Λ CARDINAL SPLINES

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CHAPTER I

The problem of cardinal spline interpolation for polynomial splines was thoroughly investigated by Schoenberg in a series of papers and a beautiful account is given in his monograph [9] .

Let

(1·1) (y_ν) $\quad \nu = 0, \pm 1, \pm 2, \ldots$

be a prescribed doubly infinite sequence of real or complex numbers. A problem of cardinal interpolation is to find a function f(x) in a given function space such that

(1·2) $f(\nu) = y_\nu$ for all integers ν .

The analogue of the Lagrange interpolation formula for this problem is the cardinal series.

(1·3) $$f(x) = \sum_{\nu=-\infty}^{\infty} y_\nu \frac{\sin \pi (x-\nu)}{\pi (x-\nu)}$$

The piecewise linear analogue of (1·3) is well known. If M(x) is the roof-function such that $M(x) = 1 + x$ in $[-1, 0]$, $M(x) = 1-x$ in $[0, 1]$ and $M(x) = 0$ elsewhere, then

(1·4) $$S(x) = \sum_{\nu=-\infty}^{\infty} y_\nu M(x-\nu)$$

is clearly the piecewise linear interpolant for the sequence (1·1). The purpose of the cardinal spline

<u>interpolation</u>, as pointed out by Schoenberg, is to bridge the gap between the linear spline (1.4) and the cardinal series (1.3).

Let n be a natural number and let

$$S_n = \{ S(x) \}$$

denote the class of functions satisfying the two conditions

$$S(x) \in C^{n-1}(R)$$

$$S(x) \in \Pi_n \text{ in each interval } (\nu, \nu+1)$$

for all integers ν, where Π_n stands for the class of polynomials of degree not exceeding n and $C^{n-1}(R)$ is the class of functions on the real line having the $(n-1)^{st}$ derivative continuous. The elements of S_n are called <u>cardinal splines of degree</u> n with simple knots at the integers. The cardinal spline interpolation problem is then : given the sequence (1.1) to find $S(x) \in S_n$ such that

$$S(\nu) = y_\nu \qquad \text{for all } \nu .$$

Clearly cardinal splines of degree n are locally annihilated (between integers) by the operator D^{n+1}.

Various considerations led Micchelli [6] to study cardinal splines for the differential equations $Ly = 0$ where

$$(1.5) \qquad L = D \prod_{\nu=1}^{n} (D - \gamma_\nu)$$

where γ_ν's are real constants. The simplest case when all $\gamma_\nu = 0$ leads to the cardinal polynomial splines studied by Schoenberg. Thus given the differential operator (1.5). We define the class

$$S(L,n) = \left\{ S(x) \right\}$$

of functions satisfying the two conditions

$$S(x) \in C^{n-1}(R)$$

$$LS(x) = 0 \quad \text{for } \nu < x < \nu + 1$$

for all integers ν and the elements of $S(L,n)$ are called <u>cardinal L-splines</u> with simple knots at the integers.

Our motivation for the present work lies in the open question raised by Schoenberg on page 7 in his monograph :- 'Which of the properties of polynomial B-splines will carry over to more general B-splines ?' These are splines having minimal support, that is the support having the smallest number of consecutive unit intervals.

We consider the linear differential operator Λ of order k given by

$$(1.6) \quad \Lambda = D^k + \sum_{j=0}^{k-1} a_j D^j$$

where $a_j \in C^j(R)$. A function $S(x)$ is called a <u>Λ-cardinal spline</u> with simple knots at the integers if it satisfies

(i) $S(x) \in C^{k-2}(R)$

(ii) $S(x) \in C^k\left(\bigcup_\nu (\nu, \nu+1)\right)$

(iii) $\Lambda S(x) = 0$ in $(\nu, \nu+1)$

for all integers ν. The set of all such <u>Λ-cardinal splines</u> is denoted by $\mathcal{S}_{\Lambda,1}$.

In this thesis we study the various properties of Λ-cardinal splines and obtain a solution for the Λ- cardinal spline interpolation problem which may be stated as follows : Given the sequence (1.1) it is required to find an $S \in \mathcal{S}_{\Lambda,1}$ satisfying

$$S(\nu) = y_\nu \quad \text{for all integers } \nu .$$

Our approach here closely follows the work of Schoenberg [9] and [10] and also uses the results of Karlin [4] on total positivity. We also study Λ-cardinal t-perfect splines generalizing the work of Sharma and Tzimbalario [11] .

There are seven chapters in this thesis, the first chapter being the introduction. In chapter II, we introduce the notion of a Λ-cardinal spline where Λ is a linear differential operator of order k whose coefficients are continuous functions on the real line R. We also consider the generalized B-splines M(x) which are Λ-splines with minimal support on R and obtain the representation of a Λ-cardinal spline in terms of the generalised B-splines.

In chapter III, various properties of the basic spline M(x) are investigated. After introducing the notion of extended Chebyshev system (ECT system), we obtain the decomposition of the operator Λ. Then the exponential Λ-splines are introduced and their properties are investigated. Finally under the additional assumption that the basis $u_1 \ldots . u_k$ of the nullspace $k(\Lambda)$ of the operator Λ form an ECT system on R, it is proved that the basis spline $M(x)$ is positive for $0 < x < k$.

Chapter IV is concerned with the problem of -Cardinal spline interpolation. After considering a preliminary answer, it is proved that the set of solutions of our interpolation problem form a linear manifold of dimension k-2 in $\mathcal{S}_{\Lambda,1}$. After obtaining a basis for the space of the so called nullsplines, we study the conditions for uniqueness of the solution. We prove the existence and uniqueness of the solution to the interpolation problem and also obtain it in an explicit form when the given data is of power growth.

In chapter V, we study what are termed Λ-cardinal t-perfect splines. These will generalize the concept of perfect splines of Glaeser [2] and the t-perfect splines of Sharma and Tzimbalario [11]. We also consider an extremal problem of determining the element having the least t-norm.

In the last two chapters we apply the method of functionals in the study of extremal problems. In their study of t-perfect splines, Sharma and Tzimbalario [11] obtain a solution to the problem of finding the polynomial of least deviation on [0,1] which satisfies certain boundary condition. We apply the method of functionals to give an independent and alternate proof. This is done in chapter VI. In the last chapter, the method of functionals is applied to an extremal problem in trigonometric polynomials which was earlier proved by Boas [1] using variational methods.

CHAPTER II

Λ - CARDINAL SPLINES

2.0 In this chapter we shall introduce the notion of a Λ - cardinal spline where Λ is a linear differential operator of order k whose coefficients are continuous functions on the real line R. We also consider the generalized B-splines and obtain the representation of a Λ - cardinal spline in terms of the generalized B-splines.

2.1 Cardinal splines

Let n be a natural number and let

$$S_n = \{ S(x) \}$$

denote the class of functions satisfying the two conditions

$$S(x) \in C^{n-1}(R)$$

$$S(x) \in \pi_n \text{ in each of the intervals } (\nu, \nu+1)$$

for all integers ν where π_n stands for the class of polynomials of degree not exceeding n and $C^{n-1}(R)$ is the class of functions on the real line R having the $(n-1)^{st}$ derivative continuous. Schoenberg calls the elements of S_n to be cardinal splines of degree n with simple knots at the integers.

Different considerations led Micchelli [6] to study in depth cardinal splines for the differential equations $L y = 0$ where the differential operator L is given by

$$L = D \prod_{\nu=1}^{n} (D - \gamma_\nu) \quad (\gamma_\nu \text{ are real constants}) \qquad (2.1.1)$$

The simplest case where all $\gamma_\nu = 0$ leads to the cardinal polynomial splines studied by Schoenberg. Thus, given the differential operator (2.1.1) we define the class

$$S(L, n) = \{ S(x) \}$$

cntd.

of functions satisfying the two conditions

$$S(x) \in C^{n-1}(R)$$

$$L\,S(x) = 0 \text{ for } \nu < x < \nu+1 \text{ for all integers } \nu$$

and the elements of $S(L, n)$ are called cardinal L - splines with simple knots at the integers.

2.2. Λ - Cardinal splines

We shall now introduce the notion of cardinal Λ -splines where Λ is the most general linear differential operator of order k.

Let $C^j[a, b]$ denote the class of continuous real valued functions on $[a, b]$ having the j th derivative continuous. Let Λ be a linear differential operator on $[a, b]$ with real coefficients given by

$$\Lambda = D^k + \sum_{j=0}^{k-1} a_j D^j \qquad k \geq 1$$

where $a_j \in C^j[a, b]$ for $0 \leq j < k$. If $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a, b]$, the function $s \in C^{k-2}[a, b]$ is called a Λ-spline with simple knots at the points $\{x_i\}$ if $s \in C^k\left(\bigcup_i (x_i, x_{i+1})\right)$ and $\Lambda s(x) = 0$ if $x \neq x_i \cdot 0 \leq i \leq n$

Let us now consider the linear differential operator Λ of order k defined by

$$\Lambda = D^k + \sum_{j=0}^{k-1} a_j D^j \tag{2.2.1}$$

where $a_j \in C^j(R)$, the class of real valued functions on the real line R having the j th derivative continuous. The null space $K(\Lambda)$ of Λ is a linear space of dimension k. For each $\xi_1 \in R$, if we define $\theta(x, \xi_1) \in K(\Lambda)$ by

$$\left[\frac{d^j}{dx^j}\theta(x, \xi_1)\right]_{x=\xi_1} = \delta_{j,k-1}, \quad j = 0, 1, \ldots, k-1 \tag{2.2.2}$$

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it is known [3; p.145] that $\Theta(x,\xi_1)$ has a unique representation

$$\Theta(x,\xi_1) = \sum_{i=1}^{k} u_i(x)\,u_i^*(\xi_1) \qquad (2.2.3)$$

where $u_1\ u_2 \cdots u_k$ form the basis of $K(\Lambda)$ and $(u_i^*)_1^k$ are the elements in the last column of $W[u_1, \cdots u_k]^{-1}$, where the Wronskian $W[u_1, \cdots u_k]$ is given by

$$(W_{ij}) = (u_j^{(i-1)}) \quad 1 \le i \le k,\ 1 \le j \le k$$

Since $u_1\ u_2, \cdots, u_k$ are linearly independent, the Wronskian is not zero. Moreover $u_1^*, \cdots, u_k^*$ form the basis of $K(\Lambda^*)$ where Λ^* is the adjoint of Λ and is given by

$$\Lambda^* f = (-1)^k D^k f + \sum_{j=0}^{k-1} (-1)^j D^j (a_j f) \qquad (2.2.4)$$

<u>Definition 2.2.1</u> A function $S(x)$ is called a <u>Λ cardinal spline</u> with simple knots at the integers if it satisfies the conditions

(i) $S(x) \in C^{k-2}(\mathbb{R})$

(ii) $S(x) \in C^k\left(\bigcup_{\nu} (\nu, \nu+1)\right)$

(iii) $\Lambda S(x) = 0 \quad \text{in } (\nu, \nu+1)$

for all integers ν. The set of all such Λ cardinal splines is denoted by $\mathcal{S}_{\Lambda,1}$.

If we define $\hat{\Theta}(x,\xi_1)$ by

$$\hat{\Theta}(x,\xi_1) = \begin{cases} \Theta(x,\xi_1), & x \geq \xi_1 \\ 0 & x < \xi_1 \end{cases}$$

then $\hat{\Theta}(x,\xi_1)$ is a Λ-spline with a single knot at ξ_1.

2.3. <u>The generalized B - splines</u>

For the study of Λ cardinal splines, we need a convenient representation for the elements of the linear space $\mathcal{S}_{\Lambda,1}$. The most desirable basis would consist of splines having finite support, that is, the support consisting of the smallest possible number of intervals between

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the knots. We shall therefore recall the notion of generalized B-splines.

Since K (Λ) is a linear space of dimension k, we can, without loss of generality, determine (k+1) numbers $\beta_0, \beta_1, \ldots, \beta_k$ with $\beta_0 > 0$ so that the relation

$$\beta_0 \,\theta(x,0) + \beta_1 \,\theta(x,1) + \cdots + \beta_k \,\theta(x,k) = 0 \tag{2.3.1}$$

holds. If we set

$$M(x) = \sum_{j=0}^{k} \beta_j \hat{\theta}(x,j) \tag{2.3.2}$$

then $M(x;0,1,\ldots k) = M(x)$ is a Λ spline with knots at 0, 1, 2, .. k. Moreover the support of M (x) is the interval (o, k). In fact M (x) = 0 for $x \leq 0$ by the definition of $\hat{\theta}(x,\xi_1)$ and M (x) = 0 for $x \geq k$ by virtue of (2.3.1). M (x) is the basic spline required for the purpose of representation of elements of $\mathcal{S}_{\Lambda,1}$.

4. Representation of Λ-cardinal splines

We shall now show that if $S \in \mathcal{S}_{\Lambda,1}$ then S has the unique representation

$$S(x) = \sum_{j=-\infty}^{\infty} c_j M(x-j),$$

where the C_j 's are constants and this justifies the name basis spline for M (x). This is achieved by a series of results.

<u>Theorem 2.4.1</u> <u>Let</u> S (x) <u>be a</u> Λ-<u>spline with simple knots at</u> $\xi_1 < \xi_2 < \cdots \xi_n$. <u>Then</u> S <u>can be represented in the form</u>

$$S(x) = P(x) + \sum_{j=1}^{n} b_j \hat{\theta}(x,\xi_j) \tag{2.4.1}$$

<u>where</u> $P(x) \in K(\Lambda)$ <u>and</u> b_j's <u>are constants</u>

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Proof: Consider the expression

$$P(x) = S(x) - \sum_{j=1}^{n} b_j \hat{\theta}(x, \xi_{\ell j})$$

where b_j s are constants. Since $\hat{\theta}(x, \xi_\ell)$ is a $\wedge$ spline with a single knot at $x = \xi_\ell$, we see that

(i) $P(x) \in C^k (R \setminus \{\xi_{\ell 1}, \xi_{\ell 2}, \ldots \xi_{\ell n}\})$

(ii) $P(x) \in C^{k-2}(R)$

and

(iii) $\wedge P(x) = 0$ for $x \neq \xi_{\ell i}$, $i = 1, \ldots n$.

To establish our theorem, it is enough to show that we can choose b_j 's so that $P(x) \in C^k(R)$. $P(x)$ will then belong to $K(\wedge)$. When $x < \xi_{\ell 1}$, we have $P(x) = S(x)$. If $\xi_{\ell 1} < x < \xi_{\ell 2}$, then

$$P(x) = S(x) - b_1 \theta(x, \xi_{\ell 1})$$

If we set

$$b_1 = S^{(k-1)}(\xi_{\ell 1}+0) - S^{(k-1)}(\xi_{\ell 1}-0) = \text{jump of } S^{(k-1)} \text{ at } x = \xi_{\ell 1},$$

it easily follows that $P^{(k-2)}(x)$ is continuous at $x = \xi_{\ell 1}$. We assert that $P^{(k)}(x)$ is also continuous at $x = \xi_{\ell 1}$. Using the relation $\wedge P(x) = 0$ for $x \neq \xi_{\ell i}$, we have

$$P^{(k)}(x) = -\sum_{j=0}^{k-1} a_j P^{(j)}(x), \quad x \neq \xi_{\ell i}.$$

using the continuity of $P^{(j)}(x)$ at $x = \xi_{\ell 1}$ for $j = 0, 1, \ldots, k-1$ it follows that

$$P^{(k)}(\xi_{\ell 1}+0) = P^{(k)}(\xi_{\ell 1}-0)$$

and the continuity of $P^{(k)}(x)$ at $x = \xi_{\ell 1}$ is obvious.

Now setting

$$S_{j-1}(x) = S_{j-2}(x) - b_{j-1} \hat{\theta}(x, \xi_{\ell j-1}),$$

where

$$S_0(x) = S(x)$$

contd..

We see that

$$P(x) = S_{j-1}(x) - b_j\,\hat{\theta}(x, \xi_{ij}) \text{ for } x < \xi_{ij+1}$$

Supposing that b_1, b_2, b_{j-1} have already been chosen so that $P^{(k)}(x)$ is continuous at $\xi_{i1}, \xi_{i2}, \ldots, \xi_{ij-1}$ we choose b_j such that

$$b_j = \text{jump of } S_{j-1}^{(k-1)}(x) \text{ at } x = \xi_{ij}$$

Then $P^{(k)}(x)$ will be continuous at $x = \xi_{ij}$. Thus choosing b_j for $j = 1, 2, \ldots, n$ inductively, we have

$$P(x) = S(x) - \sum_{j=1}^{n} b_j\,\hat{\theta}(x, \xi_{ij}) \in K(\Lambda).$$

The uniqueness of the coefficients b_j follows from the very construction. This completes the proof of the theorem.

__Corollary 2.4.2__ __If__ $S(x) = 0$ __for__ $x < \xi_{i1}$ __and__ $S(x)$ __is__ __a__ Λ __spline with simple knots at__ $\xi_{i1} < \xi_{i2} < \cdots < \xi_{in}$ __then__ $S(x)$ __can be represented in the form__

$$S(x) = \sum_{j=1}^{n} b_j\,\hat{\theta}(x, \xi_{ij}) \qquad (2.4.2)$$

__where__ b_j__'s are constants__.

__Proof:__ Since $S(x)$ is a Λ spline with simple knots at $\xi_{i1} < \xi_{i2} < \cdots < \xi_{in}$, by Theorem 2.3.1., we have the representation

$$S(x) = P(x) + \sum_{j=1}^{n} b_j\,\hat{\theta}(x, \xi_{ij})$$

where $P(x) \in K(\Lambda)$. The hypothesis $S(x) = 0$ for implies $P(x) = 0$ for all $x < \xi_{i1}$. Since $u_1, \ldots, u_k$ form a basis of the null space $K(\Lambda)$, there exist constants $a_1, a_2, \ldots, a_k$ such that

$$P(x) = \sum_{i=1}^{k} a_i u_i(x) = 0 \text{ for } x < \xi_{i1} \qquad (2.4.3)$$

contd..

Now the Wronskian of k solutions of a differential operator of order k is either identically zero or is never vanishing. But $u_1, u_2, \ldots, u_k$ are linearly independent and hence its Wronskian is not zero. Thus we conclude from (2.4.3) that $a_i \equiv 0$ for $i = 1, \ldots k$ and the desired representation follows.

<u>Theorem</u> 2.4.3 <u>If</u> $S \in \mathcal{S}_{\Lambda,1}$ <u>then the support of</u> S <u>cannot be shorter than</u> k <u>consecutive unit intervals unless</u> $S(x) \equiv 0$.

<u>Proof</u>: Let us assume that the support of S has atmost (k-1) consecutive unit intervals, say (r, r + k-1) where r is an integer. Then $S(x) = 0$ for $x \leq r$ and for $x \geq r + k-1$. If we set $S_1(x) = S(x + r-1)$, we see that $S_1(x) = 0$ for $x \leq 1$ and for $x \geq k$ so that it has the representation.

$$S_1(x) = \sum_{j=1}^{k} b_j \hat{\theta}(x, j)$$

which gives for $x \geq k$,

$$S_1(x) = \sum_{j=1}^{k} b_j \sum_{i=1}^{k} u_i(x) u_i^*(j) = \sum_{i=1}^{k} u_i(x) \sum_{j=1}^{k} b_j u_i^*(j)$$

Because $S_1(x) = 0$ for $x \geq k$ and $u_1, \ldots u_k$ form a basis of $K(\Lambda)$, we must have

$$\sum_{j=1}^{k} b_j u_i^*(j) = 0, \quad i = 1, 2, \ldots, k \qquad (2.4.4)$$

This gives a system of k homogeneous equations in k unknowns $b_1, b_2, \ldots, b_k$. Since each β_i in (2.3.2) is obtained in terms of β_0 and β_0 is chosen to be positive, it follows that the determinant

contd.-

$$\begin{vmatrix} u_1^*(1) & u_1^*(2) & --- & u_1^*(k) \\ u_2^*(1) & u_2^*(2) & -- & u_2^*(k) \\ - & - & - & - \\ u_k^*(1) & u_k^*(2) & -- & u_k^*(k) \end{vmatrix} \neq 0$$

This implies that the system of equations (2.4.4) has only the trivial solution viz $b_1 = b_2 = --- = b_k = 0$. Thus, $S_1(x) = 0$ for all $x \in R$ and hence $S(x) = 0$ for $x \in R$.

Theorem 2.4.4 Suppose $S \in \mathcal{S}_{\Lambda,1}$. If $S(x) = 0$ in $(p-1, p)$ and $(p-1+k, p+k)$ for any integer p, then $S(x) = 0$ in $(p-1, p+k)$.

Proof: Define $S_1(x) = S(x)$ if $p \leq x \leq p+k-1$ and $S_1(x) = 0$ otherwise. Then $S_1(x)$ is a Λ spline and its support is atmost $(p, p+k-1)$ which contains only $(k-1)$ consecutive unit intervals. By Theorem 2.4.3 we see that $S_1(x) \equiv 0$ which implies that $S(x) = 0$ in $[p, p+k-1]$. Hence $S(x) = 0$ in $(p-1, p+k)$.

Theorem 2.4.5 The k functions

$M(x), M(x-1), - - - - - , M(x-k+1)$

are linearly independent in $(k-1, k)$

Proof: Suppose

$$\sum_{j=0}^{k-1} a_j M(x-j) = 0 \quad \text{for } k-1 < x < k$$

where a_j's are constants. We need to show that $a_j = 0$ for $j = 0, 1, 2 --- k-1$. Set

$$S(x) = \sum_{j=0}^{k-1} a_j M(x-j)$$

contd..

Then $S(x) = 0$ in $(k-1, k)$. Moreover by the definition of the functions $M(x-j)$ it follows that $S(x) = 0$ for $x<0$. In particular we have $S(x) = 0$ in $(-1, 0)$. Because $S(x) = 0$ in $(-1, 0)$ and in $(k-1, k)$, theorem 2.4.4 asserts that $S(x) = 0$ in $(-1, k)$ so that $S(x) = 0$ for $j<x<j+1$ for each $j = 0, 1, \cdots k-1$.

Thus, when $0<x<1$, we have

$$0 = S(x) = a_0 M(x)$$

which implies that $a_0 = 0$

when $1<x<2$, we have

$$0 = S(x) = a_0 M(x) + a_1 M(x-1) = a_1 M(x-1)$$

which shows that $a_1 = 0$ also. The proof that $a_j = 0$ for all j is completed by induction.

__Theorem 2.4.6__ __If__ $S \in \mathcal{S}_{\Lambda,1}$, __and__ $S(x) = 0$ __for__ $x<0$, __then__ $S(x)$ __has the unique representation__

$$S(x) = \sum_{j=0}^{\infty} c_j M(x-j) \text{ for all } x \in R$$

__where__ c_j__'s are constants__

__Proof:__ Since $S(x) = 0$ for $x<0$, it will have only one knot viz $x = 0$ in $(-\infty, 1)$ so that it has the representation

$$S(x) = a_0 \hat{\theta}(x, 0) \text{ in } (0, 1)$$

for some constant a_0. We can then choose c_0 so that

$$S(x) - c_0 M(x) = 0 \text{ for } x<1.$$

In fact,

$$S(x) - c_0 M(x) = a_0 \hat{\theta}(x,0) - c_0 \sum_{j=0}^{k} \beta_j \hat{\theta}(x,j)$$
$$= a_0 \hat{\theta}(x,0) - c_0 \beta_0 \hat{\theta}(x,0) \text{ if } x<1.$$

Since β_0 is not zero, $c_0 = \frac{a_0}{\beta_0}$ is the required constant. We can now define c_1 uniquely so that

$$S(x) - c_0 M(x) - c_1 M(x-1) = 0 \text{ for } x<2$$

This can be seen as follows. If $x<2$, then for any constant C_1 we have

$$S(x) - c_0 M(x) - c_1 M(x-1)$$
$$= a_0 \hat{\theta}(x,0) + a_1 \hat{\theta}(x,1) - c_0 \sum_{j=0}^{k} \beta_j \hat{\theta}(x,j) - c_1 \sum_{j=0} \beta_j \hat{\theta}(x-1,0)$$
$$= (a_1 - c_0 \beta_1) \hat{\theta}(x,1) - c_1 \beta_0 \hat{\theta}(x-1,0).$$

Now $\hat{\theta}(x, 1)$ and $\hat{\theta}(x-1, 0)$ are two $\wedge$ splines having the same single knot at $x = 1$ and vanishing for $x<1$. Then by virtue of corollary 2.4.2 there exists a constant λ_0 such that

$$\hat{\theta}(x, 1) = \lambda_0 \hat{\theta}(x-1, 0)$$

Then we have

$$S(x) - c_0 M(x) - c_1 M(x-1) = \left[(a_1 - c_0\beta_1)\lambda_0 - c_1 \beta_0\right] \hat{\theta}(x-1, 0)$$

If we choose

$$C_1 = \frac{(a_1 - c_0 \beta_1)}{\beta_0}$$

our assertion follows.

Now suppose that $c_0, c_1, \ldots, c_{n-1}$ have already been chosen so that

$$S(x) - c_0 M(x) - \cdots - c_{n-1} M(x-n+1) = 0 \text{ for } x<n.$$

If $x<n+1$, we will have for any constant C_n

contd..

$$S(x) - C_0 M(x) - C_1 M(x-1) - \cdots - C_{n-1} M(x-n+1) - C_n M(x-n)$$
$$= \begin{cases} a_n \hat{\theta}(x,n) - c_{n-k}\beta_k \hat{\theta}(x-\overline{n-k},k) - \cdots - c_n \beta_0 \hat{\theta}(x-n,0) & \text{if } n > k \\ a_n \hat{\theta}(x,n) - c_0 \beta_n \hat{\theta}(x,n) - \cdots - c_n \beta_0 \hat{\theta}(x-n,0) & \text{if } n \leq k \end{cases} \quad (2.4.5)$$

Since $\hat{\theta}(x, n)$, $\hat{\theta}(x-1, n-1), \ldots, \hat{\theta}(x-n, 0)$ are all Λ splines having the same single knot at $x = n$, by corollary 4.2 we can find constants $\mu_1, \mu_2, \ldots \mu_n$ so that

$$\hat{\theta}(x-j, n-j) = \mu_j \hat{\theta}(x, n), \quad j = 1, 2, \ldots n.$$

C_n is then chosen by equating the coefficient of $\hat{\theta}(x,n)$ in (2.4.5) to zero. All the coefficients C_j in the expansion are now determined by induction. The uniqueness follows from the very construction of C_j.

<u>Remark</u> 2.4.7 For any $x \in R$, the sum

$$\sum_{j=0}^{\infty} c_j M(x-j)$$

contains almost (k+1) terms and hence is convergent.

<u>Theorem</u> 2.4.8 <u>If</u> $S \in \mathcal{G}_{\Lambda,1}$ <u>and</u> $S(x) = 0$ <u>for</u> $x > k-1$, <u>then</u> $S(x)$ <u>has the unique representation</u>.

$$S(x) = \sum_{j=-\infty}^{-1} c_j M(x-j)$$

<u>where</u> C_j's <u>are constants</u>

<u>Proof</u>: Following the arguments as in Theorem 2.4.6, we see that if $S(x) = 0$ for $x > k-1$, then $S(x)$ has the unique representation

$$S(x) = \sum_{j=-\infty}^{k-1} c_j M(x-j) \qquad (2.4.6)$$

contd..

Let

$$I_1 = \sum_{j=-\infty}^{-1} c_j M(x-j) \; ; \; I_2 = \sum_{j=0}^{k-1} c_j M(x-j)$$

Now the support of $M(x-j)$ is $(j, j+k)$. Thus $I_1 = 0$ for $x > k-1$. Since, by hypothesis, $S(x) = 0$ for $x > k-1$, it follows that $I_2 = 0$ for $x > k-1$. We shall now show that $c_0 = c_1 = \cdots = c_{k-1} = 0$ which will complete the proof. To this end, let us consider

$$I_2 = c_0 M(x) + c_1 M(x-1) + \cdots + c_{k-1} M(x-k+1) \quad (2.4.7)$$

If $2k-2 < x < 2k-1$, then $M(x-j) = 0$ for $j = 0, 1, 2, \ldots, k-2$ and $M(x-k+1) \neq 0$. But $I_2 = 0$ for these values of x and (2.4.7) then gives $c_{k-1} = 0$. Thus

$$I_2 = c_0 M(x) + c_1 M(x-1) + \cdots + c_{k-2} M(x-k+2)$$

Repeating the process we see that $c_0 = c_1 = \cdots = c_{k-2} = 0$. The required representation then is obvious.

Our representation theorem for Λ-cardinal splines can be stated as follows.

__Theorem__ 2.4.9 __If__ $S \in \mathcal{S}_{\Lambda,1}$, __then__ S __has the unique representation__

$$S(x) = \sum_{j=-\infty}^{\infty} c_j M(x-j)$$

__where__ c_j's __are constants__

__Proof:__ The k functions $M(x), M(x-1), \ldots, M(x-k+1)$ are linearly independent in $(k-1, k)$ by Theorem 2.4.5 and hence we can find unique constants c_j such that

$$S(x) = \sum_{j=0}^{k-1} c_j M(x-j) \quad \text{in } (k-1, k)$$

contd..

Define

$$S_1(x) = S(x) - \sum_{j=0}^{k-1} C_j M(x-j)$$

$$S_2(x) = \begin{cases} S_1(x) & \text{for } x < k-1 \\ 0 & \text{for } x \geq k-1 \end{cases}$$

$$S_3(x) = \begin{cases} 0 & \text{for } x \leq k \\ S_1(x) & \text{for } x > k \end{cases}$$

Then both $S_2(x)$ and $S_3(x)$ vanish in $(k-1, k)$ and so is $S_1(x)$. It is easy to see that

$$S_1(x) = S_2(x) + S_3(x) \text{ for all } x \in R$$

Now $S_2(x) = 0$ for $x \geq k-1$ and so by Theorem 2.4.8 we have

$$S_2(x) = \sum_{j=-\infty}^{-1} C_j M(x-j)$$

where C_j's are uniquely determined. Since $S_3(x) = 0$ for $x \leq k$, Theorem 2.4.6 gives

$$S_3(x) = \sum_{j=k}^{\infty} C_j M(x-j)$$

Thus

$$S(x) - \sum_{j=0}^{k-1} c_j M(x-j) = \sum_{j=-\infty}^{-1} c_j M(x-j) + \sum_{j=k}^{\infty} c_j M(x-j)$$

from which follows the desired representation.

CHAPTER III

PROPERTIES OF THE BASIS SPLINE

3.0 In this chapter we study the properties of the basis spline $M(x)$ introduced in the last chapter. In particular, we prove that, under the additional assumption that the basis $u_1, u_2, \ldots, u_k$ of the null space $K(\Lambda)$ form an ECT system on R, $M(x) > 0$ for $0 < x < k$.

3.1 The ECT System

A real function $K(x, y)$ of two variables ranging over linearly ordered sets X and Y respectively is said to be totally positive of order r if for all

$$x_1 < x_2 < \cdots < x_m, \; y_1 < y_2 < \cdots < y_m, \; x_i \in X, \; y_i \in Y \qquad (3.1.1)$$

We have the inequalities

$$K\begin{pmatrix} x_1, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \cdots & K(x_1, y_m) \\ K(x_2, y_1) & K(x_2, y_2) & \cdots & K(x_2, y_m) \\ K(x_m, y_1) & K(x_m, y_2) & \cdots & K(x_m, y_m) \end{vmatrix} \geq 0$$

for m = 1, 2, .. r. If strict inequality holds then we say that K is strictly totally positive of order r. 3.1.2

If a totally positive function of order r is of the form $K(x, y) = f(x-y)$ where X and Y are each the real line, $f(u)$ is said to be a Polya frequency function of order r.

We recall that a sequence of continuous functions $\varphi_1(x), \varphi_2(x), \ldots \varphi_n(x)$ is said to constitute a Chebyshev

contd..

system (T - system) on $a < x < b$ if, for any set of real constants $\{c_k\}$ not all zero, the function $\sum_{k=1}^{n} c_k \varphi_k(x) = \varphi x$ does not vanish more than n-1 times on the interval (a, b). This implies that the determinant

$$\begin{vmatrix} \phi_1(x_1) & \varphi_1(x_2) & \cdots & \varphi_1(x_n) \\ \varphi_2(x_1) & \varphi_2(x_2) & \cdots & \varphi_2(x_n) \\ - & - & - & - \\ \varphi_n(x_1) & \varphi_n(x_2) & \cdots & \varphi_n(x_n) \end{vmatrix} \qquad (3.1.3)$$

for all $a < x_1 < x_2 < \cdots < x_n < b$ never vanishes and therefore maintains a fixed sign. By multiplying the final function by a factor +1 or -1, we may without loss of generality specify the sign (3.1.3) as positive.

In terms of positivity, we have the following definition of a Chebyshev system.

Definition (3.1.1) Let $u_0, u_1, \ldots, u_n$ denote continuous real valued functions on a closed finite interval $[a, b]$. These functions will be called a Chebyshev system over $[a, b]$ provided the $(n + 1)^{st}$ order determinants

$$U\begin{pmatrix} 0, 1, \ldots, n \\ t_0, t_1, \ldots, t_n \end{pmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_n) \\ - & - & - & - \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_n) \end{vmatrix}$$

are strictly positive whenever $a \leq t_0 < t_1 < \cdots < t_n \leq b$. The functions $u_0, u_1, \ldots, u_n$ will be referred to as a complete Chebyshev system (or CT-system) if $\{u_0, u_1, \ldots, u_r\}$ is a Chebyshev system for each $r = 0, 1, \ldots, n$.

cont..

We shall now explain how the definition in (3.1.2) can be extended to obtain the 'derived' determinant to allow for equalities occuring among the y_i values. Suppose $Y = [a, b]$ and for each $x \in X$, the function $K(x, \cdot) \in C^{p-1}[a, b]$, $p \geq 1$ that is $K(x, y)$ possesses $p-1$ continuous derivatives in Y. We now extend the definition in (3.1.2) to allow for equalities occurring among at most p of the y_i values as follows. For each set of equal y_i, we replace successive columns by their successive derivatives. More specifically, if $x_1 < x_2 < \cdots < x_k$, $a \leq y_1 \leq y_2 \leq \cdots \leq y_k \leq b$ and $y_{i-1} < y_i = y_{i+1} = \cdots = y_{i+q}$, $0 \leq q \leq p-1$ then

$$K^*\begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ y_1 & y_2 & \cdots & y_k \end{pmatrix}$$

is defined as the determinant in (3.1.2) where $(i + 1 + j)^{th}$ column, $0 \leq j \leq q$, is replaced by the column vector

$$\left(\frac{\partial^j}{\partial y_i^j} K(x_1, y_i), \frac{\partial^j}{\partial y_i^j} K(x_2, y_i), \cdots, \frac{\partial^j}{\partial y_i^j} K(x_k, y_i)\right)$$

For example, if $K(i, t) = u_i(t)$ and $a \leq t_0 = t_1 = \cdots = t_q < t_{q+1} \leq \cdots < t_{n-1} = t_n$ then

$$U^*\begin{pmatrix} 0 & 1 & n \\ t_0 & t_1 & t_n \end{pmatrix}$$

$$= \begin{vmatrix} u_0(t_0) & u_0'(t_0) \cdots & u_0^{(q)}(t_0) & u_0(t_{q+1}) \cdots & u_0(t_{n-1}) & u_0'(t_{n-1}) \\ u_1(t_0) & u_1'(t_0) \cdots & u_1^{(q)}(t_0) & u_1(t_{q+1}) \cdots & u_1(t_{n-1}) & u_1'(t_{n-1}) \\ - & - & - & - & - & - \\ u_n(t_0) & u_n'(t_0) \cdots & u_n^{(q)}(t_0) & u_n(t_{q+1}) \cdots & u_n(t_{n-1}) & u_n'(t_{n-1}) \end{vmatrix}$$

In the case $t_0 = t_1 = \cdots = t_n = t$ i.e. $q = n$ the above

contd..

determinant reduces to the Wronskian of the functions $u_0, u_1, \ldots, u_n$.

<u>Definition</u> (3.1.2) The functions will be called an <u>extended Chebyshev system</u> of order p, or an ET-system of order p, if a, b , i = 0, 1, 2, n and

$$U^*\begin{pmatrix} 0 & 1 & \cdots & n \\ t_0 & t_1 & \cdots & t_n \end{pmatrix} > 0$$

for all choices $t_0 \leq t_1 \leq \cdots \leq t_n \; (t_i \in [a,b])$ where equality occurs in groups of almost p consecutive t_i values. An extended Chebyshev system of order $(n + 1)$ will be simply referred to as an ET-system. If $u_0, u_1, \ldots, u_n$ is a system of $(n + 1)$ functions such that $u_0, u_1, \ldots, u_k$ is an extended Chebyshev system on $[a, b]$ for each $k = 0, 1, \ldots, n$ then it is called an <u>extended complete Chebyshev system</u> or an ECT - system.

These difinitions can be easily modified to include infinite intervals $[0, \infty)$ and $(-\infty, \infty)$. For example a system of (n+1) functions $u_0, u_1, \ldots, u_n$ is a Chebyshev system on $[0, \infty$ if $u_0, u_1, \ldots, u_n$ is a Chebyshev system for every finite interval $[0, A]$ where $A > 0$ See [5].

On the ECT systems we have the following useful proposition.

<u>Theorem</u> (3.1.3) (Theorem 1.1, p.376 [5]).
Let $u_0, u_1, \ldots, u_n$ <u>be of class</u> C^n $[a, b]$. Then

contd..

$u_0, u_1, \ldots, u_n$ is an ECT system on $[a, b]$ if and only if for $k = 0, 1, \cdots, n$ we have $W(u_0, \cdots u_k) > 0$ on $[a, b]$ where the Wronskian $W(u_0, u_1, \cdots, u_k)$ is given by

$$W(u_0, \ldots u_k)(t) = \begin{vmatrix} u_0(t) & u_0'(t_1) & \cdots & u_0^{k}(t) \\ u_1(t) & u_1'(t_1) & \cdots & u_1^{k}(t) \\ - & - & - & - \\ u_k(t) & \ldots u_k'(t) & \cdots & u_k^{(k)}(t) \end{vmatrix}$$

3.2 Decomposition of the operator Λ

In this and the remaining sections we assume that the basis $u_1, u_2, \ldots u_k$ of $K(\Lambda)$ form an ECT system on R. That is $W[u_1, u_2 \cdots u_j](x) > 0$ for all $x \in R$ for $j = 1, 2, \ldots, k$. It is known that $u_1, u_2, \ldots u_k$ also form a complete Chebyshev system on R. That is

$$\sum_{i=1}^{j} C_i u_i(x)$$

has almost (j-1) zeros on R for $j = 1, 2, \cdots, k$. where C_i's are arbitrary real numbers not all vanishing simultaneously.

Define

$$w_1(x) = u_1(x)$$
$$w_2(x) = \frac{W[u_1, u_2](x)}{[u_1(x)]^2}$$

and

$$w_j(x) = \frac{W[u_1, \cdots u_{j-2}](x)\, W[u_1, \cdots u_j](x)}{W^2[u_1, \ldots u_{j-1}](x)}$$

for $j = 3, 4, \cdots k$.

contd..

Since $u_1, u_2, \ldots, u_k$ form an ECT system, it follows that $W_j(x) > 0$ for each $x \in R$, $j = 1, 2, \ldots, k$. Also $W_j \in C^{k+1-j}(R)$, $j = 1, 2 \ldots k$. In particular $W_j(x) \in C[0,1]$ and hence bounded.

Defining the first order differential operators

$$D_j = \frac{d}{dx} \frac{1}{W_j(x)}$$

we see that Λ has the factorization

$$\Lambda = \prod_{i=1}^{k} w_i(x) D_k D_{k-1} \ldots D_1$$

and the equation

$$\Lambda f = 0$$

is equivalent to

$$D_k D_{k-1} \ldots D_1 f = 0$$

3.3 The exponential Λ- spline of base t

Let t be a constant such that $t \neq 0$, $t \neq 1$ and t is real. Consider the function f defined by

$$f(x) = t^x \quad \text{for all } x \in R$$

Then $f(x)$ satisfies the functional equation

$$f(x+1) = t f(x) \quad \text{for all } x \in R \tag{3.3.1}$$

The general element of $\mathcal{S}_{\Lambda,1}$ satisfying the functional equation (3.3.1) is given by the following.

contd..

Theorem 3.3.1 ***The most general element $S(x)$ of $\mathcal{S}_{\Lambda,1}$ satisfying the functional equation***

$$S(x+1) = t\,S(x) \text{ for } x \in R \tag{3.3.2}$$

is given by

$$S(x) = C_0 \sum_{j=-\infty}^{\infty} t^j M(x-j) \tag{3.3.3}$$

where C_0 is a constant.

Proof: If $S \in \mathcal{S}_{\Lambda,1}$, it has the representation

$$S(x) = \sum_{-\infty}^{\infty} c_j M(x-j) \tag{3.3.4}$$

by Theorem (2.4.9). If $S(x)$ satisfies the equation (3.3.2) then we must have

$$\sum_{j=\infty}^{\infty} c_j M(x+1-j) = t \sum_{j=-\infty}^{\infty} c_j M(x-j) \tag{3.3.5}$$

which gives by the uniqueness of the representation in Theorem (2.4.9)

$$c_{j+1} = t\,c_j \qquad \text{for all } j$$

Therefore

$$c_j = t^j c_0 \qquad \text{for all } j$$

and then 3.3.4 gives

$$S(x) = C_0 \sum_{j=-\infty}^{\infty} t^j M(x-j)$$

which is the most general form of $S \in \mathcal{S}_{\Lambda,1}$ satisfying (3.3.2).

contd..

Definition 3.3.2 We define the function

$$\Phi(x, t) = \sum_{j=-\infty}^{\infty} t^j M(x-j) \qquad (3.3.6)$$

and call it the exponential Λ-spline of base t.

Lemma 3.3.3 *If* $t < 0$, *then*

$$\Phi(x, t) \notin C^{k-1}(R)$$

Proof: If possible, assume that $\Phi(x, t) \in C^{k-1}(R)$. Since $\Phi(x, t) \in \mathcal{S}_{\Lambda,1}$, it has the representation

$$\Phi(x,t) = P(x) + \sum_{\nu=1}^{\infty} a_\nu \hat{O}(x,\nu) + \sum_{\nu=-\infty}^{0} a_\nu \hat{O}(-x,\nu)$$

where

$$P(x) = \sum_{i=1}^{k} c_i u_i(x) \in K(\Lambda)$$

If $\Phi(x, t) \in C^{k-1}(R)$, we easily see that $a_\nu = 0$ for all integers ν so that

$$\Phi(x, t) = P(x) = \sum_{i=1}^{k} c_i u_i(x), \quad x \in R$$

Since $u_1, u_2, \ldots, u_k$ form a complete chebyshev system on R, $\Phi(x, t)$ has almost (k-1) zeros on R. On the other hand

$$\Phi(x+1, t) = t\,\Phi(x, t)$$

and since $t<0$, $\Phi(x, t)$ must have infinitely many zeros on R. This gives a contradiction. Hence $\Phi(x, t) \notin C^{k-1}(R)$. This proves Lemma 3.3.3.

Let

$$\Phi(x) = \Phi(x, t) \qquad 0 \leq x \leq 1 \qquad (3.3.7)$$

contd..

be the restriction of $\Phi(x, t)$ for $0 \leq x \leq 1$. Define

$$\phi_{k-1}(x) = D_{k-1} D_{k-2} \cdots D_1 \Phi(x) \qquad 0 \leq x \leq 1 \qquad (3.3.8)$$

Then

$$D_k \phi_{k-1}(x) = D_k D_{k-1} \cdots D_1 \Phi(x) = 0 \text{ in } [0, 1]$$

That is

$$\frac{d}{dx}\left\{\frac{\phi_{k-1}(x)}{W_k(x)}\right\} = 0$$

so that

$$\phi_{k-1}(x) = C\, W_k(x) \qquad 0 \leq x \leq 1 \qquad (3.3.9)$$

where C is a constant and $W_k(x) > 0$ for all $x \in R$.

<u>Lemma</u> 3.3.4 <u>The constant</u> C <u>in</u> (3.3.9) <u>is</u> $\neq 0$.

<u>Proof</u>: Suppose the constant C in (3.3.9) is zero. Then we have

$$\phi_{k-1}(x) = D_{k-1} D_{k-2} \cdots D_1 \Phi(x) = 0 \text{ in } [0, 1]$$

That is, the restriction of $\Phi(x, t)$ to $[0, 1]$ satisfies the differential equation

$$D_{k-1} D_{k-2} \cdots D_1 Y = 0 \text{ in } [0, 1]$$

now using the relation

$$\Phi(x + 1, t) = t\, \Phi(x, t) \quad \text{for all } x \in R$$

it follows that

$$D_{k-1} D_{k-2} \cdots D_1 \Phi(x, t) = 0 \text{ for all } x \in R$$

contd..

which implies that $\Phi(x, t) \in C^{k-1}(R)$ contradicting Lemma 3.3.3.

<u>Lemma</u> 3.3.5 If $t<0$ <u>and if</u> $\Phi(x)$ <u>has atleast two zeros in</u> $[0, 1)$, <u>then</u> $\varphi_{k-1}(x)$ <u>defined by</u> (3.3.8) <u>has a zero in</u> $[0, 1)$.

<u>Proof</u>: Consider the interval $[0, 1]$ and define

$$\varphi_0(x) = \Phi(x)$$

$$\varphi_j(x) = D_j\, \varphi_{j-1}(x) \qquad j = 1, 2, \ldots k-1.$$

where

$$D_j\, f(x) = \frac{d}{dx}\, \frac{f(x)}{w_j(x)} \qquad j = 1, 2, \ldots k-1$$

Now, if $x \in [0, 1]$ then

$$\varphi_0(x+1) = t\, \varphi_0(x)$$

so that

$$\varphi_1(x+1) = D_1\, \varphi_0(x+1) = D_1\, t\, \varphi_0(x) = t\, \varphi_1(x)$$

and inductively we have

$$D_j \cdot D_{j-1} \cdots D_1\, \varphi_0(x+1) = t\, D_j\, D_{j-1} \cdots D_1\, \varphi_0(x)$$

That is,

$$\varphi_j(x+1) = t\, \varphi_j(x) \qquad x \in [0, 1] \text{ for } j = 1, 2, \ldots, k-1$$

In particular this gives

$$\varphi_j(1) = t\, \varphi_j(0) \quad \text{for } j = 1, 2, \ldots k-1 \qquad (3.3.10)$$

We now assert that

$$\frac{\varphi_j(x)}{w_{j+1}(x)} \qquad j = 0, 1, \ldots, k\ 2$$

has atleast two zeros in $[0, 1)$.

contd..

Our assertion is proved by induction on j. Since the functions $w_j(x)$ are strictly positive and continuous on $[0,1]$ the zeros of $\frac{\varphi_j}{w_{j+1}}$ are precisely those of φ_j. By hypothesis, $\frac{\varphi_0(x)}{w_1(x)}$ has at least two zeros in $[0, 1)$ and our assertion is therefore true for $j = 0$. Now assuming that $\frac{\varphi_{j-1}}{w_j}$ has atleast two zeros in $[0, 1)$, we shall show that $\frac{\varphi_j(x)}{w_{j+1}(x)}$ has at least two zeros in $[0, 1)$.

Let us recall that

$$\frac{d}{dx}\left\{\frac{\varphi_{j-1}(x)}{w_j(x)}\right\} = \varphi_j(x) \qquad j = 1, 2, .. k-1$$

Case (i) $\underline{\varphi_{j-1}(0) \neq 0}$. By (3,3,10) then $\varphi_{j-1}(1) \neq 0$. Again by (3.3.10) because $t < 0$, it follows that $\varphi_{j-1}(1)$ and $\varphi_{j-1}(0)$ have opposite signs. Thus $\frac{\varphi_{j-1}(1)}{w_j(1)}$ and $\frac{\varphi_j(0)}{w_j(0)}$ have opposite signs. This together with the assumption that φ_{j-1}/w_j has atleast two zeros in $[0, 1)$ imply that φ_{j-1}/w_j has atleast three zeros in $[0, 1)$. Hence by Rolle's theorem we conclude that $\varphi_j = \frac{d}{dx}\left(\frac{\varphi_{j-1}}{w_j}\right)$ has atleast two zeros in $[0, 1)$.

Case (ii) $\underline{\varphi_{j-1}(0) = 0 \text{ and } \varphi_{j-1}'(0) \neq 0}$

Now by (3.3.10), we see that $\varphi_{j-1}(1) = 0$. Thus $\frac{\varphi_{j-1}}{w_j}$ vanishes both at 0 and 1. Since $\frac{\varphi_{j-1}}{w_j}$ has

contd..

atleast two zeros in $[0, 1)$. we see that $\frac{\varphi_{j-1}}{W_j}$ has atleast three zeros in $[0, 1]$ and an application of Rolle's theorem gives the desired result.

Case (iii) $\frac{\varphi_{j-1}(0)}{W_j(0)} = 0$ and $\frac{d}{dx}\left(\frac{\varphi_{j-1}(x)}{W_j(x)}\right)_{x=0} = 0$

i.e. $0 = \varphi_j(0)$: Then by (3.3.10) $\frac{\varphi_{j-1}(1)}{W_j(1)} = 0$ so that there exists $\xi \in (0, 1)$ at which $\varphi_j = 0$. Thus φ_j and hence $\frac{\varphi_j}{W_{j+1}}$ has zeros at 0 and ξ in $[0, 1)$.

Our assertion is thus proved. In particular we see that $\frac{\varphi_{k-2}}{W_{k-1}}$ has atleast two zeros in $[0, 1)$ which implies again by Rolle's theorem that φ_{k-1} has atleast one zero in $[0, 1)$. This completes the proof of our lemma.

Concerning the zeros of $\Phi(x, t)$ we have the following.

<u>Theorem</u> 3.3.6: <u>If</u>

$$t < 0$$

<u>then</u> $\Phi(x, t)$ <u>has a single simple zero in the half open interval</u> $0 \leq x \leq 1$.

<u>Proof</u>: From the defining relation of $\Phi(x, t)$, it follows that $\Phi(0)$ and $\Phi(1)$ have opposite signs and hence it has atleast one zero in $[0, 1)$. If possible

contd..

assume that for $t<0$, $\Phi(x)$ has atleast two zeros in $[0, 1)$. By Lemma 3.3.5, then $\varphi_{k-1}(x)$ has a zero in $[0, 1)$. By (3.3.9), we have $\varphi_{k-1}(x) = C W_k(x)$ for all $x \in [0, 1]$ where C is a constant and $W_k(x) > 0$ for all $x \in R$. From this we conclude that $C = 0$ while Lemma 3.3.4 ensures that $C \neq 0$. This contradiction leads to the conclusion that $\Phi(x)$ cannot have more than one zero in $[0, 1)$ for $t<0$. Hence $\Phi(x)$ has exactly one simple zero in $[0, 1)$.

3.4 Main result on basis spline

We shall now establish the main result of this chapter that $M(x) > 0$ for $0 < x < k$.

First we recall some more properties of totally positive kernels constructed by application of generalized differencing operations to the fundamental solution of differential operators.

Let $\{w_i(x)\}$ $(i = 1, 2, ..n)$ be a set of n positive functions of the class $C^n[a, b]$ and associate with them the first order differential operators

$$(D_i\varphi)(x) = \frac{d}{dx}\frac{1}{w_i(x)}\varphi(x) \quad i = 1, 2, .. n$$

and the nth order differential operator

$$L_n\varphi = D_n D_{n-1} \cdots D_1\varphi$$

contd..

Let $\varphi_k(x)$ $(k = 1, 2, .. n)$ denote the solutions of $L_n\varphi = 0$ satisfying the initial conditions $\varphi_k(a) = \delta_{k,1}\, w_1(a)$ and $D_j \ldots\ldots D_2 D_1 \varphi_k(a) = \delta_{k,\, j+1}\, W_k(a)$ $(j = 1, 2, .. n-1)$ where δ_{kj} denotes the standard Kronecker delta symbol. Then $\{\varphi_k(x)\}$ $(k = 1, 2 .. n)$ form a basis of solutions of $L_n\varphi = 0$ and so every solution of $L_n\varphi = 0$ can be written as a linear combination of the elements in $\{\varphi_1, \varphi_2 \cdots \varphi_n\}$. Let $\varphi_n(x, t)$ denote the fundamental solution of the differential operator L_n. This means that fo r each t, $\varphi_n(x, t)$ satisfies the differential equation $L_n\varphi = 0$ on each of the intervals $a \leqslant x < t$ and $t < x \leqslant b$ and exhibits the characteristic discontinuity in the $(n-1)^{st}$ derivative at the point $x = t$.

Now let $\varphi_1(x), \varphi_2(x), \cdots, \varphi_{n+1}(x)$ be an ECT system of $(n+1)$ functions on $(-\infty, \infty)$ generated by the functions $w_1(x), \ldots, W_{n+1}(x)$ of the form

$$\varphi_i(x) = w_1(x)\int_a^x w_2(\xi_1) \cdots \int_a^{\xi_{i-2}} w_i(\xi_{i-1})\, d\xi_{i-1} \cdots d\xi_1 \tag{3.4.1}$$

$$i = 1, 2, \cdots, n+1$$

These functions constitute a basis of the solutions for the differential operator $L_{n+1} = D_{n+1} D_n \cdots D$. Let us now concentrate on the differential operator of one lower order $L_n = D_n D_{n-1} \cdots D_1$ and let $\varphi_n(x, t)$ be

contd..

its fundamental solution. A basis of solutions for $L_n u = 0$ is the system $\left\{ \varphi_i(x) \right\}$, $i = 1, 2, \cdots, n$.

Let

$$- - - < x_{-2} < x_{-1} < x_0 < x_1 < x_2 - \cdots \cdots \tag{3.4.2}$$

be a sequence of fixed real numbers such that

$$x_\nu \longrightarrow \pm\infty \text{ as } \nu \longrightarrow \pm\infty$$

Define

$$M_i(\xi) = \frac{\begin{vmatrix} \varphi_1(x_i) & \varphi_2(x_i) & \cdots & \varphi_n(x_i) & \varphi_n(x_i, \xi) \\ \varphi_1(x_{i+1}) & \varphi_2(x_{i+1}) & \cdots & \varphi_n(x_{i+1}) & \varphi_n(x_{i+1}, \xi) \\ - & - & - & - & - \\ \varphi_1(x_{i+n}) & \varphi_2(x_{i+n}) & \cdots & \varphi_n(x_{i+n}) & \varphi_n(x_{i+n}, \xi) \end{vmatrix}}{\begin{vmatrix} \varphi_1(x_i) & \varphi_2(x_i) & \cdots & \varphi_n(x_i) & \varphi_{n+1}(x_i) \\ \varphi_1(x_{i+1}) & \varphi_2(x_{i+1}) & \cdots & \varphi_n(x_{i+1}) & \varphi_{n+1}(x_{i+1}) \\ - & - & - & - & - \\ \varphi_1(x_{i+n}) & \varphi_2(x_{i+n}) & \cdots & \varphi_n(x_{i+n}) & \varphi_{n+1}(x_{i+n}) \end{vmatrix}} \tag{3.4.3}$$

for $i = \cdots, -2, -1, 0, 1, 2, \cdots, -\infty < \xi < \infty$

Then we have the following.

<u>Theorem</u> 3.4.1 (Theorem 4.1, p.527 [4])

<u>The kernel</u> $M_i(\xi)$ <u>is totally positive on</u> $I \times T$ <u>where</u> $I = \left\{ \cdots \cdots -2, -1, 0, 1, 2, \cdots\cdots \right\}$ and $T = (-\infty, \infty)$.

We now prove

<u>Theorem</u> 3.4.2: <u>If</u> $0 \leqslant \alpha < 1$, <u>then the algibraic equation</u>

$$M(\alpha)\lambda^{k-1} + M(\alpha+1)\lambda^{k-2} \cdots + M(\alpha+k-1) = 0 \tag{3.4.4}$$

contd..

has only simple and negative roots given by

$$\lambda_{k-1}(\alpha) < \lambda_{k-2}(\alpha) < \cdots < \lambda_1(\alpha) < 0$$

To establish Theorem 3.4.2, we first consider the sequence

$$\left\{ M(\alpha + \nu) \right\}_{\nu=-\infty}^{\infty} \tag{3.4.5}$$

of the coefficients of the equation (3.4.4). By Theorem 3.4.1 which establishes the total positivity of a sequence of B-splines for Chebyshev system, when translated into our case, we see that $\left\{ M(\alpha+\nu) \right\}_{\nu=-\infty}^{\infty}$ is a totally positive sequence. The left hand side of (3.4.4) is the generating function of $\left\{ M(\alpha+\nu) \right\}$ and concerning the zeros of generating functions of totally positive sequence, we have the following results.

Lemma 3.4.3: For $0 < \alpha < 1$, the equation

$$M(\alpha)\lambda^{k-1} + M(\alpha)\lambda^{k-2} + \cdots M(\alpha+k-1) = 0$$

has all its roots real and negative; they are labelled as.

$$\lambda_{k-1}(\alpha) \leq \lambda_{k-2}(\alpha) \leq \cdots \leq \lambda_1(\alpha) \leq 0 \tag{3.4.6}$$

Lemma 3.4.4 : The equation

$$M(1)\lambda^{k-2} + M(2)\lambda^{k-3} \cdots + M(k-1) = 0 \tag{3.4.7}$$

contd..

has all its roots $\bar{\lambda}_\nu$ negative and real: they are

$$\bar{\lambda}_{k-2} \leqslant \bar{\lambda}_{k-3} \leqslant \cdots \leqslant \bar{\lambda}_1 \leqslant 0 \qquad (3.4.8)$$

These lemmas are a very special case of Theorem 5.3 on p. 412 of [4].

Thus to complete the proof of Theorem 3.4.2, we need to prove strict inequality in (3.4.6) and (3.4.8). Clearly the roots $\lambda_\nu(\alpha)$ and $\bar{\lambda}_\nu$ do not vanish since $M(\alpha+k-1)\neq 0$ for $0\leqslant\alpha<1$. Moreover $\lambda_\nu(\alpha)\in C(0,1)$. We shall extend this definition to $[0, 1]$ by

$$\lambda_\nu(0) = \bar{\lambda}_\nu, \; \nu=1,\dots k-2; \; \lambda_{k-1}(0) = -\infty$$
$$\lambda_\nu(1) = \bar{\lambda}_{\nu-1}, \; \nu=2,\dots k-1, \; \lambda_1(1) = 0 \qquad (3.4.9)$$

Then $\lambda_\nu(\alpha)$ are continuous functions on $[0, 1]$ such that $\lambda_{k-1}(\alpha) \to -\infty$ as $\alpha \to 0^+$

Lemma 3.4.5: If $0\leqslant\alpha_1<\alpha_2<1$ or $0<\alpha_1<\alpha_2\leqslant 1$ then $\lambda_i(\alpha_1) \neq \lambda_j(\alpha_2)$ (3.4.10) whether or not i and j are distinct.

Proof: Assume that (3.4.10) does not hold and suppose that

$$\lambda_i(\alpha_1) = \lambda_j(\alpha_2) = \lambda_0 \text{ (say)} \qquad (3.4.11)$$

We shall arrive at a contradiction. Let $0<x<1$. Then $0<x+j<k$ because $0\leqslant j\leqslant k-1$.

contd..

Now

$$\Phi(x,\tau_0) = \sum_{j=-\infty}^{\infty} \tau_0^{\,j} M(x-j) = \sum_{j=-\infty}^{\infty} \tau_0^{\,-j} M(x+j)$$
$$= \sum_{j=0}^{k-1} \tau_0^{\,-j} M(x+j)$$

$$= \tau_0^{\,-(k-1)} \sum_{j=0}^{k-1} \tau_0^{\,k-1-j} M(x+j) \qquad (3.4.12)$$

Since $\tau_i(\alpha_1)$ is a root of the equation (3.4.4) we find on letting $x = \alpha_1$ in (3.4.12), that

$$\Phi(\alpha_1,\tau_0) = \Phi(\alpha_1, \tau_i(\alpha_1))$$
$$= \tau_i(\alpha_1)^{-(k-1)} \sum_{j=0}^{k-1} (\tau_i(\alpha_1))^{k-1-j} M(\alpha_1+j) = 0$$

and similarly, letting $x = \alpha_2$ in (3.4.12), we get

$$\Phi(\alpha_2,\tau_0) = \tau_j(\alpha_2)^{-(k-1)} \sum_{j_1=0}^{k-1} \tau_j(\alpha_2)^{k-1-j_1} M(\alpha_2+j_1) = 0$$

Thus we have

$$\Phi(\alpha_1,\tau_0) = \Phi(\alpha_2,\tau_0) = 0$$

where $\tau_0 < 0$ and $0 \le \alpha_1 < \alpha_2 < 1$ giving two distinct zeros for Φ (x, t) in $[0, 1)$. This contradicts Theorem 3.3.6 and our lemma is established.

We shall now complete the proof of Theorem 3.4.2. It follows from Lemma 3.4.5 that $\tau_j(\alpha)$ is strictly monotonic in the closed interval $[0, 1]$. Also by (3.4.9), we have $\tau_1(0) = \bar{\tau}_1 < 0$ while $\tau_1(1) = 0$

contd..2

so that $\lambda_1(0) < \lambda_1(1)$ and hence $\lambda_1(\alpha)$ is strictly increasing. Further $\lambda_2(0) = \bar{\lambda}_2 \leq \bar{\lambda}_1 = \lambda_2(1)$ By Lemma 3.4.5, $\lambda_2(0) < \lambda_2(1)$ so that $\bar{\lambda}_2 < \bar{\lambda}_1$ and $\lambda_2(\alpha)$ is strictly increasing. Continuing in this manner, we see that $\lambda_\nu(\alpha)$ is strictly increasing and that $\bar{\lambda}_\nu < \bar{\lambda}_{\nu-1}$ for each ν. Thus

$$\lambda_{k-1}(\alpha) < \lambda_{k-2}(\alpha) \cdots \lambda_1(\alpha) < 0 \quad \text{for} \quad 0 < \alpha < 1$$

and

$$\bar{\lambda}_{k-2} < \bar{\lambda}_{k-3} < \cdots \cdots \bar{\lambda}_1 < 0$$

Letting $\alpha \to 0$ in Theorem 3.4.2 we have the following.

<u>Corollary 3.4.6</u>. <u>The equation</u>

$$M(1)\lambda^{k-2} + M(2)\lambda^{k-3} + \quad + M(k-1) = 0$$

<u>has only negative and simple roots given by</u>

$$\bar{\lambda}_{k-2} < \bar{\lambda}_{k-3} < \cdots \cdots < \bar{\lambda}_1 < 0$$

We shall now establish the final result in this chapter.

<u>Theorem</u> 3.4.7: $M(x) > 0$ <u>for</u> $0 < x < k$

<u>Proof</u>: We have proved that strict inequalities hold in (3.4.6) and (3.4.8). Since all the zeros of the polynomial

$$p(\lambda) = M(\alpha)\lambda^{k-1} + M(\alpha+1)\lambda^{k-2} + \cdots + M(\alpha+k-1)$$

contd..

are simple and negative, it is easy to verify that all the coefficients $M(x+j)$ have the same sign. Now

$$M(\alpha) = \beta_0 \hat{\theta}(\alpha, 0) \text{ for } 0<\alpha<1$$

and we have already chosen β_0 to be positive. It is known (See Lemma 9.2 p. 437 [5]) that if $u_1, \dots u_k$ forming the basis of $K(\Lambda)$ is an ECT system on R and $\hat{\theta}(x, \xi)$ is the fundamental solution of the operator of order k, then for

$$x_1 < x_2 < \cdots < x_p$$

$$\xi_{p_1} < \xi_{p_2} < \cdots < \xi_{p_k}$$

where p is any natural number holds the inequality

$$\det \| \hat{\theta}(x_i, \xi_{p_j}) \|_{i,j=1}^{k_1} \geq 0$$

always and

$$\det \| \hat{\theta}(x_i, \xi_{p_j}) \|_{i,j=1}^{k_1} > 0$$

if and only if $x_{i-k} < \xi_{p_i} < x_i$, $i=1, \dots k_1$; $k_1 = 1, \dots p$.

When p = 1, this reduces to

$$\hat{\theta}(x, \xi) > 0$$

if and only if $\xi < x$. Thus we conclude that if $0<\alpha<1$, then $M(\alpha)>0$ and hence $M(\alpha+j)>0$ for $j = 1, 2, \dots, k-1$. This shows that $M(x) > 0$ for $j < x < j+1$ for $j = 1, 2, \dots k-1$. Considering the limiting case $\alpha \to 0$, we have the desired result.

CHAPTER IV

THE INTERPOLATION PROBLEM

4.1 Statement of the problem and a preliminary answer

The problem of Λ-cardinal spline interpolation can be stated as follows:

Given a sequence of numbers

$$y = (y_\nu), \quad \nu = 0, \pm 1, \pm 2, \cdots \qquad (4.1.1)$$

being real or complex it is required to find an $S \in \mathcal{S}_{\Lambda,1}$ satisfying

$$S(\nu) = y_\nu \text{ for all integers } \nu \qquad (4.1.2)$$

We have already seen that any $S \in \mathcal{S}_{\Lambda,1}$ has the unique representation

$$S(x) = P(x) + \sum_{\nu=1}^{\infty} a_\nu \hat{\theta}(x,\nu) + \sum_{\nu=-\infty}^{0} a_\nu \hat{\theta}(-x,\nu)$$

where

$$P(x) = \sum_{i=1}^{k} b_i u_i(x) \in K(\Lambda)$$

and $u_1, \cdots, u_k$ form the basis of $K(\Lambda)$

If $0 \leq x \leq 1$, then

$S(x) = P(x)$

In particular,

$S(0) = P(0)$ and $S(1) = P(1)$

contd..

We select $P(x) \in K(\Lambda)$ arbitrarily such that $P(0) = y_0$ and $P(1) = y_1$. Having chosen $P(x)$ satisfying these two conditions, for a solution to our problem, we must have

$$y_2 = S(2) = P(2) + a_1 \hat{\Theta}(2,1) = P(2) + \Theta(2,1)\, a_1$$

which determines a_1 uniquely. Thus for $\nu = 2, 3, \ldots$ we determine $a_2, \ldots,$ successively and uniquely using the interpolation condition

$$S(\nu) = y_\nu \quad \text{for all integers } \nu$$

Similarly for $\nu = -1, -2, \ldots,$ the coefficients $a_0, a_{-1}, \ldots,$ are uniquely determined.

Since $P(x) \in K(\Lambda)$ satisfying $P(0) = y_0$ and $P(1) = y_1$ depends on $(k-2)$ parameters we have proved the following.

<u>Theorem 4.1.1:</u> <u>The set of solutions of our interpolation problem form a linear manifold of dimension</u> $(k-2)$ in $\mathcal{S}_{\Lambda,1}$

4.2 <u>A basis for the space of null-splines</u>

Let

$$\mathcal{S}^{\circ}_{\Lambda,1} = \left\{ S \in \mathcal{S}_{\Lambda,1} : S(\nu) = 0 \text{ for all integers } \nu \right\} \tag{4.2.1}$$

Then $\mathcal{S}^{\circ}_{\Lambda,1}$ is a linear subspace of $\mathcal{S}_{\Lambda,1}$ of dimension $(k-2)$ and its elements are referred to as <u>null-splines.</u>

contd..

We shall now obtain a basis of $\overset{\circ}{\mathcal{S}}_{\Lambda,1}$.

We have already seen that

$$\Phi(x, t) = \sum_{\nu=-\infty}^{\infty} t^{\nu} M(x-\nu) \tag{4.2.2}$$

is the most general element of $\mathcal{S}_{\Lambda,1}$ satisfying

$$\Phi(x+1, t) = t\,\Phi(x, t) \quad \text{for all } x \in R$$

In particular

$$\Phi(\nu, t) = t^{\nu}\,\Phi(0, t) \tag{4.2.3}$$

for all integers ν . If we define

$$F(x, t) = \frac{\Phi(x, t)}{\Phi(0, t)}$$

for those t, for which $\Phi(0, t) \neq 0$, then

$$F(\nu, t) = t^{\nu} \quad \text{for all integers } \nu \ .$$

Thus for t satisfying $\Phi(0, t) \neq 0$ we see that $F(x,t) \in \mathcal{S}_{\Lambda,1}$ and interpolates $f(x) = t^{x}$ at the integers ν.

If $\Phi(0, t) = 0$ for some t, then $\Phi(\nu, t) = 0$ for all integers ν by (4.2.3).

Now

$$\begin{aligned}\Phi(0,t) &= \sum_{\nu=-\infty}^{\infty} t^{\nu} M(-\nu)\\ &= \sum_{j=1}^{k-1} t^{-j} M(j)\\ &= t^{-(k-1)} \sum_{j=0}^{k-2} t^{j} M(k-1-j)\end{aligned}$$

Since $|t| < \infty$, $t \neq 0, 1$, we have if $\Phi(0, t) = 0$ then

$$\sum_{j=0}^{k-2} t^{j} M(k-1-j) = 0$$

This equation has negative zeros given by

contd..

$t_{k-2} < t_{k-3} < \cdots < t_1 < 0$

Thus we have

$\Phi(0, t) = 0$ for $t = t_1, t_2, \cdots t_{k-2}$

Define

$$P_r(x) = \Phi(x, t_r) \qquad r = 1, 2, \cdots k-2. \qquad (4.2.4)$$

Theorem 4.2.1 $\{P_r(x)\}_{r=1}^{k-2}$ *form a basis of* $\mathring{\mathcal{S}}_{\Lambda,1}$

Proof: By definition,

$$P_r(\nu) = \Phi(\nu, t_r) = t_r^{\nu}\,\Phi(0, t_r) = 0$$

for all integers ν and hence $P_r(x) \in \mathring{\mathcal{S}}_{\Lambda,1}$ for $r = 1, 2, \ldots k-2$. Since $\mathring{\mathcal{S}}_{\Lambda,1}$ is of dimension $k-2$, to establish our theorem it is enough to prove that $P_1(x)$, $P_2(x)$ ..... $P_{k-2}(x)$ are linearly independent. To this end, let

$$\sum_{j=1}^{k-2} b_j\, P_j(x) = 0 \text{ for all } x \in R \qquad (4.2.5)$$

where b_j's are constants. We have to prove that

$b_1 = \cdots = b_{k-2} = 0$

From (4.2.5), we have

$$\begin{aligned} 0 &= \sum_{j=1}^{k-2} b_j P_j(x) = \sum_{j=1}^{k-2} b_j \Phi(x, t_j) \\ &= \sum_{j=1}^{k-2} b_j \sum_{\nu=-\infty}^{\infty} t_j^{\nu} M(x-\nu) \\ &= \sum_{\nu=-\infty}^{\infty} M(x-\nu) \sum_{j=1}^{k-2} b_j t_j^{\nu} \end{aligned}$$

contd..

for all x, implying that

$$\sum_{j=1}^{k-2} b_j \, t_j^{\nu} = 0 \quad \text{for all integers } \nu .$$

In particular,

$$\sum_{j=1}^{k-2} b_j \, t_j^{\nu} = 0, \quad \nu = 0, 1, 2, \ldots k-3 \qquad (4.2.6)$$

The determinant of the coefficients is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_{k-2} \\ t_1^2 & t_2^2 & \cdots & t_{k-2}^2 \\ \cdots & \cdots & \cdots & \cdots \\ t_1^{k-3} & t_2^{k-3} & \cdots & t_{k-2}^{k-3} \end{vmatrix}$$

which is not zero because $t_{k-2} < t_{k-3} < \cdots t_1 < 0$.

Hence the homogeneous system of equations (4.2.6) has only trivial solution. Thus

$$b_1 = b_2 = \cdots = b_{k-2} = 0$$

and our assertion is proved.

Since $\overline{\Phi}(0, t) = 0$ for exactly $(k-2)$ distinct negative values of t, from what is discussed in the earlier part of this section, we also have the following.

contd..

Theorem 4.2.2: *If $t > 0$ and $t \neq 1$, then the unique element of $\mathcal{S}_{n,1}$ which interpolates the data $\{t^{\nu}\}_{\nu=-\infty}^{\infty}$ at the nodes $\{\nu\}_{-\infty}^{\infty}$ is uniquely determined as*

$$F(x, t) = \frac{\Phi(x,t)}{\Phi(0,t)} \qquad x \in R$$

where $\Phi(x, t)$ is defined by (4.2.2)

4.3 Condition for uniqueness

Consider the function $\Phi(x, t)$. By Theorem 3.3.6, we know that for $t < 0$, $\Phi(x, t)$ has exactly one simple zero in $[0, 1)$. In particular, choosing $t = -1$ and setting

$$E(x) = \Phi(x, -1)$$

we see that $E(x)$ has exactly one simple zero, say ξ_1, $0 \leq \xi_1 < 1$. Clearly $E(x+1) = -E(x)$ for all $x \in R$ so that $E(x)$ is periodic with period 2 and hence bounded on R. From the relation

$$E(x + \nu) = (-1)^{\nu} E(x)$$

it follows that

$$E(\xi_1 + \nu) = 0 \text{ for all integers } \nu . \qquad (4.3.1)$$

contd..

Considering the data of power growth we have

Theorem 4.3.1 : Suppose

$y_\nu = O(|\nu|^s)$ as $\nu \to \pm\infty$ for some $s>0$

and suppose our interpolation problem has a solution $S \in \mathcal{S}_{\Lambda,1}$ satisfying the two conditions

(i) $S(\nu) = y_\nu$ for all integers ν

(ii) $S(x) = O(|x|^s)$ as $x \to \pm\infty$

Then the solution is unique if and only if $\xi_1 \neq 0$ where ξ_1 is the zero of $E(x)$ mentioned above.

Proof: Necessity. If possible let $\xi_1 = 0$. Then $E(\nu) = 0$ for all integers ν by (4.3.1). Being periodic, $E(x)$ is also bounded on R. Let $S \in \mathcal{S}_{\Lambda,1}$ be the unique solution to (i) satisfying (ii). Now consider $S(x) + c\,E(x)$ where c is any constant. Then $S(x) + c\,E(x) \in \mathcal{S}_{\Lambda,1}$ and

$$S(\nu) + c\,E(\nu) = S(\nu) = y_\nu \text{ for all integers } \nu.$$

Moreover $S + cE$ is of power growth since S is of power growth and E is bounded. Thus we see that for any constant c, $S(x) + cE(x)$ is a solution to the interpolation problem (i) satisfying (ii) which contradics the uniqueness. Hence $\xi_1 \neq 0$.

contd..

<u>Sufficiency</u>: Assume $\xi_q \neq 0$. We shall now prove the uniqueness. We have seen that the zeros of $\Phi(0,t)$ are given by

$$\overline{\tau}_{k-2} < \overline{\tau}_{k-3} < \dots < \overline{\tau}_1 < 0$$

If $\overline{\tau}_i = -1$ for some i, then $\Phi(0, -1) = 0$ implying $z(0) = 0$ which is not possible because the only zero of $z(x)$ in $[0, 1)$ is different from zero by our assumption. Hence $\overline{\tau}_i \neq -1$ for any i. We can therefore find a p, $1 < p < k-2$ such that

$$\overline{\tau}_{k-2} < \overline{\tau}_{k-3} < \dots < \overline{\tau}_p < -1 < \overline{\tau}_{p-1} < \dots < \overline{\tau}_1 < 0$$

If possible let S_1 and S_2 be two distinct elements of satisfying (i) and (ii). Now let $S(x) = S_1(x) - S_2(x)$. Then

$S(x) \in \mathcal{S}_{n,1}$

$S(\nu) = 0$ for all integers ν

and

$S(x) = O(|x|^s)$ as $x \to \pm\infty$

We have already seen that $\{P_j(x)\}_{j=1}^{k-2}$ form a basis of $\mathring{\mathcal{S}}_{n,1}$ where

$P_j(x) = \Phi(x, \overline{\tau}_j)$ $j = 1, 2, \dots k-2$

Hence, $S(x)$ can be represented in the form

$$S(x) = \sum_{i=1}^{k-2} a_i P_i(x)$$

contd..

where the a_i's are uniquely determined constan ts.

Let us now consider the behaviour of $P_i(x)$ as $x \to \pm\infty$. First we notice that for each $i = 1, 2, \ldots k-2$ we have

$$P_i\,(x+j) = \tau_i^{\,j}\; P_i\,(x) \tag{4.3.2}$$

for each integer j and all x.

Let $1 \leq i \leq p-1$. Then by the choice of p, we have $|\tau_i| < 1$ so that $\left|\frac{1}{\tau_i}\right| > 1$

Then there exists a positive integer n_1 such that

$$\left|\frac{1}{\tau_i}\right|^{n} > n^{S+1} \quad \text{for all } n \geq n_1$$

and using the relation

$$P_i\,(x-n) = \tau_i^{-n}\; P_i(x) = \left(\frac{1}{\tau_i}\right)^{n} P_i\,(x)$$

we see that

$$|P_i(x-n)| > n^{S+1}\; |P_i\,(x)| \quad \text{for all } n \geq n_1$$

Fixing x, we can now choose an integer n_2 so large that $|x-n| \leq 2n$ for all $n \geq n_2$. Set $N_1 = \max\,(n_1, n_2)$.

Then

$$|P_i\,(x-n)| > n^{S+1}\; |P_i\,(x)| \geq |x-n|^{S+1} \frac{|P_i(x)|}{2^{S+1}} \quad \text{for all } n \geq N_1$$

which shows that

$$|P_i(y)| > |y|^{S+1} K_1 \quad \text{as } y \to -\infty \text{ for all} \tag{4.3.3}$$

where K_1 is a constant and $i = 1, 2, \ldots p-1$. But

$$S(x) = \sum_{i=1}^{K-2} a_i\; P_i\,(x)$$

and

$$S(x) = 0\,(|x|^{S}) \text{ as } x \to \pm\infty$$

contd..

This shows that, by virtue of (4.3.3) a_i must be zero for $i = 1, 2, -- p-1$. Hence $S(x)$ reduces to

$$S(x) = \sum_{i=p}^{k-2} a_i P_i(x)$$

Now suppose $p \leq i \leq k-2$. Then $|\lambda_i| > 1$ and so there exists an integer n_3 such that $|\lambda_i|^n > n^{s+1}$ for all $n \geq n_3$ using the relation

$$P_i(x+n) = \lambda_i^n P_i(x)$$

it follows that

$$|P_i(x+n)| > n^{s+1} |P_i(x)| \quad \text{for all } n \geq n_3$$

Choose n_4 sufficiently large so that $|x+n| \leq 2n$ for $n \geq n_4$. Setting $N_2 = \max(n_3, n_4)$ we see as before

$$|P_i(x+n)| > |x+n|^{s+1} \frac{|P_i(x)|}{2^{s+1}} \quad \text{for all } n \geq N_2$$

from which we deduce that

$$|P_i(y)| > |y|^{s+1} K_2 \quad \text{as } y \to \infty$$

where K_2 is a constant, for all $i = p, p+1, \ldots k-2$. This together with the fact

$$|S(x)| < K |x|^s \quad \text{as } x \to \pm\infty$$

shows that $a_i = 0$ for $i = p, p+1 \ldots k-2$. Hence $S = 0$ and so $S_1 = S_2$ proving the uniqueness as desired.

4.4 Solution of the interpolation problem

Our result on Λ-cardinal spline interpolation may be stated as follows.

contd..

Theorem 4.4.1: <u>Suppose the given data of bisequence</u> (y_ν) <u>satisfies</u>

$$y_\nu = O(|\nu|^s) \qquad \text{as } \nu \to \pm\infty \text{ for some } s > 0$$

<u>Then there exists a unique function</u> $s(x)$ <u>satisfying the following conditions</u>

(i) $s(\nu) = y_\nu$ <u>for all integers</u> ν

(ii) $s(x) \in \mathcal{S}_{\Lambda,1}$

(iii) $s(x) = O(|x|^s)$ <u>as</u> $x \to \pm\infty$ <u>if and only if</u> $\xi_1 \neq 0$ <u>where</u> ξ_1 <u>is the zero mentioned in Theorem</u> 4.3.1.

Proof: Let us suppose that the solution to our interpolation problem (if it exists) is unique so that $\xi_1 \neq 0$

Then

$$\overline{\lambda}_{k-2} < \overline{\lambda}_{k-3} < \cdots < \overline{\lambda}_p < -1 < \overline{\lambda}_{p-1} < \cdots < \overline{\lambda}_1 < 0$$

where p is defined in the sufficiency part of the proof of Theorem 4.3.1. These are precisely the zeros of

$$f(z) = \sum_{j=0}^{k-2} z^j M(k-1-j)$$

Since $\lambda_i \neq -1$ for any i, it follows that $f(z)$ has no zeros on $|z| = 1$ and so $\frac{1}{f(z)}$ has no poles there. Then we have

$$\frac{1}{f(z)} = \sum_\nu \omega_\nu z^\nu \qquad (4.4.1)$$

the Laurent series on the right hand side converging in the annulus $|\overline{\lambda}_{p-1}| < |z| < |\lambda_p|$ containing $|z| = 1$

This implies the existence of an inequality of the form

contd..

$$|\omega_\nu| \le A e^{-B|\nu|} \quad \text{for all integers } \nu . \qquad (4.4.2)$$

Where A and B are constants.

Clearly

$$\sum_\nu \omega_{-\nu} M(k-1-\nu) = 1$$

and

$$\sum_\nu \omega_{-\nu} M(k-1-\nu-j) = 0 \text{ for integers } j \neq 0$$

Define

$$L(x) = \sum_{\nu=-\infty}^{\infty} \omega_{-\nu} M(k-1-\nu-x) \qquad (4.4.3)$$

Then $L(x) \in \mathcal{S}_{\Lambda,1}$ and

$$L(j) = \delta_j = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{for } j \neq 0 \end{cases}$$

Now

$$\begin{aligned} |L(x)| &\le \sum_\nu |\omega_\nu| \, |M(k-1-\nu-x)| \\ &\le \sum_{\nu > -x-1} |\omega_{-\nu}| \, M(k-1-\nu-x) \\ &\le \sum_{\nu > -x-1} |\omega_{-\nu}| \, M(k-1) \\ &< A M(k-1) \sum_{\nu, x-1 \ge 0} e^{-B\nu} \\ &< K_1 e^{Bx} \quad \text{if } x \le -1 \end{aligned}$$

And if $x > k-1$, we have

$$|L(x)| \le A M(1) \sum_{\nu \le k-1-x} e^{-B|\nu|} < K_2 e^{-Bx}$$

Thus, we see that for appropriate constants K and B we have the inequality

$$|L(x)| < K e^{-B|x|} \quad \text{for all } x \in R \qquad (4.4.4)$$

contd..

Since $\xi_1 \neq 0$, L is also unique.

Thus L (x) defined by (4.4.3) is the unique element of $\mathcal{S}_{\Lambda,1}$ satisfying (i) and (iii) where

$$y_\nu = \delta_\nu = \begin{cases} 1 & \nu = 0 \\ 0 & \nu \neq 0 \end{cases}$$

which is a bounded sequence. L is the basic function for our interpolation problem.

Now define with the given sequence (y_ν)

$$S(x) = \sum_\nu y_\nu L(x-\nu) \qquad (4.4.5)$$

Then it is clear that $S(j) = y_j$ for all j and that $S \in \mathcal{S}_{\Lambda,1}$.

Now we prove that S (x) defined by (4.4.5) converges locally uniformly and that S is of power growth.

Since the given sequence y_ν is of power growth, we have

$$|y_\nu| \leq A\,(|\nu|^s + 1) \quad \text{for all integers } \nu \qquad (4.4.6)$$

so that

$$|S(x)| \leq \sum_\nu |y_\nu|\,|L(x-\nu)|$$

gives, using (4.4.4) and (4.4.6), that

$$|S(x)| \leq AK \sum_\nu (|\nu|^s+1)\,e^{-B|x-\nu|} \quad \text{for all } x \in R \qquad (4.4.7)$$

and the series on the right hand side converges uniformly on every compact subset of R. Thus S (x) converges locally uniformly.

contd..

It remains to show that

$$\sum_{\nu} y_{\nu} L(x-\nu) = O(|x|^{s}) \text{ as } x \to \pm\infty \quad (4.4.8)$$

By virtue of (4.4.7) it is enough to show that

$$\sum_{\nu} |\nu|^{s} e^{-B|x-\nu|} = O(|x|^{s}) \quad (4.4.9)$$

We will only prove that

$$\sum_{\nu=1}^{\infty} \nu^{s} e^{-B|x-\nu|} = O(x^{s}) \text{ as } x \to \infty \quad (4.4.10)$$

The case when $x \to -\infty$ follows in a similar way.

Now

$$\sum_{\nu=1}^{\infty} \left(\frac{\nu}{x}\right)^{s} e^{-B|x-\nu|} = \sum_{\nu \le x+1} \left(\frac{\nu}{x}\right)^{s} e^{-B|x-\nu|} + \sum_{\nu > x-1} \left(\frac{\nu}{x}\right)^{s} e^{-B|x-\nu|}$$

$$< \sum_{\nu \le x+1} e^{-B|x-\nu|} + \sum_{\nu > x+1} \nu^{s} x^{-s} e^{-B|x-\nu|}$$

$$= O(1) + x^{-s} \sum_{\nu > x+1} \nu^{s} e^{-B(\nu-x)}$$

$$= O(1) + x^{-s} e^{Bx} \sum_{\nu > x+1} \nu^{s} e^{-B\nu}$$

If we restrict x to the range $x < \xi_1$ in which the function $x^{s} e^{-Bx}$ is decreasing and convex, then we may replace the last sum by an integral and obtain

$$x^{-s} e^{Bx} \sum_{\nu > x+1} \nu^{s} e^{-B\nu} = x^{-s} e^{Bx} \int_{x}^{\infty} t^{s} e^{-Bt}\, dt$$

$$= \int_{0}^{\infty} \left(1 + \frac{u}{x}\right)^{s} e^{-Bu}\, du$$

contd..

by the change of variable $t = x + u$. The last integral being $O(1)$ as $x \to \infty$, we see that (4.4.10) holds. This completes the proof of the theorem.

CHAPTER V

t- PERFECT Λ-CARDINAL SPLINES

5.1 An extremal problem for t perfect Λ cardinal splines

The concept of Λ cardinal splines, where Λ is a linear differential operator of order k with coefficients continuous real-valued functions on R has been introduced in Chapter II. Here we consider a subset of this class of Λ- cardinal splines. These will generalise the concept of perfect splines studied by Sharma and Tzimbalario in [11] . We further dtermine the element having the least t norm (defined below) for a given real t (which is non-zero) in our subclass.

For the sake of completeness, let us recall the linear differential operator Λ , of order k, defined by

$$\Lambda = D^k + \sum_{j=0}^{k-1} a_j D^j \qquad (5.1.1)$$

where $a_j \in C^j(R)$, $j = 0, .. k-1$. Here $C^j(R)$ is the class of real valued functions on R having the j^{th} derivative continuous. The null space $K(\Lambda)$ of Λ is a linear space of dimension k. Let $u_1, ---, u_k$ basis of $K(\Lambda)$ and further suppose that $u_1, u_2, --, u_k$ form an ECT system on R. With the notations as earlier, we see that if we set

$$D^{(j)} = D_j D_{j-1} \cdots D_1 \qquad j = 1, 2, \cdots k$$

then

$$\Lambda = W_1(x) W_2(x) \dots W_k(x) D^{(k)} = \chi(x) D^{(k)}$$

contd..

where

$$\nu(x) = W_1(x)\, W_2(x) \ldots W_k(x) > 0 \quad \text{for } x \in R.$$

It is clear that $f \in K(\Lambda)$ if and only if $D^{(k)} f = 0$.

Further more

$$\left.\begin{array}{ll} D^{(j)} U_i = 0 & i \leq j\,;\ i, j = 1, 2 \,.\, k \\ D^{(j)} U_{j+1} = W_{j+1} & j = 1, 2, \ldots k-1 \end{array}\right\} \quad (5.1.2)$$

For $r = -1, 0, \ldots k-2$, we define the class $S^r_\Lambda = \{S(x)\}$ consisting of all functions with the following two properties.

(i) $S(x) \in C^r(R)$

(ii) $S(x) \in K(\Lambda)$ for $\nu < x < \nu+1$ for all integers ν.

Let t be a given real number. Define

$S^r_{\Lambda,t} = \{ S \in S^r_\Lambda : D^{(k-1)} S(x) = t^\nu \omega_k(x) \text{ in } (\nu, \nu+1)$ for all integers $\nu \}$

and set the following problem.

Determine $S(x) \in S^r_{\Lambda,t}$ having the least t norm

where

$$\| S \|_{t,\infty} = \sup_{x \in R} \left| \frac{S(x)}{t^{[x]}} \right|$$

where $[x]$ denotes the integral part of x.

5.2 <u>A property of the extremal solution</u>

We shall now prove the following result analogous to Theorem 1 of [11]

<u>Theorem 5.2.1: If $F \in S^r_{\Lambda,t}$ with finite t - norm ρ, then there exists another element $F^* \in S^r_{\Lambda,t}$ such that</u>

contd..

$$\| F^* \|_{t,\infty} \le \rho$$

and

$$F^*(x+1) = t\,F^*(x) \ \underline{\text{for all}}\ \ x \in R.$$

<u>Proof</u>: Consider the sequence $\{F_n(x)\}$ of functions given by

$$F_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} t^{-j} F(x+j) \quad n = 1, 2, 3 \cdots \tag{5.2.1}$$

Then

$$\frac{F_n(x)}{t^{[x]}} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{F(x+j)}{t^{[x+j]}} \tag{5.2.2}$$

from which we deduce that $F_n(x) \in S^{r}_{\wedge,t}$. Moreover the representation (5.2.2) also gives

$$\| F_n \|_{t,\infty} \le \rho$$

Using the standard diagonal process, we can select a subsequence converging on R and uniformly on each finite interval. If the limit is denoted by $F^*(x)$ then it is fairly obvious that $F^*(x) \in S^{r}_{\wedge,t}$ and $\| F^* \|_{t,\infty} \le \rho$. Denoting this convergent subsequence again by $\{F_n\}$ and observing that

$$\frac{F_n(x+1)}{t^{[x+1]}} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{F(x+j+1)}{t^{[x+1+j]}}$$

$$= \frac{1}{n}\left\{ \frac{F(x+1)}{t^{[x+1]}} + \frac{F(x+2)}{t^{[x+2]}} + \cdots + \frac{F(x+n)}{t^{[x+n]}} \right\}$$

$$= \frac{1}{n}\left\{ \sum_{j=0}^{n-1} \frac{F(x+j)}{t^{[x+j]}} - \frac{F(x)}{t^{[x]}} + \frac{F(x+n)}{t^{[x+n]}} \right\}$$

$$= \frac{1}{n} \sum_{j=0}^{n-1} \frac{F(x+j)}{t^{[x+j]}} - \frac{1}{n}\frac{F(x)}{t^{[x]}} + \frac{1}{n}\frac{F(x+n)}{t^{[x+n]}}$$

Since the t-norm of F is bounded, we see that the last two terms in the right hand side of the last line above tend to zero as $n \to \infty$. Thus taking the limit as $n \to \infty$, we have

$$\frac{F^*(x+1)}{t^{[x+1]}} = \frac{F^*(x)}{t^{[x]}}$$

which implies that

$$F^*(x+1) = t\, F^*(x)$$

This completes the proof of the theorem.

5.3 Some Special Cases

We shall now consider cases for particular values of r and which are of special interest.

<u>Case I</u> : <u>When r = -1</u>. In this case there are no continuity requirements. Since $u_1, u_2, \ldots, u_k$ is a complete Chebyshev system we can determine $c_1, \ldots, c_{k-1}$ uniquely so that

$$F_1(x) = \sum_{i=1}^{k-1} c_i u_i(x) + u_k(x)$$

is the unique function having the least supremium norm on $[0, 1]$. If $0 \leq x \leq 1$, then

contd..

$$D^{(k-1)} F_1(x) = W_k(x) \text{ by the definition of } S^r_{\Lambda,t}$$

Now, define

$$\begin{aligned} F^*(x) &= F_1(x) & 0 \le x \le 1 \\ &= t^{\nu} F_1(x-\nu) & \nu < x < \nu+1 \end{aligned} \qquad (5.3.1)$$

for all integers ν. Then one can easily verify that $F^*(x)$ defined by (5.3.1) is the unique solution to the problem.

Case II $r = k-2$. In this case

$$S^r_{\Lambda,t} = \Big\{ S \in C^{k-2}(R) : D^{(k-1)} S(x) = t^{\nu} W_k(x) \text{ in } (\nu, \nu+1)$$
for all integers . for all integers ν $\Big\}$

Consider the exponential spline of order k to base t defined in Chapter III and is given by

$$\Phi(x,t) = \sum_{j=-\infty}^{\infty} t^j M(x-j)$$

Clearly, $\Phi(x,t) \in S^{k-2}_{\Lambda,t}$ and satisfies

$$\Phi(x+1,t) = t\Phi(x,t) \text{ for all } x \in R.$$

Let us denote the restriction of $\Phi(x,t)$ to $0 \le x \le 1$ by $\Phi(x)$ and let

$$\varphi_{(k-1)}(x) = D^{(k-1)} \Phi(x)$$

Evidently

$$D_k \varphi_{k-1}(x) = 0$$

so that

$$\varphi_{k-1}(x) = cW_k(x) \text{ in } (0,1)$$

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We have already seen that $C \neq 0$. Without loss of generality we may choose $C = 1$ and then we have

$$\phi_{k-1}(x) = W_k(x) \text{ in } (0, 1)$$

If we define

$$\begin{aligned} F^*(x) &= \Phi(x) \qquad 0 \leq x \leq 1 \\ F^*(x+1) &= tF^*(x) \text{ for all } x \in R \end{aligned} \qquad (5.3.2)$$

then $F^*(x) \in S^{k-2}_{\Lambda,t}$ and is the unique element of $S^{k-2}_{\Lambda,t}$ satisfying

$$F^*(x+1) = tF^*(x) \text{ for all } x \in R$$

<u>Theorem</u> (5.3.1) <u>When $r = k-2$, then</u> $F^*(x)$ <u>defined by</u> (5.3.2) <u>is the unique element of</u> $S^{k-2}_{\Lambda,t}$ <u>that minimizes the t-norm, viz.,</u>

$$\|S\|_{t,\infty} = \sup_{x \in R} \left| \frac{S(x)}{t^{[x]}} \right|$$

<u>Proof</u> : <u>Case (i) $t < 0$.</u> Let $G \in S^{k-2}_{\Lambda,t}$ with $\|G\|_{t,\infty} \leq \|F^*\|_{t,\infty}$. Setting $S(x) = F^*(x) - G(x)$ we see that

$$D^{k-1}S(x) = 0 \text{ in } (\nu, \nu+1) \text{ for all integers } \nu.$$

Also $S \in C^{k-1}(R)$ so that it can be represented as

$$S(x) = \sum_{i=1}^{k-1} C_i \cdot U_i \cdot (x) \text{ for all } x \in R$$

the constants C_i's being uniquely determined. Since $U_1, U_2 \ldots U_k$ is a complete Chebyshev system on R, S has atmost k-2 zeros on R. From the relations

$$F^*(x+1) = tF^*(x)$$

and

$$t < 0$$

it follows that F^* has oscillatory behaviour in successive intervals and hence S must have infinitely many zeros on R. Thus we must have $S(x) \equiv 0$ for all $x \in R$ which proves the result.

<u>Case (ii)</u> $t > 1$. Consider $\Phi(x,t)$ for $0 \le x \le 1$. We know that $\Phi(1,t) = t\,\Phi(0,t)$. We have already proved that $\Phi(0,t)$ is positive for positive t. Since $t > 0$, we conclude that $\Phi(1,t)$ is also positive. Now from the representation.

$$\Phi(x,t) = \sum_{\nu=-\infty}^{\infty} t^{\nu} M(x-\nu)$$

We see that, if $0 < x < 1$, then

$$\Phi(x,t) = \sum_{\nu=-(k-1)}^{0} t^{\nu} M(x-\nu) = t^{-(k-1)} \sum_{j=0}^{k-1} t^j M(x+k-1-j)$$

so that, because $|t| < \infty$, $\Phi(x,t) = 0$ for $0 < x < 1$ implies that

$$\sum_{j=0}^{k-1} t^j M(x+k-1-j) = 0$$

But by Theorem 3.4.2 the equation

$$M(\alpha)\lambda^{k-1} + M(\alpha+1)\lambda^{k-2} + \dots + M(\alpha+k-1) = 0$$

for $0 \le \alpha < 1$ has only simple and negative roots and by theorem 3.4.7 $M(x) > 0$ for $0 < x < k$. Thus if $t > 0$ and $0 < x < 1$ we must have

$$\sum_{j=0}^{k-1} t^j M(x+k-1-j) > 0$$

which shows that $\Phi(x,t) > 0$ for $t > 0$ and $0 < x < 1$. Using the relation $\Phi(x+1,t) = t\,\Phi(x,t)$, we conclude that

$$\Phi(x,t) > 0 \quad \text{for} \quad t > 0 \text{ and } x \in R.$$

Hence $F^*(x)$ defined by (5.3.2) is positive for all $x \in R$. If $F^*(x)/t^{[x]}$ attains its maximum value ρ in $[0,1]$ at x_0, then since this function is periodic with period 1, we have

$$\frac{F^*(x_0-\nu)}{t^{[x_0-\nu]}} = \rho \qquad \text{for all integers } \nu .$$

Now let $G \in S^{k-2}_{\Lambda,t}$ with $\| G \|_{t,\infty} \leq \| F^* \|_{t,\infty} = \rho$ and set $S(x) = F^*(x) - G(x)$ as before. Then it is clear that

$$0 \leq \| S \|_{t,\infty} \leq 2\rho$$

and in particular, we have

$$0 \leq \left| \frac{S(x_0-\nu)}{t^{[x_0-\nu]}} \right| \leq 2\rho \qquad \text{for all integers } \nu .$$

Now $t^{[x_0-\nu]} \to 0$ as $\nu \to \infty$ since $t > 1$. This would imply that $S(x) \equiv 0$. That is $F^*(x) = G(x)$ for all $x \in R$.

<u>Case (iii)</u> $0 < t < 1$. Here again, since $t > 0$, as in the Case (ii), it follows that $F^*(x) > 0$ for all $x \in R$ and that

$$0 \leq \left| \frac{S(x_0-\nu)}{t^{[x_0-\nu]}} \right| \leq 2\rho \qquad \text{for all integers } \nu .$$

Now we may let $\nu \to -\infty$ to obtain $t^{[x_0-\nu]} \to 0$, since $t < 1$ and then we get $S(x) \equiv 0$. Hence $F^*(x) = G(x)$. This completes the proof of Theorem (5.3.1).

<u>Case III</u> When $r = 0$ and $a_j(x) = 0$ for $j = 1, 2, \ldots k-1$ in the definition of Λ so that $\Lambda = D^k$. In this case we easily observe that

$$U_i(x) = x^{i-1} \quad i = 1, 2, \ldots k$$

$$W_j(x) = j-1 \quad j = 1, 2, \ldots k$$

and

$$D_j = \frac{1}{j-1} D \text{ and } D^{(j)} = \frac{1}{\underline{|j-1}} D^j \quad j = 1, 2, \ldots k.$$

We obtain the extremal spline in an explicit form.

<u>Theorem</u> 5.3.2. <u>If $|t| \neq 1$ and</u> $\operatorname{Sgn} t = (-1)^{k-1}$, <u>then the unique</u> t-<u>perfect spline</u> $S(x)$ <u>with the minimum</u> t-<u>norm is given by</u>

$$S(x) = \frac{(1+\alpha)^{k-1}}{2^{2k-3}} \operatorname{Cos}(k-1)\operatorname{Cos}^{-1}\left(\frac{2x+\alpha-1}{1+\alpha}\right) \qquad 0 \le x \le 1$$

$$S(x+1) = tS(x) \text{ for all } x \notin [0, 1]$$

<u>Where</u>

$$\alpha = \operatorname{Cot}^2 \frac{1}{2(k-1)} \operatorname{Cos}^{-1} \frac{1}{t}$$

The proof here reduces to that given by Sharma and Tzimbalario and hence omitted. However, since it reduced to a problem of best approximation for polynomials on the interval $[0, 1]$, we shall give in the next chapter an independent and alternate proof using the method of functionals.

CHAPTER VI

ON A THEOREM OF SHARMA AND TZIMBALARIO.

6.0 In their study of cardinal t perfect splines, Sharma and Tzimbalario [11] proved the following result on best approximation.

Suppose t is a given nonzero real number. If $|t| \neq 1$ and $\operatorname{sgn} t = (-1)^n$ then the monic polynomial $P(x)$ of degree n satisfying $P(1) = tP(0)$ and having the least deviation from zero on $[0,1]$ from the class of all such polynomials is given by

$$P(x) = \frac{(1+\alpha)^n}{2^{2n-1}} \cos n \cos^{-1}\left(\frac{2x+\alpha-1}{1+\alpha}\right) \qquad 0 \leq x \leq 1$$

where

$$\alpha = \cot^2\left(\frac{1}{2n} \cos^{-1} \frac{1}{t}\right)$$

We shall now present an alternate proof of the above result using the method of functionals successfully employed in the study of extremal problems by Voronovskaya [13].

6.1 Method of functionals

We shall now recall the relevant results from the theory of Voronovskaya.

Let $C[0,1]$ denote the class of all continuous real valued functions on $[0,1]$ endowed with the norm

$$\| f \| = \max_{0 \leq x \leq 1} |f(x)| \quad , \quad f \in C[0,1] .$$

Then Riesz representation theorem asserts that every continuous linear functional F on the Banach space $C[0,1]$ has the representation

$$F(f) = \int_0^1 f(t)\,d\alpha(t) \,, \qquad f \in C[0,1] \qquad (6.1.1)$$

where α is a function of bounded variation and

$$\| F \| = \int_0^1 |d\alpha(t)| = \text{total variation of } \alpha \,.$$

It is clear that F is completely determined by the moments

$$\mu_k = \int_0^1 x^k d\alpha(x) \qquad k=0,1,2 \ldots \qquad (6.1.2)$$

In particular, if α happens to be a step function with discontinuities at $\sigma_1, \ldots, \sigma_s$ having the corresponding jumps $\delta_1, \delta_2, \ldots, \delta_s$ then

$$F(f) = \sum_{i=1}^{s} f(\sigma_i)\,\delta_i$$

and

$$\mu_k = \sum_{i=1}^{s} \sigma_i^{\,k}\,\delta_i$$

A function $\phi \in C[0,1]$ is said to be an <u>extremal function</u> for the functional F or for the sequence (6.1.2) if $\max\limits_{[0,1]} |\phi(x)| = 1$ and $F(\phi) = \| F \|$.

The class $\mathcal{P}_n$ of all polynomials of degree atmost n

$$p_n(x) = \sum_{i=0}^{n} a_i \cdot x^i$$

with real coefficients is a finite dimensional subspace of $C[0,1]$. A necessary and sufficient condition that F_n be a linear functional on $\mathcal{P}_n$ is that there exist a set of $(n+1)$ real numbers.

$$\mu_0, \mu_1, \ldots \mu_n \tag{6.1.3}$$

such that

$$F_n(p_n) = \sum_{i=0}^{n} \mu_i \cdot a_i \tag{6.1.4}$$

where $p_n(x) = \sum_{i=0}^{n} a_i \cdot x^i$. Moreover $F_n(x^k) = \mu_k$ for $k=0,1,\ldots n$. Since $\mathcal{P}_n$ is finite dimensional, F_n is always continuous and the norm of F_n is attained for some polynomial Q_n with $\| Q_n \| = 1$ and so the extremal polynomials always exist. By Hahn Banach extension theorem, every continuous linear functional defined on $\mathcal{P}_n$ can be extended to $C[0,1]$ with the preservation of norm. Sometimes we also write $p_n(\bar{\mu})$ for $F_n(p_n)$.

The following results are valid.

Theorem 6.1.1 (Lemma 1, p. 304, [7]). The function $\varphi(x) \equiv \pm 1$ is extremal for F if and only if $\alpha(t)$ is monotonic.

Theorem 6.1.2 (Theorem 1, p.305 [7]; Theorem 1, p.14 [13]). In order that a polynomial $p_n(x) \not\equiv \pm 1$ is extremal for F_n, it is necessary and sufficient that the integrating function $\alpha(t)$ is a step function with a finite number of discontinuities on $[0,1]$. Moreover the points of discontinuity of α are among the points in $[0,1]$ where the extremal polynomial takes the value ± 1. If $\sigma_1, \sigma_2, \ldots \sigma_s$ are the points of dis-

<u>continuities of α (hereafter they will be called nodes) and $\delta_1, \delta_2, \ldots \delta_s$ are the corresponding jumps, then any polynomial p such that</u>

$$\max_{[0,1]} |p(x)| = 1, \quad |p(\sigma_k)| = 1, \quad p(\sigma_k)\delta_k > 0 \qquad k = 1, 2, \ldots s$$

<u>is an extremal polynomial.</u>

If F_n is the segment functional given by (6.1.3) and $\sigma_1, \ldots \sigma_s$ are the points of discontinuity of the integrating function α with the corresponding jumps $\delta_1, \ldots \delta_s$, we have the defining system of linear equations

$$\mu_k = \sum_{i=1}^{s} \sigma_i^k \delta_i \qquad k=0,1,2,\ldots n \quad (6.1.5)$$

If the number s of nodes is greater than $\frac{n}{2}+1$, then the extremal polynomial is unique (Theorem 4, p. 308 [7]). Moreover in this case (when $s > \frac{n}{2} + 1$), we say that the segment (6.1.3) is of class II.

Let Q_n be a polynomial of class II. We pick on $[0,1]$ all the points $(\sigma_i)^s$, at which $Q_n(x) = \pm 1$ and associate with each σ_i the sign + or - according as $Q_n(\sigma_i) = +1$ or -1. An interval between the nodes (σ_k, σ_{k+1}) is an interval of repetition (alternation) if the signs associated with σ_k and σ_{k+1} are identical (different). If p and q respectively denote the number of these intervals, then $p + q = s-1$. Noticing that in any interval of repetition, there is a point of extrema of Q_n which means that there is a zero of the derivative, it follows that

$$s + p \leq n + 1 \qquad (6.1.6)$$

The triple $[n, s, p]$ is called the passport of the polynomial Q_n.

Consider the class $\{ Q_n(x) \}$ of polynomials of of class II. The Chebyshev polynomials $\pm T_n(x)$ are the only ones from this class of passport $[n, n+1, 0]$. We recall that the Chebyshev polynomials on $[0,1]$ are defined by

$$T_n(x) = \cos n \arccos(2x-1) \qquad 0 \leq x \leq 1$$

However from the polynomials $Q_n(x)$ of class II, one can obtain a family $Q_n(ax+b)$ of such polynomials with $0 \leq ax + b \leq 1$ for $0 \leq x \leq 1$ which also imply that $|a| \leq 1$ and $0 \leq b \leq 1$. Such a construction is known as a transformation.

<u>Definition</u> 6.1.3 : A polynomial Q_n of class II is called <u>primitive</u> if

$$|Q_n(0)| = 1 \; ; \quad |Q_n(0-\varepsilon)| > 1$$
$$|Q_n(1)| = 1 \; ; \quad |Q_n(1+\varepsilon)| > 1$$

for all sufficiently small $\epsilon > 0$. If the conditions of primitivity holds only at one end of the interval, the polynomial is said to be _semiprimitive_.

Theorem 6.1.4 (Theorem 38, p. 82 [13]). _All polynomials of passport_ $[n,n,0]$ _or_ $[n,n,1]$ _are either primitive or else semiprimitive Chebyshev transformations_.

There is a close connection between the extremal polynomials of segment functionals and the polynomials of least deviation whose coefficients satisfy linear relations.

Theorem 6.1.5 (Theorem 55, p. 133 [13]). _If among the polynomials_ $\{P_n(x)\}$ _of degree atmost n with real coefficients subjection to the condition_

$$\sum_{i=0}^{n} p_i \alpha_i = A \quad (>0)$$

where p_i _are the coefficients and_ $(\alpha_i)_0^n$ _and A are given real numbers._ $Y_n(x)$ _denotes the polynomial of least deviation on_ $[0,1]$ _and_ $Q_n(x)$ _is the extremal polynomial for the segment functional_ $(\alpha_i)_0^n$, _then_

$$Q_n(x) = \frac{Y_n(x)}{L} \quad \text{and} \quad N = \frac{A}{L}$$

where N is the norm of the functional and L denotes the deviation of $Y_n(x)$.

Similarly among the polynomials $\{P_n(x)\}$ whose coefficients satisfy two consistent conditions
$\sum_{i=0}^{n} p_i \mu_i = A$ and $\sum_{i=0}^{n} p_i \nu_i = B$, the one deviating least from zero on $[0,1]$ is given by

<u>Theorem 6.1.6</u> (Theorem 56, p.138 [13]).
<u>If</u> $A \neq 0$ <u>and</u> $B \neq 0$ <u>and if</u> $Q_n(x,\Omega)$ <u>is the family of extremal polynomials of the segment</u> $\mu_0 + \Omega\nu_0, \mu_1 + \Omega\nu_1, \ldots\ldots, \mu_n + \Omega\nu_n$ <u>with</u> $-\infty < \Omega < \infty$ <u>and</u> $L_n(x)$ <u>with deviation</u> M_L <u>is the required polynomial, then a necessary and sufficient condition that</u> $\frac{L_n(x)}{M_L}$ <u>belongs to the</u> $Q_n(x,\Omega)$ <u>is as follows</u> :
<u>The equation</u>

$$\frac{Q_n(\bar{\mu},\Omega)}{A} = \frac{Q_n(\bar{\nu},\Omega)}{B}$$

<u>has atleast one real root</u> $\Omega = \Omega_0$. <u>In that case</u>

$$\frac{L_n(x)}{M_L} = Q_n(x,\Omega_0)$$

<u>Remark</u> 6.1.7 : Theorem 6.1.6 cannot be applied if one of A and B is zero. In that case, let

$$\sum_{i=0}^{n} p_i \mu_i = A(\neq 0) \; ; \quad \sum_{i=0}^{n} p_i \nu_i = 0 \; ;$$

putting $\tau_i = \mu_i + \nu_i$, we replace these two conditions by the equivalent conditions

$$\sum_{i=0}^{n} p_i \mu_i = A \quad \text{and} \quad \sum_{i=0}^{n} p_i \tau_i = A$$

Then Theorem 6.1.6 is applicable.

6.2. Reduction to extremal problem

Our problem is to obtain a monic polynomial of degree n having the least deviation from zero on $[0,1]$ subject to the conditions that $P(1) = tP(0)$ where $\operatorname{sgn} t = (-1)^n$. We shall first convert this problem into a problem of finding the extremal polynomial of a segment functional. The two conditions $P(1) = tP(0)$ and the coefficient of x^n is 1 give the two linear relations

$$p_0 + p_1 + \ldots\ldots + p_n = tp_0 \qquad (6.2.1.)$$

$$p_n = 1 \qquad (6.2.2)$$

These are given by the segment functionals $(1-t, 1, 1, \ldots 1)$ and $(0, 0, \ldots 0, 1)$ respectively. By the remark 6.1.7 above, we replace the conditions (6.2.1) and (6.2.2) by equivalent conditions.

$$(1-t)p_0 + p_1 + \ldots + 2p_n = 1$$

$$p_n = 1$$

The corresponding segment functionals are given by

$$(\tau_i)_0^n : (1-t, 1, \ldots, 1, 2)$$

and

$$(\mu_i)_0^n : (0, \ldots\ldots 0, 1)$$

We are now in a position to apply Theorem 6.1.6. Set the new segment functional

$$(\tau_i + \omega\mu_i)_0^n ; \quad (1-t, 1, \ldots 1, 2+\omega)$$

Let Q_n be the extremal polynomial of this segment. We shall show that the passport of this polynomial is necessarily $[n, n, 0]$.

Suppose

$$Q_n(x, \omega) = \sum_{i=0}^{n} q_i(\omega) x^i .$$

Since this is extremal for the functional $F = (1-t, 1, \ldots 1, 2+\omega)$, we have

$$(1-t)q_0 + q_1 + \cdots + q_{n-1} + (2+\omega)q_n = \| F \| \quad (6.2.3)$$

Applying (6.2.1) to $Q_n(x,\omega)$ we get

$$(1-t)q_0 + q_1 + \cdots + q_{n-1} + q_n = 0 \quad (6.2.4)$$

From (6.2.3) and (6.2.4) we obtain

$$\| F \| = (1+\omega)q_n > 0 \quad (6.2.5)$$

which implies that $\omega \neq -1$.

Let $\sigma_1, \sigma_2, \ldots \sigma_s$ be the nodes of Q_n and $\delta_1, \ldots \delta_s$ be the corresponding jumps of the integrating function in the representation of F. If $s = n+1$, since Q_n is a polynomial of degree n, both 0 and 1 are necessarily its nodes. But from (6.2.4), $Q_n(1) = t\, Q_n(0)$ and $|t| \neq 1$ by hypothesis. We cannot have therefore both 0 and 1 as nodes at the sametime. This excludes the case $s = n+1$. If $s = n$, then by (6.1.6), the polynomial may be of passport $[n,n,1]$ or $[n,n,0]$. Every polynomial of passport $[n,n,1]$ has both 0 and 1 as nodes (see [13] p. 103) and therefore Q_n cannot be of passport $[n,n,1]$ also. We now claim that the passport of Q_n is $[n,n,0]$. It is enough to prove $s \nleq n-1$

To see this, we consider the defining system of linear equations

$$\begin{array}{lll}
\delta_1 + \cdots & + \delta_s & = 1-t \\
\sigma_1\delta_1 + & + \sigma_s\delta_s & = 1 \\
\cdots\cdots\cdots\cdots & & \\
\sigma_1^{n-1}\delta_1 + \cdots & + \sigma_s^{n-1}\delta_s & = 1 \\
\sigma_1^{n}\delta_1 + \cdots & + \sigma_s^{n}\delta_s & = 2+\omega.
\end{array} \qquad (6.2.6)$$

<u>Case (i)</u> : $0<\sigma_1<\sigma_2<\cdots<\sigma_s<1$. In this case all nodes lie in the open interval $(0,1)$ and since each node is a point of extrema and hence a zero of the derivative of the extremal polynomial which is of degree n, we must have $s \leq n-1$.

Excluding the first and the last, there are $(n-1)$-equations in (6.2.6) in s unknowns, $\delta_1, \ldots \delta_s$. If $s \leq n-2$, we can eliminate $\delta_1 \ldots \delta_s$ from $(s+1)$ of these equations and the resulting determinant should be zero if these equations are consistent. But eliminant is a

Vandermonde determinant with entries in the last column being 1. This determinant cannot be zero because all the σ_i's lie in (0,1).

On the other hand, if $s = n-1$, eliminating $\delta_1, \ldots, \delta_{n-1}$ from the first n equations of (6.2.6), we obtain

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1-t \\ \sigma_1 & \sigma_2 & & \sigma_{n-1} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_1^{n-1} & \sigma_2^{n-1} & \cdots & \sigma_{n-1}^{n-1} & 1 \end{vmatrix} = 0$$

Expanding by the elements of the last column and simplifying we get

$$\prod_{i>j} (\sigma_i - \sigma_j) \left[\prod_i (1-\sigma_i) + (-1)^n t \sigma_1 \cdots \sigma_{n-1} \right] = 0$$

This is not possible because $(-1)^n t > 0$ and $0 < \sigma_1 < \cdots < \sigma_{n-1} < 1$. Thus the system of equations (6.2.6) is not consistent when $s \leqslant n-1$.

<u>Case (ii)</u> : $0 = \sigma_1 < \sigma_2 < \cdots < \sigma_s < 1$. The system of equations (6.2.6) now reduces to

$$\begin{aligned} \delta_1 + \delta_2 + \cdots + \delta_s &= 1-t \\ \sigma_2 \delta_2 + \cdots + \sigma_s \delta_s &= 1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ \sigma_2^{n-1} \delta_2 + \cdots + \sigma_s^{n-1} \delta_s &= 1 \\ \sigma_2^n \delta_2 + \cdots + \sigma_s^n \delta_s &= 2 + \omega \end{aligned} \qquad (6.2.7)$$

If $s \leq n-1$, then eliminating the $(s-1)$ unknowns $\delta_2 \dots \delta_s$ from the first s of the middle $(n-1)$ equations in (6.2.7), we get

$$\begin{vmatrix} \sigma_2 & \cdots & \sigma_s & 1 \\ \sigma_2^2 & \cdots & \sigma_s^2 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sigma_2^s & \cdots & \sigma_s^s & 1 \end{vmatrix} = 0$$

which gives a contradiction, since the above determinant is nonvanishing.

<u>Case (iii)</u> : $0 < \sigma_1 < \sigma_2 \;\; \dots < \sigma_s = 1$. In this case, (6.2.6) becomes

$$\begin{aligned} &\delta_1 + \delta_2 + \quad + \delta_{s-1} + \delta_s = 1-t \\ &\sigma_1\delta_1 + \sigma_2\delta_2 + \quad + \sigma_{s-1}\delta_{s-1} + \delta_s = 1 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (6.2.8) \\ &\sigma_1^{n-1}\delta_1 + \sigma_2^{n-1}\delta_2 + \dots + \sigma_{s-1}^{n-1}\;{}_{s-1} + \delta_s = 1 \\ &\sigma_1^{n}\delta_1 + \sigma_2^{n}\delta_2 + \dots + \sigma_{s-1}^{n}\delta_{s-1} + \delta_s = 2+\omega \end{aligned}$$

If $s \leq n-1$, then the middle $(n-1)$ equations of (6.2.8) give the unique solution

$$\delta_1 = \delta_2 = \dots = \delta_{s-1} = 0 \text{ and } \delta_s = 1.$$

Substituting this in the last equation we see that $\omega = -1$. But using (6.2.5) we observed that $1+\omega \neq 0$. We have therefore a contradiction.

We thus conclude that for the extremal polynomial Q_n to exist we must have $s \geq n$ and our claim is established.

6.3 The solution

We shall prove the following theorem.

Theorem 6.3.1 . Suppose t is a given nonzero real number. If $|t| \neq 1$ and $\operatorname{sgn} t = (-1)^n$ then the monic polynomial $P(x)$ of degree n satisfying $P(1) = tP(0)$ and having the least deviation from zero on $[0,1]$ from the class of all such polynomials is given by

$$P(x) = \frac{(1+\alpha)^n}{2^{2n-1}} \cos n \cos^{-1}\left(\frac{2x+\alpha-1}{1+\alpha}\right) \quad 0 \leq x \leq 1$$

where

$$\alpha = \cot^2\left(\frac{1}{2n}\cos^{-1}\frac{1}{t}\right)$$

Proof : We have already seen that the corresponding extremal polynomial is of passport $[n,n,0]$ and hence can be obtained by a semiprimitive Chebyshev tranformation. Let $Q(x)$ be the required extremal polynomial. Then $Q(x) = T_n(ax+b)$ where for $0 \leq x \leq 1$, we have $0 \leq ax+b \leq 1$, $|a| \leq 1$ and $0 \leq b \leq 1$. Since 1 is a node for Q_n, it follows that $a + b = 1$. Then

$$Q(x) = T_n(ax+1-a) = \cos n \cos^{-1}(2ax-2a+1)$$

The boundary condition $Q(1) = t\,Q(0)$ gives

$$1 = t \cos n \cos^{-1}(1-2a).$$

from which we deduce that

$$a = \sin^2 \frac{1}{2n}(\cos^{-1}\frac{1}{t})$$

Since $0 < a < 1$, we can take a to be of the form

$a = \frac{1}{1+\alpha}$ with $\alpha > 0$. Then

$$\alpha = \frac{1-a}{a} = \frac{1-\sin^2(\frac{1}{2n}\cos^{-1}\frac{1}{t})}{\sin^2(\frac{1}{2n}\cos^{-1}\frac{1}{t})} = \cot^2(\frac{1}{2n}\cos^{-1}\frac{1}{t})$$

and

$$2ax - 2a + 1 = \frac{2x}{1+\alpha} - \frac{2}{1+\alpha} + 1 = \frac{2x+\alpha-1}{1+\alpha}$$

Hence

$$Q(x) = \cos n \cos^{-1}\left(\frac{2x+\alpha-1}{1+\alpha}\right) \qquad 0 \le x \le 1,$$

where

$$\alpha = \cot^2(\frac{1}{2n}\cos^{-1}\frac{1}{t}).$$

Now the coefficient of x^n in $Q(x)$

$$= \lim_{x\to\infty} \frac{Q(x)}{x^n}$$

$$= \lim_{x\to\infty} \frac{1}{2x^n}\left[\frac{2x+\alpha-1}{1+\alpha} + \sqrt{\left(\frac{2x+\alpha-1}{1+\alpha}\right)^2 - 1}\right]^n + \left[\frac{2x+\alpha-1}{1+\alpha} - \sqrt{\left(\frac{2x+\alpha-1}{1+\alpha}\right)^2 - 1}\right]^n$$

$$= \frac{2^{2n-1}}{(1+\alpha)^n}$$

Hence the required monic polynomial P(x) is given by

$$P(x) = \frac{(1+\alpha)^n}{2^{2n-1}} \cos n \cos^{-1}\left(\frac{2x+\alpha-1}{1+\alpha}\right) \quad 0 \le x \le 1$$

where

$$\alpha = \cot^2\left(\frac{1}{2n} \cos^{-1} \frac{1}{t}\right)$$

We remark that though $\cos^{-1} \frac{1}{t}$ is many valued, it can be easily seen that there is exactly one value of $\cos^{-1} \frac{1}{t}$ and hence α for which $Q(x)$ has exactly n nodes in $[0,1]$. For the other values of $\cos^{-1} \frac{1}{t}$, either the corresponding nodes repeat or lie outside $[0,1]$.

Remark 6.3.2. We now consider also the case when $\operatorname{sgn} t \neq (-1)^n$. If

$$\prod_i (1-\sigma_i) + (-1)^n t \sigma_1 \cdots\cdots \sigma_{n-1} \neq 0$$

(See case (i)) the system of equations (6.2.6) is inconsistent and then s = n. In particular, we saw that if $\operatorname{sgn} t = (-1)^n$ then s = n.

If

$$\prod_i (1-\sigma_i) + (-1)^n t \sigma_1 \cdots\cdots \sigma_{n-1} = 0$$

they system of equations (6.2.6) remain consistent with s = n-1. $\sigma_1, \cdots\cdots \sigma_{n-1}$ can be found from the middle n-1 equations and ω and t are then given by

$$\omega = -1 - \prod_i (1-\sigma_i) \text{ and } t = (-1)^{n-1} \frac{\prod_i (1-\sigma_i)}{\sigma_1 \sigma_2 \cdots \sigma_{n-1}}$$

If $[n, s, p]$ is the passport of the polynomial, we have $s = n-1$ and $p = 0, 1$ or 2. Since all the nodes lie in the open interval $(0, 1)$ which are also the zeros of the derivative of a polynomial of degree n, the cases $p=1$ and $p=2$ will provide more than $(n-1)$-zeros for the derivative which is a polynomial of degree $n-1$. Hence p must be zero and the extremal polynomial is of passport $[n, n-1, 0]$.

Thus when $\operatorname{sgn} t = (-1)^n$ or when case (i) holds, the extremal polynomial is of passport $[n, n, 0]$. When $\operatorname{sgn} t \neq (-1)^n$ and the case (i) does not hold, we must have $[n, n-1, 0]$ as the passport of the extremal polynomial. In this case neither 0 nor 1 is a node. Setting $P(0) = \delta$ and $P(1) = t\delta$ where $|\delta| < 1$ and $|t\delta| < 1$ we have the linear relations among the coefficients given by

$$p_0 = \delta$$

$$p_0 + p_1 + \cdots + p_n = t\delta$$

$$p_n = 1$$

Here we have the case of Ahiezer's polynomials. By virtue of Theorem 57 ([13] p. 143) and the remark in para 3 from below in ([13], p.123) we see that the extremal polynomial is given by the bilateral Chebyshev transformation $-T_n(\alpha x + \beta)$ with $(n-1)$ nodes. That is

$$Q(x) = -T_n(\alpha x + \beta)$$

where for $0 \le x \le 1$, $0 \le \alpha x + \beta \le 1$, $|\alpha| \le 1$, $0 \le \beta \le 1$.

Now $Q(0) = -\delta$ gives

$$\delta = \cos n \cos^{-1}(2\beta - 1)$$

which gives

$$2\beta - 1 = \cos \frac{1}{n} \cos^{-1} \delta.$$

Since $Q(1) = t\,Q(0)$, we have

$$-t\delta = -\cos n \cos^{-1}(2\alpha + 2\beta - 1)$$

which gives

$$2\alpha + 2\beta - 1 = \cos \frac{1}{n} \cos^{-1} t\delta$$

Thus

$$2\alpha = \cos \frac{1}{n} \cos^{-1} t\delta - \cos \frac{1}{n} \cos^{-1} \delta$$

Hence

$$Q(x) = -\cos n \cos^{-1} \left(2\alpha x + \cos\left(\frac{1}{n} \cos^{-1} \delta\right)\right)$$

Where

$$\alpha = \frac{1}{2} \left[\cos\left(\frac{1}{n} \cos^{-1} t\delta\right) - \cos \frac{1}{n} \cos^{-1} \delta\right]$$

Since the coefficient of x^n in $Q(x)$ is $2^{2n-1} \alpha^n$ we see that the required polynomial $P(x)$ is given by

$$P(x) = -\frac{1}{\alpha^n 2^{2n-1}} \cos n \cos^{-1} \left(2\alpha x + \cos\left(\frac{1}{n} \cos^{-1} \delta\right)\right)$$

CHAPTER VII

ON A RESULT OF BOAS

7.0 Boas [1] considered the following extremal problem.

Let $f(x)$ be a trigonometric polynomial and consider a linear functional L defined by

$$L(f) = \sum_{\nu=1}^{m} \sum_{j=0}^{n_\nu} \alpha_\nu^{(j)} f^{(j)}(x_\nu)$$

where $x_\nu, \alpha_\nu^{(j)}$ are given real numbers, $0 \leq x_\nu < 2\pi$ with the x_ν all different. Suppose further that $\alpha_\nu^{(n_\nu)} \neq 0$ and that $n_\nu > 0$ for atleast one ν. Then

$$\ell = n_1 + n_2 + \dots + n_m + m$$

is called the order of L. The problem is to determine f which maximizes $|L(f)|$ as f runs through the class of trigonometric polynomials of type n which satisfy $|f(x)| \leq 1$ for real x. A trigonometric polynomial is of type n if it is of order atmost n and a trigonometric polynomial of type n is an entire function of exponential type n.

Using variational methods, Boas obtained a solution to the above problem and then applied to the special functional $\lambda n^2 f(0) + f''(0)$. In looking for the maximum, it was further pointed out that one has to look for polynomials which are real for real x.

We shall below consider the functional

$$L(f) = \lambda n^2 f(0) + f''(0)$$

and prove the result of Boas using the method of functionals as in Chapter VI.

7.1 Method of functionals for trigonometric polynomials

Let C_n denote the space of all trigonometric polynomials of order n, with real coefficients

$$t_n(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta) \qquad (7.1.1)$$

with the metric of the space C of continuous periodic functions of period 2π. The general form a linear functional L on C_n is given by

$$L(t_n) = \sum_{k=0}^{n} a_k \lambda_k + \sum_{k=1}^{n} b_k \mu_k \qquad (7.1.2)$$

By the finite dimensionality of C_n, such an L is always bounded.

Considering C_n as a subspace of $C[0,2\pi]$, using Hahn Banach extension theorem we see that every bounded linear functional L on C_n can be extended as a bounded linear functional on $C[0,2\pi]$ with preservation of norm. Riesz theorem then says that L can be represented in the form

$$L(t) = \int_0^{2\pi} t(\theta)\, d\mu(\theta)$$

where μ is a function of bounded variation on $[0,2\pi]$.

A polynomial $T \in C_n$ is extremal for L if $\|T\| = 1$ and $L(T) = \|L\|$.

Then a necessary and sufficient condition that a polynomial $T \in C_n$, $T \not\equiv$ a constant, be extremal for L is that there exists a representation of L of the form

$$L(t) = \sum_{p=1}^{N} A_p t(\theta_p) \qquad (7.1.3)$$

where $\theta_p (p=1,2,\cdots,N)$ are the points in $[0,2\pi)$ at which $|T(\theta)| = 1$ and

$$A_p T(\theta_p) \geqslant 0 \qquad (7.1.4)$$

(See [12] Lemma 4, p. 14 Equation 1.14, p.18 or [8] Theorem 1, p. 260).

If $t_p \in C_n$ such that $t_p(\theta_p) = 1$ and $t_p(\theta_j) = 0$ for $j \neq p$, we see that $A_p = L(t_p)$ and (7.1.4) becomes $L(t_p)T(\theta_p) \geqslant 0$.

7.2 Result of Boas

Consider the linear functional L defined by

$$Lf = \lambda n^2 f(0) + f''(0)$$

We are to find the maximum of $|Lf|$ when f runs through the class C_n of trigonometric polynomials of order not exceeding n and satisfying the relation $|f(x)| \leq 1$ for all real x. As is pointed out earlier, one has to consider only real trigonometric polynomials. The result of Boas may be stated as follows.

<u>Theorem</u> 7.2.1. <u>Let</u> $L(f) = \tau n^2 f(0) + f''(0)$. <u>If</u> $n \geq 1$, <u>the largest value of</u> $|L(f)|$ <u>for</u> $f \in C_n$ <u>is furnished by</u> $\pm \cos n\theta$ <u>if</u> $\tau \leq \frac{1}{3} + \frac{1}{6n^2}$ <u>and by a function of the form</u>

$$\pm \cos\ n \cos^{-1}(\omega \cos\theta + \omega - 1) \quad , \quad 0 < \omega < 1 \qquad (7.2.1)$$

<u>if</u> $\tau > \frac{1}{3} + \frac{1}{6n^2}$

<u>Proof of Theorem 7.2.1</u> : <u>First Part</u>

Consider the linear function

$$L(f) = \tau n^2 f(0) + f''(0).$$

Suppose $t(\theta)$ is given by (7.1.1). Then

$$L(t) = \sum_{k=0}^{n} (\tau n^2 - k^2) a_k$$

This is of the form (7.1.2) where

$$\tau_k = \tau n^2 - k^2 \qquad k = 0, 1, \ldots n$$

and

$$\mu_k = 0 \qquad k = 1, 2, \ldots n$$

The trigonometric polynomial $-\cos n\theta$ is extremal for L if $\|L\| = L(-\cos n\theta) = n^2(1-\tau)$. We shall prove that this is indeed the case if $\tau \leq \frac{1}{3} + \frac{1}{6n^2}$ using condition (7.1.4).

Let $T(\theta) = -\cos n\theta = \cos(n\theta - \pi)$ so that

$$\theta_p = \frac{(p+1)\pi}{n} \qquad p = 0, 1, 2, \ldots 2n-1.$$

and

$$T(\theta_p) = (-1)^p.$$

Let

$$\phi(\theta) = \frac{\sin n\theta}{2n\tan \theta/2} = \frac{1}{n}\left[\frac{1}{2} + \sum_{k=1}^{n-1} \cos k\theta + \tfrac{1}{2}\cos n\theta\right]$$

Then ϕ is a trigonometric polynomial of order n such that $\phi(0) = 1$ and $\phi(\theta_p) = 0$ for $p = 0, 1, 2, \dots 2n-2$. if $t_p(\theta) = \phi(\theta - \theta_p)$, then t_p is a trigonometric polynomial such that

$$t_p(\theta_p) = 1 \text{ and } t_p(\theta_j) = 0 \text{ for } j \neq p.$$

Now

$$A_p = L(t_p) = \lambda n^2\, t_p(0) + t_p''(0) = \lambda\, n^2 \phi(\theta_p) + \phi''(\theta_p)$$

(i) When $p = 0, 1, 2, \dots 2n-2$, we have $\phi(\theta_p) = 0$

and

$$\phi''(\theta_p) = \frac{n(-1)^p}{2\sin^2 \frac{\theta_p}{2}}$$

so that

$$A_p = \frac{n(-1)^p}{2\sin^2 \frac{\theta_p}{2}}$$

which implies that

$$A_p T(\theta_p) > 0$$

(ii) When $p = 2n-1$, we have $\theta_p = 2\pi$ and so

$$\phi(\theta_p) = 1$$

and

$$\phi''(\theta_p) = \frac{1}{n}\left[-\sum_{k=1}^{n-1} k^2 - \frac{n^2}{2}\right] = -\frac{2n^2+1}{6}$$

which gives

$$\lambda_p = \gamma n^2 - \frac{2n^2+1}{6}$$

so that the condition $\lambda_p T(\theta_p) \geqslant 0$ gives

$$(-1)^{2n-1}\left(\gamma n^2 - \frac{2n^2+1}{6}\right) \geqslant 0$$

or equivalently

$$\gamma \leq \frac{1}{3} + \frac{1}{6n^2}$$

Thus a necessary and sufficient condition for $-\cos n\theta$ to be extremal for L is $\gamma \leq \frac{1}{3} + \frac{1}{6n^2}$.

<u>Second Part of the theorem</u>: since $\gamma > \frac{1}{3} + \frac{1}{6n^2}$ we see that $\pm \cos n\theta$ cannot be extremal for L. A look at the theorem suggests that the solution may be obtained by means of Chebyshev transformation. As pointed out by Boas ([1] p.6) we need confine only to even extremal functions. An even trigonometric polynomial is of the form

$$t(\theta) = \sum_{k=0}^{n} a_k \cos k\theta = C_n(\theta)$$

a function of $\cos\theta$ alone. There is also a one-to-one correspondence between $\{ Q_n(x) \}$ of class II on the interval $[0,1]$ and the extremal polynomials $C_n(\theta)$ given by the following theorem.

Theorem 7.2.2 (Theorem 1, [14]) : If $\{Q_n(x)\}$ is the set of polynomials of class II ($\max_{0 \le x \le 1} |Q_n(x)| = 1$ and the number of nodes $s > \cancel{\tfrac{1}{2}}n+1$) then between each polynomial $Q_n(x)$ and each polynomial $C_n(\theta)$ with number of nodes $s^* > n$ on $(-\pi, \pi]$ it is possible to establish a one to one correspondence

$$Q_n\left(\frac{1+\cos\theta}{2}\right) = C_n(\theta) \quad \text{on} \quad [0, \pi]$$

$$C_n(-\theta) = C_n(\theta) \text{ for } 0 \le \theta \le \pi$$

$$C_n\left(\cos^{-1}(2x-1)\right) = Q_n(x) \quad 0 \le x \le 1.$$

We shall therefore convert our extremal problem in trigonometric polynomials into a problem for algebraic polynomials. It also turns out that the corresponding segment functional is easier to handle.

Making the substitution $x = \dfrac{1-\cos\theta}{2}$ and setting $t(\theta) = P(x)$ we see that

$$\lambda n^2 t(0) + t''(0) = \lambda n^2 P(0) + \tfrac{1}{2}P'(0)$$

If $P(x) = \sum_{i=0}^{n} p_i \cdot x^i$, then

$$L(P) = \lambda n^2 P(0) + \tfrac{1}{2}P'(0) = \lambda n^2 p_0 + \tfrac{1}{2}p_1 \qquad (7.2.2)$$

This is represented by the segment functional

$$(\lambda n^2, \tfrac{1}{2}, 0, \ldots, 0) \qquad (7.2.3)$$

Let us now consider this segment functional in detail. If s is the number of nodes of this segment functional in $[0,1]$, then $s \leq n+1$. Suppose $0 \leq \sigma_1 < \dots < \sigma_s \leq 1$ are the nodes in $[0,1]$ and $\delta_1, \dots, \delta_s$ are the jumps at these nodes of the integrating function in the representation of the functional. The defining system of linear equations is then given by

$$\sum_{j=1}^{s} \sigma_j^{k} \delta_j = \tau_k \qquad k = 0,1, \dots n \qquad (7.2.4)$$

where

$$(\tau_k)_0^n : \quad (\tau n^2, \tfrac{1}{2}, 0, \dots 0).$$

It is easily seen that (7.2.4) is inconsistent for $s \leq n-1$. Thus $s = n$ or $n+1$. The case $s = n+1$ gives the Chebyshev polynomial which is considered in the first part of the theorem. Thus we need to consider only the case $s = n$. The extremal polynomial $P(x)$ for the segment functional $(\tau n^2, \tfrac{1}{2}, 0, \dots 0)$ is then of passport $[n,n,0]$ or $[n,n,1]$ and hence is obtained by a Chebyshev transformation. If $\sigma_1 = 0$ then the system of equations (7.2.4) can be easily seen to be inconsistent. In fact, when $\sigma_1 = 0$, the system of equations (7.2.4) can be written as

$$\delta_1 + \delta_2 + \dots + \delta_n = \tau n^2$$

$$\sigma_2 \delta_2 + \dots + \sigma_n \delta_n = \tfrac{1}{2}$$

$$\sigma_2^k \delta_2 + \dots + \sigma_n^k \delta_n = 0 \qquad k = 2, 3, \dots n$$

The last (n-1) homogeneous equations in (n-1) unknowns with nonvanishing determinant give the trivial solution viz.,

$$\delta_2 = \delta_3 = \dots = \delta_n = 0$$

This solution does not satisfy the second equation. We thus conclude that the nodes are given by

$$0 < \sigma_1 < \sigma_2 < - \dots < \sigma_n \leq 1.$$

If T denotes the nth Chebyshev polynomial on $[0,1]$, then the extremal polynomial P(x) is given by

$$P(x) = T(\omega x + \beta)$$

where $|\omega| \leq 1$, $0 \leq \beta \leq 1$ and $0 \leq \omega x + \beta \leq 1$ for $0 \leq x \leq 1$. Since the open interval (0,1) contains atmost (n-1) nodes of P(x) which are the zeros of the derivative P'(x), we should have $\sigma_n = 1$ so that $\beta = 1 - \omega$. Thus $P(x) = T(\omega x + 1 - \omega)$ where $0 \leq \omega \leq 1$ and it will be unique for $n \geq 3$.

If $\omega = 0$, the extremal polynomial is a constant and when $\omega = 1$ the extremal polynomial is the Chebyshev polynomial. If $0 < \omega < 1$, the extremal polynomial is

$$P(x) = T(\omega x + 1 - \omega) = (-1)^n T\big(\omega(1-x)\big)$$

Since P(x) is extremal for (7.2.3), the extremal polynomial for L is

$$P\left(\frac{1-\cos\theta}{2}\right) = (-1)^n T\left(\omega \frac{1+\cos\theta}{2}\right)$$

$$= (-1)^n \cos n \cos^{-1}(\omega \cos\theta + \omega - 1)$$

We have already seen that $\pm \cos n\theta$ is extremal for L if and only if $\lambda \leq \frac{1}{3} + \frac{1}{6n^2}$. Hence if

$$T(\theta) = \pm \cos n \cos^{-1}(\omega \cos \theta + \omega - 1) \quad 0 < \omega < 1$$

is extremal for L we should have $\lambda > \frac{1}{3} + \frac{1}{6n^2}$.

We also remark that in order to have exactly n nodes in $[0, 1]$, the inequality

$$\cos^2 \frac{\pi}{2n} < \omega < 1$$

should be satisfied.

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