ON CONTINUOUS AND DISCRETE INEQUALITIES

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by

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PREFACE

This thesis is based on the work done by me during the period 1977-1980 On continuous and discrete inequalities under the guidance of Professor Ravi P.Agarwal, Matscience, The Institute of Mathematical Sciences, Madras.

I am greatly indebted to Professor Ravi P.Agarwal who has been more than a guide to me. He has taken me deep into the various branches of mathematics and guided me in the problems discussed here.

Many of the results reported here have already appeared in the form of research papers in journals of international repute listed in the Appendix and the available reprints are attached herewith.

I am extremely grateful to Professor V.Lakshmikantham, University of Texas, U.S.A. for offering detailed criticism and suggestions on Chapters 1 and 3. I am much obliged to Professor Roberto Conti, Universita Degli Studi, Italy, for having gone through Chapters 1 and 6 and for having offered several useful suggestions.

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C O N T E N T S

CHAPTER 0

SYNOPSIS

It is well recognised that the inequalities furnish a very general comparison principle in studying many qualitative as well as quantitative properties of solutions of related equations. 'Everyone knows' that Gronwall inequality is but one example of an inequality for a monotone operator k in which the exact solution $w = a + kw$ provides an upper bound on all solutions of $u \le a + ku$. On the basis of various motivations this inequality has been extended and used in various contexts. The first nonlinear version of this inequality is due to Bihari which has been further generalised in several different directions. An extensive survey of these generalizations is given recently by Beesack [7,8] , Deo et. al [9] . An important feature of these results is that these inequalities are reducible to differential inequalities.

For this finite system of integral inequalities

$$\phi(t) \le b(t) + \int_0^t A(s)\,\phi(s)\,ds \tag{1}$$

where the components of $\phi(t)$, $b(t)$ and the elements of $A(t)$ are non-negative, the upper estimate for $\phi_i(t)$, $1 \le i \le n$, the i-th component of $\phi(t)$, can be obtained by using Gronwall inequality provided $A(t)$ is an upper or lower triangular matrix. For the general $A(t)$ two different approaches have been used:

(i) Using any convenient norm, one get the following inequality

$$\|\phi(t)\| \le \|b(t)\| + \int_0^t \|A(s)\| \|\phi(s)\| \, ds$$

Combining this with Gronwall inequality estimates follow for all the components of $\phi(t)$ uniformly e.g. see Theorem 1 (13 , p.34).

(ii) Connect $\phi(t)$ with the maximal solution of the system of related differential equations e.g. as in Theorem 1.9.3. [10] or the solution of the related integral equations [12].

Method (ii) is more of theoretical interest since for the related differential equations the maximal solution is infact the only solution which is in the general case, impossible to find. Still, in almost all the known results method (ii) is used indirectly e.g. adding some positive terms in the right hand side so that the solution of the related differential system can be obtained in terms of the known functions.

For the second order system

$$\phi_1(t) \le p_1(t) + \int_0^t a_{11}(s)\phi_1(s)\,ds + \int_0^t a_{12}(s)\phi_2(s)\,ds$$

$$\phi_2(t) \le p_2(t) + \int_0^t a_{21}(s)\phi_1(s)\,ds .$$

One can find the explicit genuine upper estimate for $\phi_1(t)$ and $\phi_2(t)$ provided an upper estimate for the inequality

$$\phi_1(t) \le p_1(t) + \int_0^t a_{11}(s)\,\phi_1(s)\,ds$$
$$+ \int_0^t a_{12}(s)\Big[p_2(s) + \int_0^s a_{21}(\tau)\phi_1(\tau)\,d\tau\Big]\,ds \qquad (2)$$

is known. The inequality (2) appears in a natural way in the study of integrodifferential systems. Thus, if $A(t)$ is of some particular nature, the study of integral inequalities of the type (2) involving n-integrals is required e.g. in 3 x 3 matrix $A(t)$ with $a_{22} \equiv a_{32} \equiv a_{33} \equiv 0$ leads to an inequality of type (2) involving three integrals.

For the Volterra integral equation

$$u(t) = w(t) + \int_0^t K(t,s)\,u(s)\,ds$$

the famous method of Courant: replacing $K(t,s)$ by approximate degenerate kernel also leads to inequality of the type (1) with particular $A(t)$, e.g., if

$$|K(t,s)| \le \sum_{r=1}^{n} g_r(t)\,h_r(s)$$

then

$$|u(t)| \le |w(t)| + \sum_{r=1}^{n} g_r(t) \int_0^t h_r(s)\,|u(s)|\,ds \qquad (3)$$

which can be written as

$$\phi(t) \le \int_0^t b(s)\,ds + \int_0^t A(s)\phi(s)\,ds \qquad (4)$$

where

$$\phi(t) = (R_1(t), \cdots, R_n(t))^T, b(t) = |w(t)|(h_1(t), \cdots, h_n(t))^T$$

$$A(t) = (h_i(t) g_j(t)), \quad R_r(t) = \int_0^t h_r(s)|u(s)|ds.$$

A conversion from (4) to (3) is also simple, for this note that

$$\phi_i(t) \leq \int_0^t h_i(s)\Big[|w(s)| + \sum_{j=1}^n g_j(s)\phi_j(s)\Big]ds. \qquad (5)$$

Multiply (5) by $g_i(t)$ and summing for all i, to obtain

$$\sum_{i=1}^n g_i(t)\phi_i(t) \leq \sum_{i=1}^n g_i(t)\int_0^t h_i(s)\Big[|w(s)| + \sum_{j=1}^n g_j(s)\phi_j(s)\Big]ds. \qquad (6)$$

Adding both sides $|w(t)|$, and defining

$$\psi(t) = |w(t)| + \sum_{j=1}^n g_j(t)\phi_j(t)$$

(6) leads to inequality (3) in $\psi(t)$. Thus if the estimate for $|u(t)|$ is known from (3) one has estimates for $\phi(t)$ and if the estimates for $\phi(t)$ are known from (4) one has estimates for $|u(t)|$.

In a similar way for the nonlinear Volterra integral equations it is necessary to study the inequalities of the type

$$|u(t)| \leq |w(t)| + \sum_{r=1}^n g_r(t)\int_0^t h_r(s) H(|u(s)|)ds \quad (7)$$

or perhaps inequalities involving several integrals.

In Chapter 1, mainly these types of inequalities are studied, the known results are deduced or compared as remark following the main results. These results are used to study asymptotic behaviour and oscillation of solutions of functional differential equations.

Chapter 2 deals with the discrete analogue of the results presented in Chapter 1, several known results are improved and some applications to discrete stochastic models are given.

Usually in the literature inequalities involving higher order derivatives have been converted into inequivalent systems and then the estimates are obtained in terms of maximal solutions of the related differential equations e.g. see [11]. In Chapter 3 we deal directly with these types of inequalities and obtain the estimates in terms of known functions. Some applications are also given.

The discrete inequalities involving higher order differences are important to study error estimates, convergence etc. in discrete methods for solving differential equations. These inequalities are presented in Chapter 4 and some applications are given.

In Chapter 5, we have generalized the results of Chapter 1 to n-independent variables. Snow's [101] method of Riemann function is extended and Wendroff's estimates are improved. These results are used to study several properties of the solutions of partial differential and integral equations in n-independent variables.

In Chapter 6, the discrete analogue of some results obtained in Chapter 5 have been established. These inequalities are applied to study various properties of solutions of summary difference equations in n-independent variables.

Lastly in Chapter 7 we establish several existence, uniqueness results for hyperbolic delay differential equations. Some basic inequalities have been derived which are used to study the approximate solutions. An iterative scheme is provided which converges to the maximal solution of the related system which is the basis to study several properties of the solutions of the original system.

CHAPTER 1

CONTINUOUS GRONWALL TYPE INEQUALITIES

1. INTRODUCTION.

It is well recognised that the integral inequalities furnish a veral general comparison principle in studying many qualitative as well as quantitative properties of solutions of differential equations. The celebrated Gronwall inequality known now as Gronwall-Bellman-Reid inequality [1], [2], [3], [4] provides explicit bounds on solutions of a class of linear integral inequalities. On the basis of various motivations this inequality has been extended and used in various contexts. The first non-linear version of this inequality is due to Bihari [5,6]* which has been further generalized in several different directions. An extensive survey of these generalizations is given recently by Beesack [7,8], Deo et. al [9].

For the finite system of integral inequalities

$$\phi(t) \le b(t) + \int_0^t A(s)\,\phi(s)\,ds$$

where the components of $\phi(t)$, b(t) and the elements of A(t) are non-negative, then the upper estimate for $\phi_i(t)$, $1 \le i \le m$ the i-th component of $\phi(t)$ can be obtained by using GBR inequality provided A(t) is upper or lower triangular matrix.

For the general A(t) two different approaches have been used: one connects with the maximal solution of the

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* In a recent letter, Prof.J.P.Lasalle has informed us that he gave the same inequality much earlier in the year 1949 'Annals of Math.', 50, p.722-730

system of differential equations e.g., theorem 1.9.3 [10], another using any convenient norm, resulting the inequality

$$\|\phi(t)\| \leq \|b(t)\| + \int_0^t \|A(s)\| \, \|\phi(s)\| \, ds$$

and GBR inequality provides uniform estimate for all the components of $\phi(t)$, e.g. theorem 1 (p.34,[13]). First method is more of theoretical interest since for the related differential equations the maximal solution is in fact the only solution which is in the general case impossible to find.

For the second order system

$$\phi_1(t) \leq p_1(t) + \int_0^t [a_{11}(s)\phi_1(s) + a_{12}(s)\phi_2(s)] \, ds$$
$$\phi_2(t) \leq p_2(t) + \int_0^t a_{21}(s)\phi_1(s) \, ds.$$

One can find the explicit genuine upper estimates for $\phi_1(t)$ and $\phi_2(t)$ provided an upper estimate for the inequality.

$$\phi_1(t) \leq p_1(t) + \int_0^t [a_{11}(s)\phi_1(s) + a_{12}(s)\,(p_2(s) + \int_0^s a_{21}(\tau)\phi_1(\tau)\,d\tau)] \, ds \qquad (1)$$

is known. The inequality (1) appears in a natural way in the study of integrodifferential systems. Thus, if the matrix $A(t)$ is of some particular nature, the study of integral inequalities of the type (1) involving m-integrals is required e.g. in 3 x 3 matrix $A(t)$ with $a_{22} \equiv a_{32} \equiv a_{33} \equiv 0$ leads to an inequality of type (1) involving three integrals.

In this Chapter, we shall study mainly this type of integral inequalities in Section 2 and in Section 3 we present some nonlinear generalizations. In Section 4 several applications are given. After each result several remarks are in order which deduce or compare the known results.

Throughout we shall consider all the functions appearing in the inequalities are real-valued, non-negative and continuous on $I = [0, \infty)$ without further mention.

2. LINEAR GENERALIZATION.

THEOREM 1. Let the following inequality be satisfied

$$u(t) \leq p(t) + q(t) \sum_{r=1}^{m} E_r(t, u) \tag{2}$$

where

$$E_r(t,u) = \int_0^t f_{r1}(t_1) \int_0^{t_1} f_{r2}(t_2) \cdots \int_0^{t_{r-1}} f_{rr}(t_r)\, u(t_r)\, dt_r\, dt_{r-1} \cdots dt_1 .$$

Then

$$u(t) \leq p(t) + q(t) \int_0^t \sum_{r=1}^{m} E_r'(s, p) \times \exp\left(\int_s^t \sum_{r=1}^{m} E_r'(\tau, q)\, d\tau \right) ds \tag{3}$$

PROOF. Define a differentiable function $R(t)$, by

$$R(t) = \sum_{r=1}^{m} E_r(t, u), \quad R(0) = 0.$$

Then, on differentiation, we obtain

$$R'(t) = \sum_{r=1}^{m} E_r'(t,u) \tag{4}$$

where

$$E_r'(t,u) = f_{r1}(t)\int_0^t f_{r2}(t_2)\cdots\int_0^{t_{r-1}} f_{rr}(t_r)\,u(t_r)\,dt_r\,dt_{r-1}\cdots dt_2 .$$

From the assumptions on the functions, R(t) is non-decreasing hence using (2) in (4), we find

$$\begin{aligned} R'(t) &\le \sum_{r=1}^{m} E_r'(t, p+qR) \\ &= \sum_{r=1}^{m} E_r'(t,p) + \sum_{r=1}^{m} E_r'(t,qR) \\ &\le \sum_{r=1}^{m} E_r'(t,p) + R(t)\sum_{r=1}^{m} E_r'(t,q) . \end{aligned}$$

On integrating the above inequality, we obtain

$$R(t) \le \int_0^t \sum_{r=1}^{m} E_r'(s,p)\exp\left(\int_s^t \sum_{r=1}^{m} E_r'(\tau,q)\,d\tau\right) ds .$$

Substituting this estimate for R(t) in (2), the result follows.

COROLLARY 2. In (2), let p(t) be nondecreasing. Then

$$u(t) \le p(t)\left[1+q(t)\int_0^t \sum_{r=1}^{m} E_r'(s,1)\times \exp\left(\int_s^t E_r'(\tau,q)\,d\tau\right) ds\right] \tag{5}$$

also, if $q(t) \equiv 1$

$$u(t) \le p(t) \exp\left(\sum_{r=1}^{m} E_r(t,1) \right).$$

COROLLARY 3. Let the following inequality be satisfied

$$u(t) \le p(t) + \sum_{r=1}^{m} g_r(t) E_r(t,u) \tag{6}$$

where $g_i(t) \geq 1$, $1 \le i \le m$. Then

(a) $$u(t) \le p(t) + \prod_{r=1}^{m} g_r(t) \int_0^t \sum_{r=1}^{m} E_r'(s,p) \times$$
$$\exp\left(\int_s^t \sum_{r=1}^{m} E_r'\left(\tau, \prod_{i=1}^{m} g_i\right) d\tau \right) ds.$$

(b) If $p(t)$ is nondecreasing, then

$$u(t) \le p(t) \left[1 + \prod_{r=1}^{m} g_r(t) \int_0^t \sum_{r=1}^{m} E_r'(s,1) \times \right.$$
$$\left. \exp\left(\int_s^t \sum_{r=1}^{m} E_r'\left(\tau, \prod_{i=1}^{m} g_i\right) d\tau \right) ds \right]$$
$$\le p(t) \prod_{r=1}^{m} g_r(t) \exp\left(\sum_{r=1}^{m} E_r\left(t, \prod_{i=1}^{m} g_i\right) \right).$$

PROOF. The inequality (6) can be written as

$$u(t) \le p(t) + \left(\prod_{r=1}^{m} g_r(t) \right) \sum_{r=1}^{m} E_r(t,u).$$

Rest of the proof is same as in Theorem 1.

REMARK 1. If $p(t) = u_0$, $q(t) \equiv 1$, $m = 1$ then Theorem 1 reduces to Gronwall's original result.

REMARK 2. If $q(t) = 1$, $m = 1$ Corollary 2 reduces to Bellman's {p. 58,[14]} result.

REMARK 3. If $m = 1$, Corollary 2 is same as Theorem 1 of [15].

REMARK 4. If $m = 2$, $p(t) = u_0$, $q(t) \equiv 1$ and $f_{11}(t) = f_{21}(t)$ Corollary 2 gives the following estimate

$$u(t) \le u_0 \exp\left[\int_0^t f_{11}(t_1)\left(1+\int_0^{t_1} f_{22}(t_2)\,dt_2\right)dt_1\right] \tag{7}$$

which is not comparable with

$$u(t) \le u_0\left[1+\int_0^t f_{11}(t_1)\exp\left(\int_0^{t_1}\left[f_{11}(t_2) + f_{22}(t_2)\right]dt_2\right)dt_1\right] \tag{8}$$

obtained in [16].

REMARK 5. Several other particular cases of Theorem 1 have been considered in [15-23], but the results are not comparable as in Remark 4.

In the next result we shall unify several results of Pachpatte given in [15-23].

THEOREM 4. Let the inequality (2) be satisfied, where $f_{ii} = f_i$, $1 \le i \le m$; $f_{i+1,i} = f_{i+2,i} = \cdots = f_{mi} = g_i$, $1 \le i \le m-1$.
Then

$$u(t) \le p(t) + q(t)\,P_j(t), \quad 1 \le j \le m \tag{9}$$

where

$$p_1(t) = \int_0^t [p(t_1)\sum_{r=1}^{m} f_r(t_1)] \exp(\int_{t_1}^{t} (\sum_{r=1}^{m} q(t_2) f_r(t_2) \bigcup_{i=1}^{m-1} g_i(t_2)) dt_2) dt_1 \tag{10}$$

$$P_j(t) = \int_0^t [p(t_1)\sum_{r=1}^{m-j+1} f_r(t_1) + g_{m-j+1}(t_1) P_{j-1}(t_1)] \times \exp(\int_{t_1}^{t} (\sum_{r=1}^{m-j+1} q(t_2) f_r(t_2) \bigcup_{i=1}^{m-j} g_i(t_2) - g_{m-j+1}(t_2)) dt_2) dt_1, \; 2 \le j \le m. \tag{11$_j$}$$

In (10) and (11)$_j$, $2 \le j \le m$ the term $q(t)\sum_{r=1}^{r_1} f_r(t) \bigcup_{i=1}^{r_2} g_i(t)$ represents the sum of all functions except when

$q(t) f_k(t) = g_\ell(t)$ for some $1 \le k \le r_1,\ 1 \le \ell \le r_2$;

then $g_\ell(t)$ is taken to be zero, also $\bigcup_{i=1}^{0} g_i(t) = 0$.

PROOF. The inequality (2) with these functions is equivalent to the following system

$$u_1(t) \le p(t) + q(t)\int_0^t [f_1(t_1) u_1(t_1) + g_1(t_1) u_2(t_1)] dt_1 \tag{12}$$

$$u_{j-1}(t) \le \int_0^t [f_{j-1}(t_1) u_1(t_1) + g_{j-1}(t_1) u_j(t_1)] dt_1, \quad 3 \le j \le m \tag{13$_j$}$$

$$u_m(t) \le \int_0^t f_m(t_1) u_1(t_1) dt_1. \tag{14}$$

Define

$$R_1(t) = \int_0^t [f_1(t_1) u_1(t_1) + g_1(t_1) u_2(t_1)] dt_1$$

$$R_{j-1}(t) = \int_0^t [f_{j-1}(t_1) u_1(t_1) + g_{j-1}(t_1) u_j(t_1)] dt_1, \; 3 \le j \le m$$

$$R_m(t) = \int_0^t f_m(t_1) u_1(t_1) dt_1.$$

Then, from (12), $(13)_j$, (14) it follows that

$$R_1'(t) \le f_1(t)[p(t)+q(t)R_1(t)] + g_1(t)R_2(t) \tag{15}$$

$$R_{j-1}'(t) \le f_{j-1}(t)[p(t)+q(t)R_1(t)] + g_{j-1}(t)R_j(t), \quad 3 \le j \le m \tag*{$(16)_j$}$$

$$R_m'(t) \le f_m(t)[p(t)+q(t)R_1(t)]. \tag{17}$$

Adding (15), $(16)_j$, $3 \le j \le m$, (17) to get

$$\Big(\sum_{r=1}^{m} R_r(t)\Big)' \le p(t)\sum_{r=1}^{m} f_r(t) + q(t)\sum_{r=1}^{m} f_r(t)R_1(t) + \sum_{r=1}^{m-1} g_r(t)R_{r+1}(t)$$

and hence

$$\Big(\sum_{r=1}^{m} R_r(t)\Big)' - \Big(\sum_{r=1}^{m} q(t)f_r(t) \bigcup_{i=1}^{m-1} g_i(t)\Big)\Big(\sum_{r=1}^{m} R_r(t)\Big) \le p(t)\sum_{r=1}^{m} f_r(t). \tag{18}$$

Integrating (18), we obtain

$$\sum_{r=1}^{m} R_r(t) \le P_1(t) \tag{19}$$

and hence

$$R_m(t) \le P_1(t) - \sum_{r=1}^{m-1} R_r(t). \tag{20}$$

Adding (15), $(16)_j$, $3 \le j \le m$ and making use of (20) we find

$$\Big(\sum_{r=1}^{m-1} R_r(t)\Big)' \le p(t)\sum_{r=1}^{m-1} f_r(t) + q(t)\sum_{r=1}^{m-1} f_r(t)R_1(t) + \sum_{r=1}^{m-2} g_r(t)R_{r+1}(t) + g_{m-1}(t)\Big[P_1(t) - \sum_{r=1}^{m-1} R_r(t)\Big].$$

Thus, it follows that

$$\left(\sum_{r=1}^{m-1} R_r(t)\right)' - \left(\sum_{r=1}^{m-1} q(t) f_r(t) \bigcup_{i=1}^{m-2} g_i(t) - g_{m-1}(t)\right) \times$$
$$\left(\sum_{r=1}^{m-1} R_r(t)\right) \le p(t) \sum_{r=1}^{m-1} f_r(t)$$
$$+ g_{m-1}(t)\, P_1(t) \tag{21}$$

from which, on integrating, we get

$$\sum_{r=1}^{m-1} R_r(t) \le P_2(t) \tag{21}$$

or

$$R_{r-1}(t) \le P_2(t) - \sum_{r=1}^{m-2} R_r(t). \tag{22}$$

Adding (15), $(16)_j$, $3 \le j \le m-1$, and using (22) to find

$$\left(\sum_{r=1}^{m-2} R_r(t)\right)' - \left(\sum_{r=1}^{m-2} q(t) f_r(t) \bigcup_{i=1}^{m-3} g_i(t) - g_{m-2}(t)\right) \times$$
$$\left(\sum_{r=1}^{m-2} R_r(t)\right) \le p(t) \sum_{r=1}^{m-1} f_r(t) + g_{m-2}(t)\, P_2(t).$$

Thus, on integrating we obtain

$$\sum_{r=1}^{m-2} R_r(t) \le P_3(t). \tag{23}$$

Continuing in this way, we find

$$\sum_{r=1}^{m-j+1} R_r(t) \le P_j(t), \quad 4 \le j \le m. \tag*{$(24)_j$}$$

Since $u(t) = u_1(t) \le p(t) + q(t)\, R_1(t)$ and

$R_1(t) \le \sum_{r=1}^{m-j+1} R_r(t)$, $1 \le j \le m$ the result (9) follows from (19), (21), (23), $(24)_j$.

In the next result we shall show that the estimate (8) can be improved uniformly.

THEOREM 5. Let $m = 2$, $p(t) = u_0$, $q(t) = 1$ and $f_{11}(t) = f_{21}(t)$ in (2). Then

$$u(t) \le u_0 \Big(1 + \int_0^t f_{11}(t_1)\,(1 - \phi_1(t_1)) \times \exp\Big(-\int_0^{t_1} [f_{11}(t_2) + f_{22}(t_2)]\,dt_2\Big)\,dt_1\Big) \tag{25}$$

where

$$\phi_1(t) = \int_0^t f_{22}(t_1)\Big(\int_0^{t_1} f_{22}(t_2)\,dt_2\Big) \times \exp\Big(-\int_0^{t_1} [f_{11}(t_2) + f_{22}(t_2)]\,dt_2\Big)\,dt_1 .$$

PROOF. Define $R_1(t)$ as the right member of (2). Then, we find

$$R_1'(t) = f_{11}(t)\Big[u(t) + \int_0^t f_{22}(t_2)\,u(t_2)\,dt_2\Big],$$

$$R_1(0) = u_0$$

or

$$R_1'(t) \le f_{11}(t)\Big[R_1(t) + \int_0^t f_{22}(t_2)\,R_1(t_2)\,dt_2\Big]. \tag{26}$$

Define

$$R_2(t) = R_1(t) + \int_0^t f_{22}(t_2) R_1(t_2)\,dt_2 .$$

Then, it follows from the nondecreasing nature of $R_1(t)$, the inequality

$$R_2'(t) \le f_{11}(t) R_2(t) + f_{22}(t)\left[R_2(t) - u_0 \int_0^t f_{22}(t_2)\,dt_2\right],$$

$$R_2(0) = u_0 .$$

Thus, we find

$$R_2(t) \le u_0 \exp\left(\int_0^t [f_{11}(t_1) + f_{22}(t_1)]\,dt_1\right)(1 - \phi_1(t)).$$

Substituting this estimate in (26) and integrating from 0 to t, we obtain the desired result.

REMARK 6. Following the proof of Theorem 5, it is trivial to write improved versions of most of the results obtained in [15-23] .

The next result deals with an inequality with deviating argument of the following form

$$u(t) \le p(t) + q(t) \sum_{r=1}^{m} Q_r(t,u) \tag{27}$$

where

$$Q_r(t,u) = E_r(t,u) + e_r(t,u). \tag{28}$$

In (28), $E_r(t,u)$ is same as in Theorem 1 and $e_r(t,u)$ is defined as follows

$$e_r(t,u) = \int_0^t h_{r_1}(t_1)\int_0^{t_1} h_{r_2}(t_2)\cdots\int_0^{t_{r-1}} h_{r_r}(t_r)\, u(g(t_r))\, dt_r\, dt_{r-1}\cdots dt_1 .$$

The function $g(t) \leq t$ and continuous on I. We shall assume that $-\tau = \inf_I g(t)$ where τ is finite and $g(t^*)=0$, $g(t) \geq 0$ for $t \in [t^*, \infty)$. The functions $u(t)$, $p(t)$ and $q(t)$ here are real-valued, non-negative and continuous on $[-\tau, \infty)$. The function $u(t)$ is defined as $\phi(t)$ in the initial interval $[-\tau, 0]$.

THEOREM 6. Let the inequality (27) be satisfied. Then

$$u(t) \leq p(t) + q(t)\int_0^t \sum_{r=1}^m [E_r'(s,p) + e_r'(s,\phi)] \times \exp\left(\int_s^t \sum_{r=1}^m E_r'(\tau,q)\, d\tau\right) ds \qquad 0 \leq t \leq t^*$$

and

$$u(t) \leq p(t) + q(t)\int_0^t \sum_{r=1}^m Q_r'(s,p) \times \exp\left(\int_s^t \sum_{r=1}^m Q_r'(\tau,q)\, d\tau\right) ds, \quad (t \geq t^*).$$

The proof of theorem 6 is similar to Theorem 1.

REMARK 7. The above result can be generalized for several delays also.

REMARK 8. Very particular case ($m = 1$) of this result has been extensively used to study several properties of solutions of differential equations with deviating arguments e.g. see [24, 25].

In the next theorem, we shall consider the following inequality

$$u(t) \leq w(t) + \int_0^t K(t,s)\,u(s)\,ds \tag{29}$$

where the function $K(t,s)$ is continuous, differentiable with respect to t on $I \times I$. Also $K(t,t) \geq 0$, $\frac{\partial K(t,s)}{\partial t} \geq 0$ but $K(t,s)$ or $\frac{\partial K(t,s)}{\partial t}$ not necessarily separable as considered by Willett [26].

THEOREM 7. Let the inequality (29) be satisfied. Then

$$u(t) \leq w(t) + \int_0^t \phi_2(t_1) \exp\left(\int_{t_1}^t \phi_3(t_2)\,dt_2\right) dt_1$$

where

$$\phi_2(t) = K(t,t)\,w(t) + \int_0^t \frac{\partial K(t,s)}{\partial t}\,w(s)\,ds$$

and

$$\phi_3(t) = K(t,t) + \int_0^t \frac{\partial K(t,s)}{\partial t}\,ds .$$

PROOF. The proof is similar to Theorem 1.

REMARK 9. Let $K(t,s) = \sin(t-s)$, $0 \le s \le t \le \frac{\pi}{2}$, then $K(t,t) = 0$, $\frac{\partial K(t,s)}{\partial t} = \cos(t-s)$, now if $w(t) = t$, we get $\phi_2(t) = \phi_3(t) = \sin t$. Hence Theorem 7 provides the estimate $u(t) \le \exp(1-\cos t)$. Obviously Willett's result cannot be used here directly, also if we consider $u(t) \le 1 + \int_0^t u(s)\,ds$, $(\sin(t-s) \le 1)$ then GBR inequality provides $u(t) \le e^t$. Thus, in several situations Theorem 7 provides better estimate also.

In the next result we shall consider the case

$K(t,s) \le \sum_{r=1}^{m} g_r(t) h_r(s)$ in (29).

THEOREM 8. Let the following inequality be satisfied

$$u(t) \le w(t) + \sum_{r=1}^{m} g_r(t) \int_0^t h_r(s) u(s)\,ds \qquad (30)$$

where

(i) $w(t)$ is nondecreasing (ii) $g_i(t) \geq 1$, $1 \le i \le m$ and are nondecreasing for $2 \le i \le m$.

Then

$$u(t) \le F_m w \qquad (31)$$

where F_m is defined inductively as follows.

$$F_0 w = w$$

$$F_k w = w(F_{k-1} g_k) \exp\left(\int_0^t h_k F_{k-1} g_k\,ds\right)$$

$$k = 1, 2, \cdots, m.$$

PROOF. The proof is by finite induction. For m=1, the result follows from Corollary 3. Let us assume that (31) be true for given k, $1 < k \leq m+1$. For m = k+1, we are given

$$u(t) \leq [w(t) + g_{k+1}(t)\int_0^t h_{k+1}(s)u(s)\,ds] + \sum_{r=1}^{k} g_r(t)\int_0^t h_r(s)u(s)\,ds.$$

Since the part in the bracket is nondecreasing, we have

$$u(t) \leq F_k[w(t) + g_{k+1}(t)\int_0^t h_{k+1}(s)u(s)\,ds]$$

$$\leq F_k w(t) + g_{k+1}(t)\,F_k w(t)\int_0^t h_{k+1}(s)\,\frac{u(s)}{w(s)}\,ds.$$

Thus, we find

$$\frac{u(t)}{w(t)} \leq \frac{F_k w(t)\,g_{k+1}(t)}{w(t)}\left[1 + \int_0^t h_{k+1}(s)\,\frac{u(s)}{w(s)}\,ds\right]$$

$$\leq F_k\, g_{k+1}(t)\left[1 + \int_0^t h_{k+1}(s)\,\frac{u(s)}{w(s)}\,ds\right].$$

Now it is easy to show that

$$u(t) \leq w(t)F_k\, g_{k+1}(t)\exp\left(\int_0^t h_{k+1}(s)F_k\, g_{k+1}(s)\,ds\right)$$

$$= F_{k+1}\, w.$$

Thus the result is true for all k. This completes the proof.

REMARK 10. Willett [26] has studied the same inequality (30) under a more general hypothesis, but the present form is very useful in several applications e.g. see [27], this also correct the inductive proof given by them in their theorem 1. In fact it appears without monotonic nature on $g_i(t)$, $2 \le i \le m$ their result may not be valid.

THEOREM 9. Let the inequality (30) be satisfied where $g_i(t) \geq 1$, $1 \le i \le m$. Then

(a) $$u(t) \le w(t) + \prod_{r=1}^{m} g_r(t) \int_0^t w(s) \sum_{r=1}^{m} h_r(s) \times \exp\left(\int_s^t \prod_{r=1}^{m} g_r(\tau) \sum_{r=1}^{m} h_r(\tau)\, d\tau\right) ds . \tag{32}$$

(b) If $w(t)$ is nondecreasing, then

$$u(t) \le w(t)\left[1 + \prod_{r=1}^{m} g_r(t) \int_0^t \sum_{r=1}^{m} h_r(s) \times \exp\left(\int_s^t \prod_{r=1}^{m} g_r(\tau) \sum_{r=1}^{m} h_r(\tau)\, d\tau\right) ds\right] \tag{33}$$

$$\le w(t) \prod_{r=1}^{m} g_r(t) \exp\left(\int_s^t \prod_{r=1}^{m} g_r(s) \sum_{r=1}^{m} h_r(s)\, ds\right) . \tag{34}$$

PROOF. The proof is similar to Corollary 3.

REMARK 11. Estimates obtained in Theorem 9 cannot be compared with (31), but if $g_i(t)=1$, $1 \le i \le m$, then (34) is the natural generalization of Bellman's lemma whereas (31) is a crude estimate.

REMARK 12. Estimate (31) depends on particular ordering i.e. for different orderings for the summation the results also differ, whereas estimates obtained in Theorem 9 are uniform.

3. NONLINEAR GENERALIZATION.

Our first result is connected with the following inequality

$$u(t) \le p(t)\left[u_0 + \sum_{r=1}^{m} H_r(t,u)\right] \tag{35}$$

where

$$H_r(t,u) = \int_0^t f_{r1}(t_1)\, u^{\alpha_{r1}}(t_1) \cdots \int_0^{t_{r-1}} f_{rr}(t_r)\, u^{\alpha_{rr}}(t_r)\, dt_r \cdots dt_1$$

and α_{ri}, $1 \le i \le r$, $1 \le r \le m$ are nonnegative real numbers and the constant $u_0 > 0$. We shall denote

$$\alpha_r = \sum_{i=1}^{r} \alpha_{ri} \quad \text{and} \quad \alpha = \max_{1 \le r \le m} \alpha_r .$$

THEOREM 10. Let the inequality (35) be satisfied. Then

$$u(t) \le u_0 p(t) \exp\left(\int_0^t \phi_4(s)\, ds\right), \ \text{if } \alpha = 1 \tag{36}$$

and

$$u(t) \le p(t)\left[u_0^{1-\alpha} + (1-\alpha)\int_0^t \phi_4(s)\, ds\right]^{1-\alpha}, \ \text{if } \alpha \ne 1 \tag{37}$$

where

$$\phi_4(t) = \sum_{r=1}^{m} H'_r(t,p)\, u_0^{\alpha_r - \alpha}.$$

PROOF. Following the proof of Theorem 1, we obtain

$$R'(t) \le \sum_{r=1}^{m} H'_r(t,p)\,[R(t)]^{\alpha_r}$$

where

$$R(t) = u_0 + \sum_{r=1}^{m} H_r(t,u).$$

Thus, it follows from $R(t) \geq u_0$, that

$$R'(t) \le R^{\alpha}(t) \sum_{r=1}^{m} H'_r(t,p)\,[u_0^{\alpha_r - \alpha}]$$

$$= R^{\alpha}(t)\,\phi_4(t)$$

and now the result follows immediately.

REMARK 13. In (37) where $\alpha > 1$, we assume

$$u_0^{1-\alpha} + (1-\alpha)\int_0^t \phi_4(s)\,ds > 0.$$

REMARK 14. For $m = 1$, $p(t) = 1$, $k_{11} = 2$, Theorem 10, reduces to first result in this direction by Freedman [28].

REMARK 15. If $m = 2$, $\alpha_{11} = 1$, $\alpha_{21} = 0$, $p(t) = 1$,

$f_{11}(t) = f_{21}(t)$ and $\alpha_{22} \le 1$,

Theorem 10 gives the following estimate

$$u(t) \le u_0 \exp\Big(\int_0^t f_{11}(t_1)\Big[1 + u_0^{\alpha_{22}-1}\int_0^{t_1} f_{22}(t_2)\,dt_2\Big]dt_1\Big)$$

which is not comparable with the result

$$u(t) \le u_0 + \int_0^t f_{11}(t_1)\exp\Big(\int_0^{t_1} f_{11}(t_2)\,dt_2\Big)\Big[u_0^{1-\alpha_{22}} + (1-\alpha_{22}) \times \int_0^t f_{22}(t_2)\exp\Big(1-(1-\alpha_{22})\int_0^{t_2} f_{11}(t_3)\,dt_3\Big)dt_2\Big]^{1/1-\alpha_{22}} dt_1$$

obtained in [16].

REMARK 16. If $m = 2$, $\alpha_{11} = 2$, $\alpha_{21} = 1$, $\alpha_{22} = 1$, $p(t) = 1$, $f_{11}(t) = f_{21}(t)$, Theorem 10 gives

$$u(t) \le u_0 \left[1 - u_0 \int_0^t f_{11}(t_1)\left(1 + \int_0^{t_1} f_{22}(t_2)\, dt_2\right) dt_1\right]^{-1}$$

which is not comparable with the result

$$u(t) \le u_0\, \phi_5(t) \left[1 - \phi_5(t)\right]^{-1}$$

where

$$\phi_5(t) = u_0 \int_0^t f_{11}(t_1) \exp\left(\int_0^{t_1} f_{22}(t_2)\, dt_2\right) dt_1. \qquad (38)$$

The estimate (38) is the actual form obtained after simplification of the result obtained in [23].

In the next result, we shall show that the estimates (38) can be improved uniformly.

THEOREM 11. Let $m = 2$, $\alpha_{11} = 2$, $\alpha_{21} = 1$, $\alpha_{22} = 1$, $p(t) = 1$, $f_{11}(t) = f_{21}(t)$ in (35).

Then

$$u(t) \le u_0\, \overset{*}{\phi}_5(t) \left[1 - \overset{*}{\phi}_5(t)\right]^{-1}$$

where

$$\overset{*}{\phi}_5(t) = u_0 \int_0^t f_{11}(t_1) \exp\left(\int_0^{t_1} \phi_6(t_2)\, dt_2\right) dt_1$$

$$\phi_6(t) = f_{22}(t) - u_0 f_{11}(t) \int_0^t f_{22}(t_1)\, dt_1 .$$

PROOF. Following the proof of Theorem 5, we find

$$R_1'(t) \le f_{11}(t) R_1(t) R_2(t).$$

Thus, it follows that

$$R_2'(t) \le f_{11}(t) R_2^2(t) + \left(f_{22}(t) - u_0 f_{11}(t) \int_0^t f_{22}(t_1)\, dt_1\right) R_2(t)$$

and now the result follows easily.

REMARK 17. Several other results obtained in [15-2] can be easily improved following similar lines as in Theorem 11.

For the next result we shall require the following class of functions.

DEFINITION. A function $w : [0,\infty) \to (0,\infty)$ is said to belong to the class S if

(i) $w(u)$ is nondecreasing and continuous for $u \geq 0$

(ii) $\frac{1}{v} w(u) \leq w(u/v)$, $u \geq 0$, $v \geq 1$.

This class S has been modified here to avoid the triviality $w(u) = u w(1)$ as defined and used previously in several nonlinear generalizations by Deo. et. al [29] also see [8].

In the next result we are concerned with the following inequality

$$u(t) \leq p(t) + q(t) \sum_{r=1}^{m} E_r(t,u) + \sum_{r=1}^{l} g_r(t) \int_0^t h_r(s) w_r(u(s))\, ds. \tag{39}$$

THEOREM 12. Let the inequality (39) be satisfied where (i) $p(t) \geq 1$ and nondecreasing (ii) $g_i(t) \geq 1$, $1 \leq i \leq l$ (iii) $w_r \in S$, $1 \leq r \leq l$. Then

$$u(t) \leq p(t)\, \phi_7(t) \prod_{i=1}^{l} g_i(t) \prod_{r=1}^{l} F_r(t), \qquad 0 < t \leq b < \infty \tag{40}$$

where

$$\Phi_7(t) = 1 + q(t)\int_0^t \sum_{r=1}^m E_r'(s, \prod_{i=1}^l g_i) \times$$
$$\exp\left(\int_s^t \sum_{r=1}^m E_r'(\tau, q\prod_{i=1}^l g_i)\,d\tau\right) ds$$

$$F_1(t) = G_1^{-1}\left[G_1(1) + \int_0^t h_1(s)\,\Phi_7(s) \prod_{i=1}^l g_i(s)\,ds\right]$$

$$F_k(t) = G_k^{-1}\left[G_k(1) + \int_0^t h_k(s)\,\Phi_7(s) \prod_{i=1}^l g_i(s) \prod_{r=1}^{k-1} F_r(s)\,ds\right], 2 \le k \le l$$

$$G_k(u) = \int_{u_0}^u \frac{ds}{w_k(s)}, \quad 0 < u_0 \le u, \; 1 \le k \le l.$$

In (40), t is in the subinterval $(0,b]$ of I so that

$$G_k(1) + \int_0^t h_k(s)\,\Phi_7(s) \prod_{i=1}^l g_i(s) \prod_{r=1}^{k-1} F_r(s)\,ds \in \mathrm{Dom}(G_k^{-1}),$$
$$1 \le k \le l$$

$$F_0(t) = 1.$$

PROOF. Since $g_i(t) \ge 1$, we have from inequality (39)

$$\frac{u(t)}{\prod_{i=1}^l g_i(t)} \le p^*(t) + q(t) \sum_{r=1}^m E_r\left(t, \prod_{i=1}^l g_i \frac{u}{\prod_{i=1}^l g_i}\right)$$

where

$$p^*(t) = p(t) + \sum_{r=1}^l \int_0^t h_r(s)\,w_r(u(s))\,ds.$$

Since $p^*(t)$ is nondecreasing, we obtain on using Corollary 2

$$\frac{u(t)}{\prod_{i=1}^l g_i(t)} \le p^*(t)\,\Phi_7(t)$$

and hence, since $w_k \in S$, we find

$$\frac{u(t)}{\Phi_7(t)\prod_{i=1}^l g_i(t)} \le p(t) + \sum_{r=1}^l \int_0^t h_r(s)\,\Phi_7(s) \prod_{i=1}^l g_i(s)\, w_r\left(\frac{u(s)}{\Phi_7(s)\prod_{i=1}^l g_i(s)}\right) ds$$

or

(41)

$$v(t) \le p(t) + \sum_{r=1}^{l} \int_0^t h_r(s)\, a(s)\, w_r(v(s))\, ds$$

where

$$v(t) = \frac{u(t)}{a(t)}, \quad a(t) = \phi_7(t) \prod_{i=1}^{l} g_i(t).$$

Now, the result follows by induction as in Theorem 8. For $l = 1$, we have

$$\frac{v(t)}{p(t)} \le 1 + \int_0^t h_1(s)\, a(s)\, w_1\left(\frac{v(s)}{p(s)}\right) ds.$$

Let R(t) be the right member of the above inequality, then

$$R'(t) \le h_1(t)\, a(t)\, w_1(R(t)), \quad R(0) = 1$$

and hence

$$\frac{R'(t)}{w_1(R(t))} \le h_1(t)\, a(t).$$

Integrating the above inequality from 0 to t and using the definition of G_1, we obtain after simplification

$$v(t) \le p(t)\, F_1(t)$$

thus, the result is true. Next, let (41) gives

$$v(t) \le p(t) \prod_{i=1}^{k} F_i(t)$$

for some k, $1 < k \le l-1$, then it suffices to show

$$v(t) \le p(t) \prod_{i=1}^{k+1} F_i(t) \quad \text{for } l = k+1.$$

For this, we are given

$$v(t) \le \left[p(t) + \int_0^t h_{k+1}(s)\, a(s)\, w_{k+1}(v(s))\, ds \right]$$
$$+ \sum_{r=1}^{k} \int_0^t h_r(s)\, a(s)\, w_r(v(s))\, ds$$

and hence, since the part inside the bracket is nondecreasing, we find

$$V(t) \le \prod_{i=1}^{k} F_i(t)\left[p(t) + \int_0^t h_{k+1}(s)\, a(s) \times \omega_{k+1}(V(s))\, ds\right]$$

or

$$\frac{V(t)}{p(t)\prod_{i=1}^{k} F_i(t)} \le 1 + \int_0^t h_{k+1}(s)\, a(s) \prod_{i=1}^{k} F_i(s)\, W_{k+1}\left(\frac{V(s)}{p(s)\prod_{i=1}^{k} F_i(s)}\right) ds$$

and from this it is easy to show as for $\ell = 1$, that

$$V(t) \le p(t) \prod_{i=1}^{k} F_i(t) F_{k+1}(t) = p(t) \prod_{i=1}^{k+1} F_i(t).$$

This completes the proof.

REMARK 18. If $q = 0$, $\ell = 1$, $g_1(t) = 1$ and $p(t) = u_o$, Theorem 12 reduces to Bihari's [5] original lemma. For $q = o$, $g_1(t) = 1$, Theorem 12 is same as Theorem 1 of [30]. For several other particular cases of Theorem 12 see [29], [31-44].

THEOREM 13. In Theorem 12, assume $g_i(t)$, $1 \le i \le \ell$ are nondecreasing also. Then

$$u(t) \le p(t)\, \phi_8(t) \prod_{i=1}^{\ell} F_i(t)$$

where, now

$$F_1(t) = g_1(t)\, G_1^{-1}\left[G_1(1) + \int_0^t h_1(s)\, g_1(s)\, \phi_8(s)\, ds\right]$$

$$F_k(t) = g_k(t)\, G_k^{-1}\left[G_k(1) + \int_0^t h_k(s)\, g_k(s)\, \phi_8(s) \times \prod_{i=1}^{k-1} F_i(s)\, ds\right]$$

and

$$\phi_8(t) = 1 + q(t) \int_0^t \sum_{r=1}^{m} E_r'(s,1) \times \exp\left(\int_s^t \sum_{r=1}^{m} E_r'(\tau,q)\,d\tau\right) ds.$$

THEOREM 14. Let the following inequality be satisfied

$$u(t) \le p(t) + q(t) \sum_{r=1}^{m} E_r(t,u) + g(t) \sum_{r=1}^{l} E_r(t, w(u)) \qquad (42)$$

where

(i) $p(t) \geq 1$ and nondecreasing (ii) $g(t) \geq 1$

(iii) $w \in S$. Then

$$u(t) \le p(t)\, g(t)\, \phi_9(t)\, G^{-1}\left[G(1) + \int_0^t \sum_{r=1}^{l} E_r'(s, g\phi_9)\,ds\right],$$

$$0 < t \le b < \infty \qquad (43)$$

where

$$G(u) = \int_{u_0}^{u} \frac{ds}{w(s)}, \quad 0 < u_0 \le u$$

$$\phi_9(t) = 1 + q(t) \int_0^t \sum_{r=1}^{m} E_r'(s,g) \times \exp\left(\int_s^t \sum_{r=1}^{m} E_r'(\tau, gq)\,d\tau\right) ds.$$

In (43), t is in the subinterval $(0,b]$ of I so that

$$G(1) + \int_0^t \sum_{r=1}^{l} E_r'(s, g\,\phi_9)\,ds \in \mathrm{Dom}(G^{-1}).$$

The proofs of Theorems 13 and 14 are similar to that of Theorem 12.

THEOREM 15. Let the inequality (42) be satisfied where (i) p(t) is positive and nondecreasing (ii) $g(t) \geq 1$ (iii) w is positive, nondecreasing, continous and submultiplicative. Then

$$u(t) \le g(t)\,\phi_q(t)\,p(t)\,G^{-1}\Big[G(1)+\int_0^t \sum_{r=1}^{\ell} E_r'\Big(s, \frac{w(g\phi_q p)}{p}\Big)\,ds\Big] \tag{44}$$

where G and $\phi_q(t)$ are same as in Theorem 14 and in (44), t is in the subinterval $(o,b]$ so that

$$G(1)+\int_0^t \sum_{r=1}^{\ell} E_r'\Big(s, \frac{w(g\phi_q p)}{p}\Big)\,ds \in \mathrm{Dom}\,(G^{-1}).$$

PROOF. It is easy to show that

$$u(t) \le \phi_q(t)\,g(t)\,p^*(t)$$

where

$$p^*(t) = p(t) + \sum_{r=1}^{\ell} E_r(t, w(u)).$$

Now since w is submulticative, we find

$$\frac{u(t)}{p(t)\phi_q(t)g(t)} \le 1 + \sum_{r=1}^{\ell} E_r\Big(t, w\Big(\frac{u}{p\phi_q g}\,p\phi_q g\Big)/p\Big)$$

$$\le 1 + \sum_{r=1}^{\ell} E_r\Big(t, w\Big(\frac{u}{p\phi_q g}\Big)\frac{w(p\phi_q g)}{p}\Big).$$

Let $R(t)$ be the right side of the above inequality, then

$$R'(t) \le \sum_{r=1}^{\ell} E_r'\Big(t, w(R)\,w\Big(\frac{p\phi_q g)}{p}\Big)$$

$$\le \sum_{r=1}^{\ell} E_r'\Big(t, \frac{w(p\phi_q g)}{p}\Big)\,w(R(t)),\quad R(o)=1.$$

From the definition of G, it follows that

$$R(t) \le G^{-1}\Big[G(1)+\int_0^t \sum_{r=1}^{\ell} E_r'\Big(s, \frac{w(p\phi_q g)}{p}\Big)\,ds\Big]$$

and hence (44) holds.

REMARK 19. Several particular cases of Theorems 14 and 15 have been discussed in [31-45].

4. SOME APPLICATIONS.

As we have mentioned in the introduction, for several particular systems of integral inequalities, explicit upper estimates can be obtained on using the results of Sections 2 and 3. For example, for two dimensional integral inequalities

$$|x_i(t)| \leq |k_i| + \int_0^t |f_i(s, x_1(s), x_2(s))| ds, \quad i = 1, 2$$

which appear in the study of two dimensional differential system. If

$$|f_i(t, x_1, x_2)| \leq b_i(t) + a_{i1}(t)|x_1| + a_{i2}(t)|x_2|$$

where $b_i(t)$ and $a_{ij}(t)$ are continuous and non-negative, then it follows from Theorem 1.9.3 [10] that $|x_i(t)| \leq u_i(t)$ where $u_i(t)$ be the solution of the differential system

$$u_i'(t) = b_i(t) + a_{i1}(t)u_1(t) + a_{i2}(t)u_2(t)$$
$$u_i(0) = |k_i|, \quad i = 1, 2. \tag{45}$$

From (45), we find

$$u_2(t) = \exp\left(\int_0^t a_{22}(t_1)dt_1\right)\left[|k_2| + \int_0^t (b_2(t_1) + a_{21}(t_1) u_1(t_1)) \exp\left(-\int_0^{t_1} a_{22}(t_2)dt_2\right)dt_1\right]$$

substituting this in the first equation of (45), we have for $u_1(t)$ the exact form as considered in Theorem 1, with $m = 2$, $p(t)$ nondecreasing and $q(t) = 1$. Thus Corollary 2 gives the upper estimate.

For the Volterra integral equation

$$u(t) = w(t) + \int_0^t K(t,s)\, u(s)\, ds$$

the famous method of Courants replacing $K(t,s)$ by approximate degenerate kernel i.e. if $K(t,s) \approx \sum_{r=1}^{m} g_r(t)\, h_r(s)$, then

$$|u(t)| \leq |w(t)| + \sum_{r=1}^{m} |g_r(t)| \int_0^t |h_r(s)|\, |u(s)|\, ds$$

and Theorems 8 and 9 provide upper estimate for $|u(t)|$. These results are also important in studying several properties of solutions of the original equations e.g. see [27] .

In a similar way for the nonlinear Volterra integral equations it is necessary to study the inequalities of the type (39), (42). Besides providing the upper estimate for the differential, integral and integro-differential equations or systems, the inequalities obtained here are useful in proving uniqueness, continuous dependence, stability, asymptotic behaviour and oscillation etc. of the solutions. Since the theory is almost similar to as given in several references we have provided, the details are not repeated here, except we remark that our results will be useful for more general equations.

To show the real advantage of our results, we shall make a comparative study of some results obtained in perturbations of nonlinear differential systems. Following the same notations as in [45] we consider the system

$$x'(t) = f(t, x(t)) \tag{46}$$

and the perturbed system including an operator T as

$$y'(t) = f(t, y(t)) + g(t, y(t), Ty(t)). \tag{47}$$

His Theorem 1 [45] can be restated as follows:

THEOREM 15. Suppose that

$$|\phi(t, s, y)\, g(s, y, z)| \le f_{11}(s)|y| + f_{21}(s)|z|, \quad t, s \in I$$

where $f_{11}, f_{21} \in C[I, R_+]$. Further, suppose that the operator T satisfies the inequality

$$|Ty(t)| \le \int_0^t f_{22}(s)\,|y(s)|\,ds$$

where $f_{22} \in C[I, R_+]$. Then for every bounded solution $x(t) = x(t, t_0, x_0)$ of (46) on I, the corresponding solution $y(t, t_0, x_0)$ of (47) is bounded on I provided

$$\int_0^\infty [f_{11}(t_1) + f_{21}(t_1) \int_0^{t_1} f_{22}(t_2)\,dt_2]\,dt_1 < \infty. \tag{48}$$

PROOF. The proof is similar as in [45] with an application of Corollary 2.

REMARK 20. If $f_{11}(t) = f_{21}(t)$, condition (48) becomes

$$\int_0^\infty f_{11}(t_1)\,(1 + \int_0^{t_1} f_{22}(t_2)\,dt_2)\,dt_1 < \infty. \tag{49}$$

Condition (49) is automatically satisfied if

$$\int_0^\infty f_{11}(t_1)\,dt_1 < \infty, \quad \int_0^\infty f_{22}(t_1)\,dt_1 < \infty \tag{50}$$

as required in his theorem. In several cases (49) is more general than (50), e.g. $f_{11}(t) = e^{-2t}$, $f_{22}(t) = e^{t}$, condition (49) is satisfied whereas (50) does not satisfy. Thus using the results of this Chapter, almost all the results given in [45-59] can be improved.

Next we shall consider the following functional differential equation and discuss asymptotic behaviour and oscillation of its solutions.

$$x^{(m)}(t) + \sum_{i=1}^{m} f_i(t, x(t), x'(t), \ldots, x^{(n-1)}(t), x(g_{i1}(t)), \ldots, x^{(n-1)}(g_{in}(t))) + h(t, x(t), x'(t) \ldots x^{(n-1)}(t), x'(\tau_1(t)), \ldots x^{(n-1)}(\tau_n(t))) = 0 \quad (51)$$

where

$$\begin{aligned} &h, f_i \in C[I \times R^{2n}] \quad i = 1, 2, \ldots, m \\ &g_{ik}, \tau_k \in C[I \times R] \quad k = 1, 2, \ldots, n \end{aligned} \quad (52)$$

and

$$\begin{aligned} &g_{ik}(t) \leq t, t \geq 0; \lim_{t\to\infty} g_{ik}(t) = \infty \\ &\tau_k(t) \leq t, t \geq 0; \lim_{t\to\infty} \tau_k(t) = \infty. \end{aligned} \quad (53)$$

We shall assume that under the initial conditions

$$x(t) = \phi(t), t \leq t_0 \text{ and } x^{(j)}(t_0) = x_j, j = 0, 1, \cdots n-1 \quad (54)$$

the equation (51) has a solution which exists for all $t \geq t_0$. A solution x(t) of equation (51) is called oscillatory if it has no last zero, i.e. if $x(t_1) = 0$ for some t_1 then there is a $t_2 > t_1$ with $x(t_2) = 0$. Equation (51) is called oscillatory if every solution is oscillatory. We call a solution x(t) of equation (51) nonoscillatory if it is eventually of constant sign.

First we shall study the asymptotic behaviour of solutions of equation (51).

THEOREM 16. Let the functions f_i, g_{ik} and τ_k, $i = 1,2,\ldots,m$, $k = 1,2,\ldots,n$ satisfy the conditions (52) and (53) and in addition suppose that

(i) $$|f_i(t, x_1,\ldots,x_n, z_1,\ldots,z_n)| \le \sum_{j=1}^{n} [p_{ij}(t)|x_j|^{\alpha_{ij}} + q_{ij}(t)|z_j|^{\beta_{ij}}] \tag{55}$$

for $t \geq 0, (x_1,\ldots,x_n, z_1,\ldots,z_n) \in R^{2n}$, p_{ij}, q_{ij} are continous functions and α_{ij}, β_{ij} are positive constants.

(ii) $$|h(t, x_1,\ldots,x_n, z_1,\ldots,z_n)| \le H(t). \tag{56}$$

Then, if

$$\int^{\infty} H(s)\,ds < \infty$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\int^{\infty}[|p_{ij}(s)|\, s^{(n-j)\alpha_{ij}} + |q_{ij}(s)|][(g_{ij}(s))]^{(n-j)\beta_{ij}}]\,ds < \infty \tag{58}$$

equation (51) has solutions which are asymptotic to $\sum_{i=0}^{n-1} a_i t^i$ as $t \to \infty$ where $a_{n-1} \neq 0$.

PROOF. Choose $t_0 \geq 1$ so large that $g_{ij}(t) > 0$, $\tau_j(t) > 0$, $t \geq t_0$. Integrating (51), (n-k) times from t_0 to t we obtain

$$x^{(k)}(t) - \sum_{j=0}^{n-k-1} \frac{x^{(j)}(t_0)}{j!}(t-t_0)^j + \frac{1}{(n-k-1)!}\int_{t_0}^{t}(t-s)^{n-k-1}\sum_{i=1}^{m} f_i(s, x(s),\ldots, x^{(n-1)}(s), x(g_{i1}(s)),\ldots, x^{(n-1)}(g_{in}(s)))\,ds$$
$$+ \frac{1}{(n-k-1)!}\int_{t_0}^{t}(t-s)^{n-k-1} h(s, x(s),\ldots, x^{(n-1)}(g_{in}(s)))\,ds = 0.$$

Using (55) , (56) and (57) we get for $t \geq t_0$ the estimate

$$|x^{(k)}(t)| \leq A_k t^{n-k-1} + B_k t^{n-k-1} \int_{t_0}^{t} \sum_{i=1}^{m} \sum_{j=1}^{n} [\,|p_{ij}(s)|\,|x^{(j-1)}(s)|^{\alpha_{ij}} + |q_{ij}(s)|\,|x^{(n-1)}(g_{ij}(s))|^{\beta_{ij}}]\,ds \tag{59}$$

where

$$A_k = t^{-n+k+1} \sum_{j=0}^{n-k+1} \frac{|x^{(j)}(t_0)|}{j!} t^j + \frac{1}{(n-k-1)!} \int_{t_0}^{\infty} H(s)\,ds$$

and

$$B_k = \frac{1}{(n-k-1)!}$$

Let

$A = \max\limits_{0 \leq k \leq n-1} A_k$, $B = \max\limits_{0 \leq k \leq n-1} B_k$, then we get

$$|x^{(k)}(t)| \leq t^{n-k-1} R(t), \quad t \geq t_0 \tag{60}$$

where

$$R(t) = A + B \int_{t_0}^{t} \sum_{i=1}^{m} \sum_{j=1}^{n} [\,|p_{ij}(s)|\,|x^{(j-1)}(s)|^{\alpha_{ij}} + |q_{ij}(s)|\,|x^{(j-1)}(g_{ij}(s))|^{\beta_{ij}}]\,ds. \tag{61}$$

We shall show that $R(t)$ is bounded. For this, choose $t_1 \geq t_0$ so large that $g_{ij}(t) \geq t_0$, $\tau_j(t) \geq t_0$ for $t \geq t_1$. Since $R(t)$ is nondecreasing we have from (60)

$$|x^{(k)}(g_{ij}(t))| \leq [g_{ij}(t)]^{n-k-1} R(t), \quad t \geq t_0. \tag{62}$$

From (61) and in view of (60) and (62) we obtain

$$R(t) \leq A^* + B \int_{t_1}^{t} \sum_{i=1}^{m} \sum_{j=1}^{n} [\,|p_{ij}(s)|\, s^{(n-j)\alpha_{ij}} R^{\alpha_{ij}}(s) +$$

$$+|q_{ij}(s)|(g_{ij}(s))^{(n-j)\beta_{ij}}R^{\beta_{ij}}(s)]\,ds$$

where A^* is a proper positive constant depends on A.

If $R(t) \leq 1$, the conclusion follows; otherwise we have

$$R(t) \leq A^* + B\int_{t_1}^{t}\sum_{i=1}^{m}\sum_{j=1}^{n}[|p_{ij}(s)|\,s^{(n-j)\alpha_{ij}} + |q_{ij}(s)|(g_{ij}(s))^{(n-j)\beta_{ij}}]R^{\gamma}(s)\,ds \tag{63}$$

where

$$\gamma = \max_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}(\alpha_{ij},\beta_{ij}).$$

Now a proper application of Theorem 10 yields

$$R(t) \leq A^*\exp\int_{t_1}^{t}B\sum_{i=1}^{m}\sum_{j=1}^{n}[|p_{ij}(s)|\,s^{(n-j)\alpha_{ij}} + |q_{ij}(s)|(g_{ij}(s))^{(n-j)\beta_{ij}}]\,ds,\ \text{if } \gamma=1 \tag{64}$$

and

$$(R(t))^{1-\gamma} \leq A^{*1-\gamma} + B(1-\gamma)\int_{t_1}^{t}\sum_{i=1}^{m}\sum_{j=1}^{n}[|p_{ij}(s)|\,s^{(n-j)\alpha_{ij}} + |q_{ij}(s)|(g_{ij}(s))^{(n-j)\beta_{ij}}]\,ds\quad \text{if } \gamma\leq 1$$

and is bounded from (58). For $\gamma > 1$ we have

$$[R(t)]^{\gamma-1} \leq [A^{*1-\gamma} - B(\gamma-1)\int_{t_1}^{t}\sum_{i=1}^{m}\sum_{j=1}^{n}[|p_{ij}(s)|\,s^{(n-j)\alpha_{ij}} + |q_{ij}(s)|(g_{ij}(s))^{(n-j)\beta_{ij}}]\,ds]^{-1}.$$

Hence $R(t)$ is bounded provided

$$A^{*1-\gamma} > (\gamma-1)B\int^{\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}[|p_{ij}(s)|\,s^{(n-j)\alpha_{ij}} + |q_{ij}(s)|(g_{ij}(s))^{(n-j)\beta_{ij}}]\,ds. \tag{65}$$

Since $R(t_0) = A$, this corresponds to an appropriate choice of initial conditions for the solutions of (51).

Thus we have shown that $R(t) \le C_1$ where C_1 is some finite positive constant. The inequalities (60) and (62) now become

$$|x^{(k)}(t)| \le C_1 t^{n-k+1}$$
$$|x^{(k)}(g_{ij}(t))| \le C_1 (g_{ij}(t))^{n-k+1}, \quad t \ge t_1. \tag{66}$$

Integrating (51) from t_1 to t we obtain

$$x^{(n-1)}(t) = x^{(n-1)}(t_1) - \int_{t_1}^{t} \sum_{i=1}^{m} f_i(s, \ldots, x^{(n-1)}(g_{in}(s)))\, ds$$
$$- \int_{t_1}^{t} h(s, \ldots, x^{(n-1)}(\tau_n(s)))\, ds\,. \tag{66a}$$

In view of (57), (58) and (66) the integrals in (66a) converge as $t \to \infty$ and therefore the $\lim_{t\to\infty} x^{(n-1)}(t)$ exists and finite number. To ensure that this limit is not zero choose t_0 sufficiently large s.t. $A > 0$ so that (65) holds. This condition on A guarentees s.t. $R(t)$ is bounded for all γ. Choose t_1 so large that for $0 < \epsilon < A$

$$\left| \int_{t_1}^{\infty} h(s, \ldots, x^{(n-1)}(\tau_n(s)))\, ds + \int_{t_1}^{\infty} \sum_{i=1}^{m} f_i(s, \ldots, x^{(n-1)}(g_{in}(s)))\, ds \right| < \epsilon.$$

Then the solution $x(t)$ with the condition $x^{(n-1)}(t_1) = A$ has the desired type of asymptotic behaviour.

COROLLARY 17. If $0 < \gamma \le 1$ and the hypothesis of Theorem 16, the equation (51) has nonoscillatory solutions. If $\gamma > 1$, for the appropriate choice of initial conditions satisfying (65) the equation (51) has nonoscillatory solutions.

Next we shall give oscillation theorem for bounded solutions of the equation

$$x^{(n)}(t) + \sum_{i=1}^{m} p_i(t) F_i(x(t), x'(t), \cdots, x^{(n-1)}(t), x(g_{i1}(t)), \cdots, x^{(n-1)}(g_{in}(t)))$$
$$+ h(t, x(t), \cdots, x^{(n-1)}(t), x(\tau_1(t)), \cdots, x^{(n-1)}(\tau_n(t))) = 0 \quad (67)$$

THEOREM 18. Let the functions g_{ik}, τ_k satisfy the condition (53) and in addition, suppose that

(i) $p_i(t) \geq 0 \quad i = 1, 2, \cdots, m$ for sufficiently large t

(ii) if x_i and z_i have the same sign then $F(x_1, \cdots, x_n, z_1, \cdots, z_n)$ $i = 1, 2, \ldots, n$ has that sign for all sufficiently large t

(iii) if x_i and z_i have the same sign then $h(t, x_1, \cdots, x_n, z_1, \cdots, z_n)$ has that sign for all sufficiently large t

(iv) there exists an index j such that

$$\int^{\infty} t^{n-1} p_j(t)\, dt = \infty \quad (68)$$

Then, (a) for n even all bounded solutions of (67) are oscillatory (b) for n odd, all bounded solutions of (67) are either oscillatory or tend to zero monotonically.

PROOF. Let $x(t)$ be a bounded nonoscillatory solution of (67) and hence we may assume that there is a t_0 such that $x(t) > 0$ for $t \geq t_0$. The case $x(t) < 0$ is treated similarly. Since $\lim_{t\to\infty} g_{i1}(t) = \infty$, $i = 1, 2, \ldots, m$ there exists a $t_1 \geq t_0$ such that $x(g_{i1}(t)) > 0$ for $t \geq t_1$. Therefore by (67) and conditions (i) - (iii) we have $x^{(n)}(t) \leq 0$ for $t \geq t_1$.

Since $x(t)$ is a bounded positive solution of (67) it follows that

$$x^{(j)}(t)\, x^{(j+1)}(t) \leq 0$$

for each $j = 1,2,\ldots,n-1$ and sufficiently large t_2 say $t \geq t_2 \geq t_1$. It follows that $(-1)^{j+1} x^{(n-j)}(t) \geq 0$, $j = 1,2,\cdots n-1$ for $t \geq t_2$, also

$$\lim_{t\to\infty} x^{(j)}(t) = 0, \quad j = 1,2,\cdots,n-1. \tag{69}$$

Once again from (53), we may assume that there is a $t_3 \geq t_2$ such that for $t \geq t_3 \geq t_2$ we have

$$(-1)^{j+1} x^{(n-j)}(g_{i(n-j+1)}(t)) \geq 0, \quad j=1,2,\cdots,n-1$$
$$i = 1,2,\cdots,m.$$

Also,

$$\lim_{t\to\infty} x^{(j)}(g_{i(j+1)}(t)) = 0. \tag{70}$$

Since $x(t)$ is of fixed sign for $t \geq t_3$, it follows that $\lim_{t\to\infty} x(t) = x(\infty)$ exists and it is bounded (since $x(t)$ is bounded). If n is even then $x(\infty) > 0$ because in this case $x'(t) > 0$. If n is odd either $x(\infty)$ is zero which proves (b) or $x(\infty) > 0$. So we assume that $x(\infty) > 0$ for n even or odd and we shall prove a contradiction.

Since $x(g_{11}(t)) > 0$ for $t \geq t_3$, we have from equation (67) and conditions (i) - (iii)

$$x^{(n)}(t) + p_j(t) F_j(x(t), \cdots, x^{(n-1)}(t), x(g_{j1}(t)), \cdots, x^{(n-1)}(g_{jn}(t))) < 0. \tag{71}$$

Using (69), (70) and the continuity of F_j it follows that

$$\lim_{t\to\infty} F_j(x(t),\dots,x^{(n-1)}(t),\ x(g_{j1}(t)),\dots,x^{(n-1)}(g_{jn}(t))) = L$$

exists and is a finite positive number. Therefore for sufficiently large $t_4 \geq t_3$

$$\frac{L}{2} \leq F_j(x(t),\dots,x^{(n-1)}(t),\ x(g_{j1}(t)),\dots,x^{(n-1)}(g_{jn}(t))),\qquad t \geq t_4$$

and from the inequality (71), we find

$$x^{(n)}(t) + \frac{L}{2}\,p_j(t) < 0. \tag{72}$$

Multiplying both sides of (72) by t^{n-1} and integrating from t_4 to t we obtain

$$\int_{t_4}^{t} s^{n-1}x^{(n)}(s)\,ds + \frac{L}{2}\int_{t_4}^{t} s^{n-1}p_j(s)\,ds \leq 0. \tag{73}$$

Successive integration by parts of the first integral in (73) gives

$$\int_{t_4}^{t} s^{n-1}x^{(n)}(s)\,ds = P(t) - P(t_4) + (-1)^{n+1}n!\,[x(t) - x(t_4)] \tag{74}$$

where

$$P(t) = \sum_{k=1}^{n-1}(-1)^{k+1}(n-1)(n-2)\dots(n-k+1)\,x^{(n-k)}(t)$$

which is positive. Since x(t) is bounded and because of the hypothesis (iv) the inequality (73) is impossible. This completes the proof.

Finally we shall consider the equation

$$(r(t)\,x'(t))^{(2n-1)} + \sum_{i=1}^{m} p_i(t)\,F_i(x_{\tau_i}(t), x'_{\sigma_i}(t), \dots, x^{(2n-1)}_{\sigma_i}(t))$$
$$+ h(x_{a_1}(t), x'_{a_2}(t), \dots, x^{(2n-1)}_{a_{2n}}(t)) = 0 \tag{75}$$

where

$$x_{\tau_i}(t) = x(t-\tau_i(t)) \quad i = 1, 2, \dots, m$$

$$x^{(j)}_{\sigma_i}(t) = x^{(j)}(t-\sigma_i(t)) \quad j = 1, 2, \dots, 2n-1$$

$$x^{(r)}_{a_k}(t) = x^{(r)}(t-a_k(t)) \quad r = 0, 1, \dots, 2n-1, \; k = 1, 2, \dots, 2n.$$

The delays $\tau_i(t)$, $\sigma_i(t)$ and $a_k(t)$ are bounded by a common constant M, nonnegative, nondecreasing and continuous real valued functions of t. The functions

$$\begin{aligned} p_i(t) &: R \to R \text{ and continuous for each } i, \\ F_i &: R^{2n} \to R \text{ and continuous for each } i, \\ h &: R^{2n} \to R \text{ is continuous.} \end{aligned} \tag{76}$$

The following lemma of Kiguradze [60] will be needed.

LEMMA 19. If u(t) is a function such that it and all its derivatives upto order (2n-2) inclusive, are absolutely continuous and of constant sign in the interval (t_0, ∞) and $u^{(2n-1)}(t)\,u(t) \leq 0$ then there is an integer l, $0 \leq l \leq 2n-1$ which is even so that for $t \geq t_0$ we have

$$u^{(k)}(t)\,u(t) \geq 0, \quad k = 0, 1, \dots, l$$

$$(-1)^{2n+k-2}\,u^{(k)}(t)\,u(t) \geq 0, \quad k = l+1, \dots, 2n-1$$

and

$$|u(t)| \geq \frac{(t-t_0)^{2n-2}}{(2n-2)\cdots(2n-l-1)}\left|u^{(2n-2)}(2^{2n-l-2}t)\right|.$$

THEOREM 20. Suppose in addition to (76) the following conditions hold

(i) $p_i(t) \geq 0$ for every $t \in [t_0, \infty)$

(ii) $r(t) \in C^{2n-1}[t_0, \infty)$, $r(t)$ is bounded and satisfies

$$r(t) > 0,\ r'(t) > 0,\ (-1)^{j+1}(r^{(j)}(t)) \geq 0,\ j = 2, 3, \cdots, 2n-1$$

(iii) sgn $F_i(x_1, x_2, \cdots, x_{2n}) = \operatorname{sgn} x_1$ and

$$F_i(-x_1, -x_2, \cdots, -x_{2n}) = -F_i(x_1, \cdots, x_{2n}) \text{ for all } i$$

(iv) $\operatorname{sgn} h(x_1, x_2, \cdots, x_{2n}) = \operatorname{sgn} x_1$ and

$$h(-x_1, -x_2, \cdots, -x_{2n}) = -h(x_1, x_2, \cdots, x_{2n})$$

(v) $I \neq \phi$ where I denotes the set of all indices for which the function $F_i(x_1, x_2 \ldots, x_{2n})$ is nondecreasing with respect to each variable $x_1, x_3, x_5, \ldots, x_{2n-1}$ and decreasing with respect to $x_2, x_4, \cdots, x_{2n}$ as well as the functions $\frac{h(x, 0, \cdots 0)}{x}$ and $F_i(x, 0, \cdots, 0)/x$ which are nonincreasing on $(0, \infty)$.

(vi) there exists a positive and differentiable function $\phi(t)$, $t \geq t_1$ for some t_1 such that

$$\int^{\infty} [\phi(s) \sum_{i \in I} \frac{p_i(2^{2n-2}s) F_i(s^{2n-1}, 0, \cdots, 0) + h(s^{2n-1}, 0, \cdots, 0)}{k\, s^{2n-1}}$$

$$- \frac{1}{4} \frac{r(s)\, \phi'^2(s)\, (2n-2)!}{\phi(s)\, (2s)^{2n-2}}]\, ds = \infty$$

for every $k \geq 1$.

Then all bounded solutions of equation (75) are oscillatory.

PROOF. Let $x(t)$ be a bounded nonoscillatory solution of (75). Conditions (i) - (iv) imply that $-x(t)$ is again a solution of (75). Therefore we can assume that $x(t) > 0$ eventually. Following the results obtained in [61], [62] we find that

$$(-1)^j (r(t)\, x'(t))^{(j)} \geq 0 , \quad j = 0, 1, 2, \cdots, 2n-1 \tag{77}$$

$$x(t) > 0,\ x'(t) > 0,\ x''(t) \leq 0,\ x'''(t) \geq 0, \cdots, x^{(2n)}(t) \leq 0 \tag{78}$$

for sufficiently large $t \geq t_0$. Also

$$\lim_{t \to \infty} x^{(j)}(t) = 0, \quad j = 2, 3, \cdots, 2n-1. \tag{79}$$

Now define the transformation

$$Z(t) = -\frac{(r(2^{2n-2}t)\, x'(2^{2n-2}t))^{(2n-2)}}{x(t-M)}\, \phi(t) , \quad t \geq t_0 \tag{80}$$

to obtain on using (v), (78)

$$\begin{aligned} Z'(t) \geq \phi(t) \sum_{i \in I} & \frac{P_i(2^{2n-2}t)\, F_i(x(t-M), 0, \cdots, 0)}{x(t-M)}\, 2^{2n-2} \\ & + \frac{\phi(t)\, h(x(t-M), 0, \cdots, 0)}{x(t-M)}\, 2^{2n-2} + \frac{Z(t)\, \phi'(t)}{\phi(t)} \\ & - \frac{Z(t)\, x'(t-M)}{x(t-M)} . \end{aligned} \tag{81}$$

Now in the lemma 19 we take $u(t) = r(t)x'(t)$ and obtain on using (77) and (78) also the behaviour of $r(t)$

$$r(t)\, x'(t) \geq \frac{t^{2n-2}}{(2n-2)!} (r(2^{2n-2}t)\, x'(2^{2n-2}t))^{(2n-2)}$$

or

$$x'(t-M) \geq \frac{t^{2n-2}}{(2n-2)!} \frac{1}{r(t)} (r(2^{2n-2}t)\, x'(2^{2n-2}t))^{(2n-2)} , \quad t \geq t_0 + M. \tag{82}$$

Using (82) in (81) we obtain the last two terms as

$$\frac{z(t)\,\phi'(t)}{\phi(t)} - \frac{z(t)\,x'(t-M)}{x(t-M)} \geq \frac{z(t)\,\phi'(t)}{\phi(t)} + \frac{z^2(t)\,t^{2n-2}}{\phi(t)\,r(t)\,(2n-2)!}$$

$$= \frac{t^{2n-2}}{\phi(t)\,r(t)\,(2n-2)!}\left[z(t) + \frac{1}{2}\,\frac{r(t)\,\phi'(t)}{t^{2n-2}} \times (2n-2)!\right]^2 - \frac{1}{4}\,\frac{r(t)\,\phi'^2(t)\,(2n-2)!}{\phi(t)\,t^{2n-2}}$$

$$\geq -\frac{1}{4}\,\frac{r(t)\,\phi'^2(t)\,(2n-2)!}{\phi(t)\,t^{2n-2}}, \quad t \geq t_0+M$$

and hence

$$z'(t) \geq \phi(t)\sum_{i\in I}\frac{p_i(2^{2n-2}t)\,F_i(x(t-M),0,\cdots,0)}{x(t-M)}\,2^{2n-2} + \frac{\phi(t)\,h(x(t-M),0,\cdots,0)}{x(t-M)}\,2^{2n-2} - \frac{1}{4}\,\frac{r(t)\,\phi'^2(t)\,(2n-2)!}{\phi(t)\,t^{2n-2}}, \quad t \geq t_0+M. \tag{83}$$

Since $x^{(2n)}(t) \leq 0$ for $t \geq t_0+M$, we have on using Taylor's formula

$$x(t-M) \leq P(t)$$

where

$$P(t) = \sum_{j=0}^{2n-1}\frac{x^{(j)}(t_0+M)}{j!}\,(t-t_0-2M)^j.$$

Since $\displaystyle\lim_{t\to\infty}\frac{P(t)}{t^{2n-1}} = \frac{x^{(2n-1)}(t_0+M)}{(2n-1)!}$ is finite, it follows that there exists a $t_1 \geq t_0+M$ and an appropriate constant $k \geq 1$ such that

$$x(t-M) \leq k\,t^{2n-1}, \quad t \geq t_1. \tag{84}$$

Using (84) in (83) and using (v), we find that

$$Z'(t) \geq \phi(t) \sum_{i \in I} \frac{P_i(2^{2n-2}t) F_i(t^{2n-1}, 0, \cdots, 0)}{k t^{2n-1}} 2^{2n-2}$$

$$+ \frac{\phi(t) h(t^{2n-1}, 0, \cdots, 0)}{k t^{2n-1}} 2^{2n-2}$$

$$- \frac{1}{4} \frac{r(t) \phi'^2(t)}{\phi(t) t^{2n-2}} (2n-2)!, \quad t \geq t_1. \qquad (85)$$

On integrating (85) from t_1 to t, we obtain

$$Z(t) \geq Z(t_1) + 2^{2n-2} \int_{t_1}^{t} \Big[\phi(s) \sum_{i \in I} \frac{P_i(2^{2n-2}s) F_i(s^{2n-1}, 0, \cdots, 0)}{k s^{2n-1}}$$

$$+ \frac{h(s^{2n-1}, 0, \cdots, 0)}{k s^{2n-1}}$$

$$- \frac{1}{4} \frac{r(s) \phi'^2(s) (2n-2)!}{\phi(s) (2s)^{2n-2}} \Big] ds.$$

From (vi), we find Z(t) eventually positive which is a contradiction. Hence the result follows.

REMARK 21. Dahiya [62] has obtained similar result to our Theorem 20 by using extra condition on $\phi(t)$ i.e. $\phi'(t) \leq 0$. He has also given an extra condition on F_j :

$$F_j(\lambda x_1, \cdots, \lambda x_{2n}) = \lambda F_j(x_1, x_2, \cdots, x_{2n}) \qquad (86)$$

for every $(x_1, x_2, \cdots, x_{2n}) \in R^{2n}$ and $\lambda \in R$ which is being never used in proving the main theorem. Also the condition

$$h(\lambda x_1, \cdots, \lambda x_{2n}) = \lambda h(x_1, \cdots, x_{2n}) \qquad (87)$$

for every $(x_1, x_2, \cdots, x_{2n}) \in R^{2n}$, $\lambda \in R$ is used in his Theorem 3 without mention.

Now we shall state our next theorem which also does not require $\phi'(t) \leq 0$ as used in [62].

THEOREM 21. Suppose equation (75) satisfy conditions (i) - (v) of Theorem 20 also (86) and (87) holds. There exists a positive and differentiable function $\phi(t)$, $t \geq t_1$ for some t_1 such that

$$\int^{\infty} \Big[\phi(s) \sum_{i \in I} \left\{ P_i(2^{2n-2} s) F_i(1,0,\cdots,0) + h(1,0,\cdots,0) \right\} - \frac{1}{4} \frac{r(s)\, \phi'^2(s)\, (2n-2)!}{\phi(s)\, (2s)^{2n-2}} \Big] ds = \infty .$$

Then all bounded solutions of equations (75) are oscillatory.

PROOF. The proof follows from the Theorem 20.

CHAPTER 2

DISCRETE GRONWALL TYPE INEQUALITIES

1. INTRODUCTION.

The theory of difference equations is in a process of continuous development and it has become significant for its various applications in numerical analysis, control systems, engineering and so on. The role played by the discrete inequalities in the theory of difference equations is well known. One of the most used discrete inequality is the analogue of GBR inequality established by Jones [63] and Sugiyama [64] and it varients [64 and references therein].

In this Chapter we shall study the discrete analogue of the results presented in Chapter 1. Several known results are improved and some applications to discrete stochastic models are given.

The following notations we shall use throughout this Chapter. N_0 denotes the set $\{0,1,\cdots\}$. The expression $\sum_{s=0}^{t-1} b(s)$ represents a solution of the linear difference equation $\Delta u(t) = b(t)$ for all $t \in N_0$, under the initial condition $x(0) = 0$, where Δ is the operator defined by $\Delta u(t) = u(t+1) - u(t)$. It is supposed that $\sum_{s=0}^{-1} b(s) = 0$. The expression $\prod_{s=0}^{t-1} c(s)$ represents the solution of the linear difference equation $u(t+1) = c(t)u(t)$ for all $t \in N_0$ under the initial condition $u(0) = 1$. It is supposed that $\prod_{s=0}^{-1} C(s) = 1$.

In what follows we shall assume that all the functions appearing in the inequalities are real-valued, nonnegative and defined in N_0.

2. LINEAR GENERALIZATION.

THEOREM 1. Let the following inequality be satisfied

$$u(t) \leq p(t) + q(t) \sum_{r=1}^{m} E_r(t,u) \tag{1}$$

where

$$E_r(t,u) = \sum_{t_1=0}^{t-1} f_{r1}(t_1) \sum_{t_2=0}^{t_1-1} f_{r2}(t_2) \cdots \sum_{t_r=0}^{t_{r-1}-1} f_{rr}(t_r)\, u(t_r)$$

for all $t \in N_0$. Then

$$u(t) \leq p(t) + q(t) \sum_{s=0}^{t-1} \Big(\sum_{r=1}^{m} \Delta E_r(s,p) \Big) \times \prod_{\tau=s+1}^{t-1} \Big[1 + \sum_{r=1}^{m} \Delta E_r(\tau, q) \Big]. \tag{2}$$

PROOF. Define $R(t)$ as follows

$$R(t) = \sum_{r=1}^{m} E_r(t,u), \quad R(0) = 0$$

then it follows that

$$\Delta R(t) = \sum_{r=1}^{m} \Delta E_r(t,u)$$
$$\leq \sum_{r=1}^{m} \Delta E_r(t,p) + \sum_{r=1}^{m} \Delta E_r(t, qR).$$

Since $\Delta R(t) \geq 0$, $R(t)$ is nondecreasing on N_0, and we find

$$R(t+1) - \Big[1 + \sum_{r=1}^{m} \Delta E_r(t,q) \Big] R(t) \leq \sum_{r=1}^{m} \Delta E_r(t,p).$$

Multiply the above inequality by $\prod_{s=0}^{t}\left[1+\sum_{r=1}^{m}\Delta E_r(t,q)\right]^{-1}$,

and summing over from 0 to t-1, we obtain

$$R(t)\prod_{s=0}^{t-1}\left[1+\sum_{r=1}^{m}\Delta E_r(s,q)\right]^{-1}\leq\sum_{s=0}^{t-1}\left(\sum_{r=1}^{m}\Delta E_r(s,p)\right)\times\prod_{\tau=0}^{s}\left[1+\sum_{r=1}^{m}\Delta E_r(\tau,q)\right]^{-1}$$

hence

$$R(t)\leq\sum_{s=0}^{t-1}\left(\sum_{r=1}^{m}\Delta E_r(s,p)\right)\times\prod_{\tau=s+1}^{t-1}\left[1+\sum_{r=1}^{m}\Delta E_r(\tau,q)\right]^{-1}.$$

Substituting this estimate in (1), the result (2) follows.

COROLLARY 2. In (1), let p(t) be nondecreasing.

Then

$$u(t)\leq p(t)\left[1+q(t)\sum_{s=0}^{t-1}\left(\sum_{r=1}^{m}\Delta E_r(s,1)\right)\times\prod_{\tau=s+1}^{t-1}\left[1+\sum_{r=1}^{m}\Delta E_r(\tau,q)\right]\right]\tag{3}$$

also, if $q(t)\equiv 1$

$$u(t)\leq p(t)\prod_{s=0}^{t-1}\left[1+\sum_{r=1}^{m}\Delta E_r(s,1)\right].\tag{4}$$

PROOF. (3) follows from (2) without any difficulty.

To show (4), we note that

$$\begin{aligned}\sum_{s=0}^{t-1}r(s)\prod_{\tau=s+1}^{t-1}[1+r(\tau)] &= \sum_{s=0}^{t-1}\left\{\prod_{\tau=s}^{t-1}(1+r(\tau))-\prod_{\tau=s+1}^{t-1}(1+r(\tau))\right\}\\ &= \prod_{\tau=0}^{t-1}(1+r(\tau))-\prod_{\tau=t}^{t-1}(1+r(\tau))\\ &= \prod_{\tau=0}^{t-1}(1+r(\tau))-1.\end{aligned}$$

Now substituting $q(t)\equiv 1$ in (3) and $r(t)=\sum_{r=1}^{m}\Delta E_r(t,1)$

and using the above relation the result (4) follows.

COROLLARY 3. Let the following inequality be satisfied

$$u(t) \le p(t) + \sum_{r=1}^{m} g_r(t) E_r(t,u)$$

where $g_i(t) \geq 1, 1 \le i \le m$. Then

(a) $$u(t) \le p(t) + \prod_{r=1}^{m} g_r(t) \sum_{s=0}^{t-1} \left(\sum_{r=1}^{m} \Delta E_r(s,p)\right) \prod_{\tau=s+1}^{t-1} \left[1 + \sum_{r=1}^{m} \Delta E_r(\tau, \prod_{r=1}^{m} g_r)\right].$$

(b) If $p(t)$ is nondecreasing. Then

$$u(t) \le p(t)\left[1 + \prod_{r=1}^{m} g_r(t) \sum_{s=0}^{t-1} \left(\sum_{r=1}^{m} \Delta E_r(s,1)\right) \times \prod_{\tau=s+1}^{t-1} \left[1 + \sum_{r=1}^{m} \Delta E_r(\tau, \prod_{r=1}^{m} g_r)\right]\right.$$

$$\le p(t) \prod_{r=1}^{m} g_r(t) \prod_{s=0}^{t-1} \left[1 + \sum_{r=1}^{m} \Delta E_r(s, \prod_{r=1}^{m} g_r)\right].$$

REMARK 1. In the inequality (1), let $m = 1$, $q = 1$, then (2) is same as obtained by Sugiyama [64].

REMARK 2. In the inequality (1), let $m = 1$, then (2) is same as obtained by Pachpatte [65].

A discrete version of Theorem 4 (Chapter 1) which covers almost all the results obtained in [65-71] is the following:

THEOREM 4. Let the inequality (1) be satisfied, where $f_{ii} = f_i, 1 \le i \le m$; $f_{i+1,i} = f_{i+2,i} = \dots = f_{mi} = g_i, 1 \le i \le m-1$ for all $t \in N_0$. Then if $g_i(t) < 1$, $1 \le i \le m-2$

$$u(t) \le p(t) + q(t) P_j(t), \quad 1 \le j \le m \tag{5}$$

where

$$P_1(t) = \sum_{t_1=0}^{t-1} p(t_1) \sum_{r=1}^{m} f_r(t_1) \prod_{t_2=t_1+1}^{t-1} [1 + \sum_{r=1}^{m} q(t_2) f_r(t_2) \bigcup_{i=1}^{m-1} g_i(t_2)]$$

$$P_j(t) = \sum_{t_1=0}^{t-1} [p(t_1) \sum_{=1}^{m-j+1} f_r(t_1) + g_{m-j+1}(t_1) P_{j-1}(t_1)] \times$$
$$\prod_{t_2=t_1+1}^{t-1} [1 + \sum_{r=1}^{m-j+1} q(t_2) f_r(t_2) \bigcup_{i=1}^{m-j} g_i(t_2) - g_{m-j+1}(t_2)],$$
$$2 \le j \le m.$$

PROOF. The proof is similar to that of Theorem 4 (Chapter 1) and Theorem 1.

REMARK 3. Instead of $g_i(t) < 1$, $1 \le i \le m-2$, it is sufficient to assume

$$1 + \sum_{r=1}^{m-j+1} q(t_2) f_r(t_2) \bigcup_{i=1}^{m-j} g_i(t_2) > g_{m-j+1}(t_2), \; t_2 \in N_0, \quad (6)$$
$$2 \le j \le m.$$

REMARK 4. If (6) is not satisfied, then

$$u(t) \le p(t) + q(t) P_j(t), \; j = 1, 2 \quad (7)$$

where $P_1(t)$ is same as in Theorem 4 and $P_2(t)$ is defined as

$$P_2(t) = \sum_{s=0}^{t-1} [p(s) f_1(s) + (q(s) f_1(s) \cup g_1(s)) P_1(s)].$$

A discrete analogue of Theorem 5 (Chapter 1) is the following:

THEOREM 5. Let $m = 2$, $p(t) = u_0$, $q(t) = 1$ and $f_{11}(t) = f_{21}(t)$ in (1). Then

$$u(t) \le u_0 [1 + \sum_{s=0}^{t-1} f_{11}(s) (1 - \phi(s)) \prod_{\tau=0}^{s-1} (1 + f_{11}(\tau) + f_{22}(\tau))] \quad (8)$$

where

$$\phi(t) = \sum_{s=0}^{t-1} f_{22}(s) (\prod_{\tau=0}^{s} [1 + f_{11}(\tau) + f_{22}(\tau)]^{-1}) \sum_{\tau=0}^{s-1} f_{22}(\tau).$$

The proof of the above theorem is similar to that of Theorem 5 (Chapter 1) and Theorem 1.

REMARK 5. If $\phi(t) = 0$ in (8), the estimate is same as given in [66], also obtained from Remark 4. Using the similar lines it is easy to improve all the results obtained in [65-71].

Our next result is the discrete analogue of Willett's inequality [26].

THEOREM 6. Let the following inequality be satisfied

$$u(t) \leq p_0(t) + \sum_{i=1}^{m} p_i(t) \Big(\sum_{s=0}^{t-1} v_i(s) u(s) \Big) \tag{9}$$

for all $t \in N_0$. Then for all $t \in N_0$

$$u(t) \leq F_m p_0(t) \tag{10}$$

where

$$F_i = D_i D_{i-1} \cdots D_0$$

$$D_0 w = w$$

$$D_j w = w + (F_{j-1} p_j) \Big(\sum_{s=0}^{t-1} v_j w \prod_{\tau=s+1}^{t-1} (1 + v_j F_{j-1} p_j) \Big), \quad j = 1, 2, \cdots, m.$$

PROOF. For $m = 1$, we obtain from Theorem 1

$$u(t) \leq p_0(t) + p_1(t) \sum_{s=0}^{t-1} v_1(s) p_0(s) \prod_{\tau=s+1}^{t-1} (1 + v_1(\tau) p_1(\tau))$$

$$= F_1 p_0(t).$$

Now, assume that the result is true for some k such that $1 < k \leq m-1$, then for k+1, we are given

$$u(t) \leq p_0^*(t) + \sum_{i=1}^{k} p_i(t) \sum_{s=0}^{t-1} v_i(s) u(s) \tag{11}$$

where

$$p_0^*(t) = p_0(t) + p_{k+1}(t)\sum_{s=0}^{t-1} v_{k+1}(s)\,u(s).$$

From (11), we find

$$u(t) \le F_k\, p^*(t).$$

From the definition of F_k, we have

$$u(t) \le F_k\, p_0(t) + (F_k\, p_{k+1}(t))\Big(\sum_{s=0}^{t-1} v_{k+1}(s)\,u(s)\Big)$$

which is obtained on using the fact $\sum_{s=0}^{t-1} v_{k+1}(s)\,u(s)$ is nondecreasing for all $t \in N_0$. .

Once again on using Theorem 1, we obtain

$$u(t) \le F_k\, p_0(t) + (F_k\, p_{k+1}(t))\sum_{s=0}^{t-1} v_{k+1}(s) F_k\, p_0(s)\prod_{\tau=s+1}^{t-1}[1+v_{k+1}(\tau)F_k\, p_{k+1}(\tau)]$$

$$= D_{k+1}(F_k\, p_0(t))$$

$$= F_{k+1}\, p_0(t).$$

Hence the result follows from finite induction.

Next result is the discrete version of Theorem 8 (Chapter 1).

COROLLARY. 7. Let the inequality (9) be satisfied for all $t \in N_0$, where (i) $p_0(t)$ is nondecreasing (ii) $p_i(t) \geq 1$ $1 \le i \le m$ and are nondecreasing for $2 \le i \le m$. Then

$$u(t) \le F_m\, p_0(t) \tag{12}$$

where

$$F_0\,\omega = \omega$$

$$F_k\,\omega = \omega(F_{k-1}\, p_k)\prod_{s=0}^{t-1}[1 + v_k(s)\, F_{k-1}(p_k(s))],$$

$$k = 1, 2, \cdots, m.$$

3. NONLINEAR GENERALIZATION.

THEOREM 8. Let the following inequality be satisfied

$$u(t) \le p(t)\Big[u_0 + \sum_{r=1}^{m-1} E_r(t,u) + E_m(t,u^{\alpha})\Big] \tag{13}$$

for all $t \in N_0$, where $u_0 \geq 0$ and $0 \leq \alpha < 1$. Then for all $t \in N_0$

$$u(t) \leq p(t)\, e^{-1}(t)\Big\{u_0^{1-\alpha} + (1-\alpha)\sum_{s=0}^{t-1} \Delta E_m(s, p^{\alpha})\{e(s+1)\}^{1-\alpha}\Big\}^{1/1-\alpha} \tag{14}$$

where

$$e(t) = \prod_{s=0}^{t-1} \Big[1 + \sum_{r=1}^{m-1} \Delta E_r(s, p)\Big]^{-1}.$$

PROOF. Let R(t) be the term inside the bracket of right side of (13), then as in Theorem 1, it follows that

$$R(t+1) - \Big[1 + \sum_{r=1}^{m-1} \Delta E_r(t, p)\Big] R(t) \leq \Delta E_m(t, p^{\alpha}) R^{\alpha}(t).$$

On multiplying the above inequality by e(t+1), we obtain

$$\Delta[R(t)\,e(t)] \leq \Delta E_m(t, p^{\alpha})\, e^{1-\alpha}(t+1)\,[R(t)\,e(t+1)]^{\alpha}. \tag{15}$$

For all those $t \in N_0$ when $\Delta[R(t)\,e(t)] \geq 0$, we have

$$\frac{\Delta[R(t)\,e(t)]^{1-\alpha}}{1-\alpha} = \int_t^{t+1} \frac{d[R(s)\,e(s)]}{[R(s)\,e(s)]^{\alpha}} \leq \frac{\Delta[R(t)\,e(t)]}{[R(t)\,e(t+1)]^{\alpha}}$$

and from (15), we have

$$\Delta[R(t)\,e(t)]^{1-\alpha} \leq (1-\alpha)\,\Delta E_n(t, p^{\alpha})\, e^{1-\alpha}(t+1). \tag{16}$$

Similarly for all those $t \in N_0$ when $\Delta[R(t)\,e(t)] \leq 0$, we have $\dfrac{\Delta[R(t)\,e(t)]^{1-\alpha}}{1-\alpha} \leq 0$, and hence (16) is obviously satisfied. Summing over (16) from 0 to t-1, the result (14) follows.

COROLLARY 9. Let the following inequality be satisfied

$$u(t) \leq p(t)\Big[u_0 + \sum_{r=1}^{m} H_r(t, u)\Big]$$

where

$$H_r(t,u) = \sum_{t_1=0}^{t-1} f_{r1}(t_1)\, u^{\alpha_{r1}}(t_1) \cdots \sum_{t_r=0}^{t_{r-1}-1} f_{rr}(t_r)\, u^{\alpha_{rr}}(t_r)$$

for all $t \in N_0$. Then

$$u(t) \le u_0\, p(t) \prod_{s=0}^{t-1} \Big[1 + \sum_{r=1}^{m} \Delta H_r(s,p)\, u_0^{\alpha_r-\alpha}\Big] \text{ if } \alpha = 1$$

$$u(t) \le p(t)\Big[u_0^{1-\alpha} + (1-\alpha)\sum_{s=0}^{t-1}\sum_{r=1}^{m} \Delta H_r(s,p)\, u_0^{\alpha_r-\alpha}\Big]^{1/1-\alpha}, \text{ if } \alpha \ne 1$$

where if $\alpha > 1$, we assume $u_0^{1-\alpha} + (1-\alpha)\sum_{s=0}^{t-1}\sum_{r=1}^{m} \Delta H_r(s,p)\, u_0^{\alpha_r-\alpha} > 0$.

PROOF. The proof is similar to that of Theorem 10 (Chapter 1) and Theorem 8.

REMARK 6. Several particular cases of Theorem 8 and Corollary 9 have been discussed in [65-71], however the results are not comparable, but following the same lines as in Theorem 5 most of his results can be improved uniformly.

Next result is the discrete analogue of Theorem 12 (Chapter 1), here also we shall follow the same definition of Class S.

THEOREM 10. Let the following inequality be satisfied

$$u(t) \le p(t) + q(t)\sum_{r=1}^{m} E_r(t,u) + \sum_{r=1}^{l} g_r(t) \times \sum_{s=0}^{t-1} h_r(s)\, w_r(u(s))\, ds$$

for all $t \in N_0$, where

(i) $p(t) \geqslant 1$ and nondecreasing (ii) $g_i(t) \geqslant 1$, $1 \le i \le l$ (iii) $w_r \in S$, $1 \le r \le l$.

Then

$$u(t) \le p(t)\,\phi(t) \prod_{i=1}^{l} g_i(t) \prod_{r=1}^{l} F_r(t)$$

where

$$\phi(t) = 1 + q(t) \sum_{s=0}^{t-1} \Big(\sum_{r=1}^{m} \Delta E_r(s, \prod_{i=1}^{l} g_i) \Big) \prod_{\tau=s+1}^{t-1} \Big[1 + \sum_{r=1}^{m} \Delta E_r(\tau, q \prod_{i=1}^{l} g_i) \Big]$$

$$F_k(t) = G_k^{-1}\Big[G_k(1) + \sum_{s=0}^{t-1} h_r(s)\,\phi(s) \prod_{i=1}^{l} g_i(s) \times \prod_{r=1}^{k-1} F_r(s) \Big],\ F_0(t) = 1,\ 1 \le k \le l$$

$$G_k(u) = \int_{u_0}^{u} \frac{ds}{w_k(s)}, \quad 0 < u_0 \le u,\ 1 \le k \le l$$

as long as

$$G_k(1) + \sum_{s=0}^{t-1} h_r(s)\,\phi(s) \prod_{i=1}^{l} g_i(s) \prod_{r=1}^{k-1} F_r(s) \in \mathrm{Dom}\,(G_k^{-1}),\quad 1 \le k \le l.$$

PROOF. The proof is similar to that of Theorem 12 (Chapter 1) with an application of Corollary 2.

It is easy to write down discrete analogues of Theorems 13 and 14 (Chapter 1), since the proofs are similar and the details are not repeated here.

4. SOME APPLICATIONS.

Like in continuous case, for several particular systems of discrete inequalities, explicit upper estimates can be obtained on using the results of sections 2 and 3. For example, for two dimensional discrete inequalities

$$|x_i(t)| \le |k_i| + \sum_{s=0}^{t-1} |f_i(s, x_1(s), x_2(s))|,\ i = 1, 2$$

which appear in the study of two dimensional systems using Euler's method.

If

$$|f_i(t, x_1(t), x_2(t))| \le g_i(t, |x_1(t)|, |x_2(t)|), \quad i = 1, 2$$

where $g_i(t, u_1, u_2)$ is nondecreasing in u_1 and u_2 then it follows from [73] that $|x_i(t)| \le u_i(t)$, where $u_i(t)$ are solutions of

$$\Delta u_i(t) = g_i(t, u_1(t), u_2(t))$$
$$u_i(0) = |k_i|.$$

Now for several different g_i's our results provide explicit upper estimates in terms of known functions, e.g., if g_i are linear Theorem 1 is applicable.

Next we shall make a comparative study of some known results as given in [74]. Consider the linear stochastic discrete system

$$y_{n+1}(\omega) = A(\omega)\, y_n(\omega) \qquad (17)$$
$$y_0(\omega) = x_0, \; n \in N_0$$

and the perturbedd systems including an operator T as

$$x_{n+1}(\omega) = A(\omega)\, x_n(\omega) + f_n(\omega, x_n(\omega), (Tx_n)(\omega)) \qquad (18)$$
$$x_0(\omega) = x_0, \; n \in N_0.$$

Here x_n and y_n are stochastic processes, $A(\omega)$ is an r x r matrix whose elements are measurable functions for each n, f_n is a vector valued function defined on $\Omega \times R^r \times R^r \to R^r$, R^r is an Euclidean r space and T is an operator which maps $R^r \to R^r$. Let $Y_n(\omega)$ denotes the stochastic fundamental matrix solutions of (17) such that $Y_0(\omega)$ is the unit matrix.

The following modified versions of his theorems 2-4 which require weaker conditions can be proved using the results of section 2 and the parallel arguments he has used.

THEOREM 2'. Suppose that

$$|Y_n(\omega)\, Y_{s+1}^{-1}(\omega)\, f_s(\omega, x_s(\omega), (Tx_s)(\omega))| \le a_s(\omega)|x_s(\omega)| + b_s(\omega)|(Tx_s)(\omega)|$$

where $a_s(\omega)$, $b_s(\omega)$ are non-negative random functions defined for $s \in N_0$, $\omega \in \Omega$. Further, suppose that the operator T satisfies the inequality

$$|(Tx_n)(\omega)| \le \sum_{s=0}^{n-1} c_s(\omega)|x_s(\omega)|$$

where $c_n(\omega)$ is a non-negative random function defined for $n \in N_0$, $\omega \in \Omega$. Then to every bounded random solution $y_n(\omega)$ of (17) on N_0, the corresponding random solution $x_n(\omega)$ of (18) is bounded on N_0 provided

$$\prod_{s=0}^{\infty}\left[1 + a_s(\omega) + b_s(\omega)\sum_{\tau=0}^{s-1} c_\tau(\omega)\right] < \infty .$$

THEOREM 3'. Let us assume

$$|Y_n(\omega)\, Y_{s+1}^{-1}(\omega)| \le M e^{-\alpha(n-s)}, \quad |Y_n(\omega)| \le M e^{-\alpha n}$$

$$|f_n(\omega, x_n(\omega), (Tx_n)(\omega)| \le a_n(\omega)|x_n(\omega)| + b_n(\omega)|(Tx_n)(\omega)|$$

$$|(Tx_n)(\omega)| \le e^{-\alpha n}\sum_{s=0}^{n-1} c_s(\omega)|x_s(\omega)|$$

where $M > 0$, $\alpha > 0$ are constants and $a_n(\omega)$, $b_n(\omega)$, $c_n(\omega)$ are defined in Theorem 2'. Then all random solutions of (18) approach zero as $n \to \infty$, provided

$$K = \prod_{s=0}^{\infty}\left[1 + a_s(\omega) + b_s(\omega)\sum_{\tau=0}^{s-1} c_\tau(\omega)\, e^{-\alpha\tau}\right] < \infty .$$

THEOREM 4'. In Theorem 3' let $-\alpha = \epsilon$ and $K \le C$ where $C > 0$ is a constant, then the conclusion of his Theorem 4 follows.

CHAPTER 3

INTEGRO-DIFFERENTIAL INEQUALITIES

1. INTRODUCTION.

In this Chapter we shall discuss some new integro-differential inequalities involving higher order derivatives. The estimates are in terms of known functions and applicable directly to higher order differential equations. In Section 2 we shall consider the linear case and in Section 3 we shall present several nonlinear inequalities. Some applications are given in section 4.

Throughout we shall assume all the functions and their derivatives appearing in the inequalities are real-valued, non-negative and continuous on $I = [0,\infty)$.

2. LINEAR INEQUALITIES.

THEOREM 1. Let the following inequality be satisfied

$$u^{(k)}(t) \le p(t) + q(t) \sum_{j=0}^{k} \int_0^t h_j(s) u^{(j)}(s)\,ds, \quad 0 \le k. \tag{1}$$

Then

$$u^{(k)}(t) \le p(t) + q(t) \int_0^t \phi_1(s) \exp\left(\int_s^t \phi_2(\tau)\,d\tau\right) ds \tag{2}$$

where

$$\phi_1(t) = p(t) h_k(t) + \sum_{j=0}^{k-1} \sum_{i=0}^{j} u^{(j)}(0) h_j(t) \frac{t^{j-i}}{(j-i)!} + \sum_{j=0}^{k-1} \frac{h_{k-j-1}(t)}{j!} \int_0^t (t-x)^j p(x)\,dx$$

$$\phi_2(t) = q(t) h_k(t) + \sum_{j=0}^{k-1} \frac{h_{k-j-1}(t)}{j!} \int_0^t (t-x)^j q(x)\,dx.$$

PROOF. Let us define

$$R(t) = \sum_{j=0}^{k} \int_0^t h_j(s)\, u^{(j)}(s)\, ds.$$

Then from (1), we have

$$u^{(k)}(t) \le p(t) + q(t) R(t), \quad R(0) = 0. \tag{3}$$

Integrating (3) (k-j) times to obtain

$$u^{(j)}(t) \le \sum_{i=j}^{k-1} u^{(i)}(0) \frac{t^{i-j}}{(i-j)!} + \frac{1}{(k-j-1)!} \int_0^t (t-x)^{k-j-1} \left(p(x) + q(x) R(x)\right) dx,$$
$$0 \le j \le k-1. \tag{4}$$

Thus, from the definition of R(t), we have

$$\begin{aligned} R'(t) &= \sum_{j=0}^{k} h_j(t)\, u^{(j)}(t) \\ &\le \sum_{j=0}^{k-1} h_j(t) \Big[\sum_{i=j}^{k-1} u^{(i)}(0) \frac{t^{i-j}}{(i-j)!} + \frac{1}{(k-j-1)!} \int_0^t (t-x)^{k-j-1} \big(p(x) \\ &\qquad + q(x) R(x)\big) dx \Big] + h_k(t) \left[p(t) + q(t) R(t) \right]. \end{aligned}$$

On using nondecreasing nature of R(t) and arranging the terms, we obtain

$$R'(t) \le \phi_1'(t) + \phi_2(t) R(t).$$

Integrating the above inequality and substituting the resulting estimate for R(t) in (3) produces the result.

COROLLARY 2. In the inequality (1), let $u^{(j)} \ge 0$, $0 \le j \le k-1$ and p(t) be nondecreasing. Then

$$u^{(k)}(t) \le p(t) \left[1 + q(t) \int_0^t \phi_3(s) \exp\left(\int_s^t \phi_4(\tau)\, d\tau \right) ds \right]$$

where

$$\phi_3(t) = \sum_{j=0}^{k} \frac{h_{k-j}(t)}{j!} t^j, \quad \phi_4(t) = \phi_2(t).$$

COROLLARY 3. In the inequality (1), let p(t) be
differentiable, $p'(t) \geq 0$ and $q(t) = 1$, then

$$u^{(k)}(t) \leq \exp\left(\int_0^t \phi_3(s)\,ds\right)\left[p(0)+\int_0^t \phi_5(s) \times \exp\left(-\int_0^s \phi_3(\tau)\,d\tau\right)ds\right]$$

where

$$\phi_5(t) = p'(t) + \sum_{j=0}^{k-1} \sum_{i=0}^{j} u^{(j)}(0)\, h_i(t) \frac{t^{j-i}}{(j-i)!}\,.$$

REMARK 1. For $k = 1$, $h_0(t) = h_1(t) = h(t)$ in the inequality (1), we have

$$u'(t) \leq p(t) + q(t) \int_0^t \phi_6(s) \exp\left(\int_s^t \phi_7(\tau)\,d\tau\right) ds$$

where

$$\phi_6(t) = h(t)\left[u(0) + p(t) + \int_0^t p(\tau)\,d\tau\right]$$

$$\phi_7(t) = h(t)\left[q(t) + \int_0^t q(\tau)\,d\tau\right]$$

which is not comparable with the result obtained in [79] (his theorem 1). In our next result we shall generalize his theorem for any k.

THEOREM 4. In the inequality (1), let $h_j(t) = h(t)\ (0 \leq j \leq k)$ $q(t) \geq 1$. Then

$$u^{(k)}(t) \leq p(t) + q(t)\int_0^t h(s)\left[A(s) + B_k(s)\right] \times \exp\left(\int_s^t h(\tau)[q(\tau)-1]\,d\tau\right) ds \tag{5}$$

where

$$B_1(t) = \int_0^t h(s) A(s) \exp\left(\int_s^t [h(\tau) q(\tau) + h(\tau) + k q(\tau) + (k-1)]\,d\tau\right) ds$$

$$B_i(t) = \int_0^t \left(h(s) A(s) + B_{i-1}(s)\right) \exp\left(\int_s^t [h(\tau) q(\tau) + h(\tau) + (k-i+1) q(\tau) + (k-i-1)]\,d\tau\right) ds,\quad 2 \leq i \leq k$$

$$A(t) = p(t) + \sum_{j=0}^{k-1} \sum_{i=0}^{j} \frac{u^{(j)}(0)\, t^{i}}{(i)!} + \sum_{j=0}^{k-1} \frac{1}{j!}\int_0^t (t-x)^j p(x)\,dx\,.$$

PROOF. Let us define

$$R_1(t) = \int_0^t h(s) \sum_{j=0}^{k} u^{(j)}(s)\, ds.$$

Then, as in Theorem 1, we have

$$R_1'(t) + h(t) R_1(t) \le h(t) A(t) + h(t) q(t) R_1(t) + h(t) R_2(t)$$

where

$$R_2(t) = R_1(t) + \sum_{j=0}^{k-1} \frac{1}{j!} \int_0^t (t-\tau)^j q(\tau) R_1(\tau)\, d\tau.$$

Again, as in Theorem 1, we find on using $R_1(t) \le R_2(t)$

$$R_2'(t) + R_2(t) \le h(t) A(t) + [h(t) q(t) + h(t) + q(t)] R_2(t) + R_3(t)$$

where

$$R_3(t) = R_2(t) + \sum_{j=0}^{k-2} \frac{1}{j!} \int_0^t (t-\tau)^j q(\tau) R_2(\tau)\, d\tau.$$

Once again, we find on using $R_2(t) \le R_3(t)$

$$R_3'(t) + R_3(t) \le h(t) A(t) + [h(t) q(t) + 2q(t) + 1] R_3(t) + R_4(t)$$

where

$$R_4(t) = R_3(t) + \sum_{j=0}^{k-3} \frac{1}{j!} \int_0^t (t-\tau)^j q(\tau) R_3(\tau)\, d\tau.$$

Continuing in this way, we finally get

$$R_{k+1}'(t) \le h(t) A(t) + [h(t) q(t) + h(t) + k q(t) + (k-1)] R_{k+1}(t)$$

where

$$R_i(0) = 0, \quad i = 1, 2, \cdots, k+1.$$

Integrating the last above inequality, We find

$$R_{k+1}(t) \le B_1(t)$$

and hence

$$R_k'(t)+R_k(t)\le h(t)A(t)+[h(t)q(t)+h(t)+(k-1)q(t)$$
$$+(k-2)]R_k(t)+B_1(t).$$

From this we get, on integration

$$R_k(t)\le B_2(t).$$

Continuing in this way, we obtain

$$R_1'(t)\le h(t)[A(t)+B_k(t)]+h(t)[q(t)-1]R_1(t)$$

and from this (5) follows.

REMARK 2. For the case $q(t)\le 1$, an estimate can be obtained from the inequality

$$R_1'(t)\le h(t)[A(t)+B_k(t)]+h(t)q(t)R_1(t).$$

The estimate obtained in Theorem 4 can be improved uniformly and to justify this we shall consider the case of Remark 1.

THEOREM 5. Let $k=1$, $h_0(t)=h_1(t)=h(t)$, $q(t)\geq 1$ in the inequality (1). Then

$$u'(t)\le p(t)+q(t)\int_0^t h(s)[\phi_8(s)+\phi_9(s)]\exp\left(\int_s^t h(\tau)[q(\tau)-1]\,d\tau\right)ds$$

where

$$\phi_8(t)=u(0)+p(t)+\int_0^t p(s)\,ds$$
$$\phi_9(t)=\int_0^t (h(s)\phi_8(s)-\psi(s))\exp\left(\int_s^t [h(\tau)q(\tau)+h(\tau)+q(\tau)]\,d\tau\right)ds$$
$$\psi(t)=u(0)q(t)(1+h(t))\int_0^t q(s)\left(\int_0^s h(\tau)\,d\tau\right)ds.$$

PROOF. As in the Proof of Theorem 4, we find

$$R_1'(t)+h(t)R_1(t)\le h(t)\phi_8(t)+h(t)q(t)R_1(t) \tag{6}$$
$$+h(t)R_2(t)$$

where

$$R_2(t) = R_1(t) + \int_0^t q(s) R_1(s)\, ds.$$

Thus, we find

$$R_2'(t) \le h(t)\,\phi_8(t) + [h(\tau)q(\tau) + q(\tau)]R_1(t) + h(t)R_2(t).$$

Now, since

$$R_1(t) \le R_2(t) - \int_0^t q(s)[u(0)\int_0^s h(\tau)\,d\tau]\,ds$$

we obtain

$$R_2'(t) \le [h(t)\,\phi_8(t) - \psi(t)] + [h(t)q(t) + q(t) + h(t)]R_2(t)$$

and hence

$$R_2(t) \le \phi_9(t).$$

Substituting this in (6) and integrating, we obtain an estimate for $R_1(t)$. The result follows from the definition of $R_1(t)$.

The next result is concerned with the following inequality

$$u^{(k)}(t) \le p(t) + q(t)[\sum_{r=1}^{m} E_r(t, \sum_{j=0}^{k} u^{(j)}) + E_m(t, u^{(k)})] \qquad (7)$$

where $E_r(t,\cdot)$ are same as defined in Chapter 1.

THEOREM 6. Let the inequality (7) be satisfied. Then

$$u^{(k)}(t) \le p(t) + q(t)\int_0^t [\sum_{r=1}^{m-1} E_r'(s, \phi_{10}) + E_m'(s,p)] \times$$
$$\exp(\int_s^t [\sum_{r=1}^{m-1} E_r'(\tau, \phi_{11}) + E_m'(\tau, q)]\,d\tau)\,ds$$

where $\phi_{10}(t)$ and $\phi_{11}(t)$ are same as $\phi_1(t)$ and $\phi_2(t)$ with $h_i(t) = 1$, $0 \le i \le k$ respectively.

PROOF. Let $R(t)$ be the part of right side appearing in the bracket of (7). Then we have

$$R'(t) = \sum_{r=1}^{m-1} E_r'(t, \sum_{j=0}^{k} u^{(j)}) + E_m'(t, u^{(k)})$$
$$= \sum_{r=1}^{m-1} E_r'(t, \sum_{j=0}^{k-1} u^{(j)}) + \sum_{r=1}^{m} E_r'(t, u^{(k)}).$$

Further as in Theorem 1, we obtain

$$R'(t) \le \sum_{r=1}^{m-1} E_r'(t, \phi_{10} + \phi_{11}R - p - qR) + \sum_{r=1}^{m} E_r'(t, p+qR)$$

$$\le \sum_{r=1}^{m-1} E_r'(t, \phi_{10} + \phi_{11}R) + E_m'(t, p+qR)$$

$$\le \Big[\sum_{r=1}^{m-1} E_r'(t, \phi_{10}) + E_m'(t,p)\Big] + \Big[\sum_{r=1}^{m-1} E_r'(t, \phi_{11}) + E_m'(t,q)\Big] R(t)$$

and now the result follows after integrating the above inequality.

REMARK 3. Several particular cases of Theorem 6 have been considered by Pachpatte [40], [79, 80] , however, the results obtained here cannot be compared with his results but as in Theorem 5 all his results can be improved uniformly and for this in our next theorem we shall give the improved version of his theorem 1 [80] .

THEOREM 7. In the inequality (7), let $n = 2$, $k = 1$, $p(t) = u(o)$, $q(t) = 1$, $f_{11}(t) = f_{21}(t) = a(t)$, $f_{22}(t) = b(t)$. Then

$$u'(t) \le u(o)\Big[1 + \int_0^t (2 - \phi_{12}(s))\, a(s) \exp\Big(\int_0^s [1 + a(\tau) + b(\tau)]\, d\tau\Big) ds\Big]$$

where

$$\phi_{12}(t) = \int_0^t (1+b(s))\Big(1 + s + \int_0^s b(\tau)\, d\tau\Big) \exp\Big(-\int_0^s [1 + a(\tau) + b(\tau)]\, d\tau\Big) ds.$$

PROOF. For this particular case, let $R_1(t)$ be the right part of (7). Then $R_1(o) = u(o)$ and

$$R_1'(t) = a(t)\Big[u(t) + u'(t) + \int_0^t b(\tau)\, u'(\tau)\, d\tau\Big].$$

Since

$$u'(t) \le R_1(t), \quad u(t) \le u(o) + \int_0^t R_1(s)\, ds$$

we have

$$R_1'(t) \le a(t) R_2(t) \tag{8}$$

where $R_2(0) = 2u(0)$ and

$$R_2(t) = u(0) + R_1(t) + \int_0^t R_1(s)\,ds + \int_0^t b(s) R_1(s)\,ds. \tag{9}$$

From (9), we find

$$R_1(t) \le R_2(t) - u(0)\left[1 + t + \int_0^t b(s)\,ds\right]$$

also

$$R_2'(t) \le a(t) R_2(t) + R_2(t) - u(0)\left[1 + t + \int_0^t b(s)\,ds\right](1 + b(t)) + b(t) R_2(t).$$

Integrating the above inequality, we obtain

$$R_2(t) \le u(0)\left[2 - \phi_{12}(t)\right] \exp\left(\int_0^t [1 + a(\tau) + b(\tau)]\,d\tau\right).$$

Substituting this estimate in (8) and integrating the obtained inequality produces the desired result.

3. NONLINEAR INEQUALITIES.

THEOREM 8. Let the following inequality be satisfied

$$u^{(k)}(t) \le p(t) + \sum_{j=0}^{k} \int_0^t h_j(s)\, u^{(k)}(s)\, u^{(j)}(s)\,ds \tag{10}$$

where p(t) is positive and nondecreasing. Then

$$u^{(k)}(t) \le \frac{p(t) \exp\left(\int_0^t \phi_{13}(s)\,ds\right)}{1 - \int_0^t \phi_{14}(s) \exp\left(\int_0^s \phi_{13}(\tau)\,d\tau\right) ds}$$

where

$$\phi_{13}(t) = \sum_{j=0}^{k-1} \sum_{i=0}^{j} u^{(j)}(0)\, h_j(t) \frac{t^{j-i}}{(j-i)!}$$

$$\phi_{14}(t) = p(t) \sum_{j=0}^{k} h_{k-j}(t) \frac{t^j}{(j)!}$$

as long as $1-\int_0^t \phi_{14}(s)\exp(\int_0^s \phi_{13}(\tau)d\tau)ds > 0$.

PROOF. Since p(t) is positive and nondecreasing we have

$$\frac{u^{(k)}(t)}{p(t)} \leq 1+\sum_{j=0}^{k}\int_0^t h_j(s)\frac{u^{(k)}(s)}{p(s)}u^{(j)}(s)\,ds. \tag{11}$$

Let us define the right side of (11) as R(t). Then on differentiation, we get

$$R'(t) \leq \sum_{j=0}^{k} h_j(t)R(t)u^{(j)}(t)$$

$$\leq h_k(t)p(t)R^2(t)+\sum_{j=0}^{k-1} h_j(t)R(t)\Big[\sum_{i=j}^{k-1} u^{(i)}(0)\frac{t^{i-j}}{(i-j)!}$$

$$+\frac{1}{(k-j-1)!}\int_0^t (t-\tau)^{k-j-1}p(\tau)R(\tau)d\tau\Big].$$

Now using the nondecreasing nature of R(t) and p(t), we find

$$R'(t) \leq \phi_{13}(t)R(t)+\phi_{14}(t)R^2(t).$$

Integration of the above inequality proves the theorem.

REMARK 4. As in Theorem 8, it is easy to find the estimates in terms of known functions for the following inequalities

(a) $u^{(k)}(t) \leq p(t)+\sum_{j=0}^{k} g_j(t)\int_0^t h_j(s)u^{(k)}(s)u^{(j)}(s)\,ds,$

$g_j(t) \geq 1, \ 0 \leq j \leq k.$

(b) $u^{(k)}(t) \leq p(t)+\sum_{j=0}^{k}\int_0^t h_j(s)u^{(l)}(s)u^{(j)}(s)ds, \ 0 \leq l \leq k-1.$

REMARK 5. A particular case k = 1, $h_0(t) = h_1(t) = b(t)$, p(t) = a of Theorem 8 has been considered in Theorem 3 [79], but this is not comparable with our result. However, his estimate can be improved uniformly to

$$u'(\ ) \le \frac{a(u(0)+a)\int_0^t b(s)\exp(\int_0^s \phi_{15}(\tau)d\tau)ds}{1-(u(0)+a)\int_0^t b(s)\exp(\int_0^s \phi_{15}(\tau)d\tau)ds}$$

where

$$\phi_{15}(t) = 1-(u(0)+at)b(t).$$

REMARK 6. Several other results given in [80], which are particular cases of Theorem 8 and Remark 4 can be improved uniformly as in Theorem 4.

THEOREM 9. Let the following inequality be satisfied,

$$u^{(k)}(t) \le C + \sum_{j=0}^{k}\int_0^t h_j(s)[u^{(k)}(s)]^{\alpha}[u^{(j)}(s)]^{\alpha_j}ds \tag{12}$$

where $\alpha, \alpha_j, 0 \le j \le k$ are nonnegative numbers and C is a positive constant. Then

$$u^{(k)}(t) \le \frac{C}{[1-(\alpha+\beta-1)\int_0^t \phi_{16}(s)ds]^{1/\alpha+\beta-1}} \tag{13}$$

where $\beta = \max\{\alpha_j : 0 \le j \le k\}$ and $\alpha+\beta > 1$, also

$$\phi_{16}(t) = C^{\alpha-1}\left[C^{\alpha_k}h_k(t) + \sum_{j=0}^{k-1}h_j(t)\left(\sum_{i=j}^{k-1}u^{(i)}(0)\frac{t^{i-j}}{(i-j)!} + C\frac{t^{k-j}}{(k-j)!}\right)^{\alpha_j}\right]$$

as long as $1-(\alpha+\beta-1)\int_0^t \phi_{16}(s)ds > 0$.

PROOF. The inequality (12) can be rewritten as

$$\frac{u^{(k)}(t)}{C} \le R(t)$$

where

$$R(t) = 1 + \sum_{j=0}^{k}\int_0^t \frac{h_j(s)}{C}[u^{(k)}(s)]^{\alpha}[u^{(j)}(s)]^{\alpha_j}ds.$$

Then, we find on using the fact that R(t) $\geq$ 1 and
nondecreasing

$$R'(t) \le C^{\alpha-1} R^{\alpha}(t)\Big[h_k(t) C^{\alpha_k} R^{\alpha_k}(t) + \sum_{j=0}^{k-1} h_j(t)\Big[\sum_{i=j}^{k-1} u^{(i)}(0)\frac{t^{i-j}}{(i-j)!} + \frac{1}{(k-j-1)}\int_0^t (t-\tau)^{k-j-1} C R(\tau)\,d\tau\Big]^{\alpha_j}$$

$$\le \phi_{16}(t) R^{\alpha+\beta}(t).$$

Now on integrating this we get the desired result.

THEOREM 10. Let the following inequality be satisfied

$$u^{(k)}(t) \le C + \sum_{j=0}^{k} \int_0^t h_j(s)[u^{(\ell)}(s)]^{\alpha}[u^{(j)}(s)]^{\alpha_j}\,ds, \qquad 0 \le \ell \le k-1$$

where α, α_j, C are same as in Theorem 9. Then the estimate is same as (13) replacing $\phi_{16}(t)$ by $\phi_{17}(t)$ where

$$\phi_{17}(t) = \phi_{16}(t)\frac{1}{C}\Big[\sum_{i=\ell}^{k-1} u^{(i)}(0)\frac{t^{i-\ell}}{(i-\ell)!} + C\frac{t^{i-\ell}}{(i-\ell)!}\Big]^{\alpha}.$$

THEOREM 11. Let the following inequality be satisfied

$$u^{(k)}(t) \le C + \sum_{r=1}^{m} E_r\Big(t, \sum_{j=0}^{k} u^{(j)}\Big) + E_m(t, (u^{(k)})^{\alpha})$$

where $E_r(t,\cdot)$ are same as defined in Chapter 1 and the constant C > 0, also the number $\alpha \geq 0\ (\neq 1)$. Then

$$u^{(k)}(t) \le C\exp\Big(\int_0^t \sum_{r=1}^{m-1} E'_r(s,\phi_{18})\,ds\Big)\Big[1+(1-\alpha)\times \int_0^t E'_m(s, C^{\alpha-1})\exp\Big(1-(1-\alpha)\int_0^s \sum_{r=1}^{m-1} E'_r(\tau,\phi_{18})\,d\tau\Big)ds\Big]^{1/1-\alpha}$$

where

$$\phi_{18}(t) = \frac{1}{c}\sum_{j=0}^{k-1}\sum_{i=0}^{j} \frac{u^{(j)}_{(0)}t^{(j-i)}}{(j-i)!} + \sum_{j=0}^{k}\frac{t^{j}}{j!} .$$

The proof of Theorems 10 and 11 are similar to that of Theorem 9.

THEOREM 12. Let the following inequality be satisfied

$$u^{(k)}(t) \le p(t) + \sum_{j=1}^{m} g_j(t)\int_0^t h_j(s)\,w\Big(\sum_{i=0}^{k} u^{(i)}(s)\Big)\,ds \qquad (14)$$

where

(i) $p(t)$ is positive and nondecreasing (ii) $g_i(t) \geq 1$, $1 \le i \le m$ (iii) w is positive, continuous, nondecreasing and sub-multiplicative. Then

$$u^{(k)}(t) \le p(t)\prod_{j=1}^{m} g_j(t)\, G^{-1}\Big[G(1) + \int_0^t \sum_{i=1}^{m}\frac{h_i(s)}{p(s)}\, w(\phi_{19}(s))\,ds\Big]$$

where

$$\phi_{19}(t) = \sum_{j=0}^{k-1}\sum_{i=0}^{j} u^{(j)}(0)\frac{t^{j-i}}{(j-i)!} + \sum_{j=0}^{k-1}\frac{1}{j!}\int_0^t (t-\tau)^{j}\, p(\tau)\prod_{j=1}^{m} g_j(\tau)\,d\tau + p(t)\prod_{j=1}^{m} g_j(t)$$

$$G(u) = \int_{u_0}^{u}\frac{ds}{w(s)}, \quad 0 < u_0 \le u$$

as long as

$$G(1) + \int_0^t \sum_{j=1}^{m}\frac{h_j(s)}{p(s)}\, w(\phi_{19}(s))\,ds \in \mathrm{Dom}(G^{-1}).$$

PROOF. The inequality (14) can be rewritten as

$$u^{(k)}(t) \le p(t) \prod_{j=1}^{m} g_j(t) R(t)$$

where

$$R(t) = 1 + \sum_{j=1}^{m} \int_0^t \frac{h_j(s)}{p(s)} \, w\Big(\sum_{i=0}^{k} u^{(i)}(s)\Big) ds .$$

Thus, we find on using $R(t)$ nondecreasing and ≥ 1

$$R'(t) = \sum_{j=1}^{m} \frac{h_j(t)}{p(t)} \, w\Big(\sum_{i=0}^{k} u^{(i)}(t)\Big)$$

$$\le \sum_{j=1}^{m} \frac{h_j(t)}{p(t)} \, w(\phi_{19}(t)) \, w(R(t)) .$$

Integration of the above inequality with the help of the definition of G leads to the desired result.

THEOREM 13. Let $u \in C^{(n-1)}[a,b]$ be such that $u^{(i)}(a)=0$ for $i = 0,1,\ldots,n-1$ where $n \geq 1$. Let $u^{(n-1)}$ be absolutely continuous and $\int_a^b |u^{(n)}(s)|^2 ds < \infty$. Then

$$\int_a^x |u^{(i)}(s)\, u^{(n)}(s)| ds \le K_i (x-a)^{n-i} \int_a^x |u^{(n)}(s)|^2 ds \qquad (15)$$

where

$$K_i = \frac{1}{2((n-1)!)} \left(\frac{n-i}{2n-2i-1}\right)^{1/2} .$$

PROOF. In view of the assumptions on u, for any s such that $a \le s \le x \le b$

$$u^{(i)}(s) = \frac{1}{(n-i-1)!} \int_a^s (s-t)^{n-i-1} u^{(n)}(t)\, dt .$$

Therefore

$$|u^{(i)}(s)u^{(n)}(s)| \leq \frac{|u^{(n)}(s)|}{(n-i-1)!}\int_a^s (s-t)^{n-i-1}|u^{(n)}(t)|dt$$

and by Schwarz's inequality

$$|u^{(i)}(s)\,u^{(n)}(s)| \leq \frac{|u^{(n)}(s)|}{(n-i-1)!}\,\frac{(s-a)^{n-i-\frac{1}{2}}}{(2n-2i-1)^{\frac{1}{2}}}\left(\int_a^s |u^{(n)}(t)|^2 dt\right)^{\frac{1}{2}}.$$

Thus, integrating from a to x and applying Schwarz's inequality to the right side again

$$\int_a^x |u^{(i)}(s)u^{(n)}(s)|ds \leq \frac{1}{(n-i-1)!\,(2n-2i-1)^{\frac{1}{2}}}\left(\int_a^x (s-a)^{2n-2i-1} ds\right)^{\frac{1}{2}} \times$$

$$\left(\int_a^x |u^{(n)}(s)|^2\left(\int_a^s |u^{(n)}(t)|^2 dt\right)ds\right)^{\frac{1}{2}}$$

$$= \frac{1}{(n-i-1)!\,(2n-2i-1)^{\frac{1}{2}}}\,\frac{(x-a)^{n-i}}{(2n-2i-1)^{\frac{1}{2}}}\,\frac{1}{\sqrt{2}}\int_a^x |u^{(n)}(s)|^2 ds$$

The above is, in fact (15). This completes the proof.

REMARK 7. For n = 1, i = 0, (15) is Opial's original result [84]. For $n \geq 2$. i = 0, x = b (15) is sharper than Willett's inequality [85] and reduces to same as obtained in [83]. Also for the best bound in this case see [82].

THEOREM 14. Let p, q be both positive constants satisfying $p+q > 1$. Let u be as in Theorem 13 and let $\int_a^b |u^{(n)}(s)|^{p+q} ds < \infty$. Then

$$\int_a^x |u^{(i)}(s)|^p |u^{(n)}(s)|^q ds \leq M_i (x-a)^{(n-i)p}\int_a^x |u^{(n)}(s)|^{p+q} ds \quad (17)$$

where

$$M_i = \alpha\, q^{q\alpha}\left[\frac{(n-i)(1-\alpha)}{n-i-\alpha}\right]^{p(1-\alpha)}\left((n-i)!\right)^{-p} \quad (18)$$

$$\alpha = \frac{1}{p+q}.$$

PROOF. As in Theorem 13, the result follows on using Holder's inequality instead of Schwarz's inequality two times with proper indices.

REMARK 8. For n = 1, i = 0, x = b (17) is sharper result similar to Yang's [86]. For $n \geq 1$, i = 0, x = b (17) reduces to same as obtained in [83].

THEOREM 15. Let u be as in Theorem 13 and let f(x), g(x) be non-negative continuous functions on $[a,b]$ for which the inequality

$$|u^{(n)}(x)| \leq C + \int_a^x f(s)|u^{(n)}(s)|\,ds + \int_a^x g(s)\Big(\sum_{i=0}^{n-1}\int_a^s |u^{(i)}(\tau)\, u^{(n)}(\tau)|\,d\tau\Big)\,ds \tag{19}$$

holds, where C is a positive constant. Then

$$|u^{(n)}(x)| \leq \frac{C\exp\left(\int_a^x f(s)\,ds\right)}{1 - C\int_a^x Q(s)\exp\left(\int_a^s f(\tau)\,d\tau\right)ds} \tag{20}$$

as long as $1 - C\int_a^x Q(s)\exp\left(\int_a^s f(\tau)\,d\tau\right)ds > 0$,

where

$$Q(x) = g(x)\sum_{i=0}^{n-1} K_i\,(x-a)^{n-i+1}.$$

PROOF. Define $\phi(x)$ as the right side of (19), then

$$\phi'(x) = f(x)|u^{(n)}(x)| + g(x)\sum_{i=0}^{n-1}\int_a^x |u^{(i)}(\tau)\, u^{(n)}(\tau)|\,d\tau, \qquad \phi(0) = C. \tag{21}$$

Using Theorem 13, in (21) to obtain

$$\phi'(x) \leq f(x)|u^{(n)}(x)| + g(x)\sum_{i=0}^{n-1} K_i\,(x-a)^{n-i} \times \int_a^x |u^{(n)}(\tau)|^2\,d\tau. \tag{22}$$

Since $|u^{(n)}(x)| \leq \phi(x)$ and $\phi(x)$ is nondecreasing, it follows from (22) that

$$\phi'(x) \leq f(x)\phi(x) + Q(x)\phi^2(x).$$

Integrating the above inequality, the result (20) follows.

THEOREM 16. Let p,q be as in Theorem 14 and f(x), g(x) and C be as in Theorem 15, for which the inequality

$$|u^{(n)}(x)| \leq C + \int_a^x f(s)|u^{(n)}(s)|ds + \int_a^x g(s)\Big(\sum_{i=0}^{n-1}\int_a^s |u^{(i)}(\tau)|^p \, |u^{(n)}(\tau)|^q d\tau\Big)ds \tag{23}$$

holds. Then

$$|u^{(n)}(x)| \leq \frac{C\exp\left(\int_a^x f(s)ds\right)}{\left[1 - C^{p+q-1}(p+q-1)\int_a^x R(s)\exp\left((p+q-1)\int_a^s f(\tau)d\tau\right)ds\right]^{1/(p+q-1)}} \tag{24}$$

as long as $1 - C^{p+q-1}(p+q-1)\int_a^x R(s)\exp\left((p+q-1)\int_a^s f(\tau)d\tau\right)ds > 0$,

where $R(x) = g(x)\sum_{i=0}^{n-1} M_i (x-a)^{(n-i)p+1}$

PROOF. The proof is similar to that of Theorem 15.

REMARK 9. For p = q = 1, Theorem 14 reduces to Theorem 13 and Theorem 16 is same as Theorem 15.

4. SOME APPLICATIONS.

Here we shall point out few applications of the inequalities obtained in sections 2 and 3 to discuss boundedness, asymptotic behaviour and an upper estimate for the solutions of higher order differential equations.

First we note that the estimates given for the solutions and their derivatives of the initial value problems

$$y'''(t) = F(t, y(t), y'(t))$$
$$y(0) = c_0,\ y'(0) = c_1,\ y''(0) = c_2$$

and

$$y''(t) = y(t)\,F(t, y(t), y'(t))$$
$$y(0) = c_0,\ y'(0) = c_1$$

when F satisfies

$$|F(t, y(t), y'(t))| \le h(t)\,(|y(t)| + |y'(t)|)$$

in terms of the known function h(t) in [79], [42] can be improved uniformly on using our Theorem 5 and Remark 5 respectively.

Next we shall study the asymptotic behaviour of the solutions of the following differential equation

$$y_k^{(k+1)}(t) = f(t, y(t), y'(t), \cdots, y^{(k)}(t)) \tag{25}$$

THEOREM 17. The solutions of the differential equation (25) when the function f satisfies the condition

$$|f(t, y(t), \cdots, y^{(k)}(t))| \le \sum_{j=0}^{k} h_j(t)\,|y^{(j)}(t)| \tag{26}$$

are asymptotic to $\sum_{i=0}^{k} a_i t^i$ as $t \to \infty$ where $a_k \neq 0$ provided

$$\int^{\infty} \sum_{j=0}^{k} t^j h_{k-j}(t)\,dt < \infty .$$

PROOF. From (25) and (26) we have

$$|y^{(k)}(t)| \le |y^{(k)}(0)| + \int_0^t \sum_{j=0}^{k} h_j(s)\,|y^{(j)}(s)|\,ds.$$

Now a direct application of Corollary 3 gives

$$|y^{(k)}(t)| \le \exp\left(\int_0^t \phi_3(s)\,ds\right)\left[|y^{(k)}(0)| + \int_0^t \phi_5(s) \times \exp\left(-\int_0^s \phi_3(\tau)\,d\tau\right) ds\right]. \tag{27}$$

Let $t \geq 1$, then from the definition of $\phi_3(t)$ and $\phi_5(t)$ we find

$$\phi_3(t) \le \sum_{j=0}^{k} t^j h_{k-j}(t)$$

$$\phi_5(t) \le L \sum_{j=0}^{k} t^j h_{k-j}(t)$$

where L is some proper constant.

Now from (27) and the above inequalities, we obtain

$$|y^{(k)}(t)| \le K_1 + \exp\left(\int_1^t \sum_{j=0}^{k} s^j h_{k-j}(s)\,ds\right)\left[K_2 \exp\left(-\int_1^t s^j h_{k-j}(s)\,ds\right) + K_3\right]$$

$$\le K_4 + K_3 \exp\left(\int_1^t \sum_{j=0}^{k} s^j h_{k-j}(s)\,ds\right)$$

where K_1, K_2, K_3 and K_4 are some constants. Since $\int^{\infty} \sum_{j=0}^{k} s^j h_{k-j}(s)\,ds < \infty$, then $|y^{(k)}(t)|$ is bounded. Now it is easy to show that $\lim_{t\to\infty} |y^{(k)}(t)| \neq 0$ as in Theorem 16 (Chapter 1). Here one can also find the case when the inequality (26) with some exponents is satisfied and thus the Theorem 10 is directly applicable.

Lastly we shall show that Theorem 6 can be used to find some estimates for the position and the velocity of a particle of unit mass moving in space, whose equations of motion are

$$\ddot{X} = F(t, x, \dot{x})$$

subject to the initial position $X(0) = X_0$ and initial velocity $\dot{X}(0) = \dot{X}_0$. To simplify the matter, we shall consider that F satisfies the following inequality

$$\begin{aligned} |F_1(t,x,\dot{x})| \le b(t) + a_{11}(t)|x_1(t)| + a_{12}(t)|\dot{x}_1(t)| \\ + a_{13}(t)|\dot{x}_2(t)| + a_{14}(t)|\dot{x}_3(t)| \end{aligned} \tag{28}$$

$$|F_2(t,x,\dot{x})| \le a_{21}(t)|\dot{x}_1(t)| + a_{22}(t)|\dot{x}_3(t)| \tag{29}$$

$$|F_3(t,x,\dot{x})| \le a_{31}(t)|x_1(t)|. \tag{30}$$

From (30) we find

$$|\dot{x}_3(t)| \le |\dot{x}_3(0)| + \int_0^t a_{31}(s)|x_1(s)|\,ds \tag{31}$$

and from (29), we have on using (31)

$$\begin{aligned} |\dot{x}_2(t)| \le |\dot{x}_2(0)| + \int_0^t \{a_{21}(s)|\dot{x}_1(s)| + a_{22}(s)[|\dot{x}_3(0)| \\ + \int_0^s a_{31}(s)|x_1(\tau)|\,d\tau]\}\,ds \end{aligned} \tag{32}$$

Substituting (31) and (32) in (28), we find after integration the inequality for $|\dot{x}_1(t)|$ which is exactly same form as considered in Theorem 6. Thus it is possible to find the estimate for $\dot{x}(t)$ and $x(t)$ in terms of the known functions.

CHAPTER 4

DISCRETE INEQUALITIES INVOLVING HIGHER ORDER DIFFERENCES

1. INTRODUCTION.

In this Chapter we shall discuss some new discrete inequalities involving higher order differences which hare discrete analogue of the results given in Chapter 3. Here also we follow the same notations of Chapter 2, also we shall denote $(m)^{(n)} = m(m-1)\cdots(m-n+1)$.

In what follows, all the functions and their differences appearing in the inequalities are assumed to be real-valued non-negative and defined in N_0.

2. LINEAR INEQUALITIES.

THEOREM 1. Let the following inequality be satisfied

$$\Delta^k u(t) \le p(t) + q(t) \sum_{j=0}^{k} \sum_{s=0}^{t-1} h_j(s) \Delta^j u(s) \tag{1}$$

for all $t \in N_0$. Then

$$\Delta^k u(t) \le p(t) + q(t) \sum_{s=0}^{t-1} \phi_1(s) \prod_{\tau=s+1}^{t-1} [1 + \phi_2(\tau)] \tag{2}$$

where

$$\phi_1(t) = p(t) h_k(t) + \sum_{j=0}^{k-1} \sum_{i=0}^{j} \Delta^i u(0) h_j(t) \frac{(t)^{(j-i)}}{(j-i)!} + \sum_{j=0}^{k-1} \sum_{s=0}^{t-j-1} h_{k-j-1}(t) \frac{(t-s-1)^{(j)}}{(j)!} p(s)$$

$$\phi_2(t) = q(t) h_k(t) + \sum_{j=0}^{k-1} \sum_{s=0}^{t-j-1} h_{k-j-1}(t) \frac{(t-s-1)^{(j)}}{j!} q(s).$$

PROOF. For $0 \leq j \leq k-1$, it is easy to verify

$$\Delta^{j} u(t) = \sum_{i=j}^{k-1} \frac{(t)^{(i-j)}}{(i-j)!} \Delta^{i} u(0) + \frac{1}{(k-j-1)!} \sum_{s=0}^{t-k+j} (t-s-1)^{(k-j-1)} \Delta^{k} u(s) \quad (3)$$

Define

$$R(t) = \sum_{j=0}^{k} \sum_{s=0}^{t-1} h_{j}(s) \Delta^{j} u(s)$$

then (1) can be written as

$$\Delta^{k} u(t) \leq p(t) + q(t) R(t) \quad (4)$$

with $R(0) = 0$. From the definition of $R(t)$, we obtain on using (3) and (4)

$$\begin{aligned}
\Delta R(t) &= \sum_{j=0}^{k} h_{j}(t) \Delta^{j} u(t) \\
&= h_{k}(t) \Delta^{k} u(t) + \sum_{j=0}^{k-1} h_{j}(t) \Big[\sum_{i=j}^{k-1} \frac{(t)^{(i-j)}}{(i-j)!} \Delta^{i} u(0) \\
&\qquad + \frac{1}{(k-j-1)!} \sum_{s=0}^{t-k+j} (t-s-1)^{(k-j-1)} \Delta^{k} u(s) \Big] \\
&\leq \sum_{j=0}^{k-1} \sum_{i=0}^{j} \Delta^{j} u(0) h_{j}(t) \frac{(t)^{(j-i)}}{(j-i)!} + h_{k}(t) \left(p(t) + q(t) R(t) \right) \\
&\qquad + \sum_{j=0}^{k-1} h_{k-j-1}(t) \sum_{s=0}^{t-j-1} \frac{(t-s-1)^{(j)}}{j!} \left(p(s) + q(s) R(s) \right).
\end{aligned}$$

On using nondecreasing nature of $R(t)$, we find

$$\Delta R(t) \leq \phi_{1}(t) + \phi_{2}(t) R(t).$$

Rest of the proof is similar to the one of Theorem 1 (Chapter 2).

COROLLARY 2. In the inequality (1), let $\Delta^j u(0) = 0$, $0 \le j \le k-1$ and p(t) be nondecreasing. Then

$$\Delta^k u(t) \le p(t)[1 + q(t) \sum_{s=0}^{t-1} \phi_3(s) \prod_{\tau=s+1}^{t-1} [1 + \phi_4(\tau)]$$

where

$$\phi_3(t) = \sum_{j=0}^{k} \frac{h_{k-j}(t)}{j!} (t)^{(j)}, \quad \phi_4(t) = \phi_2(t).$$

PROOF. The proof follows from (2) and the identity

$$(t)^{(j)} = j \sum_{s=0}^{t-j} (t-s-1)^{(j-i)}.$$

REMARK 1. For $k=1$, $h_0(t) = h_1(t) = h(t)$ in the inequality (1), we have

$$\Delta u(t) \le p(t) + q(t) \sum_{s=0}^{t-1} \phi_5(s) \prod_{\tau=s+1}^{t-1} [1 + \phi_6(\tau)]$$

where

$$\phi_5(t) = h(t)[u(0) + p(t) + \sum_{s=0}^{t-1} p(s)]$$

$$\phi_6(t) = h(t)[q(t) + \sum_{s=0}^{t-1} q(s)]$$

which is not comparable with the result obtained in [70] (his theorem 3). In our next result we shall generalize his theorem for any k.

THEOREM 3. In the inequality (1), $h_j(t) = h(t)$, $0 \le j \le k$, $q(t) \ge 1$. Then

$$\Delta^k u(t) \le p(t) + q(t) \sum_{s=0}^{t-1} h(s) [A(s) + B_k(s)] \times \prod_{\tau=s+1}^{t-1} [1 + h(\tau)(q(\tau) - 1)]$$

where

$$A(t) = p(t) + \sum_{j=0}^{k-1} \sum_{i=0}^{j} \Delta^j u(0) \frac{(t)^{(i)}}{(i)!} + \sum_{j=0}^{k-1} \sum_{s=0}^{t-j-1} \frac{(t-s-1)^{(j)}}{j!} p(s)$$

$$B_1(t) = \sum_{s=0}^{t-1} h(s) A(s) \prod_{\tau=s+1}^{t-1} (1 + [h(\tau) q(\tau) + h(\tau) + k q(\tau) + k - 1])$$

$$B_i(t) = \sum_{s=0}^{t-1} (h(s) A(s) + B_{i-1}(s)) \prod_{\tau=s+1}^{t-1} (1 + [h(\tau) q(\tau) + h(\tau) + (k-i+1) q(\tau) + (k-i-1)]), \quad 2 \le i \le k.$$

PROOF. The proof is similar to the one in the continuous case Theorem 4 (Chapter 3) and Theorem 1

REMARK 2. For the case $q(t) \leq 1$, the following estimates holds

$$\Delta^k u(t) \le p(t) + q(t) \sum_{s=0}^{t-1} h(s) [A(s) + B_k(s)] \prod_{\tau=s+1}^{t-1} [1 + h(\tau) q(\tau)].$$

The discrete analogue of Theorem 5 (Chapter 3) is the following:

THEOREM 4. Let $k = 1$, $h_0(t) = h_1(t) = h(t)$, $q(t) \geq 1$ in the inequality (1). Then

$$\Delta u(t) \le p(t) + q(t) \sum_{s=0}^{t-1} [\phi_6(s) + h(s) \phi_7(s)] \times \prod_{\tau=s+1}^{t-1} [1 + q(\tau) h(\tau) - h(\tau)]$$

where

$$\phi_7(t) = \sum_{s=0}^{t-1} [\phi_6(s) - \psi(s)] \prod_{\tau=s+1}^{t-1} [1 + q(\tau) + h(\tau) + q(\tau) h(\tau)]$$

$$\psi(t) = u(0) q(t) (1 + h(t)) \sum_{s=0}^{t-1} q(s) \left(\sum_{\tau=0}^{s-1} h(\tau) \right).$$

PROOF. The proof is similar to that of Theorem 5 (Chapter 3) and Theorem 1. The next result is concerned with the following inequality.

$$\Delta^k u(t) \le p(t) + q(t)\Big[\sum_{r=1}^{m-1} E_r\big(t, \sum_{i=0}^{k} \Delta^i u\big) + E_m(t, \Delta^k u)\Big] \quad (5)$$

where $E_r(t,\cdot)$ are same as defined in Chapter 2.

THEOREM 5. Let the inequality (5) be satisfied for all $t \in N_0$. Then

$$\Delta^k u(t) \le p(t) + q(t)\sum_{s=0}^{t-1}\Big[\sum_{r=1}^{m-1}\Delta E_r(s, \phi_8) + \Delta E_m(s,p)\Big] \times \prod_{\tau=s+1}^{t-1}\Big[1 + \Big(\sum_{r=1}^{m-1}\Delta E_r(\tau, \phi_9) + \Delta E_m(\tau, q)\Big)\Big]$$

where $\phi_8(t)$ and $\phi_9(t)$ are same as $\phi_1(t)$ and $\phi_2(t)$ with $h_i(t) = 1$, $0 \le i \le k$.

PROOF. The proof is similar to that of Theorem 6 (Chapter 3) and Theorem 1.

REMARK 3. Several particular cases of Theorem 5 have been discussed by Pachpatte [70], [80], [87], [88] however, the results obtained here cannot be compared with his results but as in Theorem 4 all his results can be improved uniformly and for this in our next theorem, we shall give the improved version of his Theorem 4 [70].

THEOREM 6. In the inequality (5), let $m = 2$, $k = 1$, $p(t) = u(0)$, $q(t) \geq 1$, $f_{11}(t) = f_{21}(t) = a(t)$ and $f_{22}(t) = b(t)$. Then

$$\Delta u(t) \le u(0)\Big[1 + \sum_{s=0}^{t-1}(2 - \phi_{10}(s))\, a(s) \prod_{\tau=0}^{s-1}[2 + a(\tau) + b(\tau)]\Big]$$

where

$$\phi_{10}(t) = \sum_{s=0}^{t-1}(1 + b(s))\Big(1 + s + \sum_{\tau=0}^{s-1} b(\tau)\Big)\prod_{\tau=0}^{s}[2 + a(\tau) + b(\tau)]^{-1}$$

PROOF. The proof is similar to that of Theorem 7 (Chapter 3).

3. NONLINEAR INEQUALITIES.

Here we shall state the discrete analogue of some results given in section 3 (Chapter 3). Since the proofs are similar to the one in the continuous case,(Theorem 1 here and as in Chapter 2) the details are not repeated.

THEOREM 8'. Let the following inequality be satisfied

$$\Delta^k u(t) \leq p(t) + \sum_{j=0}^{k} \sum_{s=0}^{t-1} h_j(s)\, \Delta^k(s)\, \Delta^j u(s)$$

where p(t) is positive and nondecreasing. Then

$$\Delta^k u(t) \leq \frac{p(t)\, e^{-1}(t)}{1 - \sum_{s=0}^{t-1} \phi_{11}(s)\, e^{-1}(s+1)}$$

where

$$e(t) = \prod_{s=0}^{t-1} [1 + \phi_{12}(s)]^{-1}$$

$$\phi_{11}(t) = p(t) \sum_{j=0}^{k} h_{k-j}(t) \frac{(t)^{(j)}}{j!}$$

$$\phi_{12}(t) = \sum_{j=0}^{k-1} \sum_{i=0}^{j} \Delta^j u(0)\, h_j(t) \frac{(t)^{(j-i)}}{(j-i)!}$$

as long as $1 - \sum_{s=0}^{t-1} \phi_{11}(s)\, e^{-1}(s+1) > 0$.

REMARK 4'. As in Theorem 8', it is easy to find the estimates in terms of known functions for the following inequalities

(a) $$\Delta^k u(t) \leq p(t) + \sum_{j=0}^{k} g_j(t) \sum_{s=0}^{t-1} h_j(s)\, \Delta^k u(s)\, \Delta^j u(s),$$

$$g_j(t) \geq 1, \quad 0 \leq j \leq k.$$

(b) $\Delta^k u(t) \le p(t) + \sum_{j=0}^{k} \sum_{s=0}^{t-1} h_j(s)\, \Delta^l u(s)\, \Delta^j u(s)$, $0 \le l \le k-1$.

THEOREM 9'. Let the following inequality be satisfied

$$\Delta^k u(t) \le C + \sum_{j=0}^{k} \sum_{s=0}^{t-1} h_j(s) [\Delta^k u(s)]^{\alpha} [\Delta^j u(s)]^{\alpha_j}$$

where α, α_j, $0 \le j \le k$ are nonnegative numbers and C is a positive constant. Then

$$\Delta^k u(t) \le \frac{C}{[1-(\alpha+\beta-1)\sum_{s=0}^{t-1} \phi_{13}(s)]^{1/\alpha+\beta-1}}$$

where $\beta = \max\{\alpha_j : 0 \le j \le k\}$ and $\alpha+\beta > 1$, also

$$\phi_{13}(t) = C^{\alpha-1}[C^{\alpha_k} h_k(t) + \sum_{j=0}^{k-1} h_j(t) \Big(\sum_{i=j}^{k-1} \Delta^i u(0) \frac{(t)^{(i-j)}}{(i-j)!} + C \frac{(t)^{(k-j)}}{(k-j)!} \Big)^{\alpha_j}]$$

as long as $1-(\alpha+\beta-1)\sum_{s=0}^{t-1} \phi_{13}(s) > 0$.

THEOREM 11'. Let the following inequality be satisfied

$$\Delta^k u(t) \le p(t) + \sum_{r=1}^{m-1} E_r(t, \sum_{j=0}^{k} \Delta^j u) + E_m(t, (\Delta^k u)^{\alpha})$$

where p(t) is positive and nondecreasing, and $\alpha \geq 0\ (\neq 1)$. Then

$$\Delta^k u(t) \le p(t)\, e^{-1}(t) \Big\{ 1 + (1-\alpha) \sum_{s=0}^{t-1} \Delta E_m(s, p^{\alpha-1}) \times e^{1-\alpha}(s+1) \Big\}^{1/1-\alpha}$$

where

$$e(t) = \prod_{s=0}^{t-1} \left[1 + \sum_{r=1}^{m} \Delta E_r(s, \phi_{14})\right]^{-1}$$

$$\phi_{14}(t) = \frac{1}{p(t)} \sum_{j=0}^{k-1} \sum_{i=0}^{j} \Delta^j u(0) \frac{(t)^{(j-i)}}{(j-i)!} + \sum_{j=0}^{k} \frac{(t)^{(j)}}{j!}.$$

THEOREM 12'. Let the following inequality be satisfied

$$\Delta^k u(t) \le p(t) + \sum_{i=1}^{m} g_i(t) \sum_{s=0}^{t-1} h_i(s)\, w\left(\sum_{j=0}^{k} \Delta^j u(s)\right)$$

where

(i) $p(t)$ is positive and nondecreasing (ii) $g_i(t) \geq 1$, $1 \le i \le m$ (iii) w is positive, continuous, nondecreasing and submultiplicative. Then

$$\Delta^k u(t) \le p(t) \prod_{j=1}^{m} g_j(t)\, G^{-1}\left[G(1) + \sum_{s=0}^{t-1} \sum_{r=1}^{m} \frac{h_r(s)}{p(s)} w(\phi_{15}(s))\right]$$

where

$$\phi_{15}(t) = p(t) \prod_{i=1}^{m} g_i(t) + \sum_{j=0}^{k-1} \sum_{i=0}^{j} \Delta^j u(0) \frac{(t)^{(j-i)}}{(j-i)!} + \sum_{j=0}^{k-1} \sum_{s=0}^{t-j-1} \frac{(t-s-1)^{(j)}}{j!} p(s) \prod_{i=1}^{m} g_i(s)$$

$$G(u) = \int_{u_0}^{u} \frac{ds}{w(s)}, \quad 0 < u_0 \le u$$

as long as

$$G(1) + \sum_{s=0}^{t-1} \sum_{i=1}^{m} \frac{h_i(s)}{p(s)} w(\phi_{15}(s)) \in \mathrm{Dom}(G^{-1}).$$

4. SOME APPLICATIONS.

There are many possible applications of the inequalities obtained in sections 2 and 3, here we shall point out few which are sufficient to convey the importance of our results.

First we shall consider the following k+1th order difference equation

$$\Delta^{k+1} u(t) = f(t, u(t), \Delta u(t), \ldots, \Delta^k u(t)) \tag{6}$$

and show that Theorem 1 is directly applicable to find the upper estimates for the solutions of (6), provided

$$|f(t, u_0, u_1, \ldots, u_k)| \le \sum_{j=0}^{k} h_j(t)\,|u_j|. \tag{7}$$

In fact any solution of (6) also satisfies

$$\Delta^k u(t) = \Delta^k u(0) + \sum_{s=0}^{t-1} f(s, u(s), \Delta u(s), \ldots, \Delta^k u(s))$$

or

$$|\Delta^k u(t)| \le |\Delta^k u(0)| + \sum_{j=0}^{k} \sum_{s=0}^{t-1} h_j(s)\,|\Delta^j u(s)|.$$

Hence from Theorem 1, we obtain

$$|\Delta^k u(t)| \le |\Delta^k u(0)| + \sum_{s=0}^{t-1} \overset{*}{\phi}_1(s) \prod_{\tau=s+1}^{t-1} [1 + \overset{*}{\phi}_2(\tau)]$$

where $\overset{*}{\phi}_1(t)$ and $\overset{*}{\phi}_2(t)$ are same as $\phi_1(t)$ and $\phi_2(t)$ with $p(t) = |\Delta^k u(0)|$ and $q(t) = 1$.

Now from (3), we find

$$|u(t)| \le \sum_{i=0}^{k-1} \frac{(t)^{(i)}}{(i)!} |\Delta^i u(0)| + \frac{1}{(k-1)!} \sum_{s=0}^{t-k} (t-s-1)^{(k-1)} \Big[|\Delta^k u(0)| + \sum_{\tau_1=0}^{s-1} \overset{*}{\phi}_1(\tau_1) \prod_{\tau_2=\tau_1+1}^{s-1} [1+\overset{*}{\phi}_2(\tau_2)] \Big].$$

Similarly, if the function f satisfies

$$|f(t, u_0, \dots, u_k)| \le \sum_{j=0}^{k} h_j(s)\, w\Big(\sum_{j=0}^{k} |u_j|\Big)$$

where w is same as in Theorem 12' and $|\Delta^k u(0)| > 0$. The upper estimate for the solutions of (6) can be obtained directly on using Theorem 12'.

Several other properties like boundedness, uniqueness, asymptotic behaviour etc. of the solutions of (6) can be discussed with the help of these inequalities as in ordinary differential equations.

CHAPTER 5

INEQUALITIES IN n INDEPENDENT VARIABLES

1. INTRODUCTION.

In this Chapter we shall generalize the results of Chapter 1 to n-independent variables. Snow's [101] method of Riemann function is extended and Wendroff's ([89], p.154) estimates are improved. These results are used to study several properties of the solutions of partial differential and integral equations in n-independent variables.

Throughout this Chapter we shall use the following notations.

Let Ω be an open bounded set in R^n and let a point $(x_1^i, x_2^i, \ldots, x_n^i)$ in Ω be denoted by x^i. Let y and x $(y < x)$ to be any two points in Ω and denote by D the parallelopiped defined by $y < s < x$, that is $y_j < s_j < x_j$, $1 \le j \le n$. The $\int_y^x \cdot\, ds$ indicates the n-fold integral $\int_{y_1}^{x_1} \cdots \int_{y_n}^{x_n} \cdot\, ds_n \cdots ds_1$ and $u_x(x)$ denotes $\dfrac{\partial^n u(x)}{\partial x_1 \cdots \partial x_n}$.

In what follows we shall assume that all the functions appearing in the inequalities are real-valued, non-negative, continuous and defined in Ω.

2. LINEAR INEQUALITIES.

LEMMA 1. Let p(s) be continuous function in Ω. Then the characteristic initial value problem

$$(-1)^n v_s(s,x) - p(s)\, v(s,x) = 0 \text{ in } \Omega \tag{1}$$

$$v(s,x) = 1 \text{ on } s_i = x_i \,,\; 1 \le i \le n \tag{2}$$

has a unique solution $v(s,x)$ near to x and satisfying $\prod_{i=1}^{n} (x_i - s_i) \ge 0$.

This solution is continuous and if $p(s)$ is non-negative, so is $v(s,x)$.

PROOF. The function $v(s,x)$ is the Riemann function relative to the point x. The problem (1), (2) is equivalent to the integral equation

$$v(s,x) = 1 + \int_{s}^{x} p(t)\, v(t,x)\, dt. \tag{3}$$

The existence, uniqueness and non-negative properties of $v(s,x)$ follows by successive approximation arguments as given in [93, 94, 100, 101] for $n = 2$ and systems. An explicit representation of $v(s,x)$ is given in [102]. **Since** $v(s,x)$ is continuous and $v = 1$ on $s_i = x_i$, $1 \le i \le n$ there is a domain $D^+ \supset x$ on which $v \geq 0$ even if $p(s)$ is not nonnegative.

LEMMA 2. Suppose $p(x)$ and $q(x)$ are continuous functions in Ω. Let $v(s,x)$ be the solution of (1), (2) and let D^+ be a connected subdomain of Ω containing x such that $v \geq 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$u_x(x) - p(x)u(x) \le q(x) \tag{4}$$

where u vanishes together with all its mixed derivatives upto order $n-1$ on $x_i = y_i$, $1 \le i \le n$. Then

$$u(x) \le \int_{y}^{x} q(t)\, v(t,x)\, dt. \tag{5}$$

PROOF. The proof follows from Young's theorem [102].

THEOREM 3. Let $v(s,x)$ be the solution of the characteristic initial value problem

$$(-1)^n v_s(s,x) - \sum_{r=1}^{m} E_s^r(s,b)\, v(s,x) = 0 \text{ in } \Omega \tag{6}$$

$$v(s,x) = 1 \text{ on } s_i = x_i,\ 1 \le i \le n \tag{7}$$

and let D^+ be a connected subdomain of Ω containing x such that $v \geq 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$u(x) \leq a(x) + b(x) \sum_{r=1}^{m} E^r(x,u) \tag{8}$$

where

$$E^r(x,u) = \int_y^x f_{r_1}(x^1) \int_y^{x^1} f_{r_2}(x^2) \cdots \int_y^{x^{r-1}} f_{r_r}(x^r) u(x^r) \, dx^r \cdots dx^1. \tag{9}$$

Then

$$u(x) \leq a(x) + b(x) \int_y^x \sum_{r=1}^{m} E_s^r(s,a) \, v(s,x) \, ds. \tag{10}$$

PROOF. Define a function $\phi(x)$ such that

$$\phi(x) = \sum_{r=1}^{m} E^r(x,u).$$

Then we have

$$\phi_x(x) = \sum_{r=1}^{m} E_x^r(x,u)$$

and hence from (8)

$$\phi_x(x) \leq \sum_{r=1}^{m} E_x^r(x, a+b\phi) = \sum_{r=1}^{m} E_x^r(x,a) + \sum_{r=1}^{m} E_x^r(x, b\phi). \tag{11}$$

Using the nondecreasing nature of $\phi(x)$ in (11), we find

$$\phi_x(x) - \sum_{r=1}^{m} E_x^r(x,b)\, \phi(x) \leq \sum_{r=1}^{m} E_x^r(x,a)$$

where ϕ vanishes together with all its mixed derivatives upto order n-1 on $x_i = y_i$, $1 \leq i \leq n$.

Now an application of Lemma 2, provides

$$\phi(x) \leq \int_y^x \sum_{r=1}^{m} E_s^r(s,a) \, v(s,x) \, ds. \tag{12}$$

The result now follows from (12) and $u(x) \leq a(x) + b(x)\, \phi(x)$.

REMARK 1. Some particular cases of Theorem 3, $n = 2$ and m upto 3 have been considered recently by Pachpatte ([97] theorems 1-4 and [98], theorems 1-2), but his results cannot be compared with our results. In the next theorem we shall consider a particular case of (8), the obtained result unifies all his six theorems for the general n, also a result of Agarwal [103].

THEOREM 4. Let $v_i(s,x)$, $1 \le i \le m$, be the solutions of characteristic initial value problems

$$(-1)^n v_{1s}(s,x) - \Big(\sum_{r=1}^{m} b(s) f_r(s) \bigcup_{i=1}^{m-1} g_i(s)\Big) v_1(s,x) = 0 \text{ in } \Omega$$

$$(-1)^n v_{js}(s,x) - \Big(\sum_{r=1}^{m-j+1} b(s) f_r(s) \bigcup_{i=1}^{m-j} g_i(s) - g_{m-j+1}(s)\Big) v_j(s,x) = 0$$

$$\text{in } \Omega,\ 2 \le j \le m$$

$$v_j(s,x) = 1 \text{ on } s_i = x_i,\ 1 \le i \le n,\ 1 \le j \le m$$

and let D^+ be a connected subdomain of Ω containing x such that $v_j \geq 0$, $1 \le j \le m$ for all $s \in D^+$. If $D \subset D^+$ and (8) is satisfied where

$$f_{ii}(x) = f_i(x),\ 1 \le i \le m;\ f_{i+1,i}(x) = f_{i+2,i}(x) = \cdots =$$

$$f_{m,i}(x) = g_i(x),\ 1 \le i \le m-1,$$

then

$$u(x) \le a(x) + b(x) P_j(x),\ 1 \le j \le m \tag{13}$$

where

$$P_1(x) = \int_y^x a(x') \sum_{r=1}^{m} f_r(x') v_1(x',x)\, dx'$$

$$P_j(x) = \int_y^x \Big[a(x') \sum_{r=1}^{m-j+1} f_r(x') + g_{m-j+1}(x') P_{j-1}(x')\Big]$$

$$v_j(x',x)\, dx',\ 2 \le j \le m.$$

PROOF. The proof is same as in Theorem 4 (Chapter 1) and Theorem 3 with repeated application of Lemma 1.

REMARK 2. For the particular case $m = 2$, $b = 1$, $f_{11} = f_{21} = f_1$, $f_{22} = f_2$ in (8) the estimate (13) takes the form

$$u(x) \le a(x) + \int_y^x f_1(x')\Big[a(x') + \int_y^{x'} a(x^2)\big(f_1(x^2) + f_2(x^2)\big)v_1(x^2, x')\,dx^2\Big]dx' \qquad (14)$$

where $v_1(s,x)$ is the solution of characteristic initial value problem

$$(-1)^n v_{1s}(s,x) - (f_1(s) + f_2(s))\,v_1(s,x) = 0 \text{ in } \Omega \qquad (15)$$

$$v_1(s,x) = 1 \text{ on } s_i = x_i,\ 1 \le i \le n. \qquad (16)$$

In the next result we shall show that the estimate (14) can be improved uniformly. The improved version of Theorem 1 in [97] is the following (here we have taken $\sigma = 0$ since it does not play any role, the term $\int_y^x b(s)\sigma(s)\,ds$ can be merged in $a(x)$).

THEOREM 5. Let $v_1(s,x)$ be the solution of (15), (16) and let D^+ be a connected subdomain of Ω containing x such that $v_1 \geq 0$ for all $s \in D^+$. If $D \subset D^+$ and (8) is satisfied where $m = 2$, $b = 1$, $f_{11} = f_{21} = f_1$, $f_{22} = f_2$, then

$$u(x) \le a(x) + \int_y^x f_1(x')\Big[a(x') + \int_y^{x'} \{a(x^2)(f_1(x^2) + f_2(x^2)) - c(x^2)\}\, v_1(x^2, x')\,dx^2\Big]dx' \qquad (17)$$

where

$$c(x) = f_2(x)\int_y^x a(x')f_2(x')\,dx'.$$

PROOF. Define

$$\phi_1(x) = \int_y^x f_1(x')u(x')dx' + \int_y^x f_1(x')\int_y^{x'} f_2(x^2)u(x^2)dx^2\,dx'$$

then, from (8) it follows that

$$\phi_{1x}(x) \le f_1(x)\Big[a(x)+\phi_1(x)+\int_y^x f_2(x')[a(x')+\phi_1(x')]dx'\Big]. \tag{18}$$

Let

$$\phi_2(x) = \phi_1(x) + \int_y^x f_2(x')[a(x')+\phi_1(x')]dx' \tag{19}$$

then, we find

$$\phi_{2x}(x) = \phi_{1x}(x) + f_2(x)[a(x)+\phi_1(x)]$$

which is from (18) and (19)

$$\phi_{2x}(x) \le f_1(x)[a(x)+\phi_2(x)] + f_2(x)\Big[a(x)+\phi_2(x) - \int_y^x a(x')f(x')\,dx'\Big].$$

Using Lemma 2 to obtain

$$\phi_2(x) \le \int_y^x \{a(x')(f_1(x')+f_2(x')) - c(x')\}\, v_1(x',x)\,dx'.$$

Substituting this in (18), we find

$$\phi_1(x) \le \int_y^x f_1(x')\Big[a(x') + \int_y^{x'}\{a(x^2)(f_1(x^2)+f_2(x^2)) - c(x^2)\}\, v_1(x^2,x')\,dx^2\Big]dx'$$

and now the result (17) follows from $u(x) \le a(x) + \phi_1(x)$.

In the next theorem we shall obtain Wendroff's type estimate for (8).

THEOREM 6. Let the inequality (8) be satisfied in Ω, where (i) $a(x)$ is positive and nondecreasing (ii) $b(x) \geq 1$. Then

$$u(x) \leq a(x)\, b(x) \exp\left(\sum_{r=1}^{m} E^{r}(x,b)\right). \tag{20}$$

PROOF. The inequality (8) can be written as

$$\phi_1(x) \leq 1 + \sum_{r=1}^{m} E^{r}(x, b\phi_1) \tag{21}$$

where

$$\phi_1(x) = \frac{u(x)}{a(x)\, b(x)}.$$

Let $\phi_2(x)$ be the right member of (21), then

$$\phi_{2x}(x) = \sum_{r=1}^{m} E^{r}_{x}(x, b\phi_1) \leq \sum_{r=1}^{m} E^{r}_{x}(x, b\phi_2) \tag{22}$$

and $\phi_2(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) = 1$, all the partial derivatives upto order n-1 vanishes when $x_i = y_i$, for any i, $1 \leq i \leq n$.

Since (ϕ_2) is nondecreasing, it follows from (22) that

$$\phi_{2x}(x) \leq \sum_{r=1}^{m} E^{r}_{x}(x,b)\, \phi_2(x)$$

or

$$\frac{\phi_{2x}(x)}{\phi_2(x)} \leq \sum_{r=1}^{m} E^{r}_{x}(x,b) + \frac{(\phi_{2x_n}(x))(\phi_{2x_1\cdots x_{n-1}}(x))}{\phi_2^{2}(x)}$$

and hence

$$\left(\frac{\phi_{2x_1\cdots x_{n-1}}(x)}{\phi_2(x)}\right)_{x_n} \leq \sum_{r=1}^{m} E^{r}_{x}(x,b).$$

Keeping $x_1, \ldots, x_{n-1}$ fixed in the above inequality and setting $x_n = s_n$ and integrating with respect to s_n from y_n to x_n, we obtain

$$\left(\frac{\phi_{2\,x_1 \ldots x_{n-1}}(x)}{\phi_2(x)}\right) \leq \int_{y_n}^{x_n} \sum_{r=1}^{m} E^r_{x_1 \ldots x_{n-1}, s_n}(x_1, \ldots, x_{n-1}, s_n, b)\, ds_n$$
$$= \sum_{r=1}^{m} E^r_{x_1 \ldots x_{n-1}}(x, b).$$

Repeating the above argument for x_{n-1}, x_{n-2}, to x_2, we get

$$\frac{\phi_{2\,x_1}(x)}{\phi_2(x)} \leq \sum_{r=1}^{m} E^r_{x_1}(x, b).$$

Integrating the above inequality and using $\phi_2(y_1, x_2, \ldots, x_n) = 1$, we find

$$\phi_2(x) \leq \exp\left(\sum_{r=1}^{m} E^r(x, b)\right).$$

The result (20) now follows from $\phi_1(x) \leq \phi_2(x)$ and the definition of $\phi_1(x)$.

REMARK 3. The estimate (20) for n = 2, m = 1 is sharper than given in ([89] , p.15) and same as obtained by Kasture and Deo ([96] , theorem 9). For n = 2, b = 1, m upto 2 similar results have been obtained in [92] .

In our next result we do not require any condition on a(x) and b(x) as in Theorem 6, also the estimate (20) can be deduced.

THEOREM 7. Let the inequality (8) be satisfied in Ω . Then

$$u(x) \leq a(x) + b(x) \int_y^x \sum_{r=1}^{m} E^r_s(s, a) \times \exp\left(\int_s^x \sum_{r=1}^{m} E^r_t(t, b)\, dt\right) ds. \qquad (23)$$

PROOF. Define

$$w(s,x) = \exp\left(\int_s^x \sum_{r=1}^m E_t^r(t,b)\,dt\right)$$

then it follows that

$$(-1)^n w_s(s,x) - \sum_{r=1}^m E_s^r(s,b)\, w(s,x) \geq 0$$

$$w(s,x) = 1 \text{ on } s_i = x_i,\ 1 \leq i \leq n \tag{24}$$

Consequently, $w(s,x)$ satisfies the differential inequality (24) of which $V(s,x)$ is the exact solution (Theorem 3). It follows from ([81] , p.126, 130)) that $w(s,x) \geq V(s,x)$ and now (23) follows from (10).

In case the condition on a(x) (can be non-negative) and b(x) of Theorem 6 are satisfied, then from (23) we get

$$u(x) \leq a(x)b(x)\Big[1 + \int_y^x \sum_{r=1}^m E_s^r(s,b) \times \exp\Big(\int_s^x \sum_{r=1}^m E_t^r(t,b)\,dt\Big)ds\Big]. \tag{25}$$

Using (24) in (25), to obtain

$$u(x) \leq a(x)b(x)\Big[1 + (-1)^n \int_y^x w_s(s,x)\,ds\Big]. \tag{26}$$

Now using the fact that the partial derivatives of $w(s,x)$ upto order n-1 vanishes on $s_i = x_i$, $1 \leq i \leq n$, it follows from (26) that

$$u(x) \leq a(x)b(x)\Big[1 + (-1)^{2n-1} \int_{y_1}^{x_1} w_{s_1}(s_1, y_2, \cdots, y_n, x)\,ds_1\Big]$$

and hence

$$u(x) \leq a(x)b(x)\big[1 + (-1)^{2n-1}(w(x_1, y_2, \cdots, y_n, x) - w(y,x))\big]$$

or

$$u(x) \leq a(x)b(x)\,w(y,x)$$

which is same as (20).

3. NONLINEAR INEQUALITIES.

Here we shall state n-independent variable analogue of some results given in section 3 (Chapter 1). Since the proofs are similar to the one for n = 1 and given for the linear case here in section 2, the details are not repeated.

THEOREM 10' Let the following inequality be satisfied

$$u(x) \le a(x)\left[C + \sum_{r=1}^{m} H^r(x,u)\right] \text{ in } \Omega$$

where

$$H^r(x,u) = \int_y^x f_{r1}(x')\, u^{\alpha_{r1}}(x') \cdots \int_y^{x^{r-1}} f_{rr}(x^r)\, u^{\alpha_{rr}}(x^r)\, dx^r \ldots dx'$$

and α_{ri}, $1 \le i \le r$, $1 \le r \le m$ are non-negative real numbers and the constant $C > 0$. Then

$$u(x) \le C\, a(x) \exp\left(\int_y^x Q(s)\, ds\right), \text{ if } \alpha = 1$$

$$u(x) \le a(x)\left[C^{1-\alpha} + (1-\alpha)\int_y^x Q(s)\, ds\right]^{1/1-\alpha}, \text{ if } \alpha \ne 1$$

where $\alpha_r = \sum_{r=1}^{m} \alpha_{ri}$, $\alpha = \max\limits_{1 \le r \le m} \alpha_r$, $Q(x) = \sum_{r=1}^{m} H_x^r(x,a)\, C^{\alpha_r - \alpha}$

and when $\alpha > 1$, we assume $C^{1-\alpha} + (1-\alpha)\int_y^x Q(s)\, ds > 0$.

DEFINITION.. A function $w : [0,\infty) \to (0,\infty)$ is said to belong to the class S if (i) $w(u)$ is positive, nondecreasing, continuous and $w_{x_k}(u(x_1,\cdots,x_n)) \ge 0$ for all $2 \le k \le n$ and $u \ge 0$

(ii) $\frac{1}{v} w(u) \le w(u/v)$ for all $u \ge 0$, $v \ge 1$.

THEOREM 12'. Let the following inequality be satisfied

$$u(x) \le a(x) + \sum_{r=1}^{m} E^{r}(x,u) + \sum_{i=1}^{l} g_i(x) \int_y^x h_i(s)\, w_i(u(s))\, ds \quad \text{in } \Omega$$

where (i) $a(x) \geq 1$ and nondecreasing (ii) $g_i(x) \geq 1, 1 \le i \le l$ (iii) $w_i \in S, 1 \le i \le l$.

Then

$$u(x) \le a(x)\psi(x)e(x)\prod_{i=1}^{l} F_i(x)$$

where

$$\psi(x) = \exp\left(\sum_{r=1}^{m} E^{r}(x,e)\right) ; \quad e(x) = \prod_{i=1}^{l} g_i(x)$$

$$F_k(x) = G_k^{-1}\left[G_k(1) + \int_y^x h_k(s)\psi(s)e(s)\prod_{j=1}^{k-1} F_j(s)\, ds\right], \quad F_0(x)=1, \quad 1 \le k \le l.$$

$$G_k(\theta) = \int_{\theta_0}^{\theta} \frac{ds}{w_k(s)}, \quad 0 < \theta_0 \le \theta$$

as long as

$$G_k(1) + \int_y^x h_k(s)\psi(s)a(s)\prod_{j=1}^{k-1} F_j(s)\, ds \in \mathrm{Dom}(G_k^{-1}), \quad 1 \le k \le l.$$

THEOREM 13'. In addition to the hypothesis of Theorem 12' let $g_i(x), 1 \le i \le l$, be nondecreasing. Then

$$u(x) \le a(x)\psi_1(x)\prod_{i=1}^{l} F_i(x)$$

where

$$\psi_1(x) = \exp\left(\sum_{r=1}^{m} E^{r}(x,1)\right)$$

$$F_k(x) = g_k(x)\, G_k^{-1}\left[G_k(1) + \int_y^x h_k(s)\psi_1(s)g_k(s)\prod_{i=1}^{k-1} F_i(s)\, ds\right]$$

$$F_0(x) = 1, \quad 1 \le k \le l$$

as long as

$$G_k(1)+\int_y^x h_k(s)\,\psi_1(s)\,g_k(s)\prod_{i=1}^{k-1}F_i(s)\,ds \in \mathrm{Dom}(G_k^{-1}),$$
$1 \le k \le l.$

THEOREM 14'. Let the following inequality be satisfied

$$u(x) \le a(x)+\sum_{r=1}^{m}E^r(x,u)+\sum_{i=1}^{l}E^i(x,w(u)) \text{ in } \Omega \quad (27)$$

where (i) $a(x) \ge 1$ and nondecreasing (ii) $w \in S$.
Then

$$u(x) \le a(x)\psi_1(x)\,G^{-1}\Big[G(1)+\int_y^x\sum_{i=1}^{l}E_s^i(s,\psi_1)\,ds\Big] \quad (28)$$

where $\psi_1(x)$ is same as in Theorem 13' and the term inside the bracket of (28) $\in \mathrm{Dom}(G^{-1})$.

THEOREM 15'. Let the inequality (28) be satisifed, where (i) $a(x)$ is positive and nondecreasing (ii) w is positive, continuous, nondecreasing, submultiplicative and $w_{x_k}(u(x_1,\cdots,x_n)) \ge 0$ for all $2 \le k \le n$. Then

$$u(x) \le a(x)\,\psi_1(x)\,G^{-1}\Big[G(1)+\int_y^x\sum_{r=1}^{l}E_s^r\Big(s,\frac{w(a\psi_1)}{a}\Big)ds\Big] \quad (29)$$

where $\psi_1(x)$ is same as in Theorem 13' and the term inside the bracket of (29) $\in \mathrm{Dom}(G^{-1})$.

4. <u>SOME APPLICATIONS</u>.

Here we shall present some applications of our results obtained in sections 2 and 3 to discuss uniqueness, continuous dependence and an upper estimate on the solutions of the nonlinear hyperbolic integrodifferential equation

$$u_x(x)=F\Big(x,u(x),\int_y^x k(x,s,u(s))\,ds\Big) \quad (30)$$

together with the given boundary conditions $u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)$, $1 \le i \le n$. It is also to be remarked that these properties can be satisfied for more general equations than (30).

The functions F and K are continuous on their respective domains of definitions.

EXAMPLE 1. Any solution $u(x)$ of (30) satisfying the boundary conditions is also a solution of the Volterra integral equation

$$u(x) = a(x) + \int_y^x F(x', u(x'), \int_y^{x'} K(x', x^2, u(x^2))dx^2)\, dx' \tag{31}$$

where $a(x)$ takes care of the boundary conditions.

Let us assume that

$$|F(x, u(x), v(x) - F(x, \bar{u}(x), \bar{v}(x))| \le f_{11}(x)|u(x) - \bar{u}(x)| + f_{12}(x)|v(x) - \bar{v}(x)| \tag{32}$$

$$|K(x, s, u(s)) - K(x, s, \bar{u}(s))| \le f_{22}(s)|u(s) - \bar{u}(s)| \tag{33}$$

where f_{11}, f_{12}, f_{22} are same as appear in (8), also let $V(s,x)$ be the solution of characteristic initial value problem

$$(-1)^n V_s(s, x) - [f_{11}(s) + f_{12}(s)\int_y^s f_{22}(x')\,dx']V(s,x) = 0 \text{ in } \Omega$$
$$V(s, x) = 1 \text{ on } s_i = x_i,\ 1 \le i \le n.$$

Let D^+ be a connected subdomain of Ω containing x such that $V \ge 0$ for all $s \in D^+$. Let $u_1(x)$ and $u_2(x)$ be two solutions of (31), then we find on using (32) and (33) that

$$|u_1(x) - u_2(x)| \le \int_y^x [f_{11}(x')|u_1(x') - u_2(x')| + f_{12}(x') \times \int_y^{x'} f_{22}(x^2)|u_1(x^2) - u_2(x^2)|\,dx^2]\,dx'.$$

Application of Theorem 3 to this inequality implies that $u_1(x) = u_2(x)$ for all $x \in D^+$.

EXAMPLE 2. Let us assume that

$$|F(x, u(x), v(x))| \leq f_{11}(x)|u(x)| + f_{12}(x)|v(x)| \tag{34}$$

$$|K(x, s, u(s))| \leq f_{22}(s)|u(s)|. \tag{35}$$

Using (34) and (35) in (31), to obtain

$$|u(x)| \leq |a(x)| + \int_y^x [f_{11}(x^1)|u(x^1)| + f_{12}(x^1)\int_y^{x^1} f_{22}(x^2)|u(x^2)|dx^2]\,dx^1. \tag{36}$$

From Theorem 7, we find

$$|u(x)| \leq |a(x)| + \int_y^x [f_{11}(x^1)|a(x^1)| + f_{12}(x^1)\int_y^{x^1} f_{22}(x^2)|a(x^2)|dx^2] \exp(\int_{x^1}^x [f_{11}(x^2) + f_{12}(x^2)\int_y^{x^3} f_{22}(x^3)\,dx^3]\,dx^2)\,dx^1. \tag{37}$$

If, $|a(x)| \leq M$, where $M > 0$ is a constant, then from (37) or (36) with Theorem 6, we get

$$|u(x)| \leq M \exp(\int_y^x [f_{11}(x^1) + f_{12}(x^1)\int_y^{x^1} f_{22}(x^2)\,dx^2]\,dx^1). \tag{38}$$

Further, if $f_{11} = f_{12}$ then from (38), we obtain

$$|u(x)| \leq M \exp(\int_y^x f_{11}(x^1)[1 + \int_y^{x^1} f_{22}(x^2)\,dx^2]\,dx^1). \tag{39}$$

The estimate (39) is not comparable with

$$|u(x)| \leq M[1 + \int_y^x f_{11}(x^1) \exp(\int_y^{x^1} [f_{11}(x^2) + f_{22}(x^2)]\,dx^2)\,dx^1] \tag{40}$$

as obtained in [92] for n = 2.

In order $|u(t)|$ remains bounded in (40) it is necessary to have

$$\int_y^x [f_{11}(x^1) + f_{22}(x^1)]\,dx^1 < \infty$$

which is same as

$$\int_y^x f_{11}(x')\,dx' < \infty, \quad \int_y^x f_{22}(x')\,dx' < \infty. \tag{41}$$

In (39), we require

$$\int_y^x f_{11}(x')\left[1+\int_y^{x'} f_{22}(x^2)\,dx^2\right]dx' < \infty \tag{42}$$

which is obviously satisfied if (41) holds, but in several cases (42) is more general than (41), for example, let

$f_{22}(x) = \exp(\sum_{i=1}^{n}(x_i - y_i))$ and $f_{11}(x) = \exp(-2\sum_{i=1}^{n}(x_i - y_i))$,

for this (41) is not satisfied, where as (42) holds. Thus the results obtained here will be applicable to more general situations.

EXAMPLE 3. Following the similar lines as given in [97] it is easy to show that the solutions of (31) are continuously dependent on the initial conditions, with proper application of our results.

CHAPTER 6

DISCRETE INEQUALITIES IN n INDEPENDENT VARIABLES

1. INTRODUCTION.

In this chapter, the discrete analogue of some results obtained in Chapter 5 have been established. These inequalities are applied in studying various properties of solutions of summary difference equations in n-independent variables.

Following the notations of Chapter 2, the product $N_0 \times N_0 \times \cdots \times N_0$ (n times) be denoted by N_0^n. A point $(x_1^i, \cdots, x_n^i)$ in N_0^n is denoted by x^i. The first difference with respect to the variable x_i of the function $u(x_1, \cdots, x_n)$ is defined as

$$\Delta u_{x_i}(x_1, \cdots, x_n) = u(x_1, \cdots, x_{i-1}, x_i+1, x_{i+1}, \cdots, x_n) - u(x_1, \cdots, x_n).$$

The second difference with respect to the variables x_i, x_j is defined as

$$\begin{aligned} \Delta^2 u_{x_i x_j}(x_1, \cdots, x_n) &= \Delta u_{x_i}(x_1, \cdots, x_{j-1}, x_j+1, x_{j-1}, \cdots, x_n) - \Delta u_{x_i}(x_1, \cdots, x_n) \\ &= u(x_1, \cdots, x_{i-1}, x_i+1, x_{i+1}, \cdots, x_{j-1}, x_j+1, x_{j+1}, \cdots, x_n) \\ &\quad - u(x_1, \cdots, x_{i-1}, x_i+1, \cdots x_n) - u(x_1, \cdots, x_{j-1}, x_j+1, \cdots, \\ &\quad x_n) + u(x_1, \cdots, x_n). \end{aligned}$$

The higher order differences are defined analogously. The functions appearing in the inequalities are assumed to be real-valued, non-negative and defined in N_0^n.

2. LINEAR INEQUALITIES.

THEOREM 1. Let the following inequality be satisfied

$$u(x) \leq \sum_{i=1}^{n} a_i(x_i) + \sum_{r=1}^{m} E^r(x, u) \tag{1}$$

where

$$E^{r}(x,u)=\sum_{x^1=0}^{x-1} f_{r_1}(x^1)\sum_{x^2=0}^{x^1-1} f_{r_2}(x^2)\cdots\sum_{x^r=0}^{x^{r-1}-1} f_{r_r}(x^r)\,u(x^r)$$

for all $x \in N_0^n$ and $a_i(x_i) > 0$, $\Delta a_i(x_i) \geq 0$.

Then

$$u(x) \leq [a_1(0)+\sum_{i=2}^{n} a_i(x_i)] \prod_{s_1=0}^{x_1-1} [1+\frac{\Delta a_1(s_1)}{a_1(s_1)+a_2(0)+\sum_{i=3}^{n} a_i(x_i)(2)} + \sum_{r=1}^{m} \Delta E^{r}_{s_1}(s_1, x_2, \ldots, x_m, 1)]$$

PROOF. Let $\phi(x)$ be the right member of (1). Then

$$\Delta \phi_{x_1}(x) = \Delta a_1(x_1) + \sum_{r=1}^{m} \Delta E^{r}_{x_1}(x,u) \tag{3}$$

and

$$\Delta^n \phi_x(x) = \sum_{r=1}^{m} \Delta^n E^{r}_x(x,u) \tag{4}$$

also

$$\phi(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) = a_1(0)+\sum_{\substack{j=1\\ j\neq 1}}^{n} a_j(x_j).$$

Since $u(x) \leq \phi(x)$ and $\phi(x)$ is nondecreasing in x, from (3), we get

$$\begin{aligned}\Delta^n \phi_x(x) &\leq \sum_{r=1}^{m} \Delta^n E^{r}_x(x,\phi)\\ &\leq \sum_{r=1}^{m} \Delta^n E^{r}_x(x,1)\,\phi(x).\end{aligned} \tag{5}$$

From (5), on using the fact $\phi(x_1,\ldots,x_{n-1},x_n+1) \geq \phi(x)$, we obtain

$$\frac{\Delta^{n-1}\phi_{x_1\cdots x_{n-1}}(x_1,\ldots,x_{n-1},x_n+1)}{\phi(x_1,\ldots,x_{n-1},x_n+1)} - \frac{\Delta^{n-1}\phi_{x_1\cdots x_{n-1}}(x)}{\phi(x)} \leq \sum_{r=1}^{m} \Delta^n E^{r}_x(x,1).$$

Now keeping $x_1, \ldots, x_{n-1}$ fixed and setting $x_n = s_n$ and summing over $s_n = 0, 1, \ldots, x_{n-1}$ in the above inequality we find

$$\frac{\Delta^{n-1}\phi_{x_1\cdots x_{n-1}}(x)}{\phi(x)} \leq \sum_{s_n=0}^{x_n-1} \sum_{r=1}^{m} \Delta^n E^r_{x_1\cdots x_{n-1} s_n}(x_1, \cdots x_{n-1}, s_n, 1)$$

$$= \sum_{r=1}^{m} \Delta^{n-1} E^r_{x_1\cdots x_{n-1}}(x, 1).$$

Repeating the above arguments successively, to obtain

$$\frac{\Delta\phi_{x_1}(x)}{\phi(x)} \leq \frac{\Delta a_1(x_1)}{a_1(x_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{r=1}^{m} \Delta E^r_{x_1}(x, 1). \quad (6)$$

From (6), we have

$$\phi(x_1+1, x_2, \cdots, x_n) \leq \left[1 + \frac{\Delta a_1(x_1)}{a_1(x_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{r=1}^{m} E^r_{x_1}(x, 1)\right]\phi(x).$$

Now keeping $x_2, \ldots, x_n$ fixed and setting $x_1 = s_1$ and summing over $s_1 = 0, 1, \cdots, x_1 - 1$ in the above inequality, we find from (4)

$$\phi(x) \leq \left[a_1(0) + \sum_{i=2}^{n} a_i(x_i)\right] \prod_{s_1=0}^{x_1-1} \left[1 + \frac{\Delta a_1(s_1)}{a_1(s_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{r=1}^{m} \Delta E^r_{s_1}(s_1, x_2, \cdots, x_n, 1)\right].$$

The result (2) now follows from $u(x) \leq \phi(x)$.

REMARK 1. There are $n!$ different conclusions possible for the Theorem 1, corresponding to n-permutations of $(x_1, \cdots, x_n)$ and corresponding n-permutations of $a_1, \cdots, a_n$.

REMARK 2. For $n = 3$, $m = 1$, the estimate (2) is same as obtained in([104], Theorem 1). For $n = 3$, $m = 2$, $f_{11} = f_{21}$, the estimate (2) is not comparable to as obtained in ([104] , Theorem 2). For $n = 2$ and m upto 2 some results are given in [105].

THEOREM 2. Let the following inequality be satisfied

$$u(x) \leq a(x) + b(x) \sum_{r=1}^{m} E^{r}(x, u) \tag{7}$$

for all $x \in N_0^n$, where (i) $a(x) > 0$ and nondecreasing (ii) $b(x) \geq 1$. Then

$$u(x) \leq a(x) b(x) \prod_{s_1=0}^{x_1-1} \left[1 + \sum_{r=1}^{m} \Delta E^{r}_{s_1}(s_1, x_2, \cdots, x_n, b) \right]. \tag{8}$$

PROOF. From the assumptions on a and b, inequality (7) can be written as

$$V(x) \leq 1 + \sum_{r=1}^{m} E^{r}(x, bV)$$

where $V = u/ab$.

Rest of the proof is same as in Theorem 1.

REMARK 3. For the inequality (1) with the assumptions of Theorem 1, we have from Theorem 2

$$u(x) \leq \sum_{i=1}^{n} a_i(x_i) \prod_{s_1=0}^{x_1-1} \left[1 + \sum_{r=1}^{m} \Delta E^{r}_{s_1}(s_1, x_2, \cdots, x_n, 1) \right]. \tag{9}$$

REMARK 4. There are $n!$ different conclusions possible for Theorem 2 and also for (9).

REMARK 5. If $a_i = k$ (constant), then (2) and (9) are same. In the general case (2) and (9) are not comparable. In applications (9) requires less work to compute the estimates than (2)

THEOREM 3. Let the inequality (1) be satisfied, where $a_i(x_i)$ is same as in Theorem 1 and $f_{ii} = f_i$, $1 \le i \le m$; $f_{i+1,i} = f_{i+2,i} = \cdots = f_{m,i} = g_i$, $1 \le i \le m-1$ for all $x \in N_0^n$. Then

$$u(x) \le P_i(x), \; i = 1, 2$$

where

$$P_1(x) = [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{s_1=0}^{x_1-1} [1 + \frac{\Delta a_1(s_1)}{a_1(s_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{s_2=0}^{x_2-1} \cdots \sum_{s_n=0}^{x_n-1} (\sum_{r=1}^{m} f_r(s) \bigcup_{i=1}^{m-1} g_i(s))]$$

$$P_2(x) = \sum_{i=1}^{n} a_i(x_i) + \sum_{s=0}^{x-1} (f_1(s) \cup g_1(s)) P_1(s).$$

PROOF. The proof is similar to that of Theorem 4 (Chapter 2) and Theorem 1.

REMARK 6. As in Theorem 1, there are $n!$ different conclusions possible for Theorem 3.

REMARK 7. For n = 3, m = 1 Theorem 3 is same as given in ([104], Theorem 1). For n = 3, m = 2, $f_1 = g_1$ Theorem 3 is same as Theorem 2 of [104]. This also covers some results given in [105] for n = 2, m upto 2.

THEOREM 4. Let the following inequality be satisfied

$$u(x) \le a(x) + \sum_{r=1}^{m} g_r(x) \sum_{s=0}^{x-1} h_r(s) u(s) \tag{11}$$

for all $x \in N_0^n$, where (i) $a(x) > 0$ and nondecreasing (ii) $g_i(x) \ge 1$, $1 \le i \le m$, and nondecreasing for $2 \le i \le m$. Then

$$u(x) \le F_m a(x)$$

where

$$F_0 w = w$$

$$F_k w = w(F_{k-1} g_k) \prod_{s_1=0}^{x_1-1} [1 + \sum_{s_2=0}^{x_2-1} \cdots \sum_{s_n=0}^{x_n-1} h_k(s) F_{k-1} g_k(s)]$$

$$k = 1, 2, \cdots, m.$$

PROOF. The proof is similar to that of Theorem 6 (Chapter 2) and Theorem 2.

THEOREM 5. Let the inequality (11) be satisfied for all $x \in N_0^n$, where (i) $a(x) > 0$ and nondecreasing (ii) $g_i(x) \geq 1$ for all $1 \leq i \leq m$. Then

$$u(x) \leq a(x) \prod_{i=1}^{m} g_i(x) \prod_{s_1=0}^{x_1-1} [1 + \sum_{s_2=0}^{x_2-1} \cdots \sum_{s_n=0}^{x_n-1} \sum_{r=1}^{m} h_r(s) \prod_{i=1}^{m} g_i(s)].$$

PROOF. The inequality (11) can be written as

$$u(x) \leq a(x) + \prod_{i=1}^{m} g_i(x) \overline{\underline{\sum}}_{s=0}^{x-1} (\sum_{i=1}^{m} h_i(s)) u(s)$$

and now the result follows from Theorem 2.

3. <u>NONLINEAR INEQUALITIES.</u>

THEOREM 6. Let the following inequality be satisfied

$$u(x) \leq p(x) [c + \sum_{r=1}^{m} H^r(x, u)]$$

where

$$H^r(x,u) = \sum_{x'=0}^{x-1} f_{r1}(x') u^{\alpha_{r1}}(x') \cdots \sum_{x^r=0}^{x^{r-1}-1} f_{rr}(x^r) u^{\alpha_{rr}}(x^r)$$

where C, α_{ri}, are same as in Theorem 10'. Then

$$u(x) \le C\, p(x) \prod_{s_1=0}^{x_1-1} \Big[1+\sum_{r=1}^{m} \Delta H^r_{s_1}(s_1, x_2, \ldots, x_n, p)\, C^{\alpha_r - 1}\Big], \text{ if } \alpha = 1$$

$$u(x) \le p(x) \Big[C^{1-\alpha} + (1-\alpha) \sum_{r=1}^{m} H_r(s, p)\, C^{\alpha_r - \alpha}\Big]^{1/1-\alpha}, \text{ if } \alpha < 1.$$

PROOF. The proof is similar to that of Theorem 10' and Theorem 1.

Next result is the discrete analogue of Theorem 12' (Chapter 5), we follow the same definition of S as given in Chapter 1.

THEOREM 7. Let the following inequality be satisfied

$$u(x) \le a(x) + \sum_{r=1}^{m} E^r(x, u) + \sum_{i=1}^{l} g_i(x) \sum_{s=0}^{x-1} h_i(s)\, w_i(u(s)) \tag{12}$$

for all $x \in N_0^n$, where (i) $a(x) \geq 1$ and nondecreasing (ii) $g_i(x) \ge 1$, $1 \le i \le l$ (iii) $w_i \in S$, $1 \le i \le l$. Then

$$u(x) \le a(x)\, \psi(x)\, e(x) \prod_{i=1}^{l} F_i(x) \tag{13}$$

where

$$e(x) = \prod_{i=1}^{l} g_i(x)$$

$$\psi(x) = \prod_{s_1=0}^{x_1-1} \Big[1 + \sum_{r=1}^{m} \Delta E^r_{s_1}(s_1, x_2, \ldots, x_n, e)\Big]$$

$$F_k(x) = G_k^{-1}\Big[G_k(1) + \sum_{s=0}^{x-1} h_k(s)\, \psi(s)\, e(s) \prod_{j=1}^{k-1} F_j(s)\Big],\ F_0(x) = 1,\quad 1 \le k \le l$$

$$G_k(\theta) = \int_{\theta_0}^{\theta} \frac{ds}{w_k(s)},\quad 0 < \theta_0 \le \theta$$

as long as

$$G_k(1) + \sum_{s=0}^{x-1} h_k(s)\, \psi(s)\, e(s) \prod_{j=1}^{k-1} F_j(s) \in \mathrm{Dom}(G_k^{-1}).$$

THEOREM 8. In addition to the hypothesis of Theorem 7, let $g_i(x), 1 \le i \le \ell$ be nondecreasing. Then

$$u(x) \le a(x)\,\psi_1(x) \prod_{i=1}^{\ell} F_i(x)$$

where

$$\psi_1(x) = \prod_{s_1=0}^{x_1-1} \left[1 + \sum_{r=1}^{m} \Delta E^r_{s_1}(s_1, x_2, \cdots, x_n, 1)\right]$$

$$F_k(x) = g_k(x)\, G_k^{-1}\left[G_k(1) + \sum_{s=0}^{x-1} h_k(s)\,\psi_1(s)\, g_k(s) \prod_{i=1}^{k-1} F_i(s)\right],$$

$$F_0(x) = 1, \; 1 \le k \le \ell$$

as long as

$$G_k(1) + \sum_{s=0}^{x-1} h_k(s)\,\psi_1(s)\, g_k(s) \prod_{i=1}^{k-1} F_i(s) \in \mathrm{Dom}(G_k^{-1}).$$

THEOREM 9. Let the following inequality be satisfied

$$u(x) \le a(x) + \sum_{r=1}^{m} E^r(x, u) + \sum_{i=1}^{\ell} E^i(x, w(u)) \tag{16}$$

where (i) $a(x) \ge 1$ and nondecreasing (ii) $w \in S$. Then

$$u(x) \le a(x)\,\psi_1(x)\, G^{-1}\left[G(1) + \sum_{i=1}^{\ell} E^i(x, \psi_1)\right] \tag{17}$$

where $\psi_1(x)$ is same as in Theorem 8 and the term inside the bracket of (17) $\in \mathrm{Dom}(G^{-1})$.

THEOREM 10. Let the inequality (16) be satisfied, where (i) $a(x)$ is positive and nondecreasing (ii) w is positive, continuous, nondecreasing and submultiplicative. Then

$$u(x) \le a(x)\,\psi_1(x)\, G^{-1}\left[G(1) + \sum_{i=1}^{\ell} E^i\left(x, \frac{w(a\,\psi_1)}{a}\right)\right] \tag{18}$$

where $\psi_1(x)$ is same as in Theorem 8 and the term inside the bracket of (18) $\in \mathrm{Dom}(G^{-1})$.

The proofs of Theorems 7-10 are similar to that of our earlier results.

4. SOME APPLICATIONS.

The results obtained in sections 2 and 3 can be directly used to prove the uniqueness and continuous dependence for the solutions of discrete versions of hyperbolic partial differential equations involving n-independent variables as in the continuous case (Chapter 5) also for more general equations as given in [104-106], since the arguments are similar the details are not repeated here. To show the importance of our results we shall provide an upper bound on the solutions of difference equation of the form

$$\Delta^n u_x(x) = F(x, u(x), \sum_{s=0}^{x-1} K(x, s, u(s))) \tag{19}$$

together with the given suitable boundary conditions

$$u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n), \ 1 \le i \le n.$$

The functions F and K are defined on their respective domains of definition and

$$|F(x, u(x), v(x))| \le f_{11}(x)|u(x)| + f_{12}(x)|v(x)| \tag{20}$$

$$|K(x, s, u(s))| \le f_{22}(s)|u(s)| \tag{21}$$

where f_{11}, f_{12}, f_{22} are same as appear in (1).

Any solution u(x) of (19) satisfying the boundary conditions is also a solution of the Volterra difference equation

$$u(x) = g(x) + \sum_{x^1=0}^{x-1} F(x^1, u(x^1), \sum_{x^2=0}^{x^1-1} K(x^1, x^2, u(x^2))) \tag{22}$$

where g(x) takes care of the boundary conditions.

Using (20), (21) in (22) to obtain

$$|u(x)| \le |g(x)| + \sum_{x'=0}^{x-1} [f_{11}(x')|u(x')| + f_{12}(x') \sum_{x^2=0}^{x'-1} f_{22}(x^2)|u(x^2)|].$$

If $|g(x)| \le a(x)$ where a(x) is same as in Theorem 2, we find

$$|u(x)| \le a(x) \prod_{x_1'=0}^{x_1-1} [1 + \sum_{x_2'=0}^{x_2-1} \cdots \sum_{x_n'=0}^{x_n-1} [f_{11}(x') + f_{12}(x') \sum_{x^2=0}^{x'-1} f_{22}(x^2)]] \quad (23)$$

If $|g(x)| \le \sum_{i=1}^{n} a_i(x_i)$ where $a_i(x_i)$ are same as in Theorem 1, we find from (2)

$$|u(x)| \le [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{x_1'=0}^{x_1-1} [1 + \frac{\Delta a_1(x_1')}{a_1(x_1') + a_2(0) + \sum_{i=3}^{n} a_i(x_i)}$$
$$+ \sum_{x_2'=0}^{x_2-1} \cdots \sum_{x_n'=0}^{x_n-1} [f_{11}(x') + f_{12}(x') \sum_{x^2=0}^{x'-1} f_{22}(x^2)]] \quad (24)$$

also, in case $f_{11} = f_{12}$, from Theorem 3 it follows that

$$|u(x)| \le P_i(x), \; i = 1, 2 \quad (25)$$

where

$$P_1(x) = [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{x_1'=0}^{x_1-1} [1 + \frac{\Delta a_1(x_1')}{a_1(x_1') + a_2(0) + \sum_{i=3}^{n} a_i(x_i)}$$
$$+ \sum_{x_2'=0}^{x_2-1} \cdots \sum_{x_n'=0}^{x_n-1} [f_{11}(x') + f_{22}(x')]]$$

and

$$P_2(x) = \sum_{i=1}^{n} a_i(x_i) + \sum_{x'=0}^{x-1} f_{11}(x') P_1(x').$$

The estimate (24) cannot be obtained from (23) except when $|g(x)|$ = constant, also (25) cannot be obtained from (24). For $n = 3$, (25) is same as obtained in [105]. It appears that in general it is not possible to compare anyone of the estimates obtained here, however for a particular situation we have more flexibility to use these results.

CHAPTER 7

HYPERBOLIC DELAY DIFFERENTIAL EQUATIONS

1. INTRODUCTION

Differential equations with deviating arguments have many applications in the theory of automatic control, two body problems of classical electrodynamics, economics, biological problems and in many areas of science and technology. The abundance of applications is stimulating a rapid development of the theory of delay differential equations.

The theory of partial differential equations with deviating arguments has been very weakly worked out. The only such equations encountered in applications have deviations with respect to only one variable for example see [107-113]. Here throughout we shall consider the following normal form of the hyperbolic differential equations with deviating arguments

$$\frac{\partial^2 u}{\partial x \partial y} = f(x, y, u(x,y), u(g_1(x), y), u_x(x,y), u_x(g_2(x), y), u_y(x,y), u_y(g_3(x), y)) \qquad (1)$$

$$0 \le x \le a, \; 0 \le y \le b.$$

It will always assumed that the functions $g_i(x)$ $(i = 1, 2, 3)$ are continuous over $0 \le x \le a$ and the function f is continuous of all its arguments. We shall denote,

$$-\tau = \min\left\{ \min_{0 \le x \le a} g_i(x), \; i = 1, 2, 3 \right\}$$

$$R_0 = \{(x,y) : 0 \le x \le a, \; 0 \le y \le b\}$$

$$R_1 = \{(x,y) : -\tau \le x \le 0, \; 0 \le y \le b\}$$

and assume that $-\tau \le g_i(x) \le a \ (i=1,2,3)$ for all $0 \le x \le a$.
For the equation (1) the Cauchy problem (Goursat problem) takes the following form: Equation (1) subject to the conditions

$$
\begin{aligned}
&u(x,y) = \psi(x,y)\ , \ (x,y) \in R_1 \\
&u(x,0) = \phi(x) \ , \ 0 \le x \le a \\
&\psi(0,0) = \phi(0)\ , \ \psi_x(0,0) = \phi_x(0).
\end{aligned} \tag{2}
$$

we shall denote $\psi(0,y) = \bar{\psi}(y)$. The function $\psi(x,y)$ is continuous and $\psi_x(x,y)$, $\psi_y(x,y)$ exists and are continuous on R_1, $\phi(x)$ is continuously differentiable on $0 \le x \le a$.

By a solution of the problem we shall mean a function $u(x,y) \in C[R_1 \cup R_0, R]$, the partial derivatives u_x, u_y on $R_1 \cup R_0$ and u_{xy} in R_0 exists and are continuous, also satisfies (1) in R_0 and the initial conditions (2).

It can be easily seen that the problem (1), (2) is equivalent to the following system of integral equations

$$
\left.
\begin{aligned}
u(x,y) &= \int_0^x\int_0^y \theta(x) f(s,t,u(s,t),\dots,u_y(g_3(s,t))\,ds\,dt + p(x,y) \\
u_x(x,y) &= \int_0^y \theta(x) f(x,t,u(x,t),\dots,u_y(g_3(x),t))\,dt + p_x(x,y) \\
u_y(x,y) &= \int_0^x \theta(x) f(s,y,u(s,y),\dots,u_y(g_3(s),y))\,ds + p_y(x,y)
\end{aligned}
\right\} \tag{3}
$$

where

$$
p(x,y) = \begin{cases} \psi(x,y) & \text{if } (x,y) \in R_1 \\ \phi(x) + \bar{\psi}(y) - \phi(0) & \text{if } (x,y) \in R_0 \end{cases} \tag{4}
$$

and

$$
\theta(x) = \begin{cases} 1 & \text{if } x \in [0,a] \\ 0 & \text{if } x \notin [0,a]. \end{cases} \tag{5}
$$

In section 2, several existence and uniqueness theorems have been obtained for the solutions of the problem (1), (2). In section 3, the error bound between the solutions of (1), (2) and its approximate solutions are established. In section 4, some inequalities have been derived which may be used as a tool in the theory of hyperbolic delay differential equations. Lastly in section 5, an iterative scheme is provided which converges to the maximal solution of a suitable problem. This maximal solution is being further used to compare the solutions of the given problem (1), (2).

2. EXISTENCE AND UNIQUENESS.

The first result is local existence theorem.

THEOREM 1. Assume that $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ are given real numbers and let

(i) $M = \sup\{|f(x,y,q_1,\cdots,q_6)| : (x,y) \in R_0,\ |q_1|, |q_2| \le 2K_1,$
$|q_3|, |q_4| \le 2K_2,\ |q_5|, |q_6| \le 2K_3\}$

(ii) for all $(x,y) \in R_1 \cup R_0$, $|p(x,y)| \le K_1$, $|p_x(x,y)| \le K_2$, $|p_y(x,y)| \le K_3$.

Then the problem (1), (2) has a solution provided that

$$M \le \min\left(\frac{K_1}{ab}, \frac{K_2}{b}, \frac{K_3}{a}\right). \tag{6}$$

PROOF. The proof consists of a standard application of the Schauder-Tychnoff fixed point theorem. We shall denote B the Banach space of continuous functions with first order continuous partial derivatives exists on $R_1 \cup R_0$,.

the norm for $u(x,y) \in B$ is defined as follows

$$\|u\| = \max\left\{ \max_{R_1 \cup R_0} |u(x,y)|, \max_{R_1 \cup R_0} |u_x(x,y)|, \max_{R_1 \cup R_0} |u_y(x,y)| \right\}.$$

We define a mapping T on the Banach space B as follows:

For each $u(x,y) \in B$, let $Tu(x,y)$ be the function

$$Tu(x,y) = \int_0^x \int_0^y \theta(x) f(s,t,u(s,t),\ldots,u_y(g_3(s),t))\,ds\,dt + \psi(x,y). \quad (7)$$

It should be noticed that T has the following properties:

(a) if $(x,y) \in R_1$, then $Tu(x,y) = \psi(x,y)$

(b) $Tu(x,y)$ is continuous and first order continuous partial derivatives exists on $R_1 \cup R_0$ i.e., $T: B \to B$

(c) $(Tu)_{xy}(x,y) = f(x,y,u(x,y),\cdots,u_y(g_3(x),y)),\ (x,y) \in R_0$

(d) fixed points of T are solutions of the problem (1), (2)

(e) T is a continuous operator.

Consider the closed, convex and bounded subset B_1 of B defined by

$$B_1 = \left\{ u \in B : |u(x,y)| \le 2K_1,\ |u_x(x,y)| \le 2K_2,\ |u_y(x,y)| \le 2K_3 \right\}$$

and we shall determine the conditions such that $TB_1 \subseteq B_1$. From the condition (ii) it is clear that $TB_1 \subseteq B_1$ if $(x,y) \in R_1$. Now for $u \in B_1$ and $(x,y) \in R_0$, we have

$$|Tu(x,y)| \le K_1 + M\int_0^x\int_0^y ds\,dt \le K_1 + Mab$$

$$|(Tu)_x(x,y)| \le K_2 + M\int_0^y dt \le K_2 + Mb$$

$$|(Tu)_y(x,y)| \le K_3 + M\int_0^x ds \le K_3 + Ma$$

thus condition (6) implies $TB_1 \subseteq B_1$. Using Ascoli-Arzela theorem, it follows that TB_1 is sequentially compact. Hence by the Schauder-Tychnoff fixed point theorem T has a fixed point in B . This completes the proof of the theorem.

Next result is about global existence of the solutions.

THEOREM 2. Assume that

(i) $M = \sup\{|f(x,y,q_1,\cdots,q_6)| : (x,y)\in R_0, (q_1,\cdots,q_6)\in R^6\}$ and is finite

(ii) Condition (ii) of Theorem 1.

Then, the problem (1), (2) has a solution in $R_1 \cup R_0$.

PROOF. The proof follows from Theorem 1, on choosing K_1, K_2 and K_3 sufficiently large so that (6) is satisfied.

To prove our next two results, we shall assume that the function f satisfies the Lipschitz condition of the following form

$$|f(x,y,q_1,\cdots,q_6) - f(x,y,\bar{q}_1,\cdots,\bar{q}_6)| \le \sum_{i=1}^{6} L_i |q_i - \bar{q}_i| \tag{8}$$

for all $(x,y,q_1,\cdots,q_6), (x,y,\bar{q}_1,\cdots,\bar{q}_6) \in R_0 \times R^6$.

THEOREM 3. Assume that the function f satisfies the Lipschitz condition (8). Then there exists a unique solution of the problem (1), (2) provided

$$\sum_{i=1}^{6} L_i \max(ab, b, a) < 1. \tag{9}$$

PROOF. We shall show that the operator defined on B by equation (7) is contracting. Consider the same norm as defined in Theorem 1. For this let $u, v \in B$, then on using (8), we find

$$|Tu(x,y)-Tv(x,y)| \le \int_0^x\int_0^y B(x)\left[L_1|u(s,t)-v(s,t)| + \cdots + L_6|u_y(g_3(s),t)-v_y(g_3(s),t)|\right] ds\, dt$$

$$\le \|u-v\| \sum_{i=1}^{6} L_i\, ab$$

and similarly

$$|(Tu)_x(x,y)-(Tv)_x(x,y)| \le \|u-v\| \sum_{i=1}^{6} L_i\, b$$

$$|(Tu)_y(x,y)-(Tv)_y(x,y)| \le \|u-v\| \sum_{i=1}^{6} L_i\, a .$$

Hence we obtain

$$\|Tu-Tv\| \le \sum_{i=1}^{6} L_i \max(ab, b, a)\, \|u-v\|$$

and now the condition (9) shows that T is contracting. This completes the proof of the theorem.

Actually the solution of the problem (1), (2) exists even if condition (9) is not satisfied and this is what we shall show in our next theorem.

THEOREM 4. Assume that the function f satisfies the Lipschitz condition (8). Then there exists a unique solution of the problem (1), (2) on $R_1 \cup R_0$.

PROOF. Let $L = \max L_i \; (i = 1, 2, \cdots, 6)$

Case I. $L \geq 1$.

Define the space B as in Theorem 1, with the norm

$$\|u\|_* = \max\left\{ \max_{R_1 \cup R_0} e^{-6L(x+y)} (|u(x,y)|, |u_x(x,y)|, |u_y(x,y)|) \right\}.$$

Then for $u, v \in B$, we have

$$\begin{aligned} e^{-6L(x+y)} |Tu(x,y) - Tv(x,y)| &\leq \int_0^x \int_0^y \theta(x) e^{-6L(x+y)} e^{6L(s+t)} \sum_{i=1}^{6} L_i \|u-v\|_* \, ds\,dt \\ &\leq \sum_{i=1}^{6} L_i \|u-v\|_* \max_{R_0} (1 - e^{-6Ly} - e^{-6Lx} + e^{-6L(x+y)}) \frac{1}{36L^2} \\ &\leq \frac{1}{6L} \|u-v\|_* (1-e^{-6La})(1-e^{-6Lb}). \end{aligned}$$

Similarly we have

$$e^{-6L(x+y)} |(Tu)_x(x,y) - (Tv)_x(x,y)| \leq \|u-v\|_* (1-e^{-6Lb})$$

$$e^{-6L(x+y)} |(Tu)_y(x,y) - (Tv)_y(x,y)| \leq \|u-v\|_* (1-e^{-6La})$$

and hence

$$\|Tu - Tv\|_* \leq k \|u-v\|_*$$

where

$$k = \max\left\{ \frac{1}{6L}(1-e^{-6La})(1-e^{-6Lb}), (1-e^{-6Lb}), (1-e^{-6La}) \right\}$$

and it is obvious that $k < 1$.

Case II $L \leq 1$.

Define the space B as in Theorem 1, with the different norm

$$\|u\|_{**} = \max\left\{ \max_{R_1 \cup R_0} e^{-6\sqrt{L}(x+y)} (|u(x,y)|, |u_x(x,y)|, |u_y(x,y)|) \right\}.$$

Then for u,v $\in$ B, we have

$$e^{-6\sqrt{L}(x+y)}|Tu(x,y)-Tv(x,y)| \le \frac{1}{6}\|u-v\|_{**}(1-e^{-6\sqrt{L}a})(1-e^{-6\sqrt{L}b}).$$

$$e^{-6\sqrt{L}(x+y)}|(Tu)_x(x,y)-(Tv)_x(x,y)| \le \sqrt{L}\|u-v\|_{**}(1-e^{-6\sqrt{L}b})$$

$$e^{-6\sqrt{L}(x+y)}|(Tu)_y(x,y)-(Tv)_y(x,y)| \le \sqrt{L}\|u-v\|_{**}(1-e^{-6\sqrt{L}a})$$

and hence

$$\|Tu-Tv\|_{**} \le k_1\|u-v\|_{**}$$

where

$$k_1 = \max\left\{\frac{1}{6}(1-e^{-6\sqrt{L}a})(1-e^{-6\sqrt{L}b}), \sqrt{L}(1-e^{-6\sqrt{L}b}), \sqrt{L}(1-e^{-6\sqrt{L}a})\right\}$$

and once again $k_1 < 1$. This completes the proof.

Next we shall prove a general uniqueness theorem of Perron type which is based on those given earlier by Shanahan [114] for hyperbolic differential equations.

THEOREM 5. Assume that

(i) $g_i(x)$ $(i=1,2,3)$ map $[0,a]$ into itself.

(ii) $f \in C[R_0 \times R^6, R]$ and

$$|f(x,y,q_1,\dots,q_6)-f(x,y,\bar{q}_1,\dots,\bar{q}_6)| \le g(x,y,|q_1-\bar{q}_1|,\dots,|q_6-\bar{q}_6|)$$

where $g \in C[R_0 \times R_+^6, R_+]$, $g(x,y,0,\dots,0) \equiv 0$, $g(x,y,\phi_1,\dots,\phi_6)$ is monotonic non-decreasing in $\phi_1,\dots,\phi_6$ and bounded.

(ii) $Z(x,y) \equiv 0$ is the only solution of the hyperbolic delay differential equation

$$Z_{xy} = g(x,y,Z(x,y),\dots,Z_y(g_3(x),y)) \tag{10}$$

such that $Z(0,0)=0$, $Z(x,0)\equiv 0$, $Z(0,y)\equiv 0$. (11)

PROOF. Because of (i), recall that $R_1 = \{(0,y) : 0 \le y \le b\}$ and hence $R_1 \cup R_0 = R_0$. Let us assume that there are two solutions $u(x,y)$, $V(x,y)$ for (1), (2) on R_0, we define

$$\begin{aligned} A(x,y) &= |u(x,y) - V(x,y)| \\ B(x,y) &= |u_x(x,y) - V_x(x,y)| \\ C(x,y) &= |u_y(x,y) - V_y(x,y)|. \end{aligned} \tag{12}$$

Since we have

$$u(x,0) = V(x,0) = \phi(x),\ u_x(x,0) = V_x(x,0) = \phi_x(x),\ 0 \le x \le a$$

$$u(0,y) = V(0,y) = \psi(y),\ u_y(0,y) = V_y(0,y) = \psi_y(y),\ 0 \le y \le b$$

it follows that

$$A(0,0) = 0,\ B(x,0) \equiv 0,\ C(0,y) \equiv 0.$$

Furthermore, by induction (i), (ii) we obtain

$$A(x,y) \le \int_0^x \int_0^y g(s,t, A(s,t), A(g_1(s),t), B(s,t), B(g_2(s),t), C(s,t), C(g_3(s),t))\, ds\, dt$$

$$B(x,y) \le \int_0^y g(x,t, A(x,t), \dots, C(g_3(x),t))\, dt$$

$$C(x,y) \le \int_0^x g(s,y, A(s,y), \dots, C(g_3(s),y))\, ds.$$

Let us define the sequence of successive approximations to the solutions of (10), (11) as follows

$$\alpha_0(x,y) = A(x,y),\ \beta_0(x,y) = B(x,y),\ \gamma_0(x,y) = C(x,y)$$

and for $n \geq 0$

$$\alpha_{n+1}(x,y) = \int_0^x \int_0^y g(s,t, \alpha_n(s,t), \alpha_n(g_1(s),t), \beta_n(s,t), \beta_n(g_2(s),t), \gamma_n(s,t), \gamma_n(g_3(s),t))\, ds\, dt$$

$$\beta_{n+1}(x,y) = \int_0^y g(x,t, \alpha_n(x,t), \dots, \gamma_n(g_3(x),t))\, dt$$

$$\gamma_{n+1}(x,y) = \int_0^x g(s,y, \alpha_n(s,y), \dots, \gamma_n(g_3(s),y))\, ds.$$

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Since $\alpha_0(x,y) \le \alpha_1(x,y)$, $\beta_0(x,y) \le \beta_1(x,y)$, $\gamma_0(x,y) \le \gamma_1(x,y)$ we find on using (i)

$$\text{(i) } \alpha_0(g_1(x),y) \le \alpha_1(g_1(x),y),\ \beta_0(g_2(x),y) \le \beta_1(g_2(x),y),$$
$$\gamma_0(g_3(x),y) \le \gamma_1(g_3(x),y).$$

Now on using nondecreasing property of g, it follows by induction that

$$\alpha_n(x,y) \le \alpha_{n+1}(x,y),\ \beta_n(x,y) \le \beta_{n+1}(x,y),\ \gamma_n(x,y) \le \gamma_{n+1}(x,y).$$

Also the functions $\alpha_n(x,y)$, $\beta_n(x,y)$, $\gamma_n(x,y)$ are uniformly bounded in view of the fact that g is assumed to be bounded. Hence we get

$$\lim_{n\to\infty} \alpha_n(x,y) = \alpha(x,y),\ \lim_{n\to\infty} \beta_n(x,y) = \beta(x,y),\ \lim_{n\to\infty} \gamma_n(x,y) = \gamma(x,y) \text{ on } R_0.$$

It is easy to see, using Lebegue's monotone convergence theorem, $\alpha(x,y)$ is a solution of (10), (11). Hence we have

$$A(x,y) \le \alpha(x,y),\ B(x,y) \le \beta(x,y),\ C(x,y) \le \gamma(x,y).$$

Now by assumption (iii), $\alpha(x,y) \equiv 0$, $\beta(x,y) \equiv 0$, $\gamma(x,y) \equiv 0$ and this proves $A(x,y) \equiv 0$, $B(x,y) \equiv 0$, $C(x,y) \equiv 0$. This completes the proof of the theorem.

To prove our next results we shall assume the following conditions:

(i) $g_i(x)$ $(i = 1,2,3)$ are continuous and $g_i(x) \le x$ for all $x \in [0,a]$ also $\min\limits_{0 \le x \le a} g_i(x) = 0$.

(ii) $f \in C[R_0 \times R^6, R]$ and bounded

(iii) for $(x, y, \phi_1, \cdots, \phi_6), (x, y, \bar{\Phi}_1, \cdots, \bar{\Phi}_6) \in R_0 \times R^6$

$$|f(x, y, \phi_1, \cdots, \phi_6) - f(x, y, \bar{\Phi}_1, \cdots, \bar{\Phi}_6)| \le \frac{k}{xy}\Big[a_1|\phi_1 - \bar{\Phi}_1| + a_2|\phi_2 - \bar{\Phi}_2| + a_3\frac{x}{c_0}|\phi_3 - \bar{\Phi}_3| + a_4\frac{x}{c_0}|\phi_4 - \bar{\Phi}_4| + a_5\, y/c_0\,|\phi_5 - \bar{\Phi}_5| + a_6\, y/c_0\,|\phi_6 - \bar{\Phi}_6|\Big]$$

where $k > 0$, $\sum_{i=1}^{6} a_i \le 1$, $a_i \geq 0$ $(i = 1, 2, \cdots, 6)$ and

$$c_0 = \begin{cases} 1 & \text{if } k \le 1 \\ \sqrt{k} & \text{otherwise} \end{cases}$$

(iv)

$$|f(x, y, \phi_1, \cdots, \phi_6) - f(x, y, \bar{\Phi}_1, \cdots, \bar{\Phi}_6)| \le \frac{c}{x^\beta y^\beta}\Big[b_1|\phi_1 - \bar{\Phi}_1|^\alpha + b_2|\phi_2 - \bar{\Phi}_2|^\alpha + b_3 x^\alpha|\phi_3 - \bar{\Phi}_3|^\alpha + b_4 x^\alpha|\phi_4 - \bar{\Phi}_4|^\alpha + b_5 y^\alpha|\phi_5 - \bar{\Phi}_5|^\alpha + b_6 y^\alpha|\phi_6 - \bar{\Phi}_6|^\alpha\Big]$$

where $c > 0$, $\sum_{i=1}^{6} b_i \le 1$, $b_i \geq 0$ $(i = 1, 2, \cdots, 6)$ with $0 < \alpha < 1$, $\beta < \alpha$ and $k(1 - \alpha^2) < (1 - \beta^2)$.

The present theorem is classical Nagumo type, for a particular case of hyperbolic differential equations, one can refer to Kampen [120], Wong [118], [119].

THEOREM 6. For $k \le 1$, condition (i) - (iii) implies that there exists one and only one solution for (1), (2).

Before we prove this theorem, we shall state another uniqueness theorem in which conditions are similar to Krasonosielski and Krein type. For particular cases see Palczewski [115], Palczewski and Pawelski [116], Wong [117], [118].

THEOREM 7. For $k \geq 1$, conditions (i) - (iv) imply that there exists one and only one solution for (1), (2).

PROOF. We shall prove both the theorems simultaneously. Recall condition (i) which gives $R_1 = \{(0,y) : 0 \leq y \leq b\}$ and $R_1 \cup R_0 = R_0$. To prove existence, we have from condition (ii)

$$M = \sup\{|f(x,y,\phi_1,\cdots,\phi_6)| : (x,y) \in R_0, (\phi_1,\phi_2,\cdots,\phi_6) \in R^6\}.$$

Then, if we define

$$K_1 \geq \max\{Mab, \max_{R_0}|\phi(x)+\psi(y)-\phi(0)|\}$$

$$K_2 \geq \max\{Mb, \max_{R_0}|\phi_x(x)|\}$$

$$K_3 \geq \max\{Ma, \max_{R_0}|\psi_y(y)|\}$$

all conditions of Theorem 2 are satisfied and hence there exists a solution of the problem (1), (2).

Next we shall prove the uniqueness:
Let us assume that there are two solutions $u(x,y), v(x,y)$ for (1), (2) on R_0. For $k \leq 1$, define

$$Q(x,y) = \frac{a_1 A(x,y)}{xy} + \frac{a_2 A(g_1(x),y)}{xy} + \frac{a_3 B(x,y)}{y} + a_4\frac{B(g_2(x),y)}{y} + \frac{a_5 C(x,y)}{x} + \frac{a_6 C(g_3(x),y)}{x}$$

for $xy > 0$, where $A(x,y)$, $B(x,y)$, $C(x,y)$ are defined in (12). Since f is continuous and bounded we note that

$$|f(x,y,u(x,y),\cdots,u_y(g_3(x),y)) - f(x,y,v(x,y),\cdots,v_y(g_3(x),y))| \leq Mxy$$

where M_{xy} tends to zero as x or y tends to ze[illegible] or both.

Also

$$A(x,y) \leq \int_0^x \int_0^y M_{xy}\, ds\, dt \leq M_{xy}\, xy$$

$$A(g_1(x),y) \leq \int_0^{g_1(x)} \int_0^y M_{xy}\, ds\, dt \leq \int_0^x \int_0^y M_{xy}\, ds\, dt \leq M_{xy}\, xy$$

and, similarly

$$B(x,y) \leq M_{xy}\, y\ , \quad B(g_2(x),y) \leq M_{xy}\, y$$

$$C(x,y) \leq M_{xy}\, x, \quad C(g_3(x),y) \leq M_{xy}\, x$$

and hence $Q(x,y) \leq M_{xy}$, therefore it follows that $Q(x,y) \geq 0$ for all $(x,y) \in D$ and $\lim_{s \to s_0} Q(s) = 0$ where $s = (x,y) \in R_0$, $s_0 \in D = \{s : s \in R_0 \text{ and } x = 0 \text{ or } y = 0\}$.

Clearly Q is continuous on D if we define $Q(s_0) = 0$ for $s_0 \in D$. We shall show that $Q(x,y) \equiv 0$. Using (iii), we observe for $0 < x+y < \bar{x} + \bar{y}$, .

$$A(x,y) \leq k \int_0^x \int_0^y (st)^{-1} [a_1 A(s,t) + a_2 A(g_1(s),t) + a_3 s B(s,t) + a_4 s B(g_2(s),t) + a_5 t\, C(s,t) + a_6 t\, C(g_3(s),t)]\, ds\, dt$$

$$\leq k \int_0^x \int_0^y Q(s,t)\, ds\, dt < k r xy \leq r xy$$

and similarly,

$$B(x,y) < r y, \quad C(x,y) < r x .$$

Hence we find

$$A(\bar{x},\bar{y}) < r \bar{x}\bar{y}\ , \quad A(g_1(\bar{x}),\bar{y}) < r\bar{x}\bar{y}$$

$$B(\bar{x},\bar{y}) < r\bar{y}\ , \quad B(g_2(\bar{x}),\bar{y}) < r\bar{y}$$

$$C(\bar{x},\bar{y}) < r\bar{x}\ , \quad C(g_3(\bar{x}),\bar{y}) < r\bar{x}$$

which gives $r = Q(\bar{x},\bar{y}) < r\sum_{i=1}^{6} a_i \leq r$ which is the desired contradiction.

For $R \geq 1$, let $M = \sup_{R_0 \times R^6} |f(x,y,\phi_1,\dots,\phi_6)|$ then we have

$$A(x,y) \leq 2Mxy,\ B(x,y) \leq 2My,\ C(x,y) \leq 2Mx,\ (x,y) \in R_0$$

also from this we find

$$A(g_1(x),y) \leq 2M\,g_1(x)\,y \leq 2Mxy$$
$$B(g_2(x),y) \leq 2My$$
$$C(g_3(x),y) \leq 2M\,g_3(x) \leq 2Mx.$$

From (iv), it follows that

$$A(x,y) \leq \int_0^x\int_0^y \frac{c}{s^\beta t^\beta}\Big[\sum_{i=1}^{6} b_i (st)^\alpha\Big](2M)^\alpha\, ds\, dt$$
$$\leq \frac{c(2M)^\alpha (xy)^{(1-\beta)+\alpha}}{[(1-\beta)+\alpha]^2}$$
$$\leq c(2M)^\alpha (xy)^{(1-\beta)+\alpha},\ \text{since } (1-\beta)+\alpha > 1,$$

also,

$$B(x,y) \leq \int_0^y \frac{c}{x^\beta t^\beta}\Big[\sum_{i=1}^{6} b_i (xt)^\alpha\Big](2M)^\alpha\, dt$$
$$\leq \frac{c(2M)^\alpha x^{\alpha-\beta} y^{(1-\beta)+\alpha}}{[(1-\beta)+\alpha]} \leq c(2M)^\alpha (xy)^{\alpha-\beta}\, y.$$

Similarly

$$C(x,y) \leq c(2M)^\alpha (xy)^{\alpha-\beta}\, x.$$

Since $\alpha > \beta$ we find

$$A(g_1(x),y) \le C(2M)^{\alpha}(g_1(x)y)^{(1-\beta)+\alpha} \le C(2M)^{\alpha}(xy)^{(1-\beta)+\alpha}$$

$$B(g_2(x),y) \le C(2M)^{\alpha}(g_2(x)y)^{\alpha-\beta}y \le C(2M)^{\alpha}(xy)^{\alpha-\beta}y$$

$$C(g_3(x),y) \le C(2M)^{\alpha}(g_3(x),y)^{\alpha-\beta}g_3(x) \le C(2M)^{\alpha}(xy)^{\alpha-\beta}x.$$

Now it follows by induction that

$$\left.\begin{matrix} A(x,y) \\ A(g_1(x),y) \end{matrix}\right\} \le C^{\alpha^*}(2M)^{\alpha^{m+1}}(xy)^{(1-\beta)\alpha^* + \alpha^{m+1}}$$

$$\left.\begin{matrix} B(x,y) \\ B(g_2(x),y) \end{matrix}\right\} \le C^{\alpha^*}(2M)^{\alpha^{m+1}}(xy)^{(\alpha-\beta)\alpha^*}y$$

$$\left.\begin{matrix} C(x,y) \\ C(g_3(x),y) \end{matrix}\right\} \le C^{\alpha^*}(2M)^{\alpha^{m+1}}(xy)^{(\alpha-\beta)\alpha^*}x$$

where $\alpha^* = 1 + \alpha + \alpha^2 + \cdots + \alpha^m$ for $m = 0, 1, \ldots$.

Therefore we obtain the following estimate

$$\left.\begin{matrix} A(x,y) \\ A(g_1(x),y) \end{matrix}\right\} \le p, \quad \left.\begin{matrix} B(x,y) \\ B(g_2(x),y) \end{matrix}\right\} \le qy, \quad \left.\begin{matrix} C(x,y) \\ C(g_3(x),y) \end{matrix}\right\} \le qx \tag{13}$$

where $p = C^{1/1-\alpha}(xy)^{\frac{1-\beta}{1-\alpha}}$ and $q = C^{1/1-\alpha}(xy)^{\frac{\alpha-\beta}{1-\alpha}}$.

Define

$$R(x,y) = [\sqrt{k}\,a_1 A(x,y) + \sqrt{k}\,a_2 A(g_1(x),y) + a_3 x B(x,y) + a_4 x B(g_2(x),y) + a_5 y C(x,y) + a_6 y C(g_3(x),y)]\, k^{-\frac{1}{2}}(xy)^{-\sqrt{k}}$$

for $xy > 0$. Then from (13) it follows that

$$0 \le R(x,y) = R(s) \le C^{1/1-\alpha}(xy)^{\rho}[\sqrt{k}\,a_1 + \sqrt{k}\,a_2 + a_3 + a_4 + a_5 + a_6]\,k^{-1/2}$$

$$\le C^{1/1-\alpha}(xy)^{\rho}$$

where $\delta = [(1-\beta) - \sqrt{k}(1-\alpha)]/(1-\alpha)$. Hence we have $\lim_{s \to s_0} R(s) = 0$, where $s \in R_0$ and $s_0 \in D$. Clearly R is continuous on D if we define $R(s_0) = 0$ for $s_0 \in D$. We shall show that $R(x,y) \equiv 0$. If not, there exists a point (x_0, y_0) such that $0 < r = R(x_0, y_0) = \sup_{R_0} R(x,y)$.

On the other hand (iii) implies

$$A(x_0, y_0) \leq k \int_0^{x_0}\int_0^{y_0} (st)^{-1}\Big[a_1 A(s,t) + a_2 A(g_1(s),t) + a_3 \frac{s\,B(s,t)}{\sqrt{k}} + a_4 \frac{s\,B(g_2(s),t)}{\sqrt{k}} + a_5 \frac{t\,C(s,t)}{\sqrt{k}} + a_6 \frac{t\,C(g_3(s),t)}{\sqrt{k}}\Big] ds\,dt$$

$$\leq k \int_0^{x_0}\int_0^{y_0} (st)^{\sqrt{k}-1} R(s,t)\,ds\,dt$$

$$< kr \int_0^{x_0}\int_0^{y_0} (st)^{\sqrt{k}-1} ds\,dt < r(x_0 y_0)^{\sqrt{k}}.$$

Similarly we have $A(g_1(x_0), y_0) < r(x_0 y_0)^{\sqrt{k}}$.

Furthermore analogously

$$\left.\begin{array}{l} B(x_0, y_0) \\ B(g_2(x_0), y_0) \end{array}\right\} < \frac{r\sqrt{k}\,(x_0 y_0)^{\sqrt{k}}}{x_0}, \quad \left.\begin{array}{l} C(x_0, y_0) \\ C(g_3(x_0), y_0) \end{array}\right\} < \frac{r\sqrt{k}\,(x_0 y_0)^{\sqrt{k}}}{y_0}.$$

Hence we obtain $r = R(x_0, y_0) < r \sum_{i=1}^{6} a_i \leq r$ which is the desired contradiction. This completes the proof of the theorem.

3. APPROXIMATE SOLUTION.

By an approximate solution of the problem (1), (2), we shall mean a function $v(x,y) \in C[R_1 \cup R_0, R]$, the partial derivatives v_x, v_y in $R_1 \cup R_0$ and v_{xy} in R_0

exists and are continuous, also satisfies

$$V_{xy} = f(x, y, v(x,y), \cdots, v_y(g_3(x), y)) + \eta(x,y), \qquad (x,y) \in R_0 \tag{14}$$

$$\begin{aligned} v(x,y) &= \psi(x,y) + \delta(x,y) \quad (x,y) \in R_1 \\ v(x,0) &= \phi(x) + \sigma(x) \quad -\tau \le x \le a \\ \psi(0,0) + \delta(0,0) &= \phi(0) + \sigma(0) \\ \psi_x(0,0) + \delta_x(0,0) &= \phi_x(0) + \sigma_x(0) \end{aligned} \tag{15}$$

where the functions $\psi(x,y)$, $\phi(x)$ are as defined earlier and $\eta(x,y)$ is continuous on R_0, $\delta(x,y)$ is continuously differentiable on $0 \le x \le a$. For the functions $g_i(x) (i = 1,2,3)$ we shall assume $g_i(x) \le x$.

The following theorem which is based on those given in [120], [81] for the equations without deviating arguments, estimates the error between a solution and an approximate solution of (1), (2).

THEOREM 8. Assume that

(i) $u(x,y)$ is a solution and $v(x,y)$ is its approximate solution of (1), (2)

(ii) $|f(x,y,q_1,\cdots,q_6) - f(x,y,\bar{q}_1,\cdots,\bar{q}_6)| \le g(x,y,|q_1-\bar{q}_1|,\cdots,|q_6-\bar{q}_6|)$

where $g(x,y,\phi_1,\cdots,\phi_6) \in C[R_0 \times R_+^6, R_+]$ and is monotone nondecreasing in $\phi_1, \cdots, \phi_6$.

(iii) $Z \in C[R_1 \cup R_0, R_+]$, $Z(x,y)$ possesses continuous positive partial derivatives Z_x, Z_y in $R_1 \cup R_0$ and Z_{xy} in R_0,

$$Z_{xy} > g(x,y,Z(x,y),\cdots,Z_y(g_3(x),y)) + |\eta(x,y)|, (x,y) \in R_0$$

(iv) $|\delta(x,y)| < Z(x,y)$, $|\delta_x(x,y)| < Z_x(x,y)$, $|\delta_y(x,y)| < Z_y(x,y)$ for all $(x,y) \in R_1$, $|\sigma_x(x)| < Z_x(x,0)$ for all $x \in [0,a]$.

Then, on $R_1 \cup R_0$,

$$|v(x,y) - u(x,y)| < Z(x,y), \; |v_x(x,y) - u_x(x,y)| < Z_x(x,y)$$
$$|v_y(x,y) - u_y(x,y)| < Z_y(x,y).$$

PROOF. The claim is true in R_1 because of the assumption (iv). Now it is enough to show that in R_0

$$|v_x(x,y) - u_x(x,y)| < Z_x(x,y) \tag{16}$$

$$|v_y(x,y) - u_y(x,y)| < Z_y(x,y). \tag{17}$$

Since (16) and (17) give

$$|v(x,y) - u(x,y)| - |v(0,y) - u(0,y)| \le Z(x,y) - Z(0,y)$$

$$|v(0,y) - u(0,y)| - |v(0,0) - u(0,0)| \le Z(0,y) - Z(0,0)$$

also, we have $|v(0,0) - u(0,0)| < Z(0,0)$ from (iv).

Hence adding the three we get

$$|v(x,y) - u(x,y)| < Z(x,y).$$

Assume that both (16) and (17) do not satisfy, let t_o be the greatest lower bound of the numbers $t > x+y$, $(x,y) \in R_o$ such that two inequalities are satisfied for $(x+y) < t_o$. Then there exists a point (x_o, y_o) with $x_o + y_o = t_o$ for which one of the inequalities is not true, but in either case we have

$$|v(x,y) - u(x,y)| \leq Z(x,y), \quad |v(g_1(x),y) - u(g_1(x),y)| \leq Z(g_1(x),y)$$

$$|v_x(x,y) - u_x(x,y)| \leq Z_x(x,y), \quad |v_x(g_2(x),y) - u_x(g_2(x),y)| \leq Z_x(g_2(x),y)$$

$$|v_y(x,y) - u_y(x,y)| \leq Z_y(x,y), \quad |v_y(g_3(x),y) - u_y(g_3(x),y)| \leq Z_y(g_3(x),y)$$

for $x+y < t_o$.

Now, let $|v_x(x_o,y_o) - u_x(x_o,y_o)| = Z_x(x_o,y_o)$

but we have successively

$$\begin{aligned} |v_x(x_o,y_o) - u_x(x_o,y_o)| &\leq \int_0^{y_o} |f(x_o,t,v(x_o,t),\cdots,v_y(g_3(x_o),t) \\ &\qquad + \eta(x_o,t) - f(x_o,t,u(x_o,t),\cdots, \\ &\qquad u_y(g_3(x_o),t))| \, dt + |\sigma_x(x_o)| \\ &\leq \int_0^{y_o} [g(x_o,t,|v(x_o,t)-u(x_o,t)|,\cdots, \\ &\qquad |v_y(g_3(x_o),t) - u_y(g_3(x_o),t)|) + |\eta(x_o,t)|] \\ &\qquad dt + Z_x(x,0) \\ &< \int_0^{y_o} Z_{xy}(x_o,t)\,dt + Z_x(x_o,0) = Z_x(x_o,y_o) \end{aligned}$$

contradicting the assumption. Similar contradiction occurs in the other case $|v_y(x_o,y_o) - u_y(x_o,y_o)| = Z_y(x_o,y_o)$ also. Hence the desired result follows.

4. SOME INEQUALITIES.

The results obtained here are based on those given in [81], [120] for the equations without deviating arguments. Throughout we shall assume $g_i(x) \leq x$ $(i = 1, 2, 3)$ and the expression

$$\phi_{xy} - f(x, y, \phi(x,y), \cdots, \phi_y(g_3(x), y))$$

will be denoted by $P\phi$ in what follows.

THEOREM 9. Assume that

(i) $u, v \in C[R_1 \cup R_0, R]$, the partial derivatives u_x, u_y, v_x, v_y in $R_1 \cup R_0$ and u_{xy}, v_{xy} in R_0 exists and are continuous.

(ii) $f \in C[R_0 \times R^6, R]$, $f(x, y, \phi_1, \cdots, \phi_6)$ is monotonic, non-decreasing in $\phi_1, \cdots, \phi_6$ and in R_0, $Pu < Pv$.

(iii) $u(x,y) < v(x,y)$, $u_x(x,y) < v_x(x,y)$, $u_y(x,y) < v_y(x,y)$ in R_1

$u_x(x,0) < v_x(x,0)$, $0 \leq x \leq a$.

Then we have on $R_1 \cup R_0$,

$$u(x,y) < v(x,y),\ u_x(x,y) < v_x(x,y) \qquad (18)$$
$$u_y(x,y) < v_y(x,y).$$

PROOF. The claim (18) is true in R_1 because of the assumption (iii). Now it is enough to show that in R_0

$$u_x(x,y) < v_x(x,y),\ u_y(x,y) < v_y(x,y). \qquad (19)$$

For,

$$u_x(x,y) < v_x(x,y) \Rightarrow u(x,y) - u(0,y) < v(x,y) - v(0,y)$$

$$u_y(0,y) < v_y(0,y) \Rightarrow u(0,y) - u(0,0) < v(0,y) - v(0,0)$$

also, we have $u(0,0) < v(0,0)$. Hence adding the three implies $u(x,y) < v(x,y)$. Assume that (19) is false, and let t_0 be the greatest lower bound of numbers $t > x+y$ such that two inequalities are satisfied for $x+y < t_0$. Then there exists a point (x_0, y_0) with $x_0 + y_0 = t_0$ for which one of the inequalities is not true. Let us suppose that

$$u_x(x_0, y_0) = v_x(x_0, y_0). \tag{20}$$

Then, by assumption (iii) $y_0 > 0$ we have

$$u_x(x_0, y_0 - h) < v_x(x_0, y_0 - h), \; h > 0 \tag{21}$$

also,

$$u(x_0, y_0) \leq v(x_0, y_0), \; u(g_1(x_0), y_0) \leq v(g_1(x_0), y_0)$$

$$u_x(g_2(x_0), y_0) \leq v_x(g_2(x_0), y_0)$$

$$u_y(x_0, y_0) \leq v_y(x_0, y_0), \; u_y(g_3(x_0), y_0) \leq v_y(g_3(x_0), y_0).$$

From (20) and (21), we obtain

$$u_{xy}(x_0, y_0) \geqslant v_{xy}(x_0, y_0) \tag{22}$$

Now using monotonic non-decreasing property of f, it follows that

$$f(x_0, y_0, u(x_0,y_0), \cdots, u_y(g_3(x_0), y_0)) \leq f(x_0, y_0, v(x_0,y_0), \cdots v_y(g_3(x_0), y_0))$$

this together with (22) and the assumption $Pu < Pv$ yields

$$f(x_0, y_0, v(x_0,y_0), \cdots, v_y(g_3(x_0), y_0)) < f(x_0, y_0, v(x_0,y_0), \cdots, v_y(g_3(x_0), y_0))$$

which is an absurdity. If we assume that $u_y(x_0, y_0) = v_y(x_0, y_0)$ instead of (20) an argument similar to the foregoing leads to a contradiction. This completes the proof of the theorem.

DEFINITION. A function $u \in C[R_1 \cup R_0, R]$ possessing continuous partial derivatives u_x, u_y in $R_1 \cup R_0$, u_{xy} in R_0 and satisfying the hyperbolic delay differential inequality

$$u_{xy}(x,y) < f(x, y, u(x,y), \cdots, u_y(g_3(x), y))$$

in R_0, is said to be an under function with respect to hyperbolic delay differential equation

$$u_{xy} = f(x, y, u(x,y), \cdots, u_y(g_3(x), y)). \tag{23}$$

If, on the other hand, u satisfies the reversed inequality it is said to be an over function.

THEOREM 10. Let u, w be under and over functions with respect to the hyperbolic delay differential equation (23) respectively. Suppose that $f \in C[R_0 \times R^6, R]$, $f(x, y, \phi_1, \cdots, \phi_6)$

is monotonic nondecreasing in $\phi_1, \dots, \phi_6$. Assume that V is a solution of (23) such that

$$u_x(x,y) < v_x(x,y) < w_x(x,y) \quad \text{in } R_1$$

$$u_y(x,y) < v_y(x,y) < w_y(x,y) \quad \text{in } R_1$$

$$u_x(x,0) < v_x(x,0) < w_x(x,0)\,, \quad 0 \le x \le a$$

$$u(0,0) < v(0,0) < w(0,0).$$

Then, we have on $R_1 \cup R_0$,

$$u(x,y) < v(x,y) < w(x,y)$$

$$u_x(x,y) < v_x(x,y) < w_x(x,y)$$

$$u_y(x,y) < v_y(x,y) < w_y(x,y).$$

PROOF. This result can be proved by a repeated application of Theorem 9.

THEOREM 11. Assume that

(i) the condition (1) of Theorem 9

(ii) $f \in C[R_0 \times R^4, R]$, $f(x,y,\phi_1,\phi_2,\phi_3,\phi_4)$ is monotonic nondecreasing in ϕ_1, ϕ_2, ϕ_4 and in R_0, $Pu < Pv$.

(iii) $u(x,y) < v(x,y)$, $u_x(x,y) < v_x(x,y)$ in R_1

$$u_x(x,0) < v_x(x,0)\,, \quad 0 \le x \le a.$$

Then, we have on $R_1 \cup R_0$,

$$u(x,y) < v(x,y), \quad u_x(x,y) < v_x(x,y).$$

PROOF. The proof follows from Theorem 9.

5. MAXIMAL SOLUTION.

For the past several years to study several qualitative as well as quantitative properties of the solutions of differential and integral equations maximal solution of the related systems have been used effectively e.g. see [120], [11], [11].

Here we present a constructive iterative scheme which converges to the maximal solution of

$$u_{xy} = h(x, y, u(x,y), \cdots, u_y(g_3(x), y)), \quad (x,y) \in R_0 \qquad (24)$$

subject to conditions

$$\begin{aligned} &u(x,y) = \beta(x,y) \qquad (x,y) \in R_1 \\ &u(x,0) = \alpha(x) \qquad 0 \le x \le a \\ &\beta(0,0) = \alpha(0), \ \beta_x(0,0) = \alpha_x(0) \end{aligned} \qquad (25)$$

where $\beta(x,y)$, $\beta_x(x,y)$, $\beta_y(x,y)$ exists, non-negative and continuous on R_1, $\alpha(x)$ is continuously differentiable, also $\alpha_x(x) \geq 0$, $x \in [0,a]$ and $\alpha(0) \geq 0$.

The scheme is somewhat similar to given by Agarwal and Krishnamoorthy [121], Krishnamoorthy and Agarwal [124] for boundary value problems for ordinary differential equations with and without deviating arguments, we shall denote

$$P^*(x,y) = \begin{cases} \beta(x,y) & \text{if } (x,y) \in R_1 \\ \alpha(x) + \beta(y) - \alpha(0) & \text{if } (x,y) \in R_0 \end{cases}$$

It can easily be seen that like problem (1), (2) the problem (24), (25) is equivalent to

$$u(x,y)=\int_0^x\int_0^y \theta(x)\, h(s,t;u(s,t),\cdots,u_y(g_3(s),t))\,ds\,dt + p^*(x,y).$$

Let B denote the Banach space of continuous functions with first order continuous partial derivatives exists on $R_1 \cup R_0$ with the norm

$$\|u\| = \max\left\{\max_{R_1\cup R_0}|u(x,y)|,\ \max_{R_1\cup R_0}|u_x(x,y)|,\ \max_{R_1\cup R_0}|u_y(x,y)|\right\}.$$

Define $B_1 \subseteq B$ the closed, convex cone of non-negative functions with non-negative partial derivatives on $R_1 \cup R_0$. Let S denote those elements of B_1 which coincide identically with $\beta(x,y)$ on R_1 and $\alpha(x)$ on $0 \le x \le a$. Define the set

$$S_\rho = \{u \in S : \|u\| \le \rho\}.$$

THEOREM 12. Assume that

(i) $h \in C[R_0 \times R_+^6, R_+]$

(ii) $h(x,y,\phi_1,\cdots,\phi_6)$ is non-decreasing with respect to $\phi_1,\phi_2,\cdots,\phi_6$

(iii) for all $(x,y) \in R_1 \cup R_0$ there exists a $\rho > 0$ such that

$$u^0(x,y)=\int_0^x\int_0^y \theta(x)\, h(s,t,\rho,\cdots,\rho)\,ds\,dt + p^*(x,y) \le \rho$$

$$u_x^0(x,y)=\int_0^y \theta(x)\, h(x,t,\rho,\cdots,\rho)\,dt + p_x^*(x,y) \le \rho$$

$$u_y^0(x,y)=\int_0^x \theta(x)\, h(s,y,\rho,\cdots,\rho)\,ds + p_y^*(x,y) \le \rho.$$

For $(x, y) \in R_1 \cup R_0$ define the iterative scheme as

$$u^{n+1}(x,y) = \int_0^x \int_0^y \theta(x) h(s,t, u^n(s,t), \dots, u_y^n(g_3(s),t))\,ds\,dt + p^*(x,y)$$

$$u_x^{n+1}(x,y) = \int_0^y \theta(x) h(x,t, u^n(x,t), \dots, u_y^n(g_3(x),t))\,dt + p_x^*(x,y)$$

$$u_y^{n+1}(x,y) = \int_0^x \theta(x) h(s,y, u^n(s,y), \dots, u_y^n(g_3(s),y))\,ds + p_y^*(x,y).$$

Then, the sequence $\{u^n(x,y)\}$ converges to the maximal solution of (24), (25) in S_ρ .

PROOF. Clearly $u^0(x,y), u_x^0(x,y), u_y^0(x,y) \in S_\rho$.
Now using assumptions (i) - (iii) we find

$$\begin{aligned} u^1(x,y) &= \int_0^x \int_0^y \theta(x) h(s,t, u^0(s,t), \dots, u_y^0(g_3(s),t))\,ds\,dt + p^*(x,y) \\ &\le \int_0^x \int_0^y \theta(x) h(s,t,\rho,\dots,\rho)\,ds\,dt + p^*(x,y) \\ &= u^0(x,y) \end{aligned}$$

where we used $u^0(g_1(x),y), u_x^0(g_2(x),y), u_y^0(g_3(x),y) \le \rho$ and which follows from the assumption (iii). Similarly we get

$$u_x^1(x,y) \le u_x^0(x,y), \quad u_y^1(x,y) \le u_y^0(x,y).$$

Using an inductive argument, it is easily seen that

$$\rho \geq u^0(x,y) \geq u^1(x,y) \geq \dots \geq u^n(x,y) \geq 0$$

$$\rho \geq u_x^0(x,y) \geq u_x^1(x,y) \geq \dots \geq u_x^n(x,y) \geq 0$$

$$\rho \geq u_y^0(x,y) \geq u_y^1(x,y) \geq \dots \geq u_y^n(x,y) \geq 0$$

for all $(x,y) \in R_1 \cup R_0$. The sequence $\{u^n(x,y)\}$, $\{u^n_x(x,y)\}$ and $\{u^n_y(x,y)\}$,are uniformly bounded and monotonically decreasing and hence converges to some $u(x,y)$ $u_x(x,y)$, $u_y(x,y)$ respectively. From the continuity of h, $u(x,y)$ is a solution of (24), (25) in S_ρ . Then

$$u^0(x,y) - v(x,y) = \int_0^x\int_0^y \theta(x)[h(s,t,\rho,\cdots,\rho) - h(s,t,v(s,t),\ldots,v_y(g_3(s),t))]\,ds\,dt$$

$$\geq 0$$

and similarly $u^0_x(x,y) - v_x(x,y) \geq 0$, $u^0_y(x,y) - v_y(x,y) \geq 0$.

In fact induction shows that for each n

$$u^n(x,y) - v(x,y) \geq 0$$
$$u^n_x(x,y) - v_x(x,y) \geq 0$$
$$u^n_y(x,y) - v_y(x,y) \geq 0$$

and therefore in the limit $u(x,y) - V(x,y) \geq 0$, $u_x(x,y) - V_x(x,y) \geq 0$, $u_y(x,y) - V_y(x,y) \geq 0$. This completes the proof.

Next we shall prove the following main comparison theorem.

THEOREM 13. With respect to problem (1), (2) and (24), (25) we assume

(i) $|f(x,y,u_1,\cdots,u_6)| \leq h(x,y,|u_1|,\cdots,|u_6|)$ for all $(x,y,u_1,\cdots,u_6) \in R_0 \times R^6$

(ii) $|p(x,y)| \le p^*(x,y)$, $|p_x(x,y)| \le p_x^*(x,y)$, $|p_y(x,y)| \le p_y^*(x,y)$

for all $(x,y) \in R_1 \cup R_0$

(iii) assumptions of Theorem 12.

Then, if $Z(x,y)$ is any solution of (1), (2) in S_ρ we have for $(x,y) \in R_1 \cup R_0$

$$|Z(x,y)| \le u(x,y),\ |Z_x(x,y)| \le u_x(x,y),\ |Z_y(x,y)| \le u_y(x,y).$$

PROOF. For n = 1,2,3,..., set

$$v^{n+1}(x,y) = \int_0^x\int_0^y \theta(x) h(s,t,v^n(s,t),\dots,v_y^n(g_3(s),t))\,ds\,dt + p^*(x,y)$$

$$v_x^{n+1}(x,y) = \int_0^y \theta(x) h(x,t,v^n(x,t),\dots,v_y^n(g_3(x),t))\,dt + p_x^*(x,y)$$

$$v_y^{n+1}(x,y) = \int_0^x \theta(x) h(s,y,v^n(s,y,\dots,v_y^n(g_3(s),y))\,ds + p_y^*(x,y)$$

with

$$v^0(x,y) = |Z(x,y)|,\ v_x^0(x,y) = |Z_x(x,y)|,\ v_y^0(x,y) = |Z_y(x,y)|$$

for all $(x,y) \in R_1 \cup R_0$.

Then, using (i) and (ii) we find

$$\begin{aligned} v^1(x,y) &= \int_0^x\int_0^y \theta(x) h(s,t,|Z(s,t)|,\dots,|Z_y(g_3(s),t)|)\,ds\,dt + p^*(x,y) \\ &\geq \int_0^x\int_0^y \theta(x)|f(s,t,Z(s,t),\dots,Z_y(g_3(s),t))|\,ds\,dt + |p^*(x,y)| \\ &\geq \left|\int_0^x\int_0^y \theta(x) f(s,t,Z(s,t),\dots,Z_y(g_3(s),t))\,ds\,dt + p^*(x,y)\right| \\ &\geq |Z(x,y)| \end{aligned}$$

and similarly

$$v_x^1(x,y) \geq |Z_x(x,y)|,\ v_y^1(x,y) \geq |Z_y(x,y)|$$

for all $(x,y) \in R_1 \cup R_0$. Inductively, we have

$$|z(x,y)| \le v^1(x,y) \le v^2(x,y) \le \cdots \le v^n(x,y) \le \rho$$

$$|z_x(x,y)| \le v_x^1(x,y) \le v_x^2(x,y) \le \cdots \le v_x^n(x,y) \le \rho$$

$$|z_y(x,y)| \le v_y^1(x,y) \le v_y^2(x,y) \le \cdots \le v_y^n(x,y) \le \rho$$

for all $(x,y) \in R_1 \cup R_0$. Since the sequence $\{v^n(x,y)\}$, $\{v_x^n(x,y)\}$ and $\{v_y^n(x,y)\}$ are monotonically increasing and uniformly bounded. We get in limit these sequences converge to $V(x,y)$, $V_x(x,y)$, $V_y(x,y)$ in S_ρ respectively. From the continuity of h, $V(x,y)$ is a solution of (24), (25). Thus we have

$$|z(x,y)| \le V(x,y),\ |z_x(x,y)| \le V_x(x,y),\ |z_y(x,y)| \le V_y(x,y).$$

But the maximility of $u(x,y)$ completes the proof of the theorem.

In concluding this Chapter, we mention that the results of this Chapter can be extended for the systems without any difficulty.

APPENDIX

PAPERS COVERING THE CONTENTS OF THE THESIS

1. R.P.Agarwal and E.Thandapani, Remarks on generalizations of Gronwall's inequality (sent for publication).
2. R.P.Agarwal and E.Thandapani, Some inequalities of Gronwall's type (sent for publication).
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