PROXIMINAL SUBSPACES OF FINITE CODIMENSION IN GENERAL NORMED LINEAR SPACES

THESIS

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Introduction

This thesis deals with proximinal and chebychev subspaces of finite codimension in general normed linear spaces.

Throughout the thesis E denotes a real normed linear space and E^* its dual. Further we denote by E_1,

$$E_1 = \{ x \in E : \|x\| \leq 1 \}$$

the unit ball of E.

Let M be a closed subspace of finite codimension n in E. We have by the canonical linear isometry

$$M^{\perp} \simeq (E/M)^*$$

and hence

$$\dim M^{\perp} = \dim (E/M)^* = \dim (E/M) = n$$

where 'dim' denotes dimension. Thus $M^{\perp}$ is the linear span of n linearly independent elements $f_1, f_2, \cdots, f_n$ of E^*. We denotes this symbolically

$$M^{\perp} = [f_1, f_2, \cdots, f_n]$$

We recall that M is a proximinal subspace of E if

$$P_M(x) = \left\{ m_0 \in M : \|x - m_0\| = \inf_{m \in M} \|x - m\| \right\}$$

is nonempty for every $x \in E$. Further if $P_M(x)$ is a singleton set for each $x \in E$, M is a chebychev subspace of E.

Further for any subset A of E, we denote by $co(A)$, the convexhull of the set A, and by $B \setminus A$ the complement of the set A in the subset B of E.

The following theorem of Garkavi characterising proximinal subspaces of finite codimension in a general normed linear space is the starting point of our study.

THEOREM 1.1. Garkavi [6]. Let M be a subspace of finite codimension in E. Then M is proximinal in E if and only if for every $\Phi \in (M^{\perp})^{*}$ there exists $x \in E$ such that

$$\|\Phi\| = \|x\|$$

(A) and

$$\Phi(f) = f(x) \quad \text{for all} \quad f \in M^{\perp}$$

The problem of characterisation of proximinal subspaces of finite codimension in concrete normed linear spaces is a difficult one and Garkavi has given a complete solution for the space C(Q), the space of all continuous real valued functions on the compact Hausdroff space Q with sup norm and $L_1(T, \nu)$, the space of all real valued Lebesgue integrable functions on the σ-finite positive measure space (T, ν). His results run as follows:

THEOREM 1.2 Garkavi [3]. Let M be a closed subspace of finite codimension in C(Q). For $\mu \in C(Q)^*$ **let** $\mu = \mu^+ + \mu^-$ $(\mu^+ \geq 0,\ \mu^- \leq 0)$ **denote the Jordan decomposition of the measure** μ **and** $S(\mu)$ **its support. Then the following conditions are necessary and sufficient for M to be proximinal in C(Q):**

(α) $S(\mu^+) \cap S(\mu^-) = \phi$ **for every** $\mu \in M^\perp \setminus \{0\}$

(β) μ_2 **is absolutely continuous with respect to** μ_1 **on** $S(\mu_1)$ **for every** $\mu_1, \mu_2 \in M^\perp \setminus \{0\}$.

(γ) $S(\mu_2) \setminus S(\mu_1)$ **is closed for each pair** μ_1, μ_2 **in** $M^\perp \setminus \{0\}$.

THEOREM 1.3 Garkavi [4]. Let M be a closed subspace of finite codimension n in $L_1(T,\nu)$. **Then in order that M be proximinal in** $L_1(T,\nu)$ **it is necessary and sufficient that for every basis** $f_1, f_2, \ldots, f_n$ **the measure of the set**

$$G_n^0 = \{ t \in T : |f_i(t)| = \bar{N}_i ,\ i = 1, 2 \ldots n \}$$

is positive, where

$$\bar{N}_1 = \operatorname*{ess\cdot sup}_{T\ \nu} |f_1(t)| , \quad \bar{N}_i = \lim_{\epsilon \to 0} \operatorname*{ess\cdot sup}_{\bar{G}^\epsilon_{i-1}} |f_i(t)| \quad (i = 2, 3, \ldots, n)$$

where

$$\bar{G}^\epsilon_{i-1} = \{ t \in T : \bar{N}_j - \epsilon \leq |f_j(t)| \leq \bar{N}_j + \epsilon ,\ j = 1, 2 \ldots i-1 \}$$

The problem has also been studied in m_0 = the space of all bounded sequences of scalars [4], c_0 = the space of all sequences of scalars converging to zero [7] and ℓ^1 = the space of all sequences $\{\alpha_n\}$ of scalars satisfying $\sum_{n=1}^{\infty} |\alpha_n| < \infty$ [7]

Though the problem of characterisation of proximinal subspaces of finite codimension has been studied both in general and as well as in particular cases, there has been so far no characterisation of such subspaces which unifies the main known results in the concrete spaces. It is the object of our study to provide one which serves the purpose effectively. We give here a characterisation of proximinal subspaces of finite codimension in a general normed linear space (Theorem 2.10), which either reduces to or give rise to equivalent conditions to those of Garkavi's in the spaces C(Q) and $L_1(T,\nu)$. Both the reduction and reconcilation are far from trival and involve considerable amount of technicalities. Apart from uniting the apparently unrelated results of Garkavi, our results provide clear interpretations of his conditions developing at the same time a point of view which both abbreviates and simplifies the arguments in the special cases.

Chapter I contains our main theorem and all the preliminaries required for proving it. In Chapter II we derive a characterisation of proximinal subspaces of finite codimension in the space $L_1(T,\Sigma,\mu)$ using our characterisation theorem proved in Chapter I. This we reconcile with Theorem 1.3 of Garkavi. The final theorem of this chapter asserts that the proximinal subspaces of finite codimension n are 'dense' in the class of all subspaces of finite codimension n, in the space $L_1(T,\Sigma,\mu)$.

In Chapter III we apply the characterisation theorem of Chapter I to the space C(Q) to deduce the characterisation theorem (Theorem 1.2) of Garkavi in that space.

Chapter IV discusses the problem of characterisation of semichebychev and chebychev subspaces of finite codimension. We give a characterisation of semichebychev subspaces of finite codimension in a general normed linear space and then show that the conditions that are required for the proximinality of semichebychev subspaces are much more relaxed than those that are required for subspaces which are not semichebychev. Hence we derive a characterisation for Chebychev subspaces of finite codimension. We also give applications to the space $L_1(T,\Sigma,\mu)$.

CHAPTER I.

Proximinal Subspaces of Finite Codimension

This chapter will be mainly devoted to stating and proving our characterisation Theorem which gives a necessary and sufficient condition for a subspace of finite codimension n to be proximinal in a general normed linear space. We also derive some conditions which are equivalent to the condition of this theorem and close this chapter with our comments on the characterisation theorem of Singer (Theorem 1, [7]) on proximinal subspaces of finite codimension.

Since the subspaces under consideration are of finite codimension, finite dimensional convex sets play a significant role in our discussions and the proof of our theorem essentially involves many properties of finite dimensional convex sets. So we will begin by recalling the relevant definitions and results.

DEFINITION 2.1 Let K be a subset of a vector space. A nonempty set $S \subset K$ is called an extreme set of K if no point of S is an internal point of a linear interval whose endpoints are in K but not in S. The extreme points are extreme sets that consist of just one point.

Let E be a finite dimensional normed linear space and K be a convex subset of E. Then

DEFINITION 2.2 The dimension, dim K, of K is the dimension of the affinehull of K.

DEFINITION 2.3 The relative interior and relative frontier of K, are the interior and frontier of K relative to the affinehull of K. They will be denoted by relint K and relfr K.

Also, we have the following definitions concerning 'faces' of the convex set K.

DEFINITION 2.4 A subset F of K is a face of K if it is nonempty convex and extreme subset of K.

DEFINITION 2.5 A subset F of K is an exposed face of K if F is nonempty and there exists a nonzero linear functional g on E such that if

$$M = \{ x \in E : g(x) = \sup g(K) \}$$

then $F = M \cap K$.

In this case it will be said that g and M expose F. (Note that since an exposed face of K is a nonempty convex extreme subset of K, it is also a face of K.)

DEFINITION 2.6 If F is a face of K and $F \neq K$ then F is called a proper face of K.

We will now list some of the elementary properties of K which will be needed in the sequel.

(i) A nonempty intersection of extreme subsets of K is an extreme subset of K.

(ii) Let $x \in K$. Then there exists a smallest extreme subset of K containing x, it will be denoted by $E_K(x)$.

(For $y, z \in E$, let (y, z) and $[y, z]$ denote the open and closed line segments determined by y and z) .

(iii) $E_K(x) = \bigcup \left\{ [y, z] : y, z \in K \text{ and } x \in (y, z) \right\}$

(iv) $E_K(x)$ is convex and is the smallest face of K containing x.

(v) If E is a convex subset of K and $x \in$ relint E then $E \subseteq E_K(x)$.

(vi) A nonempty intersection of exposed faces of K is an exposed face of K.

If $x \in \text{relfr} K \cap K$ then there exists a proper exposed face F of K such that $x \in F$.

All the definitions (except 2.1) and properties are given in [2] .

We observe that if $x \in$ relint K then by (ii), (iii) and (v) x belongs to the convexhull of the relfr K. Thus we have

(viii) K is the convexhull of its relative frontier.

We shall now prove two lemmas about finite dimensional convex sets which are needed in the proof. Both these lemmas are known. The first one in a different formulation has been given by Garkavi. The second lemma is given in [5] and it is attributed to Minkowski. Also Alfsen [1] proves it as an application of Caratheodory's theorem. However, we given independent proofs of these lemmas using only the most elementary methods.

LEMMA 2.7. Let E be a n-dimensional normed linear space. Then in order that Φ_0 be an extreme point of the unit ball of E, it is necessary and sufficient that there exist linearly independent elements $f_1, f_2, \dots, f_n$ of E such that

$$(B)\quad \begin{cases} f_1(\Phi_0) = \max\limits_{\|\Phi\| \le 1} f_1(\Phi) \\ f_k(\Phi_0) = \max\limits_{\Phi \in \mathcal{N}^E_{f_1 \cdots f_{k-1}}} f_k(\Phi), \quad (k = 2, 3 \dots n) \end{cases}$$

where

$$\mathcal{N}^E_{f_1} = \left\{ \Phi \in E : \|\Phi\| = 1 \text{ and } f_1(\Phi) = \|f_1\| \right\}$$

$$\mathcal{N}^E_{f_1 \cdots f_{k-1}} = \left\{ \Phi_0 \in \mathcal{N}^E_{f_1 \cdots f_{k-2}} : f_{k-1}(\Phi_0) = \max_{\Phi \in \mathcal{N}^E_{f_1 \cdots f_{k-2}}} f_{k-1}(\Phi_0) \right\}$$

__Proof. Necessity:__ Let Φ_0 be an extreme point of E_1. Then $\Phi_0 \in \mathrm{rel\,fr}\, E_1 \cap E_1$ and so by (vii) there exists a proper exposed face $\mathcal{N}_{f_1}^{E}$ of E_1 exposed by $f_1 \in E^*$ and M_1 where

$$M_1 = \left\{ \Phi \in E_1 : f_1(\Phi) = \sup_{\Phi \in E_1} f_1(\Phi) \right\}$$ containing

Φ_0. Then $\mathcal{N}_{f_1}^{E} = E_1 \cap M_1$ and Φ_0 is an extreme point of $\mathcal{N}_{f_1}^{E}$. Proceeding thus inductively we can find a finite sequence $\left\{ \mathcal{N}_{f_1 \cdots f_i}^{E} \right\}_{i=1}^{n}$ of convex subsets of E_1 each containing Φ_0 such that

$$\mathcal{N}_{f_1 \cdots f_i}^{E} = M_i \cap \mathcal{N}_{f_1 \cdots f_{i-1}}^{E}$$

where

$$M_i = \left\{ \Phi \in E_1 : f_i(\Phi) = \sup_{\Phi \in \mathcal{N}_{f_1 \cdots f_{i-1}}^{E}} f_i(\Phi) \right\}$$

Hence $\Phi_0 = \bigcap_{i=1}^{n} \mathcal{N}_{f_1 \cdots f_i}^{E}$ and the condition of the lemma is obviously satisfied.

__Sufficiency:__ Let $f_1, f_2, \ldots, f_n$ be a basis of E^* such that $\Phi_0 \in E_1$ satisfies (B). The sets $\mathcal{N}_{f_1 \cdots f_i}^{E}$'s being exposed faces of E_1, are extreme subsets of E_1 Further since $f_1, f_2, \ldots, f_n$ constitute a basis for E^*, it follows that $\bigcap_{i=1}^{n} \mathcal{N}^{E}_{f_1 \cdots f_i}$ is a singleton set. Thus

$$\left\{ \Phi_0 \right\} = \bigcap_{i=1}^{n} \mathcal{N}^{E}_{f_1 \cdots f_i}$$

is an extreme point of E_1.

LEMMA 2.8 Let K be a compact convex subset of a finite dimensional normed linear space E. Then K is the convexhull of the set of all its extreme points.

Proof by induction on the dimension of K: If dim K is 1, then the conclusion is trivially true. Suppose that the dim K is d where $d > 1$. Then by the induction hypothesis the lemma is valid for all compact convex subsets of dimension less than d. Since K is a finite dimensional convex set it is the convexhull of its relative frontier by (viii). Further by (vii), every point of the relative frontier is contained in some proper exposed face of K and so K is the convexhull of the union of all its proper exposed faces. Let A be any proper exposed face of K, exposed by some $f \in E^*$ and

$$M = \left\{ x \in E_1 : f(x) = \sup_{y \in K} f(y) = k_1 \ (\text{say}) \right\}$$

Then $A = M \cap K$ is closed and since $A \neq K$ there exists $x_0 \in K$ such that $f(x_0) < k_1$ and thus x_0 does not belong to the affinehull of A. This implies that $\dim A < \dim K$ and since A is a compact, convex set, by the induction hypothesis, A is the convexhull of the set of all its extreme points. Moreover every extreme point of A is clearly an extreme point of K also. Thus the union of all proper exposed faces of K is contained in the convexhull of the set of all extreme points of K. Hence K itself is the convexhull of the set of all its extreme points.

Having now given all the needed informations about the finite dimensional convex sets, we will go on to the preliminaries that are required for stating the main theorem.

As mentioned earlier, throughout we assume that E denotes a real normed linear space, E^* its dual and E_1 its unit ball.

Let M be a closed subspace of codimension n and

$$M^{\perp} = [f_1, f_2, \ldots, f_n]$$

where $f_1, f_2, \ldots, f_n$ are linearly independent elements of $M^{\perp}$. We set

$$\mathcal{N}_{f_1} = \left\{ \Phi \in (M^{\perp})^* : \|\Phi\| = 1,\ \Phi(f_1) = \|f_1\| \right\}$$

and inductively define

$$\mathcal{N}^{\circ}_{f_1, f_2, \dots, f_k} = \left\{ \Phi_0 \in \mathcal{N}^{\circ}_{f_1 \cdots f_{k-1}} : \Phi_0(f_k) = \max_{\Phi \in \mathcal{N}^{\circ}_{f_1 \cdots f_{k-1}}} \Phi(f_k) \right\}$$

$$(k = 2, 3, \dots n)$$

Analogously we define

$$\mathcal{M}_{f_1} = \left\{ x \in E : \|x\| = 1, \ f_1(x) = \|f_1\| \right\}$$

and

$$\mathcal{M}_{f_1 \cdots f_k} = \left\{ x_0 \in \mathcal{M}_{f_1 \cdots f_{k-1}} : f_k(x_0) = \sup_{x \in \mathcal{M}_{f_1 \cdots f_{k-1}}} f_k(x) \right\}$$

$$(k = 2, 3, \dots, n)$$

<u>Remark 2.9</u> We notice that

$$(M^{\perp})^*_1 \supset \mathcal{N}^{\circ}_{f_1} \supset \mathcal{N}^{\circ}_{f_1 f_2} \supset \dots \dots \supset \mathcal{N}^{\circ}_{f_1 \cdots f_n}$$

is a finite chain of nonempty convex subsets of $(M^{\perp})^*_1$ where each set is an exposed face of the previous one. Further since $f_1, f_2, \dots, f_n$ is a basis of $M^{\perp}$, it follows that $\mathcal{N}^{\circ}_{f_1 \cdots f_n}$ is a singleton set. Also, we note that

$$E_1 \supset \mathcal{M}_{f_1} \supset \mathcal{M}_{f_1 f_2} \supset \dots \dots \supset \mathcal{M}_{f_1 \cdots f_n}$$

and $\mathcal{M}_{f_1 \cdots f_i}$'s need not be nonempty subsets of E_1 $(i = 1, 2, \cdots, n)$.

For $x \in E$, we will denote by $\hat{x}$, the image of x in the canonical embedding of E into its second dual E^{**} and $\theta(x)$, the restriction of the functional $\hat{x}$ to the subspace $M^{\perp}$ of E^*.

Now our characterisation theorem can be stated as follows.

THEOREM 2.10 Let E be a normed linear space. Let M be a closed subspace of finite codimension n and

$$M^{\perp} = [f_1, f_2, \cdots, f_n]$$

Then in order that M be proximinal in E, it is necessary and sufficient that for every basis $f_1, f_2, \cdots, f_n$ of $M^{\perp}$ we must have

$$\theta(\mathcal{M}_{f_1, f_2, \cdots f_n}) = \mathcal{N}_{f_1, f_2, \cdots, f_n} \qquad (1)$$

Proof. Necessity: Let M be a proximinal subspace of codimension n in E. Then the condition (A) of Theorem 1.1 hold and hence we have

$$\theta(E_1) = (M^{\perp})^*_1$$

This clearly implies that

$$\theta(\mathcal{M}_{f_1, f_2, \cdots f_n}) = \mathcal{N}_{f_1 \cdots f_n}$$

for every basis $f_1, f_2, \ldots, f_n$ of $M^{\perp}$.

<u>Sufficiency</u>: Suppose that $M^{\perp}$ satisfies condition (1). We have to show that M is proximinal. We first assert that it is enough for our purpose to prove that

$$\theta(E_1) = (M^{\perp})^*_1 \tag{2}$$

Let Φ be an arbitrary element of $(M^{\perp})^*_1$. Consider the element $\Phi' = \frac{\Phi}{\|\Phi\|}$. We have $\|\Phi'\| = 1$. Further by (2) there exists $x \in E_1$ such that

$$\theta(x) = \Phi' \tag{3}$$

Since $(M^{\perp})^*$ is a finite dimensional space there exists an $f \in M^{\perp} \setminus \{0\}$ satisfying

$$\Phi'(f) = \|f\|$$

But (3) holds and so we have

$$f(x) = \Phi'(f) = \|f\|$$

This implies that $\|x\| \geq 1$, which together with the fact that $x \in E_1$ further implies $\|x\| = 1$.

Now consider the element $x_1 = \|\Phi\| \, x$ in E. Obviously $\|x_1\| \leq 1$, and so $x_1 \in E_1$. Since θ is linear, (3) implies that

$$\theta(x_1) = \Phi$$

which gives

$$g(x_1) = \Phi(g) \quad \text{for all} \quad g \in M^{\perp} \tag{4}$$

Also, since $\|x\| = 1$ we have

$$\|x_1\| = \|\Phi\| \|x\| = \|\Phi\| \tag{5}$$

Since Φ is an arbitrary element of $(M^{\perp})^*_1$ (4) and (5) together imply that (A) is satisfied and hence our assertion is proved.

Thus to complete the proof of the theorem, we will only have to show that (1) implies

$$\theta(E_1) = (M^{\perp})^*_1$$

To this end, consider Φ_0, an extreme point of $(M^{\perp})^*_1$. Then by Lemma 2.7 there exists a basis $g_1, g_2, \ldots g_n$ of $M^{\perp}$ such that

$$\{\Phi_0\} = \mathcal{N}_{g_1, g_2, \ldots, g_n}$$

Now by condition (1) there exists an $x_0 \in \mathcal{M}_{g_1, g_2, \ldots, g_n}$ such that

$$\theta(x_0) = \Phi_0$$

Thus (1) ensures that $\theta(E_1) \subset (M^{\perp})^*_1$ contains all the extreme points of the convex set $(M^{\perp})^*_1$. But by

Lemma 2.8 $(M^{\perp})_1^*$ is the convexhull of the set of all its extreme points and hence we have

$$(M^{\perp})_1^* = co\ (\theta(E_1)) \tag{6}$$

Further E_1 is a convex set, θ is linear and so $\theta(E_1)$ is also a convex set which gives

$$\theta(E_1) = co\ (\theta(E_1)) \tag{7}$$

So from (6) and (7) we get

$$\theta(E_1) = (M^{\perp})_1^*$$

as desired and hence the proof of the theorem is completed.

We shall now derive a set of conditions which together are equivalent to condition (1) of Theorem 2.10. These will be made use of in the later chapters in interpreting condition (1) in the spaces $C(Q)$ and $L_1(T,\nu)$.

<u>PROPOSITION 2.11</u> <u>Let</u> M <u>be a closed subspace of codimension</u> n <u>in</u> E. <u>Let</u> $f_1, \dots f_n$ <u>be a basis of</u> $M^{\perp}$. <u>Then condition (1) of Theorem 2.12 is equivalent to</u>

$$\mathcal{M}_{f_1 \dots f_i} \quad \text{\underline{is nonempty for each}} \quad i = 1, 2 \dots n \tag{8}$$

and

$$\sup_{x \in \mathcal{M}_{f_1 \dots f_{i-1}}} f_i(x) = \max_{\Phi \in \mathcal{N}_{f_1 \dots f_{i-1}}} \Phi(f_i) \text{ for } i = 2, 3 \dots n \tag{9}$$

Proof: Assume that

$$\theta(\mathcal{M}_{f_1, f_2, \ldots f_n}) = \mathcal{N}_{f_1, f_2, \ldots f_n}$$

Since $\mathcal{N}_{f_1 \ldots \ldots f_n}$ is always a nonempty set. We have $\mathcal{M}_{f_1 \ldots f_n}$ also to be a nonempty set by our assumption. Further

$$E_1 \supset \mathcal{M}_{f_1} \supset \mathcal{M}_{f_1 f_2} \supset \cdots \cdots \supset \mathcal{M}_{f_1 \ldots f_n}$$

and so we get

$\mathcal{M}_{f_1 \ldots \ldots f_i}$ is nonempty for $i = 1, 2, \ldots \ldots n$ which gives (8).

To prove (9) consider Φ_0 in $\mathcal{N}_{f_1, \ldots f_n}$. By (1) there exists $x_0 \in \mathcal{M}_{f_1 \ldots f_n}$ such that

$$\theta(x_0) = \Phi_0$$

This implies

$$f_i(x_0) = \Phi_0(f_i) \text{ for } i = 1, 2, \ldots \ldots n$$

Since $x_0 \in \mathcal{M}_{f_1 \ldots f_n}$ and $\Phi_0 \in \mathcal{N}_{f_1 \ldots f_n}$ the above equality implies

$$\sup_{x \in \mathcal{M}_{f_1 \ldots f_{i-1}}} f_i(x) = f_i(x_0) = \Phi_0(f_i) = \max_{\Phi \in \mathcal{N}_{f_1 \ldots f_{i-1}}} \Phi(f_i)$$

for $i = 2, 3, \ldots \ldots, n$, which gives (9).

Now suppose that both (8) and (9) are satisfied. Let

$$\{\bar{\Phi}_0\} = \mathcal{N}_{f_1, f_2, \ldots, f_n}$$

Since (8) holds $\mathcal{M}_{f_1 \ldots f_n}$ is a nonempty set and further (9) implies that for every $x_0 \in \mathcal{M}_{f_1, f_2, \ldots, f_n}$

$$f_i(x_0) = \sup_{x \in \mathcal{M}_{f_1 \ldots f_{i-1}}} f_i(x) = \max_{\Phi \in \mathcal{N}_{f_1 \ldots f_{i-1}}} \Phi(f_i) = \bar{\Phi}_0(f_i)$$

for $i = 1, 2, \ldots, n$. Hence

$$[\theta(x_0)](f) = f(x_0) = \bar{\Phi}_0(f)$$

for each $f \in M^{\perp}$. Thus

$$\theta(x_0) = \bar{\Phi}_0$$

for every $x_0 \in \mathcal{M}_{f_1, f_2, \ldots, f_n}$ which gives

$$\theta(\mathcal{M}_{f_1 \ldots f_n}) = \mathcal{N}_{f_1 \ldots f_n}$$

<u>Remark 2.12</u>. It is clear from the proof of Proposition 2.11 that the condition (A) of Theorem 1.1 is equivalent to

$$\theta(E_1) = (M^{\perp})^*_1$$

Notation 2.13 We will denote by

$$\text{I}\quad\begin{cases} N_{f_1} = \max\limits_{\Phi \in (M^\perp)_1^*} \Phi(f_1) \\ N_{f_1 \cdots f_i} = \max\limits_{\Phi \in \mathcal{N}_{f_1 \cdots f_{i-1}}} \Phi(f_i) \\ \text{for } i = 2, 3, \ldots, n. \end{cases}$$

and

$$\text{II}\quad\begin{cases} M_{f_1} = \sup\limits_{x \in E_1} f_1(x) \\ M_{f_1 \cdots f_i} = \sup\limits_{x \in \mathcal{M}_{f_1 \cdots f_{i-1}}} f_i(x) \\ \text{for } i = 2, 3, \ldots, n \end{cases}$$

We will now conclude this chapter by giving a very simple alternate proof for Singer's characterisation theorem given in [7] using Theorems 1.1 and 2.10 and also Proposition 2.11. First, we state the theorem as in [7].

THEOREM. 2·14 Let E be a normed linear space and M a subspace of codimension n of E. The following statements are equivalent:

1. M is proximinal.
2. There exists a basis $f_1, f_2, \ldots, f_n$ of $M^\perp$ such that the set $A = \{f_1(y), f_2(y), \ldots, f_n(y) : y \in E_1\}$ is closed in the n-dimensional euclidean space $\mathbb{R}^n$.
3. For every basis $f_1, f_2, \ldots f_n$ of $M^\perp$ the set A is closed in $\mathbb{R}^n$.
4. E_1 is sequentially complete for the (locally convex, non-Hausdroff) weak topology $\sigma(E, M^\perp)$. These statements imply and if $\dim \mathcal{M}_f < \infty$ $(f \in M^\perp \setminus \{0\})$ they are equivalent to the following statement:
5. Every $f \in M^\perp \setminus \{0\}$ satisfies the following two condition:

   (a) We have

   $$\mathcal{M}_f = \{x \in E : \|x\| = 1 , f(x) = \|f\|\} \neq \phi$$

   and

   (b) if $\{x_j\} \subset E_1$, $\lim_{j \to \infty} |f(x_j)| = \|f\|$, then

   $$\lim_{j \to \infty} |h(x_j)| \leq \sup_{x \in \mathcal{M}_f} |h(x)| , h \in M^\perp .$$

Proof: We first observe that if $F=\{f_1,f_2,\ldots,f_n\}$ is a basis of $M^{\perp}$, then the map

$$T_F : (M^{\perp})^* \longrightarrow \mathbb{R}^n$$

defined by

$$T_F(\Phi) = (\Phi(f_1), \Phi(f_2), \ldots, \Phi(f_n)) \quad , \quad \Phi \in (M^{\perp})^*$$

is onto and is actually a homeomorphism. Considering the maps

$$E \xrightarrow{\theta} (M^{\perp})^* \xrightarrow{T_F} \mathbb{R}^n$$

we find that

$$\{(f_1(y), f_2(y), \ldots, f_n(y)) : y \in E_1\} = T_F(\theta(E_1))$$

and with this observation we shall proceed to prove the theorem.

$1 \Longrightarrow 3$. Suppose that M is proximinal and let $f_1, f_2, \ldots, f_n$ be a basis of $M^{\perp}$. Then Theorem 1.1 gives $\theta(E_1) = (M^{\perp})^*_1$. Also, by the preceding observation $A = T_F(\theta(E_1)) = T_F((M^{\perp})^*_1)$. Since T_F

is a homeomorphism and $(M^{\perp})^*_1$ is closed in $(M^{\perp})^*$ it follows that A is closed. Since $f_1, \ldots, f_n$ is an arbitrary basis for $M^{\perp}$, 3 is proved.

$3 \Longleftrightarrow 4$ follows immediately from the definition of sequential completeness.

$3 \longrightarrow 2$ is obvious.

$2 \Longrightarrow 1$ Suppose 2 holds. Then there exists a basis $G = (g_1, g_2, \ldots, g_n)$ of $M^{\perp}$ such that the set

$$A = \left\{ g_1(y), g_2(y), \ldots\ldots, g_n(y) : y \in E_1 \right\} \text{ is}$$

is closed in $\mathbb{R}^n$. Then $A = T_G(\theta(E_1))$. Since A is assumed to be closed and T_G is a homeomorphism it follows that $\theta(E_1)$ is closed in $(M^{\perp})^*$. But $\theta(E_1)$ is dense in $(M^{\perp})^*_1$ and hence $\theta(E_1) = (M^{\perp})^*_1$ which proves the proximinality of M. Hence 1,2,3 and 4 are equivalent.

To prove the remaining part of the theorem we first note that since 5(b) is satisfied for every $f \in M^{\perp} \setminus \{0\}$ and each $h \in M^{\perp}$, it is obviously equivalent to

5(b') For every $f \in M^{\perp} \setminus \{0\}$,

if $\{x_j\} \subset E_1$ $\lim_{j \to \infty} f(x_j) = \|f\|$, then

$$\lim_{j \to \infty} h(x_j) \leq \sup_{x \in \mathcal{N}_f} h(x), \quad h \in M^{\perp}.$$

So, we will now show that proximinality of M implies 5(a) and 5(b') and the converse is valid under the additional assumption that $\mathcal{M}_f$ is finite dimensional for $f \in M^\perp \setminus \{0\}$.

1 $\Longrightarrow$ 5. Suppose M is proximinal. Then condition (1) of Theorem 2.10 holds, Hence by Proposition 2.11 we have $\mathcal{M}_f$ is nonempty for each $f \in M^\perp \setminus \{0\}$ and

$$\sup_{x \in \mathcal{M}_{f_1}} f_2(x) = \max_{\Phi \in \mathcal{N}_{f_1}^0} \Phi(f_2) \text{ for all } f_1, f_2 \in M^\perp \setminus \{0\} \tag{10}$$

Since $\theta(E_1)$ is dense in $(M^\perp)^*_1$, 5(b') is equivalent to (10). Hence both 5(a) and 5(b') are satisfied and so 1 $\Longrightarrow$ 5.

To prove the other implication we note that $\theta(\mathcal{M}_f) \subset \mathcal{N}_f^0$ for each $f \in M^\perp \setminus \{0\}$. So, if $\theta(\mathcal{M}_f)$ is not dense in $\mathcal{N}_f^0$, there exists an $h \in M^\perp$ and $\Phi \in \mathcal{N}_f^0 \setminus \theta(\mathcal{M}_f)$ such that

$$\sup_{x \in \mathcal{M}_f} h(x) \leq \sup_{\{x_n\} \subset \mathcal{M}_f} \left\{ \lim_{n \to \infty} h(x_n) \right\} < \Phi(h)$$

which contradicts (10) and so refutes 5(b') also. Thus 5(b') implies that $\theta(\mathcal{M}_f)$ is dense in $\mathcal{N}_f^0$ for each $f \in M^\perp \setminus \{0\}$.

Further, if $\mathcal{M}_f$ is finite dimensional for each $f \in M^\perp \setminus \{0\}$ then, being a closed and bounded set it is also compact. By the continuity of θ it then follows that $\theta(\mathcal{M}_f)$ is compact and hence is closed. But we have already shown that 5(b') implies $\theta(\mathcal{M}_f)$ is dense in $\mathcal{N}_f'$ for each $f \in M^\perp \setminus \{0\}$.

Hence

$$\theta(E_1) = (M^\perp)^*_1$$

and so M is proximinal.

The Space $L_1(T,\nu)$

In this chapter we apply Theorem 2.10 to the space, $L_1(T,\nu)$, of all real valued Lebesgue integrable functions on the σ-finite positive measure space (T,ν), and hence obtain the characterisation of proximinal subspaces of finite codimension in that space. This we prove to be equivalent to the characterisation theorem of Garkavi in Theorem 3.5. The crucial part of this procedure is interpreting the quantities $N_{f_1 \ldots f_i}$'s given by I (Notation 2.13 of Chapter I) in the space $L_1(T,\nu)$. This is achieved in Proposition 3.3 using the fact that $\hat{E}$ is ω^* dense in E^{**}.

Proposition 3.3 is similar to a result of Garkavi which is given in the course of the proof of his characterisation Theorem (Theorem 2 in [7]). But the method of proof that is employed here to prove the Proposition is totally different from the one which is used by Garkavi to prove his result.

First we will give some definitions and obtain the preliminary results that are required for proving Proposition 3.3 and Theorem 3.5.

Let M be a closed subspace of codimension n in $M^\perp$ Since $L_\infty(T,\nu)$ is the dual of $L_1(T,\nu)$ we have

$M^\perp = [f_1, f_2, \ldots, f_n]$ where $f_1, \ldots f_n \in L_\infty(T,\nu)$

Let χ_A denote the characteristic function of the set A.

If $f_1, f_2, \ldots, f_n$ is a basis of $M^{\perp}$ we set

$$\left.\begin{aligned} \tilde{N}_{f_1} &= \operatorname*{ess\cdot sup}_{T\ \nu} |f_1(t)| = \|f_1\| \\ &\text{and} \\ \tilde{N}_{f_1 \cdots f_k} &= \lim_{\rho \to 0} \tilde{N}^{\rho}_{f_1 \cdots f_k} \end{aligned}\right\} \tag{11}$$

for $k = 2, 3, \ldots n$, where

$$\tilde{N}^{\rho}_{f_1 \cdots f_k} = \operatorname{ess\cdot sup} f_k(t) \left[\chi_{G^{\rho+}_{f_1 \cdots f_{k-1}}} - \chi_{G^{\rho-}_{f_1 \cdots f_{k-1}}} \right]$$

$$G^{\rho+}_{f_1 \cdots f_{k-1}} = \left\{ t \in T : \tilde{N}_{f_1 \cdots f_i} - \rho \leq f_i(t) \leq \tilde{N}_{f_1 \cdots f_i} + \rho,\ i = 1, 2 \cdots k-1 \right\}$$

$$G^{\rho-}_{f_1 \cdots f_{k-1}} = \left\{ t \in T : -\tilde{N}_{f_1 \cdots f_i} - \rho \leq f_i(t) \leq -\tilde{N}_{f_1 \cdots f_i} + \rho,\ i = 1, 2 \cdots k-1 \right\}$$

and

$$G^{\rho}_{f_1 \cdots f_{k-1}} = G^{\rho+}_{f_1 \cdots f_{k-1}} \cup G^{\rho-}_{f_1 \cdots f_{k-1}}$$

Similarly we define

$$\left.\begin{aligned} \bar{N}_{f_1} &= \operatorname*{ess\cdot sup}_{T\ \nu} |f_1(t)| = \|f_1\| \\ \bar{N}_{f_1 \cdots f_k} &= \lim_{\rho \to 0} \operatorname*{ess\cdot sup}_{\bar{G}^{\rho}_{f_1 \cdots f_{k-1}}} |f_k(t)|,\ (k = 2, 3 \cdots n) \end{aligned}\right\} \tag{12}$$

where

$$\overline{G}^{\rho}_{f_1 \cdots f_{k-1}} = \left\{ t \in T : \overline{N}_{f_1 \cdots f_i} - \rho \le |f_i(t)| \le \overline{N}_{f_1 \cdots f_i} + \rho ,\ i = 1, 2 \ldots k-1 \right\}$$

Also we set

$$G^{0+}_{f_1 \cdots f_i} = \left\{ t \in T : \ f_j(t) = \widetilde{N}_{f_1 \cdots f_j} ,\ j = 1, 2 \ldots i \right\}$$

$$G^{0-}_{f_1 \cdots f_i} = \left\{ t \in T : f_j(t) = -\widetilde{N}_{f_1 \cdots f_j} ,\ j = 1, 2 \ldots i \right\}$$

and

$$G^{0}_{f_1 \cdots f_i} = G^{0+}_{f_1 \cdots f_i} \cup G^{0-}_{f_1 \cdots f_i} \tag{13}$$

for $i = 1, 2 \ldots n$.

Further we have

$$\overline{G}^{0}_{f_1 \cdots f_i} = \left\{ t \in T : \ |f_j(t)| = \overline{N}_{f_1 \cdots f_j} ,\ j = 1, 2 \ldots i \right\} \tag{14}$$

for $i = 1, 2 \ldots n$.

<u>Remark 3.1</u> Whenever we are dealing with a fixed basis $f_1, f_2, \ldots, f_n$ we will replace the suffix $f_1 \cdots f_i$ by i $(i = 1, 2 \ldots n)$ for brevity's sake as there will be no confusion. For example $\widetilde{N}_i$ will denote $\widetilde{N}_{f_1 \cdots f_i}$ and G^{ρ}_i will denote $G^{\rho}_{f_1 \cdots f_i}$.

Now for the basis $f_1 \ldots f_n$ of $M^{\perp}$ let $\mathfrak{S}_k$ $(k = 2, 3, \ldots n)$ denote the class of all sequences $\{\phi_i\}$ in $L_1(T,\nu)$ such that $\|\phi_i\| \leq 1$ for all i and

$$\lim_{i\to\infty} \int_T \phi_i f_j \, d\nu = \tilde{N}_j \qquad (\tilde{N}_j\text{'s given by (11)})$$

for $j = 1, 2, \ldots, k$

Also we set for $x \in L_1(T,\nu)$

$$x^+ = \max[x, 0] \quad , \quad x^- = \min[x, 0]$$

and note that

$$x^+ \geq 0 \;,\; x^- \leq 0 \;,\; x = x^+ + x^- \quad \text{and} \quad |x| = x^+ - x^-$$

Then for any measurable subset A of T and $\{\phi_i\} \in S_k$,

$k = 1, 2, \ldots n$, we have

$$\int_A \phi_i^+ \, d\nu \leq \sup_i \int_A |\phi_i| \, d\nu \leq \sup_i \|\phi_i\| \leq 1$$

and

$$\int_A \phi_i^- \, d\nu \leq \sup_i \int_A |\phi_i| \, d\nu \leq \sup_i \|\phi_i\| \leq 1$$

so that both

$$\limsup_i \int_A \phi_i^+ \, d\nu$$

and

$$\limsup_i \int_A \phi_i^- \, d\nu$$

exist. Hence for arbitrary $\epsilon > 0$ we can define the following quantities for a sequence

$$\alpha_j^\epsilon = \limsup_i \int_{G_j^{\epsilon+}} \phi_i^+ d\nu \quad , \quad \beta_j^\epsilon = \limsup_i \int_{T\setminus G_j^{\epsilon+}} \phi_i^+ d\nu$$

$$\gamma_j^\epsilon = -\limsup_i \int_{G_j^{\epsilon-}} \phi_i^- d\nu \quad , \quad \delta_j^\epsilon = -\limsup_i \int_{T\setminus G_j^{\epsilon-}} \phi_i^- d\nu \; .$$

Then

$$\left.\begin{array}{l} \alpha_j^\epsilon \geq 0 \; , \; \beta_j^\epsilon \geq 0 \; , \; \gamma_j^\epsilon \geq 0 \; , \; \delta_j^\epsilon \geq 0 \\ \text{and} \\ \alpha_j^\epsilon + \beta_j^\epsilon + \gamma_j^\epsilon + \delta_j^\epsilon \leq 1 \end{array}\right\} \qquad (15)$$

for each $\epsilon > 0$ and all $j = 1, 2 \cdot \cdots n$.

We note that the above quantities vary for different sequences $\{\phi_i\}$. Still we have chosen not to distinguish them for varying sequences $\{\phi_i\}$ since we will be dealing with only one sequence at a time and there will be no room for ambiguity.

Now we can prove the following result about the sequence $\{\phi_i\}$ in S_k.

LEMMA 3.2 **For each** $k = 1, 2, \ldots, n-1$ **, if** $\{\phi_i\} \in S_k$ **then** for every $\rho > 0$

$$\limsup_i \int_{T \setminus G_k^{\rho +}} \phi_i^+ \, d\nu = \limsup_i \int_{T \setminus G_k^{\rho -}} \phi_i^- \, d\nu = 0$$

Proof by the method of induction: To begin with we have for $\rho > 0$ and $\{\phi_i\} \in S_1$,

$$\tilde{N}_1 = \lim_{i \to \infty} \int_T \phi_i f_1 \, d\nu = \lim_{i \to \infty} \int_T \phi_i^+ f_1 \, d\nu + \lim_{i \to \infty} \int_T \phi_i^- f_1 \, d\nu$$

$$= \lim_{i \to \infty} \int_{G_1^{\rho +}} \phi_i^+ f_1 \, d\nu + \lim_{i \to \infty} \int_{T \setminus G_1^{\rho +}} \phi_i^+ f_1 \, d\nu$$

$$+ \lim_{i \to \infty} \int_{G_1^{\rho -}} \phi_i^- f_1 \, d\nu + \lim_{i \to \infty} \int_{T \setminus G_1^{\rho -}} \phi_i^- f_1 \, d\nu$$

$$\le \tilde{N}_1 \limsup_i \int_{G_1^{\rho +}} \phi_i^+ \, d\nu + (\tilde{N}_1 - \rho) \limsup_i \int_{T \setminus G_1^{\rho +}} \phi_i^+ \, d\nu$$

$$+ \tilde{N}_1 \limsup_i \int_{G_1^{\rho -}} \phi_i^- \, d\nu + (-\tilde{N}_1 + \rho) \limsup_i \int_{T \setminus G_1^{\rho -}} \phi_i^- \, d\nu$$

$$= \tilde{N}_1 \alpha_1^\rho + (\tilde{N}_1 - \rho) \beta_1^\rho - \tilde{N}_1 (-\gamma_1^\rho) + (-\tilde{N}_1 + \rho)(-\delta_1^\rho)$$

$$= \tilde{N}_1 (\alpha_1^\rho + \beta_1^\rho + \gamma_1^\rho + \delta_1^\rho) - \rho (\beta_1^\rho + \delta_1^\rho)$$

$$\le \tilde{N}_1 - \rho (\beta_1^\rho + \delta_1^\rho) \qquad \text{by (15)}$$

This implies that $\beta_1^{\rho} + \delta_1^{\rho} = 0$, which again by (15) further implies that $\beta_1^{\rho} = \delta_1^{\rho} = 0$. Thus we have proved for $\{\phi_i\} \in \mathcal{G}_1$,

$$\limsup_i \int_{T \setminus G_1^{\rho+}} \phi_i^{+} d\nu = \limsup_i \int_{T \setminus G_1^{\rho-}} \phi_i^{-} d\nu = 0$$

We will now assume that for arbitrary $\rho > 0$ and $\{\phi_i\} \in \mathcal{G}_j$ we have

$$\limsup_i \int_{T \setminus G_j^{\epsilon+}} \phi_i^{+} d\nu = \limsup_i \int_{T \setminus G_j^{\epsilon-}} \phi_i^{-} d\nu = 0 \tag{16}$$

for $j = 1, 2, \ldots, n-2$. Consider any $\{\phi_i\}$ in $\mathcal{G}_{n-1}$. Since $\mathcal{G}_{n-1} \subset \mathcal{G}_{n-2}$ by induction hypothesis we have for each $\rho > 0$

$$\tilde{N}_{n-1} = \lim_{i \to \infty} \int_T \phi_i f_{n-1} d\nu = \lim_{i \to \infty} \int_{G_{n-2}^{\rho+}} \phi_i^{+} f_{n-1} d\nu + \lim_{i \to \infty} \int_{G_{n-2}^{\rho-}} \phi_i^{-} f_{n-1} d\nu$$

Thus given $\epsilon > 0$

$$\tilde{N}_{n-1} = \lim_{i \to \infty} \int_{G_{n-2}^{\rho+}} \phi_i^{+} f_{n-1} d\nu + \lim_{i \to \infty} \int_{G_{n-2}^{\rho-}} \phi_i^{-} f_{n-1} d\nu$$

$$= \lim_{i\to\infty} \int_{G_{n-2}^{\rho+} \cap G_{n-1}^{\epsilon+}} \phi_i^+ f_{n-1}\, d\nu + \lim_{i\to\infty} \int_{G_{n-2}^{\rho+} \setminus G_{n-1}^{\epsilon+}} \phi_i^+ f_{n-1}\, d\nu$$

$$+ \lim_{i\to\infty} \int_{G_{n-2}^{\rho-} \cap G_{n-1}^{\epsilon-}} \phi_i^- f_{n-1}\, d\nu + \lim_{i\to\infty} \int_{G_{n-2}^{\rho-} \setminus G_{n-1}^{\epsilon-}} \phi_i^- f_{n-1}\, d\nu$$

$$\leq \tilde{N}_{n-1}^{\rho} \limsup_i \int_{G_{n-2}^{\rho+} \cap G_{n-1}^{\epsilon+}} \phi_i^+ d\nu + (\tilde{N}_{n-1} - \epsilon) \limsup_i \int_{G_{n-2}^{\rho+} \setminus G_{n-1}^{\epsilon+}} \phi_i^+ d\nu$$

$$- \tilde{N}_{n-1}^{\rho} \limsup_i \int_{G_{n-2}^{\rho-} \cap G_{n-1}^{\epsilon-}} \phi_i^- d\nu + (-\tilde{N}_{n-1} + \epsilon) \limsup_i \int_{G_{n-2}^{\rho-} \setminus G_{n-1}^{\epsilon-}} \phi_i^- d\nu$$

$$= \tilde{N}_{n-1}^{\rho} \alpha_{n-1}^{\epsilon} + (\tilde{N}_{n-1} - \epsilon)\beta_{n-1}^{\epsilon} - \tilde{N}_{n-1}^{\rho}(-\gamma_{n-1}^{\epsilon}) + (-\tilde{N}_{n-1} + \epsilon)(-\delta_{n-1}^{\epsilon})$$

by (16). Since $\rho > 0$ was arbitrarily chosen, we can let $\rho \to 0$ in the above equality to obtain

$$\tilde{N}_{n-1} = \tilde{N}_{n-1} \alpha_{n-1}^{\epsilon} + (\tilde{N}_{n-1} - \epsilon)\beta_{n-1}^{\epsilon} + \tilde{N}_{n-1} \gamma_{n-1}^{\epsilon} + (\tilde{N}_{n-1} - \epsilon)(\delta_{n-1}^{\epsilon})$$

$$= \tilde{N}_{n-1} (\alpha_{n-1}^{\epsilon} + \beta_{n-1}^{\epsilon} + \gamma_{n-1}^{\epsilon} + \delta_{n-1}^{\epsilon}) - \epsilon(\beta_{n-1}^{\epsilon} + \delta_{n-1}^{\epsilon})$$

from which we conclude as before using (15)

$$\beta_{n-1}^{\epsilon} = \delta_{n-1}^{\epsilon} = 0 \text{ for each } \epsilon > 0$$

Hence for $\{\phi_i\} \in S_{n-1}$, we have proved,

$$\limsup_i \int_{T \setminus G_{n-1}^{\epsilon+}} \phi_i^+ d\nu = \limsup_i \int_{T \setminus G_{n-1}^{\epsilon-}} \phi_i^- d\nu = 0$$

for every $\epsilon > 0$, which completes the proof of this lemma.

PROPOSITION 3.3 Let M be a closed subspace of codimension n in $L_1(T, \nu)$ and $M^{\perp}$ its annihilator. Then for every basis $f_1, f_2, \ldots, f_n$ of $M^{\perp}$ we have

$$N_i = \widetilde{N}_i \quad \text{for } i = 1, 2, \ldots, n$$

Proof by the method of induction. To start with we have

$$N_1 = \widetilde{N}_1 = \|f_1\|$$

Applying induction we assume that

$$N_i = \widetilde{N}_i \text{ for } i = 1, 2 \ldots n-1$$

We will now have to show that

$$N_n = \widetilde{N}_n$$

To this end, we observe that since $\widehat{E}_1$ is ω^* dense in E_1^{**} we have for $k = 2, 3, \ldots, n$,

$$N_k = \sup \left\{ \limsup_i \int_T \phi_i f_k \, d\nu \right\}$$

where the supremum is taken over all sequences $\{\phi_i\}$ satisfying $\|\phi_i\| \le 1$ for all i and

$$\lim_{i \to \infty} \int_T \phi_i f_j \, d\nu = N_j$$

for $j = 1, 2, \ldots, k-1$. Since by the induction hypothesis

$$N_i = \tilde{N}_i \quad \text{for} \quad i = 1, 2, \ldots, n-1$$

we have

$$N_k = \sup_{\{\phi_i\} \in S_{k-1}} \left\{ \limsup_i \int_T \phi_i f_k \, d\nu \right\} \tag{17}$$

for $k = 2, 3, \ldots n$.

Now consider $\{\phi_i\} \in S_{n-1}$. Then by Lemma 3.2 we have for each $\epsilon > 0$

$$\limsup_i \int_{T \setminus G_{n-1}^{\epsilon +}} \phi_i^+ \, d\nu = \limsup_i \int_{T \setminus G_{n-1}^{\epsilon -}} \phi_i^- \, d\nu = 0$$

This implies that

$$\limsup_i \int_T \phi_i f_n \, d\nu = \limsup_i \int_{G_{n-1}^{\epsilon +}} \phi_i^+ f_n \, d\nu + \limsup_i \int_{G_{n-1}^{\epsilon -}} \phi_i^- f_n \, d\nu$$

$$\leq \tilde{N}_n^{\epsilon} \alpha_{n-1}^{\epsilon} - \tilde{N}_n^{\epsilon} \left(-\delta_{n-1}^{\epsilon} \right)$$

$$= \tilde{N}_n^{\epsilon} \left(\alpha_{n-1}^{\epsilon} + \delta_{n-1}^{\epsilon} \right)$$

$$\leq \tilde{N}_n^{\epsilon} \qquad \text{by (15)}$$

so that taking the limit as $\epsilon \to 0$ in the above inequality we get

$$\limsup_{i} \int_T g_i f_n \, d\nu \le \tilde{N}_n$$

for every $\{g_i\} \in S_{n-1}$. This together with (17) gives

$$N_n \le \tilde{N}_n \tag{18}$$

We shall now prove the opposite inequality. To this end let $\{\epsilon_i\}$ be a sequence of positive numbers tending to zero and consider the functions $x_i \in L_1(T, \nu)$ given by

$$x_i(t) = \begin{cases} \dfrac{\chi_{G_n^{\epsilon_i +}} - \chi_{G_n^{\epsilon_i -}}}{\nu(G_n^{\epsilon_i})} & , \quad t \in G_n^{\epsilon_i} \\ 0 & , \quad t \notin G_n^{\epsilon_i} \end{cases}$$

Then $\|x_i\| = 1$ for all i and

$$\lim_{i \to \infty} \int_T x_i f_j \, d\nu = \lim_{i \to \infty} \int_{G_n^{\epsilon_i}} x_i f_j \, d\nu$$

$$= \lim_{i\to\infty} \int_{G_n^{\epsilon_i +}} x_i^+ f_j \, d\nu + \lim_{i\to\infty} \int_{G_n^{\epsilon_i -}} x_i^- f_j \, d\nu$$

$$\geq \lim_{i\to\infty} \left[\frac{1}{\nu(G_n^{\epsilon_i})} \left\{ (\tilde{N}_j - \epsilon_i)\, \nu(G_n^{\epsilon_i +}) + (-\tilde{N}_j + \epsilon_i)\, \nu(G_n^{\epsilon_i -}) \right\} \right]$$

$$= \lim_{i\to\infty} \frac{1}{\nu(G_n^{\epsilon_i})} (N_j - \epsilon_i)\, \nu(G_n^{\epsilon_i})$$

$$= \lim_{i\to\infty} (\tilde{N}_j - \epsilon_i) = \tilde{N}_j$$

for $j = 1, 2, \cdots n$. Thus we have $\{x_i\} \in S_{n-1}$

and

$$\limsup_i \int_T \phi_i f_n \, d\nu = \tilde{N}_n .$$

By (17) this clearly implies that

$$N_n \geq \tilde{N}_n$$

which together with (18) gives

$$N_n = \tilde{N}_n$$

and completes the proof.

The following proposition will also be needed in the proof of Theorem 3.5.

PROPOSITION 3.4 **Given** $f_1, f_2, \ldots, f_n$**, a basis of** **it is possible to choose another basis** $g_1, g_2, \ldots, g_n$ **of** $M^\perp$ **such that,**

(i) $g_i = \sum_{j=1}^{i} \alpha_j f_j \quad , \alpha_j > 0$

(ii) $\tilde{N}_{g_1 \ldots g_i} = \bar{N}_{g_1 \ldots g_i}$

(iii) $\{t \in T : |g_i(t)| = \bar{N}_{g_1 \ldots g_i}\} = G^0_{f_1 \ldots f_i}$

for $i = 1, 2, \ldots n$ **.** $\bar{N}_{f_1 \ldots f_i}$'s **being given by (12)** **and** $G^0_{f_1 \ldots f_i}$'s **by (13).**

Proof. Let $f_1, f_2, \ldots f_n$ be a given basis of $M^\perp$. We will now choose inductively another set of n linearly independent elements $g_1 \ldots g_n$ of $M^\perp$ satisfying (i), (ii) and (iii) for $i = 1, 2, \ldots, n$.

To start with $g_1 = f_1$ obviously satisfies (i), (ii) and (iii) for $i = 1$. Also we have for $\epsilon > 0$

$$\bar{G}^\epsilon_{g_1} = G^\epsilon_{g_1} = G^{\epsilon +}_{g_1} \cup G^{\epsilon -}_{g_1} = G^\epsilon_{f_1} \tag{19}$$

Now we choose $\alpha_2 \geq 1$ such that

$$\alpha_2 \lim_{\epsilon \to 0} \text{ess} \cdot \inf f_2(t) \left[\chi_{G^{\epsilon +}_{g_1}} - \chi_{G^{\epsilon -}_{g_1}} \right] \geq 0 \tag{20}$$

and take $g_2 = f_1 + \alpha_2 f_2$. We note that

$$\lim_{\epsilon \to 0} \text{ess}\cdot\inf \; g_2(t) \left[\chi_{G_{g_1}^{\epsilon+}} - \chi_{G_{g_1}^{\epsilon-}} \right]$$

$$= \| g_1 \| + \alpha_2 \lim_{\epsilon \to 0} \text{ess}\cdot\inf f_2(t) \left[\chi_{G_{g_1}^{\epsilon+}} - \chi_{G_{g_1}^{\epsilon-}} \right]$$

$$> 0$$

by (20). Hence

$$\lim_{\epsilon \to 0} \text{ess}\cdot\sup \quad -g_2(t) \left[\chi_{G_{g_1}^{\epsilon+}} - \chi_{G_{g_1}^{\epsilon-}} \right] < 0 \qquad (21)$$

Also, it is easy to see that (20) implies

$$\lim_{\epsilon \to 0} \text{ess}\cdot\sup \; g_2(t) \left[\chi_{G_{g_1}^{\epsilon+}} - \chi_{G_{g_1}^{\epsilon-}} \right] > 0 \qquad (22)$$

Thus

$$\lim_{\epsilon \to 0} \underset{\bar{G}_{g_1}^{\epsilon}}{\text{ess}\cdot\sup} \; |g_2(t)|$$

$$= \lim_{\epsilon \to 0} \underset{G_{g_1}^{\epsilon}}{\text{ess}\cdot\sup} \; |g_2(t)| \qquad \text{by (19)}$$

$$= \max\left[\lim_{\epsilon\to 0} \text{ess.sup}\ g_2(t)\left\{\chi_{G_{g_1}^{\epsilon+}} - \chi_{G_{g_1}^{\epsilon-}}\right\},\right.$$

$$\left.\lim_{\epsilon\to 0} \text{ess.sup}\ -g_2(t)\left\{\chi_{G_{g_1}^{\epsilon+}} - \chi_{G_{g_1}^{\epsilon-}}\right\}\right]$$

$$= \lim_{\epsilon\to 0} \text{ess.sup}\ g_2(t)\left\{\chi_{G_{g_1}^{\epsilon+}} - \chi_{G_{g_1}^{\epsilon-}}\right\} \text{ (by (21) and (22))}$$

$$= \tilde{N}_{g_1 g_2}$$

Further from (19) we have

$$\tilde{N}_{g_1 g_2} = \lim_{\epsilon\to 0} \alpha_2\ \text{ess.sup}\ g_2(t)\left\{\chi_{G_{f_1}^{\epsilon+}} - \chi_{G_{f_1}^{\epsilon-}}\right\}$$

$$= \tilde{N}_{f_1} + \alpha_2 \tilde{N}_{f_1 f_2}$$

Since $\alpha_2 > 0$ this implies

$$\left\{t\in T : |g_2(t)| = \bar{N}_{g_1 g_2}\right\} = \left\{t\in T : |g_2(t)| = \tilde{N}_{g_1 g_2}\right\}$$

$$= G^{0+}_{f_1 f_2} \cup G^{0-}_{f_1 f_2} = G^{0}_{f_1 f_2}$$

and hence g_1 and g_2 satisfy (i), (ii) and (iii) for $i = 1, 2$. Further since

$$\tilde{N}_{g_1 g_2} = \tilde{N}_{f_1} + \alpha_2 \tilde{N}_{f_1 f_2}$$

holds, we have

$$G^{\epsilon+}_{g_1g_2} = \left\{ t \in G^{\epsilon+}_{g_1} : \tilde{N}_{g_1g_2} - \epsilon \le g_2(t) \le \tilde{N}_{g_1g_2} + \epsilon \right\}$$

$$= G^{\epsilon/2+}_{f_1f_2}$$

and similarly

$$G^{\epsilon-}_{g_1g_2} = G^{\epsilon/2-}_{f_1f_2}$$

Also, (21) clearly implies that for small enough $\epsilon > 0$,

$$g_2 > 0 \qquad \text{on } G^{\epsilon+}_{g_1}$$

$$g_2 < 0 \qquad \text{on } G^{\epsilon-}_{g_1}$$

and hence

$$\bar{G}^{\epsilon}_{g_1g_2} = G^{\epsilon}_{g_1g_2}$$

Thus we have obtained for sufficiently small $\epsilon > 0$

$$\bar{G}^{\epsilon}_{g_1g_2} = G^{\epsilon}_{g_1g_2} = G^{\epsilon/2}_{f_1f_2}$$

so that we can proceed as before to choose g_3 with g_1 g_2 and g_3 satisfying (i), (ii) and (iii) for Continuing thus we end up with n linearly independent functionals $g_1, \ldots, g_n$ of $M^{\perp}$ satisfying (i), (ii) and (iii) for $i = 1, 2, \ldots, n$.

THEOREM 3.5. *Let M be a closed subspace of codimension n in $L_1(T,\nu)$. Then the following conditions are equivalent:*

1. M *is proximinal*

2. $\nu(G^{0}_{f_1,\dots f_n}) > 0$ *for every basis* $f_1, f_2, \dots, f_n$ *of* $M^{\perp}$.

3. $\nu(\overline{G}^{0}_{f_1,\dots f_n}) > 0$ *for every basis* $f_1, f_2, \dots, f_n$ *of* $M^{\perp}$, *where* $G^{0}_{f_1,\dots f_n}$ *and* $\overline{G}^{0}_{f_1,\dots f_n}$ *are given by* (13) and (14) respectively.

Proof: 2 $\Longrightarrow$ 1. Let $f_1, f_2, \dots, f_n$ be a given basis of $M^{\perp}$. Assume that $\nu(G^{0}_{n}) > 0$. Then $x(t)$ given by

$$x(t) = \begin{cases} \dfrac{\chi_{G_n^{0+}} - \chi_{G_n^{0-}}}{\nu(G_n^{0})}, & t \in G_n^{0} \\[2ex] 0, & t \notin G_n^{0} \end{cases}$$

is in $L_1(T,\nu)$ and $\|x\| = 1$. Further

$$\begin{aligned} \int_T x f_j \, d\nu &= \int_{G_n^{0+}} x^{+} f_j \, d\nu + \int_{G_n^{0-}} x^{-} f_j \, d\nu \\ &= \frac{1}{\nu(G_n^{0})} \left\{ \tilde{N}_j \, \nu(G_n^{0+}) - \tilde{N}_j \left(-(\nu(G_n^{0-}))\right) \right\} \\ &= \tilde{N}_j \quad \text{for } j = 1, 2, \dots, n \end{aligned}$$

But $N_j = \tilde{N}_j$ for $j = 1, 2 \cdots n$ by Proposition 3.3

Hence we have $x \in \mathcal{M}_{f_1 \cdots f_n}$ and

$$\theta(x) = \mathcal{N}^0_{f_1 \cdots f_n}$$

This clearly implies any $y \in \mathcal{M}_{f_1 \cdots f_n}$ should satisfy

$$\int_T y f_j \, d\nu = \tilde{N}_j = N_j \quad \text{for} \quad j = 1, 2, \cdots, n$$

and thus we get

$$\theta(\mathcal{M}_{f_1 \cdots f_n}) = \mathcal{N}^0_{f_1 \cdots f_n}$$

Then by Theorem 1.2, M is proximinal in $L_1(T, \nu)$.

1 $\Longrightarrow$ 2. (Proof using the method of induction).

Let $f_1, \cdots, f_n$ be a given basis of $M^{\perp}$. Since M is proximinal condition (1) of Theorem 2.10 holds and so by Proposition 2.11, we have

$$\sup_{x \in \mathcal{M}_{f_1 \cdots f_{i-1}}} f_i(x) = \max_{\Phi \in \mathcal{N}_{f_1 \cdots f_{i-1}}} \Phi(f_i) = N_i \quad \text{for } i = 1, 2 \cdots n$$

But by Proposition 3.3

$$N_i = \tilde{N}_i \text{ for } i = 1, 2 \cdots n$$

Thus we have

$$\sup_{x \in \mathcal{M}_{f_1 \cdots f_{i-1}}} f_i(x) = \tilde{N}_i \quad \text{for} \quad i = 1, 2, \cdots, n \tag{23}$$

Further $\mathcal{M}_f$ is nonempty and so

$$\nu(G_1^0) > 0$$

Thus for $x \in \mathcal{M}_f$, if we define

$$\alpha_i = \int_{G_i^{0+}} x^+ f_1 \, d\nu \quad , \quad \beta_i = \int_{T \setminus G_i^{0+}} x^+ f_1 \, d\nu$$

$$\gamma_i = -\int_{G_i^{0-}} x^- f_1 \, d\nu \quad , \quad \delta_i = -\int_{T \setminus G_i^{0-}} x^- f_1 \, d\nu$$

then we have

$$\left.\begin{array}{l} \alpha_i \geq 0, \ \beta_i \geq 0, \ \gamma_i \geq 0, \ \delta_i \geq 0 \\ \text{and} \\ \alpha_i + \beta_i + \gamma_i + \delta_i \leq 1 \qquad (i = 1, 2 \ldots n) \end{array}\right\} \qquad (24)$$

Now

$$\int_{G_1^{0+}} x^+ f_1 \, d\nu + \int_{G_1^{0-}} x^- f_1 \, d\nu = \|f_1\| \alpha_1 - \|f_1\| (-\gamma_1)$$

$$= \|f_1\| \, (\alpha_1 + \gamma_1)$$

and

$$\int_{T \setminus G_1^{0+}} x^+ f_1 \, d\nu + \int_{T \setminus G_1^{0-}} x^- f_1 \, d\nu < \|f_1\| \beta_1 - \|f_1\| (-\delta_1)$$

$$= \|f_1\| \, (\beta_1 + \delta_1)$$

Hence

$$\|f_1\| = \int_T x f_1 \, d\nu$$

$$= \int_{G_1^{0+}} x^+ f_1 \, d\nu + \int_{G_1^{0-}} x^- f_1 \, d\nu + \int_{T \setminus G_1^{0+}} x^+ f_1 \, d\nu + \int_{T \setminus G_1^{0-}} x^- f_1 \, d\nu$$

$$\leq \|f_1\| (\alpha_1 + \beta_1 + \gamma_1 + \delta_1) \leq \|f_1\| \qquad \text{by (24)}$$

This implies that $\beta_1 + \delta_1 = 0$, which by (24) further gives $\beta_1 = \delta_1 = 0$. So we have for any $x \in \mathcal{M}_{f_1}$

$$\int_{T \setminus G_1^{0+}} x^+ f_1 \, d\nu = \int_{T \setminus G_1^{0-}} x^- f_1 \, d\nu = 0$$

We will now assume that

$$\nu(G_i^0) > 0 \text{ for } i = 1, 2, \cdots n-1$$

and that for every $x \in \mathcal{M}_{f_1 \cdots f_i}$, $(i = 1, 2 \cdots n-2)$

$$\int_{T \setminus G_i^{0+}} x^+ \, d\nu = \int_{T \setminus G_i^{0-}} x^- \, d\nu = 0$$

Consider any $x \in \mathcal{M}_{f_1 \cdots f_{n-1}}$. Since $\mathcal{M}_{f_1 \cdots f_{n-1}} \subset \mathcal{M}_{f_1 \cdots f_{n-2}}$, we have by the induction hypothesis,

$$\int_{T \setminus G_{n-2}^{0+}} x^+ \, d\nu = \int_{T \setminus G_{n-2}^{0-}} x^- \, d\nu = 0$$

Hence

$$\int_{G^{0+}_{n-2}\setminus G^{0+}_{n-1}} x^+ f_{n-1}\,d\nu + \int_{G^{0-}_{n-2}\setminus G^{0-}_{n-1}} x^- f_{n-1}\,d\nu = \int_{T\setminus G^{0+}_{n-1}} x^+ f_{n-1}\,d\nu + \int_{T\setminus G^{0-}_{n-1}} x^- f_{n-1}\,d\nu$$

$$< \tilde{N}_{n-1}\,\beta_{n-1} - \tilde{N}_{n-1}\,(-\delta_{n-1})$$

$$= \tilde{N}_{n-1}\,(\beta_{n-1} + \delta_{n-1})$$

Thus

$$\tilde{N}_{n-1} = \int_T x f_{n-1}\,d\nu$$

$$= \int_{G^{0+}_{n-2}} x^+ f_{n-1}\,d\nu + \int_{G^{0-}_{n-2}} x^- f_{n-1}\,d\nu$$

$$= \int_{G^{0+}_{n-1}} x^+ f_{n-1}\,d\nu + \int_{G^{0+}_{n-2}\setminus G^{0+}_{n-1}} x^+ f_{n-1}\,d\nu$$

$$+ \int_{G^{0-}_{n-1}} x^- f_{n-1}\,d\nu + \int_{G^{0-}_{n-2}\setminus G^{0-}_{n-1}} x^+ f_{n-1}\,d\nu$$

$$< \tilde{N}_{n-1}\,\alpha_{n-1} + \tilde{N}_{n-1}\,\beta_{n-1} - \tilde{N}_{n-1}\,(-\gamma_{n-1}) - \tilde{N}_{n-1}\,(-\delta_{n-1})$$

$$= \tilde{N}_{n-1}\,(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} + \delta_{n-1})$$

$$\leq \tilde{N}_{n-1} \quad \text{by (24)}.$$

So we get $\beta_{n-1} + \delta_{n-1} = 0$ which again by (24) gives $\beta_{n-1} = \delta_{n-1} = 0$. Hence for every $x \in \mathcal{M}_{f_1 \cdots f_{n-1}}$

$$\int_{T \setminus G_{n-1}^{0+}} x^+ d\nu = \int_{T \setminus G_{n-1}^{0-}} x^- d\nu = 0 \tag{25}$$

Now suppose that $\nu(G_n^0) = 0$. Then $\nu(G_n^{0+}) = \nu(G_n^{0-}) = 0$. Consider any $x \in \mathcal{M}_{f_1 \cdots f_{n-1}}$. Then we have by (25)

$$\int_T x f_n d\nu = \int_{G_{n-1}^{0+}} x^+ f_n d\nu + \int_{G_{n-1}^{0-}} x^- f_n d\nu$$

$$= \int_{G_{n-1}^{0+} \setminus G_n^{0+}} x^+ f_n d\nu + \int_{G_{n-1}^{0-} \setminus G_n^{0-}} x^- f_n d\nu$$

$$< \tilde{N}_n \beta_n - \tilde{N}_n (-\delta_n)$$

$$= \tilde{N}_n (\beta_n + \delta_n)$$

$$\leq \tilde{N}_n$$

Thus for every $x \in \mathcal{M}_{f_1 \cdots f_{n-1}}$. $f_n(x) < \tilde{N}_n$ which contradicts (23). Hence $\nu(G_n^0) > 0$ which completes the proof for $1 \Longleftrightarrow 2$.

$2 \Longrightarrow 3.$ Suppose that $\mathcal{V}(G_n^\circ) > 0$ for all bases of $M^\perp$. We shall show that $\mathcal{V}(\bar{G}_n^\circ) > 0$ for all bases of $M^\perp$.

Let $f_1, f_2, \ldots, f_n$ be a given basis of $M^\perp$. We will now prove by the method of induction that there exists another basis $k_1, \ldots, k_n$ of $M^\perp$ satisfying

$$G^\circ_{k_1 \ldots k_n} \subset \bar{G}^\circ_{f_1 \ldots f_n}$$

Since $\mathcal{V}(G^\circ_{k_1 \ldots k_n}) > 0$ by assumption, this would imply $\mathcal{V}(\bar{G}^\circ_{f_1 \ldots f_n}) > 0$ as desired.

To this end, we first show that we can select a basis $k_1 \ldots, k_n$ of $M^\perp$ such that

$$k_i \quad \text{is either } f_i \text{ or } -f_i$$

and

$$\tilde{N}_{k_1 \ldots k_i} = \bar{N}_{k_1 \ldots k_i} \quad \text{for } i = 1, 2, \ldots n$$

To begin with we have

$$\tilde{N}_{f_1} = \bar{N}_{f_1} = \| f_1 \|$$

and we take $k_1 = f_1$.

We will now assume that we can choose $n-1$ linearly independent functionals $k_1, \cdots k_{n-1}$ of $M^\perp$ satisfying

$$k_i \qquad \text{is either } f_i \text{ or } -f_i \qquad (26)$$

and

$$\tilde{N}_{k_1 \cdots k_i} = \bar{N}_{k_1 \cdots k_i} \quad (i = 1, 2, \cdots, n-1) \qquad (27)$$

Let A be the collection of all such $(n-1)$ tuples of linearly independent functionals. A is nonempty since $k_1, k_2, \cdots, k_{n-1}$ is in A and further by (26) we have A to be a finite collection. Also, for any $g_1 \cdots g_{n-1}$ in A, by (26) and (27) we have

$$\bar{N}_{f_1 \cdots f_{n-1}} = \bar{N}_{k_1 \cdots k_{n-1}} = \tilde{N}_{k_1, \cdots k_{n-1}} = \tilde{N}_{g_1 \cdots g_{n-1}} \qquad (28)$$

and so for every $\epsilon > 0$

$$\bar{G}^{\epsilon}_{f_1 \cdots f_{n-1}} = \left\{ t \in T : \bar{N}_{f_1 \cdots f_i} - \epsilon \le |f_i(t)| \le \bar{N}_{f_1 \cdots f_i} + \epsilon, i = 1, 2 \cdots n-1 \right\}$$

$$= \left\{ t \in T : \tilde{N}_{k_1 \cdots k_i} - \epsilon \le |f_i(t)| \le \tilde{N}_{k_1 \cdots k_i} + \epsilon, i = 1, 2 \cdots n-1 \right\}$$

$$= \bigcup_{A} G^{\epsilon}_{g_1 \cdots g_{n-1}} \text{ by (26) and (28)}$$

Hence

$$\bar{N}_{f_1 \cdots f_n} = \lim_{\epsilon \to 0} \operatorname*{ess \cdot sup}_{\bar{G}^{\epsilon}_{f_1 \cdots f_{n-1}}} |f_n(t)|$$

$$= \lim_{\epsilon \to 0} \operatorname*{ess \cdot sup}_{\bigcup_A G^{\epsilon}_{g_1 \cdots g_{n-1}}} |f_n(t)|$$

$$= \max_{A}\left[\max\left\{\lim_{\varepsilon\to 0}\ \text{ess}\cdot\sup f_n(t)\left(\chi_{G^{\varepsilon+}_{g_1\cdots g_{n-1}}} - \chi_{G^{\varepsilon-}_{g_1\cdots g_{n-1}}}\right),\right.\right.$$

$$\left.\left.\lim_{\varepsilon\to 0}\ \text{ess}\cdot\sup -f_n(t)\left(\chi_{G^{\varepsilon+}_{g_1\cdots g_{n-1}}} - \chi_{G^{\varepsilon-}_{g_1\cdots g_{n-1}}}\right)\right\}\right]$$

Since A is a finite collection, this maximum will be attained for some member of A. Thus there exists a member $h_1, h_2, \cdots, h_{n-1}$ of A satisfying

$$\overline{N}_{f_1\cdots f_n} = \lim_{\varepsilon\to 0}\ \text{ess}\cdot\sup h_n(t)\left\{\chi_{G^{\varepsilon+}_{h_1\cdots h_{n-1}}} - \chi_{G^{\varepsilon-}_{h_1\cdots h_{n-1}}}\right\}$$

$$= \widetilde{N}_{h_1\cdots h_n}$$

where h_n is either f_n or $-f_n$. Thus for the basis $h_1, h_2, \cdots, h_n$ of $M^\perp$, for $i = 1, 2, \cdots, n$

$$h_i \text{ is either } f_i \text{ or } -f_i \tag{29}$$

and

$$\widetilde{N}_{h_1\cdots h_i} = \overline{N}_{h_1\cdots h_i} = \overline{N}_{f_1\cdots f_i} \tag{30}$$

Hence

$$G^{o}_{h_1\cdots h_n} = \left\{t\in T:\ h_i(t) = \widetilde{N}_{h_1\cdots h_i},\ i = 1, 2, \cdots n\right\}$$

$$= \left\{t\in T:\ h_i(t) = \overline{N}_{f_1\cdots f_i},\ i = 1, 2, \cdots n\right\} \text{ by (30)}$$

$$\subset \left\{t\in T:\ |f_i(t)| = \overline{N}_{f_1\cdots f_i},\ i = 1, 2 \cdots n\right\} \text{ by (29)}$$

$$= \overline{G}^{o}_{f_1\cdots f_n}$$

which completes the proof for the implication $2 \Longrightarrow 3$.

$3 \Longrightarrow 2$. We will suppose that $\nu(\bar{G}_n^0) > 0$ for all bases of $M^\perp$ and thus show that $\nu(G_n^0) > 0$ for all bases of $M^\perp$.

To this end we consider an arbitrary basis $f_1, \cdots f_n$ of $M^\perp$ and show that there exists another basis $h_1 \cdots h_n$ of $M^\perp$ such that

$$\bar{G}^0_{h_1 \cdots h_n} \subset G^0_{f_1 \cdots f_n}$$

Since by assumption $\nu(\bar{G}^0_{h_1 \cdots h_n}) > 0$, this would imply $\nu(G^0_{f_1 \cdots f_n}) > 0$ and thus complete the proof.

By Proposition 3.4 we can get another basis $h_1 \cdots h_n$ of $M^\perp$ such that

$$\tilde{N}_{h_1 \cdots h_n} = \bar{N}_{h_1 \cdots h_n} \tag{31}$$

and

$$\left\{ t \in T : |h_n(t)| = \tilde{N}_{h_1 \cdots h_n} \right\} = G^0_{f_1 \cdots f_n} \tag{32}$$

It is clear from (31) and (32) that

$$\bar{G}^0_{h_1 \cdots h_n} \subset \left\{ t \in T : |h_n(t)| = \bar{N}_{h_1 \cdots h_n} \right\} = G^0_{f_1 \cdots f_n}$$

So our claim is proved and $3 \Longrightarrow 2$.

Hence $1 \Longleftrightarrow 2 \Longleftrightarrow 3$ and the proof of the theorem is completed.

Remark 3.6. The equivalence $1 \to 3$ has been given by Garkavi in [7] .

Remark 3.7. We observe that a result similar to Proposition 3.3, evaluating the M_i's can also be proved. If we denote by

$$\tilde{M}_{f_1} = \operatorname*{ess.\,sup}_{T\ \nu} |f_1(t)| = \|f_1\|$$

$$\tilde{M}_{f_1 \cdots f_i} = \operatorname{ess.\,sup} f_i(t) \left\{ \chi_{\tilde{G}^{0+}_{f_1 \cdots f_{i-1}}} - \chi_{\tilde{G}^{0-}_{f_1 \cdots f_{i-1}}} \right\}$$

where

$$\tilde{G}^{0+}_{f_1 \cdots f_{i-1}} = \left\{ t \in T : f_j(t) = \tilde{M}_{f_1 \cdots f_j},\ j = 1, 2 \cdots i \right\}$$

$$\tilde{G}^{0-}_{f_1 \cdots f_{i-1}} = \left\{ t \in T : f_j(t) = -\tilde{M}_{f_1 \cdots f_j},\ j = 1, 2 \cdots i-1 \right\}$$

then we will have

$$M_{f_1 \cdots f_i} = \tilde{M}_{f_1 \cdots f_i} \quad \text{for} \quad i = 1, 2 \cdots n$$

Proof follows in similar lines as given in the equivalence $1 \Longleftrightarrow 2$ of Theorem 3.5.

We will now conclude this chapter by stating the following theorem which asserts the density of proximinal subspaces of finite codimension n in the class of all subspaces of finite codimension n in the space $L_1(T,\nu)$.

THEOREM 3.8. Let $1 \leq n < \infty$. Then given n linearly independent elements $f_1, f_2, \ldots, f_n \in L_\infty(T,\nu)$ and $\epsilon > 0$, we can select $g_1 \ldots g_n$, n simple functions on (T,ν) such that

(i) $\|f_i - g_i\| < \epsilon$ for $i = 1, 2 \ldots n$

(ii) $\bigcap_{i=1}^{n} g_i^{-1}(0)$ is proximinal in $L_1(T,\nu)$.

Proof: We first observe that given $f \in L_\infty(T,\nu)$ and $\epsilon > 0$ there exists a simple function $g \in L_\infty(T,\nu)$ with $\|f - g\| < \epsilon$. Hence given $f_1, \ldots, f_n$ and $\epsilon > 0$ we can select n simple functions $g_1 \ldots g_n \in L_\infty(T,\nu)$ such that

$$\|f_i - g_i\| < \epsilon \quad \text{for } i = 1, 2 \ldots n$$

Since each g_i takes only finite set of values on T, it is easy to see that $M^\perp = [g_1, \ldots, g_n]$ satisfies condition 2 of Theorem 3.5, and thus $\bigcap_{i=1}^{n} g_i^{-1}(0)$ is proximinal in $L_1(T,\nu)$. Thus both (i) and (ii) are satisfied.

CHAPTER III.

The Space C(Q)

In this chapter we consider the space, C(Q), of all real valued continuous functions on the compact Hausdroff space Q with sup norm and derive the characterisation theorem of Garkavi given below:

THEOREM 4.1 Garkavi. [3] Let M be a closed subspace of finite codimension n in C(Q) and

$$M^{\perp} = [\mu_1, \mu_2, \ldots, \mu_n] \quad , \mu_1, \mu_2, \ldots, \mu_n \in C(Q)^*$$

Also for any $\mu \in C(Q)^*$, let $\mu = \mu^+ + \mu^-$ denote its Jordan decomposition and $S(\mu)$ its support.

(α) $S(\mu^+) \cap S(\mu^-) = \phi$ for every $\mu \in M^{\perp} \setminus \{0\}$

(β) μ_2 is absolutely continuous with respect to μ_1 on $S(\mu_1)$ for every $\mu_1, \mu_2 \in M^{\perp} \setminus \{0\}$.

(γ) $S(\mu_2) \setminus S(\mu_1)$ is closed for each $\mu_1, \mu_2 \in M^{\perp} \setminus \{0\}$.

For this purpose, we consider the following 3 conditions which are contained in the 2n-1 conditions given by (8) and (9) and exhibit their equivalence to (α), (β) and (γ) in a series of propositions.

For $A \subset Q$, we denote by $\bar{A}$, the closure of the set A in Q and A^c its complement in Q.

Let M be a closed subspace of finite codimension n in C(Q) and $M^{\perp}$ its annihilator. Then

PROPOSITION 4.2: (I) $\Longleftrightarrow$ (α)

PROPOSITION 4.3: **If** (I) **or** (α) **is satisfied, then**

(II) $\Longleftrightarrow$ (β)

PROPOSITION 4.4: **If** (I) **or** (α) **is satisfied, then**

(III) $\Longleftrightarrow$ (γ)

Proof of Proposition 4.2. $\mathcal{M}_\mu$ is nonempty if and only if there exists $f \in C(\Omega)$ such that

$$f = \begin{cases} 1 & \text{on} \quad S(\mu^+) \\ -1 & \text{on} \quad S(\mu^-) \end{cases}$$

This is possible if and only if

$$S(\mu^+) \cap S(\mu^-) = \phi$$

Proof of Proposition 4.3. First we show that (II) $\Longrightarrow$ (β). Assume that (II) holds. We need to show that

$$A \subset S(\mu_1), \quad |\mu_1|(A) = 0 \qquad \text{implies} \quad \mu_2(A) = 0$$

Suppose not. Then there exists $A \subset S(\mu_1)$ such that $|\mu_1|(A) = 0$ but $\mu_2(A) \neq 0$. We can assume without loss of generality that $A \subset S(\mu_1^-) \cap S(\mu_2^+)$ so that $\mu_2(A) > 0$. Setting

$$\nu = \sum_{i=1}^{n} |\nu_i|$$

for some fixed basis $\nu_1, \nu_2, \ldots, \nu_n$ of $M^\perp$, we see that to each $\Phi \in (M^\perp)^*$, there exists a unique $\alpha \in L_\infty(\Omega, \nu)$

such that

$$\Phi(\mu) = \int_Q \alpha(q)\,d\mu(q) \qquad \text{for all } \mu \in M^\perp$$

and

$$\|\Phi\| = \operatorname*{ess\cdot sup}_{Q \quad \nu} |\alpha(q)|$$

If in addition $\Phi \in \mathcal{N}_{\mu_1}$, we also have

$$\alpha = \begin{cases} 1 & \mu_1 \text{ a.e on } S(\mu_1^+) \\ -1 & \mu_1 \text{ a.e on } S(\mu_1^-) \end{cases} \tag{33}$$

Let B be the union of all the sets F such that $\mu_1(F) = 0$
Then $A \subset B \cap S(\mu_2^+)$. Further we set

$$C_1 = \{S(\mu_2^-) \setminus S(\mu_1)\} \cup \{B \cap S(\mu_2^-)\}$$

$$C_2 = \{S(\mu_2^+) \setminus S(\mu_1)\} \cup \{B \cap S(\mu_2^+)\}$$

Then

$$C_1 \cup C_2 = \{S(\mu_2) \setminus S(\mu_1)\} \cup \{B \cap S(\mu_2)\}$$

Now we define $\alpha_0 \in L_\infty(Q,\nu)$ as follows:

$$\alpha_0 = \begin{cases} 1 & \text{on } \{S(\mu_1^+) \setminus B\} \cup C_2 \\ -1 & \text{on } \{S(\mu_1^-) \setminus B\} \cup C_1 \end{cases} \tag{34}$$

Now consider any $\alpha \in L_\infty(\Theta_2, \nu)$ representing a $\Phi \in \mathcal{N}_{\mu_1}$. We have

$$\operatorname*{ess\cdot sup}_{\Theta_2 \ \mu_2} |\alpha(q)| \leq \operatorname*{ess\cdot sup}_{\Theta_2 \ \nu} |\alpha(q)| = \|\Phi\| = 1$$

Then

$$\begin{aligned}\int_{S(\mu_2)} (\alpha_0 - \alpha)(q)\, d\mu_2(q) &= \int_{S(\mu_1)\setminus B} (\alpha_0 - \alpha)(q)\, d\mu_2(q) + \int_{C_1} (\alpha_0 - \alpha)(q)\, d\mu_2(q) \\ &\quad + \int_{C_2} (\alpha_0 - \alpha)(q)\, d\mu_2(q) \\ &\geq 0\end{aligned}$$

for the first term on the R.H.S. is zero and the remaining terms are nonnegative since by (33) and (34)

$$\begin{aligned}\alpha_0 - \alpha &\leq 0 \quad \text{on} \quad C_1 \subset S(\mu_2^-) \\ \alpha_0 - \alpha &\geq 0 \quad \text{on} \quad C_2 \subset S(\mu_2^+)\end{aligned}$$

Thus if Φ_0 is the functional represented by α_0, we have for any $\Phi \in \mathcal{N}_{\mu_1}$,

$$\Phi_0(\mu_2) - \Phi(\mu_2) = \int_{S(\mu_2)} (\alpha_0 - \alpha)(q)\, d\mu_2(q) \geq 0$$

and so

$$\Phi_0(\mu_2) = \sup_{\Phi \in \mathcal{N}_{\mu_1}} \Phi(\mu_2) \tag{35}$$

Choose ϵ such that $0 < \epsilon < \mu_2(A)$. Since (II) holds there exists $f_0 \in \mathcal{F}_{\mu_1}$ such that

$$\mu_2(f_0) > \Phi_0(\mu_2) - \epsilon \tag{36}$$

Since $f_0 \in \mathcal{F}_{\mu_1}$

$$f_0 = \begin{cases} 1 & \text{on } S(\mu_1^+) \\ -1 & \text{on } S(\mu_1^-) \end{cases} \tag{37}$$

and

$$|f_0(q)| \leq 1 \quad \text{on } \Theta$$

Since $A \subset S(\mu_1^-)$, $\mu_1(A)$ is also equal to zero. Thus

$$\alpha_0 = 1 \text{ on } A \subset S(\mu_2^+) \tag{38}$$

Further we note that

$$f_0 = \alpha_0 \text{ on } S(\mu_1) \setminus B \tag{39}$$

Now

$$\Phi_0(\mu_2) - \mu_2(f_0) = \int_{S(\mu_2)} (\alpha_0 - f_0)(q)\, d\mu_2(q)$$

$$= \int_A (\alpha_0 - f_0)(q)\, d\mu_2(q) + \int_{S(\mu_2)\cap(S(\mu_1)\setminus B)} (\alpha_0 - f_0)(q)\, d\mu_2(q)$$

$$+ \int_{G_1\setminus A} (\alpha_0 - f_0)(q)\, d\mu_2(q) + \int_{G_2\setminus A} (\alpha_0 - f_0)(q)\, d\mu_2(q)$$

The first term on the R.H.S. is easily seen to be equal to $2\mu_2(A)$ from (37) and (39). The second term is zero by (38). Using (34) and (36) we note that

$$\alpha_0 - f_0 \geq 0 \quad \text{on} \quad C_2 \subset S(\mu_2^+)$$

$$\alpha_0 - f_0 \leq 0 \quad \text{on} \quad C_1 \subset S(\mu_2^-)$$

and thus last two terms are nonnegative. Hence

$$\Phi_0(\mu_2) - \mu_2(f_0) \geq 2\mu_2(A) > 2\epsilon$$

which contradicts (35). This proves (II).

We will now prove $(\beta) \Longrightarrow (\mathrm{II})$. Assume that (β) holds, so that $A \subset S(\mu_1)$, $|\mu_1|(A) = 0$ would imply $\mu_2(A) = 0$. Define $\Phi_0 \in \mathcal{N}_{\mu_1}$ by

$$\Phi_0(\mu) = \int_{\Theta} \alpha_0(q)\, d\mu(q) \qquad , \quad \mu \in M^{\perp}$$

where $\alpha_0 \in L_\infty(\Theta, \mathcal{D})$ is given by

$$\alpha_0 = \begin{cases} 1 & \text{on } \{S(\mu_1^+)\} \cup \{S(\mu_2^+) \setminus S(\mu_1)\} \\ -1 & \text{on } \{S(\mu_1^-)\} \cup \{S(\mu_2^-) \setminus S(\mu_1)\} \end{cases}$$

Then for any $\alpha \in L_\infty(\Theta, \mathcal{D})$ representing a $\Phi \in \mathcal{N}_{\mu_1}$

$$\alpha = \alpha_0 \quad \mu_1 \text{ a.e on } S(\mu_1)$$

and hence, since and are satisfied,

$$\alpha = \alpha_0 \quad \mu_2 \text{ a.e on } S(\mu_1)$$

Also

$$\int_{S(\mu_2^+)\setminus S(\mu_1)} (\alpha-\alpha_0)(q)\,d\mu_2(q) + \int_{S(\mu_2^-)\setminus S(\mu_1)} (\alpha-\alpha_0)(q)\,d\mu_2(q) \le 0$$

and hence

$$\int_{S(\mu_2)} (\alpha-\alpha_0)(q)\,d\mu_2(q) \le 0$$

So we have

$$\Phi_0(\mu_2) = \sup_{\Phi\in\mathcal{N}_{\mu_1}} \Phi(\mu_2)$$

Now since $\mathcal{M}_{\mu_1} \subset \mathcal{N}_{\mu_1}$

$$\sup_{f\in\mathcal{M}_{\mu_1}} \mu_2(f) \le \max_{\Phi\in\mathcal{N}_{\mu_1}} \Phi(\mu_2) \tag{40}$$

Let U be an open set containing $S(\mu_1)$. Because (α) is satisfied, using Urysohn's lemma, we can get an $f\in C(Q)$ such that

$$f(x) = \begin{cases} 1 & \text{for } x\in S(\mu_1^+) \cup (S(\mu_2^+)\setminus U) \\ -1 & \text{for } x\in S(\mu_1^-) \cup (S(\mu_2^-)\setminus U) \end{cases}$$

and

$$\|f\| = 1$$

Then $f \in \mathcal{T}_{\mu_1}$. Also

$$|\Phi_0(\mu_2) - \mu_2(f)| = 2\mu_2(U \setminus S(\mu_1))$$

By the regularity of μ_2, the R.H.S. can be made as small as we please and hence

$$\sup_{f \in \mathcal{T}_{\mu_1}} \mu_2(f) \geq \Phi_0(\mu_2)$$

This together with (40) implies (II).

<u>Proof of Proposition 4.4.</u> We will start with the implication $(\gamma) \longrightarrow (\mathrm{III})$. By assumption $S(\mu_2) \setminus S(\mu_1)$ is closed for every pair $\mu_1, \mu_2 \in M^{\perp} \setminus \{0\}$.
Since (α) is satisfied we can infer from our assumption that both the sets $S(\mu_2^+) \setminus S(\mu_1)$ and $S(\mu_2^-) \setminus S(\mu_1)$ are also closed for all $\mu_1, \mu_2, \in M^{\perp} \setminus \{0\}$.
Hence f_0 defined by

$$f_0 = \begin{cases} 1 & \text{on} \quad (S(\mu_1^+)) \cup (S(\mu_2^+) \setminus S(\mu_1)) \\ -1 & \text{on} \quad (S(\mu_1^-)) \cup (S(\mu_2^-) \setminus S(\mu_1^-)) \end{cases}$$

has continuous extension to the whole of Ω with $\|f_0\| = 1$ by Urysohn's lemma. Also $f_0 \in \mathcal{T}_{\mu_1}$ and further

$$\mu_2(f_0) = \sup_{f \in \mathcal{T}_{\mu_1}} \mu_2(f)$$

which gives (III).

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Now we shall prove the converse. We show that if (α) and (3) hold then

$$\overline{S(\mu_2^-)\setminus S(\mu_1)} \cap S(\mu_1^+) = \phi \tag{41}$$

and assert that (41) implies (γ) . To see this, we change μ_2 to $-\mu_2$ in (41) to get

$$\overline{S(\mu_2^+)\setminus S(\mu_1)} \cap S(\mu_1^+) = \phi \tag{42}$$

Again changing μ_1 to $-\mu_1$ in both (41) and (42) we obtain

$$\overline{S(\mu_2^-)\setminus S(\mu_1)} \cap S(\mu_1^-) = \phi \tag{43}$$

and

$$\overline{S(\mu_2^+)\setminus S(\mu_1)} \cap S(\mu_1^-) = \phi \tag{44}$$

It is easy to observe that (41) and (43) imply $S(\mu_2^-)\setminus S(\mu_1)$ is closed and (42) and (44) imply $S(\mu_2^+)\setminus S(\mu_1)$ is closed. Thus $S(\mu_2)\setminus S(\mu_1)$ is closed and our assertion is proved.

Hence to complete the proof it only remains to show that (41) holds. Assume that (α) and (III) are satisfied. Then there exists $f_0 \in \mathcal{T}_{\mu_1}$ such that

$$\mu_2(f_0) = \sup_{f\in \mathcal{T}_{\mu_1}} \mu_2(f)$$

Now if (41) does not hold, there exists a p_0 in $\overline{S(\mu_2^-)\setminus S(\mu_1)} \cap S(\mu_1^+)$. Since $f_0 \in \mathcal{M}_{\mu_1}$ and $p_0 \in S(\mu_1^+)$ we have

$$f_0(p_0) = 1$$

Let $U_1 = f_0^{-1}((0,\infty))$. Then

$$U_1 \supset S(\mu_1^+)\ ,\ U_1 \cap S(\mu_1^-) = \phi \quad \text{and} \quad f_0(U_1) > 0$$

Set for any $U \supset S(\mu_1^+)$

$$U' = U \cap (S(\mu_2^-) \setminus S(\mu_1))$$

U' is nonempty for any open set $U \supset S(\mu_1^+)$ since U contains $p_0 \in S(\mu_1^+)$, a limit point of $S(\mu_2^-)\setminus S(\mu_1)$. Also

$$\mu_2(U') = \mu_2\Big[\{U \cap S(\mu_1)\}^c \cap \{S(\mu_2^-)\}\Big] < 0$$

because $U \cap S(\mu_1)^c$ is an nonempty open set and $S(\mu_2^-)$ is the support of the measure μ_2^-. Using the regularity of μ_2 we can select another open set U_2 such that

$$U_1 \supset U_2 \supset S(\mu_1^+)$$

and

$$|\mu_2(U_1' \setminus U_2')| > 2\,|\mu_2(U_2' \setminus S(\mu_1^+))|$$

Now define

$$f = \begin{cases} f_0 & \text{on } \; S(\mu_1) \cup S(\mu_2^+) \\ -1 & \text{on } \; U_2^c \cap \{S(\mu_2^-) \setminus S(\mu_1)\} \end{cases} \tag{46}$$

By Tietze extension theorem f can be extended to the whole of Q continuously with norm 1 and it is easy to prove that $f \in \mathfrak{M}_{\mu_1}$. Consider

$$\mu_2(f_0) - \mu_2(f) = \int_{S(\mu_2)} (f_0 - f)(q)\, d\mu_2(q)$$

$$= \int_{S(\mu_2^-) \setminus S(\mu_1)} (f_0 - f)(q)\, d\mu_2(q) \qquad \text{by } (46)$$

Again

$$\int_{S(\mu_2^-) \setminus S(\mu_1)} (f_0 - f)(q)\, d\mu_2(q) = \int_{U_2' \setminus S(\mu^+)} (f_0 - f)(q)\, d\mu_2(q)$$

$$+ \int_{U_1' \setminus U_2'} (f_0 - f)(q)\, d\mu_2(q)$$

$$+ \int_{U_1^c \cap (S(\mu_2^-) \setminus S(\mu_1))} (f_0 - f)(q)\, d\mu_2(q)$$

We note that $f_0 - f \geq 0$ on $\{U_1^c \cap (S(\mu_2^-) \setminus S(\mu_1))\} \subset S(\mu_2^-)$ and thus the third term is nonpositive. Also, since $|f_0 - f| \leq 2$ on Q the first term is less than or equal to $2|\mu_2(U_2' \setminus S(\mu_1^+))|$. Further $f_0 - f \geq 1$ on $U_1' \setminus U_2' \subset S(\mu_1^-)$ and so the second term is less than or equal to $\mu_2(U_1' \setminus U_2')$ which is strictly less than zero. Thus

$$\int_{S(\mu_2^-) \setminus S(\mu_1)} (f_0 - f)(q)\, d\mu_2(q) \leq 2|\mu_2(U_2' \setminus S(\mu_1^+))| + \mu_2(U_1' \setminus U_2')$$

$$< 0 \qquad \text{by (45)}$$

This gives a contradiction to the fact that

$\mu_2(f_0) = \sup_{f \in \mathcal{M}_{\mu_1}} \mu_2(f)$ and hence completes the proof.

We will now show that the $2n-1$ conditions given by (8) and (9) reduce to the 3 conditions (I), (II) and (III) (or equivalently (α), (β) and (γ) by the above propositions) in the space C(Q).

<u>PROPOSITION 4.5</u>: (α), (β) <u>and</u> (γ) $\Longleftrightarrow$ (8) and (9) <u>in the space</u> C(Q).

Proof: Clearly (I), (II) and (III) are contained in the conditions given by (8) and (9).

But (I), (II) and (III) together are equivalent to (α), (β) and (γ) by propositions 4.2, 4.3 and 4.4. Hence (8) and (9) imply (α), (β) and (γ).

To prove the converse, we will assume that $M^{\perp}$ satisfies (α), (β) and (γ).

Since (γ) is satisfied, we have $S(\mu_2)\setminus S(\mu_1)$ is closed for each $\mu_1, \mu_2 \in M^{\perp}\setminus\{0\}$. Further (α) is satisfied and this implies $S(\mu_2^{+})\setminus S(\mu_1)$ and $S(\mu_2^{-})\setminus S(\mu_1)$ are also closed for each $\mu_1, \mu_2 \in M^{\perp}\setminus\{0\}$. Hence

$$S(\mu_i^{+})\setminus \bigcup_{j=1}^{i-1} S(\mu_j) = \bigcap_{j=1}^{i-1}\{S(\mu_i^{+})\setminus S(\mu_j)\}$$

and

$$S(\mu_i^{-})\setminus \bigcup_{j=1}^{i-1} S(\mu_j) = \bigcap_{j=1}^{i-1}\{S(\mu_i^{-})\setminus S(\mu_j)\}$$

are closed sets for $2\le i\le n$.

Define f by

$$f = \begin{cases} 1 & \text{on } S(\mu_k^{+})\setminus \bigcup_{j=1}^{k-1} S(\mu_j) \quad 1\le k\le i \\ -1 & \text{on } S(\mu_k^{-})\setminus \bigcup_{j=1}^{k-1} S(\mu_j) \quad 1\le k\le i \end{cases}$$

Then f has continuous extension to the whole of Q with $\|f\|=1$ by Urysohn's lemma and it is easy to observe that $f\in \mathcal{M}_{\mu_1\cdots\mu_i}$. Thus (α) and (γ) together imply

$\mathcal{M}_{\mu_1\cdots\mu_i}$ is nonempty for $1\le i\le n$

and this gives all the conditions given by (8)

Now we shall show that (9) is also satisfied.

Let $\Phi \in (M^{\perp})^{*}$ be in $\mathcal{N}_{\mu_1 \cdots \mu_i}$ $(1 \le i \le n)$ and

$\alpha \in L_\infty(Q, \mathcal{D})$ represent the functional Φ

Then

$$\alpha = \begin{cases} 1 & \mu_k \text{ a.e on } S(\mu_k^+) \setminus \bigcup_{j=1}^{k-1} S(\mu_j) \text{ for } 1 \le k \le i \\ -1 & \mu_k \text{ a.e on } S(\mu_k^-) \setminus \bigcup_{j=1}^{k-1} S(\mu_j) \text{ for } 1 \le k \le i \end{cases}$$

We note that any μ_k null set contained in $S(\mu_k^+)$ or $S(\mu_k^-)$ is also a $|\mu_k|$ null set. Since (β) is satisfied it is also a μ_i null set for all i, $1 \le i \le n$. Thus

$$\alpha = \begin{cases} 1 & \mu_i \text{ a.e } (1 \le i \le n) \text{ on } S(\mu_k^+) \setminus \bigcup_{j=1}^{k-1} S(\mu_j) \text{ for } 1 \le k \le i \\ -1 & \mu_i \text{ a.e } (1 \le i \le n) \text{ on } S(\mu_k^-) \setminus \bigcup_{j=1}^{k-1} S(\mu_j) \text{ for } 1 \le k \le i \end{cases}$$

and

$$\alpha = f \quad \mu_i \text{ a.e } (1 \le i \le n) \text{ on } \bigcup_{j=1}^{n} S(\mu_j)$$

So

$$\int_Q \alpha(q)\, d\mu_j(q) = \int_Q f(q)\, d\mu_j(q) \quad \text{for } 1 \le j \le i$$

which gives

$$\Phi(\mu_j) = \mu_j(f) \quad \text{for } 1 \le j \le i .$$

Since $\Phi \in \mathcal{N}_{\mu_1 \cdots \mu_i}$, this implies

$$\max_{\Phi \in \mathcal{N}_{\mu_1 \cdots \mu_{i-1}}} \Phi(\mu_i) = \sup_{f \in \mathcal{M}_{\mu_1 \cdots \mu_{i-1}}} \mu_i(f) \quad \text{for } 2 \le i \le n$$

and hence all the conditions given by (9) are also satisfied. Thus (α), (β) and (γ) together imply (8) and (9) .

THEOREM 4.5: <u>Let M be a closed subspaces of finite codimension n in C(Q) and $M^{\perp}$ its annihilator. Then</u>

$$(1) \Longleftrightarrow (\alpha), (\beta), \text{ and } (\gamma)$$

<u>Proof</u>: By Proposition 4.5

$$(\alpha), (\beta) \text{ and } (\gamma) \Longleftrightarrow (8) \text{ and } (9)$$

But by Proposition 2.11

$$(1) \Longleftrightarrow (8) \text{ and } (9)$$

Hence $(1) \Longleftrightarrow (\alpha), (\beta)$ and (γ) .

<u>Remark 4.6</u>. It is clear from Theorem 4.1 and the equivalence $1 \Longleftrightarrow 2$ of Theorem 3.5 that the proximinal characterisation of the subspaces M of finite codimension in $L_1(T,\nu)$ differs in nature from the corresponding characterisation of such subspaces in C(Q).

But if for the subspace $M^\perp$, G_f° consists of finite union of atoms (See p. 79) for each $f \in M^\perp \setminus \{0\}$, then we have $\mathcal{M}_f$ to be finite dimensional for every $f \in M^\perp \setminus \{0\}$.

Hence the equivalence $1 \Longleftrightarrow 5$ of Theorem 2.14 holds and thus we can conclude that

$$\nu(G^\circ_{f_1 f_2}) > 0 \quad \text{for every pair} \quad f_1, f_2 \in M^\perp \setminus \{0\}$$

$$\Longleftrightarrow \nu(G^\circ_{f_1 \dots f_n}) > 0 \ \text{for every basis} \ f_1, f_2, \dots, f_n \ \text{of} \ M^\perp.$$

For using Proposition 3.3 it can easily be seen that $\nu(G^\circ_{f_1 f_2}) > 0 \ (f_1, f_2 \in M^\perp \setminus \{0\})$ is equivalent to

(I) $\mathcal{M}_f$ is nonempty for each $f \in M^\perp \setminus \{0\}$

(II) $\sup\limits_{x \in \mathcal{M}_{f_1}} f_2(x) = \max\limits_{\Phi \in \mathcal{N}_{f_1}} \Phi(f_2)$ for each $f_1, f_2 \in M^\perp \setminus \{0\}$.

and

(III) $\sup\limits_{x \in \mathcal{M}_{f_1}} f_2(x)$ is attained for every $f_1, f_2 \in M^\perp \setminus \{0\}$.

Since (I), (II) and (III) characterise proximinal subspaces of finite codimension in the space $C(Q)$ (follows from Theorem 4.1) and Propositions 4.2, 4.3 and 4.4) we infer that proximinal subspaces M of $L_1(T, \nu)$ behave like the proximinal subspaces in $C(Q)$ under the additional assumption that G_f° is a finite union of atoms for each $f \in M^\perp \setminus \{0\}$.

CHAPTER IV.

Semichebychev and Chebychev Subspaces of Finite Codimension

Let M be a subspace of a normed linear space E. Then M is semichebychev if

$$P_M(x) = \left\{ m_0 \in M : \|x - m_0\| = \inf_{m \in M} \|x - m\| \right\}$$

contains at most one single element for every $x \in E$.

In this chapter we give a characterisation of semichebychev subspaces in a general normed linear space and derive from this a characterisation of chebychev subspaces using Theorem 2.10. Further we apply both these theorems to the space $L_1(T, \nu)$ to obtain the corresponding characterisation theorems in that space.

Garkavi has given the following result on semichebychev subspaces which will be needed in the sequel.

THEOREM 5.1 [6] (Garkavi) Let M be a subspace of finite codimension in E. Then M is semichebychev if and only if for every $\Phi \in (M^\perp)^*$, there exists at the most only one element x in E such that

$$\|\Phi\| = \|x\|$$

and

$$\Phi(f) = f(x) \text{ for every } f \in M^\perp$$

We remark that if $\theta|_{\mathcal{M}_f}$ denotes the restriction of the map θ to the set $\mathcal{M}_f$, the condition of the above theorem is equivalent to saying that $\theta|_{\mathcal{M}_f}$ is one to one to one for each $f \in M^{\perp} \setminus \{0\}$.

Now we will give our characterisation of semichebychev subspaces.

<u>THEOREM 5.2.</u> <u>Let M be a subspace of finite codimension n in E. Then M is semichebychev if and only if</u>

b(i) $\mathcal{M}_{f_1,\ldots,f_n}$ <u>is at most a singleton set for every basis</u> $f_1,\ldots,f_n$ <u>of</u> $M^{\perp}$

b(ii) <u>For each</u> $f \in M^{\perp} \setminus \{0\}$ <u>either</u> $\nu(G_f^0) = 0$ <u>or</u>

$G_f^0 = co(A_f)$.

<u>Proof.</u> <u>Necessity.</u> Let M be a semichebychev subspaces in E and $f_1,\ldots,f_n$ be a basis of $M^{\perp}$. If $\mathcal{M}_{f_1,\ldots,f_n}$ is empty b(i) is clearly satisfied. Assume that $\mathcal{M}_{f_1,\ldots,f_n}$ is nonempty. Since $(M^{\perp})^*$ is of dimension n and $f_1,\ldots,f_n$ is a basis of $M^{\perp}$ we have, $\theta(\mathcal{M}_{f_1,\ldots,f_n}) \subset (M^{\perp})^*$ to be a singleton set. Also, M is semichebychev and so by the remark earlier $\theta|_{\mathcal{M}_{f_1}}$ is one-to-one. This implies

$\mathcal{M}_{f_1,\ldots,f_n} \subset \mathcal{M}_{f_1}$ is a singleton set.

Now let $f \in M^{\perp} \setminus \{0\}$ be given. If $\mathcal{M}_f$ is at the most a singleton set $b(ii)$ is easily seen to be satisfied. Suppose that $\mathcal{M}_f$ contains more than one element. $\mathcal{M}_f$ is convex, θ is linear and so $\theta(\mathcal{M}_f)$ is also a convex set.

Further $\theta(\mathcal{M}_f) \subset (M^\perp)^*$ is a finite dimensional convex set and so by Lemma 2.8 $\theta(\mathcal{M}_f)$ is convexhull of the set of all its extreme points. But by Lemma 1.7 $\tilde{B}_f = B_f \cap \theta(\mathcal{M}_f)$, is the set of all the extreme points of $\theta(\mathcal{M}_f)$. Hence

$$\theta(\mathcal{M}_f) = co(\tilde{B}_f) \tag{46}$$

Let $\bar{\Phi}_0 \in \tilde{B}_f$. Then $\{\bar{\Phi}_0\} = \mathcal{N}^0_{f, g_2, \ldots, g_n}$ for some basis $f, g_2, \ldots, g_n$ of $M^\perp$. From (26) we have an $x_0 \in \mathcal{M}_f$ such that

$$\theta(x_0) = \bar{\Phi}_0$$

which implies $x_0 \in \mathcal{M}_{f, g_2, \ldots, g_n}$. So $x_0 \in A_f$ which gives A_f is nonempty.

Suppose that $b(ii)$ does not hold for f in $M^\perp$. Then there exists $x_1 \in \mathcal{M}_f \setminus co(A_f)$. Consider the element $\theta(x_1)$. Since $\theta(x_1) \in \theta(\mathcal{M}_f)$ we see that $\theta(x_1) \in co(\tilde{B}_f)$ by (46). Since $\theta^{-1}(\tilde{B}_f) \subset A_f$ and θ is linear this implies there exists an $x_2 \in co(A_f)$ satisfying

$$\theta(x_1) = \theta(x_2)$$

Since both x_1 and x_2 are in $\mathcal{M}_f$, this is a contradiction to our assumption that $\theta/\mathcal{M}_f$ is one to one. Since both x_1 and x_2 are in $\mathcal{M}_f$ this is a contradiction to our assumption that $\theta/\mathcal{M}_f$ is one to one.

<u>Sufficiency</u>. Suppose that $b(i)$ and $b(ii)$ are satisfied for the annihilator space $M^\perp$. We will show that $\theta/\mathcal{M}_f$ is one-to-one for each $f \in M^\perp \setminus \{0\}$. Let f be an arbitrary element of $M^\perp \setminus \{0\}$. Then by $b(i)$ $\mathcal{M}_{f, g_2, \ldots, g_n}$ is a singleton set whenever $f, g_2, \ldots, g_n$ is a basis of $M^\perp$. Hence $x_1, x_2 \in A_f$ and $x_1 \neq x_2$ implies $\theta(x_1) = \theta(x_2)$ and thus θ / A_f is one-to-one.

Since $\mathcal{N}_f^{\circ}$ is finite dimensional, $\theta(\mathcal{M}_f) \subset \mathcal{N}_f^{\circ}$ contains only a finite number of linearly independent elements of $(M^\perp)^*$. Further θ/A_f is one-to-one and thus θ/A_f too contains only finite number of linearly independent elements of E. Hence $\mathcal{M}_f = co(A_f)$ is finite dimensional. Since θ/A_f is one-to-one this gives is also one-to-one and completes the proof of this theorem.

<u>Remark 5.3</u>. It is clear from the above proof that b(i) and $b(ii)$ together imply $\mathcal{M}_f$ is finite dimensional.

The following theorem of Garkavi on semichebychev subspaces is given in [6].

<u>THEOREM</u> [6] (Garkavi). <u>Let</u> M <u>be a subspace of finite codimension n in E. Then M is semichebychev if and only</u> if for every $f \in M^\perp \setminus \{0\}$, the set $\mathcal{M}_f$ is of dimension $r \leq n-1$ and for any $r+1$ linearly independent elements $x_0, x_1, \ldots, x_r$ of $\mathcal{M}_f$ we have

$$\operatorname{rank}\{f_k(x_i)\}_{\substack{k=1,2\ldots n\\ i=0,1,\ldots r}} = r+1$$

Equivalently the above theorem can be stated as

THEOREM 5.4. *Let M be a subspace of finite codimension n in E. Then M is semichebychev in E if and only if for every* $f \in M^{\perp} \setminus \{0\}$, *the set* $\mathcal{M}_f$ *is of dimension less than or equal to* $n-1$ *and if* $r \leq n-1$ *and* $x_1, \ldots, x_r$ *are any* r *linearly independent elements of* $\mathcal{M}_f$ *then we have*

$$\left\{ (f(x_1), f(x_2), \ldots, f(x_r)) : f \in M^{\perp} \right\} = \mathbb{R}^r$$

where $\mathbb{R}^r$ *is the* r*-dimensional euclidean space.*

We now show that Theorem 5.4 can be derived by a much easier and simpler method than given in from Theorem 5.1 using the map θ.

Proof of Theorem 5.4. Necessity. Let M be semichebychev in E. We have $\theta(\mathcal{M}_f) \subset \mathcal{N}_f^{\circ}$ for any $f \in M^{\perp} \setminus \{0\}$ and so dimension of $\theta(\mathcal{M}_f) \leq$ dimension of $\mathcal{N}_f^{\circ} \leq n-1$. since M is semichebychev we have by Theorem 5.1, $\theta / \mathcal{M}_f$ is one to one, which implies

dimension of $\mathcal{M}_f =$ dimension of $\theta(\mathcal{M}_f) \leq n-1$.

Further if $r \leq n-1$, and $x_1, \ldots, x_r$ are any set of linearly independent elements of $\mathcal{M}_f$, then

$$\left\{ \theta(x_1), \theta(x_2), \ldots, \theta(x_r) \right\}$$

is also a set of linearly independent elements. Hence if

$$\Phi_i = \theta(x_i) \quad , i = 1, 2, \ldots, n \text{ , then}$$

$$\{(\Phi_1(f), \Phi_2(f), \ldots \ldots, \Phi_n(f)) : f \in M^\perp\} = \mathbb{R}^n$$

This implies

$$\{(f(x_1), f(x_2), \ldots \ldots \ldots, f(x_n)) : f \in M^\perp\} = \mathbb{R}^n$$

<u>Sufficiency</u>: Assume that the condition of Theorem 5.4 holds. Suppose that there exists $f \in M^\perp \setminus \{0\}$ such that $\theta / \mathcal{N}_f$ is not one to one. Then we can get x_1 and x_2 in $\mathcal{N}_f$ satisfying $x_1 \neq x_2$, but

$$\theta(x_1) = \theta(x_2) = \Phi \text{ for some } \Phi \in \mathcal{N}_f^\circ$$

Then x_1 and x_2 are two linearly independent elements and so by Theorem 5.4 we have

$$\{(f(x_1), f(x_2)) : f \in M^\perp\} = \{\Phi(f), \Phi(f) : f \in M^\perp\} = \mathbb{R}^2$$

which is not true. Hence $\theta / \mathcal{N}_f$ is one to one and this completes the proof of this theorem.

We now characterise the Chebychev subspaces of finite codimension in the following theorem.

THEOREM 5.5. Let M be a closed subspace of codimension n in E. Then in order that M be a chebychev subspace of E it is necessary and sufficient that the following conditions are satisfied.

(a_1) $\mathcal{M}_{f_1 \ldots f_n}$ is a singleton set for every basis $f_1 \ldots f_n$ of $M^\perp$.

(a_2) $\mathcal{M}_f = co(A_f)$ for every $f \in M^\perp \setminus \{0\}$.

(a_3) $\sup_{x \in \mathcal{M}_{f_1}} f_2(x) = \max_{\Phi \in \mathcal{N}_{f_1}} \Phi(f_2)$ for each pair $f_1, f_2 \in M^\perp \setminus \{0\}$.

Proof. Necessity. Assume that M is a chebychev subspace of E. Then M is proximinal and so by Theorem 2.10 (1) holds. Further by Proposition 2.11 (1) implies (9) which gives (a_3). Also M is semichebychev and since $\mathcal{M}_{f_1 \ldots f_n}$ is nonempty (a_1) and (a_3) are also satisfied by Theorem 5.2.

Sufficiency: Assume that (a_1), (a_2) and (a_3) hold. Then by Theorem 5.2 we have M to be semichebychev. Further by remark 5.3 (a_1) and (a_2) imply that $\mathcal{M}_f$ is nonempty and is finite dimensional for each $f \in M^\perp \setminus \{0\}$. Further (a_3) is also satisfied. Hence by the implication $5 \Longrightarrow 1$ of Theorem 2.14 we conclude that M is proximinal.

Remark 5.6. We note that in the case of semichebychev subspaces the requisition for proximinality reduces to

(i) $\mathcal{M}_f$ is nonempty for each $f \in M^\perp \setminus \{0\}$ and

(ii) $\sup_{x \in \mathcal{M}_{f_1}} f_2(x) = \max_{\Phi \in \mathcal{N}_{f_1}} \Phi(f_2)$ for every

pair $f_1, f_2 \in M^{\perp} \setminus \{0\}$

since $\mathcal{M}_f$ is finite dimensional for each $f \in M^{\perp} \setminus \{0\}$.

We will now give the application of Theorem 5.2 to the space $L_1(T, \nu)$. We will first give a proposition which is needed in the sequel.

If

$$\tilde{G}^{0+}_{f_1 \ldots f_n} = \left\{ t \in T : f_i(t) = \tilde{M}_{f_1 \ldots f_i},\ i = 1, 2, \ldots, n \right\}$$

$$\tilde{G}^{0-}_{f_1 \ldots f_n} = \left\{ t \in T : f_i(t) = -\tilde{M}_{f_1 \ldots f_i},\ i = 1, 2, \ldots, n \right\}$$

and

$$\tilde{G}^{0}_{f_1 \ldots f_n} = \tilde{G}^{0+}_{f_1 \ldots f_n} \cup \tilde{G}^{0-}_{f_1 \ldots f_n}$$

(where the $\tilde{M}_{f_1 \ldots f_i}$'s are as given in Remark 3.6)
Then we have

<u>PROPOSITION 5.6.</u> <u>Let</u> $f_1, \ldots, f_n$ <u>be a basis of</u> $M^{\perp}$ <u>Then x in</u> $L_1(T, \nu)$ <u>belongs to</u> $\mathcal{M}_{f_1 \ldots f_n}$ <u>if and only if</u>

(b$_1$) $\|x\| = 1$

(b$_2$) x vanishes almost everywhere outside $\tilde{G}^{0}_{f_1 \ldots f_n}$.

(b$_3$) $x \geq 0$ on $\tilde{G}^{0+}_{f_1 \ldots f_n}$

$x \leq 0$ on $\tilde{G}^{0-}_{f_1 \ldots f_n}$

Proof. Let $x \in \mathcal{M}_{f_1 \cdots f_n}$. (b_1) follows from the definition of the set $\mathcal{M}_{f_1 \cdots f_n}$. (b_2) can be proved in exactly the same way as given in the proof of the implication $1 \Longrightarrow 2$ in Theorem 3.4

To show (b_3) we will assume that there exists a subset A of $\tilde{G}^{o}_{f_1 \cdots f_n}$ such that

$$\nu(A) > 0$$

and

$$\begin{aligned} x < 0 \quad &\text{on} \quad A^{+} = A \cap \tilde{G}^{o+}_{f_1 \cdots f_n} \\ x > 0 \quad &\text{on} \quad A^{-} = A \cap \tilde{G}^{o-}_{f_1 \cdots f_n} \end{aligned} \tag{47}$$

Then

$$\begin{aligned} \int_T x f_n d\nu &= \int_{\tilde{G}^{o}_{f_1 \cdots f_n}} x f_n d\nu \\ &= \int_{A \cap \tilde{G}^{o+}_{f_1 \cdots f_n}} x f_n d\nu + \int_{\tilde{G}^{o}_{f_1 \cdots f_n} \setminus A} x f_n d\nu \\ &= \int_{A^{+}} x f_n d\nu + \int_{A^{-}} x f_n d\nu + \int_{\tilde{G}^{o}_{f_1 \cdots f_n} \setminus A} x f_n d\nu \\ &= \tilde{M}_n \int_{A^{+}} x d\nu - \tilde{M}_n \int_{A^{-}} x d\nu + \tilde{M}_n \int_{\tilde{G}^{o}_{f_1 \cdots f_n} \setminus A} x d\nu \\ &= \tilde{M}_n \left[\int_{A^{+}} x d\nu - \int_{A^{-}} x d\nu + \int_{\tilde{G}^{o}_{f_1 \cdots f_n} \setminus A} x d\nu \right] \\ &< \tilde{M}_n \end{aligned}$$

by (b_1) and (47). Since $M_n = \tilde{M}_n$ by Remark 3.6 and $x \in \mathcal{M}_{f_1 \cdots f_n}$ this contradicts the fact that $\int_T x f_n d\nu = \tilde{M}_n$ and thus (b_3) is also satisfied.

Assume that for x in $L_1(T, \nu)$, (b_1), (b_2) and (b_3) hold. Then by (b_2) and (b_3)

$$\int_T x f_i d\nu = \int_{\tilde{G}^0_{f_1 \cdots f_n}} x f_i d\nu$$

$$= \int_{\tilde{G}^{0+}_{f_1 \cdots f_n}} x^+ f_i d\nu + \int_{\tilde{G}^{0-}_{f_1 \cdots f_n}} x^- f_i d\nu$$

$$= \tilde{M}_i \int_{\tilde{G}^{0+}_{f_1 \cdots f_n}} x^+ d\nu - \tilde{M}_i \int_{\tilde{G}^{0-}_{f_1 \cdots f_n}} x^- d\nu$$

$$= \tilde{M}_i \left[\int_{\tilde{G}^{0+}_{f_1 \cdots f_n}} x^+ d\nu - \int_{\tilde{G}^{0-}_{f_1 \cdots f_n}} x^- d\nu \right]$$

$$= \tilde{M}_i \qquad \text{by } (b_1).$$

for $i = 1, 2, \cdots, n$. This together with Remark 3.6 proves $x \in \mathcal{M}_{f_1 \cdots f_n}$.

<u>DEFINITION</u> [6] . <u>Let A be a measurable subset of</u> <u>Then A is called an atom if</u>

(i) $\nu(A) > 0$

(ii) If B is a measurable subset of A , Then

$$\nu(B)=0 \quad \text{or} \quad \nu(A\setminus B)=0$$

We will now give the characterisation semichebychev subspaces of finite codimension.

THEOREM 5.7. Let M be a closed subspace of finite codimension n in $L_1(T,\nu)$. Then in order that M be semichebychev in $L_1(T,\nu)$ it is necessary and sufficient that the following conditions hold.

(d_1) For every $f_1,\dots,f_n$ of $M^\perp$, either $\nu(\tilde{G}^0_{f_1\dots f_n})=0$ or $\tilde{G}^0_{f_1\dots f_n}$ is an atom.

(d_2) For every $f\in M^\perp\setminus\{0\}$, whenever $\nu(G^0_f)>0$, there exists r (r finite) sets of bases

$$\{f, g_2^i, g_3^i, \dots, g_n^i\}_{i=1}^{r} \text{ of } M^\perp \text{ such that}$$

$$G^0_f = \bigcup_{i=1}^{r} \tilde{G}^0_{f_1, g_2^i\dots g_n^i}.$$

Proof. Necessity: Let M be a semichebychev subspace of codimension n in $L_1(T,\nu)$. Then by Theorem 5.2 $h(i)$ and $h(ii)$ hold.

Let $f_1,\dots,f_n$ be an arbitrary basis of $M^\perp$. Suppose that $\nu(\tilde{G}^0_{f_1\dots f_n})>0$ but $\tilde{G}^0_{f_1\dots f_n}$ is not an atom. Then there exists A a proper subset of $\tilde{G}^0_{f_1,\dots f_n}$ satisfying,

$$0<\nu(A)<\nu(\tilde{G}^0_{f_1\dots f_n})$$

so that we can define x_1 and x_2 in $L_1(T, \nu)$ with $x_1 \neq x_2$ as follows:

$$x_1(t) = \begin{cases} \dfrac{\chi_{G^{\gamma_0 +}_{f_1 \cdots f_n}} - \chi_{G^{\gamma_0 -}_{f_1 \cdots f_n}}}{\nu(A)}, & t \in A \\ 0, & t \notin A \end{cases}$$

$$x_2(t) = \begin{cases} \dfrac{\chi_{G^{\gamma_0 +}_{f_1 \cdots f_n}} - \chi_{G^{\gamma_0 -}_{f_1 \cdots f_n}}}{\nu(\tilde{G}^0_{f_1 \cdots f_n})}, & t \in \tilde{G}^0_{f_1 \cdots f_n} \\ 0, & t \notin \tilde{G}^0_{f_1 \cdots f_n} \end{cases}$$

It is easy to see that $\|x_i\| = 1 \quad (i = 1, 2)$ and

$$\int_T x_i f_j \, d\nu = \tilde{M}_j = M_j \quad \text{for} \quad j = 1, 2 \cdots n$$

Thus both x_1 and x_2 are in $\sigma_{f_1 \cdots f_n}$ but $x_1 \neq x_2$ which contradicts $b(i)$. Hence $\tilde{G}^0_{f_1 \cdots f_n}$ is an atom.

Let $f \in M^{\perp} \setminus \{0\}$ be such that $\nu(G_f^0) > 0$

Consider $x \in \mathcal{M}_f$ given by

$$x(t) = \begin{cases} \dfrac{\chi_{G_f^{0+}} - \chi_{G_f^{0-}}}{\nu(G_f^0)}, & t \in G_f^0 \\ 0, & t \notin G_f^0 \end{cases}$$

Then by $b(ii)$ there exists r (r finite) sets of bases

$\{f, g_2^i, \cdots g_n^i\}_{i=1}^{r}$ of $M^{\perp}$ such that

$$x \in co\{x_1, x_2, \cdots, x_n\} \tag{48}$$

where $x_i \in \mathcal{M}_{f, g_2^i, \cdots g_n^i}$ $(i = 1, 2, \cdots, r)$.

But by Proposition 3.6 each x_i vanishes a.e. outside $\widetilde{G}^0_{f, g_2^i, \cdots g_n^i}$ and so (48) implies that

$$G_f^0 = \bigcup_{i=1}^{r} \widetilde{G}^0_{f, g_2^i, \cdots, g_n^i}$$

<u>Sufficiency</u>. Assume that both the conditions of the theorem hold. Let $f_1, \cdots f_n$ be an arbitrary basis of $M^{\perp}$. If $\nu(\widetilde{G}^0_{f_1 \cdots f_n}) = 0$, then by Proposition 3.5 $\mathcal{M}_{f_1 \cdots f_n}$ is an empty set. Suppose that $\nu(\widetilde{G}^0_{f_1 \cdots f_n}) > 0$. Then $\widetilde{G}^0_{f_1 \cdots f_n}$ is an atom

by assumption. Also $\overset{x}{\wedge}$ satisfies (b_1), (b_2) and (b_3). Further if

$$B_1 = \left\{ t \in \tilde{G}^{o}_{f_1 \cdots f_n} : |x(t)| > \frac{1}{v(\tilde{G}^{o}_{f_1 \cdots f_n})} \right\}$$
$$B_2 = \left\{ t \in \tilde{G}^{o+}_{f_1 \cdots f_n} : |x(t)| > \frac{1}{v(\tilde{G}^{o}_{f_1 \cdots f_n})} \right. \qquad (49)$$

then, since $\tilde{G}^{o}_{f_1 \cdots f_n}$ is an atom, either

$$v(B_1) = v(\tilde{G}^{o}_{f_1 \cdots f_n}) \quad \text{and} \quad v(\tilde{G}^{o}_{f_1 \cdots f_n} \setminus B_1) = 0$$

or

$$v(B_1) = v(\tilde{G}^{o}_{f_1 \cdots f_n}) \quad \text{and} \quad v(\tilde{G}^{o}_{f_1 \cdots f_n} \setminus B_2) = 0 \qquad (50)$$

In either case it is easy to see that

$$\int_T x f_n \, dv = \int_{\tilde{G}^{o}_{f_1 \cdots f_n}} x f_n \, dv \neq M_n .$$

using (30), (31) (b_1) and (b_3). Hence

$$|x(t)| = \frac{1}{v(\tilde{G}^{o}_{f_1 \cdots f_n})} \quad v \text{ a.e on } \tilde{G}^{o}_{f_1 \cdots f_n} \qquad (51)$$

and this together with (b_2) and (b_3) implies that $\mathcal{M}_{f_1 \cdots f_n}$ is a singleton set.

Now let $f \in M^{\perp} \setminus \{0\}$ be given. If $\nu(G_f^0)=0$ then clearly $\mathcal{M}_f$ is an empty set. Suppose that $\nu(G_f^0)>0$. Then by assumption there exists r (r finite) sets of bases $\{f, g_2^i, \ldots g_n^i\}_{i=1}^{r}$ of $M^{\perp}$ such that

$$G_f^0 = \bigcup_{i=1}^{r} \tilde{G}^0_{f, g_2^i, \ldots, g_n^i} \tag{52}$$

where each $\tilde{G}^0_{f, g_2^i, \ldots, g_n^i}$ is an atom. Since

$$G_f^{0+} \supset \tilde{G}^{0+}_{f, g_2^i, \ldots, g_n^i}$$

$$G_f^{0-} \supset \tilde{G}^{0-}_{f, g_2^i, \ldots, g_n^i}$$

for $i=1,2,\ldots,r$ and $G_f^{0+} \cap G_f^{0-} = \phi$ (33) gives

$$\begin{aligned} G_f^{0+} &= \bigcup_{i=1}^{r} \tilde{G}^{0+}_{f, g_2^i, \ldots, g_n^i} \\ G_f^{0-} &= \bigcup_{i=1}^{r} \tilde{G}^{0-}_{f, g_2^i, \ldots, g_n^i} \end{aligned} \tag{53}$$

Now consider any $x \in \mathcal{M}_f$. x vanishes a.e. outside G_f^0 and since (52) holds this implies x vanishes a.e. outside $\bigcup_{i=1}^{r} \tilde{G}^0_{f, g_2^i, \ldots, g_n^i}$. Further, since $\tilde{G}^0_{f, g_2^i, \ldots, g_n^i}$ is an atom for each $i=1,2,\ldots,r$,

x is a constant a.e. on $\tilde{G}^0_{f, g_2^i, \ldots, g_n^i}$ $(i=1,2\ldots r)$ (54)

Also

$$\begin{aligned} x &\geq 0 \quad \text{on } G_f^{0+} \\ x &\leq 0 \quad \text{on } G_f^{0-} \end{aligned} \tag{55}$$

Now using (b_3) (54) and (55) we have

x and x_i are of the same sign for $i = 1, 2 \ldots r$

This together with (51), (52) and (54) imply that there exists positive scalars α_i, $i = 1, 2 \ldots r$ such that

$$\sum_{i=1}^{r} \alpha_i = 1$$

and

$$x = \sum_{i=1}^{r} \alpha_i x_i$$

So $x \in co\{x_1, \ldots, x_r\} \subset co(A_f)$.

Hence by Theorem 5.3 M is semichebychev.

<u>THEOREM 5.8</u>. <u>Let M be a closed subspace of codimension n in</u> $L_1(T, \nu)$. <u>Then M is a chebychev subspace if and only if</u>

$G^{\circ}_{f_1 \ldots f_n}$ <u>is an atom for every basis</u> $f_1, \ldots, f_n$ <u>of</u> $M^{\perp}$.

For every $f \in M^{\perp} \setminus \{0\}$, there exists r (r finite) sets of bases $\{f, g_2^i, g_3^i \ldots, g_n^i\}_{i=1}^{r}$ such that

$$G^{\circ}_f = \bigcup_{i=1}^{r} G^{\circ}_{f, g_2^i, \ldots, g_n^i}$$

We observe that for a proximinal subspace M of $L_1(T, \nu)$

$$\tilde{G}^{\circ}_{f_1 \ldots f_n} = G^{\circ}_{f_1 \ldots f_n}$$

for every basis $f_1, \ldots, f_n$ of $M^{\perp}$ and hence the above theorem follows easily from Theorem 5.7 and the equivalence $1 \Longleftrightarrow 2$ of Theorem 3.4.

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