A STUDY OF MULTIPLIER SPACES FOR SEGAL ALGEBRAS

THESIS

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INTRODUCTION

This thesis deals with the study of multiplier spaces of Segal algebras. Special attention is paid to the Segal algebras on the real line. In this context, the method of characterising multipliers of various classes of functions on the real line given by Doss [4] appears to be a relevant approach. On the real line, the classical Wiener algebra W , a similar class V, introduced by Burnham and Goldberg [2] , and a new Segal algebra D will be of special interest to us. Complete characterisations and relationships of several multiplier spaces among the spaces W,V,D and the classes F_i , i = 1,2,3,4,5,6 introduced by Doss [4] , are given here. Observing that the space V is actually a normed ideal containing the Wiener algebra W and proving that V is the dual of a certain Banach space, we are motivated to formulate a result on the space of multipliers for Segal algebras and the normed ideals. When the normed ideal is a dual space, the space of multipliers into the corresponding Segal algebra turns out to be a dual space. Many interesting special cases are deduced. Using a completion technique, we characterize the multipliers from a Segal algebra into $L^1(G)$, where G is a locally compact abelian group, and the result is given in a different form on the real line . Finally, we have considered a generalization for the multiplier on translation invariant spaces.

While dealing with the Segal algebras on the real line, we have chosen to employ very simple techniques in the proofs and have found them nevertheless effective to produce worth-while results. In the characterization of multiplier spaces of Segal algebras as dual spaces, the main result follows from simple arguments. Moreover it turns out that our method not only simplifies and unites various known results proved using different techniques, but also generalizes to envelop a larger domain of multiplier results.

Before moving on to expose the chapterwise summary we shall collect the necessary material to provide the background.

Let G denote a locally compact abelian group and $\hat{G}$ its character group. We denote by dx and $d\gamma$ the respective normalised Haar measures on G and $\hat{G}$. $C_c(G)$ stands for the space of all continuous functions on G with compact support, while $C_0(G)$ is the space of continuous functions which vanish at infinity. For $1 \leq p < \infty$, $L^p(G)$ is the Lebesgue space of equivalence classes of complex valued functions f on G such that

$$\|f\|_p = \left(\int_G |f(x)|^p \, dx \right)^{1/p} < \infty .$$

$L^\infty(G)$ denotes the space of all essentially bounded functions f on G and

$$\|f\|_\infty = \underset{x \in G}{\text{ess. sup.}} |f(x)| .$$

M (G) denotes the convolution measure algebra of bounded regular Borel measures on G . We use * to denote the convolution operation. For example, let $f, g \in L^1(G)$,

then

$$f * g(x) = \int_G f(x-y)\, g(y)\, dy$$

where the group operation on G is given by + .

The Fourier stieltjes transform of a measure $\mu \in M(G)$ is given by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma)\, d\mu(x) \quad , \quad \gamma \in \hat{G} .$$

and the Fourier transform of a function f in L^1 (G) is given by

$$\hat{f}(\gamma) = \int_G (-x, \gamma)\, f(x)\, dx \quad , \quad \gamma \in \hat{G}$$

where (x, γ) denotes the functional value $\gamma(x)$ of the character $\gamma \in \hat{G}$ at $x \in G$.

A (G) stands for the subspace of $C_o(G)$ of functions given by Fourier transforms of all the functions in $L^1(\hat{G})$. A (G) is a Banach space under the norm

$$\|\hat{f}\|_A = \|f\|_1 , \qquad f \in L^1(\hat{G})$$

The dual space P (G) of A (G) is called the space of pseudo-measures on G. There exists an isometry between P (G) and $L^\infty(\hat{G})$, induced by

$$\langle \hat{f}, \sigma \rangle = \langle f, \hat{\sigma} \rangle , \qquad f \in L^1(\hat{G})$$

and $\sigma \in P(G)$.

$\langle \cdot , \cdot \rangle$ stands for the functional representation or dual space pairing through out this thesis.

We denote by B (G) the space of all functions in $L^1(G)$ whose Fourier transforms have compact support in $\hat{G}$.

If $y \in G$, the translation operator τ_y is defined on a space of functions X on G by the formula

$$\tau_y f(x) = f(x-y)$$

A space X is translation-invariant if $\tau_y f \in X$ for each $f \in X$ and $x \in G$.

For any function g on G, the reflection function $\tilde{g}$ is given by $\tilde{g}(x) = \overline{g(-x)}$, $x \in G$. We write supp g to denote the support of g. We use the notation $\cong$ to mean isometric isomorphism.

$\hat{A}$ stands for the set of all $\hat{f}$ where $f \in A$.

If R denotes the additive group of real numbers, the Lebesgue integral is normalized as to give

$$\|f\|_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t)| \, dt$$

and the Fourier transform of $f \in L^1(R)$ takes the form

$$\hat{f}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iut} f(t) \, dt$$

BV denotes the set of all functions μ which are of bounded variation on R and normalized by

$$\mu(-\infty) = \mu(-\infty + 0) = 0$$

$$\mu(x) = \frac{1}{2} \left[\mu(x-0) + \mu(x+0) \right]$$

We set

$$\|\mu\| = \text{Total variation of } \mu .$$

Also, Z denotes the set of all integers.

Whenever, for a function space, the underlying group is evident, we drop it in the notation, for convenience. For example, we may write L^1 instead of $L^1(G)$.

DEFINITION 0.1 A linear subspace $S(G)$ of $L^1(G)$, with G locally compact abelian, is called a Segal algebra if the following four conditions are satisfied

S1. $S(G)$ is dense in $L^1(G)$.

S2. $S(G)$ is a Banach space under some norm $\| \ \|_S$

and $\|f\|_S \geqslant \|f\|_1$, $f \in S(G)$.

S3. For $y \in G$, $\tau_y f \in S(G)$ for every f in $S(G)$ and

$$\|\tau_y f\|_S = \|f\|_S .$$

S4. The mapping $y \longrightarrow \tau_y f$ is continuous from G into $S(G)$, for every $f \in S(G)$.

On non abelian groups, Segal algebras are defined using left (right) translations.

Segal algebras are defined and studied by Hans Reiter in 1965, though their origin can be traced to the work of Segal

[28] in 1947 . The Wiener algebra (See 0.8) is the first example of a Segal algebra.

Various properties of Segal algebras that are used in the sequel are stated below as lemmas and can be found in Reiter [22] .

<u>LEMMA 0.2</u> For every $f \in S(G)$ and arbitrary h in $L^1(G)$ the vector valued integral

$$\int_G h(y)\, \tau_y f \, dy$$

exists as an element of $S(G)$ and

$$h * f = \int_G h(y)\, \tau_y f \, dy$$

Moreover,

$$\| h * f \|_S \leq \| h \|_1 \, \| f \|_S \, ,$$

It follows immediately that if $h \in S(G)$, from S2 , then

$$\| h * f \|_S \leq \| h \|_S \, \| f \|_S$$

which shows that $S(G)$ is actually a Banach algebra, in addition to being an ideal in $L^1(G)$.

<u>LEMMA 0.3</u> Let $\mu \in M(G)$. Then for any $f \in S(G)$ the vector valued integral

$$\int_G \tau_y f \, d\mu(y)$$

exists as an element of $S(G)$ and

$$\mu * f = \int_G \tau_y f \, d\mu(y) .$$

Further

$$\| \mu * f \|_S \leq \|\mu\| \, \|f\|_S .$$

Thus $S(G)$ is an ideal in $M(G)$.

<u>LEMMA 0.4</u>: The space $B(G)$ is contained in every Segal algebra $S(G)$.

<u>LEMMA 0.5</u>: To every compact set K in $\hat{G}$, there is a constant $C_K > 0$ such that every $f \in S(G)$ whose Fourier transform vanishes outside K satisfies the inequality

$$\|f\|_S \leqslant C_K \|f\|_1 .$$

From lemma 0.5, we can show that if G is discrete, every Segal algebra $S(G)$, coincides with $L^1(G)$. Hence, to avoid trivialities, we shall always assume G to be non-discrete.

<u>LEMMA 0.6</u>: Given any $f \in S(G)$, there is, for every $\varepsilon > 0$ a $g \in S(G)$ such that the Fourier transform $\hat{g}$ of g has compact support and

$$\| g * f - f \|_S < \epsilon$$

<u>LEMMA 0.7</u>: Every Segal algebra has approximate units of L^1 - norm 1.

We shall now give a few examples of Segal algebras which will be of interest to us.

<u>EXAMPLE 0.8</u>: The space W of all continuous functions f on R such that

$$\| f \|_W = \sum_{k=-\infty}^{\infty} \max_{k \leq x \leq k+1} | f(x) | < \infty$$

is the space introduced by Wiener [33] and it has been extensively studied by Goldberg [7] . (See also Hewitt and Ross [10, p.506]). Since $\| \tau_y f \|_W \leq 2 \| f \|_W$, the translation operator has a norm bounded by 2 . Introducing an equivalent norm

$$||| f |||_W = \sup_{m \in R} \sum_{k=-\infty}^{\infty} \max_{k \leq x \leq k+1} | f(x+m) |$$

W is a Segal algebra under $||| \cdot |||_W$.

<u>EXAMPLE 0.9</u> The space $L^A(R)$ of all functions f in $L^1(R)$ which are absolutely continuous on R with the derivative f' belonging to $L^1(R)$ is a Segal algebra if we set the norm as

$$\|f\|_{L^A} = \|f\|_1 + \|f'\|_1 .$$

<u>EXAMPLE 0.10</u> $L^1 \cap C_0(G)$ the space of all $L^1(G)$ functions which are in $C_0(G)$ is a Segal algebra under the norm

$$\|f\|_{L^1\cap C_0} = \|f\|_1 + \|f\|_\infty , \quad f \in L^1 \cap C_0 .$$

Notice that here $\|f\|_\infty$ stands for the supremum norm of f.

<u>EXAMPLE 0.11</u> $L^1 \cap L^p(G)$ is also a Segal algebra, when $p < \infty$ and the norm is given by

$$\|f\|_{L^1\cap L^p} = \|f\|_1 + \|f\|_p , \quad f \in L^1 \cap L^p .$$

Closely related to the Segal algebras we have the concept of normed ideals studied by Cigler [3] .

<u>DEFINITION 0.12</u> An ideal $N(G)$ in $L^1(G)$ is called <u>normed ideal</u> if the following conditions hold:

N1. $N(G)$ is dense in $L^1(G)$.

N2. $N(G)$ is a Banach space under some norm $\| \; \|_N$ such that

$$\|f\|_1 \le \|f\|_N \quad , \quad f \in N(G).$$

for all $f \in N(G)$.

N3. $$\|h * f\|_N \le \|h\|_1 \, \|f\|_N ,$$

for all $h \in L^1(G)$ and $f \in N(G)$.

The most important normed ideals are the Segal algebras. Here we shall state important properties of normed ideals due to Cigler [3].

<u>LEMMA 0.13</u> A normed ideal $N(G)$ in $L^1(G)$ is a Segal algebra if and only if $B(G)$ is dense in $N(G)$.

<u>REMARK 0.14</u> It follows from lemma 0.13 and the fact that $B(G)$ is an ideal in $L^1(G)$ that every normed ideal $N(G)$ contains a unique closed subspace $N_0(G)$, the closure of $B(G)$ in $N(G)$, which is a Segal algebra. Hence, a normed ideal $N(G)$ is a Segal algebra if and only if $N(G) = N_0(G)$.

<u>LEMMA 0.15</u> Let $N(G)$ be a normed ideal and $f \in N(G)$. Then $f \in N_0(G)$ if and only if

$$y \longrightarrow \tau_y f$$

is a continuous map from G into N (G) . (N_o (G) is as given in remark 0.14).

LEMMA 0.16 If N (G) is any normed ideal, N_o (G) is the smallest closed subspace of N (G) containing all elements of the form

$$h * f \quad , \quad h \in L^1(G), \quad f \in N(G) .$$

We shall now mention an example of a normed ideal which we shall deal with in the sequel.

EXAMPLE 0.17 The space V defined by Burnham and Goldberg [2] consists of all those functions f in L^1 (R) which are bounded on R and for which

$$\|f\|_V = \sum_{k=-\infty}^{\infty} \operatorname{ess.sup.}_{k \leq x \leq k+1} |f(x)| < \infty$$

Under the equivalent norm

$$|||f|||_V = \sup_{m \in R} \sum_{k=-\infty}^{\infty} \operatorname{ess.sup.}_{k \leq x \leq k+1} |f(x+m)|$$

it can be proved that V is a normed ideal (See theorem 2.1 and Proposition 9.2)

The essential difference between a normed ideal and a Segal algebra is the continuity in translation. We thus make the following definition.

<u>DEFINITION 0.18</u> Let $\mathcal{B}$ be a translation invariant Banach space of functions on G . An element $f \in \mathcal{B}$ is said to translate continuously if the mapping $x \to \tau_x f$ is continuous from G into the Banach space $\mathcal{B}$. The set of all continuously translating functions in $\mathcal{B}$ is called the <u>continuously translating subspace</u> of $\mathcal{B}$ and it is denoted by $\mathcal{B}_0$.

Thus $N_0(G)$ is the continuously translating subspace of $N(G)$.

We also have occasion to use the concept of Banach modules and their tensor products.

<u>DEFINITION 0.19</u> Let A be a Banach algebra and X a Banach space. Then X is a <u>A - Banach module</u> if X is a module over A in the algebraic sense, satisfying

$$\|ax\|_X \leq \|a\|_A \|x\|_X , \quad a \in A, \ x \in X .$$

The <u>essential part</u> X_e of a Banach module X is the closed linear span of

$$\{ ax : a \in A, \ x \in X \} .$$

X is an essential module if $X = X_e$. If X is a Banach module, so is X^* under the adjoint action.

DEFINITION 0.20 The A - module tensor product of the two Banach modules X and Y over a Banach algebra A is denoted by

$$X \underset{A}{\otimes} Y$$

and is defined as the quotient space

$$X \otimes_{\gamma} Y / M$$

where $X \otimes_{\gamma} Y$ is the projective tensor product of X and Y and M is the closed linear span of elements of the form

$$ax \otimes y - x \otimes ay, \quad a \in A, x \in X \text{ and } y \in Y.$$

Each element Φ of $X \underset{A}{\otimes} Y$ can be expressed in the form

$$\Phi = \sum_{i=1}^{\infty} x_i \otimes y_i$$

where $x_i \in X$, $y_i \in Y$ and $\sum_{i=1}^{\infty} \|x_i\|_X \|y_i\|_Y < \infty$.

The norm of Φ is defined by

$$\|\Phi\| = \inf \left\{ \sum_{i=1}^{\infty} \|x_i\|_X \|y_i\|_Y \right\}$$

where the infimum is taken over all possible representations of Φ. For various properties of Banach modules, one refers to Rieffel [24].

NOTATION 0.21 If A is a Banach algebras and X and Y are A - modules, then

$$\mathrm{Hom}_A (X, Y)$$

stands for the space of all continuous module homomorphisms T from A into B with operator norm, satisfying

$$T(ax) = aT(x), \quad a \in A, \ x \in X.$$

REMARK 0.22 We shall immediately note that any normed ideal $N(G)$ and Segal algebra $S(G)$ are $L^1(G)$ - modules. In fact, every Segal algebra is an essential $L^1(G)$ - module. Lemma 0.16 shows that $N_o(G)$ is the essential part of $N(G)$.

DEFINITION 0.23 Let A, B be any two translation invariant Banach spaces of functions on G. Then a bounded linear transformation $T : A \longrightarrow B$ is a multiplier if for each $y \in G$, T satisfies

$$\tau_y T = T \tau_y$$

That is, T commutes with translations.

For the case of Segal algebras, in the strength of [31] one can formulate the following proposition giving equivalent forms for multipliers.

<u>PROPOSITION 0.24</u> Let S_1 and S_2 be any two Segal algebras on a locally compact abelian group G and let T be a linear operator from S_1 into S_2. Then the following are equivalent:

(i) T is continuous from S_1 into S_2 which commutes with translations.

(ii) T commutes with convolutions. i.e. $T(f * g) = Tf * g$ for $f \in L^1(G)$, $g \in S_1(G)$.

(iii) T belongs to $\mathrm{Hom}_{L^1}(S_1, S_2)$.

(iv) There exists a unique pseudomeasure σ in $P(G)$ such that

$$Tf = \sigma * f \quad , \quad f \in S_1(G).$$

(v) There exists a unique bounded continuous function ϕ on $\hat{G}$ such that

$$(Tf)^\wedge = \phi \hat{f} \qquad \text{for all } f \in S_1.$$

In the light of the above Proposition we set different notations, to deal with different types of multipliers.

NOTATION 0.25

(i) (A,B) : If A and B are subsets of $L^1(G)$ or more generally subsets of pseudo-measures, (A,B) will denote the set of all functions ϕ on $\hat{G}$ such that $\phi \hat{A} \subset \hat{B}$.

(ii) $M(A,B)$: If A and B are translation invariant Banach spaces of functions on G, $M(A,B)$ denotes the space of all bounded linear transformations from A into B that commutes with translations.

(iii) $\mathrm{Hom}_{L^1}(A,B)$: If A and B are Banach $L^1(G)$-modules, then $\mathrm{Hom}_{L^1}(A,B)$ is the space of all continuous module homomorphisms from A into B.

We shall use the term 'multiplier' to mean an element of any of the spaces given above in 0.25 (i), (ii) and (iii) and the space we mean would be evident from the context.

With this, we shall summarize the contents of each chapter.

In Chapter I, we define the various subclasses F_i, $i = 1, 2, \ldots 6$, on the real line, introduced by Doss [4].

We add to this collection the Wiener algebra W, the space V and a new Segal algebra D. We first prove that V is a Banach space and the space ~~is~~ $L^1 \cap BV(R)$ is contained in V. That D is a Segal algebra is then established. Also, D is identified in a known form. We close the chapter after providing necessary and sufficient conditions for a function to be a Fourier transform of an element in L^A, W, V and D.

Chapter II contains the various results on multipliers among the classes W, V, D and F_i, $i = 1, 2, \dots 6$. We derive inspiration and information from the work of Doss [4] for the treatment of the problems in this chapter. To make our exposition self-contained we have included a coincise table of the plentiful results of Doss [4]. We prove that for the Segal algebra D, the space of multipliers into itself coincides with that of W. We derive several relationships between the multipliers of V and W and characterize their multiplier spaces in the fashion of Doss [4]. In the last part, we prove the analogue of Homander's lemma [11] for W and this enables us to characterize some more multiplier spaces for W and V.

Chapter III deals with the characterization of multiplier spaces of Segal algebras as dual spaces. If N is a normed ideal, then it contains a unique Segal algebra N_o (0.14). We show that every multiplier from a Segal algebra S into N is essentially a multiplier from S into No and whenever N is a dual space, the multiplier space M (S,No) is isometrically isomorphic to a dual space. This result can further be generalized to homogeneous Banach spaces. But the special cases are more interesting. By proving that V is actually a dual space, we deduce the result of Krogstad [14] that M (W,W) is isometrically isomorphic to a dual space. The result of Burnham and Goldberg [2] that M $(L^1,W) \cong V$ also becomes a byproduct of this result. From Liu and Wang [17] , $L^1 \cap L^\infty(G)$ is seen to be a dual space. This enables us to characterize $M(S, L^1 \cap C_0)$ as a dual space for any Segal algebras S . Further, we show that the class of all complex valued functions on the real line, with period 1, whose one sided n^{th} difference satisfy certain Lifschitz conditions is actually a Segal algebra and obtain its multiplier space as a dual space. Certain Segal algebras introduced by Burnham [1] and Riemersma [25] are also discussed.

In chapter IV we study the multipliers from a Segal algebra into L^1 (G) where G is a locally compact abelian group. The weakstar closure in P(G) of a Segal algebra S(G) with the convolution operator norm is found to be isometrically isomorphic to M (S , L^1) . Whenever G is compact abelian, this result can be carried over to $M(S_1, S_2)$ where S_1, S_2

are any two Segal algebras. On the real line, we characterise $M(S, L^1)$ as a space of sequences of $L^1(R)$ - functions and derive the result of Pigno [20] for Segal algebras.

The final chapter deals with the study of a generalization for an operator commuting with translations. We show that the definition of a multiplier pair introduced by Nandakumar [19] can be carried over to difference spaces,

$L^p(G)$, $(1 \leq p \leq \infty)$, $M(G)$, $C_0(G)$ and so on. We shall present only typical results since the methods of proof are analogous to those given by Larsen in [15] .

CHAPTER I

SEGAL ALGEBRAS AND RELATED CLASSES ON THE REAL LINE

1. Various classes of functions

Let R denote the additive group of real numbers. The following Segal algebras are of interest to us. We recall the definitions from 0.8., 0.9. and 0.10.

1. The Banach algebra $L^A(R)$ of all functions f in $L^1(R)$ which are absolutely continuous on R with $f' \in L^1(R)$ is a Segal algebra with the norm given by

$$\|f\|_{L^A} = \|f\|_1 + \|f'\|_1 .$$

2. The space $W(R)$ of all continuous functions f on R such that

$$\sum_{k=-\infty}^{\infty} \max_{k \leq x \leq k+1} |f(x)| < \infty$$

is the Wiener space.

If we set

$$|||f|||_W = \sup_{m \in R} \sum_{k=-\infty}^{\infty} \max_{k \leq x \leq k+1} |f(x+m)|$$

then $W(R)$ is a Segal algebra under $|||\cdot|||_W$.

3. The Banach space $L^1 \cap C_0(R)$ is a Segal algebra under the norm

$$\|f\|_{L^1\cap C_0} = \|f\|_1 + \|f\|_\infty , \quad f \in L^1 \cap C_0(R).$$

In addition, we consider the following classes of functions on R which are always assumed to be absolutely integrable in the sense of Lebesgue.

F_1 : Absolutely integrable functions on R in the sense of Lebesgue. This is our usual space $L^1(R)$.

F_2 : All functions in $L^1(R)$ which are bounded on R. This is $L^1 \cap L^\infty(R)$.

F_3 : All functions in $L^1(R)$ which are bounded on R and integrable in the sense of Riemann in every finite interval.

F_4 : All functions in $L^1(R)$ which are uniformly continuous on R. This is $L^1 \cap C_0(R)$

F_5 : All functions in $L^1(R)$ which are of bounded variation on R. This is $L^1 \cap BV$.

F_6 : All functions in $L^1(R)$ which are absolutely continuous on R and of bounded variation on R. This is $L^A(R)$.

V : All functions in $L^1(R)$ which are bounded and such that

$$\sum_{k=-\infty}^{\infty} \operatorname*{ess.\,sup.}_{k \le x \le k+1} |f(x)| < \infty .$$

We assign a norm in V, similar to W.

$$|||f|||_V = \sup_{m \in R} \sum_{k=-\infty}^{\infty} \operatorname{ess.sup}_{k \le x \le k+1} |f(x+m)|$$

The classes F_1 to F_6 were considered by Doss [4] in his study of multiplicators of the subclasses of $L^1(R)$, the class W was introduced by Wiener [33] in his study of Tauberian theorems and the class V was introduced by Burnham and Goldberg [2].

2. Inclusion relations

Among the classes defined above the following inclusion relations hold

(1) $F_4 \subset F_3 \subset F_2 \subset F_1$.

(2) $F_6 \subset F_5 \subset F_3$

(3) $L^A(R) = F_6 \subset W(R) \subset L^1 \cap C_0(R) = F_4$

(4) $W(R) \subset V \subset F_2$; $F_5 \subset V \subset F_2$.

The inclusions (1) and (2) are stated by Doss [4] and easily follow from the definitions. The inclusions in (3) were proved by Wang [32]. We remark that W(R) is nothing but the class of functions in V which are continuous. The only nontrivial inclusion in (4) is that $F_5 \subset V$ which we shall now prove independently though it is implied from the results on multipliers given by Burnham and Goldberg [2].

Frist, we establish the completeness of the space V.

THEOREM 2.1 The space V is complete under the norm

$$\|f\|_V = \sum_{k=-\infty}^{\infty} \underset{k \leq x \leq k+1}{\text{ess. sup.}} |f(x)|, \quad f \in V.$$

PROOF: Let $\{f_n\}$ be a Cauchy sequence in V. That is,

$$\sum_{k=-\infty}^{\infty} \underset{k \leq x \leq k+1}{\text{ess. sup.}} |f_m(x) - f_n(x)| \longrightarrow 0$$

as $m, n \to \infty$. It can easily be seen that $\|\cdot\|_V$ majorizes the L^∞ - norm and the L^1 - norm. Hence there exists a set E_1 and a function f in $L^\infty(R)$ such that the Lebesgue measure of $(R \setminus E_1)$ is zero and $f_n(x) \to f(x)$ on E_1. Using the L^1 -norm similarly, we can find a set E_2 and a function g in $L^1(R)$ such that

$$\lim_{n \to \infty} f_n(x) = g(x) \qquad \text{on } E_2$$

and $(R \setminus E_2)$ has measure zero. Let $E = E_1 \cap E_2$. Then $(R \setminus E)$ has measure zero and $f(x) = g(x)$ for each $x \in E$ and hence $f \in L^1(R)$. Thus $f_n(x) \to f(x)$ a.e. on R. Now

$$\sum_{k=-\infty}^{\infty} \underset{x \in [k,k+1]}{\text{ess. sup.}} |f(x)| = \sum_{k=-\infty} \underset{x \in [k,k+1]}{\text{ess. sup.}} \lim_{n \to \infty} |f_n(x)|$$

$$\leq \liminf_{n \to \infty} \sum_{k=-\infty}^{\infty} \underset{x \in [k,k+1]}{\text{ess. sup}} |f_n(x)| \quad \text{by Fatou's lemma}$$

$$= \liminf_{n \to \infty} \|f_n\|_V < \infty.$$

Hence $f \in V$. We now claim that $f_n \to f$ in V. First we notice that

$$\underset{x\in[k,k+1]}{\text{ess. sup.}}\ |f_n(x)-f(x)| \le \lim_{m\to\infty}\ \underset{x\in[k,k+1]}{\text{ess. sup.}}\ |f_n(x)-f_m(x)|$$

so that

$$\|f_n-f\|_V \le \sum_{k=-\infty}^{\infty} \lim_{m\to\infty}\ \underset{x\in[k,k+1]}{\text{ess. sup.}}\ |f_n(x)-f_m(x)|$$

$$\le \liminf_{m\to\infty} \sum_{k=-\infty}^{\infty} \underset{x\in[k,k+1]}{\text{ess. sup.}}\ |f_n(x)-f_m(x)|$$

$$\le \liminf_{m\to\infty} \|f_n-f_m\|_V \longrightarrow 0 \text{ as } n\to\infty .$$

Thus $f\in V$ and $\|f_n-f\|_V \to 0$ as $n\to\infty$.

REMARK 2.2 Notice that the norm of V given in section 1 and the norm given in theorem 2.1 are equivalent. We shall not distinguish between the two norms unless the necessity arises.

THEOREM 2.3 The space V contains the space $F_5 = L^1 \cap BV$.

PROOF:- Let $f\in F_5$ and V_f denote the total variation of f. Let $\varepsilon>0$ be arbitrary and define for each $k\in Z$, $\varepsilon_k = \dfrac{\varepsilon}{2^{|k|}}$ and set

$$m_k = \underset{x\in[k,k+1]}{\text{ess. inf}}\ |f(x)| .$$

Then there exists $x_0\in[k,k+1]$ such that

(5) $$|f(x_0)| < m_k + \varepsilon_k$$

Then for any $x\in[k,k+1]$ we have

$$|f(x)| \leq |f(x) - f(x_0)| + |f(x_0)|$$

$$\leq |f(x) - f(x_0)| + m_K + \varepsilon_K$$

by virtue of (5)

$$\leq |f(x) - f(x_0)| + \int_K^{K+1} |f(y)|\,dy + \varepsilon_K$$

$$\leq V_f[K, K+1] + \int_K^{K+1} |f(y)|\,dy + \varepsilon_K$$

Then

$$\operatorname*{ess.\,sup.}_{x \in [K, K+1]} |f(x)| \leq V_f[K, K+1] + \int_K^{K+1} |f(y)|\,dy + \varepsilon_K$$

Hence summing up with respect to K, we get

$$\|f\|_V \leq V_f + \|f\|_1 + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\|f\|_V \leq V_f + \|f\|_1 \qquad \text{for} \quad f \in F_5 .$$

3. A new Segal algebra

We shall now introduce a Segal algebra which is properly contained in W and will have the same multiplier space as that of W. (See Theorem 6.5)

THEOREM 3.1 Let D denote the set of all functions $f \in L^A(R)$ such that $f' \in W$. Define a norm on D by

(6) $$\|f\|_D = \|f\|_1 + |||f'|||_W ,$$

where $|||\cdot|||_W$ is the Segal norm of W. Then with the norm defined by (6), D is a Segal algebra.

PROOF:- First observe that

(7) $$\|f\|_D \leq \|f\|_{L^A} + |||f'|||_W \leq 2\|f\|_D$$

To prove that D is a Segal algebra, it is enough to prove (i) completeness (ii) the set B(R) of all functions in $L^1(R)$ whose Fourier transform have compact support is contained in D. (iii) $y \to \tau_y f$ is continuous from R into D. That (iii) holds is obvious from the properties of $L^A(R)$ and W and by virtue of (3). (i) Completeness:- Let $\{f_n\}$ be a Cauchy sequence in D. Then $\|f_m - f_n\|_D \longrightarrow 0$ as $m, n \longrightarrow \infty$ so that $\|f_m - f_n\|_{L^A} \to 0$ and $|||f'_m - f'_n|||_W \to 0$. By the completeness of the spaces L^A and W there exists $f \in L^A$, $g \in W$ such that

$$\|f_n - f\|_{L^A} \to 0, \quad |||f'_n - g|||_W \longrightarrow 0, \text{ as } n \to \infty.$$

Then

$$\|g - f'\|_1 \leq \|g - f'_n\|_1 + \|f'_n - f'\|_1$$

$$\leq |||g - f'_n|||_W + \|f_n - f\|_{L^A} \to 0.$$

Take $f =$ integral of g. Then $f \in D$ and $\|f_n - f\|_D \to 0$. (ii) Let $f \in B(R)$. Since $B(R) \subset L^A(R)$ it follows that $f' \in L^1(R)$. But $\hat{f}'(u) = iu\hat{f}(u)$ which shows that $\hat{f}'$ has also compact support. Hence $f' \in B(R)$ and afortiori to W. Hence $B(R) \subset D$.

REMARK 3.2 The space D can also be defined by

$$D = \{ f \in L^1(R) : f' \in W \}$$

PROPOSITION 3.3. Let k be a function on R defined by

$$k(x) = \begin{cases} -2\pi e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

so that $\hat{k}(u) = \frac{i}{u-i}$ for $u \in \hat{R}$, where $i = \sqrt{-1}$. Then the Segal algebra D can be identified with the space

$$k * W = \{ k * f : f \in W \}.$$

PROOF. That $\hat{k}(u) = \frac{i}{u-i}$ for $u \in \hat{R}$ is a straightforward verification. Let $f \in D$. Then $f' \in W$ so that

$$iu\hat{f}(u) \in \hat{W}.$$

Since $\hat{f} \in \hat{D} \subset \hat{W}$, we see that $\frac{u-i}{i}\hat{f}(u) \in \hat{W}$; which is the same as $\hat{f}(u) \in \frac{i}{u-i}\hat{W}$.

Equivalently, $f \in k * W$.

Conversely, suppose $f \in k * W$. Since $k * L^1 = L^A$ (a result of Doss which is stated in Proposition 4.6) we see that $f \in L^A(R)$. Then

(8) $$\hat{f'}(u) = iu\hat{f}(u)$$

By assumption, there exists $g \in W$ such that so that

(9) $$\hat{f}(u) = \frac{i}{u-i}\hat{g}(u)$$

and therefore, using (8) and (9),

$$\hat{f'}(u) = iu\,\hat{f}(u) = \frac{i^2}{u-i}\,\hat{g}(u) = -\hat{g}(u) - \frac{i}{u-i}\,\hat{g}(u) \in \hat{W}.$$

This implies $f \in D$.

4. Conditions for a function to be a Fourier transform

If $A \subset L^1(R)$, we write $\hat{A} = \{\hat{f} : f \in A\}$ where $\hat{f}$ stands for the Fourier transform of f . Let

$$\delta(t) = \frac{1}{\pi}\,\frac{1-\cos t}{t^2}, \quad t \in R$$

and

$$\Delta(x) = \begin{cases} 1-|x| & \text{if } |x| \le 1 \\ 0. & \text{if } |x| > 1 \end{cases}$$

Then $\delta \in L^1(R)$ and $\Delta = \hat{\delta}$. Moreover, $\delta = \Delta^{\vee}$ is the inverse Fourier transform of Δ . We set

$$\delta_n(t) = n\,\delta(nt) = \frac{1}{\pi}\,\frac{1-\cos nt}{nt^2}$$

and

$$\Delta_n(t) = \Delta(t/n) = \hat{\delta}_n(t)$$

Then $\delta_n \geqslant 0$, $\|\delta_n\|_1 = 1$ and $\|f * \delta_n - f\|_1 \to 0$ as $n \to \infty$ for each $f \in L^1(R)$. That is $\{\delta_n\}$ is an approximate identity for $L^1(R)$.

Given a function λ defined on $\hat{R}$, we denote by Λ_n the function defined by

$$\Lambda_n(x) = \lambda(x)\Delta_n(x) , \quad x \in \hat{R}$$

and set

$$\sigma_n(x) = \check{\Lambda}_n(x) , \quad x \in R .$$

the inverse Fourier transform of $\Lambda_n(x)$. Then the following results are proved by Doss [4] giving the conditions for a function to be the Fourier transform in the specified class of functions.

PROPOSITION 4.1 (Doss): A necessary and sufficient condition that λ is the Fourier transform of a function in $L^1(R)$ is that Λ_n be continuous for each positive integer n and that the functions σ_n satisfy the two conditions:

(i) There exists a constant A such that

$$\|\sigma_n\|_1 \leq A$$

for every n.

(ii) To every $\varepsilon > 0$ there exists a $\eta > 0$ such that, for every set E whose measure is less than η,

$$\int_E |\sigma_n(x)|\, dx \leq \varepsilon$$

for every n.

PROPOSITION 4.2 (Doss): A necessary and sufficient condition that λ is the Fourier transform of a function in the class F_2 is that Λ_n is continuous for every positive integer n and that the functions σ_n satisfy the two conditions:

(i) There exists a constant A such that

$$\|\sigma_n\|_1 \leq A$$

for every n .

(ii) There exists a constant B such that

$$|\sigma_n(x)| \leq B$$

for every n and x.

PROPOSITION 4.3 (Doss) A necessary and sufficient condition that $\lambda \in \hat{F}_3$ is that the conditions of proposition 4.2 are satisfied and that the function σ_n are uniformly. Riemann integrable in $(-\infty, \infty)$.

PROPOSITION 4.4 (Doss) A necessary and sufficient condition that $\lambda \in \hat{F}_4$ is that the conditions of proposition 4.2 are realized and that the functions $\sigma_n(x)$ are uniformly (in n and x) continuous.

PROPOSITION 4.5 (Doss) A necessary and sufficient condition that $\lambda \in \hat{F}_5$ is that the functions Λ_n be continuous for every positive integer n and that the functions

$$\tau_n(x) = \int_{-\infty}^{\infty} \lambda(u) \frac{u-i}{i} \Delta_n(u) e^{iux} du$$

satisfy the condition: There exists a constant C such that, for every n

$$\int_{-\infty}^{\infty} |\tau_n(x)| \, dx \leq C$$

PROPOSITION 4.6 (Doss): The two relations

$$\lambda(u) \in \hat{F}_6$$

and

$$\frac{u-i}{i}\,\lambda(u) \in \hat{F}_1$$

are equivalent.

We shall now proceed to establish the necessary and sufficient conditions for a function to be a Fourier transform of a function belonging to the classes L^A, W, V and D. This is done for the sake of completeness in the light of the above results.

THEOREM 4.7: $\lambda \in \widehat{L^A(R)}$ if and only if

(i) Λ_n is continuous for each positive integer n

(ii) For some constant $C > 0$, $\|\sigma_n\|_{L^A} \leq C$ for all n.

(iii) $\{\sigma_n(x)\}$ is uniformly (in n and x) continuous.

PROOF: Let $\lambda \in \widehat{L^A(R)}$. Then $\sigma \in L^A(R)$ where $\hat{\sigma} = \lambda$.

Then $\sigma_n(x) = \sigma * \delta_n(x)$ for all $x \in R$.

Hence

$$\|\sigma_n\|_{L^A} = \|\sigma * \delta_n\|_1 + \|\sigma' * \delta_n\|_1$$

$$\leq (\|\sigma\|_1 + \|\sigma'\|_1)\,\|\delta_n\|_1 = \|\sigma\|_{L^A}.$$

Now consider $x_1, x_2 \in R$. Then

$$|\sigma_n(x_1) - \sigma_n(x_2)| = |\sigma * \delta_n(x_1) - \sigma * \delta_n(x_2)|$$

$$= \left|\int_R \sigma(x_1 - y)\,\delta_n(y)\,dy - \int_R \sigma(x_2 - y)\,\delta_n(y)\,dy\right|$$

$$\leq \int_R |\sigma(x_1 - y) - \sigma(x_2 - y)|\,\delta_n(y)\,dy$$

Since σ is uniformly continuous for arbitrary $\varepsilon > 0$ We can find a $\eta > 0$ such that whenever $|x_1 - x_2| < \eta$ then $|\sigma(x_1) - \sigma(x_2)| < \epsilon$. Hence for this $\eta > 0$ we get if $|x_1 - x_2| < \eta$, $|\sigma_n(x_1) - \sigma_n(x_2)| < \epsilon$ since $\int_R \delta_n(y)dy = 1$ for all n. And η is independent of n. Thus $\{\sigma_n\}$ are uniformly continuous in n and x.

Now let (i), (ii) and (iii) hold. We shall prove that $\lambda \in L^{\hat{A}}(R)$. If we define

$$\tau_n(x) = -\sigma_n(x) - \sigma_n'(x) \qquad \text{for all } n$$

Then the hypothesis (ii) gives $\|\tau_n\|_1 \leq C$. Since (i) holds, by Proposition 4.5, this implies that λ is the Fourier transform of a function σ in L^1 with bounded variation. Hence σ' exists a.e. and $\sigma' \in L^1(R)$. Now to prove that σ is absolutely continuous.

First note that $\sigma_n(x) = \sigma * \delta_n(x)$ and $\|\sigma_n - \sigma\|_1 \to 0$ or $\sigma_n(x) \longrightarrow \sigma(x)$ a.e. But the condition that $\{\sigma_n(x)\}$ is uniformly continuous in n and x implies $\sigma_n(x) \longrightarrow \sigma(x)$ everywhere in R. For every n, $\sigma_n \in L^A$ and hence is absolutely continuous. Now given $\varepsilon > 0$, $N > 0$ there exists n_0 such that

$$(10) \qquad |\sigma_{n_0}(x) - \sigma(x)| < \varepsilon/4N \qquad \text{for all } x \in R$$

Let $\{(\alpha_i, \beta_i)\}_{i=1}^{N}$ be any partition on the real line and consider

$$\sum_{i=1}^{N} |\sigma(\beta_i) - \sigma(\alpha_i)| \le \sum_{i=1}^{N} |\sigma(\beta_i) - \sigma_{n_0}(\beta_i)|$$

$$+ \sum_{i=1}^{N} |\sigma_{n_0}(\beta_i) - \sigma_{n_0}(\alpha_i)| + \sum_{i=1}^{N} |\sigma_{n_0}(\alpha_i) - \sigma(\alpha_i)|$$

By the absolute continuity of σ_{n_0}, there exists a $\eta > 0$ such that whenever $\sum_{i=1}^{N} |\beta_i - \alpha_i| < \eta$ then

(11) $$\sum_{i=1}^{N} |\sigma_{n_0}(\beta_i) - \sigma_{n_0}(\alpha_i)| < \varepsilon/2$$

Hence from (10) and (11) we obtain σ to absolutely continuous.

<u>THEOREM 4.8</u> $\lambda \in \widehat{W(R)}$ if and only if the following conditions hold:

(i) Λ_n is continuous for each positive integer n

(ii) For some constant $C > 0$, $\|\sigma_n\|_W \le C$ for all n.

(iii) $\{\sigma_n(x)\}$ are uniformly (in n and x) continuous.

<u>PROOF</u>: Let $\lambda \in (\widehat{W(R)})$. Then $\sigma \in W(R)$ where $\hat{\sigma} = \lambda$ so that (i) is obvious. Now $\sigma_n(x) = \sigma * \delta_n(x)$ for all $x \in R$ so that

$$\|\sigma_n\|_W = \|\sigma * \delta_n\|_W \le \|\sigma\|_W \|\delta_n\|_1 = \|\sigma\|_W$$

From inclusion (3) and Proposition 4.4 (iii) follows.

Now suppose (i), (ii) and (iii) are valid. Then $\|\sigma_n'\|_1 \le C$ and $|\sigma_n(x)| \le C$ for all x and the conditions of Proposition

4.4 are satisfied. Hence there exists a function σ in $L^1(R)$ such that σ is uniformly continuous on R and $\sigma_n(x) \to \sigma(x)$ everywhere in R. Now

$$\sum_{k=-\infty}^{\infty} \max_{k \le x \le k+1} |\sigma(x)| = \sum_{k=-\infty}^{\infty} \max_{[k,k+1]} \lim_n |\sigma_n(x)|$$

$$\le \liminf_{n\to\infty} \sum_{k=-\infty}^{\infty} \max_{x\in[k,k+1]} |\sigma_n(x)|$$

$$= \liminf_{n\to\infty} \|\sigma_n\|_W \le C$$ using Fatou's lemma.

Hence $\sigma \in W(R)$. This completes the proof.

<u>THEOREM 4.9</u> $\lambda \in \hat{V}$ if and only if

(i) Λ_n is continuous for every positive integer n

(ii) there exists a constant B such that $|\sigma_n(x)| \le B$ for every n and x

(iii) $\|\sigma_n\|_V \le C$ for some $C > 0$

<u>PROOF</u>: Suppose $\lambda = \hat{\sigma}$ where $\sigma \in V$. Then (i) and (ii) are satisfied by Proposition 4.2 and inclusion (4). Now (iii) follows from the fact that $\sigma_n = \sigma * \delta_n$ and

$$\|\sigma_n\|_V = \|\sigma * \delta_n\|_V \le 2\|\sigma\|_V \|\delta_n\|_1 = 2\|\sigma\|_V$$

Suppose the conditions are satisfied. Then $\|\sigma_n\|_1 \le \|\sigma_n\|_V \le C$ shows that all the conditions of Proposition 4.2 are valid and hence $\sigma_n \to \sigma$ a.e. where $\sigma \in F_2$. The proof that $\|\sigma\|_V < \infty$ follows exactly as in the previous theorem.

THEOREM 4.10 $\lambda \in \hat{D}$ if and only if

(i) Λ_n is continuous for each positive integer n.

(ii) $\|\sigma_n\|_D \leq C$ for all n.

(iii) $\{\sigma_n(x)\}$ and $\{\sigma_n'(x)\}$ are uniformly continuous in n and x.

PROOF: If $\lambda = \hat{\sigma}$, $\sigma \in D$, then clearly (i) holds and

$$\|\sigma_n\|_D = \|\sigma * \delta_n\|_D \leq \|\sigma\|_D \|\delta_n\|_1 = \|\sigma\|_D .$$

Also from $\sigma \in L^A$ and $\sigma' \in W$ and Theorems 4.7 and 4.8 (iii) is obvious.

If (i), (ii) and (iii) hold, $\|\sigma_n\|_{L^A} \leq \|\sigma_n\|_D \leq C$ implies from Theorem 4.7, that $\sigma_n = \sigma * \delta_n$ for some $\sigma \in L^A$. Then clearly $\sigma_n' = \sigma' * \delta_n$. Again $\|\sigma_n'\|_W \leq \|\sigma_n\|_D \leq C$ and $\{\sigma_n'(x)\}$ is uniformly continuous in n and x imply from Theorem 4.8 that $\sigma' \in W$ or $\sigma \in D$.

Doss has extensively made use of propositions 4.1 to 4.5 in his study of multiplicators. But here, we do not depend on theorems 4.7 to 4.10 to characterize the multipliers. We have included them as they are of independent interest. For example, using theorem 4.7 we are able to make the following important observation on the spaces F_4, F_5 and F_6.

THEOREM 4.11: The space F_6 is precisely the intersection of F_4 and F_5.

PROOF: Let $h \in F_4 \cap F_5$. Since h is in F_5, h' exists a.e. and $h' \in L^1$. Set

$$\sigma_n = h * \delta_n$$

for each integer n. Then

$$\sigma_n' = h' * \delta_n \quad \text{a.e.}$$

Also, since $h \in F_4$, from proposition 4.4, $\{\sigma_n(x)\}$ is uniformly continuous in n and x. Now

$$\|\sigma_n\|_{L^A} = \|\sigma_n\|_1 + \|\sigma_n'\|_1 = \|h*\delta_n\|_1 + \|h'*\delta_n\|_1$$

$$\leq (\|h\|_1 + \|h'\|_1)\,\|\delta_n\|_1 = \|h\|_1 + \|h'\|_1 < \infty$$

That $\Lambda_n = \hat{h}\,\Delta_n$ is continuous for each n is also clear. Hence from theorem 4.7, we conclude that $h \in F_6$. We have thus proved

$$F_4 \cap F_5 \subset F_6$$

The opposite inclusion is trivial.

CHAPTER II

MULTIPLIERS OF SEGAL ALGEBRAS ON THE REAL LINE

In this chapter we shall study the multiplier space for the Segal algebras D and W and for the space V, related to the various classes F_j, $j = 1, 2, \dots 6$, defined in chapter I.

5. The results of Doss

Let us recall that the space (A,B) is the set of all functions ϕ such that $\phi \hat{A} \subset \hat{B}$. Using the characteristic properties of the classes $\hat{F}_j$, and by classical methods, Doss [4] has successfully obtained the conditions for a function ϕ to belong to the space (F_j, F_k), $j, k = 1, 2 \dots 6$. His conditions express the fact that one or other of the functions

$$\phi(u), \quad \frac{i\,\phi(u)}{u-i}, \quad \frac{\phi(u)}{(u-i)^2}$$

belongs to the class $\hat{F}_k$. It turns out that the function

$$k(x) = \begin{cases} -2\pi e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

which has the nonvanishing Fourier transform, $\hat{k}(u) = \frac{i}{u-i}$, $u \in \hat{R}$, where $i = \sqrt{-1}$ plays an important role in his characterizations.

The spaces under consideration here, W, V and D, also being subclasses of $L^1(R)$, it appears natural to take up the approach of Doss, while studying their multipliers. Thus the multiplier spaces for W, V and D among themselves and in relation with F_j, $j = 1, 2, \dots 6$ are effectively dealt with in this chapter.

To make a convenient reference in our proofs we have enclosed here a table containing all the results of Doss in [4]

$\phi \in (F_1,F_1) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$	$\phi \in (F_2,F_1) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$
$\phi \in (F_1,F_2) \iff \phi \in \hat{F}_2$	$\phi \in (F_2,F_2) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$
$\phi \in (F_1,F_3) \iff \phi \in \hat{F}_2$	$\phi \in (F_2,F_3) \iff \phi \in \hat{F}_1$
$\phi \in (F_1,F_4) \iff \phi \in \hat{F}_2$	$\phi \in (F_2,F_4) \iff \phi \in \hat{F}_1$
$\phi \in (F_1,F_5) \iff \phi \in \hat{F}_5$	$\phi \in (F_2,F_5) \iff \phi \in \hat{F}_5$
$\phi \in (F_1,F_6) \iff \phi \in \hat{F}_5$	$\phi \in (F_2,F_6) \iff \phi \in \hat{F}_5$
$\phi \in (F_3,F_1) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$	$\phi \in (F_4,F_1) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$
$\phi \in (F_3,F_2) \iff \frac{i\phi(u)}{u-i} \in F_5$	$\phi \in (F_4,F_2) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$
$\phi \in (F_3,F_3) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$	$\phi \in (F_4,F_3) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$
$\phi \in (F_3,F_4) \iff \phi \in \hat{F}_1$	$\phi \in (F_4,F_4) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$
$\phi \in (F_3,F_5) \iff \phi \in \hat{F}_5$	$\phi \in (F_4,F_5) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$
$\phi \in (F_3,F_6) \iff \phi \in \hat{F}_5$	$\phi \in (F_4,F_6) \iff \phi \in \hat{F}_5$
$\phi \in (F_5,F_1) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_1$	$\phi \in (F_6,F_1) \iff \frac{i^2\phi(u)}{(u-i)^2} \in \hat{F}_5$
$\phi \in (F_5,F_2) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_2$	$\phi \in (F_6,F_2) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_2$
$\phi \in (F_5,F_3) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_3$	$\phi \in (F_6,F_3) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_2$
$\phi \in (F_5,F_4) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_4$	$\phi \in (F_6,F_4) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_2$
$\phi \in (F_5,F_5) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$	$\phi \in (F_6,F_5) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$
$\phi \in (F_5,F_6) \iff \phi \in \hat{F}_1$	$\phi \in (F_6,F_6) \iff \frac{i\phi(u)}{u-i} \in \hat{F}_5$

The result on (F_2,F_3) is obtained by Pigno [21] . The rest are due to Doss [4] . The symbol $\iff$ stands for if and only if.

In the first three sections of this chapter, we shall confine ourselves to treat the multiplier space (A,B) where A,B very over the classes F_1 to F_6 , V , W and D. However, we observe that, in view of the equivalence of the various multiplier definitions given in Proposition 0.24, for Segal algebras, the aforementioned characterization of Doss reduces to the characterization of pseudomeasures σ such that $\sigma * k$ belongs to a known class. For example, note that the result proved in Proposition 7.1 can also be formulated as:

M (L^A, W) is isomorphic to the space $\{ \sigma \in P(R) : \sigma * k \in V \}$

As observed earlier, the classes F_4 and F_6 are the well known Segal algebras $L^1 \cap C_0(R)$ and $L^A(R)$ respectively. While the multipliers on $L^1 \cap C_0$ have been obtained as bounded measures using different techniques, the characterization for multipliers on L^A (R) has not been proved elsewhere. It is only from the results of Doss, we ascertain that the multipliers from $L^A(R)$ into itself are bounded measures. For, it is easy to see from the table that

$$(F_1, F_1) = (F_6, F_6)$$

and we know that $(F_1, F_1) = \widehat{M(R)}$. (See, for example [15])

We shall now list below the relevant results that will used in the proof of our multiplier results.

PROPOSITION 5.1: $L^A(R) = k * L^1(R)$.

This follows from Proposition 4.6

PROPOSITION 5.2: $F_5 = k * M(R)$.

This follows from the fact that $\phi \in (F_1, F_1)$ if and only if $\frac{i\phi(u)}{u-i} \in \hat{F}_5$ (See the table) and the known result

$$(F_1, F_1) = \hat{M(R)} .$$

PROPOSITION 5.3: $(L^1, W) = \hat{V}$.

This is an equivalent form of the result of Burnham and Goldberg [2] that

$$M(L^1, W) \cong V .$$

An alternate proof of this result is given by us. See Corollary 11.7 (i).

PROPOSITION 5.4. If $A \subset B$, then

$$(C, A) \subset (C, B)$$

and

$$(B, C) \subset (A, C) .$$

This is an easy consequence of the definition of the multiplier space. See Notation 0.25 (i).

PROPOSITION 5.5. Let A,B be subspaces of L^1 such that

(i) A and B are Banach spaces with norms $\| \|_A$ and $\| \|_B$ respectively.

(ii) $\| \|_A \geq \| \|_1$ and $\| \|_B \geq \| \|_1$. Let ϕ be a bounded function in (A,B). Then the mapping T_ϕ defined by

$$T_\phi(f) = g \quad \text{where} \quad \hat{g} = \phi \hat{f},\ f \in A.$$

is a bounded linear operator.

__PROOF:__ Linearity being easily verified, we shall prove the boundedness. Suppose $f_n \to 0$ in A and $T_\phi f_n \to g$ in B. Then

$$\begin{aligned}
\|\hat{g}\|_\infty &\leq \|\hat{g} - \widehat{T_\phi f_n}\|_\infty + \|\widehat{T_\phi f_n}\|_\infty \\
&\leq \|g - T_\phi f_n\|_1 + \|\phi \hat{f}_n\|_\infty \\
&\leq \|g - T_\phi f_n\|_1 + \|\phi\|_\infty \|f_n\|_1 \\
&\leq \|g - T_\phi f_n\|_B + \|\phi\|_\infty \|f_n\|_A \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $\hat{g} = 0$ and thus $g = 0$ a.e. Now by closed graph theorem, we see that T_ϕ is bounded.

Since we have identified the spaces F_1 with $L^1(R)$, F_5 with $L^1 \cap$ BV, F_4 with $L^1 \cap C_0(R)$ and F_6 with $L^A(R)$ we use any convenient notation without forewarning.

6. __Multipliers for D:__

We shall first obtain a relationship between the multipliers of W and those of D which enables us to establish that D is another Segal algebra on the real line which contains unbounded measures as multipliers.

PROPOSITION 6.1:

Let A,B $\subset L^1(R)$ and $k(x)$ be the function on R, defined in section 5. Then

(i) $\phi \in (A,B)$ if and only if $\frac{i\phi(u)}{u-i} \in (A, k * B)$

(ii) $\phi \in (k*B,A)$ if and only if $\frac{i\phi(u)}{u-i} \in (B,A)$.

PROOF:- It is an easy consequence of the fact that

$$\hat{k}(u) = \frac{i}{u-i}, \quad u \in \hat{R}.$$

(i) and (ii) hold.

COROLLARY 6.2: Let $A \subset L^1(R)$. Then

(i) $\phi \in (A,W)$ if and only if $\frac{i\phi(u)}{u-i} \in (A,D)$

(ii) $\phi \in (D,A)$ if and only if $\frac{i\phi(u)}{u-i} \in (W,A)$.

PROOF: By Proposition 3.3

$$D = k * W.$$

And so the corollary is immediate.

COROLLARY 6.3: Let $A \subset L^1(R)$. Then

(i) $\phi \in (A,L^1)$ if and only if $\frac{i\phi(u)}{u-i} \in (A,L^A)$

(ii) $\phi \in (L^A,A)$ if and only if $\frac{i\phi(u)}{u-i} \in (L^1,A)$

PROOF: By proposition 5.1

$$L^A = k * L^1.$$

and the corollary follows.

REMARK 6.4 Though proposition 6.1 appears rather trivial, the corollaries 6.2 and 6.3 are of no less importance. Putting $A = F_i$, i = 1,2,...6 in Corollary 6.3, we get the corresponding results of Doss. For A = V,W in 6.3, would give us the important relationships of the multipliers among the spaces L^A, W and V. This we shall deal with separately. Corollary 6.2 connects all the multipliers between the Segal algebras D and W. In particular, we arrive at the following interesting result.

__THEOREM 6.5__ The space of multipliers from D into itself is identical with the space of multipliers from W into itself.

__PROOF:__ Taking $A = W$ in 6.2 (i) we have

(12) $\phi \in (W, W)$ if and only if $\frac{i\phi(u)}{u-i} \in (W, D)$.

with $A = D$, 6.2 (ii) gives

(13) $\phi \in (D, D)$ if and only if $\frac{i\phi(u)}{u-i} \in (W, D)$. (12) and (13) together give

$$(D, D) = (W, W)$$

This completes the proof.

Since the space of multipliers on W strictly contains bounded measures the same is true for D also.

__REMARK 6.6__ The construction of D suggests, if one defines again,

$$D_1 = \{ f \in L^A(R) : f' \in D \}$$

Then D_1 is a Segal algebra such that

$$M(D_1, D_1) = M(D, D) = M(W, W).$$

__REMARK 6.7__ In [14] Krogstad has considered the generalization of the Wiener algebra to general locally compact (noncompact) groups and has mentioned this to be the only known Segal algebra on noncompact groups where the multipliers strictly contain the bounded measures. Here we have by 6.5 and 6.6 exhibited that there exist infinitely many such Segal algebras on the real line itself distinct from W having multipliers spaces larger than the space of bounded measures.

We now proceed to obtain another multiplier space for D.

<u>THEOREM 6.8</u>: $(L^1, D) = \widehat{K * V}$

<u>PROOF</u>: Putting $A = L^1$ in 6.2 (i) we get

$$(L^1, D) = \frac{i}{u-i} \quad (L^1, W)$$

By Proposition 5.3,

$$(L^1, W) = \hat{V}$$

and the result follows.

<u>REMARK 6.9</u> Using the result given in [2] we can prove that the space

$$\{ f \in L^A(R) : f' \in V \}$$

endowed with the norm

$$\|f\|_1 + \|f'\|_V$$

is the relative completion of D and is isometrically isomorphic to $M(L^1, D)$. We find that in theorem 6.8 we have arrived at this result in a simplified manner, as we can easily show that

$$k * V = \{ f \in L^A(R) : f' \in V \}$$

similar to the Proposition 5.3.

7. Multipliers for W and V

We shall now characterize some multiplier spaces for W and V and simultaneously study the relationship between the multipliers of W and those of V. Proposition 6.3 will be an essential tool in our arguments.

PROPOSITION 7.1

$$\phi \in (L^A, W) \text{ if and only if } \frac{i\phi(u)}{u-i} \in \hat{V}$$

PROOF: Put $A = W$ in 6.3 (ii) and note that

$$(L^1, W) = \hat{V} \; .$$

PROPOSITION 7.2

(i) $\phi \in (L^A, V)$ if and only if $\frac{i\phi(u)}{u-i} \in \hat{V}$.

(ii) $\phi \in (F_5, V)$ if and only if $\frac{i\phi(u)}{u-i} \in \hat{V}$.

PROOF: By virtue of Proposition 7.1 to prove (i) it is enough to show that

$$(L^A, W) = (L^A, V)$$

That $(L^A, W) \subset (L^A, V)$ follows from Proposition 5.4. To prove the opposite inclusion, let $\phi \in (F_6, V)$. Then $\phi \in (F_6, F_2)$ which gives $\frac{i\phi(u)}{u-i} \in \hat{F_2}$ (Table on page 39). But then $\phi \in (F_6, F_4)$, (again from table). Thus $\phi \in (F_6, V)$ implies $\phi \in (F_6, V \cap F_4)$. But $F_4 = L^1 \cap C_0$ and so $V \cap F_4 = W$.

Thus $(F_6, V) = (F_6, W)$. This proves (i).

(ii) $\phi \in (F_5, V)$ implies $\frac{i\phi(u)}{u-i} \in \hat{V}$ because $k \in F_5$. To prove the converse, suppose $\frac{i\phi(u)}{u-i} \in \hat{V}$. Then there exists $h \in V$ such that

(14) $$\frac{i\phi(u)}{u-i} = \hat{h}(u) \quad , u \in \hat{R} .$$

Let, $f \in F_5$. We claim that $\phi \hat{f} \in \hat{V}$. For any $g \in L^1$, $\hat{h}\hat{g} \in \hat{W}$ as $(L^1, W) = \hat{V}$. From Proposition 5.2 we see that

$$\hat{F}_5 = \frac{i}{u-i} \widehat{M(R)} .$$

Hence, there exists $\mu \in M(R)$ corresponding to $f \in F_5$ such that

(15) $$\frac{u-i}{i} \hat{f}(u) = \hat{\mu}(u) \quad , u \in \hat{R} .$$

Hence

$$\phi(u) \hat{f}(u) \hat{g}(u) = \frac{u-i}{i} \hat{h}(u) \hat{f}(u) \hat{g}(u) \qquad \text{from (14)} .$$

$$= \frac{u-i}{i} \hat{f}(u) \hat{h}(u) \hat{g}(u) = \hat{\mu}(u) \hat{h}(u) \hat{g}(u) \qquad \text{from (15)} .$$

$$= \hat{\mu}(u) \left(\hat{h}(u) \hat{g}(u) \right) \in \hat{W} ,$$

as $\widehat{M(R)} \subset (W, W)$. Hence for each $g \in L^1$,

$$\phi \hat{f} \hat{g} \in \hat{W} .$$

This means that $\phi \hat{f} \in (L^1, W) = \hat{V}$. We have thus proved $\phi \in (F_5, V)$.

<u>COROLLARY 7.3</u>: $(L^1, V) = \hat{V}$.

PROOF: With $A = V$ in 6.3 (ii) we get

$\phi \in (L^A, V)$ if and only if $\frac{i\phi(u)}{u-i} \in (L^1, V)$. Then 7.2 (i) gives the corollary.

PROPOSITION 7.4: (i) $(F_2, V) = (F_4, W)$

(ii) $(F_6, V) = (F_6, W)$

PROOF: Notice that (ii) is already proved in the course of the proof of Proposition 7.2 (i). We shall now prove (i). To this end, let $\phi \in (F_2, V)$. Then by Proposition 5.4 $\phi \in (F_2, F_2)$ which is the same as (F_4, F_4) from the table. Since $\phi \in (F_2, V)$, ϕ takes $\hat{F}_4$ into $\hat{V}$. Hence

$$\phi \in (F_4, F_4 \cap V)$$

As already observed, $F_4 \cap V = W$. Hence

$$(F_2, V) \subset (F_4, W)$$

If, on the other hand, $\phi \in (F_4, W)$, let $f \in F_2$. Using the relation

$$\hat{F}_2 = (F_1, F_4) \quad \text{(See table p. 39)}$$

we can derive

$$\phi \hat{f} \in (F_1, W) = \hat{V}$$

and so

$$(F_4, W) \subset (F_2, V).$$

COROLLARY 7.5

(i) $(F_2,V) = (F_3,V) = (F_4,V) = (F_4,W)$

(ii) $(F_2,W) = (F_3,W)$.

PROOF: (i) Since $F_2 \supset F_3 \supset F_4$ (See inclusion (1)) and $(F_2,V) = (F_4,W)$ from proposition 7.4 , it is enough to show that

$$(F_4,V) = (F_4,W).$$

But clearly, $(F_4,V) \subset (F_4,F_2) = (F_4,F_4)$ from the table. Hence it follows that

$$(F_4,V) \subset (F_4\ V \cap F_4) = (F_4,W)$$

because, $V \cap F_4 = W$. Combining this with the trivial inclusion $(F_4,W) \subset (F_4,V)$, (i) follows: .
To obtain (ii) it is enough to prove that

$$(F_3,W) \subset (F_2,W)$$

Let $\phi \in (F_3,W)$. Since

$$(F_3,W) \subset (F_3,F_4) = (F_2,F_4) \text{ (from the table p. 39)}$$

and

$$(F_3,W) \subset (F_4,W) = (F_2,V) \text{ from proposition 7.4 we get}$$

$$\phi \in (F_2,\ V \cap F_4) = (F_2,W).$$

REMARK 7.6 In section 8, we prove that all the multiplier spaces considered in 7.4, 7.5(i) and (ii) are identical with V.

PROPOSITION 7.7

$$\phi \in (F_5, W) \text{ if and only if } \frac{i\phi(u)}{u-i} \in \hat{W}.$$

PROOF: Let $\phi \in (F_5, W)$. We notice that the function $k \in F_5$ and hence

$$\frac{i\phi(u)}{u-i} = \phi(u)\hat{k}(u) \in \hat{W}.$$

Conversely, suppose there exists $h \in W$ such that

$$\frac{i\phi(u)}{u-i} = \hat{h}(u), \quad u \in \hat{R}.$$

Let $g \in F_5$. Then, as before, by Proposition 5.2, there exists $\mu \in M(R)$ such that

$$\frac{u-i}{i}\hat{g}(u) = \hat{\mu}(u), \quad u \in \hat{R}.$$

Now

$$\phi(u)\hat{g}(u) = \frac{i\phi(u)}{u-i}\frac{u-i}{i}\hat{g}(u) = \hat{h}(u)\hat{\mu}(u)$$

$$= \hat{\mu}(u)\hat{h}(u) \in \hat{W}.$$

as $\widehat{M(R)} \subset (W,W)$. (See lemma 0.3). Hence $\phi \in (F_5, W)$, proving the proposition.

PROPOSITION 7.8: $(W, F_5) = (W, F_6)$.

PROOF: Obviously, $(W, F_6) \subset (W, F_5)$. Let $\phi \in (W, F_5)$. But, from Proposition 5.4 and the table

$$(W,F_5) \subset (F_6,F_5) = (F_1,F_1) = \widehat{M}(R)$$

Hence, there exists $\mu \in M(R)$ such that

$$\phi(u) = \hat{\mu}(u) \quad , \quad u \in \hat{R} .$$

As $\widehat{M}(R) \subset (W,W)$, we see that $\phi \in (W,W)$ and hence we have

$$\phi \in (W, W \cap F_5)$$

But, since $W \subset F_4$, from theorem 4.12, we get

$$\phi \in (W,F_6).$$

<u>PROPOSITION 7.9</u>: $(V,F_5) = (W,F_6) = (W,F_5)$.

<u>PROOF</u>: In the light of the Proposition 7.8 it is enough to prove that

$$(W,F_6) \subset (V,F_5)$$

Let $\phi \in (W,F_6)$ and $f \in V$. Then $\hat{f} \in (L^1,W)$, by Proposition 5.3. Choosing an arbitrary function in F_1 and using the properties of ϕ and $\hat{f}$ it is easy to see that

$$\phi \hat{f} \in (F_1,F_6)$$

But $(F_1,F_6) = \hat{F}_5$ from (the Table, p. 39). Hence

$$\phi \hat{f} \in \hat{F}_5 \qquad \text{for every } f \in V .$$

That is, $\phi \in (V,F_5)$.

LEMMA 7.10: If $\phi \in (W,F_1)$ then $\frac{i\phi(u)}{u-i} \in \hat{F}_1$.

PROOF: Let $\phi \in (W,F_1)$. First, consider the two functions defined by

$$K_1(x) = \begin{cases} -2\pi e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$K_2(x) = \begin{cases} 0 & x > 0 \\ 2\pi e^{x} & x \leq 0 \end{cases}$$

Then K_1 and K_2 belong to $L^1 \cap BV$ and have nonvanishing Fourier transforms as follows

$$\hat{K}_1(u) = \frac{i}{u-i}, \quad u \in \hat{R}.$$

and

$$\hat{K}_2(u) = \frac{i}{u+i}, \quad u \in \hat{R}.$$

Then, the function K_1-K_2 is a continuous function which belongs to W. In fact, $K_1-K_2 \in L^A$. But this fact is unnecessary for our purposes. Define

$$K_3(x) = -\tfrac{1}{2}(K_1-K_2)(x), \quad x \in R.$$

Then $K_3 \in W$ with

$$\hat{K}_3(u) = \frac{1}{1+u^2}, \quad u \in \hat{R}$$

Hence, since $\phi \in (W,F_1)$,

(16) $$\frac{\phi(u)}{1+u^2} \in \hat{F}_1 = (L^1(R))^\wedge.$$

Also, since ϕ is bounded on $\hat{R}$ being a multiplier from the Segal algebra W into the Segal algebra $L^1(R)$, (Proposition 0.24) we obtain

(17) $$\frac{\phi(u)}{1+u^2} \in L^1(\hat{R})$$

From (16) and (17) we can use the inversion theorem to obtain

(18) $$\frac{\phi(u)}{1+u^2} \in \hat{F}_4$$

Now $\phi \in (W,F_1)$ implies $\phi \in (F_6,F_1)$ which in turn implies that

(19) $$\frac{i\phi(u)}{u-i} \in \widehat{M(R)}$$ (See the table, p. 39.)

Since $\frac{-i}{u+i} = -\hat{k}_2(u) \in \hat{F}_5$

and $(F_5,F_5) = \widehat{M(R)}$, we get

(20) $$\frac{\phi(u)}{1+u^2} = \frac{-i}{u+i}\,\frac{i\phi(u)}{u-i} \in \hat{F}_5$$ using (19).

(18) and (20) put together give

(21) $$\frac{\phi(u)}{1+u^2} \in \hat{F}_4 \cap \hat{F}_5$$

But from Theorem 4.12, $F_4 \cap F_5 = F_6$. Hence

(22) $$\frac{\phi(u)}{1+u^2} \in \hat{F}_6$$

Now, using proposition 4.6, one can derive, (22) implies that

$$\frac{i\,\phi(u)}{u-i} \in \hat{F}_1$$

which is what we wished to show.

COROLLARY 7.11: $M(W,L^A) \subset L^1(R)$.

PROOF: From, putting $A = W$ in 6.3 (i) we get

$\phi \in (W,L^A)$ if and only if $\frac{u-i}{-i}\phi(u) \in (W,L^1)$

But, from lemma 7.10 above, this implies

$$\phi \in \hat{F}_1 = \hat{L^1}(R)$$

Since W and L^A are Segal algebras, appealing to proposition 0.24 the corollary follows.

PROPOSITION 7.12 $(V,F_5) = (V,F_6)$.

PROOF: Clearly $(V,F_6) \subset (V,F_5)$. So, let $\phi \in (V,F_5)$. First, we observe that $\phi \in (W,F_6)$ from proposition 7.9. Then from the corollary 7.11,

$$\phi(u) = \hat{f}(u)$$

for some $f \in L^1(R)$. Thus for any $g \in V$, $\phi\hat{g} = \hat{f}\hat{g} \in \hat{W}$

But $\phi \in (V,F_5)$. Hence from theorem 4.12 again,

$$\phi \in (V,F_6)$$

COROLLARY 7.13 $(W,F_1) = (V,F_1)$.

PROOF: From proposition 7.9 and 7.12,

$$(W,F_6) = (V,F_6) \tag{23}$$

In 6.3 (i), put $A = W,V$. Then from (23) the corollary follows.

PROPOSITION 7.14 $(W,F_2) = (V,F_2)$.

PROOF: It suffices to prove $(W,F_2) \subset (V,F_2)$. Let $\phi \in (W,F_2)$ and f any function in V. Then, for every $g \in L^1(R)$, we see that

$$\phi \hat{f} \hat{g} \in \hat{F_2}$$

as $\hat{f}\hat{g} \in \hat{W}$. Thus $\phi \hat{f} \in (F_1,F_2)$ which is $\hat{F}_2$. (See the table on p. 39). This means that

$$\phi \in (V,F_2) \quad .$$

This completes the proof.

We shall now proceed to get the important identification of the space of multipliers of V into itself with that of W into itself. We do this by identifying both the spaces (W,W) and (V,V) with the multiplier space (W,V).

PROPOSITION 7.15: $(W,W) = (V,V) = (W,V)$.

PROOF: First, clearly, $(V,V) \subset (W,V)$. If $\phi \in (W,V)$ and $f \in V$, let $g \in F_1$. Then

$$\hat{f}\hat{g} \in \hat{W} \qquad \text{as } (L^1,W) = \hat{V} \quad .$$

Therefore,

$$\phi \hat{f}\, \hat{g} \in \hat{V} \quad \text{for all } g \in F_1$$

Or,

$$\phi \hat{f} \in (L^1, V) \text{ for } f \in V .$$

But $(L^1, V) = \hat{V}$ from corollary 7.3. Hence $\phi \in (V, V)$.

We shall now establish $(W, W) = (W, V)$. Obviously, by Proposition 5.4, $(W, W) \subset (W, V)$. If $\phi \in (W, V)$, then $\phi \in (F_6, V)$, again by Proposition 5.4 . Then by Proposition 7.4 (ii) we have

(24) $$(W, V) \subset (F_6, W)$$

Let T_ϕ be the operator associated to ϕ given in Proposition 5.5. Then

$$T_\phi : W \longrightarrow V$$

is given by $T_\phi f = g$ for each f in W where $\hat{g} = \phi \hat{f}$. Since T_ϕ is bounded, there exists a constant $C > 0$ such that

(25) $$\| T_\phi f \|_V \leq C \, \| f \|_W , \quad f \in W.$$

But, from (24), $\phi \hat{f} \in \hat{W}$ whenever $f \in F_6$. Hence $T_\phi f \in W$ for each $f \in F_6$. As

$$\|f\|_V = \|f\|_W \quad \text{for} \quad f \in W \subset V,$$

the inequality (25) gives

$$(26) \qquad \|T_\phi f\|_W \leq C\|f\|_W, \quad f \in F_6.$$

Since $F_6 = L^A(R)$ is dense in W, we can prove that can be extended to W with

$$\|T_\phi f\|_W \leq C\|f\|_W, \quad f \in W.$$

Hence we conclude $\phi \in (W,W)$.

<u>COROLLARY 7.16:</u> $(V,F_4) \cap (W,W) = (V,W)$.

<u>PROOF:</u> $(V,F_4) \cap (W,W) = (V,F_4) \cap (V,V)$

$$= (V,F_4 \cap V) = (V,W).$$

<u>COROLLARY 7.17:</u> $(V, k*V) = (W,D) = (W, k*V)$.

<u>PROOF:</u> Putting $B = V$, $A = V,W$ in proposition 6.1 (i) and $A = W$ in 6.2 (i) and using the above proposition 7.15, we get the corollary.

<u>PROPOSITION 7.18</u>

$$\text{(i)} \ (V,W) \supsetneqq \widehat{L^1}(R)$$

and

$$\text{(ii)} \ (V,W) \cap \widehat{M}(R) = \widehat{L^1}(R).$$

<u>PROOF:</u> (i) That $\widehat{L^1}(R) \subset (V,W)$ is clear. Put $\hat{X} = (V,W)$. Since $(V,W) \subset (W,W)$, X is a subspace of

tempered distributions on R and hence $\hat{X}$ is meaningful. By the kind of arguments used in the foregoing, it can be proved that

$$(L^1, X) = (V, V)$$

Since $(V,V) = (W,W)$ and $\hat{M(R)} \subsetneqq (W,W)$ (See, for example [14]), X cannot be $L^1(R)$.

(ii) Let $\phi \in (V,W) \cap \hat{M(R)}$. Since $k(x) \in V$,

$$\frac{i\phi(u)}{u-i} = \phi(u)\hat{k}(u) \in \hat{W}.$$

But $\phi \in \hat{M(R)}$ implies $\frac{i\phi(u)}{u-i} \in \hat{F}_5$ by Proposition 5.2. Hence

$$\frac{i\phi(u)}{u-i} \in \hat{W} \cap \hat{F}_5 = \hat{F}_6, \tag{27}$$

from Theorem 4.11. Now, using proposition 4.6 (27) gives $\phi \in \hat{F}_1$. Hence (ii) follows.

PROPOSITION 7.19 :

(i) $\hat{F}_5 \subsetneqq (W,D) \subset \hat{V}$.

(ii) $\hat{F}_6 \subsetneqq (V,D) \subset \hat{W}$.

PROOF:

(i) Let $\phi \in (W,D)$. From corollary 7.17, $\phi \in (V, k*V)$

Since $k \in V$, $\phi\hat{k} \in \hat{k*V} = \frac{i\hat{V}}{u-i}$ which implies that $\phi \in \hat{V}$.

In 6.2, put $A = W$. Since $\hat{M}(R) \underset{\neq}{\subset} (W,W)$ and Proposition 5.2 gives

~~says~~ $\frac{u-i}{i}\hat{F}_5 = \widehat{M(R)}$, we get (i)

In 6.2(i), put $A = V$ and use Propositions 7.18(i) and 5.1 to get

$$\hat{F}_6 \underset{\neq}{\subset} (V,D)$$

If $\phi \in (V,D)$, then $\frac{i\phi(u)}{u-i} \in \hat{D} = \widehat{k * W}$ since $\frac{i}{u-i} \in \hat{V}$. Hence $\phi \in \hat{W}$. Or, (ii) holds.

In the foregoing proofs, one observes that the techniques used are extremely simple and the results obtained are nevertheless significant. Before closing this chapter, we add one more important multiplier result for W, using a different technique.

8. Hormander's lemma for the Wiener algebra

In his study of translations invariant operators [11] Hormander proved the following lemma.

LEMMA 8.1 (Hormander): Let $y \in R^n$ and $f \in L^p(R^n)$. If $1 \leq p < \infty$, then

$$\| \tau_y f + f \|_p \longrightarrow 2^{1/p} \| f \|_p \quad \text{as } y \to \infty .$$

The result is valid for $p = \infty$ whenever f vanishes at infinity.

We shall now state and prove its analogue for the space W as it will be used in our investigations.

<u>THEOREM 8.2</u> (Hormander's lemma for W): Let $f \in W$. Then

$$|||f + \tau_y f|||_W \longrightarrow 2|||f|||_W, \text{ as } y \longrightarrow \infty.$$

<u>PROOF</u>: Let $C_c(R)$ denote the space of continuous functions on R with compact support. Then $C_c(R)$ is dense in W. Let $\varepsilon > 0$ be given and $f \in W$. Then f can be expressed in the form $f = g + e$ where $|||e|||_W < \varepsilon$ and g has compact support, and

(28) $$\left| \, |||f|||_W - |||g|||_W \right| < \varepsilon,$$

(29) $$\left| \, |||\tau_y f + f|||_W - |||\tau_y g + g|||_W \right| < 2\varepsilon.$$

If $|y|$ is sufficiently large, then the supports of g and $\tau_y g$ have empty intersection and so

(30) $$|||g + \tau_y g|||_W = |||g|||_W + |||\tau_y g|||_W$$

Since ε is arbitrary, (28), (29) and (30) give the desired result.

THEOREM 8.3: The multiplier space $M(L^1 \cap C_0, W)$ coincides with the multiplier space $M(L^1, W)$ and so $M(L^1 \cap C_0, W)$ is isometrically isomorphic to the space V.

PROOF: Since $L^1 \cap C_0 \subset L^1$, we need to prove only that $M(L^1 \cap C_0, W) \subset M(L^1, W)$. To this end, let $T \in M(L^1 \cap C_0, W)$. Then there exists $C > 0$ such that, for $f \in L^1 \cap C_0(R)$,

$$|||Tf|||_W \leq C(\|f\|_1 + \|f\|_\infty)$$

Since T is linear and translation invariant, for $y \in R$ we have

$$|||Tf + \tau_y Tf|||_W = |||T(f + \tau_y f)|||_W$$
$$\leq C(\|f + \tau_y f\|_1 + \|f + \tau_y f\|_\infty)$$

Now, letting $y \to \infty$ and applying theorems 8.2 and 8.2, we get

$$2\,|||Tf|||_W \leq C(2\|f\|_1 + \|f\|_\infty)$$

so that

$$|||Tf|||_W \leq C(\|f\|_1 + 2^{-1}\|f\|_\infty)$$

Repeating the argument n times, this yields

$$|||Tf|||_W \leq C(\|f\|_1 + 2^{-n}\|f\|_\infty).$$

Since the left hand side is independent of n , we let $n \to \infty$ on the right to get

$$\|Tf\|_W \leq C\|f\|_1$$

for all $f \in L^1 \cap C_0(R)$. Since $L^1 \cap C_0(R)$ is dense in $L^1(R)$, we can extend T as a multiplier from L^1 to W. That $M(L^1,W)$ is isometrically isomorphic to V is a result of Burnham and Goldberg [2] . This completes the proof.

<u>COROLLARY 8.4</u> $(F_j, V) = (F_j, W) = \hat{V}$ for $j = 1,2,3,4$.

<u>PROOF</u>: Since $F_1 = L^1(R)$ and $F_4 = L^1 \cap C_0(R)$, are Segal algebras, Proposition 0.24 is applicable. Hence we get immediately from the above theorem,

$$(F_1,W) = (F_4,W) = \hat{V} \qquad (31)$$

Since the inclusion $F_1 \supset F_2 \supset F_3 \supset F_4$ holds, from (31) we can write, using Proposition 5.4,

$$(F_j,W) = \hat{V} \quad \text{for} \quad j = 1,2,3,4 .$$

From corollaries 7.3 and 7.5 (i) it follows that

$$(F_j,V) = \hat{V} \quad \text{for} \quad j = 1,2,3,4 .$$

<u>THEOREM 8.5</u>: For $1 < p < \infty$, the multiplier space $M(L^1 \cap L^p, W)$ is isometrically isomorphic to the space V .

Proof is analogous to that of theorem 8.3

CHAPTER III

MULTIPLIER SPACES OF SEGAL ALGEBRAS AS DUAL SPACES

In this chapter our object is to characterise the multiplier spaces for Segal algebras as dual spaces. The main result is an identification of the multiplier spaces for Segal algebras and normed ideals on general locally compact abelian groups, (Theorem 10.4). As a consequence, we are able to identify the multipliers from certain normed ideals into themselves (Theorem 10.6). Furthermore, whenever a normed ideal is a dual space, the space of multipliers into the corresponding Segal algebra turns out to be a dual space (Theorem 10.7). This enables us to characterize the multiplier spaces as dual spaces for several special cases. Many known results ([2] . [14] . [4]) that have been proved using different techniques are easily derived from our result.

9. Several examples of normed ideals and Segal algebras

Let us briefly recall the properties of a normed ideal $N(G)$, on G locally compact abelian. (See definition 0.12 and lemmas 0.13, 0.14, 0.15 and 0.16)

$N(G)$ contains a unique Segal algebra $N_o(G)$ which is the continuously translating subspace of $N(G)$. (0.18) $N_o(G)$ is actually the closure of $B(G)$ in $N(G)$.

We shall now collect some examples for $N(G)$ and $N_o(G)$.

PROPOSITION 9.1 On the real line R, the space $L^1 \cap BV$ endowed with the norm

$$\|f\|_{L^1 \cap BV} = \|f\|_1 + V_f \quad , f \in L^1 \cap BV$$

(V_f stands for the total variation of f) is a normed ideal. The space $L^A(R)$ is the continuously translating subspace of $L^1 \cap BV$ and hence is a Segal algebra.

PROOF: That $L^1 \cap BV$ satisfies the properties N_1, N_2 and N_3 can be easily verified. Let f be a function in $L^1 \cap BV$ satisfying

$$\| \tau_y f - f\|_{L^1 \cap BV} \longrightarrow 0 \quad \text{as } y \longrightarrow 0 \text{ in } R$$

Then $V_{(\tau_y f - f)} \longrightarrow 0$ as $y \longrightarrow 0$. But $\|\tau_y f - f\|_\infty \leq V_{(\tau_y f - f)}$. Hence $\|\tau_y f - f\|_\infty \to 0$ as $y \to 0$ which shows that f is uniformly continuous. From theorem 4.11 we can immediate see that $f \in L^A(R)$. Hence the proposition follows. Notice that for $f \in L^A(R)$,

$$\|f\|_{L^1 \cap BV} = \|f\|_1 + \|f'\|_1$$

which is the L^A - norm given in 0.9.

PROPOSITION 9.2 The space V with the norm

$$|||f|||_V = \sup_{m \in R} \sum_{k=-\infty}^{\infty} \operatorname{ess.sup.}_{k \leq x \leq k+1} |f(x+m)| , \quad f \in V.$$

is a normed ideal and possesses the Wiener algebra W as its continuously translating subspace.

PROOF: Again, from the definition of the norm in V, N1, N2 and N3 are satisfied. If, for any $f \in V$,

$$|||\tau_y f - f|||_V \longrightarrow 0 \quad \text{as} \quad y \to 0.$$

then $|||\tau_y f - f||_\infty \longrightarrow 0$ as $y \to 0$.

Hence f is uniformly continuous. This immediately gives $f \in W$. For any $f \in W$, the essential supremum over $[K, K+1]$ reduces to the maximum value and hence the norm $|||\cdot|||_V$ in W gives the Segal norm of W defined in 0.8 .

PROPOSITION 9.3 If G is locally compact abelian, $L^1 \cap L^\infty(G)$ is a normed ideal having $L^1 \cap C_0(G)$ as its continuously translating subspace. This shows that $L^1 \cap C_0(G)$ is a Segal algebra under the norm

$$\|f\|_{L^1\cap C_0} = \|f\|_1 + \|f\|_\infty .$$

The proof follows easily.

PROPOSITION 9.4 Let T denote the circle group $(-\pi, \pi]$. Then define

$$N_1(T) = \left\{ f \in L^1(T) : \hat{f}(n) = O\left(\frac{1}{\log|n|}\right) \right\}$$

and set the norm in $N_1(T)$ by

$$\|f\|_{N_1(T)} = \|f\|_1 + \max_{1<|n|\leq\infty} |\hat{f}(n) \log|n||$$

PROPOSITION 9.5 $N_1(T)$ is a normed ideal on the circle group and if we define

$$S_1(T) = \left\{ f \in L^1(T) : \hat{f}(n) = o\left(\frac{1}{\log|n|}\right) \right\}$$

then $S_1(T)$ is the continuously translating subspace of $N_1(T)$ and hence is a Segal algebra under the norm $\|\cdot\|_{N_1}$.

PROOF: That $N_1(T)$ is a normed ideal can be proved easily. Now let $f \in N_1(T)$ and

$$\|\tau_y f - f\|_{N_1(T)} \longrightarrow 0 \quad \text{as} \quad y \to 0 \text{ in } T .$$

Then

$$\max_{1<|n|\leq\infty} |e^{-iny}-1|\,|\hat{f}(n)\log|n|| \to 0 \tag{32}$$

as $y \to 0$. This would imply that given $\varepsilon > 0$ there exists $\delta > 0$ such that for $y \in (-\delta, \delta)$

$$|e^{-iny}-1|\;|\hat{f}(n)\log|n|| < \varepsilon \tag{33}$$

for all $n \in Z \setminus \{-1, 0, 1\}$. Now choose an integer $n_0 > 1$ such that

$$\delta > \pi / n_0$$

and set $\delta_n = \frac{\pi}{n}$ for all n with $|n| \geq n_0$. Then, by the choice of n_0, $|\delta_n| = \frac{\pi}{|n|} \leq \frac{\pi}{n_0} < \delta$, so that $\delta_n \in (-\delta, \delta)$ for all n , such that $|n| \geq n_0$. Thus δ_n satisfy (33) giving

$$|e^{-in\delta_n}-1|\;|\hat{f}(n)\log|n|| < \varepsilon \tag{34}$$

for $|n| \geq n_0$. But $|e^{-in\delta_n}-1| = 2\sin\frac{n\delta_n}{2}$ $= 2\sin\pi/2 = 2$. Hence (34) becomes

$$|\hat{f}(n)\log|n|| < \epsilon/2 < \varepsilon$$

for $|n| \geq n_0$ which shows that $f \in S_1(T)$. Hence the proposition.

We mention that $S_1(T)$ is given as an example of a Segal algebra on T by Goldberg (See [1]) with a slight variation in the norm.

DEFINITION 9.6 Let G be a locally compact abelian group and $\hat{G}$ its character group. If α is a function on $\hat{G}$ such that α is locally bounded and

$$\alpha(\xi) \geq 1 \qquad \text{for all } \xi \in \hat{G}$$

then define

$$N(\alpha) = \left\{ f \in L^1(G) : \sup_{\xi \in \hat{G}} |\alpha(\xi)\hat{f}(\xi)| < \infty \right\}$$

Then $N(\alpha)$ is a Banach space under the norm

(35) $$\|f\|_{N(\alpha)} = \|f\|_1 + \sup_{\xi \in \hat{G}} |\alpha(\xi)\hat{f}(\xi)| , \quad f \in N(\alpha).$$

Define a subspace $S(\alpha)$ of $N(\alpha)$ by

$$S(\alpha) = \{ f \in L^1(G) : \alpha\hat{f} \text{ vanishes at infinity} \}$$

PROPOSITION 9.7 $N(\alpha)$ is a normed ideal on the locally compact abelian group G and $S(\alpha)$ is the continuously translating subspace of $N(\alpha)$. Hence $S(\alpha)$ is a Segal algebra under the norm of $N(\alpha)$ given in (35).

The proof essentially takes the lines of that of Proposition 9.5 . Notice that $S(\alpha)$ is the Segal algebra defined by Riemersma in [25] .

DEFINITION 9.8 If I: denotes the real numbers modulo 1, then the functions on I are the same as the functions on the real line R having period 1 . Now, let $0 < \alpha < n$ where n is a positive integer. Also, let $\Lambda(\alpha, n)$ denote the class of all complex valued functions f on the real line R with period 1 such that there exists a constant $K > 0$ satisfying the condition

$$\sup_{x \in R} |\Delta_t^n f(x)| \leq K |t|^\alpha, \quad \text{as } t \to 0 \tag{36}$$

where the one-sided n^{th} difference is given by

$$\Delta_t^n f(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x + \overline{n-r}\, t) \tag{37}$$

We denote by $\lambda(\alpha, n)$ the subset of $\Lambda(\alpha, n)$ consisting of those functions f which satisfy the condition

(38) $$\sup_{x \in R} |\Delta_t^n f(x)| = o(|t|^\alpha)$$

Set

$$\|f\|_\infty = \sup_{x \in R} |f(x)|$$

$$\|f\|_\alpha = \sup_{x, t \in R} \frac{|\Delta_t^n f(x)|}{|t|^\alpha}$$

and define

(39) $$\|f\| = \max \{ \|f\|_\infty , \|f\|_\alpha \}$$

for each $f \in \Lambda(\alpha, n)$. Then the following result is established in [30] .

PROPOSITION 9.9 [30]

(i) $\Lambda(\alpha, n)$ is a Banach space and $\lambda(\alpha, n)$ is a closed linear subspace of $\Lambda(\alpha, n)$.

(ii) The continuously translating subspace of $\Lambda(\alpha, n)$ is $\lambda(\alpha, n)$.

(iii) $\lambda(\alpha, n)$ is the closed linear span of trigonometric polynomials in $\Lambda(\alpha, n)$.

(iv) $\Lambda(\alpha, n)$ is isometrically isomorphic to the second dual of $\lambda(\alpha, n)$.

By virtue of (i), (ii) and (iii) of the above proposition and definition of the norm in (39) we can easily obtain the following proposition.

PROPOSITION 9.10 $\Lambda(\alpha, n)$ is a normed ideal on the compact group I of real numbers modulo 1, and $\lambda(\alpha, n)$ is a Segal algebra uniquely contained in $\Lambda(\alpha, n)$.

10. Multipliers of Segal algebras in relation to normed ideal

Before giving the main result of this chapter, we shall refer to the definitions of a Banach module, module homomorphism and module tensor product given in 0.19, 0.20 and 0.21. We mention a few results of Rieffel [24] which are to be used in the sequel. In fact, theorem 10.3 below plays important role in our investigations.

THEOREM 10.1 (Rieffel) If A is a Banach algebra with approximate identity and X is an A - module, then

$$A \underset{A}{\otimes} X \cong X_e$$

where X_e is the essential part of X and $\cong$ denotes isometric isomorphism.

THEOREM 10.2 (Rieffel) If A is a Banach algebra with an approximate identity bounded by 1. If X is an essential

A - module and Y is an A - module then

$$\mathrm{Hom}_A(X, Y) \cong \mathrm{Hom}_A(X, Ye) .$$

THEOREM 10.3 (Rieffel) If A is a Banach algebra with approximate identity bounded by 1 and X and Y are A - modules, then

$$\mathrm{Hom}_A(X, Y^*) \cong (X \otimes_A Y)^*$$

where Y^* stands for the dual space of Y .

Observing that the normed ideals considered in section 9 are all translation invariant, we assume the general normed ideal $N(G)$ considered here to be closed with respect to translations. This facilitates the definition of the multiplier, which commutes with translations, for $N(G)$.

We shall now state and prove the main result.

THEOREM 10.4 Let G be a locally compact abelian group and $N(G)$ be any normed ideal on G . If $N_0(G)$ is the closure of $B(G)$ in $N(G)$, then the following result holds for any Segal algebra $S(G)$:

$$(S, N) = (S, N_0) \approx M(S, N) = M(S, N_0) \cong \mathrm{Hom}_{L^1}(S, N) = \mathrm{Hom}_{L^1}(S, N_0)$$

$\approx$ stands for isomorphism.

PROOF: First, observing that $N_o(G)$ is a Segal algebra, from Proposition 0.24 it follows that

(40) $$(S, N_o) \cong M(S, N_o) \cong \mathrm{Hom}_{L^1}(S, N_o) \; .$$

Again, we recall that S and N are L^1-modules, where S is in addition, essential. Also N_o is the essential part of N (See 0.16). Hence, using theorem 10.2, we get immediately

(41) $$\mathrm{Hom}_{L^1}(S, N) \cong \mathrm{Hom}_{L^1}(S, N_o) \; .$$

Now, to prove $M(S,N) = M(S,N_o)$, it is enough to prove that

$$M(S, N) \subset M(S, N_o) \; .$$

Since N_o is a closed subspace of N . Let $T \in M(S,N)$.

Then

(42) $$\|Tf\|_N \leq \|T\| \, \|f\|_S \; , \qquad f \in S \; .$$

Since T commutes with translations and is linear,

(43) $$\|\tau_y Tf - Tf\|_N = \|T(\tau_y f - f)\|_N \leq \|T\| \, \|\tau_y f - f\|_S$$

from (42) . Since $f \in S$, by the property S4 of Segal algebras (See 0.1), if $y \to 0$ the R.H.S. of (43) tends to zero. Hence

$$\| \tau_y Tf - Tf \|_N \longrightarrow 0 \qquad \text{as} \quad y \to 0$$

which implies that $Tf \in N_0(G)$ from (0.18). Thus we have established

(44) $$M(S, N) = M(S, N_0)$$

Now, let $\phi \in (S,N)$. This means that to each $f \in S$ there exists $g \in N$ such that $\phi \hat{f} = \hat{g}$.

Define

$$T_\phi : S \longrightarrow N ,$$

by

$$T_\phi f = g ,$$

where $\phi \hat{f} = \hat{g}$. Then T_ϕ is clearly linear and commutes with convolutions. To see that T_ϕ is bounded, we appeal to the closed graph theorem. See Proposition 5.5.

The mapping $\phi \longrightarrow T_\phi$ is clearly an isomorphism of (S,N) into Hom_{L^1} (S,N) . Now the obvious inclusion (S,N_0) $\subset$ (S,N) together with (40), (41) and (44) yield the result. This completes the proof.

REMARK 10.5 In the proof of the above theorem we ntoce that the density of the Segal algebra S is not used anywhere. Hence the result is valid for any homogeneous Banach space which is an L^1-module, in the place of S. Similarly, the role of N and N_0 relies mainly on the continuity of translation and the fact that

$$(N)_e = N_0$$

Hence the result can be easily extended to spaces other than normed ideals and Segal algebras. For example, if B is any homogeneous Banach space which is an $L^1(G)$ - module then we can obtain

$$M(B, M) = M(B, L^1)$$

$$\cong \mathrm{Hom}_{L^1}(B, M) \cong \mathrm{Hom}_{L^1}(B, L^1).$$

where M stands for the convolution measure algebra $M(G)$.

THEOREM 10.6 If N is a normed ideal on G, such that

$$M(L^1, N) \cong N$$

then

$$M(N, N) = M(N_0, N) = M(N_0, N_0).$$

(We have omitted the other equivalent multiplier spaces for the sake of brevity).

PROOF: That $M(N_0, N) = M(N_0, N_0)$ follows from theorem 10.4, putting $S = N_0$. Clearly, $M(N,N) \subset M(N_0,N)$. Hence, let $T \in M(N_0,N)$. Since, again from 10.4,

$$M(L^1,N) = M(L^1,N_0)$$

for $f \in N$, $g \in L^1$, $f * g \in N_0$ and also,

$$\|f * g\|_{N_0} \leq \|f\|_N \|g\|_1 . \tag{45}$$

Then, extend T to N by defining, $\{e_\alpha\}$ being the approximate identity,

$$Tf = \lim_\alpha T(f * e_\alpha) \quad , \quad f \in N.$$

From (45), T satisfies

$$\|Tf\|_N \leq \|T\| \|f\|_N \quad , \quad f \in N .$$

As T commutes with translations for elements in N_0 and N_0 is dense in L^1, we can prove that $T \in M(N,N)$.

THEOREM 10.7: If S is a Segal algebra on G and N is any normed ideal on G such that N is a dual space B^* of an L^1-module B, then the space of multipliers from S into the Segal algebra N_0 is a dual space given by

$$M(S, N_0) \cong (S \underset{L^1}{\otimes} B)^*$$

In particular,

$$M(N_0, N_0) \cong (N_0 \underset{L^1}{\otimes} B)^*$$

Theorem 10.7 is a direct consequence of theorem 10.4, if we use theorem 10.3 of Rieffel.

11. Application to Special Cases

THEOREM 11.1

(i) $M(L^1, L^1 \cap BV) = M(L^1, L^A)$.

(ii) $M(L^1 \cap C_0, L^1 \cap BV) = M(L^1 \cap C_0, L^A)$.

(iii) $M(L^1 \cap BV, L^1 \cap BV) = M(L^A, L^1 \cap BV) = M(L^A, L^A)$.

(iv) $M(W, L^1 \cap BV) = M(W, L^A)$.

PROOF: We first realize that, by Proposition 9.1, we can get, for any Segal algebra $S(R)$ on the real line R

$$M(S, L^1 \cap BV) = M(S, L^A)$$

as a consequence of Theorem 10.4. Hence, when $S_1 = L^1, W$ and $L^1 \cap C_0$, we get (i), (iv) and (ii). Observe that in [2] Burnham and Goldberg have obtained the result,

$$M(L^1, L^A) \cong L^1 \cap BV.$$

Hence Theorem 10.6 is applicable, so that we get (iii).

Notice that (i), (ii) and (iii) follow from the results of Doss [4] . (See the table given in Section 5). But his proofs are based on classical methods. Compare Proposition 7.8 with (iv).

In [17] Liu and Wang have proved that the space $L^1 \cap L^p(R)$ for $1 \leq p \leq \infty$ is a dual space. The result can easily be extended to G, any locally compact abelian group. Here, we state the theorem for $L^1 \cap L^\infty(G)$.

<u>THEOREM 11.2 (Liu and Wang)</u> Let P be the space of functions on G defined by

$$P = \left\{ g : g = g_1 + g_2 \text{ where } g_1 \in C_0(G) \text{ and } g_2 \in L^1(G) \right\}$$

For $g \in P$, define the norm as

$$\|g\| = \inf \left\{ \sup_{x \in G} |g_1(x)| + \|g_2\|_1 \right\}$$

where the infimum is taken over all such decompositions of g in the form $g_1 + g_2$. Then P becomes a Banach space in this norm. Further, the conjugate space P^* of P is isometrically isomorphic to $L^1 \cap L^\infty(G)$, the operation of f in $L^1 \cap L^\infty(G)$ on $g \in P$ being given by

$$\langle g, f \rangle = \int_G g(x)\, f(x)\, dx$$

THEOREM 11.3 Let S be any Segal algebra on G . Then the space of multipliers $M(S, L^1 \cap C_0)$ is isometrically isomorphic to the dual space $(S \underset{L^1}{\otimes} P)^*$

PROOF: By Proposition 9.3, $L^1 \cap C_0$ is the continuously translating subspace of the normed ideal $L^1 \cap L^\infty(G)$ which is a dual space as seen from theorem 11.2. Hence the theorem is a direct consequence of 10.7 .

COROLLARY 11.4

(i) $M(L^1, L^1 \cap L^\infty) = M(L^1, L^1 \cap C_0) \cong L^1 \cap L^\infty$.

(ii) $M(L^1 \cap L^\infty, L^1 \cap L^\infty) = M(L^1 \cap C_0, L^1 \cap C_0) = M(L^1 \cap C_0, L^1 \cap L^\infty)$

(iii) $M(W, L^1 \cap L^\infty) = M(W, L^1 \cap C_0)$ on the real line.

(iv) On the real line R, the multiplier spaces $M(L^1, L^1 \cap C_0)$ and $M(W, L^1 \cap C_0)$ are dual spaces.

PROOF: (i) By theorem 10.4, clearly, it follows that

$$M(L^1, L^1 \cap L^\infty) = M(L^1, L^1 \cap C_0) .$$

In the theorem above, put $S = L^1$. Then

$$M(L^1, L^1 \cap C_0) \cong (L^1 \underset{L^1}{\otimes} P)^*$$

As it can be easily verified that P is an essential L^1-module appealing to theorem 10.1, we get

$$L^1 \underset{L^1}{\otimes} P \cong P$$

so that, $M(L^1, L^1 \cap L^\infty) \cong P^* = L^1 \cap L^\infty$. Therefore (i) holds.

Since (i) is true, theorem 10.6 can be applied to give

$$M(L^1 \cap L^\infty, L^1 \cap L^\infty) = M(L^1 \cap C_0, L^1 \cap L^\infty) = M(L^1 \cap C_0, L^1 \cap C_0).$$

This is (ii). To get (iii) put S = W, $N = L^1 \cap L^\infty(R)$ and $N_0 = L^1 \cap C_0(R)$ in theorem 10.4 . (iv) obviously follows from theorem 11.3 . Hence the corollary ; notice that (i) and (ii) are also given by Doss [4] . See the table of section 5.

We shall now prove that the normed ideal V defined in 9.2 is a dual space under the equivalent norm

$$\|f\|_V = \sum_{k=-\infty}^{\infty} \underset{k \leq x \leq k+1}{\text{ess. sup.}} |f(x)|, \quad f \in V.$$

We shall not distinguish between the equivalent norms to avoid excess of details.

THEOREM 11.5 Let E denote the space of all locally integrable functions f on R such that, $k \in Z$,

$$\int_k^{k+1} |f(t)|\, dt \longrightarrow 0 \qquad \text{as } k \to \infty$$

With norm defined by $\|f\|_E = \sup_{k \in Z} \int_k^{k+1} |f(t)|\, dt$, $f \in E$,

E becomes a Banach space. Moreover the space V is the dual of E. Every bounded linear functional L on E has the representation

$$(46) \qquad L(f) = \int_R f(t)\, g(t)\, dt, \qquad f \in E.$$

where $g \in V$ and $\|L\| = \|g\|_V$.

PROOF:- To see the completeness of E, let $\{f_n\}$ be a Cauchy sequence in E. Then, for each $k \in Z$, there exists, $f^k \in L^1[k, k+1]$ such that

$$\int_k^{k+1} |f_n(x) - f^k(x)|\, dx \to 0 \quad \text{as } n \to \infty.$$

Put $f = f^k$ on $[k, k+1]$, $k \in Z$. Then, clearly $f \in E$.

We shall now prove the duality.

We recall that

$$\|g\|_V = \sum_{k=-\infty}^{\infty} \underset{x \in [k,k+1]}{\text{ess. sup.}} |g(x)|, \; g \in V.$$

For $g \in V$, it is easy to see that (46) represents a bounded linear functional on E because

$$|L(f)| \leq \int_R |f(t)|\,|g(t)|\,dt\,.$$

$$= \sum_{k=-\infty}^{\infty} \int_k^{k+1} |f(t)|\,|g(t)|\,dt\,.$$

$$\leq \left(\sup_{k \in Z} \int_k^{k+1} |f(t)|\,dt \right) \left(\sum_{k=-\infty}^{\infty} \underset{x \in [k,k+1]}{\text{ess. sup.}} |g(x)| \right)$$

$$= \|f\|_E \, \|g\|_V\,,$$

and $\|L\| \leq \|g\|_V$.

To prove the converse we proceed as follows. Let I_k denote the closed interval $[K, K+1]$. Given $f \in E$, let f_k be the function that agrees with f on I_k and is zero outside of I_k. Then $f = \sum_k f_k$. Now

$$\|f_k\|_E = \int_k^{k+1} |f_k(t)|\,dt$$

That is the space E_k of all such f_k with E-norm can be regarded as the space of all integrable functions on I_k

with the L^1-norm. By Riesz representation theorem every continuous linear functional on E_k is given by an L^∞ function on I_k. Hence if L is a continuous linear functional on E, then its restriction to E_k is a continuous linear functional on E_k and has the form

$$(47) \quad L(f_k) = \int_{I_k} f_k(t)\, g_k(t)\, dt \quad \text{for all } f_k \in E_k .$$

where $g_k \in L^\infty(I_k)$. Let g be defined on R by setting

$$g = g_k \text{ on } I_k, \quad k \in Z .$$

We now claim that $g \in V$ and that

$$L(f) = \int_R f(t)\, g(t)\, dt \qquad \text{for all } f \in E .$$

From (47) we observe that given $\varepsilon > 0$ there exists h_k in $L^1(I_k)$ such that

$$\int_{I_k} |h_k(t)|\, dt = 1, \quad \left| \int_{I_k} h_k(t)\, g_k(t)\, dt \right| > \|g_k\|_\infty - \frac{\varepsilon}{2^{|k|}}$$

We can find a constant θ_k of absolute value 1 such that

$$\theta_k \int_{I_k} h_k(t)\, g_k(t)\, dt = \left| \int_{I_k} h_k(t)\, g_k(t)\, dt \right|$$

Then

$$\sum_{k=-m}^{n} \left(\| g_k \|_\infty - \varepsilon/_{2^{|k|}} \right)$$

$$< \sum_{k=-m}^{n} \theta_k \int_{I_k} h_k(t)\, g_k(t)\, dt$$

$$= \sum_{k=-m}^{n} \theta_k L(h_k) = L\left(\sum_{k=-m}^{n} \theta_k h_k \right)$$

$$\leq \| L \| \, \| \sum_{k=-m}^{n} \theta_k h_k \|_E = \| L \| ,$$

Since $\| \sum_{k=-m}^{n} \theta_k h_k \|_E = 1$. This gives

$\sum_{k=-m}^{n} \| g_k \|_\infty \leq \| L \| + \sum_{k=-\infty}^{\infty} \frac{\varepsilon}{2^{|k|}}$. This gives

$$\sum_{k=-m}^{n} \| g_k \|_\infty \leq \| L \| + 3\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\sum_{k=-m}^{n} \| g_k \|_\infty \leq \| L \|$$

for all m and n from which we conclude that

$$\sum_{k=-\infty}^{\infty} \| g_k \|_\infty \leq \| L \| .$$

Thus $\|g\|_V \leq \|L\|$ and $g \in V$. Now the required representation is obvious. For if $f \in E$, then $f = \sum_k f_k$ and

$$L(f) = L(\sum_k f_k) = \sum_k L(f_k)$$

$$= \sum_k \int_{I_k} f_k(t) g_k(t)\,dt = \sum_k \int_{I_k} f(t)g(t)\,dt = \int_R f(t)g(t)\,dt$$

This completes the proof.

<u>THEOREM 11.6</u> Let S (R) be any Segal algebra on the real line R. Then the multiplier space M (S,W) is isometrically isomorphic to the dual space $(S \underset{L^1}{\otimes} E)^*$.

This is a direct consequence of Theorems 11.5 and 10.7. Following important results are derived from theorem 11.6.

<u>COROLLARY 11.7</u>

(i) $M(L^1, V) = M(L^1, W) \cong V$.

(ii) $M(V, V) = M(W, V) = M(W, W) \cong (W \underset{L^1}{\otimes} E)^*$

(iii) $M(L^1 \cap C_0, V) = M(L^1 \cap C_0, W)$.

(iv) $M(L^A, W) \cong (L^A \underset{L^1}{\otimes} E)^*$.

PROOF: (i) From theorem 10.4, we get

$$M(L^1, V) = M(L^1, W)$$

Since, we can prove that E is an essential L^1-module, from theorem 10.1,

$$L^1 \otimes_{L^1} E \cong E.$$

Hence from theorem 11.6, we get (i) since $E^* = V$. Notice that $M(L^1, W) \cong V$ is a result of Burnham and Goldberg [2] using entirely different techniques.

(ii) As (i) holds, theorem 10.6 is applicable. Also applying the above theorem, we get

$$M(V, V) = M(W, V) = M(W, W) \cong (W \otimes_{L^1} E)^*$$

In [14] , Krogstad has obtained $M(W,W)$ as a dual space of a tensor product $W \otimes_{L^1} W_0^c$. We notice that our space E is isometrically isomorphic to W_0^c defined in [14] . Hence, we have arrived at Krogstad's result through a simpler method.

(iii) From theorem 10.4, we write

$$M(L^1 \cap C_0, V) = M(L^1 \cap C_0, W).$$

Notice that $M(L^1 \cap C_0, V) \cong M(L^1 \cap C_0, W)$ is proved in 7.5, using a different method.

(iv) is an obvious deduction from Theorem 11.6 .

<u>REMARK 11.8</u> On a general locally compact group G, the Wiener algebra can be defined. See for example [23] . We can define V in a similar way and the foregoing results for W and V can be extended without much difficulty.

Thus we see that theorems 10.4 and 10.7 are, though simple, powerful enough to obtain several known results at a stroke.

We shall now turn our attention to some normed ideals and Segal algebras on compact groups. We shall first consider the Lipschitz classes $\lambda(\alpha, n)$ and $\Lambda(\alpha, n)$ defined in 9.8 .

<u>THEOREM 11.9</u> For $0 < \alpha < n$, let $\lambda(\alpha, n)$ and $\Lambda(\alpha, n)$ be defined on the compact group I, real numbers modulo 1, as given in 9.8 . Then the following hold:

(i) $M(L^1, \lambda(\alpha, n)) = M(L^1, \Lambda(\alpha, n))$

(ii) $M(\lambda(\alpha, n), \lambda(\alpha, n)) = M(\lambda(\alpha, n), \Lambda(\alpha, n))$

$= \Gamma(\lambda(\alpha, n), \lambda(\alpha, n))$

and the multiplier space $M(\lambda(\alpha, n), \lambda(\alpha, n))$ is a dual space.

PROOF: By Proposition 9.10 we see that $\Lambda(\alpha, n)$ is a normed ideal on the compact group I and $\lambda(\alpha, n)$ is the continuously translating subspace of $\Lambda(\alpha, n)$. Hence, in theorem 10.4, if we put $S = L^1$ and $\lambda(\alpha, n)$, $N_1 = \Lambda(\alpha, n)$ and $N_0 = \lambda(\alpha, n)$, clearly (i) and (ii) follow.

Now Proposition 9.9 (iv) says that $\Lambda(\alpha, n)$ is the dual space $(\lambda(\alpha, n))^{**}$. Therefore, from theorem 10.7 we obtain

$$M(\lambda(\alpha, n), \lambda(\alpha, n)) \cong (\lambda(\alpha, n) \underset{L^1}{\otimes} \lambda(\alpha, n)^*)^*$$

We note that as $\lambda(\alpha, n)$ is proved to be a Segal algebra, it is an L^1 - module. Hence, its dual space $\lambda(\alpha, n)^*$ is also an L^1 - module, by the adjoint action.

In fact, the space of multipliers from any Segal algebra on the compact group I into $\lambda(\alpha, n)$ is a dual space.

We shall now prove that the spaces of multipliers given in 11.9 (i) are actually identical with $\Lambda(\alpha, n)$. To do this, we resort to the following result of Burnham and Goldberg [2].

LEMMA 11.10 (Burnham and Goldberg)

Let A be any Segal algebra on G under a norm $\|\cdot\|_A$ Define B_x as the ball in the Segal algebra A given by

$$B_x = \{ f \in A : \|f\|_A \le x \}$$

Also, we define,

$$\tilde{B_x} = \{ f \in L^1(G) : \text{There exists } \{f_n\} \subset B_x$$

such that $\|f_n - f\|_1 \to 0$ as $n \to \infty \}$. Now, set

$$\tilde{A} = \bigcup_{x>0} \tilde{B_x}$$. Then under the norm

$\|f\|_{\sim} = \inf_{f \in \tilde{B_x}} x$, $f \in \tilde{A}$, $\tilde{A}$ becomes a Banach space.

Moreover, whenever

$$M(L^1, A) \subset L^1(G) ,$$

then $M(L^1, A)$ is isometrically isomorphic to $\tilde{A}$. $\tilde{A}$ is called the relative completion of A .

<u>THEOREM 11.11</u> For $\lambda(\alpha, n)$ and $\Lambda(\alpha, n)$ as in 11.9 .

(i) $M(L^1, \lambda(\alpha, n)) \cong \Lambda(\alpha, n)$.

(ii) $M(\lambda(\alpha, n), \lambda(\alpha, n)) = M(\Lambda(\alpha, n), \Lambda(\alpha, n))$.

<u>PROOF</u>: Note that (ii) follows immediately, by theorem 10.6, if (i) is established.

To prove (i), first observe that

$$M(L^1, \lambda(\alpha,n)) \subset M(L^1, L^2) \cong L^2(I) \subset L^1(I)$$

since $\lambda(\alpha,n) \subset L^2(I)$. Therefore, by lemma 11.10, it is enough to prove that $\Lambda(\alpha,n)$ is the relative completion of $\lambda(\alpha,n)$. Therefore, let $\{f_k\} \subset \lambda(\alpha,n)$ such that

(48) $\quad \|f_k\| \leq A \quad$ for all k, for some $A > 0$

and

(49) $\quad \|f_k - f\|_1 \to 0 \quad$ as $k \to \infty$

Now, from (49) it follows that

(50) $\quad f_k(x) \longrightarrow f(x) \quad$ a.e.

We recall the norm in $\Lambda(\alpha,n)$ to be, for any $g \in \Lambda(\alpha,n)$

(51) $$\|g\| = \max\left\{ \sup_x |g(x)|, \sup_{x,t} \frac{|\Delta_n^t g(x)|}{|t|^\alpha} \right\}$$

Hence, using (48) and (50) in (51) we find that $f \in \Lambda(\alpha,n)$.

On the other hand, let $f_m = f * \delta_m$ where f_m is any element in $\Lambda(\alpha, n)$ and $\{\delta_m\}$ is the Fejer's kernel, the approximate identity in $L^1(I)$. Then clearly

$$\|f_m\| \leq \|f\| \, \|\delta_m\|_1 = \|f\|$$

by the normed ideal property of $\Lambda(\alpha, n)$. That $\|f_m - f\|_1 \longrightarrow 0$ as $m \to \infty$ is also obvious. Hence we see that $\Lambda(\alpha, n)$ is the relative completion of $\lambda(\alpha, n)$. Thus we have proved (i).

<u>COROLLARY 11.12</u> Let $0 < \alpha < 1$. A necessary and sufficient condition for T to be a multiplier belonging to $M(\lambda(\alpha,1), \lambda(\alpha,1))$ is that

$$\Phi(x) = \sum_{m \neq 0} (im)^{-1} \varphi(m) e^{2\pi i m x}, \quad m \in Z$$

be a function satisfying the condition

$$\sup_{a>0} \frac{1}{a} \|\tau_a \Phi + \tau_{-a} \Phi - 2\Phi\|_1 < \infty$$

where $\varphi(m)$ is a function on the integers Z given by

$$\hat{Tf}(m) = \varphi(m) \hat{f}(m), \qquad f \in \lambda(\alpha, n).$$

PROOF: Since theorem 11.11 (ii) gives

$$M(\lambda(\alpha,1), \lambda(\alpha,1)) = M(\Lambda(\alpha,1), \Lambda(\alpha,1))$$

The stated condition is proved by Zygmund in [34] whenever

$T \in M(\Lambda(\alpha,1), \Lambda(\alpha,1))$. See, for example, [6]

This completes the proof.

So far, we dealt with cases where the normed ideals are dual spaces. In the following we see that theorem 10.3 can be used to give in some cases, the multiplier spaces of Segal algebras as dual spaces even when the corresponding normed ideals need not necessarily be dual spaces.

THEOREM 11.13 Suppose

$$S_1(T) = \{ f \in L^1(T) : \hat{f}(n) = o\left(\frac{1}{\log|n|}\right) \}$$

and

$$N_1(T) = \{ f \in L^1(T) : \hat{f}(n) = O\left(\frac{1}{\log|n|}\right) \}$$

as defined in 9.4 and 9.5 . Define

$$M_1(T) = \{ \mu \in M(T) : \hat{\mu}(n) = O\left(\frac{1}{\log|n|}\right) \}$$

Then $M_1(T)$ is a Banach space under the norm

$$\|\mu\|' = \|\mu\| + \max_{1<|n|\leq\infty} |\hat{\mu}(n) \log|n||$$

Further,

$$M(L^1, S_1) = M(L^1, N_1) \cong M_1(T).$$

PROOF: Using theorem 10.4 and Proposition 9.5, clearly

$$M(L^1, S_1) = M(L^1, N_1) .$$

Hence it is enough to show that

(52) $$M(L^1, N_1) \cong M_1(T).$$

First, let $T \in M(L^1, N_1) \subset M(L^1, L^1)$. Hence there exists $\mu \in M(T)$ such that

$$Tf = \mu * f , \qquad f \in L^1(T).$$

Let $\{\delta_n\}$ be an approximate identity in $L^1(T)$ with $\|\delta_n\|_1 = 1$ for all n .

Then

$$\|T\delta_n\|_{N_1} \leq \|T\| \, \|\delta_n\|_1 \qquad \text{for each } n .$$

Hence

$$(53) \quad \|\mu * \delta_n\|_1 + \max_{1<|m|\leq\infty} |\hat{\mu}(m)\, \hat{\delta}_n(m) \log|m||$$

is bounded by $\|T\|$. If we choose $\{\delta_n\}$ to be the Fejer's kernel, for all $m \in Z$,

$$\hat{\mu}(m)\, \hat{\delta}_n(m) \longrightarrow \hat{\mu}(m) \qquad \text{as } n \longrightarrow \infty .$$

Also, from (53)

$$\mu * \delta_n \longrightarrow \nu \qquad \text{weak star in } M(T) \cdot$$

Hence $\nu = \mu$, so that we get from (53)

$$(54) \quad \|\mu\| + \max_{1<|m|\leq\infty} |\hat{\mu}(m) \log|m|| \leq \|T\| .$$

If, on the other hand, $\mu \in M_1(T)$. Then clearly, for each $f \in L^1(T)$,

$$\|\mu * f\|_1 + \max_{1<|m|\leq\infty} |\hat{\mu}(m)\, \hat{f}(m) \log|m||$$

$$\|\mu * f\|_1 + \max_{1<|m|\le\infty} |\hat{\mu}(m)\hat{f}(m)\log|m||$$

$$\le \|\mu\|\,\|f\|_1 + \max_{1<|m|\le\infty} |\hat{\mu}(m)\log|m||\,\|\hat{f}\|_\infty$$

$$\le \Big(\|\mu\| + \max_{1<|m|\le\infty} |\hat{\mu}(m)\log|m||\Big)\|f\|_1$$

so that $T \in M(L^1, N_1)$ where

$$Tf = \mu * f \quad , \qquad f \in L^1(T)$$

and

(55) $$\|T\| \le \|\mu\| + \max_{1<|m|\le\infty} |\hat{\mu}(m)\log|m||$$

From (54) and (55) the isometry follows.

We shall now prove that the space $M_1(T)$ is actually a dual space. For this, we need the following result of Liu and Van Rooij [16] .

<u>LEMMA 11.14 (Liu and Van Rooij)</u> : Let E and F be normed linear spaces and H , a closed subspace of $E \times F$. Put

$$K = \left\{ (\sigma, \tau) \in E^* \times F^* : \sigma(x) + \tau(y) = 0 \text{ for every } (x, y) \in H \right\}$$

where E^* and F^* are the corresponding dual spaces of E and F. Then there exists a natural isometry between K and $(E \times F / H)^*$.

<u>THEOREM 11.15</u> $M_1(T)$, with the equivalent norm given by

$$\|\mu\|' = \max \left\{ \|\mu\|, \max_{1 < |m| \leq \infty} |\hat{\mu}(m) \log |m|| \right\}$$

is a dual space.

<u>PROOF:</u> Consider the weight function defined on the space of integers Z,

$$\omega(n) = \log |n| \qquad \text{for } |n| > 1$$

$$\omega(-1) = \omega(0) = \omega(1) = 1 .$$

Then the weighted Lebesgue space $L^\infty_\omega(Z)$ on Z is the dual space of $L^1_{\omega^{-1}}(Z)$, the corresponding L^1-space, weighted with $\omega^{-1}(n)$. (See, for example [13]).

Let $M(T) \times L^{\infty}_{\omega}(Z)$ be the Cartesian product and define a closed subspace K of $M(T) \times L^{\infty}_{\omega}(Z)$ by

$$K = \{ (\mu, \hat{\mu}) : \mu \in M_1(T) \}$$

Then K can be identified with $M_1(T)$. Now define

$H \subset C(T) \times L^{1}_{\omega^{-1}}(Z)$ as

$$H = cl \{ (f, -\hat{f}) : f \in \mathcal{P}(T) \}$$

where cl stands for the closure and $\mathcal{P}(T)$ is the space of all trigonometric polynomials on T.

We assign the norm

$$\sup_{x \in T} |f(x)| + \sum_{n \in Z} |a_n \omega^{-1}(n)|$$

for any $(f, \{a_n\}) \in C(T) \times L^{1}_{\omega^{-1}}(Z)$. Then we can prove

$$K = \{ (\sigma, \tau) \in M(T) \times L^{\infty}_{\omega}(Z) : \sigma(f) - \tau(\hat{f}) = 0 \text{ for } (f, -\hat{f}) \in H \}$$

Hence an immediate application of lemma 11.14 gives us that K is isometric to

$(C(T) \times L^1_{\omega^{-1}}(Z)/_H)^*$. But we know that

K is identified with $M_1(T)$. Therefore we have proved that $M_1(T)$ is a dual space.

<u>THEOREM 11.16</u> $M(S_1, S_1) = M(S_1, N_1)$ is a dual space.

<u>PROOF:</u> From theorem 10.4, $M(S_1, S_1) = M(S_1, N_1)$

and

$$M(S_1, N_1) \subset M(S_1, M_1) \subset M(S_1, M) .$$

But, as we have mentioned in remark 10.5 . We can prove that

$$M(S_1(T), M(T)) = M(S_1(T), L^1(T)) .$$

Hence for any $T \in M(S_1, M_1) \subset M(S_1, M_1)$ we get

$T \in M(S_1, L^1)$. This implies that $T \in M(S_1, M_1)$.

Or, $M(S_1, N_1) \subset M(S_1, M_1) \subset M(S_1, N_1)$.

Hence we can write , as in theorem 10.4,

$$M(S_1, S_1) = M(S_1, N_1) = M(S_1, M_1) \cong Hom_{L^1}(S, M_1)$$

Let $M_1(T) = Q^*$. Hence from theorem 10.3,

$$M(S_1, S_1) \cong (S_1 \otimes_{L^1} Q)^* .$$

This completes the proof of the theorem.

REMARK 11.17 We observe that theorem 11.15 together with (52) characterises the multiplier space $M(L^1, S_1)$ to be a dual space.

THEOREM 11.18

Let G be a locally compact abelian group with $\hat{G}$ as dual group. Let α be a locally bounded function on $\hat{G}$ with

$$\alpha(\xi) \geqslant 1 \quad , \quad \xi \in \hat{G}$$

then, denote by $M(\alpha)$, the sub space of bounded measures on G defined by

$$M(\alpha) = \left\{ \mu \in M(G) : \sup_{\xi \in \hat{G}} |\alpha(\xi)\, \hat{\mu}(\xi)| < \infty \right\}$$

Then $M(\alpha)$ is a Banach space under the norm

$$\|\mu\|_\alpha = \|\mu\| + \sup_{\xi \in \hat{G}} |\alpha(\xi)\, \hat{\mu}(\xi)|\ , \mu \in M(\alpha).$$

Then for $S(\alpha)$ and $N(\alpha)$ defined in 9.6 we have

$$M(L^1, S(\alpha)) = M(L^1, N(\alpha)) \cong M(\alpha).$$

Proof is analogous to that of theorem 11.13

THEOREM 11.19 $M(\alpha)$ is a dual space.

PROOF: Again, analogous to the proof in theorem 11.15, we can prove that $M(\alpha)$ is the dual space

$$\left(C_0(G) \times L^1_{\omega^{-1}}(\hat{G}) / H \right)^* = U^*$$

where the weight function ω is taken to be α itself and

$$H = Cl \{ (f, -\hat{f}) : f \in B(G) \}$$

Hence, analogous to theorem 11.16, we can state for the Riemersma Segal algebra the following result:

THEOREM 11.20 : $M(S(\alpha), S(\alpha)) \cong (S(\alpha) \otimes_{L^1} U)^*$.

12. A characterization for $M(W,W)$

This section contains a characterization for $M(W,W)$ viewed as a subspace of W^*, the dual space of W.

Our characterization is actually obtained for $M(V,V)$. But from 11.7 (ii), $M(V,V) = M(W,W)$. We have also included a similar characterization for the multiplier space $M(W, L^1 \cap C_0)$.

THEOREM 12.1 $M(W,W) = M(V,V)$ is isomorphic to the set of all $\lambda \in W^*$ such that $f \longrightarrow \lambda * f$ is continuous from $L^\infty(K)$ into V for each compact set K of R.

PROOF: Since $M(W, L^\infty) \cong Hom_{L^1}(W, L^\infty)$ $\cong (W \otimes_{L^1} L^1)^* \cong W^*$ it follows that every element of $M(W,W) \subset M(W, L^\infty)$ is given by an element of W^*. (See also [12]). If $T \in M(W,W)$ then $T \in M(V,V)$ by 11.7 (ii) which implies $M(V,V) = M(W,W)$. Hence there exists $\lambda \in W^*$ such that

$$Tf = \lambda * f \qquad f \in V$$

and the mapping $f \longrightarrow \lambda * f$ is continuous from V into V. Let K be a compact subset of R and let $f \in L^\infty(K)$. If F is a function which agrees with f on K and zero outside, then $F \in V$. Moreover, there exists $N > 0$ such that

$K \subset [-N, N]$ so that

$$\|F\|_V \leq (2N+1) \|f\|_\infty$$

Hence

$$\|\lambda * f\|_V = \|Tf\|_V = \|TF\|_V \leq \|T\| \|F\|_V \leq (2N+1)\|T\| \|f\|_\infty$$

so that $f \longrightarrow \lambda * f$ is continuous from $L^\infty(K)$ into V for each compact subset K of R.

On the other hand, suppose that $\lambda \in W^*$ is such that $f \longrightarrow \lambda * f$ is conttinuous from $L^\infty(K)$ into V for each compact $K \subset R$. Since λ is a measure on R satisfying the condition that

$$|\lambda|(I_k) \leq \eta \qquad \text{for all } k \in Z$$

where $I_k = [k, k+1]$ and η is independent of k, it follows that for each $f \in V$, we have

$$\|Tf\|_\infty = \|\lambda * f\|_\infty \leq \eta \|f\|_V$$

so that $T \in M(V, L^\infty)$. Now let $f \in V$. If $f_n = f \chi_n$ where χ_n is the characteristic function of I_n we have

$$f = \sum_{n=-\infty}^{\infty} f_n \quad \text{and} \quad \|f\|_V = \sum_{n=-\infty}^{\infty} \|f_n\|_\infty .$$

Since $g \longrightarrow \lambda * g$ is continuous from $L^\infty(K)$ into V for any compact set $K \subset R$ there exists a constant such that

$$\|\lambda * g\|_V \leq C_K \|g\|_\infty , \qquad g \in L^\infty(K)$$

This inequality remains valid if $g \in L^\infty(x+K)$ for arbitrary $x \in R$. Hence taking $K = [0,1]$, we have a constant C_1 such that

$$\|\lambda * f_n\|_V \leq C_1 \|f_n\|_\infty$$

so that

$$\|\lambda * f\|_V = \|\lambda * \sum_{n=-\infty}^{\infty} f_n\|_V$$

$$\leq \sum_{n=-\infty}^{\infty} \|\lambda * f_n\|_\infty \leq C_1 \sum_{n=-\infty}^{\infty} \|f_n\|_\infty$$

$$= C_1 \|f\|_V .$$

for all $f \in V$. Hence $T \in M(V,V) = M(W,W)$.
This completes the proof.

THEOREM 12.2. The space of multipliers $M(W, L^1 \cap C_0)$ is isomorphic to the space $\{ \lambda \in W^* : \text{the mapping } f \longrightarrow \lambda \times f$ is continuous from $L^\infty(K)$ into $L^1(R)$ for each compact $K \subset R \}$

PROOF: We first notice that from 11.4 (iii)

$$M(W, L^1 \cap C_0) = M(W, L^1 \cap L^\infty)$$

Also, by Proposition 7.14, it can be proved that

$$M(W, L^1 \cap L^\infty) \approx M(V, L^1 \cap L^\infty)$$

using Proposition 5.5. Hence the proof of the theorem is evidently along the lines of the previous theorem.

REMARK 12.3. Edwards [5] has characterised $M(W, L^1)$ as a space of pseudomeasures $P^1(R) = \{ \sigma \in P(R)$; the mapping $f \longrightarrow \sigma \times f$ is continuous from $C_c(R)$ into $L^1(R)\}$ where $C_c(R)$ is endowed with the internal inductive limit topology. Theorems 12.1 and 12.2 are analogous versions in the space W^*.

CHAPTER IV

MULTIPLIERS FROM A SEGAL ALGEBRA INTO $L^1(G)$

In this chapter, motivated by Goldberg and Seltzer [8] we characterise the multiplier space $M(S,L^1)$ as a completion of the Segal algebra $S(G)$. We consider the convolution operator norm on the Segal algebra and define a completion as the weak star closure in $P(G)$. We identify this completion with the multiplier space $M(S,L^1)$. We give $M(S, L^1)$ separately for the case $G = R$, as a space of sequences of $L^1(R)$ - functions, so as to derive Pigno's result [20] .

13. A weak star completion for Segal algebras

We first recall that $P(G)$ denotes the space of all pseudomeasures on G, a locally compact abelian group. $P(G)$ is actually the dual space of $A(G)$, the space of Fourier transforms of all functions in $L^1(\hat{G})$, endowed with the norm

$$\|f\|_A = \|F\|_1$$

where $F \in L^1(\hat{G})$ with $\hat{F} = f$. $A(G)$ contains the space

$$B(G) = \left\{ f \in L^1(G) : \hat{f} \text{ has compact support} \right\}$$

DEFINITION 13.1 Let $S(G)$ be a Segal algebra on G, a locally compact abelian group, with a norm $\|\cdot\|_S$. Define another norm $|||\cdot|||$ on S by

$$|||f||| = \sup\{ \|f*g\|_1 : \|g\|_S \leq 1\}$$

and denote, for $x > 0$,

$$B_x = \{ f \in S : |||f||| \leq x \}$$

and define

$\tilde{B_x} = \{ \sigma \in P(G)$: There exists a net $\{h_\alpha\} \subset B_x$ such that $h_\alpha \to \sigma$ weak star in $P(G) \}$.

Let $\tilde{S} = \bigcup_{x>0} \tilde{B_x}$ and define for $\sigma \in \tilde{S}$

$$\|\sigma\|_\sim = \inf_{\sigma \in \tilde{B_x}} x$$

Clearly, $\|\cdot\|_\sim$ is a semi norm on $\tilde{S}$. To prove it is actually a norm, let $\|\sigma\|_\sim = 0$ for some $\sigma \in \tilde{S}$. Let $\epsilon > 0$ be given. Suppose $f, g \in B(G)$ such that $\|f * g\|_A \leq 1$. Set $r = \dfrac{\epsilon}{\|g\|_\infty \|f\|_S}$.

Then there exists $\{h_\alpha\} \subset B_r$ such that $h_\alpha \to \sigma$ weak star in $P(G)$. Now

$$|\langle \tilde{f}*g, \sigma\rangle| = \lim_\alpha |\langle \tilde{f}*g, h_\alpha\rangle|$$

$$= \lim_{\alpha} | \langle g, h_\alpha * f \rangle |$$

$$\leq \| g \|_\infty \lim_{\alpha} \| h_\alpha * f \|_1$$

$$\leq \| g \|_\infty \, \eta \, \| f \|_S = \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\langle h, \sigma \rangle = 0$ for $h \in B(G) * B(G)$, with $\| h \|_A \leq 1$ and hence σ vanishes on $B(G) * B(G)$. But $B(G) * B(G)$ is dense in A(G) and σ is a bounded linear functional on A(G). This proves that $\sigma = 0$.

Hence $\tilde{S}$ is actually a normed linear space.

We now prove the following theorem.

<u>THEOREM 13.2</u>: The multiplier space $M(S, L^1)$ is isometrically isomorphic to $\tilde{S}$.

<u>PROOF</u>: Let $T \in M(S, L^1)$. Then, using Proposition 0.24, we see that there exists a unique pseudomeasure $\sigma \in P(G)$ such that

$$Tf = \sigma * f \qquad \text{for all} \quad f \in S$$

Now, let $\{ e_\alpha \} \subset B(G)$ be an approximate identity for S (G) with $\| e_\alpha \|_1 = 1$ for each α and set

$$h_\alpha = \sigma * e_\alpha$$

Then $h_\alpha \in S$ for all α. For any $f \in S$, we have

$$\| \sigma * e_\alpha * f \|_1 \leq \|T\| \; \| e_\alpha * f \|_S \leq \|T\| \; \| f \|_S ,$$

which implies that $||| h_\alpha ||| \leq \|T\|$ for all α.

For any $g \in A(G)$, considering the reflection function $\tilde{g}$ defined by

$$\tilde{g}(x) = \bar{g}(-x), \quad x \in G,$$

we have a pseudomeasure $\tilde{\sigma}$ given by

$$\langle g, \tilde{\sigma} \rangle = \langle \tilde{g}, \sigma \rangle$$

Now suppose $g \in B(G)$ and $K = \operatorname{supp} \hat{g}$. From the relations

$$\| e_\alpha * g - g \|_A \leq C_K \| \hat{e}_\alpha \hat{g} - \hat{g} \|_\infty \leq C_K \| e_\alpha * g - g \|_1$$

where C_K is a constant depending on K, we see that

$$\lim_\alpha \| e_\alpha * g - g \|_A = 0 \qquad \text{for each } g \in B(G).$$

Then if $g \in B(G)$, we have

$$\langle g, \sigma * e_\alpha \rangle = \langle \tilde{g} * e_\alpha , \tilde{\sigma} \rangle$$
$$\longrightarrow \langle \tilde{g} , \tilde{\sigma} \rangle = \langle g , \sigma \rangle .$$

we have thus proved that

$\langle g, h_\alpha \rangle \longrightarrow \langle g , \sigma \rangle$ for all $g \in B(G)$

Using the density of $B(G)$ in $A(G)$ we conclude that $h_\alpha \longrightarrow \sigma$ weak star in $P(G)$. Thus $\sigma \in \tilde{S}$ with

(56) $\|\sigma\|_\sim \leq \|T\|$.

Suppose, now $\sigma \in \tilde{S}$. We can find $\{h_\alpha\}$ in $S(G)$, such that

$$|||h||| < \|\sigma\|_\sim + \varepsilon$$

and $h_\alpha \longrightarrow \sigma$ weak star in $P(G)$. Let $f \in B(G)$ and $g \in A(G)$, then $\tilde{f} * g \in B(G)$. Using the definition of the pseudomeasure $\sigma * f$ we get

$$\langle g, \sigma * f \rangle = \langle \tilde{f} * g, \sigma \rangle$$

$$= \lim_\alpha \langle \tilde{f} * g, h_\alpha \rangle = \lim_\alpha \langle g , h_\alpha * f \rangle$$

so that

$$|\langle g, \sigma * f\rangle| = \lim_{\alpha} |\langle g, h_\alpha * f\rangle|$$

$$\leq \lim_{\alpha} \|g\|_\infty \|h_\alpha * f\|_1$$

(57) $$\leq \|g\|_\infty (\|\sigma\|_\sim + \varepsilon) \|f\|_S$$

Since $A(G)$ is dense in $C_o(G)$, it follows from (57) that $\sigma * f \in C_o(G)^* = M(G)$, for each $f \in B(G)$ and the measure norm satisfies

(58) $$\|\sigma * f\| \leq (\|\sigma\|_\sim + \varepsilon) \|f\|_S \quad .$$

On the other hand, $f \in B(G)$ so that $\sigma * f \in B(G)$. Or, $\widehat{\sigma * f}$ has compact support. Choose a $g \in L^1$ such that $\hat{g} = 1$ on the support of $\widehat{\sigma * f}$ and so we can write

$$\sigma * f = (\sigma * f) * g$$

hence $\sigma * f \in L^1(G)$. Then (58) gives

(59) $$\|\sigma * f\|_1 \leq (\|\sigma\|_\sim + \varepsilon) \|f\|_S$$

Defining the operator T on $B(G)$ by

$$Tf = \sigma * f \quad \text{for} \quad f \in B(G)$$

it follows easily from (59) that T can be extended as a multiplier from S into L^1 with

(60) $$\|T\| \leq \|\sigma\|_{\sim} + \varepsilon .$$

Thus $T \in M(S, L^1)$ and since $\varepsilon > 0$ is arbitrary, (60) gives

(61) $$\|T\| \leq \|\sigma\|_{\sim} .$$

But $\|\sigma\|_{\sim} \leq \|T\|$ by (56). Thus the mapping $T \to \sigma$ given by

$$Tf = \sigma * f$$

is an isometric isomorphism between $M(S,L^1)$ and $\tilde{S}$.

<u>COROLLARY 13.3</u> If $T \in M(S, L^1)$, then the associated pseudomeasure σ is a bounded measure if and only if

$$\sup_{\alpha} \|\sigma * e_{\alpha}\|_1 < \infty .$$

<u>REMARK 13.4</u> $M(S,L^1)$ can also be considered as a dual space of a certain Banach space of continuous functions. This follows from observing that $M(S(G), L^1(G))$ is the same as

$M(S(G), M(G))$. For, $L^1(G)$ is the continuously translating subspace of $M(G)$ (See remark 10.5). Hence by (Rieffel [24, p. 452]) theorem 10.3,

$$M(S, L^1) = M(S(G), M(G)) \cong (S(G) \underset{L^1(G)}{\otimes} C_0(G))^*$$

When G is a compact abelian group, the above theorem will be the special case of the following more general result.

<u>THEOREM 13.5</u> Let G be a compact abelian group and $(S_1, \|\ \|_{S_1})$ and $(S_2, \|\ \|_{S_2})$ be any two Segal algebras on G. Defining on S_1, a norm given by

$$|||f||| = \sup\left\{ \|f * g\|_{S_2} : g \in B(G), \|g\|_{S_1} \leq 1 \right\}$$

denoting as before, for $x > 0$,

$$B_x = \{ f \in S_1 : |||f||| \leq x \}$$

and $B_x^\sim = \{ \sigma \in P(G)$: there exists a net $\{h_\alpha\}$ in B_x such that $h_\alpha \longrightarrow \sigma$ weak star in $P(G) \}$ and setting $\tilde{S}_{12} = \bigcup_{x>0} B_x^\sim$ normed by, for $\sigma \in \tilde{S}_{12}$

$$\|\sigma\|_\sim = \inf_{\sigma \in B_x^\sim} x .$$

the space $M(S_1, S_2)$ of multipliers from S_1 into S_2 is isometrically isomorphic to $\tilde{S}_{12}$.

PROOF: That $M(S_1, S_2)$ is contained in $\tilde{S}_{12}$ is proved exactly in the same way as in the proof of Theorem 13.2 and the inequality (56) also holds.

Suppose now $\sigma \in \tilde{S}_{12}$. Then there exists $\{h_\alpha\} \subset B_r$ such that $h_\alpha \to \sigma$ weak star in $P(G)$, where $r = \|\sigma\|_\sim + \varepsilon$. Then we can say that

(62) $\hat{h}_\alpha(\gamma) \longrightarrow \hat{\sigma}(\gamma)$ for every $\gamma \in \hat{G}$.

Let $f \in B(G)$. Since G is compact, $\hat{G}$ is discrete. Therefore support of $\hat{f}$ is finite. As, from (62),

$$\hat{h}_\alpha(\gamma)\,\hat{f}(\gamma) \longrightarrow \hat{\sigma}(\gamma)\,\hat{f}(\gamma), \quad \gamma \in \hat{G}.$$

it follows that

$$\hat{h}_\alpha \hat{f} \longrightarrow \hat{\sigma}\,\hat{f} \quad \text{in} \quad L^1(\hat{G})$$

Hence $h_\alpha * f \longrightarrow \sigma * f$ in $L^\infty(G)$

and hence in $L^1(G)$ as G is compact. Thus

$\|h_\alpha * f - \sigma * f\|_1 \longrightarrow 0.$ Let $K = \mathrm{Supp}\,\hat{f}$.

Then, using lemma 0.5,

$$\|h_\alpha * f - \sigma * f\|_{S_2} \leq C_K \|h_\alpha * f - \sigma * f\|_1$$

so that

$$\|\sigma * f\|_{S_2} = \lim_\alpha \|h_\alpha * f\|_{S_2} \leq r \|f\|_{S_1} .$$

Hence we can define $T : B(G) \longrightarrow S_2(G)$ given by

$$Tf = \sigma * f$$

satisfying

$$\|Tf\|_{S_2} \leq (\|\sigma\|_\sim + \varepsilon) \|f\|_{S_1} , \quad f \in B(G).$$

Since $B(G)$ is dense in S_1 and $\varepsilon > 0$ is arbitrary, we conclude that $T \in M(S_1, S_2)$ and that $\|T\| \leq \|\sigma\|_\sim$. This together with (56) gives the isometry. This completes the proof.

From the above theorem, we now derive result of Goldberg and Seltzer [8]

<u>COROLLARY 13.6</u>: Let S be a Segal algebra on a compact abelian group G. Then a measure $\mu \in M(G)$ induces a multiplier $T \in (L^1, S)$ given by

$$Tf = \mu * f \quad , \quad f \in L^1(G)$$

if and only if there exists a sequence $\{h_n\}$ in S such that $\|h_n\|_S \leq K < \infty$ for $n = 1,2 \ldots\ldots$ and $h_n \to \mu$ weak star in M (G).

The above kind of weak star completion can also be carried over to the characterization of the multipliers from the Wiener algebra into itself. We recall that W(R) is a Segal algebra on the real line R under the norm

$$|||f|||_W = \sup_{m \in R} \sum_{K=-\infty}^{\infty} \max_{K \leq x \leq K+1} |f(x+m)|$$

From [12] it is known that the space of multipliers M (W , W) is isometrically isomorphic to a subspace of W^*, the dual space of W. Hence, in this case, instead of considering the pseudomeasures, the weak star completion is taken in W^* itself.

<u>DEFINITION 13.7</u> If W^* denotes the dual space of W, then

$$W \subset L^1(R) \subset M(R) \subset W^*$$

Setting

$$|||f||| = \sup \{ |||f * g|||_W : |||g|||_W \leq 1 \} ,$$

We define, as before, for $x > 0$,

$$B_x = \{ f \in W : |||f||| \leq x \} .$$

Now we define

$\tilde{B_x} = \{ \lambda \in W^* :$ there exists a sequence $\{h_n\} \subset B_x$ such that $h_n \to \lambda$ weak star in $W^* \}$ and the completion is denoted by

$$\tilde{W} = \bigcup_{x>0} \tilde{B_x} .$$

We introduce a norm in $\tilde{W}$ by setting

$$\| \lambda \|_{\sim} = \inf_{\lambda \in \tilde{B_x}} x$$

With this, we state the characterisation of the space of multipliers.

THEOREM 13.8 $M(W, W)$ is isometrically isomorphic to $\tilde{W}$.

PROOF: First, let $\lambda \in \tilde{W}$. Then there exists a sequence $\{h_n\}$ in W with $h_n \to \lambda$ weak star in W^* and

$$|||h_n||| < \|\lambda\|_\sim + \varepsilon,$$

$\varepsilon > 0$ is arbitrary. Now let $f \in W$ and $x \in R$.

$$h_n * f(x) = \int f(x-y)\, h_n(y)\, dy$$

$$= \int_R \tau_x \tilde{f}(y)\, h_n(y)\, dy \rightarrow \int_R \tau_x \tilde{f}(y)\, d\lambda(y)$$

$$= \int_R f(x-y)\, d\lambda(y)$$

by the weak star convergence of h_n to λ. Hence

$$h_n * f(x) \longrightarrow \lambda * f(x) \qquad \text{for all } x \in R$$

We shall show that $\lambda * f \in W$. To this end, consider

$$\sup_{m\in R} \sum_{k=-\infty}^{\infty} \max_{x\in[k,k+1]} |\lambda * f(x+m)|$$

$$= \sup_{m\in R} \sum_{k=-\infty}^{\infty} \max_{x\in[k,k+1]} \lim_{n\to\infty} |h_n * f(x+m)|$$

$$\leq \sup_{m\in R} \sum_{k=-\infty}^{\infty} \lim_{n\to\infty} \max_{x\in[k,k+1]} |h_n * f(x+m)|$$

By Fatou's lemma,

$$\leq \liminf_{n \to \infty} \sup_{m \in R} \sum_{k=-\infty}^{\infty} \max_{x \in [k, k+1]} |h_n * f(x+m)|$$

$$= \liminf_{n \to \infty} |||h_n * f|||_W \leq (\|\lambda\|_\sim + \varepsilon)\, |||f|||_W < \infty .$$

Since $\varepsilon > 0$ is arbitrary, we get $T \in M(W,W)$, defined by

$$Tf = \lambda * f \quad , \quad f \in W,$$

such that

$$\|T\| \leq \|\lambda\|_\sim$$

If $T \in M(W,W)$, by Unni and Keshavamurthy [12] we get a $\lambda \in W^*$ such that

$$Tf = \lambda * f \quad , \quad f \in W.$$

Take any approximate identity $\{e_n\} \subset B(R)$ for W, with $\|e_n\|_1 = 1$ for all n. For example, we can take the Fejer's kernel $\left\{ \frac{1-\cos nt}{\pi n t^2} \right\}$. Let $h_n = \lambda * e_n$ for each n. Following the same argument as before, it follows that $h_n \to \lambda$ weak star in W^* and $\{h_n\} \subset B_x$ where $x = \|T\|$. The rest of the proof is as before.

REMARK 13.9 We have obtained other characterisations for $M(W, W)$. See Corollary 11.7 (ii) and theorem 12.1

14. The multiplier space $M(S,L^1)$ on the real line R

In [20] Pigno defines the convolution approximation property for a subspace $D(G)$ of $L^1(G)$ with $D(G) \supset B(G)$ and proves that if $D(G)$ has this property, then $(D,L^1) = M(G)$ (Theorem 14.2 below). Here we characterize $M(S, L^1)$ for any Segal algebras on the real line R and derive Pigno's result [20]. Also, we specify the multipliers when $S(R)$ does not possess the said property.

DEFINITION 14.1 If $D(G) \subset L^1(G)$ is such that $D(G) \supset B(G)$, $D(G)$ is said to have the convolution approximation property with respect to $L^1(G)$ if for each sequence $\{\sigma_n\} \subset L^1(G)$ such that $\limsup_{n\to\infty} \|\sigma_n\|_1 = \infty$ there exists a $g \in D(G)$ such that $\limsup_{n\to\infty} \|\sigma_n * g\|_1 = \infty$.

THEOREM 14.2 [20] If $D(G)$ has the convolution approximation property then $\phi \in (D, L^1)$ if and only if $\phi = \hat{\mu}$ for some $\mu \in M(G)$.

We shall now restrict our attention to $G = R$, the real line. Let $S(R)$ be any Segal algebra on the real line.

DEFINITION 14.3 Let E denote the class of all sequences $\{g_n\} \subset L^1(R)$ such that

$$\limsup_{n \to \infty} \|g_n * f\|_1 < \infty \quad \text{for each } f \in S(R)$$

Define $\|\{g_n\}\|_E = \sup_n \left\{ \dfrac{\|g_n * f\|_1}{\|f\|_S} : f \in S \right\}$

Then $\|\cdot\|_E$, is a norm on E. For, let $\|\{g_n\}\|_E = 0$. Then we must have $\|g_n * f\|_1 = 0$ for all n and each $f \in S(R)$. This means that

$$|\hat{g}_n(u)\hat{f}(u)| = 0 \quad \text{for all n, for all } f \in S(R),$$

and each $u \in \hat{R}$. But for each $u \in \hat{R}$, there exists an $f \in S(R)$ such that $\hat{f}(u) \neq 0$. Hence $\hat{g}_n(u) = 0$ for each $u \in \hat{R}$. This implies $g_n = 0$ a.e. for each n.

THEOREM 14.4 $M(S, L^1)$ is isometrically isomorphic to E.

PROOF: Let $\{g_n\} \in E$. Then $\lim\limits_{n \to \infty} \| g_n * f \|_1 < \infty$ for each $f \in S$. Using Alaoglu's theorem, we see that $\{g_n * f\}$ has a weak star convergent subsequence in $M(R)$ converging to μ_f in $M(R)$. That is, for $g \in C_0(R)$,

$$(63) \qquad \int_R g(x)\, g_{n_i} * f(x)\, dx \xrightarrow{i} \int_R g(x)\, d\mu_f(x)$$

Then

$$\|\mu_f\| = \sup_{\substack{g \in C_0(R) \\ \|g\|_\infty \leq 1}} \left\{ \left| \int_R g(x)\, d\mu_f(x) \right| \right\}$$

$$= \sup \left\{ \left| \lim_i \int_R g(x)\, g_{n_i} * f(x)\, dx \right| : g \in C_0(R),\ \|g\|_\infty \leq 1 \right\}$$

$$\leq \lim_i \| g_{n_i} * f \|_1 \leq \|\{g_n\}\|_E \, \|f\|_S$$

Therefore

$$(64) \qquad \|\mu_f\| \leq \|\{g_n\}\|_E \, \|f\|_S .$$

Hence, defining $Tf = \mu_f , f \in S$, we get a bounded linear operator T from $S(R)$ into $M(R)$. To prove T is actually a multiplier let $y \in R , g \in C_0(R)$. Then

$$\langle g , \tau_y T f \rangle = \langle g , \tau_y \mu_f \rangle = \langle \tau_{y^{-1}} g , \mu_f \rangle$$

$$= \lim_i \langle \tau_{y^{-1}} g , g_{n_i} * f \rangle = \lim_i \langle g , \tau_y (g_{n_i} * f) \rangle$$

$$= \lim_i \langle g , g_{n_i} * \tau_y f \rangle = \langle g , \mu_{\tau_y f} \rangle$$

$$= \langle g , T \tau_y f \rangle .$$

Hence $T \in M(S(R), M(R))$. But $M(S(R) , M(R))$ $= M(S(R), L^1(R))$. See remark 13.4 under theorem 13.2. Hence $T \in M(S, L^1)$. Moreover it is clear from (64) that

(65) $$\| T \| \leq \| \{ g_n \} \|_E$$

Conversely let $T \in M(S, L^1)$. Then there exists $\sigma \in P(R)$ such that

$$Tf = \sigma * f \quad , \quad f \in S(R) .$$

Put $\sigma_n = \sigma * e_n$ where $\{e_n\}$ is an approximate identity for S (R) with $\|e_n\|_1 = 1$, and $\hat{e}_n$ has compact support. Then $\{\sigma_n\} \subset L^1(R)$ and

$$\|\sigma_n * f\|_1 = \|\sigma * e_n * f\|_1 \leq \|T\| \, \|f\|_S \quad \text{for all } n \text{ and all } f \in S(R).$$

Thus $\{\sigma_n\} \in E$ with

(66) $$\|\{\sigma_n\}\|_E \leq \|T\| .$$

Since $\lim\limits_{n \to \infty} \sigma_n * f = Tf$ in L'-norm, we have

$$\lim_n \langle g, \sigma_n * f \rangle = \langle g, Tf \rangle \quad \text{for all } g \in C_0(R).$$

Hence from (65)

$$\|T\| \leq \|\{\sigma_n\}\|_E$$

Together with (66) the isometry follows.

REMARK 14.5 The proof of the above theorem can be obtained from Theorem 13.2 also. In fact, we can extend theorem 14.4 to any G, locally compact abelian, with E consisting of nets instead of sequences. This is so because only in the case of the real line R, $S(R)$ and $L^1(R)$ possess the approximate identity as a sequence.

COROLLARY 14.6 If $S(R)$ has the convolution approximation property, then $M(S,L^1) = M(R)$. This is theorem 14.2 for $D = S(R)$.

PROOF: Suppose convolution approximation property holds in S. If $\{g_n\} \in E$, then for each $f \in S$,

$\limsup_{n\to\infty} \|g_n * f\|_1 < \infty$ implies that

$$\lim_{n\to\infty} \|g_n\|_1 < \infty .$$

That is, there exists $K > 0$ such that, for all n,

$$\|g_n\|_1 \leq K$$

Then for each $f \in L^1(R)$, we have

$$\|g_n * f\|_1 \leq K \|f\|_1$$

Thus T corresponding to $\{g_n\}$ given in the proof, becomes a multiplier of L^1 into L^1 and so is given by a measure.

Thus $M(S, L^1) = M(R)$ and hence the corollary.

COROLLARY 14.7 : Suppose convolution approximation property does not hold for $S(R)$. Then the elements $\{g_n\} \in E$ satisfying

(67) $$\limsup_{n \to \infty} \|g_n\|_1 = \infty ,$$

are precisely the ones which correspond to multipliers which are not bounded measures.

PROOF:- Let $T \in M(S, L^1)$. Then

$$Tf = \sigma * f , \quad f \in S ,$$

for some $\sigma \in P(R)$. Then putting $\sigma_n = \sigma * e_n$ we see that $\{\sigma_n\}$ satisfies (67) whenever σ is not a bounded measure, from corollary 13.3.

If $\{g_n\} \in E$ satisfies (67) then, again from corollary 13.3, the corresponding pseudomeasure cannot be in $M(R)$.

CHAPTER V

A GENERALIZATION FOR MULTIPLIERS ON TRANSLATION INVARIANT SPACES

In this chapter we define a more general commutation of translations for operators on translation invariant spaces of functions on locally compact abelian groups and characterize such operators on well known spaces. Our methods of proof closely follow the methods given in the book of Larsen [15].

15 Multiplier pairs

NOTATION 15.1 Throughout this chapter, G will denote a locally compact abelian group and $\hat{G}$ its character group. λ and η denote the respective Haar measures on G and $\hat{G}$. We use multiplication to denote the group operation in keeping with the notations of Larsen [15] . Hence the translation operator τ_y will be defined as

$$\tau_y f(x) = f(x y^{-1})$$

for $x, y \in G$ and f any function defined on G and the reflection function $\tilde{f}$ is defined as

$$\tilde{f}(x) = f(x^{-1}) \quad , \quad x \in G .$$

The definition of a multiplier pair on algebras is stated as follows.

<u>DEFINITION 15.2 [19]</u> Let A_1 and A_2 be two algebras. Let T and ψ be two linerar operators from A_1 to A_2 where ψ is an algebra homonorphism. Then T is a multiplier associated with ψ if

$$T(xy) = T(x)\,\psi(y)$$

for all $x, y \in A_1$ and we call the pair (T, ψ) a <u>multiplier pair</u> .

On the convolution group algebra $L^1(G)$ the definition of a multiplier pair takes the following equivalent forms, analogous to theorem 0.1.1 given in Larsen [15] .

<u>THEOREM 15.3 (Nandakumar)</u> Let G be a locally compact abelian group. Let T and ψ be two bounded linear operators on the convolution algebra L^1 (G) where ψ is an isomorphism of L^1 (G) onto itself. Then the following are equivalent.

(i) There exists an isomorphism β from G onto G and $\gamma_0 \in \hat{G}$ such that

$$\psi f(y) = K(y, \gamma_0)\ f(\beta(y))$$

for all $f \in L^1$ (G) and all $y \in G$; and

$$\tau_y T = (y^{-1}, \gamma_0)\, T \tau_{\beta(y)}$$

where K is given by

$$K \int_G f(\beta(y))\, d\lambda(y) = \int_G f(y)\, d\lambda(y).$$

(ii) For all $f, g \in L^1(G)$, $T(f * g) = Tf * \psi g$.

(iii) There exists a unique function ϕ defined on G and an affine map α from $\hat{G}$ to $\hat{G}$ such that

$$(Tf)^\wedge = \phi\, \hat{f} \circ \alpha$$

(iv) There exists a unique measure μ in M (G) such that

$$(Tf)^\wedge = \hat{\mu}\, \hat{f} \circ \alpha$$

for each $f \in L^1(G)$ and α as in (iii).

(v) There exists a unique measure $\mu \in M(G)$, a unique isomorphism β from G onto G and $\gamma_0 \in \hat{G}$ such that

$$Tf(y) = (y, \gamma_0)\, (f \circ \beta) * \mu(y)$$

for all $y \in G$.

(Note: For the definition of an affine map, see [27, p. 78])

We shall now extend this notion of a multiplier pair to translation invariant spaces like L^p spaces. We use the equivalence (i) of the above theorem to do this, as in the case of ordinary multipliers.

16. A generalization for operators commuting with translations

DEFINITION 16.1 Let G be a locally compact abelian group and $\hat{G}$ be the character group. Let $F(G)$ be a function space on G. If $\gamma_0 \in \hat{G}$ and β be any isomorphism on G then a bounded linear operator T on $F(G)$ is which satisfies

$$(*) \qquad \tau_y T = (y^{-1}, \gamma_0)\, T \tau_{\beta(y)}, \quad y \in G,$$

shall be our new definition of a multiplier.

If the transformation ψ defined on $F(G)$ by

$$\psi f = (\cdot, \gamma_0)\, f \circ \beta \quad , \quad f \in F(G),$$

is meaningful, we denote the space of all T satisfying (*) by $M(F(G), \psi)$.

We shall frequently make use of the duality property of the groups G and $\hat{G}$ which enables us to write

$$(\beta(x), \gamma) = (x, \alpha(\gamma)), \quad x \in G, \quad \gamma \in \hat{G},$$

where α is the isomorphism on $\hat{G}$, induced in a natural way by the isomorphism β on G.

<u>PROPOSITION 16.2</u> Let G be a locally compact abelian group and $\hat{G}$ its dual. Let T be a bounded linear operator on $L^p(G)$, $(1 \leq p \leq \infty)$. Let $\gamma_0 \in \hat{G}$ and β any isomorphism of G onto G. If

$$\tau_y T = (y^{-1}, \gamma_0)\, T \tau_{\beta(y)}, \qquad y \in G.$$

is satisfied by T, then

(68) $\quad T(f * g) = Tf * \psi g \quad$ for $f \in L^p$, $g \in L^1 \cap L^p$,

where ψ is the linear isomorphism of L^p onto L^p given by

$$\psi f = k(\cdot, \gamma_0) f \circ \beta, \qquad f \in L^p,$$

where k is a constant given by

$$k \int f(\beta(x))\, d\lambda(x) = \int f(x)\, d\lambda(x) \quad \text{for } f \in L^1(G).$$

PROOF: First we shall prove that ψ is a linear isomorphism of L^p onto L^p, $(1 \leq p \leq \infty)$. For any $f \in L^p$, $p < \infty$, consider

$$\left(\int |\psi f(x)|^p \, d\lambda(x) \right)^{1/p}$$

$$= \left(\int K^p \,|\, (x, \gamma_0) \, f \circ \beta(x) |^p \, d\lambda(x) \right)^{1/p}$$

$$= \left(K^p \int |f(\beta(x))|^p \, d\lambda(x) \right)^{1/p}$$

Put $F(x) = |f(x)|^p$. Then $F \in L^1(G)$.

Hence R.H.S. $= \left(K^p \int F(\beta(x)) \, d\lambda(x) \right)^{1/p}$

$= \left(K^{p-1} \int F(x) \, d\lambda(x) \right)^{1/p}$, by definition of K

$= K^{1 - 1/p} \left(\int |f(x)|^p \, d\lambda(x) \right)^{1/p}$

Hence

$$\| \psi f \|_p \leq K^{1 - 1/p} \| f \|_p .$$

whenever $p = \infty$, clearly, by the definition of ψ,

$$\| \psi f \|_\infty \leq K \| f \|_\infty , \quad f \in L^\infty,$$

holds. Hence, for every p such that $1 \leq p \leq \infty$, we see that ψ is a bounded operator on L^p. Clearly ψ is linear. If $\psi f_1 = \psi f_2$ for $f_1, f_2 \in L^p$ then

$$(x, \gamma_0)\, f_1(\beta(x)) = (x, \gamma_0)\, f_2(\beta(x)) \quad , \quad x \in G .$$

Or $f_1(\beta(x)) = f_2(\beta(x))$ for $x \in G$. Since β is an isomorphism of G onto G, $f_1 = f_2$. Hence ψ is one-to-one. Let $f \in L^p$. Then the function $g_1 = k_1 (\cdot, \alpha(\gamma_0)) f \circ \beta^{-1}$ belongs to $L^p(G)$ where k_1 is the constant given by

$$k_1 \int f(\beta^{-1}(x))\, d\lambda(x) = \int f(x)\, d\lambda(x) \,, \quad f \in L^1 .$$

Now, put $g = (k k_1)^{-1} g_1$. Then $\psi g = f$. Hence ψ is onto.

Also, ψ maps $L^1 \cap L^p$ onto $L^1 \cap L^p$. For

$$\int |\psi f(x)|\, d\lambda(x) = k \int |(x, \gamma_0) f(\beta(x))|\, d\lambda(x)$$

$$= \int |f(x)|\, d\lambda(x) \qquad \text{for all } f \in L^1 \cap L^p$$

Also, for any $f \in L^1 \cap L^p$, the function $g \in L^p$ such that $\psi g = f$ obtained as above also belongs to L^1.

Hence the equation (68) is meaningful. To prove (68) consider $h \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in C_c(G)$ and $g \in L^1 \cap L^p(G)$

$$\langle Tf * \psi g, h \rangle = \int Tf * \psi g(x) \, h(x^{-1}) \, d\lambda(x)$$

$$= \iint \tau_y Tf(x) \, \psi g(y) \, d\lambda(y) \, h(x^{-1}) \, d\lambda(x)$$

$$= K \iint \tau_y Tf(x) \, (y, \gamma_0) \, g(\beta(y)) \, d\lambda(y) \, h(x^{-1}) \, d\lambda(x)$$

By the assumption on T, R.H.S. becomes

$$= K \iint (y^{-1}, \gamma_0) \, T\tau_{\beta(y)} f(x) \, (y, \gamma_0) \, g(\beta(y)) \, h(x^{-1}) \, d\lambda(x) \, d\lambda(y)$$

$$= K \iint T\tau_{\beta(y)} f(x) \, g(\beta(y)) \, d\lambda(y) \, h(x^{-1}) \, d\lambda(x)$$

Since the double integral $\iint T\tau_{\beta(y)} f(x) \, h(x^{-1}) \, g(\beta(y)) \, d\lambda(x) \, d\lambda(y)$

is absolutely integrable, by Fubini's theorem change of order of integration is valid. Hence

$$\langle Tf * \psi g, h \rangle = K \iint T\tau_{\beta(y)} f(x) \, h(x^{-1}) \, d\lambda(x) \, g(\beta(y)) \, d\lambda(y)$$

$$\langle Tf * \psi g, h \rangle$$

$$= k \iint T\tau_{\beta(y)} f(x)\, h(x^{-1})\, d\lambda(x)\, g(\beta(y))\, d\lambda(y)$$

$$= \iint \tau_{\beta(y)} f(x)\, T^* \tilde{h}(x) \overset{d\lambda(x)}{\wedge} g(\beta(y))\, d\lambda(y)$$

where T^* is the adjoint of T.

Again by Fubini's theorem, we obtain

$$\langle Tf * \psi g, h \rangle \cancel{= k \iint f(x\,(\beta(y))^{-1})}$$

$$= k \iint f(x\ (\beta(y))^{-1})\, g(\beta(y))\, d\lambda(y)\, T^* \tilde{h}(x)\, d\lambda(x)$$

$$= \int k \int \tau_{x^{-1}} f(\beta(y^{-1}))\, g(\beta(y))\, d\lambda(y)\, T^* \tilde{h}(x)\, d\lambda(x)$$

Since $f \in C_c$, clearly $\tau_{x^{-1}} f \cdot g \in L^1(G)$. Hence

$$k \int \tau_{x^{-1}} f(\beta(y^{-1}))\, g(\beta(y))\, d\lambda(y)$$

$$= \int \tau_{x^{-1}} f(y^{-1})\, g(y)\, d\lambda(y).$$

So we see that

$$\langle Tf * \psi g, h\rangle = \iint f(xy^{-1})\, g(y)\, d\lambda(y)\, T^* \tilde{h}(x)\, d\lambda(x)$$

$$= \int f*g(x)\, T^*\tilde{h}(x)\, d\lambda(x)$$

$$= \int T(f*g)(x)\, h(x^{-1})\, d\lambda(x)$$

$$= \langle T(f*g), h\rangle$$

Since $h \in L^2$ **is taken arbitrarily, we can immediately arrive at**

$$Tf * \psi g = T(f*g), \; f \in C_c, \; g \in L^1 \cap L^p .$$

Now, let $f \in L^p$ **and** $\{f_n\} \subset C_c$ **such that** as $n \to \infty$, $\|f_n - f\|_p \to 0$. **Then for** $g \in L^1 \cap L^p$,

$$\| Tf * \psi g - T(f*g)\|_p$$

$$= \| Tf * \psi g - Tf_n * \psi g + T(f_n * g) - T(f*g)\|_p$$

$$\leq \| Tf * \psi g - Tf_n * \psi g\|_p + \| T(f_n * g) - T(f*g)\|_p$$

$$\leq \| Tf - Tf_n\|_p \, \|\psi g\|_1 + \|T\| \, \|f_n - f\|_p \, \|g\|_1 \to 0$$

Hence (68) holds.

THEOREM 16.3 Let G be a locally compact abelian group and $\hat{G}$ its dual. Let T be a bounded linear operator on L^2. Let $\gamma_0 \in \hat{G}$ and β any isomorphism of G onto G. If T satisfies

$$\tau_y T = (y^{-1}, \gamma_0)\, T \tau_{\beta(y)}, \qquad y \in G,$$

then there exists a unique $\phi \in L^\infty(\hat{G})$ such that

$$(Tf)^\wedge = \phi\, (\psi f)^\wedge \quad \text{for each} \quad f \in L^2(G).$$

where ψ is the linear isomorphism of $L^2(G)$ onto $L^2(G)$ given by $\psi f = K(\cdot, \gamma_0) f \circ \beta$, $f \in L^2$. Moreover, the correspondence between T and ϕ is a linear, one-to-one and onto relation.

PROOF: By the assumption on T, the above proposition implies that

$$Tf * \psi g = T(f * g) = T(g * f)$$

$$= Tg * \psi f, \qquad \text{for } f, g \in L^1 \cap L^2.$$

Or, $\widehat{Tf}\, \widehat{\psi g} = \widehat{Tg}\, \widehat{\psi f}$ for all $f, g \in L^1 \cap L^2$.

But, for every $\gamma \in \hat{G}$ **, there exists** $f \in L^1 \cap L^2$ **such that** $\widehat{\psi f}(\gamma) \neq 0$ **. This follows from Rudin [27] and the fact that** ψ **maps** $L^1 \cap L^2$ **onto** $L^1 \cap L^2$.

Define

$$\phi(\gamma) = \frac{\widehat{Tf}(\gamma)}{\widehat{\psi f}(\gamma)} \qquad \gamma \in \hat{G}.$$

ϕ **is well defined, since** $\dfrac{\widehat{Tf}(\gamma)}{\widehat{\psi f}(\gamma)} = \dfrac{\widehat{Tg}(\gamma)}{\widehat{\psi g}(\gamma)}$

for all $f, g \in L^1 \cap L^2$ **such that** $\widehat{\psi f}(\gamma) \neq 0, \widehat{\psi g}(\gamma) \neq 0$.

Hence $\widehat{Tf}(\gamma) = \phi(\gamma)\widehat{\psi f}(\gamma)$ **holds for all** $f \in L^1 \cap L^2$ **such that** $\widehat{\psi f}(\gamma) \neq 0$ **. For any** $f \in L^1 \cap L^2$ **with** $\widehat{\psi f}(\gamma) = 0$ **, the identity**

$$\widehat{Tf}(\gamma) \cdot \widehat{\psi g}(\gamma) = \widehat{Tg}(\gamma)\widehat{\psi f}(\gamma)$$

would simply give $\widehat{Tf}(\gamma) = 0$.

Hence

$$\widehat{Tf} = \phi\, \widehat{\psi f} \qquad \textbf{holds for all } f \in L^1 \cap L^2.$$

Now we claim that ϕ is unique. For, if ϕ^* is another such function satisfying this equation, then

$$(\phi - \phi^*)\, \psi\hat{f} = 0 .$$

i.e. $(\phi - \phi^*)(\gamma)\, \psi\hat{f}(\gamma) = 0$ for all $\gamma \in \hat{G}$ and for all $f \in L^1 \cap L^2$

But for each $\gamma \in \hat{G}$, there exists $f \in L^1 \cap L^2$ such that $\psi\hat{f}(\gamma) \neq 0$. Hence $\phi = \phi^*$. Thus ϕ is unique.

Define ψ_1 on $L^1(G)$ by $\psi_1 f = k(\cdot, \gamma_0)\, f \circ \beta$ for every $f \in L^1(G)$. Then ψ_1 is an isomorphism of $L^1(G)$ onto $L^1(G)$. Hence by Rudin [27] there exists an affine homeomorphism α of $\hat{G}$ onto $\hat{G}$ such that

$$\widehat{\psi_1 f} = \hat{f} \circ \alpha \qquad \text{for } f \in L^1(G)$$

Since ψ_1, coincides with ψ on $L^1 \cap L^2$, we get

$$\psi\hat{f} = \hat{f} \circ \alpha \qquad \text{for all } f \in L^1 \cap L^2 .$$

Thus $\widehat{Tf} = \phi\, \hat{f} \circ \alpha \qquad$ for all $f \in L^1 \cap L^2$.

We now claim that ϕ belongs to $L^\infty(\hat{G})$. To see this, choose $f \in L^1 \cap L^2$ such that $\hat{f} = 1$ on $\mathbb{K} \cup \alpha(\mathbb{K})$ where $\mathbb{K}$ is any compact set in $\hat{G}$ and $0 \leq \hat{f} \leq 1$ on any open set U containing $\mathbb{K} \cup \alpha(\mathbb{K})$ and $\hat{f}$ vanishes off U. Then, if $\chi_{\mathbb{K}}$ is the characteristic function of $\mathbb{K}$

$$\| \phi \chi_{\mathbb{K}} \|_2 \leq \| \phi \hat{f} \circ \alpha \|_2 = \| T\hat{f} \|_2 = \| Tf \|_2$$

$$\leq \| T \| \, \| f \|_2 = \| T \| \, \| \hat{f} \|_2 \leq \| T \| \, (\eta(U))^{1/2}$$

where η is the Haar measure on $\hat{G}$. Thus

(69) $$\| \phi \chi_{\mathbb{K}} \|_2 \leq \| T \| \, (\eta(U))^{1/2}$$

holds for any $\mathbb{K} \subset \hat{G}$ compact and U open set containing $\mathbb{K} \cup \alpha(\mathbb{K})$.

Now suppose that ϕ is not essentially bounded. Then for any $N > 0$ arbitrarily large, there exists a compact set $\mathbb{K} \subset \hat{G}$ with $\eta(\mathbb{K}) > 0$ such that $|\phi(\gamma)| > N$ on $\mathbb{K}$. Since α is an isomorphism of $\hat{G}$ onto $\hat{G}$ there exists a constant $C > 0$ such that

$$c \int_{\hat{G}} g(\gamma)\, d\eta(\gamma) = \int_{\hat{G}} g(\alpha^{-1}(\gamma))\, d\eta(\gamma), \quad g \in L^1(\hat{G}).$$

Applying this for the characteristic function $\chi_{\mathbb{K}}$ of any compact $\mathbb{K} \subset \hat{G}$, we get

$$\eta(\alpha(\mathbb{K})) = c\, \eta(\mathbb{K}).$$

Now choose $N^2 > (1+c)\, \|T\|^2$. Since for every $m > 1$ we can choose an open set $U \supset \mathbb{K} \cup \alpha(\mathbb{K})$ such that

$$\eta(U) < m\, \eta(\mathbb{K} \cup \alpha(\mathbb{K})),$$

for $m = \dfrac{N^2}{(1+c)\|T\|^2}$ choose the corresponding open set U. Then we obtain

$$\eta(U) < \frac{N^2}{\|T\|^2(1+c)}\, \eta(\mathbb{K} \cup \alpha(\mathbb{K}))$$

$$\leq \frac{N^2 (\eta(\mathbb{K}) + \eta(\alpha(\mathbb{K})))}{\|T\|^2 (1+c)} \leq \frac{N^2}{\|T\|^2}\, \frac{(\eta(\mathbb{K}) + c\,\eta(\mathbb{K}))}{(1+c)}$$

$$= \frac{N^2}{\|T\|^2}\, \eta(\mathbb{K}).$$

Therefore,

(70) $\eta(U) < \frac{N^2}{\|T\|^2} \eta(K)$ by choice of U.

Now, by the assumption on ϕ we see that

(71) $\|\phi \chi_K\|_2 > N(\eta(K))^{1/2}$.

Combining (69) and (71) we get

$$\|T\| (\eta(K))^{1/2} \geq \|\phi \chi_K\|_2 > N(\eta(K))^{1/2},$$

which implies that,

$$\eta(U) > \frac{N^2}{\|T\|^2} \eta(K)$$

contrary to (70) . Hence ϕ is essentially bounded on $\hat{G}$.

Hence, for a unique $\phi \in L^\infty(\hat{G})$ we have

$$\hat{Tf} = \phi \, \hat{\psi f} \qquad \text{for all } f \in L^1 \cap L^2$$

Now define a bounded linear operator S on $L^2(G)$ by

$$(Sf)\hat{} = \phi \, \hat{\psi f} \qquad \text{for all } f \in L^2$$

This is well defined since $\phi \in L^\infty(\hat{G})$. Also, for

$f \in L^1 \cap L^2$, $(Sf)^\wedge = (Tf)^\wedge$ from above.

Hence by uniqueness of Fourier Plancheral transform,

$$Sf = Tf \qquad \text{for} \quad f \in L^1 \cap L^2$$

Since $L^1 \cap L^2$ is dense in L^2 and T is a continuous operator on L^2, we get $S = T$. Or,

$$\hat{Tf} = \phi\, \hat{\psi f} \qquad \text{for all} \quad f \in L^2 .$$

For any $\phi \in L^\infty(\hat{G})$, define $T : L^2 \longrightarrow L^2$ by

$$\hat{Tf} = \phi\, \hat{\psi f}, \qquad f \in L^2 .$$

Then

$$\|Tf\|_2 = \|\hat{Tf}\|_2 = \|\phi\, \hat{\psi f}\|_2$$

$$\leq \|\phi\|_\infty \|\hat{\psi f}\|_2 = \|\phi\|_\infty \|\psi f\|_2$$

$$\leq \|\phi\|_\infty \|\psi\| \|f\|_2 .$$

T is also linear. For $y \in G$, $\gamma \in \hat{G}$,

$$(T\,(y^{-1},\gamma_0)\,\tau_{\beta(y)} f)^\wedge(\gamma)$$

$$= (y^{-1},\gamma)\ (T\,\tau_{\beta(y)} f)^\wedge(\gamma)$$

$$= (y^{-1},\gamma)\ \phi(\gamma)\,\psi\,(\tau_{\beta(y)} f)^\wedge(\gamma)$$

$$= \phi(\gamma)\,(\tau_y\,\psi f)^\wedge(\gamma)$$

by using the definition of ψ,

$$= \phi(\gamma)\ (y^{-1},\gamma)\ \psi\hat{f}(\gamma) = (y^{-1},\gamma)\ \phi(\gamma)\,\psi\hat{f}(\gamma),$$

$$= (y^{-1},\gamma)\ \hat{Tf}(\gamma) = \widehat{\tau_y Tf}(\gamma).$$

Hence T satisfies the required property and the correspondence $T \longrightarrow \phi$ is onto. If $\phi_1 = \phi_2$, then for all $f \in L^1 \cap L^2$,

$$\phi_1\,\widehat{\psi f} = \phi_2\,\widehat{\psi f}\,.$$

Or

$$\widehat{T_1 f} = \widehat{T_2 f}$$

for all $f \in L^2$. Hence by the uniqueness of Plancheral transform,

$$T_1 = T_2\,.$$

Linearity follows easily.

Remark 16.4 : The above theorem is a generalization of Theorem 4.1.1. of Larsen [15] . Since there exists an isometric isomorphism between $L^\infty(\hat{G})$ and the pseudomeasures we can immediately generalize theorem 4.3.2 of Larsen [15] from the above theorem.

THEOREM 16.5 : Let G be a locally compact abelian group and G its dual. If $\gamma_0 \in \hat{G}$ and β is an isomorphism of G onto G , then

$$\psi f = k(\cdot, \gamma_0) f \circ \beta$$

defines an isomorphism ψ of $L^p(G)$ onto $L^p(G)$ for any p such that $1 \le p \le \infty$. If $M(L^p, \psi)$ denotes the space of all bounded linear multipliers T satisfying

$$\tau_y T = (y^{-1}, \gamma_0) T \tau_{\beta(y)} \quad , \quad y \in G$$

then there exists a linear isomorphism of $M(L^p, \psi)$ onto $M(L^q, \psi)$ where $1/p + 1/q = 1$.

PROOF: Let $g \in C_c(G)$. Then define a functional as

$$F_g(f) = \langle Tg, f \rangle = Tg * f(0) \quad \text{for all} \quad f \in C_c(G).$$

where $T \in M(L^p, \psi)$.

If $\psi f_1 = f$ for any $f \in C_c$,

$$T g * f(0) = T g * \psi f_1(0) = T f_1 * \psi g(0)$$

Thus

$$|F g(f)| = |T f_1 * \psi g(0)| \le \| T f_1 \|_p \, \| \psi g \|_q .$$

Since ψ takes $L^1 \cap L^q$ onto $L^1 \cap L^q$ and $g \in L^1 \cap L^q$

$\| \psi g \|_q < \infty$. Also $f_1 = \psi^{-1} f$ and ψ^{-1} is an

isomorphism on $L^p(G)$ given by

$$\psi^{-1} f = k^{-1} (\cdot, \alpha(\gamma_0)^{-1}) \, f \circ \beta^{-1}$$

where

$$(x, \alpha(\gamma_0)) = (\beta(x), \gamma_0), \quad x \in G.$$

Then $\| \psi^{-1} f \|_p \le k_1^{1/q} \| f \|_p$ where $k_1 = k^{-1}$.

Hence

$$| F g(f) | \le \| T \|_p \, \| \psi^{-1} f \|_p \, \| \psi g \|_q$$

$$\le \| T \|_p \, k^{-1/q} \, \| f \|_p \, k^{1/p} \, \| g \|_q$$

Extending F_g to which of L^p, we see that

$$F g(f) = \langle T g, f \rangle$$

implies $Tg \in L^q(G)$ for each $g \in C_c$

$$\|Tg\|_q \leq \|T\|_q \, k^{1/p - 1/q} \, \|g\|_q$$

Hence T can be extended to a bounded linear operator on the whole of $L^q(G)$ with norm

$$\|T\|_q \leq \|T\|_p \, k^{1/p - 1/q} .$$

Hence reversing the role of p and q,

$$\|T\|_p \leq \|T\|_q \, k^{1/q - 1/p} .$$

Equivalently,

$$\|T\|_p \leq \|T\|_q \, k^{1/q - 1/p} \leq \|T\|_p \, k^{1/p - 1/q} \, k^{1/q - 1/p}$$

$$= \|T\|_p$$

Hence we get

$$\|T\|_p = \|T\|_q \, k^{1/q - 1/p} .$$

so that

$$k^{1/p} \|T\|_p = k^{1/q} \|T\|_q .$$

This implies clearly that the correspondence between $M(L^p, \psi)$ and $M(L^q, \psi)$ is a one-to-one, onto, continuous linear transformation.

If we redefine the norm in $M(L^p, \psi)$ by the equivalent norm $K^{1/p-1/q} \|\cdot\|_p$ then there exists an isometry between $M(L^p, \psi)$ and $M(L^q, \psi)$.

THEOREM 16.6 Let G be a locally compact abelian group and β an isomorphism of G onto G, $\gamma_0 \in \hat{G}$. Then, if T a bounded linear operator on $L^p(G)$ that satisfies

$$\tau_y T = (y^{-1}, \gamma_0)\, T \tau_{\beta(y)} \quad , \quad y \in G$$

then T is also an element of $M(L^2, \psi)$ where ψ is given by

$$\psi f = (\cdot, \gamma_0)\, f \circ \beta$$

Moreover, there exists a linear isomorphism continuous from $M(L^p, \psi)$ into $M(L^2, \psi)$.

PROOF: Notice that ψ defines an isomorphism both in L^p and L^2. Let $T \in M(L^p, \psi)$. By a form of Riesz Thorin convexity theorem the function log $\|T\|_{1/a}$ is convex on $0 \leq a \leq 1$. In particular, since $1/p \cdot p + 1/q \cdot q = 2$ and $1/p + 1/q = 1$, we have

$$\log \|T\|_2 \le {}^1\!/_p \log \|T\|_p + {}^1\!/_q \log \|T\|_q$$

$$= {}^1\!/_p \log \|T\|_p + {}^1\!/_q \log (\|T\|_p K^{1/q - 1/p}).$$

by the previous theorem.

It follows then that

$$\log \|T\|_2 \le ({}^1\!/_p + {}^1\!/_q) \log \|T\|_q + {}^1\!/_q \log K^{1/q - 1/p}$$

$$= \log (\|T\|_p (K^{1/q - 1/p})^{1/q}) .$$

Therefore we obtain

$$\|T\|_2 \le C \|T\|_p \qquad \text{where} \quad C = (K^{1/q - 1/p})^{1/q}.$$

For any characteristic function χ_E of a Borel set E

$$\|T\chi_E\|_2 \le C \|T\chi_E\|_p \le C \|T\|_p \|\chi_E\|_p$$

since $T \in M(L^p, \psi)$. Let $p > 2$. Then

$$\|T\chi_E\|_2 \le C \|T\|_p (\lambda(E))^{1/p}$$

$$\le C \|T\|_p (\lambda(E))^{1/2} = C \|T\|_p \|\chi_E\|_2 .$$

If $1 < p < 2$, then , since $q > 2$

$$\|T\chi_E\|_2 \leq C \|T\chi_E\|_p = C\, k^{1/p - 1/q} \|T\chi_E\|_q \cdot$$

by Theorem 16.5 . Hence we see that

$$\|T\chi_E\|_2 \leq C' \|T\|_q \,(\lambda(E))^{1/q}$$

$$< C' \|T\|_q \,\|\chi_E\|_2$$

$$< \|T\|_p \,\|\chi_E\|_2$$ again using Theorem 16.5 .

Therefore, for any $T \in M(L^p, \psi)$, $1 < p < \infty$, we get

$\|T\chi_E\|_2 \leq C \|\chi_E\|_2$ for some constant $C > 0$.

Hence T is a bounded linear operator on the space of all integrable simple functions . Since the space of all integrable simple functions are dense in L^2 , T is a bounded linear operator on L^2 . Also, it is easy to see that T satisfies

$$\tau_y T = (y^{-1}, \gamma_0)\, T \tau_{\beta(y)}$$

on $L^2(G)$.

Clearly the correspondence of $M(L^p, \psi)$ into $M(L^q, \psi)$ is continuous, one-to-one and linear. This completes the proof.

Now, using theorems 16.5 and 16.6 we can characterize the space $M(L^p, \psi)$ to be the dual of a certain Banach space $A_p^p(G)$ of continuous functions analogous to the result of Figa-Talamanca (See, for example, [15]).

Similar statements for the spaces $M(G)$, $M_\omega(G)$, $C_0(G)$ $L^\infty(G)$ and $L^\infty_\omega(G)$ are also valid. $M_\omega(G)$ and $L^\infty_\omega(G)$ stand for the respective spaces with the weak star topology induced by $C_0(G)$ and $L'(G)$, where G is a locally compact abelian group.

<u>THEOREM 16.7</u> Let T be a bounded linear transformation of M (G) into itself. Suppose $\gamma_0 \in \hat{G}$ and β a continuous isomorphism of G onto itself. Then if T satisfies

(72) $$\tau_y T = (y^{-1}, \gamma_0)\, T \tau_{\beta(y)} , \quad y \in G .$$

then there exists a unique $\omega \in M(G)$ such that

(73) $$T\mu = \omega * \psi\mu$$

where ψ is an isomorphism on M (G) given by

(74) $$\psi\mu = (\cdot\, , \gamma_0)\, \mu \circ \beta$$

Moreover, there exists a continuous linear map of $M(M(\omega), \psi)$ onto M (G) . In addition,

$$T(\nu * \mu) = T\nu * \psi\mu , \quad \nu \in L^1(G), \mu \in M(G).$$

<u>THEOREM 16.8</u> Let $T : M_\omega(G) \longrightarrow M_\omega(G)$ be a continuous linear transformation. If T satisfies (72) then there exists a unique $\omega \in M(G)$ such that (73) holds, where ψ is the isomorphism on $M_\omega(G)$ given by (74). Moreover, the map, $T \to \omega$ defines a linear isomorphism from $M(M_\omega(G), \psi)$ onto M (G) .

<u>THEOREM 16.9</u> If $T : L^\infty_\omega(G) \to L^\infty_\omega(G)$ is a continuous linear transformation satisfying (72) then there exists a unique $\omega \in$ M (G) such that, for

(75) $$Tf = \omega * \psi f$$

where ψ is an isomorphism on $L^\infty_\omega(G)$ given by

(76) $$\psi f = (\cdot, \gamma_0) f \circ \beta$$

Moreover, the correspondence $T \to \omega$ is a one-to-one, linear, onto isometry.

<u>THEOREM 16.10</u> If $T : L^\infty(G) \longrightarrow L^\infty(G)$ is a bounded linear transformation satisfying (72) then there exists a unique $\omega \in M(G)$ such that (75) holds for $f \in L^\infty(G)$. Moreover, there exists a continuous homomorphism of $M(L^\infty, \psi)$ onto $M(G)$ where ψ is the isomorphism on $L^\infty(G)$ given by (76).

<u>THEOREM 16.11</u> Let $T : C_0(G) \longrightarrow C_0(G)$ be a continuous linear transformation. Let $\gamma_0 \in \hat{G}$ and β be a continuous isomorphism of G onto G such that β^{-1} carries compact sets into compact sets. Then if T satisfies (72) we can find a unique $\omega \in M(G)$ such that (75) holds for all $f \in C_0(G)$, where ψ is given by (76). Moreover, the correspondence $T \rightarrow \omega$ is a one to one, linear, onto isometry.

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