha\rangle$$

$$= \mathrm{Exp}(\beta^*\alpha-\beta\alpha^*)\,e^{-\frac{|\alpha|^2}{2}}\sum\frac{\alpha^n}{\sqrt{n!}}D(\beta)|n\rangle . \tag{2.22}$$

Using relation (2.5)

$$|\beta,\alpha\rangle = \mathrm{Exp}(\beta^*\alpha-\beta\alpha^*)\,e^{-\frac{|\alpha|^2}{2}}\,e^{-\frac{|\beta|^2}{2}}\sum_{n,m}\frac{\alpha^n}{\sqrt{m!}}L_n^{(m-n)}(|\beta|^2) \times \beta^{(m-n)}|m\rangle . \tag{2.23}$$

Equ.(2.21) could also be written as

$$|\beta,\alpha\rangle = \mathrm{Exp}(\beta^*\alpha-\beta\alpha^*)\,D(\beta)\,D(\alpha)|0\rangle . \tag{2.24}$$

Therefore $|\beta,\alpha\rangle$ is just another element in the set of coherent states, which forms an invariant subset of the Hilbert space. Again equ.(2.21) could also be written as

$$|\beta,\alpha\rangle = e^{-|\beta|^2/2}\sum\frac{\beta^n}{\sqrt{n!}}|\alpha,n\rangle \tag{2.25}$$

From equs. (2.24) and (2.25), we note an interesting fact that any arbitrary coherent state could be expanded in the basis of g.c.s.

These states $|n,\alpha\rangle$ are used to get the 'generalized squeezed coherent states' of the radiation field (See Chapter 5).

The states $|n,\alpha\rangle$ have interesting (also important) non-classical attributes like antibunching and sub-poissonian statistics (See Chapter 7).

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CHAPTER 3

ANGULAR MOMENTUM COHERENT STATES *

There have been many attempts to define coherent states for other Lie groups other than the Heisenberg-Weyl group, the generalizations involving the criteria (B) and (C) in Sec. 1.1

Barut and Girardello[1] have extended the idea (criterion B) to the non-compact SU(1,1). Sharma et al[2] have defined the coherent states of the para-Bose oscillators as the eigenstates of the annihilation operator.

We have already seen that the oscillator coherent states can be defined as the displaced ground state (equ.(1.15)). Radcliffe[3] generalized this concept to the rotation group SU(2) (known as coherent spin states or atomic coherent states[5]) and Perelemov[4] extended the idea for an arbitrary Lie group.

3.1 Radcliffe States

The angular momentum algebra is

$$\left[J_z, J_\pm\right] = \pm J_\pm \quad ; \quad \left[J_-, J_+\right] = -2J_z \tag{3.1}$$

where
$$J_\pm = J_x \pm i J_y \tag{3.2}$$
are the ladder operators and J_x, J_y and J_z are the generators of the angular momentum algebra.

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* Based on

1. M. Venkata Satyanarayana, Preprint, The Institute of Mathematical Sciences 1984.
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Radcliffe made a formal analogy between the operators of the Heisenberg algebra and the generators of SU(2) as

$$\left.\begin{array}{l} |0\rangle \sim |j,j\rangle \\ J_- \sim a^\dagger \end{array}\right\} \qquad (3.3)$$

to define the angular momentum states as

$$|\mu\rangle = D(\mu)\,|j,j\rangle, \quad \mu \text{ (complex)} \qquad (3.4)$$

where

$$D(\mu) = \mathrm{Exp}(\mu J_- - \mu^* J_+) \qquad (3.5)$$

Since

$$\left.\begin{array}{l} (J_-)^p\,|j,j\rangle = \left\{\dfrac{p!\,(2j)!}{(2j-p)!}\right\}^{1/2} |j,p\rangle, \\ \qquad 0 \leq p = j-m \leq 2j \\ \qquad -j \leq m \leq j \end{array}\right\} \qquad (3.6)$$

Equ.(3.4) can be rewritten as

$$|\mu\rangle = \frac{1}{(1+|\mu|^2)^j} \sum_{p=0}^{2j} \binom{2j}{p}^{1/2} \mu^p\,|p\rangle \qquad (3.7)$$

The normalisation factor has been chosen so that

$$\langle\mu|\mu\rangle = \frac{1}{(1+|\mu|^2)^{2j}} \sum_{p=0}^{2j} \binom{2j}{p} \left(|\mu|^2\right)^p = 1$$

The states $|\mu\rangle$ are the eigenstates of the operator

$$D(\mu) J_z \overset{+}{D}(\mu) = J_z + \mu J_- \qquad (3.8)$$

with the eigenvalue j.

The characteristic properties of the coherent states (Sec.1.2) are also possessed by the Radcliffe states.

They are non-orthogonal since

$$\langle\lambda|\mu\rangle = \left\{\frac{1+\lambda^*\mu}{[(1+|\mu|^2)(1+|\lambda|^2)]^{1/2}}\right\}^{2j}. \tag{3.9}$$

The completeness relation is

$$\int |\mu\rangle\langle\mu| \mathcal{M}(|\mu|^2)\, d^2\mu = \mathbb{1} \tag{3.10}$$

where

$$\mathcal{M}(|\mu|^2) = \frac{(2j+1)}{\pi} \frac{1}{(1+|\mu|^2)^2}. \tag{3.11}$$

The states $|\mu\rangle$ are also the minimum uncertainty states in the sense they minimise the uncertainty product

$$(\Delta J_x')^2 (\Delta J_y')^2 \geq \frac{1}{4} \langle J_z' \rangle^2 \tag{3.12}$$

where J_x', J_y' and J_z' are operators rotated by $D(\mu)$.

The states $|\mu\rangle$ are also known as Bloch states since the wave functions corresponding to the states $|\mu\rangle$ are defined over the Bloch sphere S^2.

In view of equ.(3.10) the states $|\mu\rangle$ admit diagonal representation for operators.

There have been many similar attempts to define the coherent states of the rotation group and these are described by Gulshani[6].

3.2 Eigen States of Phase

The problem of defining a self-adjoint operator for phase in quantum mechanics has attracted a lot of attention.[7]

The problem is to find a self-adjoint ϕ canonically conjugate to J_z i.e.,

$$[\phi, J_z] = i\mathbb{1}. \tag{3.13}$$

As pointed out by Carruthers and Nieto,[7] equ. (3.13) is strictly not correct, since taking the matrix elements with states $|j,m\rangle$ (the eigenstates of J_z and J^2) leads to a contradiction

$$(m-m')\langle jm'|\,\phi\,|jm\rangle = i\,\delta_{mm'}. \tag{3.14}$$

This is because of the domain of the operator J_z consists of functions periodic in ϕ.[8]

On the other hand Weyl's commutation relation

$$UV = \epsilon VU \tag{3.15}$$

$$U = \mathrm{Exp}\frac{2\pi i\phi}{2j+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \cdots \end{bmatrix} \tag{3.16}$$

$$V = \mathrm{Exp}\left[i(J_z + j\mathbb{1})\right] \tag{3.17}$$

and

$$\epsilon = \mathrm{Exp}\left(\frac{2\pi i}{2j+1}\right)$$

still holds.[9] In fact ϕ and J_z satisfy the quantum mechanics in Finite Discrete Space

$$[\phi, J_z]_{mm'} = \begin{cases} \dfrac{(\log\epsilon)^2}{2\pi}(m'-m)\left[\dfrac{1}{\epsilon^{(m'-m)}-1}\right] & \text{for } m' \neq m \\ 0 & \text{for } m' = m \end{cases} \tag{3.18}$$

As $j \to \infty$

$$[\phi, J_z]_{mm'} \longrightarrow i\,\delta(m-m') \,.$$

Equ.(3.18) can be reexpressed as

$$e^{-\frac{2\pi i \phi}{2j+1}}\; J_z\; e^{\frac{2\pi i\phi}{2j+1}} = J_z + 1 - L \tag{3.19}$$

$$L = (2j+1) P_j \tag{3.20}$$

where P_j stands for the projection operator on the highest weight vector $|j,j\rangle$. The presence of L takes care of the <u>finite</u> dimensionality of the space.

The angular momentum operators can be polar decomposed as

$$\left.\begin{aligned} J_+ &= J_T\, e^{i\phi} = e^{-i\phi} J_\perp \\ J_- &= e^{-i\phi} J_T = J_\perp e^{i\phi} \end{aligned}\right\} \tag{3.21}$$

J_T and $J_\perp$ are singular hermitian operators, $e^{i\phi}$ is unitary. In the standard basis, $e^{i\phi}$ is the same as the unitary operator U of equ.(3.16) and this will satisfy equ.(3.19). Since $e^{i\phi}$ is a cyclic permutation matrix (a circulant), it is diagonalized by the Sylvester matrix (finite Fourier transform)

$$S = \frac{1}{\sqrt{2j+1}} \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \epsilon & \epsilon^2 & \cdot & \cdot & \cdot & \cdot \\ 1 & \epsilon^2 & \epsilon^4 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \epsilon^{n-1} & \epsilon^{n-2} & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{3.22}$$

Also,

$$SS^{\dagger} = S^{\dagger}S = 1 \quad \text{and} \quad S^{4} = 1 \tag{3.23}$$

Therefore the eigenstates $|\phi\rangle$ of the hermitian phase operator are given by

$$|j,\zeta\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} \epsilon^{\zeta m} |j,m\rangle \tag{3.24}$$

These states $|j,\zeta\rangle$ are eigenstates of ϕ and called by Santhanam as 'generalised coherent states' since they have a sharply defined phase.(10) These form a complete set.

For these states the uncertainty product is

$$(\Delta J_x)^2 (\Delta J_y)^2 \geq 0 \tag{3.25}$$

The minimum on the left hand side is reached for

$$\left.\begin{aligned} \phi &= 0 \bmod \frac{(2j+1)}{4} \\ &= k\,\frac{(2j+1)}{4}, \quad k = 0,1,2,\ldots \end{aligned}\right\} \tag{3.26}$$

These coherent states have also been obtained independently by Goldhirsch(11) and employed in the study of precession of spin in a magnetic field.

Now, we could see that the coherent states as defined by Santhanam are associated with the Principle of Maximum Entropy, by calculating the probability for a system described by a coherent state $|j,\zeta\rangle$ to be in a projected state $|j,m\rangle$ given by

$$P_{\langle jm|j\zeta\rangle} = \frac{1}{2j+1}, \text{ for } m = (-j, \ldots\ldots, j). \tag{3.27}$$

For the Racliffe states $|\mu\rangle$ this probability is

$$P_{\langle j,m|\mu\rangle} = \frac{\binom{2j}{m}\left(|\mu|^2\right)^m}{(1+|\mu|^2)^{2j}}, \tag{3.28}$$

which is a binomial probability.

According to information theory the distribution defined by equ.(3.27) (i.e., equal probability for all states) corresponds to maximum entropy and any other distribution like equ.(3.28) defines a more organised state and carries more information.

The situation is worth comparing with radiation field. Equ.(3.24) possesses a distribution similar to the black body distribution, (Principle of Maximum Entropy gives the Bose-Einstein distribution[12]). Distributions similar to equ.(3.28) like the Poisson distribution (equ.(1.62)) are employed to study more organised fields.

We give below in the Table, the characterisation of the coherent states of the radiation field and the angular momentum from the information theoretic approach.

	Angular Momentum	Radiation Field
Principle of Maximum Entropy	$\vert j\zeta\rangle \longrightarrow$ (uniform probability) Equ.(3.24)	Thermal states $\longrightarrow$ (geometric distribution)
rganised States	$\vert\mu\rangle \longrightarrow$ (binomial probability) Equ.(3.7)	$\vert\alpha\rangle \longrightarrow$ (Poisson distribution) Equ.(1.10)

3.3 Contraction of Coherent States

The apparent similarities in the treatment of coherent states of the harmonic oscillator and the angular momentum led Arecchi et al[5] to show that the angular momentum coherent states go over to the harmonic oscillator coherent states. They made use of an algebraic technique called the 'group contraction procedure' originally introduced by Inonu and Wigner.[13] The concept is that of obtaining one Lie algebra from another Lie algebra (usually non-isomorphic) by means of a limiting procedure. It is a generalisation of the relationship between the Lorentz group and Galilei groups. The limit involved is that the velocity of light tends to infinity. The contraction of Lie algebras has been developed by Saletan[14] and discussed by many authors.[15]

In the present discussion the contraction of angular momentum coherent states to the harmonic oscillator coherent states involves the contraction of SU(2) group to the Heisenberg-Weyl group.

Hioe[16] and Onofri[17] made certain studies concerning the coherent state representation of Lie groups.

Not much has been known about the meaning of this procedure. In this section we illustrate that this limiting procedure could be <u>understood</u> in the language of probability theory.

From equ.(1.10) and equ.(1.62) the Poisson distribution

$$f_n(\alpha) = e^{-|\alpha|^2} \frac{(|\alpha|^2)^n}{n!} \tag{3.29}$$

could be associated with the Heisenberg-Weyl algebra.

Also, from equ.(3.28) the binomial distribution

$$\pi_p(\mu) = |\langle p|\mu\rangle|^2 = \frac{\binom{2j}{p}\,(|\mu|^2)^p}{(1+|\mu|^2)^{2j}} \tag{3.30}$$

could be associated with the SU(2) algebra. Taking $|\mu|^2 = \frac{|\alpha|^2}{2j}$, equ.(3.30) could be rewritten as

$$\pi_p(\mu) = \frac{(2j)(2j-1)\ldots\ldots(2j-p+1)}{\left(1+\frac{|\alpha|^2}{2j}\right)^{2j}} \quad \frac{(|\alpha|^2)^p}{p!\,(2j)^p}\,. \tag{3.31}$$

If in equ.(3.31) we keep α fixed and let j tend to infinity then

$$\pi_p(\mu) \longrightarrow f_p(\alpha) \tag{3.32}$$

where

$$f_p(\alpha) = e^{-|\alpha|^2}\,\frac{(|\alpha|^2)^p}{p!}\,, \tag{3.33}$$

corresponding to the coherent state $|\alpha\rangle$ of the Heisenberg-Weyl algebra. This limit is also known asthe Holstein-Primakoff limit.[18] Thus the <u>contraction of Lie algebras</u> used by Arecchi et al[5] entails a <u>contraction of probability distribution</u>. Here the 'contraction of probabilities' is meant in the sense of one probability distribution going over to another probability distribution, a concept that is well known[19] in probability theory. In the case of SU(1,1) the commutation relations are

$$[J_z, J_\pm] = \pm J_\pm \;;\; [J_-, J_+] = 2J_z\,, \tag{3.34}$$

where $J_\pm$ are the ladder operators. The coherent states of SU(1,1) are defined as[1,4]

$$|n,z,\mu\rangle = \sum_{k=0}^{\infty} \binom{n+k-1}{k-1}^{1/2} z^n \mu^k |k\rangle , \tag{3.35}$$

where z and μ are complex and related by $|z|^2+|\mu|^2=1$.
The associated probability distribution is

$$P_k(\mu) = \binom{n+k-1}{k-1} (|z|^2)^n (|\mu|^2)^k , \tag{3.36}$$

which is negative binomial.

If in equ.(3.36) we let $|\mu|^2$ tend to zero, n tend to infinity and $|\mu|^2 n$ tend to λ then

$$P_k(\mu) \longrightarrow f_k(\lambda) , \tag{3.37}$$

where

$$f_k(\lambda) = e^{-|\lambda|^2} \frac{(|\lambda|^2)^k}{k!} , \tag{3.38}$$

the Poisson probability distribution associated with the coherent state $|\lambda\rangle$ of the harmonic oscillator algebra. This is the same limiting procedure employed by Barut and Girardello.[1] So the contraction of SU(1,1) to Heisenberg-Weyl algebra is same as contraction of the negative binomial distribution associated with SU(1,1) to the Poisson distribution associated with the Heisenberg-Weyl algebra.

So, we have explicitly shown that using two examples that the procedure involved in the contraction of Lie algebras is closely related to the method of obtaining one probability distribution from another involving a limiting procedure.

This relationship between Lie-algebraic contraction and 'Contraction of probabilities' could be extended to the general theory of contraction of Lie algebras. Probability distributions could be associated with arbitrary Lie algebras via defining coherent states.(For this purpose Perelemov's method[4] could be made use of). Such an association makes the study of Lie algebras interesting in the same way as special functions are associated with them.[20]

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PART II

SQUEEZED COHERENT STATES

CHAPTER 4

INTRODUCTION

In the first two chapters the coherent states of the radiation field have been discussed. As already said in Sec 1.3 these coherent states are useful for the 'classical description' of the optical fields. An ideal coherent state corresponds to the state of a one-photon laser operating well above the threshold. This could be inferred from the fact that the photon counting statistics of an ideal coherent state i.e., Poisson distribution (equ. (1.62)) agrees very well with the counting distribution from a laser operating well above the threshold. This is due to the fact that a laser signal is very close to an amplitude-stable sinusoid.(1) Thus the coherent state description of the radiation field is justified.

Recently a more general class of coherent states known as the 'squeezed coherent states' have been introduced and they are more appropriate to describe the so called 'squeezed light'.(2)

'Squeezing' and 'antibunching' are two effects which reveal the quantum properties of the radiation field and cannot be explained by treating the radiation field classically.

Photon antibunching is characterised by a quantum state of the field in which the variance of the number of photons is less than the mean number of photons. (See Chapter 7). Squeezing is characterised by a field state in which the variance of one of two noncommuting observables is less than one half of the absolute value of their commutator.

In this chapter the squeezed states are introduced and the 'non-classical' aspects of squeezing are also discussed.

4.1 Definition

Let us consider two dimensionless operators A and B satisfying the commutation relation

$$[A,B]=iC. \tag{4.1}$$

According to Heisenberg's uncertainty relation

$$(\Delta A)(\Delta B)\geq \frac{1}{2}|\langle C\rangle| . \tag{4.2}$$

A state is called '<u>squeezed</u>' when uncertainty in one observable (say A) is less than that for minimum uncertainty state, i.e.,

$$(\Delta A)^2 < \frac{1}{2}|\langle C\rangle| . \tag{4.3}$$

For a and $a^\dagger$ the annihilation and creation operators of a single-mode electromagnetic field

$$[a,a^\dagger]=1 \tag{4.4}$$

and the hermitian amplitude operators a_1 and a_2 defined as

$$a=a_1+ia_2\ ;\ a^\dagger=a_1-ia_2 \tag{4.5}$$

satisfy the commutation relation

$$[a_1,a_2]=\frac{i}{2} \tag{4.6}$$

and the corresponding uncertainty relation is

$$(\Delta a_1)(\Delta a_2)\geq 1/4 . \tag{4.7}$$

<u>Def.1</u>: A state of the radiation field is <u>squeezed</u> if one of the amplitudes $a_i, i=1,2$ satisfies

$$(\Delta a_i)^2 < 1/4 . \tag{4.8}$$

Def.2: A squeezed state is known as an '<u>ideal squeezed state</u>' or '<u>squeezed coherent state</u>' (s.c.s) if in addition to (4.8)

$$(\Delta a_1)(\Delta a_2) = 1/4 \; . \tag{4.9}$$

For the usual coherent states (equ.(1.14))

$$\begin{aligned}(\Delta a_1)^2 &= \langle\alpha|(a_1 - \langle a_1\rangle)^2|\alpha\rangle \\ &= \langle\alpha|a_1^2|\alpha\rangle - (\langle\alpha|a_1|\alpha\rangle)^2 \\ &= \frac{1}{4}\left[\langle\alpha|(a+a^\dagger)^2|\alpha\rangle - (\langle\alpha|(a+a^\dagger)|\alpha\rangle)^2\right] \\ &= \frac{1}{4}\end{aligned} \tag{4.10}$$

Also

$$(\Delta a_2)^2 = \frac{1}{4} \; . \tag{4.11}$$

Therefore

$$(\Delta a_1) . (\Delta a_2) = 1/4 \; . \tag{4.12}$$

So, a coherent state is not a squeezed state.

For the '<u>two-photon coherent states</u>'[3] defined as

$$|\alpha, z\rangle \equiv D(\alpha) S(z) |0\rangle \tag{4.13}$$

where $D(\alpha)$ is the usual displacement operator and

$$S(z) \equiv \mathrm{Exp}\left[\frac{z}{2} a^\dagger a^\dagger - \frac{z^*}{2} a a\right] \tag{4.14}$$

is known as the 'squeezing operator' and $z = r e^{i\theta}$ is a complex number,

and

$$\left.\begin{aligned}(\Delta a_1)^2 &= \frac{1}{4} e^{-2r} \\ (\Delta a_2)^2 &= \frac{1}{4} e^{2r}\end{aligned}\right\} (\text{for } \theta = 0) \tag{4.15}$$

Therefore

$$\Delta a_1 . \Delta a_2 = 1/4 \; . \tag{4.16}$$

It is clear that for $r \neq 0$, the two-photon state is an ideal squeezed state.

Before we proceed to sections 4.3 and 4.4 for more details regarding two-photon states and squeezing, in section 4.2 we introduce new squeezed states known as the logarithmic states.

4.2 Logarithmic States*

The logarithmic states(4,5) of the radiation field are defined as

$$|q\rangle = C|0\rangle + \beta \sum_{n=1}^{\infty} \left(\frac{q^n}{n}\right)^{1/2} |n\rangle \tag{4.17}$$

where $$\beta = \left[\frac{-(1-|c|^2)}{\log(1-q)} \right]^{1/2}$$

and C is a point inside a unit circle and $-1 \leq q < 1$.

The counting statistics for these states is given by

$$P_n = \left\{ \begin{array}{l} |c|^2, \text{for } n=0 \\ \beta^2 q^n/n, \text{for } n = 1,2,\ldots \end{array} \right\} \tag{4.18}$$

The states $|q\rangle$ could be viewed as a certain limiting case of the generalised Bose-Einstein distribution(4,6). For $C=0$, P_n is just the logarithmic distribution and that is the reason the states $|q\rangle$ are named as the logarithmic states.

Since $a = \frac{x+ip}{\sqrt{2}}$ and $a^\dagger = \frac{x-ip}{\sqrt{2}}$ and for C (real),

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*Based on

R. Simon and M. Venkata Satyanarayana, Preprint, The Institute of Mathematical Sciences, 1986.

$$(\Delta x)^2 = \left(\frac{1}{2} + \langle a^\dagger a\rangle\right) + \langle a^2\rangle - 2\langle a\rangle^2 \tag{4.19}$$

$$(\Delta p)^2 = \left(\frac{1}{2} + \langle a^\dagger a\rangle\right) - \langle a^2\rangle \tag{4.20}$$

and

$$\langle a^\dagger a\rangle = \beta^2\left(\frac{q}{1-q}\right) \tag{4.21}$$

$$\langle a^{\dagger 2}\rangle = \langle a^2\rangle = q\left\{\beta c + \beta^2\left[\sum_{n=1}^{\infty} q^n\left(1+\frac{1}{n}\right)^{1/2}\right]\right\} \tag{4.22}$$

$$\langle a^\dagger\rangle = \langle a\rangle = \sqrt{q}\left\{\beta c + \beta^2\left[\sum_{n=1}^{\infty} q^n/\sqrt{n}\right]\right\} \tag{4.23}$$

$(\Delta x)^2$ or $(\Delta p)^2$ should be less than $1/2$ in order that the states $|q\rangle$ are squeezed. Squeezing is exhibited by the logarithmic states both in x and p for certain ranges of the parameter q. $(\Delta x)^2$ and $(\Delta p)^2$ are plotted in Fig.1 for various values of C . For low values of C , x is not squeezed and the squeezing in p is attained at high values of q . For high values of C , x is squeezed even for low values of q and p is squeezed at a higher value than where x is squeezed.

Fig.2 gives a plot of $(\Delta x)^2$ vs $(\Delta p)^2$. The points in between the two rectangles correspond to the squeezed states.

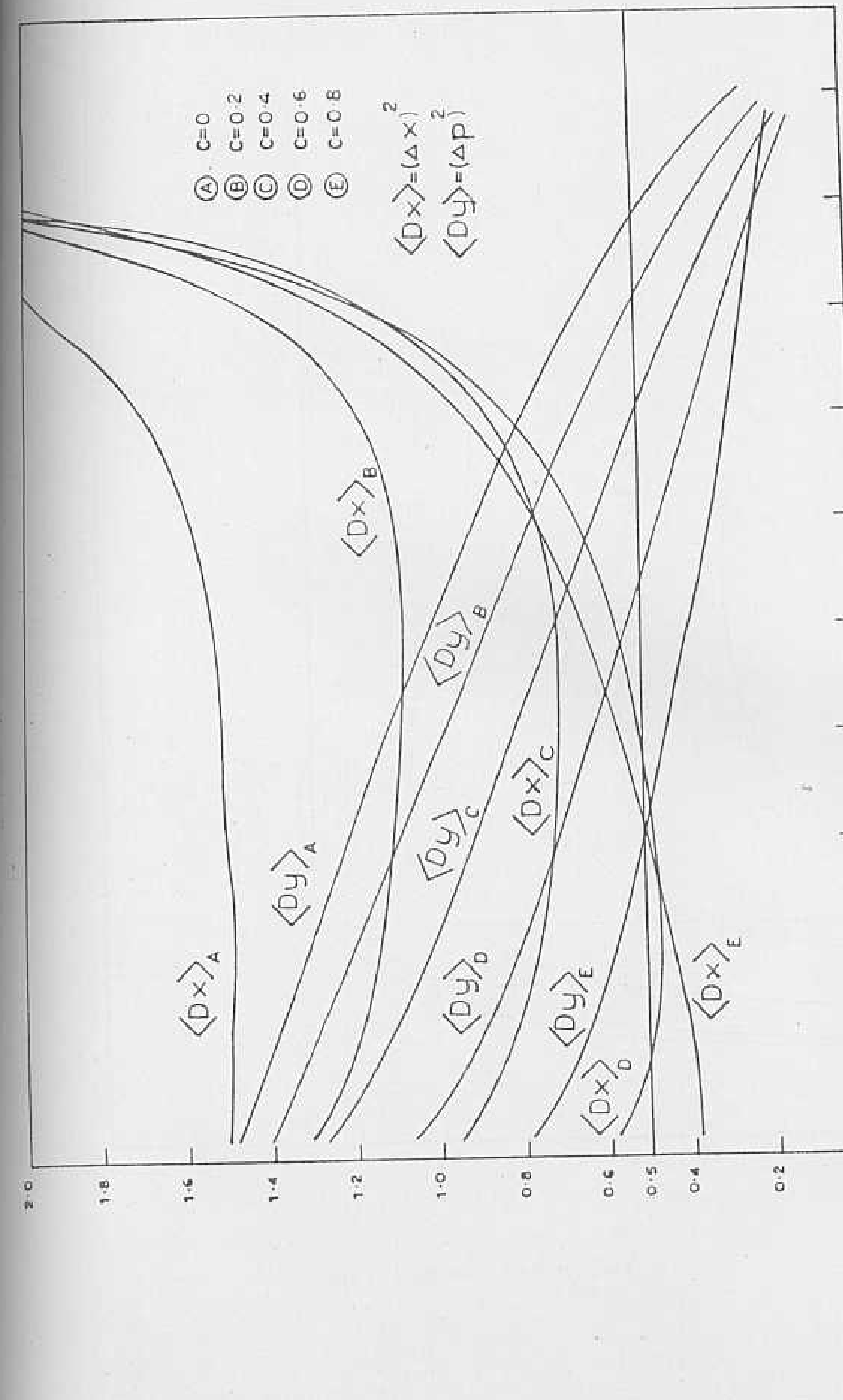

FIG.1 PLOT OF $(\Delta x)^2$ AND $(\Delta p)^2$

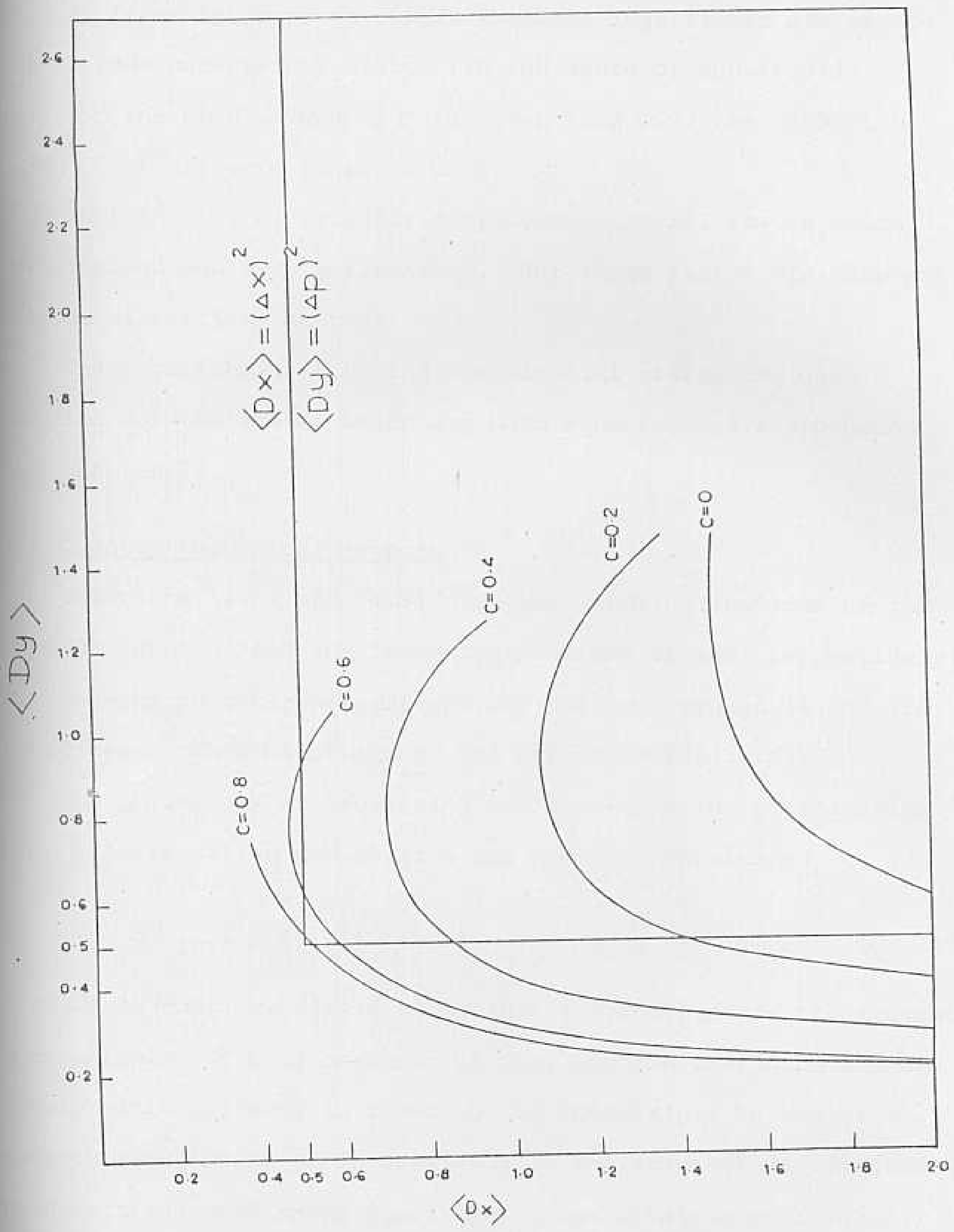

FIG.2 PLOT OF $(\Delta p)^2$ VS $(\Delta x)^2$

It is of interest to remark that the logarithmic states are not minimum uncertainty states (in the sense of equ.(4.9)). Only for the high values of C (greater than 0.9) the product $(\Delta x)^2(\Delta p)^2$ is very close to 0.25.

Stoler[7] et al recently introduced binomial states which are squeezed and also antibunched. But these states are also not minimum uncertainty states.

The logarithmic states and the binomial states are good examples for the states which are both squeezed and antibunched. (See Chapter 7).

4.3 Quantum Mechanical Aspects

Equations (4.8) and (4.9) give the exact definitions of the 'squeezed states' and the 'squeezed coherent states' respectively. The 'two-photon coherent states' (TCS) defined by equ.(4.13) are the squeezed coherent states of the harmonic oscillator.[8]

The philosophy of 'squeezing' is contained in the following scale transformation on position and momentum operators:

$$x \longrightarrow \lambda x \quad \text{and} \quad p \longrightarrow \frac{1}{\lambda} p \,, \tag{4.24}$$

λ could be complex. It is clear that the above scale transformation is canonical. It will be shown in this section that such a scale transformation effects in reducing the uncertainty in one of the quadrature variables as in equ.(4.15). The above scale transformation is in fact the well known Bogoliubov transformation employed in Superfluidity and Superconductivity.[12]

A) Bogoliubov Transformation

The squeezing operator $S(z)$ transforms $(a, a^{\dagger})$ a set of bosonic operators into $(b, b^{\dagger})$ another equivalent set of bosonic operators:

$$\left.\begin{aligned} S a S^{\dagger} &= a \cosh r + e^{i\theta} a^{\dagger} \sinh r = b \\ S a^{\dagger} S^{\dagger} &= e^{-i\theta} a \sinh r + a^{\dagger} \cosh r = b^{\dagger} \end{aligned}\right\} . \tag{4.25}$$

Since

$$[a, a^{\dagger}] = 1 \ , \ [b, b^{\dagger}] = 1 .$$

The operator $S(z)$ is unitary.

$$S S^{\dagger} = S^{\dagger} S = 1 \tag{4.26}$$

i.e., $S(z)$ is the operator which transforms a and $a^{\dagger}$ like a Bogoliubov transformation.

A more general canonical transformation could be written as

$$b = \mu a + \nu a^{\dagger} \ ; \ b^{\dagger} = \nu^{*} a + \mu^{*} a^{\dagger} \tag{4.27}$$

for a pair of c numbers μ, ν satisfying

$$|\mu|^{2} - |\nu|^{2} = 1 . \tag{4.28}$$

The structure and physical realisation of that leads to equ.(4.27) have been discussed by Yuen.[3]

If $f(a, a^{\dagger})$ is any power-series function of a and $a^{\dagger}$ then

$$S^{\dagger} f(a, a^{\dagger}) S = f(b, b^{\dagger}) . \tag{4.29}$$

If $N = a^{\dagger} a$ is the number operator in the $(a, a^{\dagger})$ system then

$$N' = b^{\dagger} b = S N S^{\dagger} . \tag{4.30}$$

N' has discrete positive eigen values n' with ground state $|0_g\rangle$.[13]

$$N'|m_g\rangle = m_g|m_g\rangle \; ; N'|0_g\rangle = 0 \tag{4.31}$$

$$|m_g\rangle = S\,|m\rangle \tag{4.32}$$

where

$$a^{+}a|m\rangle = m|m\rangle .$$

Similar to $|m\rangle$, the states $|m_g\rangle$ can be expressed as

$$|m_g\rangle = \frac{(b^{+})^{m}}{\sqrt{m!}}|0_g\rangle \; . \tag{4.33}$$

The states $|m_g\rangle$ are complete and orthornormal. b and b^{+} act as lowering and raising operators for $|m_g\rangle$.

The squeezed coherent states are defined to be the eigenstates of b :

$$b|\beta\rangle_g = \beta|\beta\rangle_g \; . \tag{4.34}$$

From equ.(4.25)

$$|\beta\rangle_g = S|\beta\rangle \tag{4.35}$$

where
$$a|\beta\rangle = \beta|\beta\rangle \; . \tag{4.36}$$

From equ.(4.35) it is clear that from an arbitrary coherent state $|\beta\rangle$, a s.c.s can be created by using the unitary transformation $S(z)$.

Therefore the usual technology of the coherent states could be applied for the states $|\beta\rangle_g$ i.e.,

$$|\beta\rangle_g \equiv D_g(\beta)|0_g\rangle \, ; \; D_g(\beta) = e^{\beta b^\dagger - \beta^* b} . \tag{4.37}$$

$$\int |\beta\rangle_g \, {}_g\langle\beta| \frac{d^2\beta}{\pi} = \mathbb{1} . \tag{4.38}$$

$${}_g\langle\beta|\beta'\rangle_g = \mathrm{Exp}\left(\beta^*\beta' - \frac{1}{2}|\beta|^2 - \frac{1}{2}|\beta'|^2\right) . \tag{4.39}$$

The Bogoliubov transformation achieves something more interesting and useful apart from the equivalence of two bosonic systems. It transforms any quadratic hamiltonian into a harmonic oscillator hamiltonian.[14]

The general quadratic hamiltonian involving two-photon process is of the form

$$\mathcal{H} = \hbar(f_1 a^\dagger a + f_2^* a^2 + f_2 a^{\dagger 2} + f_3^* a + f_3 a^\dagger) \tag{4.40}$$

where the c-numbers f_i may be time dependent and f_1 should be real. Under the condition

$$f_1 > 2|f_2| ,$$

using Bogolíubov transformation, the hamiltonian in equ.(4.40) could be changed to the following form

$$\mathcal{H}' = \hbar f_0 b^\dagger b \tag{4.41}$$

where

$$b = \mu a + \nu a^{\dagger} + \frac{1}{f_0}(\mu f_3^{*} - \nu f_3) \tag{4.42}$$

and

$$\left.\begin{aligned} \mu &= \left[2/f_0(f_1 - f_0)\right]^{1/2} f_2\, e^{i\phi_\mu} \\ \nu &= \left[(f_1 - f_0)/2f_0\right]^{1/2} e^{i\phi_\nu} \\ f_0 &= \left(f_1^2 - 4|f_2|^2\right)^{1/2} \end{aligned}\right\} \tag{4.43}$$

where ϕ_μ and ϕ_ν are arbitrary phases. Unlike equ.(4.40) the hamiltonian (4.41) does not contain the nonlinear terms such as b^2 and $(b^{\dagger})^2$

B) <u>Fock Space Representation</u>

The Fock space representation of $|\beta\rangle_g$ in $(a, a^{\dagger})$ system is given by

$$|\beta\rangle_g = \sum_{n=1}^{\infty} \langle n|\beta\rangle_g \, |n\rangle \tag{4.44}$$

<u>To determine $\langle n|\beta\rangle_g$</u>

Since $\quad b = \mu a + \nu a^{+}$,

$$b|\beta\rangle_g = \beta|\beta\rangle_g \ ,$$

and

$$\langle\alpha|F(a^{+},a)|\psi\rangle = F\left(\alpha^{*}, \frac{\alpha}{2} + \frac{\partial}{\partial\alpha^{*}}\right)\langle\alpha|\psi\rangle \ , \tag{4.45}$$

$$\frac{\partial}{\partial\alpha^{*}}\langle\alpha|\beta\rangle_g = \left(\frac{\beta}{\mu} - \frac{\alpha}{2} - \frac{\nu\alpha^{*}}{\mu}\right)\langle\alpha|\beta\rangle_g \ . \tag{4.46}$$

The solution of (4.46) is

$$\langle\alpha|\beta\rangle_g = \left(\frac{1}{\mu}\right)^{1/2} \operatorname{Exp}\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 - \left(\frac{\nu}{2\mu}\right)\alpha^{*2} + \left(\frac{\nu^{*}}{2\mu}\right)\beta^2 + \frac{1}{\mu}\alpha^{*}\beta + i\theta_0\right] . \tag{4.47}$$

For $\mu = 1$ and $\nu = 0$, equ.(4.45) reduces to the usual form of $\langle\alpha|\beta\rangle$ for $\theta_0 = 0$. (see equ.(1.25))

Using the generating function of the Hermite polynomials

$$e^{2zt - t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n, \quad |t| < \infty \tag{4.48}$$

and

$$\langle\alpha|\beta\rangle_g = \sum_{n=0}^{\infty} \langle\alpha|n\rangle\langle n|\beta\rangle_g \tag{4.49}$$

where

$$\langle\alpha|n\rangle = e^{-|\alpha|^2/2} \frac{\alpha^{*n}}{\sqrt{n!}} , \tag{4.50}$$

$\langle n|\beta\rangle_g$ is obtained as

$$\langle n|\beta\rangle_g = \left(\frac{1}{n!\,\mu}\right)^{1/2} \left(\frac{\nu}{2\mu}\right)^{n/2} H_n\left[\frac{\beta}{\sqrt{2\mu\nu}}\right]$$

$$\times \mathrm{Exp}\left[-\frac{1}{2}|\beta|^2 + \left(\frac{\nu^*}{2\mu}\right)\beta^2\right] . \tag{4.51}$$

Substituting equ.(4.51) in equ.(4.44) the Fock space representation of $|\beta\rangle_g$ is obtained.

For $\mu = 1$, $\nu = 0$ and the asymptotic forms of $H_n(z)$ equ.(4.51) reduces to

$$\langle n|\beta\rangle_g = e^{-|\beta|^2/2} \frac{\beta^n}{\sqrt{n!}} \tag{4.52}$$

as expected.

<u>Completeness</u>

From the completeness relation of the states $|\alpha\rangle$ (equ.(1.29)),

$$\frac{1}{\pi}\int d^2\alpha\ |z,\alpha\rangle\langle z,\alpha| = 1 . \tag{4.53}$$

So the states $|z,\alpha\rangle$ could be used as basis states.

c) Minimum Uncertainty States

The conditions under which the states $|z,\alpha\rangle$ (or equivalently $|\beta\rangle_g$) remain as minimum uncertainty states will be analysed in this subsection. The presentation also gives another interesting approach for viewing the squeezed states.

If $|\psi\rangle$ is the state of a system then $S(r)|\psi\rangle$ (r real) represents the same system compressed in position space by the factor $l = e^{-r}$. This is the reason for $S(z)$ to be called as 'squeezing operator'. This could be seen from the effect of $S(r)$ on the position and momentum operators. From equ.(4.25)

$$S^{+}(r)\, x\, S(r) = \left(\frac{2\hbar}{m\omega}\right)^{1/2}(b+b^{+})$$

$$= \left(\frac{2\hbar}{m\omega}\right)^{1/2}(\cosh r + \sinh r)(a+a^{+}) \tag{4.54}$$

$$\left.\begin{aligned} \text{So}\quad & S^{+}(r)\, x\, S(r) = x e^{r} = x/l \\ \text{and}\quad & S^{+}(r)\, p\, S(r) = e^{-r} p = pl \end{aligned}\right\} \tag{4.55}$$

$$\left.\begin{aligned} \text{Also,}\quad & S^{+}(r)\, x^{k} S(r) = (x/l)^{k} \\ \text{and}\quad & S^{+}(r)\, p^{k} S(r) = (pl)^{k} \end{aligned}\right\} \tag{4.56}$$

From the discussion above it is clear that the width in position space of the state

$$|r,0\rangle \equiv S(r)\,|0\rangle \tag{4.57}$$

is e^{-r} multiplied by the position spread of the state $|0\rangle$. Since the ground state $|0\rangle$ is Gaussian, its position spread is

$$\Delta x = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \tag{4.58}$$

and the position spread of $|r,0\rangle$ is given by

$$\Delta x' = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \frac{1}{\ell} . \tag{4.59}$$

Therefore, for very **large negative** values of r, the state $|r,0\rangle$ is highly localized in position space and for very large positive values of r, the state $|r,0\rangle$ is highly localized in momentum space.

The states $|r,0\rangle$ could be generalized to get the 'squeezed coherent states' by defining

$$|z,\alpha\rangle \equiv D(\alpha)\,S(z)\,|0\rangle \tag{4.60}$$

where α and z are complex. The state $|z,\alpha\rangle$ is a Gaussian packet with the same shape as $|r,0\rangle$ but displaced from the origin in position and momentum space.

The wave packet of a squeezed state could be directly written from equ.(1.9)

$$\Psi_{Squeezed} = 2\pi(\Delta x')^2 \times Exp\left\{-\left[\frac{x-\langle x\rangle}{2(\Delta x')}\right]^2 + \frac{i}{\hbar}\langle p\rangle x\right\} . \tag{4.61}$$

The time development of the states $|z,\alpha\rangle$ (under the harmonic oscillator hamiltonian) could be obtained from the temporal evolution of $S(z)$ given by

$$
\begin{aligned}
e^{-i\omega t a^{+}a} \quad S(z)\, e^{i\omega t a^{+}a} \\
&= e^{-i\omega t a^{+}a} \operatorname{Exp}\left(\frac{z}{2} a^{+2} - \frac{z^{*}}{2} a^{2}\right) e^{i\omega t a^{+}a} \\
&= \operatorname{Exp}\left(\frac{z}{2} e^{-2i\omega t} a^{+2} - \frac{z^{*}}{2} e^{2i\omega t} a^{2}\right) \\
&= S(z e^{-2i\omega t}) \quad . \qquad (4.62)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
e^{-i\omega t a^{+}a} |z,\alpha\rangle &= D(\alpha e^{-i\omega t})\, S(z e^{-2i\omega t})\, |0\rangle \\
&= |\alpha e^{-i\omega t}, z e^{-2i\omega t}\rangle \quad . \qquad (4.63)
\end{aligned}
$$

The time development of the uncertainty product is obtained from the following expectation values (for $\ell = e^{-r}$ and $\alpha = \alpha_1 + i\alpha_2$)

$$
\langle x(t)\rangle = \left(\frac{2\hbar}{m\omega}\right)^{1/2} (\alpha_1 \cos\omega t + \alpha_2 \sin\omega t) \qquad (4.64)
$$

$$
\langle p(t)\rangle = (2m\hbar\omega)^{1/2} (-\alpha_1 \sin\omega t + \alpha_2 \cos\omega t) \qquad (4.65)
$$

$$(\Delta x)^2 = \left(\frac{\hbar}{2m\omega}\right)\left[\ell^2 \sin^2\omega t + \frac{1}{\ell^2}\cos^2\omega t\right] \tag{4.66}$$

$$(\Delta p)^2 = (m\hbar\omega)\left[\frac{1}{\ell^2}\sin^2\omega t + \ell^2\cos^2\omega t\right] . \tag{4.67}$$

The uncertainty product is

$$(\Delta x)^2 (\Delta p)^2 = \frac{\hbar^2}{4}\left[1 + \frac{1}{4}\left(\ell^2 - \frac{1}{\ell^2}\right)^2 \sin^2 2\omega t\right] . \tag{4.68}$$

Now for $r = 0$ $(\ell = 1)$ the states $|0,\alpha\rangle$ are just the coherent states and the uncertainty product is constant in time. Also the states $|z,\alpha\rangle$ do not remain as a minimum-uncertainty state as time progresses. Every time $\omega t = \pi/2$ the states $|z,\alpha\rangle$ will be minimum uncertainty states.

So the state $|z,\alpha\rangle$ will be a minimum uncertainty packet only if z is real $(i.e., \theta = 0)$. Even these states $|r,\alpha\rangle$ do not remain as minimum uncertainty states as time progresses; the minimum uncertainty is attained at times $\omega t = \pi/2$.

From equs.(4.66) and (4.67) and $L = \left(\ell^2 + \frac{1}{\ell^2}\right)$,

$$\frac{[\Delta x(t)]^2}{\left[L\frac{\hbar}{2m\omega}\right]} + \frac{[\Delta p(t)]^2}{[L m\hbar\omega]} = 1 \tag{4.69}$$

So the uncertainties in x and p lie on an ellipse rather than a circle as in the case of coherent states.(see equ.(1.12)). These ideas have been pictorially depicted by Walls[16] in his review.

D) Nonclassical Nature

It has been indicated earlier that squeezing is a nonclassical effect and the 'manifestation of the squeezed states of the radiation field is a purely quantum mechanical effect.'[9] In fact this is the reason for the recent interest in the squeezed states. The nonclassical nature comes from the condition $(\Delta a_i)^2 < \frac{1}{4}$, for $i = 1$ or 2. As seen in Sec.1.2., the coherest state description of a radiation field is defined by the diagonal representation

$$\rho = \int P(\alpha)\, |\alpha\rangle\langle\alpha|\, d^2\alpha \tag{4.70}$$

Also, according to equ.(1.37) the normal ordered correlation functions of a and $a^\dagger$ can be obtained from the coherent state representation using the methods of classical statistical mechanics.

For a_1, a_2 described by equs.(4.5) and (4.6),

$$(\Delta a_1)^2 = \frac{1}{4}\left[1 + \int P(\alpha)\left[(\alpha+\alpha^*) - (\langle\alpha\rangle + \langle\alpha^*\rangle)\right]^2 d^2\alpha\right] \tag{4.71}$$

Since the quantity inside the parenthesis is real and its square is positive, the squeezing condition $(\Delta a_1)^2 < 1/4$

implies that $P(\alpha)$ is not non-negative. Such a field does not have a classical analog as a coherent field. Walls[16] defines the squeezed states as those which 'do not have positive nonsingular representation in terms of the Glauber - Sudarshan P-distribution'.[17]

Summary of this section:

In (A) equivalence of $(a, a^\dagger)$ system and $(b, b^\dagger)$ system is established. Squeezing in $(a, a^\dagger)$ system is equivalent to coherence in $(b, b^\dagger)$ system and vice versa.

In (C) it has been shown that the Heisenberg uncertainty product takes the minimum value $(\hbar^2/4)$ only at certain instances of time and this is periodic. Also the wave packets have classical motion satisfying Schrodinger's criterion. Thus the coherent states could be put at par with the squeezed coherent states as far as the classical motion of the wave packets are concerned.

The real distinction between the c.s and the s.c.s is brought out in (D) that a s.c.s does not have $P(\alpha)$ as positive definite and hence a s.c.s is nonclassical.

4.4 Quantum Optical Aspects

It has been discussed in the earlier section that the squeezed light is characterised by the nonexistence of P-representation as a non-singular positive function. So the real quantum behaviour of the optical fields occurs in a regime in which no non-singular $P(\alpha)$ exists.[18]

Our present knowledge of the squeezed states is mostly due to the work of Stoler,[10] Lu,[20] Yuen[3] and Walls.[16]

States $|\alpha, z\rangle$ were first defined (as in equ.(4.13)) by Stoler.[10] He characterises the c.s. among all other minimal uncertainty states uniquely, from identifying coherence with minimality of the uncertainty product. This is clear from Sec.4.3 C.

Lu[20] independently obtained the s.c.s and employed them to study two-photon amplification. He also showed that the fluctuation properties of these states are quite different from chaotic as well as coherent fields.

First comprehensive study of the s.c.s in the context of quantum optics is due to Yuen.[3]

Rowe[21] showed that the s.c.s could be emitted by a two-photon laser, which was also noted by Yuen.[3]

The review of Walls[16] contains various schemes for the generation and detection of the squeezed states and their applications. Walls also pointed out that the squeezing is to be considered as a 'macroscopic quantum effect.'

This section discusses certain statistical properties of the s.c.s and also various schemes to generate the s.c.s.

The probability for n photons to be found in a s.c.s $|\alpha, r\rangle$ could be calculated from equ.(4.51) (for μ, ν and β real)

$$P_n = \left(\frac{1}{n!\,\mu}\right)\left(\frac{\nu}{2\mu}\right)^n e^{-\beta^2\left(1-\frac{\nu}{\mu}\right)} H_n^2\left(\frac{\beta}{\sqrt{2\mu\nu}}\right) \tag{4.72}$$

where ν = sin hr; μ = cos hr and $\beta(\mu-\nu) = \alpha$

It is to be remarked that the counting statistics of the s.c.s 'is far from the Poisson.'[3]

Walls[16] has plotted $n \sim P_n$ for $\alpha = 7$ and $r = \pm 0.5$ and compared the distribution with a coherent state ($r = 0$). The photon statistics is super-poissonian for $r < 0$ and sub-poissonian for $r > 0$.

The expression for P_n in equ.(4.72) has been used in the study of the interaction of a two-level atom with a squeezed light by Milburn.[22]

P_n for the pure squeezed state

$$S(z)|0\rangle = (\cosh|z|)^{-1/2} \times \sum_{n=0}^{\infty} \left(\frac{z}{2|z|} \tanh|z| \right)^n \frac{[(2n)!]^{1/2}}{n!} |2n\rangle \qquad (4.73)$$

is obtained from equ.(4.72) by choosing $\alpha = 0$ as

$$\left.\begin{aligned} P_{2n} &= \frac{1}{\cosh|z|} \left(\frac{\tanh^2|z|}{4} \right)^n \binom{2n}{n} \\ P_{2n+1} &= 0 \end{aligned}\right\} . \qquad (4.74)$$

Since $\langle n \rangle = \sinh^2|z|$ for a pure squeezed state, equ.(4.74) could be written in the following form[23]

$$\left.\begin{aligned} P_{2n} &= \frac{(2n-1)!!}{(2n)!!}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n \frac{1}{(1+\langle n\rangle)^{1/2}} \\ P_{2n+1} &= 0 \end{aligned}\right\} . \qquad (4.75)$$

An interesting aspect of the s.c.s is the fact that the mean number of photons in a s.c.s varies depending on the squeezing parameter $|Z|$.

$$\langle n\rangle = \langle b^\dagger b\rangle = |\alpha|^2 + \sinh^2|Z| \quad , \qquad (4.76)$$

whereas the mean photon number in a c.s. is just $|\alpha|^2$. So squeezing increases the population level in a cavity. In fact one can view the action of operators $S(Z)$ and $D(\alpha)$ on the vacuum $|0\rangle$ as various methods of filling an optical cavity.

It is to be noted that the counting statistics of a chaotic light and a purely squeezed light are akin to each other in view of the expressions for the second order correlation function for both these radiation fields.(24) They are given below:

For a chaotic light

$$g^{(2)}(0) = 2 \qquad (4.77a)$$

For a pure squeezed light

$$g^{(2)}(0) = 2 + \coth^2 r \qquad (4.77b)$$

Equ.(4.76) implies that squeezing could be considered as a tuning mechanism for the purpose of amplifying the mean number of photons in a cavity or in an optical process. This aspect of squeezing gives ' a remarkable result'[21] for the intensity of a two-photon laser. Since I, the energy intensity of radiation is proportional to the expectation value of $a^{\dagger}a$, for the single photon laser

$$\langle I^2 \rangle_{C.S} = \langle \alpha | a^{\dagger} a a^{\dagger} a | \alpha \rangle = I(I+1) \tag{4.78a}$$

and for the two-photon laser

$$\langle I^2 \rangle_{S.C.S} = 3I^2 + 2I \,. \tag{4.78b}$$

So $\langle I^2 \rangle$ for a two-photon laser is enhanced by three times that of a single-photon laser. This is the reason for the belief that the squeezed states may be emitted by high intensity fields.[16]

Recently there has been a lot of interest in looking for the squeezed states (both theoretically and experimentally) in various optical processes like resonance fluorescence,[25] parametric amplification,[26] FEL (Free Electron Lasers),[27] higher order nonlinear optical processes,[28] Jaynes-Cummings model[29] and four-wave mixing.[30] Most of the interactions which generate the squeezed states are of the quadractic form given in equ.(4.40). We will give a brief survey of a typical squeezing hamiltonian below:

Degenerate Parametric Amplifier

In a degenerate parametric amplifier, a pumping field of frequency 2ω interacts with a nonlinear medium and gives rise to a field of frequency ω. This process is described at exact resonance by the hamiltonian

$$\mathcal{H} = 2\omega b^{+}b + \omega a^{+}a + \frac{ik}{2}(a^{2}b^{+} - b a^{+2}) \tag{4.79}$$

(b, b^{+} : pump operators; a, a^{+} : signal mode operators)
(K : coupling constant).

In the parametric approximation, pump field is treated classically and the pump depletion is neglected. Then the operator b could be replaced by $Be^{-2i\omega t}$. For B real, the resulting hamiltonian is

$$\mathcal{H} = \omega a^{+}a + \frac{ik}{2}(a^{2}e^{2i\omega t} - a^{+2}e^{-2i\omega t}). \tag{4.30}$$

In the interaction picture,

$$\mathcal{H}_{I} = \frac{iK}{2}(a^{2} - a^{+2}). \tag{4.81}$$

Thus we see that this nonlinear device generates s.c.s.

The detection of the squeezed states in an optical process is not an easy job. The reason is the highly transient nature of the squeezed states. As seen in Sec.4.3 the squeezed states are generated by quadratic interactions. Further under the time evolution these states attain the minimum value of the uncertainty product only at certain points of time. Also the generation of the squeezed states is critically sensitive to the phase stability of the driving laser field. In the conventional optical experiments one measures only the diagonal elements of the density matrix of the signal field and hence the problems associated with the phase stability disappear.[31]

Recently the Bell Laboratories group have announced the experimental observation of the squeezed states. They employed the nonlinear interaction, four-wave mixing, in an optical cavity.[32]

4.5 Conclusion

The spurt of activity in the study of the squeezed states is not only due to their nonclassical nature, also due to their utility in reducing the quantum noise in a totally different branch of physics namely gravitational wave detection. The gravitational wave detection is based on Michelson interferometry. The sensitivity of this device is limited by quantum fluctuations; the two sources of this being <u>photon counting</u> and <u>radiation pressure</u>. Recently a technique has been proposed which uses the squeezed states of the radiation field to reduce the photon counting fluctuation in the interferometer and thereby increasing the sensitivity of the interferometer.[33] To say briefly, the arriving gravitational signal interacts with an oscillator that is in a squeezed state. Various methods of producing the squeezed states in an oscillator to register the gravitational force have been studied by Grishchuk and Sazhin.[34]

Another important application of the squeezed states is the possibility of employing them in optical communications using lasers. If the state of the laser beam is a squeezed state, the information could be communicated with reduced fluctuations and thus the quantum noise level could be lowered below the zero point limit. Yuen and his collaborators have developed a theory of optical communications using the squeezed states.(35)

So, we see that the squeezed states have a potential capacity to play a much more important role in quantum mechanics, quantum optics and the detection of gravitational waves.

Some more historical details regarding the s.c.s have been given in Ref.(36).

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$$P_n^{(k)} = \frac{(n+k-1)!}{n!\,(k-1)!}\left(\frac{\bar{n}/k}{1+\bar{n}/k}\right)^n \left(\frac{1}{1+\bar{n}/k}\right)^k$$

where $\bar{n} = n_1+n_2+\cdots+n_k$.

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36) Note on historical details

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CHAPTER 5

SQUEEZING OPERATOR *

The nonclassical effects of the states generated by $S(z)$ on $|0\rangle$ have been discussed in the earlier chapter.

Mathematically $S(z)$ could be thought of as an operator acting conformally on the phase space (for z = real); since an unit cell in the phase space namely a circle would be deformed into an ellipse due to the action of $S(z)$.

In this chapter we shall be concerned about other aspects of $S(z)$ like the possibility of generalizing $S(z)$ for multiphoton production and the extension of the results to the para-Bose oscillator, which gives a surprising result that even the squeezing (i.e., the two para-Bose coherent state) is not admissible.

Fisher et al[1] in an interesting paper discussed the possibility of obtaining generalized squeezed coherent states in the Hilbert space of the harmonic oscillator. They are obtained by exponentials of polynomials a and $a^\dagger$ acting on the vacuum:

$$|\alpha, z_k\rangle \equiv D(\alpha)\, S(z_k)\,|0\rangle \tag{5.1}$$

where

$$S(z_k) \equiv \mathrm{Exp}\left[z_k (a^\dagger)^k - z_k^* a^k\right] \tag{5.2}$$

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*This chapter is based on

1) T.S. Santhanam and M. Venkata Satyanarayana, Phys. Rev. D30, 2251 (1984).

2) M. Venkata Satyanarayana, Phys. Rev. D32, 400 (1985).

and $D(\alpha)$ is the displacement operator and Z_k is a complex factor. Such a generalization of the squeezed coherent states, Fisher et al[1] named it as 'naive generalization' for reasons which are obvious.

Fisher et al[1] proved that the squeezed coherent states exist only for $k=2$, which are the s.c.s discussed in Chapter 4. For $k>2$ the squeezed coherent states do not exist. The method of their proof is to consider the vacuum expectation value of $S(Z_k)$, which has a power series expansion

$$\langle 0|S(Z_k)|0\rangle = 1 - |Z_k|^2 \frac{k!}{2!} + |Z_k|^4 \frac{[(k!)^2 + (2k)!]}{4!}$$

$$- |Z_k|^6 \frac{1}{6!}\left[(k!)^3 + 2k!(2k)! + \frac{1}{k!}((2k)!)^2 + (3k)!\right]$$

$$+ \cdots\cdots + (-)^n |Z_k|^{2n} \frac{1}{(2n)!} C_n + \cdots\cdots \tag{5.3}$$

C_n is the coefficient of $(-)^n |Z_k|^{2n}/(2n)!$ and it contains many terms and all are positive and the largest is of the order of $(kn)!$ and others are also of the same order.

Therefore,
$$\operatorname*{Lt}_{n\to\infty} |Z_k|^{2n} \frac{1}{(2n)!} C_n \neq 0 \tag{5.4}$$

for all $k>2$ and $Z_k \neq 0$. This means the series in equ.(5.3) is divergent[2] for $k>2$. For $k=2$ it marginally converges. This shows that for $k>2$, $S(Z_k)$ though unitary, it is unbounded in the Hilbert space of the simple harmonic oscillator. This is due to the fact that $|0\rangle$ is not an analytic vector[3] of the generator A_k, where A_k is given by

$$S(z_k) \equiv \mathrm{Exp}\left[i \mathcal{A}_k\right] \tag{5.5}$$

i.e., $\mathcal{A}_k$ is not self-adjoint for $k>2$.

So the result is that the squeezing disappears for the many-photon states $k>2$.

It is interesting and also instructive to look for the fields which do not even admit squeezing for $k=2$. Such an example is furnished by para-Bose fields. In Sec 5.1 it is shown that for the para-Bose oscillator the squeezed states do not exist for all orders of the statistics.[4]

5.1 Para-Bose Oscillator

The para-Bose oscillator[5,6] satisfies the equation of motion

$$\left[A, N\right] = A \tag{5.6}$$

and does not satisfy the canonical commutation relation

$$\left[A, A^{\dagger}\right] = 1 . \tag{5.7}$$

N is defined as

$$N = \frac{1}{2}\left(A^{\dagger}A + AA^{\dagger}\right) - h_0 \tag{5.8}$$

where h_0 is the lowest eigen value of the hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(A^{\dagger}A + AA^{\dagger}\right) . \tag{5.9}$$

The case $h_0 = \frac{1}{2}$ corresponds to the usual harmonic oscillator. $2h_0$ is defined as the order of the statistics of the para-Bose oscillator.

It can be easily seen that the operators A, A^+ and N (or $\mathcal{H}$) do not close to form a Lie algebra. However A^2, A^{+2} and N close and the algebra is the algebra of the Lorentz group $SO(2,1)$. This could be seen from the following symmetric operators

$$H_1 = \frac{i}{4}(A^{+2} - A^2); \; H_2 = \frac{1}{4}(A^{+2} + A^2); \; H_0 = \frac{1}{2}N \tag{5.10}$$

and they satisfy the commutation relations

$$\left.\begin{aligned} [H_0, H_1] &= iH_2 \\ [H_1, H_2] &= -iH_0 \end{aligned} \quad \text{and} \quad [H_2, H_0] = iH_1 \right\} . \tag{5.11}$$

Since the spectrum of N is positive definite the Fock space representation is given by[5,6]

$$\left.\begin{aligned} (A)_{2n,2n+1} &= [2(n+h_0)]^{1/2} \\ (A)_{2n-1,2n} &= (2n)^{1/2} \end{aligned}\right\} . \tag{5.12}$$

When $h_0 = 1/2$, the above representation reduces to the usual representation of the Bose oscillators and the distinction between the even and odd matrix elements disappears. So for every value of h_0 an infinite spectrum is obtained given by

$$h_0, h_0+1, h_0+2, \ldots\ldots\ldots \infty .$$

The physical meaning of the para-Bose oscillator may be understood if we see the Green's ansatz[5]

$$A = \sum_{j=1}^{2h_0} b^{j} \tag{5.13}$$

where b^{j}'s are the usual annihilation operators for the Bose oscillators for a given j, but anticommute for different j values.

The idea of the para-Bose oscillator was proposed under a broader scheme known as 'para-Statistics' which means the maximum occupation number of a state might be any finite (positive) integer unlike Fermi-Dirac statistics or Bose-Einstein statistics for which the occupation numbers are 0 and 1 or n(= 1,2,...)respectively. The concept of para-Statistics has played a significant role in the development of quantum field theory since the method of quantization of a field is related to the statistics of the particle associated with the field.[7]

As seen earlier the para-Bose oscillator also possesses an infinite spectrum similar to the usual harmonic oscillator spectrum. The difference between the two oscillators is manifested in the symmetry properties of the states which are different for the two oscillators.

The coherent states of a para-Bose oscillator have been defined by Sharma et al[8] and Biswas and Santhanam.[9] Considering the displacement operator $D(\alpha) \equiv \mathrm{Exp}[\alpha A^{+} - \alpha^{*} A]$ acting on the vacuum $|0\rangle_{h_0}$, Sharma et al[8] obtained coherent state $|\alpha\rangle_{h_0}$ as

$$|\alpha\rangle_{h_0} = \sum_{n=0}^{\infty} a_n \, |n\rangle_{h_0} \qquad (5.14)$$

where

$$a_n = \left[\frac{\Gamma(h_0)}{2^n \, \Gamma\left\{\left[\frac{n}{2}\right]+1\right\} \Gamma\left\{\left[\frac{n+1}{2}\right]+h_0\right\}} \right]^{1/2} \alpha^n a_0 \qquad (5.15)$$

and

$$a_0 = \left[\sum_{n=0}^{\infty} \frac{\Gamma(h_0)}{\Gamma\left\{\left[\frac{n}{2}\right]+1\right\} \Gamma\left\{\left[\frac{n+1}{2}\right]+h_0\right\}} \left(\frac{1}{2}|\alpha|^2\right)^n \right]^{-1/2}. \qquad (5.16)$$

The same coherent states have been obtained by Biswas and Santhanam[9] by using the differential operator representation for the annihilation operator. They also obtained

$$_{h_0}\langle 0| A^m A^{+m} |0\rangle_{h_0} = \frac{2^m \, \Gamma\left\{\left[\frac{m}{2}\right]+1\right\} \Gamma\left\{\left[\frac{m+1}{2}\right]+h_0\right\}}{\Gamma(h_0)} \qquad (5.17)$$

$$= C_m \ (\text{Say})$$

Proceeding along the lines of Fisher et al,[1] we consider

$$S(z_k) = Exp\left[z_k (A^{+})^k - z_k^* A^k\right] \tag{5.18}$$

and

$$
\begin{aligned}
{}_{h_0}\langle 0|S(z_k)|0\rangle_{h_0} &= 1 - |z_k|^2 \, {}_{h_0}\langle 0| A^k A^{+k} |0\rangle_{h_0} + \cdots \\
&= 1 - |z_k|^2 \frac{T_1}{2!} + |z_k|^4 \frac{T_2}{4!} + \cdots \\
&\quad + (-)^n |z_k|^{2n} \frac{T_n}{(2n)!} + \cdots
\end{aligned} \tag{5.19}
$$

where

$$T_1 = C_k \; ; \; T_2 = C_k^2 + C_{2k} \quad etc.$$

In general, as in the case of the usual harmonic oscillator (equ.5.3) each T_n has many terms, all are positive and the largest is of the order of C_{kn} and indeed the others are also of the same order, as all of them arise from the vacuum espectation value of a polynomial of degree k in A and A^{+}.

The leading term of the nth term of the series in equ.(5.19) is

$$t_n = \frac{|z_k|^{2n}}{(2n)!} \frac{2^{nk}\,\Gamma\left\{\left[\frac{nk}{2}\right]+1\right\}\Gamma\left\{\left[\frac{nk+1}{2}\right]+h_0\right\}}{\Gamma(h_0)} \tag{5.20}$$

Now we shall discuss the convergence of the series in equ.(5.19) for the various cases which arise for the various statistics.

Case 1: $h_0 = 1/2$, then

$$t_n = \frac{|z_k|^{2n}}{(2n)!} \frac{2^{nk}\,\Gamma\left\{\left[\frac{nk}{2}\right]+1\right\}\Gamma\left\{\left[\frac{nk+1}{2}\right]+\frac{1}{2}\right\}}{\Gamma\left\{\frac{1}{2}\right\}} . \tag{5.21}$$

Now two cases arise that nk could be odd $(nk = 2m+1)$ or even $(nk = 2m)$. In both the cases

$$t_n = \frac{|z_k|^{2n}}{(2n)!} (nk)! \tag{5.22}$$

For $k = 2$, the series in equ.(5.19) converges, implying that the two-photon coherent states are possible, the result obtained by Fisher et al.[1]

Case 2: $h_0 = 1$. As in earlier case again two cases arise that nk could be odd or even and we will discuss them separately.

(i) $nk = 2m$

$$t_n = \frac{|z_k|^{2n}}{(2n)!} 2^{2m}\,\Gamma(m+1)\,\Gamma(m+1)$$

$$= \frac{|z_k|^{2n}}{(2n)!} 2^{2m} (m!)^2 \tag{5.23}$$

Using the test in Ref (2), it is easy to see that $\sum (-)^n t_n$ diverges.

(ii) $nk = 2m+1$

$$t_n = \frac{|z_k|^{2n}}{(2n)!} 2^{2m+1}\,\Gamma(m+1)\,\Gamma(m+2) \tag{5.24}$$

Using the similar arguments as earlier $\sum(-)^n t_n$ diverges.

Now we consider the special case of particular interest: $h_0=1$ and $k=2$, i.e., corresponding to the two para-Bose coherent states

$$t_n = \frac{|z|^{2n}}{(2n)!} 2^{2n} (n!)^2 . \tag{5.25}$$

The vacuum espectation value of $S(z) = Exp[zA^{+^2} - z^* A^2]$ could be written as a power series of z :

$$f(z) = \sum_{n=0}^{\infty} (-)^n |z|^{2n} 2^{2n} \Big/ \binom{2n}{n} . \tag{5.26}$$

Clearly t_n does not tend to zero as n tends to ∞ and hence the divergence of $f(z)$ is established which means the nonexistence of the squeezed states for a para-Bose oscillator. Exactly similar arguments hold good to prove that the squeezed states do not exist for all orders of the statistics.

So having established that a para-Bose oscillator does not admit squeezing for all orders of the statistics we arrive at the conclusion that the para-Bose vacuum $|0\rangle_{h_0}$ is not an analytic vector of the generator G where

$$Exp[iG] = Exp[zA^{+}A^{+} - z^* AA]$$

which is a bit surprising.

5.2 Generalized Squeezed Coherent States

The 'impossibility' result of Fisher et al[1] is very interesting and it has led us to a more interesting situation where even the squeezing of the order two is not possible.

Though one cannot 'naively' generalize them for the harmonic oscillator, one can generalize them in a more useful way. The generalized squeezed coherent states (g.s.c.s) are defined as[10]

$$|n,z,\alpha\rangle \equiv D(\alpha)\, S(z)\, |n\rangle \tag{5.27}$$

where $$S(z) \equiv \mathrm{Exp}\left[\frac{z}{2} a^{+}a^{+} - \frac{z^{*}}{2} a a\right] \tag{5.28}$$

(We have chosen to put $z/2$ in the squeezing operator as a convention followed in the literature in contrast to Sec 5.1)

We proceed to get the Fock space representation for the state $|n,z,\alpha\rangle$.

We first compute $|n,z\rangle$:

$$|n,z\rangle \equiv S(z)\,|n\rangle \tag{5.29}$$

$$\equiv \sum_m |m\rangle\langle m|\, S(z)\,|n\rangle$$

$$\equiv \sum_m |m\rangle\, G_{mn}^{(z)} \tag{5.30}$$

The method of calculating the expansion coefficients is given in the Appendix A and they are given by

$$
G_{mn}(z) = \begin{cases}
e^{-i(n-m)\theta/2}\,(-)^{m+n/2}\left(\dfrac{m!\,n!}{\cosh r}\right)^{1/2}\left(\dfrac{\tanh r}{2}\right)^{(m+n)/2} \\[2ex]
\quad \times \displaystyle\sum_{\lambda} \frac{\left(-4/\sinh^2 r\right)^{\lambda}}{(2\lambda)!\left[\frac{m}{2}-\lambda\right]!\left[\frac{n}{2}-\lambda\right]!}, \\[2ex]
\qquad\qquad\qquad \text{for } m, n \text{ even} \\[4ex]
e^{-i(n-m)\theta/2}\,(-)^{m+n/2-3/2}\left(\dfrac{m!\,n!}{\cosh^3 r}\right)^{1/2}\left(\dfrac{\tanh r}{2}\right)^{\frac{(m+n)}{2}-1} \\[2ex]
\quad \times \displaystyle\sum_{\lambda} \frac{\left(-4/\sinh^2 r\right)^{\lambda}}{(2\lambda+1)!\left[\frac{m-1}{2}-\lambda\right]!\left[\frac{n-1}{2}-\lambda\right]!}, \\[2ex]
\qquad\qquad\qquad \text{for } m, n \text{ odd} \\[4ex]
0, \text{ otherwise}
\end{cases} \qquad (5.31)
$$

Only odd-odd or even-even elements of $G_{mn}(z)$ survive due to the fact that $S(z)$ essentially creates two excitations every time it acts.

In the spirit of Refs. (1) and (4), we realise that $S(z)$ has a finite expectation value in the state $|n\rangle$ (for $n=0,1,2,\ldots$) i.e., squeezing of the states $|n\rangle$ is possible. We consider an interesting case below:

$$G_{00}(z) \equiv \langle 0|S(z)|0\rangle$$

$$\equiv \frac{1}{(\cosh|z|)^{1/2}} . \qquad (5.32)$$

The authors of Refs.(1) and (11) have wrongly remarked that $G_{00}(z)$ sums as $\tanh|z|<1$ for $n<\infty$. The exact expression for $G_{00}(z)$ is given by equ.(5.32)

Now the g.s.c.s $|n,z,\alpha\rangle$ is given by

$$|n,z,\alpha\rangle \equiv D(\alpha)\,|n;z\rangle$$

$$\equiv D(\alpha)\sum_m |m\rangle\, G_{mn}(z) \qquad (5.33)$$

Using equ.(2.5)

$$|n,z,\alpha\rangle \equiv e^{-|\alpha|^2/2} \sum_{m,l} G_{mn}(z) \left(\frac{m!}{l!}\right)^{1/2}$$

$$\times L_m^{(l-m)}(|\alpha|^2)\, \alpha^{(l-m)}\, |l\rangle \qquad (5.34)$$

Equ.(5.34) gives the Fock space representation of $|n, Z, \alpha\rangle$.

Also, $|n, Z, \alpha\rangle$ could be expanded in terms of the states $|m, Z\rangle$ as illustrated below:

Consider

$$\langle m, Z | n, Z, \alpha\rangle = \langle m, Z | D(\alpha) | n, Z\rangle$$

$$= \langle m | S^{\dagger}(Z) D(\alpha) S(Z) | n\rangle$$

$$= \langle m | \mathrm{Exp}(\alpha b^{\dagger} - \alpha^{*} b) | n\rangle$$

$$= \langle m | \mathrm{Exp}(\gamma a^{\dagger} - \gamma^{*} a) | n\rangle$$

$$= \langle m | n, \gamma\rangle \tag{5.35}$$

where $\gamma = (\alpha \cosh r - \alpha^{*} \sinh r)$ and $b, b^{\dagger}$ are as given by equ.(4.25) and $|n, \gamma\rangle$ is a g.c.s.

From equ.(2.25)

$$|n, Z, \alpha\rangle = \sum_{m} |m, Z\rangle \langle m, Z | n, Z, \alpha\rangle$$

$$= \sum_{m} |m, Z\rangle \langle m | n, \gamma\rangle \tag{5.36}$$

Since $|m, Z\rangle$ is given by

$$|m, Z\rangle = S(Z) |m\rangle \tag{5.37}$$

and

$$\langle m | n, \gamma\rangle = e^{-|\gamma|^2/2} \left(\frac{n!}{m!}\right)^{1/2} L_n^{(m-n)}(|\gamma|^2) \, (\gamma)^{m-n} \tag{5.38}$$

the g.s.c.s $|n,z,\alpha\rangle$ could be written as

$$|n,z,\alpha\rangle = e^{-|\gamma|^2/2} \sum_{m} |m,z\rangle \left(\frac{n!}{m!}\right)^{1/2} L_n^{(m-n)}(|\gamma|^2)\,(\gamma)^{m-n} . \tag{5.39}$$

As an example for a g.s.c.s we give below $|1,z,\alpha\rangle$:

$$|1,z,\alpha\rangle = \frac{e^{-|\alpha|^2/2}}{(\cosh|z|)^{3/2}} \sum_{k,m=0}^{\infty} \left(\frac{z}{2|z|}\tanh|z|\right)^{k} \frac{(2k+1)!}{\sqrt{k!\,m!}}$$

$$\times\; L_{2k+1}^{(m-2k+1)}(|\alpha|^2)\;\alpha^{m-2k-1}\;|m\rangle . \tag{5.40}$$

Overcompleteness of $|n,z,\alpha\rangle$

Since $D(\alpha)$ and $S(z)$ are unitary, for a given α and z the set of states $|n,z,\alpha\rangle, n=0,1,2,\ldots$ forms a complete set just like the set $|n\rangle$. For a given n and z , the set $|n,z,\alpha\rangle$ with all complex α's forms an overcomplete set. We can obtain the resolution of the identity as

$$\mathbb{1} = \int \frac{d^2\alpha}{\pi}\, |n,z,\alpha\rangle\langle n,z,\alpha| . \tag{5.41}$$

In the spirit of equ.(2.21), we can define the squeezed coherent state of the displaced oscillator as

$$|z,\alpha\rangle_{D.O} \equiv \mathrm{Exp}\left[\frac{z}{2}(a^{+}-\alpha^{*})^{2}-\frac{z^{*}}{2}(a-\alpha)^{2}\right]|\alpha\rangle$$

$$\equiv (\cosh|z|)^{-1/2}\sum_{n=0}^{\infty}\left(\frac{z}{2|z|}\tanh|z|\right)^{n}\frac{\sqrt{(2n)!}}{n!}\,|2n,\alpha\rangle. \quad (5.42)$$

The physical interpretation of the g.s.c.s is the same as that of the two-photon coherent state of the radiation field.(12) We can consider the g.s.c.s as a coherent state formed due to two excitations on a particular state $|n\rangle$.

5.3 New Squeezed States

Recently D'Ariano et al(13) have arrived at 'a new type of two-photon squeezed coherent states' using the formalism of Brandt and Greenberg(14) and the method suggested by them overcomes the impossibility result of Fisher et al.(1)

Brandt and Greenberg(14) constructed Bose operators $b_{(k)}$ and $b^{+}_{(k)}$ for k-particles by employing certain kind of renormalization to 'dress' the k-particles:

$$b_{(k)} = \sum_{j=0}^{\infty}\alpha_j^{(k)}(a^{+})^{j}a^{(j+k)} \quad (5.43)$$

and

$$\alpha_j^{(k)} = \sum_{\ell}\frac{(-)^{j-\ell}}{(j-\ell)!}\left[\frac{1+\left[\frac{\ell}{k}\right]}{\ell!\,(\ell+k)!}\right]^{1/2} e^{i\theta_\ell} \quad (5.44)$$

where θ_ℓ arbitrary.

$b_{(k)}$ and $b^{+}_{(k)}$ satisfy the commutation relations

$$\left.\begin{aligned} \left[b_{(k)}, b^{+}_{(k)} \right] &= 1 \\ \left[N, b_{(k)} \right] &= -k\, b_{(k)} \end{aligned}\right\} \tag{5.45}$$

for $N = a^{+}a$.

From the commutation relations it is clear that $b^{+}_{(k)}$ and $b_{(k)}$ act in the Fock space as k-particle creators and annihilators:

$$\left.\begin{aligned} b_{(k)} \left| sk+\lambda \right\rangle &= \sqrt{s}\, \left| (s-1)k+\lambda \right\rangle \\ b^{+}_{(k)} \left| sk+\lambda \right\rangle &= \sqrt{s+1}\, \left| (s+1)k+\lambda \right\rangle \end{aligned}\right\} \tag{5.46}$$

where $0 \leq \lambda \leq k$.

D'Ariano et al (13) have defined the many-photon squeezed coherent state as

$$\left| \alpha, (z,\omega)_{(k)} \right\rangle \equiv D(\alpha)\; S_{(k)}(z,\omega) \left| 0 \right\rangle \tag{5.47}$$

$$S_{(k)}(z,\omega) \equiv \mathrm{Exp}\left(z\, b^{+}_{(k)} + i\omega N - z^{*} b_{(k)} \right) \tag{5.48}$$

for ω real and z complex. The minimum uncertainty product of these states is somewhat similar to that of the usual s.c.s.

5.4 Other Developments

An important problem of mathematical interest is disentangling the squeezing operators. We first give the disentangled form of $S(z)$ and then discuss the methods of obtaining it.

$$\begin{aligned} S(z) &= \mathrm{Exp}\left[\frac{z}{2} a^{\dagger} a^{\dagger} - \frac{z^{*}}{2} a a\right] \\ &= (\cosh|z|)^{-1/2} \, \mathrm{Exp}\left[\frac{z}{2|z|} \tanh|z| \, a^{\dagger 2}\right] \\ &\times \sum_{n=0}^{\infty} \frac{(\operatorname{Sech}|z|-1)^{n}}{n!} (a^{\dagger})^{n} a^{n} \\ &\times \mathrm{Exp}\left[-\frac{z^{*}}{2|z|} \tanh|z| \, a^{2}\right] \end{aligned} \tag{5.49}$$

The above normal ordered form of $S(z)$ could be obtained by 'the most straight forward and obscure'(15) method i.e., by applying the famous McCoy's theorem.(16)

Fisher et al(1) obtained it by using Lie algebra matrix technique, i.e., by the following identification procedure:

$$\left.\begin{aligned} \frac{1}{2} a^{\dagger} a^{\dagger} &\equiv L_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \\ -\frac{1}{2} a a &\equiv L_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \\ \frac{1}{2} a^{\dagger} a + \frac{1}{4} &\equiv L_{3} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}\right\} \tag{5.50}$$

Now we have a realisation of SU(1,1) algebra

$$[L_+, L_-] = 2L_3 \text{ and } [L_3, L_\pm] = \pm L_\pm . \qquad (5.51)$$

Now

$$S(z) = \mathrm{Exp}\left[\begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}\right] \qquad (5.52)$$

and expanding the exponential, the normal ordered form as in equ.(5.49) of $S(z)$ is obtained.

Disentangling of other squeezing operators is more difficult and discussed by Witschel.[17] For example the k-photon squeezing operator

$$S(z_k) = \mathrm{Exp}\left[z_k (a^\dagger)^k - z_k^* a^k\right] \qquad (5.53)$$

is yet to be written in the normal ordered form.

Before we conclude this chapter, few remarks are in order regarding other generalizations of the squeezed states.

Strictly speaking the squeezed states are nothing but the 'intelligent states' well known to group theorists.[19]

In the case of SU(2), the Radcliffe states (Sec 3.1) exhibit squeezing.[20]

Recently the concept of higher-order squeezing (similar to higher-order coherence) has been introduced by Mandel and Hong.[21]

For E_1, E_2 two canonical conjugate operators satisfying

$$[E_1, E_2] = 2ic$$

the field is squeezed to order $2n$ if

$$\langle (\Delta E_1)^{2n} \rangle < (2n-1)!!\, c^n . \tag{5.54}$$

The above definition is in view of the fact that the normally ordered moments of the deviation all vanish for a coherent state. They have also shown that the higher-order squeezing exists in a number of systems like resonance fluorescence.

Appendix A Calculation of $G_{mn}^{(z)}$

$$G_{mn}(z) = \langle m|\, S(z)\, |n\rangle \tag{A.1}$$

Direct method is to use the normal ordered form of (equ.(5.49)) and then take the projection of $S(z)|n\rangle$ on $|m\rangle$.

We use an elegant method due to Rashid[18]

$$G_{k\ell}^{(x)} = \frac{1}{(k!\,\ell!)^{1/2}} \langle 0|\, a^k \, \mathrm{Exp}\left(-\frac{x}{2}(a^{+2} - a^2)\right)(a^+)^\ell |0\rangle \tag{A.2}$$

(for x real)

$$= \sqrt{k!\,\ell!}\; H_{k,\ell}^{(x)} \tag{A.3}$$

Using $[a, a^+] = 1$ and equ.(4.25), the following recursion relations are obtained:

$$\left.\begin{aligned} k\cosh x\, H_{k,\ell}(x) &= -\sinh x\, H_{k-2,\ell}(x) + H_{k-1,\ell-1}(x) \\ \ell\cosh x\, H_{k,\ell}(x) &= \sinh x\, H_{k,\ell-2}(x) + H_{k-1,\ell-1}(x) \end{aligned}\right\} \tag{A.4}$$

and $$H_{1,1}(x) = \frac{1}{\cosh x} H_{0,0}(x) \,. \tag{A.5}$$

We define the generating function

$$H(a,b;x) = \sum_{k,\ell=0}^{\infty} H_{k,\ell}(x)\, a^k b^\ell \tag{A.6}$$

(A.4) and (A.5) give

$$\cosh x \frac{\partial}{\partial a} H(a,b;x) = \left(-a\tanh x + \frac{b}{\cosh x}\right) H(a,b;x) \tag{A.7}$$

and its solution is

$$H(a,b;x) = C(b;x)\,\mathrm{Exp}\left[-\frac{1}{2}a^2\tanh x + \frac{ab}{\cosh x}\right] \tag{A.8}$$

where $$C(b;x) = H(0,b;x) = \sum_{\ell=0}^{\infty} H_{0,2\ell}(x)\, b^{2\ell}$$

Substituting in (A.4)

$$H_{0,2\ell}(x) = \left(\frac{\tanh x}{2}\right)^{\ell} \frac{1}{\ell!} H_{0,0}(x) \tag{A.9}$$

Therefore

$$H(a,b;x) = H_{0,0}(x)\,\mathrm{Exp}\left[\frac{ab}{\cosh x} - \frac{(a^2-b^2)}{2}\tanh x\right] \tag{A.10}$$

To calculate $H_{0,0}$:

$$\frac{d}{dx} H_{0,0}(x) = H_{2,0}(x) \tag{A.11}$$

Also from (A.4)

$$H_{2,0}(x) = -\frac{1}{2} H_{0,0} \tanh x \tag{A.12}$$

From (A.11) and (A.12)

$$H_{0,0}(x) = 1/(\cosh x)^{1/2} \tag{A.13}$$

in view of $H_{0,0}(0) = 1.$ (A.14)

Therefore from (A.10)

$$G_{k,l}(x) = \left\{ \begin{array}{l} (-)^{k/2} \left(\dfrac{k!\,l!}{\cosh x} \right)^{1/2} \left(\dfrac{\tanh x}{2} \right)^{\frac{k+l}{2}} \\ \quad \times \displaystyle\sum_{\lambda} \frac{(-4/\sinh^2 x)^{\lambda}}{(2\lambda)!\,(\frac{k}{2}-\lambda)!\,(\frac{l}{2}-\lambda)!}, \text{ for } k, l \text{ even} \\ (-)^{(k-1)/2} \left(\dfrac{k!\,l!}{\cosh^3 x} \right)^{1/2} \left(\dfrac{\tanh x}{2} \right)^{\frac{k+l}{2}-1} \\ \quad \times \displaystyle\sum_{\lambda} \frac{(-4/\sinh^2 x)^{\lambda}}{(2\lambda+1)!\left[\frac{k-1}{2}-\lambda\right]!\left[\frac{l-1}{2}-\lambda\right]!}, \text{ for } k, l \text{ odd} \\ 0, \text{ otherwise} \end{array} \right\} \tag{A.15}$$

Since we need

$$G_{k,l}(z) = \langle k | \exp\left[\frac{z}{2} a^{+} a^{+} - \frac{z^{*}}{2} a a\right] | l \rangle \tag{A.16}$$

(for z complex)

we redefine a and a^{+} as

$$A = e^{-i\theta/2} a \text{ and } A^{+} = e^{i\theta/2} a^{+} \tag{A.17}$$

where $z = r e^{i\theta/2}$

The commutation relation

$$[A, A^{+}] = 1 \tag{A.18}$$

remains unaffected.

Therefore

$$S(z) = \mathrm{Exp}\left[\frac{r}{2}(A^{+2} - A^{2})\right] \tag{A.19}$$

and by applying the technique described above the $G_{mn}(z)$ coefficients are obtained as in equ.(5.31).

&& && && && &&

&& && && &&

&&

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3) A C^{∞} vector x for an operator T is called an analytic vector if the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} \|T^n x\|$$

has a positive radius of convergence.

(Note: By a C^{∞} vector x for an operator T we mean

$$x \in \bigcap_{n=1}^{\infty} D(T^n)$$, where $D(T^n)$ is the domain of the operator T^n.)

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Definition of Intelligent States

For the SU(2) algebra, the commutation relations are

$$[J_x, J_y] = i J_z \; ; \; x, y, z \text{ cyclic}$$

Using the Schwarz inequality it could be shown that the eigen states of the operator

$$A\Psi = \frac{J_x - i\lambda J_y}{\sqrt{1+\lambda^2}} \Psi = z\Psi$$

$$z = \frac{\langle J_x \rangle - i\lambda \langle J_y \rangle}{\sqrt{1+\lambda^2}} \;, \; \lambda \text{ real}$$

will give the equality

$$(\Delta J_x)^2 (\Delta J_y)^2 = \frac{1}{4} |\langle J_z \rangle|^2.$$

For $\lambda \neq 1$, A is diagonalizable and these states are called the intelligent states.

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CHAPTER 6

SQUEEZED STATES OF HYDROGEN ATOM*

The methodology of the squeezed coherent states described in Chapter 4 comes handy to define squeezed coherent states of other quantum mechanical systems especially where the harmonic oscillator could be employed. In this chapter the dynamical symmetry group of the hydrogen atom namely $O(4)$ is made use of to define its squeezed coherent states. The result is very interesting: the hydrogen atom in a squeezed coherent state has more energy than given by Bohr levels.

The method of approach is along that of Gerry.[1] He introduced oscillator like coherent states on the $O(4)$ algebra of the hydrogen atom and showed that the correct classical limit could be obtained without the correspondence limit. He also indicated that the thus introduced coherent states describe the 'elliptical orbits' anticipated in 1926 by Schrodinger.[2] More recently Bhaumik et al[3] using these coherent states constructed a wave-packet which travels on an elliptic trajectory. In fact all these works are further developments on the realisation of the connection between the hydrogen atom and a four-dimensional oscillator with a constraint which has been rediscovered by many authors[4] since the original discovery by Pauli. (See the elaborate review, incidentally the first such one by Bander and Itzykson[5]). Also Nieto[6] obtained

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*This Chapter is based on

M. Venkata Satyanarayana, J. Phys. A19, (1986) To Appear.

coherent states for the coulomb potential which are different from those introduced by Gerry[1] and Bhaumik et al.[3] Nieto[6] developed a formalism to obtain coherent states for general potentials using Schrodinger's criterion, i.e., to define coherent states as those with undistorted wave packets with classical motion. Coherent states of Gerry[1] and Bhaumik et al[3] are the minimum uncertainty states of the harmonic oscillator.
We like to recall that the harmonic oscillator hamiltonian

$$\mathcal{H} = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right) \tag{6.1}$$

will have energy in a s.c.s $|\alpha, z\rangle$ given by

$$\begin{aligned} \mathcal{H}_{Sq}^{cl} &= \langle \alpha, z | \mathcal{H} | \alpha, z\rangle \\ &= \hbar\omega\left(|\alpha|^2 + \sinh^2|z| + \frac{1}{2}\right) . \end{aligned} \tag{6.2}$$

Now we can define action variables as

$$J = h\left(|\alpha|^2 + \sinh^2|z| + \frac{1}{2}\right) \tag{6.3}$$

and

$$\mathcal{H}_{Sq}^{cl} = \nu J \tag{6.4}$$

Also

$$\nu = \frac{\partial \mathcal{H}_{Sq}^{cl}}{\partial J} . \tag{6.5}$$

The quantized energy levels are recovered by invoking Bohr-Sommerfeld rule

$$J = \hbar\left(n + \sinh^2|z| + \frac{1}{2}\right) \tag{6.6}$$

The above form of quantization condition is chosen in view of the fact that $\mathcal{H}^{cl}_{Sq}$ becomes $\hbar\omega(n+1/2)$ as the squeezing parameter z approaches zero.

We shall now proceed to discuss the s.c.s of the hydrogen atom. The hamiltonian of the hydrogen atom is given by

$$\mathcal{H} = \frac{p^2}{2\mu} - \frac{Ze^2}{r} \tag{6.7}$$

It is very well known that the angular momentum vector $\underset{\sim}{L}$ and Pauli-Runge-Lenz vector $\underset{\sim}{A}'$ given by

$$\underset{\sim}{A}' = -\frac{Ze^2\underset{\sim}{r}}{r} + \frac{1}{2\mu}(\underset{\sim}{L}\times\underset{\sim}{p} - \underset{\sim}{p}\times\underset{\sim}{L}) \tag{6.8}$$

commute with $\mathcal{H}$. Also $\underset{\sim}{A}'$ is orthogonal to $\underset{\sim}{L}$ and

$$A'^2 = \left(\frac{2}{\mu}\right)\mathcal{H}(L^2+\hbar^2) + (Ze^2)^2 \tag{6.9}$$

Using the decomposition $O(4) = SU(2)_a \otimes SU(2)_b$, Schwinger's boson realisation of SU(2) and the properties of $\underset{\sim}{A}'$ Gerry obtained

$$(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)^2 = (b_1^\dagger b_1 + b_2^\dagger b_2 + 1)^2 \;. \tag{6.10}$$

and

$$-\frac{\mu Z^2 e^4}{\hbar^2 E} = (a_1^\dagger a_1 + a_2^\dagger a_2 + 1)^2 + (b_1^\dagger b_1 + b_2^\dagger b_2 + 1)^2 \tag{6.11}$$

where $[a_i, a_i^\dagger] = 1$ and $[b_i, b_i^\dagger] = 1$ for $i = 1, 2$.

Now introducing the Fock space basis i.e., eigen states of $a_i^\dagger a_i$ as $|n_i\rangle$ and $b_i^\dagger b_i$ as $|m_i\rangle$ where $i = 1, 2$ equations (6.10) and (6.11) become

$$n_1 + n_2 = m_1 + m_2 \tag{6.12}$$

and

$$E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2} \tag{6.13}$$

where $n = n_1 + n_2 + 1 = m_1 + m_2 + 1$.

We now introduce the s.c.s. as

$$\left.\begin{aligned} |\alpha_i^a, Z_i^a\rangle &\equiv D(\alpha_i^a)\, S(Z_i^a)\, |n_i = 0\rangle \\ |\alpha_i^b, Z_i^b\rangle &\equiv D(\alpha_i^b)\, S(Z_i^b)\, |m_i = 0\rangle \end{aligned}\right\} \tag{6.14}$$

for $i = 1, 2$ and $Z_i^a = r_i^a$ and $Z_i^b = r_i^b$

The condition in equ.(6.12) reads as

$$|\alpha_1^a|^2 + |\alpha_2^a|^2 + \sinh^2 r_1^a + \sinh^2 r_2^a = |\alpha_1^b|^2 + |\alpha_2^b|^2 + \sinh^2 r_1^b + \sinh^2 r_2^b . \tag{6.15}$$

Now for the squeezing parameters r_i^a and r_i^b for $i=1,2$ tending to zero, the above equation becomes

$$|\alpha_1^a|^2+|\alpha_2^a|^2 = |\alpha_1^b|^2+|\alpha_2^b|^2 \tag{6.16}$$

the one obtained by Gerry[1] as expected.

In the spirit of equ.(6.2) the energy becomes

$$\mathcal{H}_{sq}^{cl} = \frac{-\mu Z^2 e^4}{2\hbar^2(|\alpha_1^a|^2+|\alpha_2^a|^2+\sinh^2 r_1^a+\sinh^2 r_2^a+1)^2} \tag{6.17}$$

Again for $r_1^a = r_2^a = 0$, the above equation becomes

$$\mathcal{H}^{cl} = E = \frac{-2\pi^2\mu k^2}{h^2(|\alpha_1^a|^2+|\alpha_2^a|^2+1)^2} \tag{6.18}$$

the one obtained by Gerry[1] (where $k = Ze^2$).

Further we see that squeezing does not affect the relation for the period of kepler orbit.

Using equ.(6.6) and taking $r_1^a = r_1$ and $r_2^a = r_2$ we get

$$E_{sq}^{(n)} = \frac{-\mu Z^2 e^4}{2\hbar^2(n_1+n_2+\sinh^2 r_1+\sinh^2 r_2+1)^2} \tag{6.19}$$

The above equation gives the energy levels of a hydrogen atom specified by the principal quantum number $n(=n_1+n_2+1)$ and the squeezing parameters r_1 and r_2. It is to be noted that the energy levels acquire higher values, as it happens for the harmonic oscillator.

Taking $r_1=r_2=0$ equ.(6.19) becomes

$$E^{(n)} = \frac{-\mu Z^2 e^4}{2\hbar^2(n_1+n_2+1)^2}, \qquad (6.20)$$

which gives Bohr energy levels.

We can proceed to calculate the absolute shift in energy $E_{sq}^{(n)}-E^{(n)}$:

$$E_{sq}^{(n)}-E^{(n)} = \frac{\mu Z^2 e^4}{2\hbar^2}\frac{(\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1)^2}$$

$$\times \frac{(2n_1+2n_2+2+\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1+\sinh^2 r_1+\sinh^2 r_2)^2} \qquad (6.21)$$

For $\sinh^2 r_1 \ll n_1$ and $\sinh^2 r_2 \ll n_2$,

$$E_{sq}^{(n)}-E^{(n)} = \frac{\mu Z^2 e^4}{2\hbar^2}\frac{(\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1)^4}$$

$$\times(2n_1+2n_2+2+\sinh^2 r_1+\sinh^2 r_2)$$

$$\times\left[1-\frac{2(\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1)}\right]$$

$$= \frac{\mu Z^2 e^4}{2\hbar^2} \frac{(\sinh^2 r_1 + \sinh^2 r_2)}{(n_1+n_2+1)^5}$$

$$\times \left[n_1+n_2+1 - 2\left(\sinh^2 r_1 + \sinh^2 r_2\right)\right]$$

$$\times \left[2(n_1+n_2+1) + (\sinh^2 r_1 + \sinh^2 r_2)\right] \qquad (6.22)$$

The relative shift in energy is given by

$$\frac{E_{sq}^{(n)} - E^{(n)}}{E^{(n)}} = \frac{-(\sinh^2 r_1 + \sinh^2 r_2)\left[2(n_1+n_2+1) + (\sinh^2 r_1 + \sinh^2 r_2)\right]}{(n_1+n_2+1+\sinh^2 r_1 + \sinh^2 r_2)^2} \qquad (6.23)$$

When the squeezing parameters are very small compared to n_1 and n_2 i.e., for $\sinh^2 r_1 \ll n_1$ and $\sinh^2 r_2 \ll n_2$

$$\frac{E_{sq}^{(n)} - E^{(n)}}{E^{(n)}} = \frac{-(\sinh^2 r_1 + \sinh^2 r_2)}{(n_1+n_2+1)^3}$$

$$\times \left[2(n_1+n_2+1) + \sinh^2 r_1 + \sinh^2 r_2\right]$$

$$\times \left[n_1+n_2+1 - 2\left(\sinh^2 r_1 + \sinh^2 r_2\right)\right] \qquad (6.24)$$

The spectral transition energy $\Delta E(n, r_1, r_2 \longrightarrow m, r_1, r_2)$ is given below:

$$
\left.
\begin{aligned}
\Delta E^{(n,m)}_{(r_1,r_2)} = \frac{\mu Z^2 e^4}{2\hbar^2} & \Big[(n_1 + m_1) + (n_2 + m_2) + 2 \\
& \qquad + 2\left(\sinh^2 r_1 + \sinh^2 r_2\right) \Big] \\
& \times \Big[(n_1 - m_1) + (n_2 - m_2) \Big] \\
& \times \frac{1}{\Big[(n_1 + n_2 + \sinh^2 r_1 + \sinh^2 r_2 + 1) \times (m_1 + m_2 + \sinh^2 r_1 + \sinh^2 r_2 + 1) \Big]}
\end{aligned}
\right\} \quad (6.25)
$$

Equ.(6.21) gives the squeezing correction to the spectral transitions.

So by making use of the dynamical symmetry group $O(4)$ it turns out that the squeezed states are available in the $O(4)$ group due to $O(4) \sim SU(2) \otimes SU(2)$. The essential idea of this chapter is that $O(4)$ possesses states which have more energy than the Bohr energy levels.

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CHAPTER 7

ANTIBUNCHED STATES OF RADIATION*

In the recent past the two nonclassical effects of light namely 'squeezing' and 'antibunching' have attracted a lot of theoreticians and experimentalists. So far in this thesis the discussion has been regarding squeezing; in this chapter antibunching and various states of the radiation field which exhibit antibunching are discussed. Antibunching furnishes 'a clear demonstration of the quantum nature of light which is not explained by classical theory'.[1] First we shall give a brief introduction to bunching and antibunching. These ideas arose mainly due to Hanbury-Brown and Twiss experiment,[2] described briefly below:

A quasimonochromatic light beam from a thermal source is divided by a beam splitter into two mutually coherent parts (Fig.1). The delayed coincidences are measured, i.e., we look for those events that the second detector counts a photon τ seconds later the first detector counts a photon. The coincidence rate in the detection response is then plotted against a time delay τ (Fig.2)

The coincidence counting rate as a function of the delay time exhibits a distinct peak at $\tau = 0$. This means the photons have a tendency to arrive in pairs; this tendency is known as 'photon bunching'.

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*Based on

M.H. Mahran and M. Venkata Satyanarayana, Phys. Rev. A (1986) (In Press)

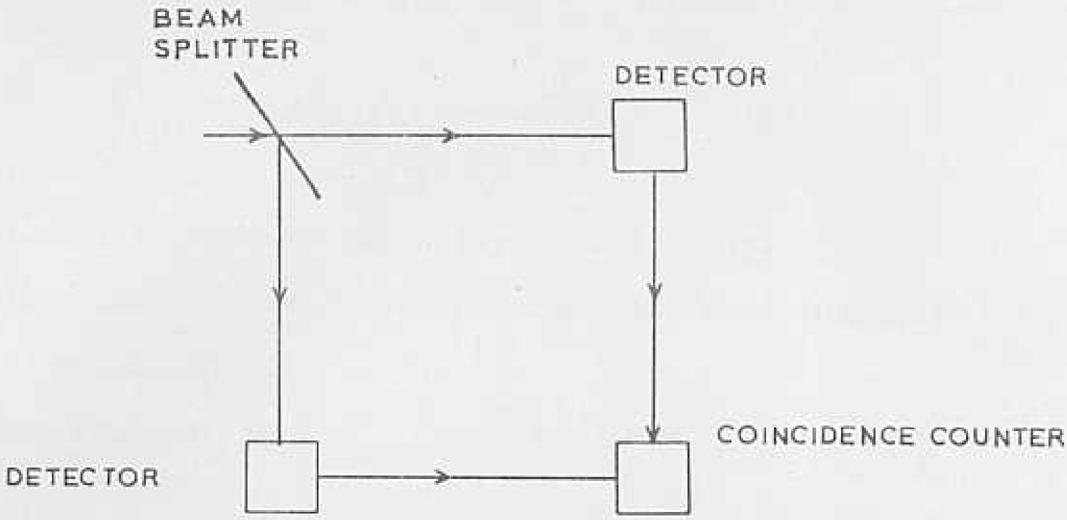

FIG. 1 EXPERIMENTAL SETUP OF HANBURY BROWN AND TWISS EXPERIMENT.

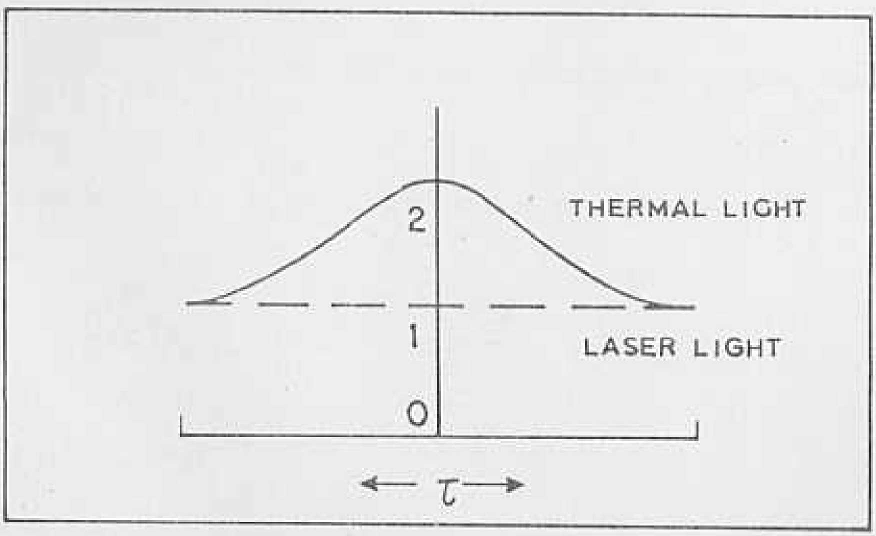

FIG. 2 COINCIDENCE RATE

As indicated in Sec.1.3 the second order correlation function

$$g^{(2)}(0) = \frac{\langle a^{\dagger} a a^{\dagger} a \rangle - \langle a^{\dagger} a \rangle}{\langle a^{\dagger} a \rangle^{2}} \qquad (7.1)$$

provides a measure of such intensity correlations.

So the outcome of the Hanbury Brown and Twiss intensity correlation experiment is

for the thermal light , $g^{(2)}(0) = 2$ (7.2)

and

for the laser light , $g^{(2)}(0) = 1$ (7.3)

A light field (or the Fock space state describing the light field) is said to be antibunched if

$$0 \leq g^{(2)}(0) < 1 \qquad (7.4)$$

It means that the probability of detecting a coincident pair of photons is less than that from coherent field described by a coherent state which has Poisson distribution for the photon counts.

The expression for $g^{(2)}(0)$ could be rewritten as

$$g^{(2)}(0) = 1 + \frac{(\sigma^{2} - \langle n \rangle)}{\langle n \rangle^{2}} \qquad (7.5)$$

where $\sigma^{2} = \langle n^{2} \rangle - \langle n \rangle^{2}$.

So antibunching means

$$\sigma^{2} < \langle n \rangle . \qquad (7.6)$$

For Poisson distribution (corresponding to ideal laser light) $\sigma^2 = \langle n \rangle$ and so $g^{(2)}(0) = 1$ agreeing with equ.(7.3). So antibunched light possesses photon distribution narrower than a Poisson distribution.

Suppose a radiation field has a binomial counting distribution then it must be antibunched since the binomial counting distribution has its mean np and variance npq (for $0 \leqslant q,\ p \leqslant 1$ and $p+q=1$). Very recently the binomial states of the radiation field have been introduced by Stoler et al.[3] They are

$$|\eta, M\rangle = \sum_{n=0}^{M} \beta_n^M |n\rangle \tag{7.7}$$

where

$$\beta_n^M = \left[\binom{M}{n} \eta^n (1-\eta)^{M-n} \right]^{1/2} \tag{7.8}$$

and η and $(1-\eta)$ are the probabilities of the two possible outcomes of a Bernoulli trial. These binomial states reduce to the coherent states and to number states in different limits. The binomial states are antibunched and squeezed for certain parameter ranges.

The method of generating antibunched states has been described by Stoler[4] and there has been a lot of theoretical and experimental activity.[5] A detailed and good review has been given by Paul.[6]

There are many states in the Fock space which are antibunched. For example, the number state $|n\rangle$ is one such since $g^{(2)}(0) = 1 - \frac{1}{n}$ which is a reflection of the fact that the state $|n\rangle$ contains a definite number of photons.

The logarithmic states of the radiation field introduced by Simon and Venkata Satyanarayana[7] (See Sec.4.2) are also antibunched for certain parameter ranges. Using equs.(4.19)to (4.23)

$$g^{(2)}(0) = \frac{-\log(1-q)}{(1-|c|^2)} \tag{7.9}$$

For $q < 1 - e^{-(1-|c|^2)}$, clearly the states are antibunched. Pure logarithmic states (i.e., for $c = 0$) are antibunched for $q < 1 - e^{-1}$ and its $g^{(2)}(0)$ is as in Fig.3.

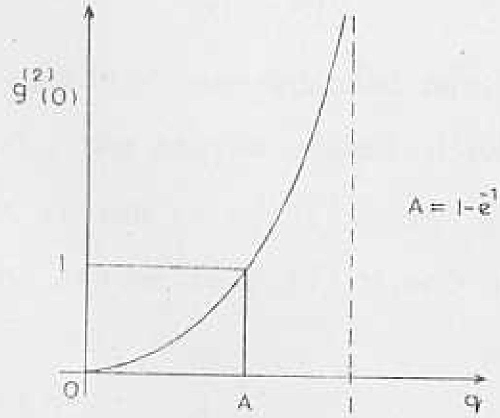

FIG.3 $g^{(2)}(0)$ OF LOGARITHMIC STATES

But the phase states $|\phi\rangle$

$$|\phi\rangle = \mathop{Lt}_{s\to\infty} (s+1)^{-1/2} \sum_{n=0}^{s} e^{in\phi} |n\rangle \tag{7.10}$$

are bunched since

$$g^{(2)}(0) = 4/3 . \tag{7.11}$$

We further proceed to investigate the bunching and antibunching properties of various coherent states since those are the states which are useful for the description of the optical fields.

First we consider the generalized coherent state $|n,\alpha\rangle$ defined in Chapter 2 (equ.2.2)

$$|n,\alpha\rangle \equiv Exp(\alpha a^{+} - \alpha^{*} a)|n\rangle$$

Its $g^{(2)}(0)$ is given by

$$g^{(2)}(0) = 1 + \frac{n(2|\alpha|^2 - 1)}{(|\alpha|^2 + n)^2} \tag{7.12}$$

which means the states $|n,\alpha\rangle$ are bunched only for $2|\alpha|^2 - 1 > 0$ and for $2|\alpha|^2 - 1 < 0$, the states clearly have antibunching. So unlike the coherent states $|\alpha\rangle$ for which $g^{(2)}(0) = 1$, the generalized coherent states (g.c.s) $|n,\alpha\rangle$ have sub-poissonian statistics for

$$2|\alpha|^2 < 1 . \tag{7.13}$$

This means if α were to lie within the unit phase cell around the origin then the corresponding states are antibunched. Here we have an interesting comment regarding the counting statistics of $|n,\alpha\rangle$. 'The appropriate generalizations of the Poisson distribution' as stated by Roy and Virendra Singh(8) also contain sub-poissonian statistics for $2|\alpha|^2<1$.

Next we consider the squeezed coherent states (s.c.s) (See Chapter 4).

$$|\alpha,z\rangle \equiv D(\alpha)\,S(z)\,|0\rangle$$

Its $g^{(2)}(0)$ is given by

$$g^{(2)}(0) = \frac{\left[(\alpha^2 - e^{-i\theta}\sinh r.\cosh r)(\alpha^{*2} - e^{i\theta}\sinh r.\cosh r) + 4|\alpha|^2\sinh^2 r + 2\sinh^4 r\right]}{(|\alpha|^2+\sinh^2 r)^2} \tag{7.14}$$

For $\theta=0$ and α real,

$$g^{(2)}(0) = 1 + \frac{2\sinh^4 r + (2\alpha^2+1)\sinh^2 r - \alpha^2\sinh 2r}{(\alpha^2+\sinh^2 r)^2} \tag{7.15}$$

The state $|\alpha,r\rangle$ is bunched only if the numerator of the second term of equ.(7.15) which could be rewritten as

$$f(\alpha) = \alpha^2(2\sinh^2 r - \sinh 2r) + (2\sinh^4 r + \sinh^2 r) \tag{7.16}$$

is positive.

$f(\alpha)$ is a quadratic expression and its analysis is simple and given below.

The roots of $f(\alpha)$ are

$$\alpha_1, \alpha_2 = \mp \left[\frac{\sinh r(1+2\sinh^2 r)}{2(\cosh r - \sinh r)} \right]^{1/2} \tag{7.17}$$

Case 1: $r > 0$: The roots are real and distinct and the coefficient of α^2 namely $2\sinh r(\sinh r - \cosh r)$ is negative. Therefore for a given value of the squeezing parameter r in order to have an antibunched state, α should be chosen such that $\alpha < \alpha_1$ or $\alpha > \alpha_2$. For $\alpha_1 < \alpha < \alpha_2$, the state $|\alpha, r\rangle$ is bunched.

Case 2: $r < 0$: The roots α_1 and α_2 are purely imaginary quantities and the coefficient of α^2 namely $2\sinh r(\sinh r - \cosh r)$ is positive which means $f(\alpha)$ is positive, and therefore for all values of α we have only bunched states.

The discussion in Case 1 and Case 2 given above are to be compared with Ref.(9) and equ.(6.7) of Yuen.(10) Our results are exact for α (real) and r (real) and fix the exact range of values for α in terms of r whereas the discussion of Walls(9) is based on the limit $|\alpha|^2 \gg \sinh^2 r$.

Now for $\alpha = 0$ i.e., for the squeezed vacuum state $|0, z\rangle$

$$g^{(2)}(0) = 2 + \coth^2 r \tag{7.18}$$

which is to be compared with $g^{(2)}(0) = 2$ for a chaotic light beam in an optical cavity. As remarked in Sec.4.4 equ.(7.18) means that the cavity filling due to squeezing is more bunched than the chaotic light and the counting statistics are similar.

All the above discussed results could be obtained as various special cases of the $g^{(2)}(0)$ of the generalized squeezed coherent states (g.s.c.s) introduced by the author (See Sec.5.2)

$$|n,Z,\alpha\rangle \equiv D(\alpha)\,S(Z)\,|n\rangle$$

and its $g^{(2)}(0)$ is given by

$$g^{(2)}(0) = \frac{\begin{aligned}\Big[&(\alpha^2-(2n+1)\sinh r.\cosh r\,e^{-i\theta})(\alpha^{*2}-(2n+1)\sinh r.\cosh r\,e^{i\theta})\\ &+4|\alpha|^2\sinh^2 r\,(n+1)+\sinh^4 r\,(n+1)(n+2)\\ &+4|\alpha|^2\cosh^2 r.n+\cosh^4 r\,.\,n(n-1)\Big]\end{aligned}}{(|\alpha|^2+\sinh^2 r+n\cosh^2 r)^2} \qquad (7.19)$$

Now,

<u>Case 1</u>: For $Z=0$, the equ.(7.19) becomes equ.(7.12)

<u>Case 2</u>: For $n=0$, the equ.(7.19) becomes equ.(7.14)

<u>Case 3</u>: For $n=0$ and $\alpha=0$, the equ.(7.19) becomes equ.(7.18)

<u>Case 4</u>: For $n=0,\ \theta=0$ and α real, the equ.(7.19) becomes equ.(7.15)

Now we proceed to get the conditions for bunching and antibunching of g.s.c.s.

$$g^{(2)}(0) = 1 + \frac{\begin{aligned}\Big[&\sinh^4 r(n^2+3n+1)+2\alpha^2\sinh^2 r+(4n^2+2n+1)\sinh^2 r.\cosh^2 r\\ &+2n\alpha^2\cosh^2 r-\cosh^4 r-2(2n+1)\alpha^2\sinh r\cosh r\Big]\end{aligned}}{(\alpha^2+\sinh^2 r+n\cosh^2 r)^2} \qquad (7.20)$$

The numerator of the second term of $g^{(2)}(0)$ in the above expression could be written as (for α and r real),

$$F(\alpha) = \alpha^2\left[2(n+1)\sinh^2 r + 2n - (2n+1)\sinh 2r\right] + \sinh^4 r\,(5n^2+4n+2) + (4n^2+1)\sinh^2 r - n \quad (7.21)$$

$$= \alpha^2 f(r) - G(r) \;\; (\text{Say}) \quad (7.22)$$

The roots of $F(\alpha)$ are

$$\alpha_1, \alpha_2 = \mp\left[G(r)/f(r)\right]^{1/2} \quad (7.23)$$

Case 1: $r<0$. In this case $f(r)>0$

Taking $x = \sinh^2 r$, $G(r)$ could be rewritten as,

$$g(x) = n - x(1+4n^2) - x^2(5n^2+4n+2) \quad (7.24)$$

Since $x > 0$, the positive root of $g(x)$ is

$$x_2 = \frac{\left\{(1+4n^2)^2 + 4n(5n^2+4n+2)\right\}^{1/2} - (1+4n^2)}{2(5n^2+4n+2)} \quad (7.25)$$

Now two cases arise, r positive and r negative.

a) For $r<0$, such that $x > x_2$; $g(x)<0$ and α_1 and α_2 are purely imaginary. Since $f(r)>0$, for all α, $F(\alpha)>0$ i.e., the states are bunched.

b) For $r<0$, such that $0<x<x_2$, the sign of $g(x)$ is opposite to that of the coefficient of x^2 i.e., positive. Therefore, α_1 and α_2 are real and distinct. For $\alpha<\alpha_1$ and $\alpha>\alpha_2$, $F(\alpha)>0$, i.e., the states are bunched and for $\alpha_1<\alpha<\alpha_2$, $F(\alpha)<0$ i.e., the states are antibunched.

Case 2: $r>0$

The positive root of $f(r)$ is given by

$$r_2 = \frac{1}{2}\ln\left[\frac{(n-1)+\left\{(n-1)^2+n(3n+2)\right\}^{1/2}}{n}\right] \tag{7.26}$$

Now similar analysis could be done as above in case 1 and the range of values for both the cases are given in Table 1.

To know whether a given $|n,z,\alpha\rangle$ is bunched or antibunched, one should just calculate x_2 (equ.(7.25)) and r_2 (equ.(7.26)) and then look at the Table 1.

To have a feeling for $F(\alpha)$ (equ.(7.21)) we have Fig.4 which gives the behaviour of the functions $f(r)$ (1a and 2a) and $G(r)$ (1b and 2b) for $n=5$ and $n=25$ respectively. At a chosen r the ratio of $G(r)$ to $f(r)$ determines α_1 and α_2.

We note that $f(r)$ is a monotonically decreasing function and the root of $f(r)$ namely r_2 tends to $(\ln 3)/2$, as n tends to infinity. The positive root of $G(r)$ given by equ.(7.25) could also be obtained from the positive zeroes of $G(r)$ from Fig.4.

Just for the sake of reference, x_2 and r_2 are given in Table 2 upto $n=25$, since the exact range of values for r and α are decided by x_2 and r_2. The last column in Table 2 gives the positive roots of $G(r)$ for different values of n.

Finally we would like to discuss why antibunching should be considered as nonclassical. It is due to the nonexistence Glauber-Sudarshan $P(\alpha)$ as a positive definite function. This could be seen from the expression for $g^{(2)}(0)$. From equ.(7.5)

$$g^{(2)}(0) = 1 + \frac{\int P(\alpha)\left(|\alpha|^2 - \langle|\alpha|^2\rangle\right)^2 d^2\alpha}{\left[\int P(\alpha)\,|\alpha|^2\, d^2\alpha\right]^2} \tag{7.27}$$

For antibunching $g^{(2)}(0) < 1$, which means $P(\alpha)$ is highly singular and takes on negative values. $P(\alpha)$ exists as a positive definite function for thermal light and coherent light; whereas $P(\alpha)$ does not exist as a well behaved function for antibunched light.[11]

Range for r		Range for α	
r	$x = \sinh^2 r$	Bunching	Antibunching
$-\infty < r < r_2$	$x > x_2$	$-\infty < \alpha < \infty$	NO
	$0 < x < x_2$	$\alpha < \alpha_1$ and $\alpha > \alpha_2$	NO
		NO	$\alpha_1 < \alpha < \alpha_2$
$r > r_2$	$x > x_2$	NO	$\alpha < \alpha_1$ and $\alpha > \alpha_2$
		$\alpha_1 < \alpha < \alpha_2$	NO

Table 1

Number	x_2	r_2
1	0.15030104	0.40235943
2	0.09999996	0.47030687
3	0.07266194	0.49556065
4	0.05669263	0.50862217
5	0.04637483	0.51658773
6	0.03919661	0.52194834
7	0.03392628	0.52580166
8	0.02989722	0.52870417
9	0.02671907	0.53096914
10	0.02414943	0.53278542
11	0.02202909	0.53427505
12	0.02025004	0.53551769
13	0.01873623	0.53657103
14	0.01743253	0.53747463
15	0.01629819	0.53825855
16	0.01530223	0.53894520
17	0.01442089	0.53955173
18	0.01363534	0.54009056
19	0.01293089	0.54057360
20	0.01229556	0.54100800
21	0.01171971	0.54140186
22	0.01119535	0.54175959
23	0.01071585	0.54208660
24	0.01027571	0.54238605
25	0.00987029	0.54266214

Table 2

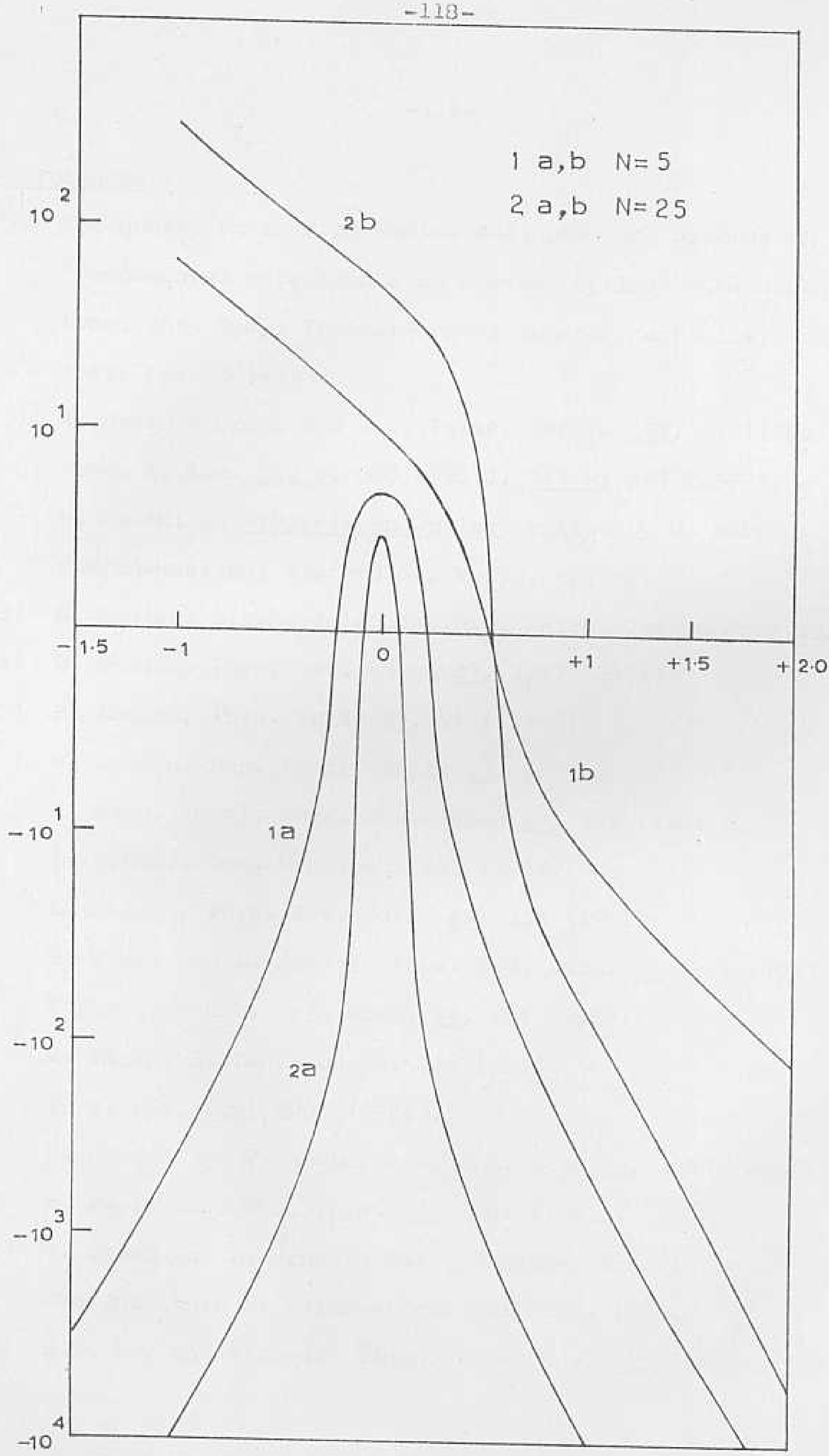

FIG.4. Behavior of f(r) (1a and 2a) and G(r) (1b and 2b) for n = 5 and 25, respectively.

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11) For thermal light:

$$P_n = \frac{1}{1+\langle n\rangle}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n$$

and

$$P(\alpha) = \frac{1}{\pi\langle n\rangle} e^{-|\alpha|^2/\langle n\rangle}$$

For laser light:

$$P_n = e^{-\langle n\rangle} \frac{\langle n\rangle^n}{n!}$$

and

$$P(\alpha) = \delta(\alpha - \alpha_0)$$

ADDENDUM

SCALING PROPERTY OF COUNTING DISTRIBUTIONS*

The quantized versions of Maxwell's equations are enough to do the 'practical electrodynamics', i.e., to calculate fields. If we stop here then we fail to get complete details regarding the radiation field. To have a knowledge of the dynamics of the radiation field we require the counting distribution of the radiation field. This is so because light is a 'randomly fluctuating excitation'.(1) Again depending on the photon counting distribution of the radiation field, we realise a particular class of states to be more useful to work with, in a given context. For example the chaotic light possesses geometric counting distribution, which is determined by the Principle of Maximum Entropy. On the other hand the coherent states have the counting distribution to be Poisson. These states could be termed as 'classical' for reasons mentioned earlier in this thesis.

There are other nonclassical states which have different counting distributions exhibiting squeezing and antibunching (Chapters 5,6 and 7) like binomial states, logarithmic states and squeezed states. In this note we briefly discuss an important property known as '<u>asymptotic scaling</u>' associated with certain counting distributions in quantum optics.

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*Based on

B.A. Bambah and M. Venkata Satyanarayana, Preprint, The Institute of Mathematical Sciences, 1986.

Definition: A counting distribution $\{P_n\}_{n=0}^{\infty}$ is said to admit scaling form (or asymptotic scaling) if

$$\mathop{\mathcal{L}t}_{\substack{n\to\infty \\ \langle n\rangle\to\infty}} \langle n\rangle P_n = \Psi\left(\frac{n}{\langle n\rangle}\right) = \Psi(z) \tag{1}$$

where $z = n/\langle n\rangle$ and Ψ is some function.

The asymptotic scaling form of counting distributions was first known to workers in quantum optics. Later it was introduced in the study of hadronic multiplicity distributions, now known as KNO scaling.(2)

The generalized Bose-Einstein distribution (See page 65) has a scaling form

$$\Psi(z) = \frac{k^k}{(k-1)!} z^{k-1} e^{-kz} \tag{2}$$

Naturally the chaotic light has a scaling form.

Coherent light does not scale because the Poisson distribution does not scale.

In this note it is shown that the pure squeezed states have the asymptotic scaling form.

Pure Squeezed States

The counting distribution is given by

$$\left.\begin{aligned} P_{2n} &= \frac{1}{\cosh r}\left(\frac{\tanh^2 r}{4}\right)^n \binom{2n}{n} \\ \text{and} & \\ P_{2n+1} &= 0 \end{aligned}\right\} \tag{3}$$

Also, $\langle n \rangle = \sinh^2 r$ (4)

and

$$\langle n \rangle P_n = \frac{2\langle n \rangle}{(1+\langle n \rangle)^{1/2}} \left(\frac{\langle n \rangle}{4(1+\langle n \rangle)}\right)^n \frac{(2n)!}{n!\,n!} \,. \tag{5}$$

As $n \longrightarrow \infty$ and $\langle n \rangle \longrightarrow \infty$ and making use of Stirling's approximation equ.(5) tends to

$$\Psi(z) = \frac{2}{\sqrt{\pi z}}\, e^{-z}, \text{ for } z = \frac{n}{\langle n \rangle} \,, \tag{6}$$

which is the asymptotic scaling form of a pure squeezed state.

Now we would like to point out the similarities between a pure squeezed state and a thermal state. Though the counting distributions are different for these two states, as already pointed out earlier, their second-order correlation functions behave somewhat identically. Both the states are bunched; in fact pure squeezed states are more bunched than a chaotic state. Both the states admit asymptotic scaling forms; for the thermal light $\langle n\rangle P_n$ goes as e^{-z} whereas for a purely squeezed light $\langle n\rangle P_n$ goes as $e^{-z}/\sqrt{z}$.

It appears ironical that while a pure squeezed state has a counting distribution 'similar' to black body distribution, squeezed coherent states are to be attributed entirely nonclassical aspects. But we must realise that squeezing defies a classical description in the sense that the Glauber-Sudarshan function fails to exist as a 'well behaved function'.

What the scaling form amounts to is as $\langle n\rangle$ is varied (it could be done by changing the counting time of the photon detector) the P_n attains a kind of stationarity for large values of n and $\langle n\rangle$.

The counting distributions which admit asymptotic scaling form arising in quantum optics have been made use of in constructing quantum optical models for particle production in hadron-hadron collisions and they have been highly successful.[3]

Biyajima and Suzuki[4] have discussed many other counting distributions and their scaling forms.

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KNO scaling states that at asymptotic energies (high energies)

$$\Psi(z) = \langle n \rangle \left(\frac{\sigma_n}{\sum_n \sigma_n} \right)$$

where $\Psi(z)$ is a universal function for a given type of reaction and σ_n is the n-prong cross section. Also, $\left(\sigma_n \Big/ \sum_n \sigma_n\right)$ is proportional to P_n, the probability for the production of n particles.

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COMMENT

A note on the contraction of Lie algebras

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Abstract. It is demonstrated that the contraction of Lie algebras can be viewed as the procedure that underlies limiting distributions in probability theory. The consequences of such an interpretation are discussed.

The contraction of Lie algebras, i.e. the method of obtaining one Lie algebra from another Lie algebra (usually non-isomorphic) by means of a limiting procedure, was originally introduced by Inonu and Wigner (1953) and later developed by Saletan (1965) and it is discussed elaborately by many authors (Venkatesan 1967, Gilmore 1974, Barut and Raczka 1980). Arecchi *et al* (1972) employed this method to obtain the harmonic oscillator coherent states from the so-called atomic coherent states. In this comment we shall explicitly show, using two examples, that the procedure involved in the contraction of Lie algebras is closely related to the well known method of obtaining one probability distribution from another probability distribution involving a limiting procedure.

The coherent states of the harmonic oscillator algebra (Klauder and Sudarshan 1968) (also known as Heisenberg–Weyl algebra) and angular momentum algebra (also known as SU(2) algebra) (Radcliffe 1971, Arecchi *et al* 1972) have been defined by different people in different contexts. The coherent state representation of Lie groups is also well studied (Hioe 1974, Onofri 1975). Coherent states that arise from the Heisenberg–Weyl algebra are known to be the eigenstates of the destruction operator a:

$$a|z\rangle = z|z\rangle \tag{1}$$

where z is a complex number and its Fock space representation is given by

$$|z\rangle = \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle. \tag{2}$$

These states have many interesting properties and applications, especially in quantum optics (Klauder and Sudarshan 1968). Also

$$f_n(z) = |\langle n|z\rangle|^2 = \exp(-|z|^2)\frac{(|z|^2)^n}{n!} \tag{3}$$

gives the probability that there are n photons in the coherent state $|z\rangle$. Due to equation (3), $|z\rangle$ is known as the Poissonian superposition of number states $|n\rangle$.

Coherent states of angular momentum are defined as (Radcliffe 1971, Arecchi *et al* 1972, Hioe 1974)

$$|\mu\rangle = \frac{1}{(1+|\mu|^2)^j} \sum_{p=0}^{2j} \binom{2j}{p}^{1/2} \mu^p|p\rangle \tag{4}$$

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where μ is a complex number and $|p\rangle$ are the projections of a single angular momentum j.

Owing to apparent similarities in the treatment of coherent states of the harmonic oscillator and angular momentum, Arecchi *et al* (1972) showed, by using contraction of Lie algebras, that the angular momentum coherent states go over to the harmonic oscillator coherent states. Not much has been known about the meaning of this contraction procedure. We propose that this limiting procedure could be *understood* in the language of probability theory as illustrated below. Now

$$\pi_p(\mu) = |\langle p|\mu\rangle|^2 = \binom{2j}{p}(|\mu|^2)^p(1+|\mu|^2)^{-2j} \tag{5}$$

gives the probability that a system described by the coherent state $|\mu\rangle$ is in the projected state $|p\rangle$ which is a binomial distribution.

Taking $|\mu|^2 = |z|^2/2j$ equation (5) can be written as

$$\pi_p(\mu) = \frac{(2j)(2j-1)\ldots(2j-p+1)}{[1+(|z|^2)/2j]^{2j}} \frac{(|z|^2)^p}{p!(2j)^p}. \tag{6}$$

If in equation (6) we keep z fixed and let j tend to infinity then $\pi_p(\mu) \to f_p(z)$ of equation (3). This is the so called Holstein–Primakoff (1940) limit used by Arecchi *et al* (1972). Thus the *contraction of Lie algebras* used by Arecchi *et al* (1972) entails a *contraction of probability distributions*. Here we mean that the 'contraction of probabilities' is the limit involved in one distribution going over to another distribution.

In the case of SU(1, 1) the commutation relations are specified by

$$\begin{aligned}
&[J_3, J_\pm] = \pm J_\pm \\
&[J_-, J_+] = 2J_3
\end{aligned} \tag{7}$$

where $J_\pm$ are the ladder operators. Using Perelemov's definition (Barut and Girardello 1971, Perelemov 1977) the coherent states are defined as

$$|n, z, \mu\rangle = \sum_{k=0}^{\infty} \binom{n+k-1}{k-1}^{1/2} z^n \mu^k |k\rangle \tag{8}$$

where z and μ are complex and related by $|z|^2 + |\mu|^2 = 1$. The associated probability distribution is given by

$$P_k(\mu) = \binom{n+k-1}{k-1}(|z|^2)^n(|\mu|^2)^k \tag{9}$$

which is negative binomial.

If in equation (9) we let $|\mu|^2$ tend to zero, n tend to infinity and $|\mu|^2 n$ tend to λ then $P_k(\mu) \to f_k(\lambda)$ of equation (3). This is the same limiting procedure employed by Barut and Girardello (1971). So the contraction of SU(1, 1) to Heisenberg–Weyl algebra is the same as the contraction of the negative binomial distribution associated with SU(1, 1) to the Poisson distribution associated with Heisenberg–Weyl algebra.

This relationship between Lie algebraic contraction and 'contraction of probabilities' could be extended to the general theory of contraction of Lie algebras. Probability distributions could be associated with arbitrary Lie algebras via defining coherent states as shown by Perelemov (1977). This makes the study of Lie algebras interesting in the same way as special functions associated with them (Miller 1968). The connection between 'contraction of probabilities' and the study of Cayley–Klein geometries (Sanjuan 1984) via the group contraction procedure will be published elsewhere.

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Impossibility of squeezed coherent states for a para-Bose oscillator

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In this note we show that squeezed coherent states do not exist for the para-Bose oscillator.

I. INTRODUCTION

In a recent paper, Fisher, Nieto, and Sandberg[1] discussed the question of obtaining generalized squeezed coherent states in the Hilbert space of the simple harmonic oscillator. These are obtained by exponentials of polynomials in $\hat{a}$ and $\hat{a}^\dagger$ acting on the vacuum $|0\rangle$ as follows:

$$|\alpha, Z_k\rangle = D(\alpha) S(Z_k)|0\rangle \quad , \tag{1}$$

where

$$S(Z_k) = \exp[Z_k(\hat{a}^\dagger)^k - Z_k^*(\hat{a})^k] \tag{2}$$

and

$$D(\alpha) = \exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}] \quad , \tag{3}$$

where Z_k is the complex squeeze factor.

Fisher *et al.*[1] proved that squeezed coherent states exist only for $k=2$, which are the familiar two-photon coherent states.[2] For $k>2$, the squeezed coherent states do not exist. The method of their proof is to consider the vacuum expectation value of

$$U_k(Z) = \exp[Z_k(\hat{a}^\dagger)^k - Z_k^*(\hat{a})^k] \quad , \tag{4}$$

which clearly diverges for $k>2$. For $k=2$, it marginally converges. This shows that for $k>2$, $U_k(Z)$ though unitary is unbounded in the Hilbert space of the simple harmonic oscillator.

In this Comment we extend the result to the para-Bose case, where we prove that squeezed coherent states do not exist for all orders of statistics.

II. PARA-BOSE COHERENT STATES

The para-Bose oscillator satisfies the equation of motion

$$[\hat{A}, \hat{N}] = \hat{A} \tag{5}$$

and does *not* satisfy the canonical commutation relation

$$[\hat{A}, \hat{A}^\dagger] = 1 \quad . \tag{6}$$

$\hat{N}$ is defined as

$$\hat{N} = \tfrac{1}{2}(\hat{A}^\dagger\hat{A} + \hat{A}\hat{A}^\dagger) - h_0 \quad , \tag{7}$$

where h_0 is the lowest eigenvalue of the Hamiltonian

$$\hat{H} = \tfrac{1}{2}(\hat{A}^\dagger\hat{A} + \hat{A}\hat{A}^\dagger) \quad . \tag{8}$$

The case $h_0 = \frac{1}{2}$ corresponds to the standard oscillator.

Generally $2h_0$ is defined as the order of the statistics of the para-Bose oscillator.

It can be easily seen that the operators $\hat{A}$, $\hat{A}^\dagger$, and $\hat{N}$ (or $\hat{H}$) do not close and form an algebra. However, $\hat{A}^2$, $\hat{A}^{\dagger 2}$, and $\hat{N}$ close and the algebra is the algebra of the Lorentz group SO(2,1). Since the spectrum of $\hat{N}$ is positive definite, one can obtain the Fock representation[3,4]

$$(\hat{A})_{2n,2n+1} = [2(n+h_0)]^{1/2} \tag{9}$$

and

$$(\hat{A})_{2n-1,2n} = (2n)^{1/2} \quad . \tag{10}$$

When $h_0 = \frac{1}{2}$, we get the usual representation of Bose oscillators and the distinction between the even and odd matrix elements disappears. We thus get an infinite spectrum $h_0, h_0+1, h_0+2, \ldots, \infty$, for every value of h_0.

The physical meaning of the para-Bose oscillator may be understood if we see the Green's ansatz[5]

$$\hat{A} = \sum_{j=1}^{2h_0} \hat{b}^{(j)} \quad , \tag{11}$$

where the $\hat{b}$'s are the usual annihilation operators for the Bose oscillators for a given j, but anticommute for different j values. Though this representation is known to be reducible, it brings out the physical meaning of $\hat{A}$ as a collection of Bose oscillators and thus an extra degree of freedom is introduced.

Coherent states of a para-Bose oscillator have been defined by Sharma, Mehta, and Sudarshan[6] and Biswas and Santhanam.[7] Considering the displacement operator $D(\alpha) = \exp[\alpha\hat{A}^\dagger - \alpha^*\hat{A}]$ acting on the vacuum $|0\rangle_{h_0}$, Sharma *et al.*[6] obtained coherent state $|\alpha\rangle_{h_0}$ as

$$|\alpha\rangle_{h_0} = \sum_{n=0}^{\infty} a_n |n\rangle_{h_0} \quad , \tag{12}$$

where

$$a_n = \left[\frac{\Gamma(h_0)}{2^n \Gamma\left(\left[\frac{n}{2}\right]+1\right)\Gamma\left(\left[\frac{n+1}{2}\right]+h_0\right)}\right]^{1/2} \alpha^n a_0 \tag{13}$$

and

$$a_0 = \left[\sum_{n=0}^{\infty} \frac{\Gamma(h_0)}{\Gamma\left(\left[\frac{n}{2}\right]+1\right)\Gamma\left(\left[\frac{n+1}{2}\right]+h_0\right)} (\tfrac{1}{2}|\alpha|^2)^n\right]^{-1/2} \quad . \tag{14}$$

The same coherent states have also been obtained by Biswas and Santhanam[7] by using the differential operator representation for the annihilation operator. Biswas and Santhanam also obtained

$$ {}_{h_0}\langle 0|\hat{A}^{m}\hat{A}^{\dagger m}|0\rangle_{h_0} = \frac{2^{m}\Gamma\left(\left[\frac{m}{2}\right]+1\right)\Gamma\left(\left[\frac{m+1}{2}\right]+h_0\right)}{\Gamma(h_0)} $$

$$ = c_m \quad \text{(say)} \ . \tag{15} $$

III. PARA-BOSE SQUEEZED COHERENT STATES

Proceeding along the lines of Fisher *et al.*[1] we consider

$$ U_k(Z) = \exp[Z_k(\hat{A}^{\dagger})^{k} - Z_k^{*}(\hat{A})^{k}] \tag{16} $$

and

$$ {}_{h_0}\langle 0|U_k(Z)|0\rangle_{h_0} = 1 - |Z_k|^2\,{}_{h_0}\langle 0|\hat{A}^{k}\hat{A}^{\dagger k}|0\rangle_{h_0} + \cdots $$

$$ = 1 - |Z_k|^2\frac{T_1}{2} + |Z_k|^4\frac{T_2}{4!} + \cdots $$

$$ + (-1)^{n}|Z_k|^{2n}\frac{T_n}{(2n)!} + \cdots \ , \tag{17} $$

where $T_1 = C_k$, $T_2 = C_k^{\,2} + C_{2k}$, etc.

In general, as in the case of the usual Bose oscillator each T_n has many terms, all are positive and the largest is of the order of C_{kn} and indeed the others are also of the same order, as all of them arise from the vacuum expectation value of a polynomial of degree k in $\hat{A}^{\dagger}$ and $\hat{A}$.

Now, we write the leading term of the nth term of the expansion on the right-hand side of Eq. (17) as

$$ t_n = \frac{|Z_k|^{2n}}{(2n)!}\,\frac{2^{nk}\Gamma\left(\left[\frac{nk}{2}\right]+1\right)\Gamma\left(\left[\frac{nk+1}{2}\right]+h_0\right)}{\Gamma(h_0)} \ . \tag{18} $$

Case 1: Take $h_0 = \frac{1}{2}$, then

$$ t_n = \frac{|Z_k|^{2n}}{(2n)!}\,\frac{2^{nk}\Gamma\left(\left[\frac{nk}{2}\right]+1\right)\Gamma\left(\left[\frac{nk+1}{2}\right]+\frac{1}{2}\right)}{\Gamma(\frac{1}{2})} \ . \tag{19} $$

Now two cases arise: $nk = 2m$ and $2m+1$. In both cases we get

$$ t_n = \frac{|Z_k|^{2n}}{(2n)!}(nk)! \ , \tag{20} $$

which is obtained by Fisher *et al.*[1] as is expected, since $h_0 = \frac{1}{2}$ corresponds to the standard oscillator. Now for $k = 2$ we get the usual two-photon coherent states.

Case 2: Take $h_0 = 1$. Again two cases arise: $nk = 2m$ and $2m+1$.

(i) $nk = 2m$:

$$ t_n = \frac{|Z_k|^{2n}}{(2n)!}2^{2m}\Gamma(m+1)\Gamma(m+1) $$

$$ = \frac{|Z_k|^{2n}}{(2n)!}2^{2m}(m!)^2 \ . \tag{21} $$

It is easy to see that $\sum(-)^{n}t_n$ diverges.

(ii) $nk = 2m+1$:

$$ t_n = \frac{|Z_k|^{2n}}{(2n)!}2^{2m+1}\Gamma(m+1)\Gamma(m+2) \ . \tag{22} $$

Again it is easy to see that $\sum(-)^{n}t_n$ diverges.

Now we consider the special case $h_0 = 1$ and $k = 2$, i.e., corresponding to the two-para-Bose coherent state.

We have

$$ t_n = \frac{|Z|^{2n}}{(2n)!}2^{2n}(n!)^2 \ . \tag{23} $$

Consider

$$ f(z) = \sum_{n=0}^{\infty}(-1)^{n}|Z|^{2n}\frac{2^{2n}}{\binom{2n}{n}} \ . \tag{24} $$

Since the series defined by Eq. (24) is an alternating series where each t_n is positive, we can apply Leibnitz's test[8] to test the convergence of $f(z)$. We note that $t_n \nrightarrow 0$ as $n \to \infty$ and $|t_n|$ is not a decreasing sequence. Therefore $f(z)$ is divergent, which proves the nonexistence of two-para-Bose coherent states.

Similarly, it could be proved that squeezed coherent states do not exist for all orders of statistics.

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Generalized coherent states and generalized squeezed coherent states

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Roy and Virendra Singh showed that the harmonic oscillator possesses an infinite string of exact shape-preserving coherent wave-packet states $|n,\alpha\rangle$ having classical motion. In this paper it is shown that the states $|n,\alpha\rangle$ could be obtained from the coherent state $|\alpha\rangle$ and it is also shown how a coherent state $|\alpha\rangle$ could be expanded in the basis of $|n,\alpha\rangle$'s. Further, the possibility of "squeezing" the state $|n\rangle$ is investigated and the "generalized squeezed coherent states" are obtained. The squeezed coherent states for the displaced oscillator are also defined. The physical meaning of squeezing is also pointed out.

I. INTRODUCTION

It is well known that Schrödinger's original motivation[1] for introducing coherent states was to look for those states with probability-density wave packet remaining unchanged in shape as time progresses and have the classical motion

$$\langle x\rangle = x_{\rm cl}(t) \equiv A\cos(\omega t+\varphi) \tag{1}$$

and

$$\langle p\rangle = p_{\rm cl}(t) \equiv M\dot{x}\ , \tag{2}$$

where

$$A=|\alpha|\left[\frac{2\hbar}{M\omega}\right]^{1/2} \tag{3}$$

and

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

is a coherent state (CS).

Nieto[2] *et al.* have used Schrödinger's criterion to define coherent states for arbitrary potentials. (For the other criteria of defining coherent states see Ref. 3.)

Recently, Roy and Virendra Singh[4] showed, by adopting Schrödinger's criterion, i.e., to define coherent states as those with *undistorted normalizable wave packets with classical motion,* that the harmonic oscillator possesses an infinite string of coherent states, hitherto not thought of. Originally they were known as "semicoherent states" introduced by Boiteux and Levelut.[4] We briefly discuss their coherent states below.

Define

$$U(\alpha(t)) \equiv \exp[\alpha(t)a^\dagger - \alpha^*(t)a]\ . \tag{4}$$

Then the "generalized coherent states" (GCS's) of the harmonic oscillator are

$$|n,\alpha\rangle \equiv U(\alpha(t))|n\rangle\exp[-i(n+\tfrac{1}{2})\omega t]\ ,\qquad n=0,1,2,\ldots\ , \tag{5}$$

where $|n\rangle$ is the nth state of the harmonic oscillator. It is easy to see that the state $|n,\alpha\rangle$ satisfies Schrödinger's criterion.

The Fock-space representation of $|n,\alpha\rangle$ is found to be

$$|n,\alpha\rangle = \exp[-\tfrac{1}{2}|\alpha(0)|^2]\sum_{m=0}^{\infty}\left[\frac{n!}{m!}\right]^{1/2}L_n^{(m-n)}(|\alpha(0)|^2)[\alpha(0)]^{m-n}|m\rangle\exp[-i\omega t(m+\tfrac{1}{2})]\ , \tag{6}$$

where $L_n^{(m-n)}(x)$ are Laguerre polynomials.

Also, the uncertainty in the state $|n,\alpha\rangle$ is given by

$$\Delta x\,\Delta p = (n+\tfrac{1}{2})\hbar\omega\ . \tag{7}$$

The above equation implies that the minimum uncertainty (i.e., $\hbar\omega/2$) is not necessary for the classical motion of a wave packet. This fact has been also noted by Ohnuki and Kamefuchi.[5]

The states $|n,\alpha\rangle$, though not explicitly stated, could be spotted in the literature.[6-11] The states $|n,\alpha\rangle$ could also be obtained when one considers the Hamiltonian[6]

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}q^2+qf(t)\ , \tag{8}$$

where $f(t)$ is an external force. If the driving term is taken to be linear in a and $a^\dagger$, then one obtains the states $|n,\alpha\rangle$.

If one considers $\langle m|n,\alpha\rangle$, then

$$|\langle m|n,\alpha\rangle|^2 = e^{-|\alpha|^2}\left[\frac{n!}{m!}\right]|\alpha|^{2(m-n)}[L_n^{(m-n)}(|\alpha|^2)]^2\ . \tag{9}$$

It has been shown by Koonin[7] that Eq. (9) is related to the S-matrix element S_{mn} by

$$|S_{mn}|^2=|\langle m|n,\alpha\rangle|^2 . \tag{10}$$

S_{mn} gives the amplitude for excitation from the initial oscillator state $|n\rangle$ to the final $|m\rangle$.

Also, Hollenhorst[8] has proved that Eq. (9) gives the matrix element for a transition from the state $|n\rangle$ to the state $|m\rangle$ under the influence of a gravity wave.

Equation (9) is also known as Schwinger's formula,[9] and is also given by Feynman.[10] (See also Refs. 11–13.)

Section II discusses the relationship between GCS's and CS's and in Sec. III generalized squeezed coherent states are introduced.

II. RELATIONSHIP BETWEEN GCS AND CS

We are interested in obtaining $|n,\alpha\rangle$ from $|\alpha\rangle$. Let

$$|n,\alpha\rangle=A(a,a^\dagger,n)|\alpha\rangle \tag{11}$$

$$\equiv A(a,a^\dagger,n)U(\alpha)|0\rangle , \tag{12}$$

where $A(a,a^\dagger,n)$ is the operator to be determined. Also

$$|n,\alpha\rangle\equiv U(\alpha)|n\rangle \tag{13}$$

$$\equiv U(\alpha)\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle . \tag{14}$$

From Eqs. (12) and (14)

$$A=U\frac{(a^\dagger)^n}{\sqrt{n!}}U^\dagger . \tag{15}$$

Since $U(\alpha)$ translates a and $a^\dagger$, it could be proved using operator calculus[14] that

$$A=\frac{(a^\dagger-\alpha^*)^n}{\sqrt{n!}} . \tag{16}$$

Therefore,

$$|n,\alpha\rangle=\frac{(a^\dagger-\alpha^*)^n}{\sqrt{n!}}|\alpha\rangle . \tag{17}$$

Now, the meaning of the state $|n,\alpha\rangle$ is very clear as the nth state of the oscillator whose ground state is $|\alpha\rangle$, a coherent state, *not* $|0\rangle$, as in the case of the usual oscillator. In other words, the GCS's are the excited states of the displaced oscillator.

The above result is clearly depicted in the following diagram:

$$\begin{tikzcd}
|\alpha\rangle \arrow[d, "A"'] & |0\rangle \arrow[l, "U(\alpha)"'] \arrow[d, "\frac{(a^\dagger)^n}{\sqrt{n!}}"] \\
|n,\alpha\rangle & |n\rangle \arrow[l, "U(\alpha)"]
\end{tikzcd}$$

$|0\rangle$: ground state of the harmonic oscillator,

$|\alpha\rangle$: ground state of the displaced harmonic oscillator .

The above method of obtaining GCS's using Eq. (17) not only establishes the relationship between GCS's and CS's and also gives a simple algebraic way to obtain the GCS's.

One can also see that the GCS $|n,\alpha\rangle$ is the "generalized coherent state" in the sense of Perelemov[11], for whom the reference state could be an arbitrary vector in the Fock space.

Since

$$a^\dagger|\alpha\rangle=\left[\frac{\partial}{\partial\alpha}+\frac{\alpha^*}{2}\right]|\alpha\rangle , \tag{18}$$

Eq. (17) could be given a differential operator representation as

$$|n,\alpha\rangle=\frac{1}{\sqrt{n!}}\sum_{m=0}^{n}\binom{n}{m}(-\alpha^*)^{n-m}\left[\frac{\partial}{\partial\alpha}+\frac{\alpha^*}{2}\right]^m|\alpha\rangle . \tag{19}$$

Therefore, we observe that the GCS $|n,\alpha\rangle$ is related to the CS $|\alpha\rangle$ just the number state $|n\rangle$ is related to the vacuum state $|0\rangle$.

Now, we illustrate below an interesting use of GCS's.

The displacement operators $U(\alpha)$'s provide a complete and orthonormal basis for the adjoint group of the Weyl group formed by $a,a^\dagger,\mathbf{1}$ with a scalar product given by

$$(U(\alpha),U(\alpha'))=\mathrm{Tr}[U(\alpha)U^\dagger(\alpha')]=\pi\delta(\alpha-\alpha') \tag{20}$$

and

$$U(\alpha)U(\beta)=\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]U(\alpha+\beta) . \tag{21}$$

In view of Eqs. (20) and (21), we think of defining coherent states of the displaced oscillator as

$$|z,\alpha\rangle\equiv\exp[z(a^\dagger-\alpha^*)-z^*(a-\alpha)]|\alpha\rangle \tag{22}$$

$$=\exp[z^*\alpha-z\alpha^*]U(z)|\alpha\rangle$$

$$=\exp[z^*\alpha-z\alpha^*]e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}U(z)|n\rangle . \tag{23}$$

Using the relation (6), we get

$$|z,\alpha\rangle\equiv e^{(z^*\alpha-z\alpha^*)}e^{-|\alpha|^2}\sum_{n,m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}L_n^{(m-n)}(|\alpha|^2)|m\rangle . \tag{24}$$

Equation (21) could be written as

$$|z,\alpha\rangle\equiv\exp(z^*\alpha-z\alpha^*)U(z)U(\alpha)|0\rangle . \tag{25}$$

In view of Eq. (21), $|z,\alpha\rangle$ is just another element in the set of coherent states, which forms an invariant subspace of the Hilbert space. Equation (21) could also be written as

$$|z,\alpha\rangle\equiv e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|\alpha,n\rangle . \tag{26}$$

From Eqs. (25) and (26), we note that any arbitrary coherent state could be expanded in terms of GCS's.

III. GENERALIZED SQUEEZED COHERENT STATES

First, we shall briefly discuss the squeezed coherent states (SCS's). The SCS is defined as[15,16]

$$|\alpha,z\rangle \equiv U(\alpha)S(z)|0\rangle , \tag{27}$$

where $U(\alpha)$ is the displacement operator given by

$$U(\alpha)\equiv\exp(\alpha a^\dagger-\alpha^* a) \tag{28}$$

and

$$S(z)\equiv\exp\left[\frac{z}{2}a^\dagger a^\dagger-\frac{z^*}{2}aa\right] \tag{29}$$

is known as the squeeze operator. Also,

$$\begin{aligned} SaS^\dagger &= a\cosh r+e^{i\theta}a^\dagger\sinh r=b ,\\ Sa^\dagger S^\dagger &= a^\dagger\cosh r+e^{-i\theta}a\sinh r=b^\dagger , \end{aligned} \tag{30}$$

where $z=re^{i\theta}$. The squeezed states correspond to Gaussian wave packets with *widths distorted from that of the vacuum state and those states follow the classical motion though the uncertainties oscillate.*[17]

Recently there has been a lot of excitement regarding SCS's, since they are considered to be useful in the detection of gravity waves.[8] It has also been proved that these states are emitted in certain nonlinear optical processes.[18] (For a review on SCS's see Refs. 19 and 20.)

Now, we define generalized squeezed coherent states (GSCS's) as

$$|n,z,\alpha\rangle\equiv U(\alpha)S(z)|n\rangle . \tag{31}$$

We first compute $|n,z\rangle$:

$$|n,z\rangle\equiv S(z)|n\rangle \tag{32}$$

$$\equiv\sum_m |m\rangle\langle m|S(z)|n\rangle$$

$$\equiv\sum_m |m\rangle G_{mn}(z) . \tag{33}$$

Making use of a slightly modified form of the technique developed by Rashid,[21] we get the expansion coefficients $G_{mn}(z)$ to be

$$G_{mn}(z)=\begin{cases} e^{-i(n-m)\theta/2}(-1)^{m+n/2}\left[\dfrac{m!n!}{\cosh r}\right]^{1/2}\left[\dfrac{\tanh r}{2}\right]^{(m+n)/2}\displaystyle\sum_\lambda\frac{\left[-\dfrac{4}{\sinh^2 r}\right]^\lambda}{(2\lambda)!\left[\dfrac{m}{2}-\lambda\right]!\left[\dfrac{n}{2}-\lambda\right]!} & \text{for } m,n \text{ even}\\[2ex] e^{-i(n-m)\theta/2}(-1)^{m+n/2-3/2}\left[\dfrac{m!n!}{\cosh^3 r}\right]^{1/2}\left[\dfrac{\tanh r}{2}\right]^{(m+n)/2-1}\displaystyle\sum_\lambda\frac{\left[-\dfrac{4}{\sinh^2 r}\right]^\lambda}{(2\lambda+1)!\left[\dfrac{m-1}{2}-\lambda\right]!\left[\dfrac{n-1}{2}-\lambda\right]!} & \text{for } m,n \text{ odd}\\[2ex] 0, \text{ otherwise} . \end{cases} \tag{34}$$

Only odd-odd or even-even elements of $G_{mn}(z)$ survive due to the fact that $S(z)$ essentially creates two excitations every time it acts.

In the spirit of Refs. 17 and 22, we realize that $S(z)$ has a finite expectation value in the state $|n\rangle$ (for $n=0,1,2,\ldots$), i.e., squeezing of the states $|n\rangle$ is possible. We consider an interesting case below:

$$G_{00}(z)=\langle 0|S(z)|0\rangle \tag{35}$$

$$\equiv\frac{1}{(\cosh|z|)^{1/2}} . \tag{36}$$

[The authors of Refs. 17 and 19 have remarked that $G_{00}(z)$ sums as $\tanh|z|<1$ for $r<\infty$. See Eq. (4.2) of Ref. 14.] Now the GSCS $|n,z,\alpha\rangle$ is given by

$$|n,z,\alpha\rangle\equiv U(\alpha)|n,z\rangle\equiv U(\alpha)\sum_m |m\rangle G_{mm}(z) . \tag{37}$$

Using Eq. (6),

$$|n,z,\alpha\rangle=e^{-|\alpha|^2/2}\sum_{m,l}G_{mn}(z)\left[\frac{m!}{l!}\right]^{1/2}\times L_m^{(l-m)}(|\alpha|^2)\alpha^{l-m}|l\rangle . \tag{38}$$

Equation (38) gives the Fock-space representation for $|n,z,\alpha\rangle$. Also, $|n,z,\alpha\rangle$ could be expanded in terms of $|m,z\rangle$'s as given below.

Consider

$$\langle m,z\,|\,n,z,\alpha\rangle=\langle m,z\,|\,U(\alpha)\,|\,n,z\rangle$$
$$=\langle m\,|\,S^\dagger(z)U(\alpha)S(z)\,|\,n\rangle$$
$$=\langle m\,|\exp(\alpha b^\dagger-\alpha^* b)\,|\,n\rangle$$
$$=\langle m\,|\exp(\gamma a^\dagger-\gamma^* a)\,|\,n\rangle$$
$$=\langle m\,|\,n,\gamma\rangle\,, \tag{39}$$

where $\gamma=(\alpha\cosh r-\alpha^*\sinh r)$ and we also note that $|\,n,\gamma\rangle$ is a GCS. From Eq. (26)

$$|\,n,z,\alpha\rangle=\sum_m |\,m,z\rangle\langle m,z\,|\,n,z,\alpha\rangle$$
$$=\sum_m |\,m,z\rangle\langle m\,|\,n,\gamma\rangle\,. \tag{40}$$

Now $|\,m,z\rangle$ is given by

$$|\,m,z\rangle\equiv S(z)\,|\,m\rangle \tag{41}$$

and $\langle m\,|\,n,\gamma\rangle$ is given by

$$\langle m\,|\,n,\gamma\rangle=e^{-|\gamma|^2/2}\left[\frac{n!}{m!}\right]^{1/2}L_n^{(m-n)}(\,|\gamma|^2)(\gamma)^{m-n}\,. \tag{42}$$

Using Eqs. (41) and (42), we get GSCS as

$$|\,n,z,\alpha\rangle=e^{-|\gamma|^2/2}\sum_m |\,m,z\rangle\left[\frac{n!}{m!}\right]^{1/2}\times L_n^{(m-n)}(\,|\gamma|^2)(\gamma)^{m-n}\,. \tag{43}$$

As an example, we give below the GSCS $|\,1,z,\alpha\rangle$:

$$|\,1,z,\alpha\rangle=\frac{e^{-|\alpha|^2/2}}{(\cosh|z|)^{3/2}}\sum_{k,m=0}^{\infty}\left[\frac{z}{2|z|}\tanh|z|\right]^k\frac{(2k+1)!}{\sqrt{k!m!}}L_{2k+1}^{(m-2k-1)}(\,|\alpha|^2)\alpha^{m-2k-1}\,|\,m\rangle\,. \tag{44}$$

Overcompleteness of $|\,n,z,\alpha\rangle$

Since $U(\alpha)$ and $S(z)$ are unitary, for a given α and z the set of states $|\,n,z,\alpha\rangle$, $n=0,1,2,\ldots$, forms a complete set just like the set $|\,n\rangle$. For a given n and z, the set $|\,n,z,\alpha\rangle$ with all complex α's forms an overcomplete set. We can obtain the resolution of the identity as

$$1=\int\frac{d^2\alpha}{\pi}\,|\,n,\alpha,z\rangle\langle n,\alpha,z\,|\,. \tag{45}$$

Using Eqs. (38), (39) and (43) the projection of $|\,n,z,\alpha\rangle$ on other states like $|\,m\rangle$ and $|\,m,z\rangle$ could be calculated. In the spirit of Eq. (21), we can define the squeezed coherent state of the displaced oscillator as

$$|\,z,\alpha\rangle_{\rm DO}\equiv\exp\left[\frac{z}{2}(a^\dagger-\alpha^*)^2-\frac{z^*}{2}(a-\alpha)^2\right]|\,\alpha\rangle \tag{46}$$

$$\equiv(\cosh|z|)^{-1/2}\sum_{n=0}^{\infty}\left[\frac{z}{2|z|}\tanh|z|\right]^n\frac{[(2n)!]^{1/2}}{n!}\,|\,2n,\alpha\rangle\,. \tag{47}$$

The physical interpretation of the GSCS is the same as that of the two-photon coherent state of the radiation field.[16] We can consider the GSCS as a coherent state formed due to two excitations on a particular state $|\,n\rangle$. It is a well-established fact that SCS's are employed in quantum nondemolition (QND) measurements to reduce the quantum noise. It is also hoped that GSCS's will find application in the QND measurements and quantum optics.

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COMMENT

Squeezed coherent states of the hydrogen atom

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Abstract. The O(4) algebra of the hydrogen atom is made use of to define its squeezed coherent states. It is shown that the effect of squeezing is to increase the energy of the atom in its levels. The corrections to the Bohr formula and the spectral transitions due to squeezing are calculated.

Recently Gerry (1984) introduced oscillator-like coherent states on the O(4) algebra of the hydrogen atom, and showed that the correct classical limit could be obtained without the correspondence limit. He also indicated that the coherent states thus introduced describe the 'elliptical orbits' anticipated in 1926 by Schrödinger (1978). More recently Bhaumik *et al* (1986) using these coherent states constructed a wavepacket which travels on an elliptic trajectory. In fact all these works are further developments on the realisation of the connection between the hydrogen atom and a four-dimensional oscillator with a constraint which has been rediscovered by many authors (Kibler and Negadi 1983, Cornish 1984, Chen and Kibler 1985) since the original discovery by Pauli (see the elaborate review, incidentally the first such one, by Bander and Itzykson (1966)). Also Nieto (1980) obtained coherent states for the Coulomb potential which are different from those introduced by Gerry (1984) and Bhaumik *et al* (1986). Nieto developed a formalism to obtain coherent states for general potentials using Schrödinger's criterion, i.e. to define coherent states as those with undistorted wavepackets with classical motion. Coherent states of Gerry (1984) and Bhaumik *et al* (1986) are the minimum uncertainty states of the harmonic oscillator (see Klauder and Sudarshan 1966). In this comment we introduce the oscillator squeezed coherent states on the O(4) algebra of the hydrogen atom. Further the effects of 'squeezing' are calculated.

First we shall briefly discuss squeezed coherent states and their importance.

Consider the harmonic oscillator Hamiltonian

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \tag{1}$$

with $[a, a^\dagger] = 1$. The eigenstates of H are given by $|n\rangle$ where $a^\dagger a|n\rangle = n|n\rangle$.

The coherent states $|\alpha\rangle$ is obtained as

$$|\alpha\rangle = D(\alpha)|0\rangle \tag{2}$$

where $D(\alpha)$ is the displacement operator given by

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{3}$$

and α a complex number.

Squeezed coherent states (SCS) are defined as

$$|\alpha, Z\rangle = D(\alpha)S(Z)|0\rangle \tag{4}$$

where $S(Z)$ is the squeezing operator given by

$$S(Z) = \exp[\tfrac{1}{2}Za^\dagger a^\dagger - \tfrac{1}{2}Z^* aa]. \tag{5}$$

Z is known as the squeezing parameter.

The squeezing operator $S(Z)$ transforms a and $a^\dagger$ into another equivalent bosonic system b and $b^\dagger$ as

$$\begin{aligned} SaS^\dagger &= a\cosh r + e^{i\theta}a^\dagger \sinh r = b \\ Sa^\dagger S^\dagger &= a^\dagger \cosh r + e^{-i\theta} a \sinh r = b^\dagger \end{aligned} \tag{6}$$

where $Z = r\,e^{i\theta}$.

The SCS have the following two remarkable properties.

(1) *The classical motion of the wavepackets.* It is easy to see that the SCS correspond to Gaussian wavepackets with widths distorted from that of the vacuum state and those states also follow classical motion but only the uncertainties oscillate (see Fisher *et al* 1984). Because of the oscillatory motion of the uncertainties Fujiwara and Wergelan (1984) named it as the 'jester'.

(2) *The non-classical nature of squeezing.* Writing $a = x_1 + ix_2$ and calculating variances $V(x_1)$ and $V(x_2)$ in the SCS $|\alpha, Z\rangle$ one obtains

$$V(x_1) = \tfrac{1}{4}e^{-2r} \qquad V(x_2) = \tfrac{1}{4}e^{2r}. \tag{7}$$

It has been pointed out by Walls (1983) that the condition $V(x_1) < \frac{1}{4}$ implies that the Glauber–Sudarshan representation P function should be a non-positive definite function. Thus squeezing is a non-classical property of the electromagnetic quanta.

Originally SCS were obtained by Stoler (1970, 1971), who showed that the states with minimum uncertainty form an equivalence class, which are the SCS.

There has been some activity on the quantum mechanical aspects of the squeezing operator (Fisher *et al* 1984, Santhanam and Satyanarayana 1984). The present author also introduced generalised squeezed coherent states for the harmonic oscillator (Satyanarayana 1985). (For reviews on SCS see Nieto (1984) and Walls (1983).)

From (1) we obtain

$$H^{\text{cl}}_{\text{sq}} = \langle \alpha, Z|H|\alpha, Z\rangle = \hbar\omega(|\alpha|^2 + \sinh^2 Z + \tfrac{1}{2}). \tag{8}$$

Now we can define the action variables as

$$J = h(|\alpha|^2 + \sinh^2 Z + \tfrac{1}{2}) \tag{9}$$

and

$$H^{\text{cl}}_{\text{sq}} = \nu J.$$

Also

$$\nu = \partial H^{\text{cl}}_{\text{sq}}/\partial J. \tag{10}$$

The quantised energy levels are recovered by invoking the Bohr–Somerfeld rule

$$J = h(n + \sinh^2 Z + \tfrac{1}{2}). \tag{11}$$

The above form of quantisation condition is chosen in view of the fact that $H^{\text{cl}}_{\text{sq}}$ becomes $\hbar\omega(n + \frac{1}{2})$ as the squeezing parameter Z approaches zero.

We shall now proceed to discuss the SCS of the hydrogen atom. We employ the notation of Gerry (1984).

The Hamiltonian of the hydrogen atom is given by

$$H = p^2/2\mu - Ze^2/r. \tag{12}$$

It is very well known that the angular momentum vector $\boldsymbol{L}$ and Pauli–Runge–Lenz vector $\boldsymbol{A}'$ given by

$$\boldsymbol{A}' = -Ze^2\boldsymbol{r}/r + (1/2\mu)(\boldsymbol{L}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{L}) \tag{13}$$

commute with H. Also A' is orthogonal to L and A'^2 has a term containing H. Using the decomposition $\mathrm{O}(4) = \mathrm{SU}(2)_a \times \mathrm{SU}(2)_b$, Schwinger's boson realisation of SU(2) and the properties of A' Gerry (1984) obtain

$$(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)^2 = (b_1^\dagger b_1 + b_2^\dagger b_2 + 1)^2 \tag{14}$$

and

$$-\mu Z^2 e^4/\hbar^2 E = (a_1^\dagger a_1 + a_2^\dagger a_2 + 1)^2 + (b_1^\dagger b_1 + b_2^\dagger b_2 + 1)^2 \tag{15}$$

where $[a_i, a_i^\dagger] = 1$ and $[b_i, b_i^\dagger] = 1$ for $i = 1, 2$.

Now introducing the Fock space basis, i.e. eigenstates of $a_i^\dagger a_i$ as $|n_i\rangle$ and $b_i^\dagger b_i$ as $|m_i\rangle$ where $i = 1, 2$, equations (14) and (15) become

$$n_1 + n_2 = m_1 + m_2 \tag{16}$$

and

$$E_n = -\mu Z^2 e^2/2\hbar^2 n^2 \tag{17}$$

where

$$n = n_1 + n_2 + 1 = m_1 + m_2 + 1.$$

We now introduce the SCS as

$$\begin{aligned} |\alpha_i^a, Z_i^a\rangle &= D(\alpha_i^a)S(Z_i^a)|n_i = 0\rangle \\ |\alpha_i^b, Z_i^b\rangle &= D(\alpha_i^b)S(Z_i^b)|m_i = 0\rangle \end{aligned} \tag{18}$$

for $i = 1, 2$ and $Z_i^a = r_i^a$ and $Z_i^b = r_i^b$.

The condition in equation (16) reads

$$|\alpha_1^a|^2 + |\alpha_2^a|^2 + \sinh^2 r_1^a + \sinh^2 r_2^a = |\alpha_1^b|^2 + |\alpha_2^b|^2 + \sinh^2 r_1^b + \sinh^2 r_2^b. \tag{19}$$

Now, for the squeezing parameters r_i^a and r_i^b for $i = 1, 2$ tending to zero, the above equation becomes equation (17) of Gerry (1984).

In the spirit of (8), the energy becomes

$$H_{\mathrm{sq}}^{\mathrm{cl}} = \frac{-\mu Z^2 e^4}{2\hbar^2(|\alpha_1^a|^2 + |\alpha_2^a|^2 + \sinh^2 r_1^a + \sinh^2 r_2^a + 1)^2}. \tag{20}$$

For $r_1^a = r_2^a = 0$, the above equation becomes equation (18) of Gerry (1984).

Further we see that squeezing does not affect the relation for the period of the Kepler orbit.

Using equation (11) and taking $r_1^a = r_1$ and $r_2^a = r_2$, we obtain

$$E_{\mathrm{sq}}^{(n)} = \frac{-\mu Z^2 e^4}{2\hbar^2()n_1 + n_2 + \sinh^2 r_1 + \sinh^2 r_2 + 1)^2}. \tag{21}$$

Equation (21) gives the energy levels of a hydrogen atom specified by the principal quantum number $n(=n_1+n_2+1)$ and the squeezing parameters r_1 and r_2. It is to be noted that the energy levels acquire higher values, as it happens for the harmonic oscillator.

Taking $r_1=r_2=0$, equation (21) becomes

$$E^{(n)} = -\mu Z^2 e^4/2\hbar^2(n_1+n_2+1)^2 \tag{22}$$

which is equation (15) of Gerry (1984).

Now, we proceed to calculate the absolute shift in energy $E_{sq}^{(n)}-E^{(n)}$:

$$E_{sq}^{(n)}-E^{(n)} = \frac{\mu Z^2 e^4}{2\hbar^2}\frac{(\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1)^2} \times\left(\frac{(2n_1+2n_2+2+\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1+\sinh^2 r_1+\sinh^2 r_2)^2}\right). \tag{23}$$

For $\sinh^2 r_1 \ll n_1$ and $\sinh^2 r_2 \ll n_2$, we have

$$\begin{aligned} E_{sq}^{(n)}-E^{(n)} &= \frac{\mu Z^2 e^4}{2\hbar^2}\frac{(\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1)^4}(2n_1+2n_2+2+\sinh^2 r_1+\sinh^2 r_2) \\ &\quad \times\left(1-\frac{2(\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1)}\right) \\ &= \frac{\mu Z^2 r^4}{2\hbar^2}\frac{(\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1)^5}[n_1+n_2+1-2(\sinh^2 r_1+\sinh^2 r_2)] \\ &\quad \times[2(n_1+n_2+1)+(\sinh^2 r_1+\sinh^2 r_2]. \end{aligned} \tag{24}$$

The relative shift in energy is given by

$$\frac{E_{sq}^{(n)}-E^{(n)}}{E^{(n)}} = \frac{-(\sinh^2 r_1+\sinh^2 r_2)[2(n_1+n_2+1)+(\sinh^2 r_1+\sinh^2 r_2)]}{(n_1+n_2+1+\sinh^2 r_1+\sinh^2 r_2)^2}. \tag{25}$$

For $\sinh^2 r_1 \ll n_1$ and $\sinh^2 r_2 \ll n_2$

$$\begin{aligned} \frac{E_{sq}^{(n)}-R^{(n)}}{E^{(n)}} &= \frac{-(\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1)^3}[2(n_1+n_2+1)+\sinh^2 r_1+\sinh^2 r_2] \\ &\quad \times[n_1+n_2+1-2(\sinh^2 r_1+\sinh^2 r_2]. \end{aligned} \tag{26}$$

The spectral transition energy $\Delta E(n, r_1, r_2 \to m, r_1, r_2)$ is given below:

$$\begin{aligned} \Delta E_{(r_1,r_2)}^{(n,m)} &= \frac{\mu Z^2 e^4}{2\hbar^2}[(n_1+m_1)+(n_2+m_2)+2(\sinh^2 r_1+\sinh^2 r_2+1)][(n_1-m_1)+(n_2-m_2)] \\ &\quad \times\frac{1}{(n_1+n_2+\sinh^2 r_1+\sinh^2 r_2+1)(m_1+m_2+\sinh^2 r_1+\sinh^2 r_2)}. \end{aligned} \tag{27}$$

Equation (23) gives the squeezing correction to the spectral transitions, which arise mainly due to quantum mechanical nature of squeezing.

In a future communication we shall investigate the nature of the motion of the wavepacket constructed using these squeezed coherent states.

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Bunching and antibunching properties of various coherent states of the radiation field

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In contrast to coherent states $|\alpha\rangle$ which have zero Hanbury Brown and Twiss Effect [i.e., $g^{(2)}(0)=1$], it is shown that generalized coherent states $|n,\alpha\rangle$ are antibunched for $|\alpha|^2<\frac{1}{2}$. The range of values for α (real) in terms of the squeezing parameter r (real) for the squeezed coherent state $|\alpha,r\rangle$ in order to exhibit bunching and antibunching are obtained. The conditions and the exact range of values for r and α for a given n for generalized squeezed coherent states $|n,\alpha,r\rangle$ to exhibit bunching and antibunching are also obtained.

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The quantity which determines bunching and antibunching of a state[1] of the radiation field is decided by the second-order correlation function $g^{(2)}(0)$ given by

$$g^{(2)}(0)=\frac{\langle a^\dagger a^\dagger aa\rangle}{\langle a^\dagger a\rangle^2} \tag{1}$$

and it could be written as

$$g^{(2)}(0)=\frac{\langle a^\dagger aa^\dagger a\rangle-\langle a^\dagger a\rangle}{\langle a^\dagger a\rangle^2}, \tag{2}$$

where $a^\dagger$ and a are the photon creation and destruction operators.

A light field (or the Fock-space state describing the light field) is said to be antibunched if $g^{(2)}(0)<1$, which means that the probability of detecting a coincident pair of photons is less than that from a coherent field described by a coherent state which has Poisson distribution for photon counts. Antibunching is considered to be "a clear demonstration of the quantum nature of light which is not explained by classical theory," since it means "anticorrelation" in the photon detection. The method of generating antibunched states has been described by Stoler[2] and the subject has been attracting a lot of theoretical and experimental activity[3] (see the review of Paul[4]).

There are many states in the Fock space which are antibunched. For example, the number state $|n\rangle$ is one such since $g^{(2)}(0)=1-1/n$, which is a reflection of the fact that the state $|n\rangle$ contains a definite number of photons. The binomial states of the radiation field recently introduced by Stoler *et al.*[5] are antibunched for certain parameter ranges. Also, Simon and Satyanarayana[6] very recently introduced the logarithmic states of the radiation field defined as

$$|q\rangle=c|0\rangle+\beta\sum_{n=1}^{\infty}\left[\frac{q^n}{n}\right]^{1/2}|n\rangle,\quad \text{for } -1\le q<1 \tag{3}$$

where

$$\beta=\left[\frac{-(1-|c|^2)}{\ln(1-q)}\right]^{1/2}, \tag{4}$$

and c is the point inside a unit circle. These states are antibunched for certain ranges of q and c. But the phase state $|\phi\rangle$ (Ref. 7) is bunched. For the coherent state $|\alpha\rangle$, $g^{(2)}(0)=1$ which means that it has null Hanbury Brown and Twiss correlation.

In this communication we discuss the bunching and antibunching properties of various coherent states, since those are the states which are useful for the description of the optical fields. First we consider the generalized coherent state[8] $|n,\alpha\rangle$ as defined as

$$|n,\alpha\rangle\equiv\exp(\alpha a^\dagger-\alpha^* a)|n\rangle, \tag{5}$$

where $|n\rangle$ is the nth state of the oscillator. These states have been studied in detail.[9] Its $g^{(2)}(0)$ is given by

$$g^{(2)}(0)=1+\frac{n(2|\alpha|^2-1)}{(|\alpha|^2+n)^2}, \tag{6}$$

which means that the states $|n,\alpha\rangle$ are bunched only for $2|\alpha|^2-1)0$, and for $2|\alpha|^2<1$ the states clearly have antibunching. So unlike the coherent states $|\alpha\rangle$ for which $g^{(2)}(0)=1$, the generalized coherent states (GCS) $|n,\alpha\rangle$ has subpoissonian statistics for $2|\alpha|^2<1$. This means if α were to lie within the unit phase cell around the origin then the corresponding states are antibunched. Here we have an interesting comment regarding the counting statistics of $|n,\alpha\rangle$. "The appropriate generalizations of the Poisson distribution" as stated by Roy and Virendra Singh also contains subpoissonian statistics for $2|\alpha|^2<1$.

Next, we consider the squeezed coherent states (SCS) defined as[1,10–12]

$$|\alpha,Z\rangle\equiv D(\alpha)S(Z)|0\rangle, \tag{7}$$

where $D(\alpha)$ is the displacement operator given by

$$D(\alpha)\equiv\exp(\alpha a^\dagger-\alpha^* a), \tag{8}$$

and

$$S(Z)=\exp\left[\frac{Z}{2}a^\dagger a^\dagger-\frac{Z^*}{2}aa\right] \tag{9}$$

is known as the squeeze operator. Also,

$$SaS^{\dagger}=a\cosh r+e^{i\theta}a^{\dagger}\sinh r\ ,$$
$$Sa^{\dagger}S^{\dagger}=e^{-i\theta}a\sinh r+a^{\dagger}\cosh r\ , \tag{10}$$

where $Z=re^{i\theta}$. SCS were also introduced by Rowe as the "breathing modes" of the radiation field in the context of two photon processes (Ref. 13).

The $g^{(2)}(0)$ of SCS is given by

$$g^{(2)}(0)=\frac{[(\alpha^2-e^{-i\theta}\sinh r\cosh r)(\alpha^{*2}-e^{-\theta}\sinh r\cosh r)+4|\alpha|^2\sinh^2 r+2\sinh^4 r]}{(|\alpha|^2+\sinh^2 r)^2}\ . \tag{11}$$

For $\theta=0$ and α (real),

$$g^{(2)}(0)=1+\frac{2\sinh^4 r+\sinh^2 r(2\alpha^2+1)-\alpha^2\sinh 2r}{(\alpha^2+\sinh^2 r)^2}\ . \tag{12}$$

The above form of $g^{(2)}(0)$ could be reduced to Eq. (2.30) of Walls and Milburn.[1] The state $|\alpha,r\rangle$ is bunched only if the numerator of the second term of Eq. (12), which could be written as

$$f(\alpha)=\alpha^2(2\sinh^2 r-\sinh 2r)+(2\sinh^4 r+\sinh^2 r)\ , \tag{13}$$

is positive. $f(\alpha)$ is a quadratic expression and its analysis is simple and given below. The roots of $f(\alpha)$ are

$$\alpha_1,\alpha_2=\pm\left|\frac{\sinh r(1+2\sinh^2 r)}{2(\cosh r-\sinh r)}\right|^{1/2}\ . \tag{14}$$

Case 1: $r>0$. The roots are real and distinct and the coefficient of α^2, namely $2\sinh r(\sinh r-\cosh r)$, is negative. Therefore for a given value of the squeezing parameter r, in order to have an antibunched state, α should be chosen such that $\alpha<\alpha_1$ or $\alpha>\alpha_2$. For $\alpha_1<\alpha<\alpha_2$, we have a bunched state.

Case 2: $r<0$. The roots α_1 and α_2 are purely imaginary quantities and the coefficient of α^2 namely $2\sinh r(\sinh r-\cosh r)$ is positive which means $f(\alpha)$ is positive, and therefore for all values of α we have only bunched states.

The discussion in case 1 and case 2 given above are to be compared with Ref. 1 and Eq. (6.7) of Yuen.[11] Our results are exact for α (real) and r (real) and fix the exact range of values for α in terms of r whereas the discussion of Walls[1] is based on the limit $|\alpha|^2\gg\sinh^2 r$.

Now for $\alpha=0$, i.e., for the squeezed vacuum state $|0,z\rangle$,

$$g^{(2)}(0)=2+\coth^2 r\ , \tag{15}$$

which is a rewritten form of Eq. (17) of Walls,[1] which is to be compared with $g^{(2)}(0)=2$ for a chaotic light beam in an optical cavity. This means that the cavity filling due to squeezing is more bunched than chaotic light and the counting statistics are similar.

All the above discussed results could be obtained as various special cases of the $g^{(2)}(0)$ of the generalized squeezed coherent states[9] (GSCS) introduced by one of the authors, earlier defined as

$$|n,Z,\alpha\rangle\equiv D(\alpha)S(z)|n\rangle\ , \tag{16}$$

and its $g^{(2)}(0)$ is given by

$$g^{(2)}(0)=|[\alpha^2-(2n+1)\sinh r\cosh re^{-i\theta}][\alpha^{*2}-(2n+1)\sinh r\cosh re^{i\theta}]+4|\alpha|^2\sinh^2 r(n+1)$$
$$+\sinh^4 r(n+1)(n+2)+4|\alpha|^2\cosh^2 rn+\cosh^4 rn(n-1)]/(|\alpha|^2+\sinh^2 r+n\cosh^2 r)^2\ . \tag{17}$$

Now, for case 1, $Z=0$, Eq. (17) becomes Eq. (6); for case 2, $n=0$, Eq. (17) becomes Eq. (11); for case 3, $n=0$ and $\alpha=0$, Eq. (17) becomes Eq. (15); for case 4, $n=0$; $\theta=0$, and α (real), Eq. (17) becomes Eq. (12).

Now we proceed to get the conditions for bunching and antibunching of GSCS

$$g^{(2)}(0)=[\sinh^4 r(n^2+3n+1)+2\alpha^2\sinh^2 r+(4n^2+2n+1)\sinh^2 r\cosh^2 r+2n\alpha^2\cosh^2 r$$
$$-\cosh^4 r-2(2n+1)\alpha^2\sinh r\cosh r]/(\alpha^2+\sinh^2 r+n\cosh^2 r)^2\ . \tag{18}$$

TABLE I. Range of values for r and α.

r		α	
r	$x=\sinh^2 r$	Bunching	Antibunching
$-\infty<r<r_2$	$x>x_2$	$-\infty<\alpha<\infty$	[illegible]
	$0<x<x_2$	$\alpha<\alpha_1$ and $\alpha>\alpha_2$	[illegible]
			$\alpha_1<\alpha<\alpha_2$
$r>r_2$	$x>x_2$		$\alpha<\alpha_1$ and $\alpha>\alpha_2$
		$\alpha_1<\alpha<\alpha_2$	

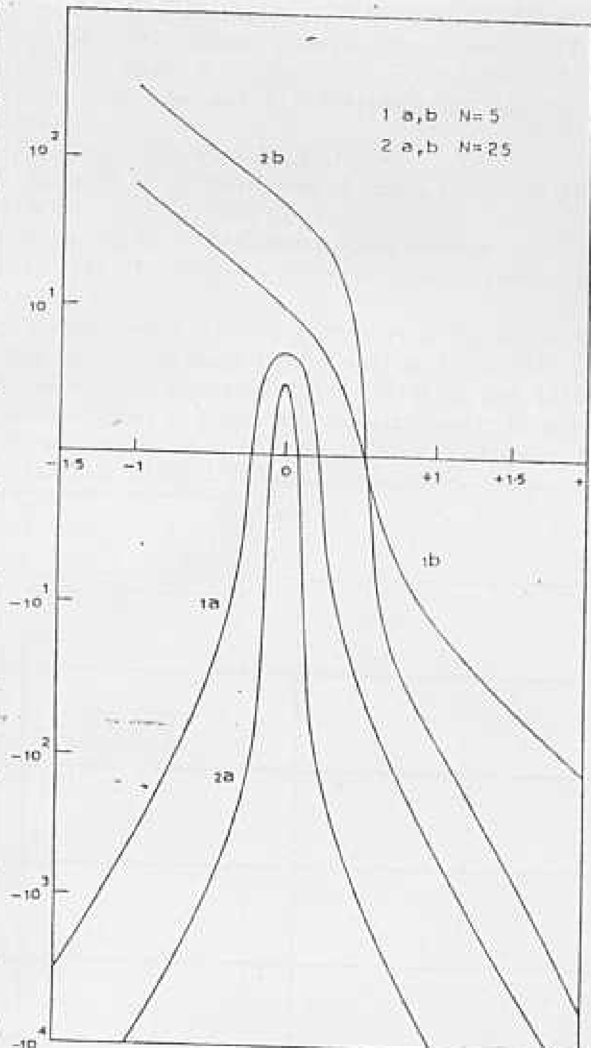

FIG. 1. Behavior of $f(r)$ (1a and 2a) and $G(r)$ (1b and 2b) for $n=5$ and 25, respectively.

The numerator of the second term of $g^{(2)}(0)$ in the above expression could be written as (for α and r real)

$$F(\alpha)=\alpha^2[2(n+1)\sinh^2 r+2n-(2n+1)\sinh 2r] + \sinh^4 r(5n^2+4n+2)+(4n^2+1)\sinh^2 r-n \quad (19)$$

$$=\alpha^2 f(r)-G(r). \quad (20)$$

The roots of $F(\alpha)$ are given by

$$\alpha_1,\alpha_2=\pm\left[\frac{G(r)}{f(r)}\right]^{1/2}. \quad (21)$$

Case 1: $r>0$. In this case $f(r)>0$. Taking $x=\sinh^2 r$, $G(r)$ could be rewritten as

$$g(x)=n-x(1+4n^2)-x^2(5n^2+4n+2). \quad (22)$$

Since $x>0$, the positive root of $g(x)$ is

$$x_2=\frac{[(1+4n^2)^2+4n(5n^2+4n+2)]^{1/2}-(1+4n^2)}{2(5n^2+4n+2)}. \quad (23)$$

(a) for $r<0$, such that $x>x_2$, $g(x)<0$ and α_1 and α_2 are purely imaginary. Since $f(r)>0$, for all α, $F(\alpha)>0$, i.e., the states are bunched.

(b) For $r<0$, such that $0<x<x_2$, the sign of $g(x)$ is opposite to that of the coefficient of x^2, i.e., positive. Therefore, α_1 and α_2 are real and distinct. For $\alpha<\alpha_1$ and $\alpha>\alpha_2$, $F(\alpha)>0$, i.e., the states are bunched and for $\alpha_1<\alpha<\alpha_2$, $F(\alpha)<0$, i.e., the states are antibunched.

Case 2: $r>0$. The positive root of $f(r)$ is given by

$$r_2=\tfrac{1}{2}\ln\left[\frac{(n+1)+[(n-1)^2+n(3n+2)]^{1/2}}{n}\right]. \quad (24)$$

A similar analysis can be done as above in case 1 and the range of values for both the cases are given in Table I. To know whether a given $|n,z,\alpha\rangle$ is bunched or antibunched, one should just calculate x_2 [Eq. (23)] and r_2 [Eq. (24)] and then look at Table I.

To have a feeling for $F(\alpha)$ [Eq. (20)] we have Fig. 1 which gives the behavior of the functions $f(r)$ (1a and 2a) and $G(r)$ (1b and 2b) for $n=5$ and $n=25$, respectively. At a chosen r, the ratio of $G(r)$ to $f(r)$ determines α_1 and α_2. We note that $f(r)$ is a monotonically decreasing function and the root of $f(r)$, namely r_2, tends to $\frac{1}{2}\ln 3$ as n tends to infinity. The positive root of $G(r)$ given by Eq. (23) could also be obtained from the positive zeroes of $G(r)$ from Fig. 1.

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