COSMOLOGICAL PERTURBATION THEORY

By

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BONAFIDE CERTIFICATE

Certified that this dissertation titled **COSMOLOGICAL PERTURBATION THOERY** is the bonafide work of **Mr.KRISHNAKUMAR SABAPATHY** who carried out the project under my supervision.

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Abstract

In this review we study cosmological perturbation theory. The theory aims to model the physical universe by perturbing about a background Friedmann-Robertson-Walker cosmological model. Two approaches are highlighted, the gauge-invariant formalism and the gauge fixed(conformal newtonian gauge) approach. Boltzmann equations for all the matter components of the universe are studied. We get a set of linear differential equations. The initial conditions and the origin of fluctuations lead us to the study of scalar field inflation. Future directions include quantum gravity corrections, alternative inflationary and non-inflationary models and the study of inhomogeneity and isotropy in concordance with current day observations.

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Chapter 1

Introduction

1.1 The Standard Model of Cosmology

The current well accepted model of cosmology is called the Standard Model of cosmology. The model is based on the Friedman-Robertson-Walker [FRW] metric [5]. There are two important aspects for the metric ansatz, homogeneity and isotropy. That the universe is homogeneous is assumed and the input of isotropy is observationally motivated [6]. This is also called the Cosmological Principle. Isotropy is observed on large scales of current cosmological interest which is around 100 - 200Mpc. These symmetry properties lead to a diagonal metric in a convenient coordinate system. The metric has one unknown function called the scale factor that describes the dynamics of the universe. This scale factor is a function of time alone and its evolution is governed by the Einstein's equation. Depending on the equation of state for the system, the scale factor evolves in a certain way. The universe is understood to have evolved through various epochs, with each epoch named after the component that has a dominant contribution to the total energy density [7]. The following figure (refer Fig.(1.1)) illustrates the evolution of the universe through the various epochs. The standard model predicts that the universe has been expanding after the onset of the 'big bang' and is currently in a matter dominant state.

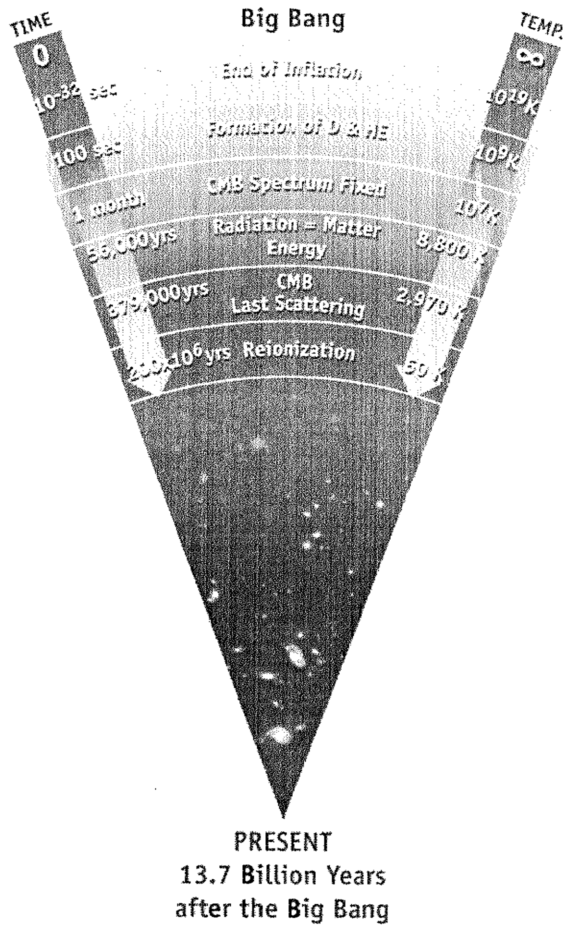

Figure 1.1: A schematic diagram depicting the time-line of the evolution of the universe - [1]

There are three key observations which constitute the success of this model:

- The verification of the Hubble's Law, a statement that the universe expands [8]. The Hubble diagram is a linear plot of velocities of galaxies versus their distances.
- Light element abundances in accordance with the Big Bang Nucleosynthesis [9].
- The existence of relic radiation that had decoupled from matter, known as the Cosmic Microwave Background Radiation (CMB) [10].

The metric for this model is setup in the comoving coordinate system. The symmetry properties of the universe is observable to a certain class of observers called the isotropic observers. These class of observers are comoving with respect to each other. To any observer who is moving with respect to these isotropic observers, will not see the universe to have these symmetry properties. In this coordinate system, each point is associated a fixed spatial coordinate and its time coordinate is decided by the clock at rest at that point. Then physical distances are related to the comoving distances through the scale factor.

$$l(t)_{physical} = a(t) \quad l_{comoving} \tag{1.1}$$

where, 'l' denotes distances and the comoving coordinate is a constant with time. The comoving coordinates can be understood as points marked on a rubber sheet. Then as the rubber sheet expands, the physical distances increase but the coordinate distances remain a constant.

1.2 Inhomogeneities and Anisotropies

But on 'smaller' scales we see specific physical structure like galaxies, galaxy clusters and other matter distributions. Due to the observational success of the standard model, we expect that these inhomogeneities and anisotropies can be incorporated in a perturbation theory with the dominant contribution being the FRW Model. We expect that these inhomogeneities have evolved from some primordial inhomogeneities. Therefore, we can attempt a perturbation theory and expect the signatures to be verifiable from the observed matter distributions and CMB data, for example. This raises several questions:

- *How do these perturbations evolve?* To address this, we need to setup and study a cosmological perturbation theory i.e. a theory that gives the dynamics of the universe which is perturbed from the background FRW universe.
- *Initial conditions:* The perturbation variables satisfy differential equations that are evolution equations. When we take the fourier transform of the differential equations, we find that each mode of the perturbation variable satisfies an ordinary differential equation. We need to supply initial conditions to solve the equations.
- *How do the initial fluctuations arise?* It is believed that scales of current cosmological importance were once small and hence affected by micro-physical processes in the early phase of the universe. The origin of these perturbations is assumed to be seeded by quantum fluctuations during the inflationary epoch which is the period of exponential expansion of the scale factor [11, 12].

- *The statistical distribution of the initial conditions:* This question is of important because observations are also at the level of statistical distributions. One such observationally important quantity is the power spectrum. This is usually the fourier transform of the relevant two point correlation functions averaged over all possible configurations.

- *On what observations can we expect the signatures of this theory?* Inhomogeneities in matter distributions manifest as anisotropies in the CMB analysis. CMB is also sensitive to the tensor perturbations during inflation. In general relativity, tensor perturbations of the metric about a flat background give rise to what are known as gravity waves.

1.3 Perturbation Theory

The two sets of perturbations are matter and gravity perturbations. There are subtle issues regarding the definition of a perturbation theory. Gravitation is a theory of general covariance i.e. the theory is invariant under arbitrary coordinate transformations. The definition of perturbation depends on the choice of coordinate system. Therefore we need to be able to define quantities in a coordinate invariant way. Another possibility is to work in a particular coordinate system throughout. This is in analogy to making a particular choice of gauge in electromagnetic theory. This method has some disadvantages like unphysical modes which are coordinate artifacts are present [13]. We will setup and work in a gauge invariant formalism of perturbation theory using fixed background functions which are invariant under coordinate transformations. But it is also sometimes easier to perform certain calculations in a specific coordinate system. For such a coordinate choice, the conformal gauge is very convenient. We will illustrate this point later but the idea is that the metric perturbation variables and the gauge invariant variables coincide in this particular coordinate system. Thereby allowing an easy change from the gauge fixed to the gauge invariant variables. There are two popular approaches for a gauge invariant formalism. Ellis et.al. have worked on a covariant approach. The other formalism is the gauge invariant formalism introduced by Bardeen [14, 15] and further developed by Brandenberger [16] et.al. The thesis is based on this formalism.

The thesis is organised as follows: We will briefly describe the standard model of cosmology in chapter 2. Then we will describe the gauge invariant formalism as developed by Bardeen et. al and discuss some simple examples in chapter 3. Then we will discuss how the physical universe is modelled[1]. In chapter 4 we will use the Boltzmann equation for every component and account for interactions to study the perturbations of the statistical phase space distribution functions. We will work in a particular gauge for this treatment. Then we will study what the initial conditions are for the perturbations variables. This will lead us to the study of fluctuations during inflation as described in chapter 5. We will end with describing parameters that are important for observational purposes.

[1]We will follow the approach as given in the textbook [17]

Chapter 2

Friedmann-Robertson-Walker Cosmological Model

As mentioned earlier, there are two important simplifications, homogeneity and isotropy which go into determining the metric. Spatial homogeneity and isotropy implies that the spatial metric is of constant curvature. This leads to the Riemann tensor having a certain form [18]:

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \tag{2.1}$$

where, K is a real number. The Ricci tensor is got by contracting the Reimann tensor in the following way,

$$R_{\beta\delta} := g^{\alpha\gamma}R_{\alpha\beta\gamma\delta} \tag{2.2}$$

Then, we get,

$$R_{\alpha\beta} = 2Kg_{\alpha\beta} \tag{2.3}$$

The three dimensional spatial metric for a spherically symmetric space can be written in the following way,

$$\begin{aligned} d\sigma^2 &= f(r)dr^2 + r^2 d\omega^2 \qquad \text{where,} & (2.4)\\ d\omega^2 &= d\theta^2 + \sin^2\theta d\phi^2 & (2.5) \end{aligned}$$

Using the above condition on the Ricci tensor, we can determine $f(r)$, a function of the magnitude of the radial coordiante alone.

$$f(r) = \frac{1}{1 - Kr^2} \tag{2.6}$$

Therefore, taking the time coordinate in the comoving system and introducing the scale factor; the full FRW metric can be written as

$$ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1-Kr^2} + r^2 d\omega^2\right) \tag{2.7}$$

Here, $a(t)$ is the scale factor which is determined from the Einsteins equation depending on the kind of matter system we are interested in.

In conformal time coordinates, the metric can be written in the following form with,

$$ds^2 = a(\eta)^2\left(d\eta^2 - \frac{dr^2}{1-Kr^2} + r^2 d\omega^2\right), \qquad \text{with } d\eta := dt/a \tag{2.8}$$

Curvature Constant K: K can in general be any real number, but we can re-scale the radial coordinate and can take the three possible values for K, K=0, +1 , -1. The three cases are called flat, open and closed respective. The names come from the range of the radial coordinate. This point can be seen easily if we make the following coordinate transformation. Define,

$$\chi = \int \frac{dr}{\sqrt{1-Kr^2}} = \begin{cases} \sin^{-1} r & (\text{for } k=+1) \\ r & (\text{for } k=0) \\ \sinh^{-1} r & (\text{for } k=-1) \end{cases} \tag{2.9}$$

Then the spatial part of the metric can be written as

$$ds^2 = a^2(d\chi^2 + f_k(\chi)(d\omega^2)), \qquad d\omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{2.10}$$

Where,

$$f_k(\chi) = \begin{cases} \sin\chi & (\text{for } k=+1) \\ \chi & (\text{for } k=0) \\ \sinh\chi & (\text{for } k=-1) \end{cases} \tag{2.11}$$

The $K = 0$ is the familiar spatially flat case. The $K = +1$ case is called a closed universe. This can be seen through the following points. The range of the coordinate χ is bounded. Hence the volume of the full spatial region is bounded. Also a two sphere in this space has a bounded surface area. This is in contrast to the $K = -1$ space which is hyperboloidal. Here, the range of χ is unbounded leading to an unbounded spatial volume. Also a two sphere in this space has a monotonically increasing surface area with the coordinate χ. Hence the name, open and closed for the corresponding cases.

The coordinate system (2.7) is also called the comoving coordinate system This is because ($x^\alpha = constant$) are geodesics. The physical distances will be the coordinate distance multiplied by the scale factor, as can be seen from the metric. The scale factor at present is increasing with time.

In (2.7), the spatial part of the metric is given in polar coordinates. An alternate coordinate system is the same metric with the spatial part now described by the cartesian coordinate system. This can be realised by the performing the following change to the radial coordinate.

$$r = \frac{\bar{r}}{1 + \frac{K\bar{r}^2}{4}} \tag{2.12}$$

Then, the metric becomes, (dropping the bar on the new radial coordinate),

$$ds^2 = dt^2 - \frac{a(t)^2}{1 + Kr^2/4}(dr^2 + r^2 d\omega^2) \tag{2.13}$$

We can see that the spatial part of the metric is conformal to the flat spatial metric. As before in conformal time,

$$ds^2 = a(\eta)^2 \left(d\eta^2 - \frac{dx^2 + dy^2 + dz^2}{1 + \frac{Kr^2}{4}} \right) \tag{2.14}$$

We then define γ_{ij} as the metric on a constant time 3D surface as follows:

$$ds^2 = a(\eta)^2 \left(d\eta^2 - \gamma_{ij} dx^i dx^j \right) \tag{2.15}$$

From hence forth, we shall use the above metric and coordinate system unless otherwise stated.

Einstein's equations relates the energy momentum tensor to the geometry of the spacetime. Therefore, the symmetry properties of homogeneity and isotropy apply to the energy momentum tensor as well. The form the energy momentum tensor then takes is the perfect fluid form which will be discussed a little later. The physical system is then fully described by the equation of state which needs to be accounted for in the energy momentum tensor. Then the evolution of the scale factor is determined by the Einstein's equation.

Finally, we end this brief summary with the equations of motion. The specific details can be extracted by pluging in the choice of energy momentum tensor. Note: From here on, prime refers to the derivative taken with respect to the conformal time η. We set $c = 1$. Starting from Einstein's equation,

$$G^{\mu}{}_{\nu} := R^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu}\frac{R}{2} = 8\pi G T^{\mu}{}_{\nu} \tag{2.16}$$

We have,

$$\begin{aligned} a'^2 + Ka^2 &= \frac{8}{3}\pi G T^0_0 a^4 && (2.17)\\ a'' + Ka &= \frac{4}{3}\pi G T a^3, \qquad \text{where } T = T^{\mu}_{\mu} && (2.18)\\ dT^0_0 &= -(4T^0_0 - T)\mathrm{d}\ \mathrm{lna}. && (2.19) \end{aligned}$$

The last of the above equations is the conservation of the energy momentum tensor. This equation is not independant and can be got from manipulating the two Einstein's equtions. We list a few examples of how the scale factor evolves when the energy momentum tensor describes a perfect fluid with the equation of state being a constant. Let $w = p/\rho$, where p is the pressure and ρ is the energy density, be a constant. In that case, Einstein's equations reduce to the following,

$$\begin{aligned} a'^2 + Ka^2 &= \frac{8}{3}\pi G T^0_0 a^4 = \frac{8}{3}\pi G\rho a^4 && (2.20)\\ a'' + Ka &= \frac{4}{3}\pi G T a^3 = \frac{4}{3}\pi G(\rho - 3p)a^3 && (2.21)\\ \rho' &= -3(\rho + p)\frac{a'}{a} && (2.22) \end{aligned}$$

Then we have the following evolution for a(t), the scale factor, as a function of both η and t. We have also taken $K = 0$.

$$\begin{aligned} a &\sim \eta^{\frac{2}{1+3w}} && (2.23)\\ &\sim t^{\frac{2}{3(1+w)}} && (2.24) \end{aligned}$$

Chapter 3

Beyond the Standard Model

We believe that the exact universe defers 'slightly' from the homogeneous model and that a linear perturbation theory is feasible. In this section, we will introduce the gauge invariant formalism of classical cosmological pertubation theory. Many advantages of this formalism will be highlighted as the discussion proceeds.

3.1 Perturbations, gauge transformation and gauge invariance

Consider a physical manifold M which is understood to be 'close' to FRW metric in a certain sense. Let there be a coordinate system on this manifold. To any tensor field Q, we associate a background function Q^0. The background functions are fixed functions of the coordinate system. They are non geometrical quantities and transform as scalars with respect to a coordinate transformation. The necessity for such a definition stems from the following. From the definition of perturbation, we see that it is coordinate dependant, leading to an ambiguous definition of perturbation. So we would like the perturbation to be invariant under arbitrary coordinate transformations (diffeomorphisms). By associating to each tensor field a background function, we in a sense are giving an absolute meaning to each point on the manifold, thereby making perturbations in different coordinate systems comparable. We make a note that all coordinate transformations are 'infinitesimal'. Mathematically, the above can be expressed in the following equations[1],

$$Q(x^\alpha(p)) = Q^0(x^\alpha(p)) + \delta Q(x^\alpha(p)) \tag{3.1}$$

[1] This chapter presents the formalism developed by Bardeen and later by Brandenberger et. al. [16]

where 'p' is a point in the manifold. Now consider a new coordinate system,

$$\tilde{x}^{\alpha} := x^{\alpha} + \xi^{\alpha} \tag{3.2}$$

where, ξ^{α} is understood as an infinitesimal vector. Then the background quantities by definition do not change under this coordinate transformation.

$$Q^0(x^{\alpha}(p)) = \tilde{Q}^0(\tilde{x}^{\alpha}(p)) \tag{3.3}$$

The tensor field changes in the following way,

$$\tilde{Q}(\tilde{x}^{\alpha}(p)) = \tilde{Q}^0(\tilde{x}^{\alpha}(p)) + \delta\tilde{Q}(\tilde{x}^{\alpha}(p)) \tag{3.4}$$

Then the change in the perturbation δQ is given by, using (3.3),

$$\triangle\delta Q = \delta\tilde{Q} - \delta Q \tag{3.5}$$

$$= \tilde{Q}(\tilde{x}^{\alpha}(p)) - Q(x^{\alpha}(p)) = L_{\xi}Q|_p \tag{3.6}$$

The above transformation for the perturbation δQ under infinitesimal coordinate transformation is called a **gauge transformation**. $L_{\xi}Q$ is the Lie derivative of Q with respect to the vector field ξ^{α}. The example for the expression of the Lie derivative of a rank two tensor is given as the following,

$$\tilde{A^{\mu}}_{\nu} - A^{\mu}{}_{\nu} = L_{\xi}A^{\mu}{}_{\nu} \tag{3.7}$$

$$= \xi^{\sigma}\partial_{\sigma}A^{\mu}{}_{\nu} - (\partial_{\sigma}\xi^{\mu})A^{\sigma}{}_{\nu} + (\partial_{\nu}\xi^{\sigma})A^{\mu}{}_{\sigma} \tag{3.8}$$

In this thesis, the background metric is the FRW metric and the background quanities for both matter and geometry are those calculated in the comoving coordinate system as described in the previous chapter.
Now going back to equation (3.6), we can apply the above expansion and calculate the change in the perturbation variables. The most important tensor for which we would like to calculate the change in perturbation variables is the metric tensor. We will give more details regarding metric perturbations in the following sections.

Gauge Invariance:

Gauge invariant perturbations are those whose Lie derivative is zero with respect to the above coordinate transformation (3.2). Such tensor fields can only be constant or can be taken to be zero. What we will do in the following section is that, we make a perturbation to the metric. These perturbations will be characterised in a certain way. We know how the metric will transform under a coordinate transformation (using the Lie derivative as given above (3.6)). We will then accommodate the change in the metric into these variables. Then from these perturbation variables, we can construct gauge invariant quantities. All the above statements will be explained in mathematical terms in the following sections.

3.2 Metric perturbation

Consider the FRW metric in cartesian coordinates in conformal time.

$$^{(0)}g_{\mu\nu} = a^2 \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \delta_{ij} \frac{1}{(1+Kr^2/4)^2} \end{array} \right) = a^2 \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \gamma_{ij} \end{array} \right) \tag{3.9}$$

Let the perturbation be characterised in general by the following,

$$\delta g_{\mu\nu} = a^2 \left(\begin{array}{c|c} A & B_i \\ \hline B_i & C_{ij} \end{array} \right) \tag{3.10}$$

where, A is a scalar. B_i a vector and C_{ij} a symmetric tensor. The quantities (A,B,C) transform as a scalar, vector or tensor under 3D spatial transformations respectively.

3.2.1 Scalar, Vector and Tensor decomposition

We can further decompose the variables B, C [19]. We will state the theorems regarding the general decomposition of vectors and tensors.

- Any covariant tensor of rank 1 can be written as the sum of two vectors. The first being the gradient of a scalar and the second a vector whose divergence is zero. $B_i = \partial_i \phi + K_i$ where the divergence of K_i is zero. In a general background, the gradient or divergence should be replaced with the 3D covariant derivative.

- Similarly a symmetric traceless tensor can be decomposed into the following:

$$T^{\alpha\beta} = \triangle^{\alpha\beta}\psi + 2D^{(\alpha}B^{\beta)} + W^{\alpha\beta}, \tag{3.11}$$

where,

$$\triangle_{\alpha\beta} := D_\alpha D_\beta - \frac{1}{3}\gamma_{\alpha\beta}\triangle_l \tag{3.12}$$

Here, D is the covariant derivative, B is a divergence free vector, $\triangle_l$ is the laplacian and W is a tracefree transverse tensor. To this tensor we can add the trace by the following way

$$T\gamma_{ij} \tag{3.13}$$

Here all covariant derivatives are with respect to the spatial background metric γ_{ij}.

We can collect and characterise the perturbations variables according to their behaviour under 3D spatial diffeomorphisms. Therefore, scalar, vector and tensor perturbations are characterised in the following way.

$$\delta g_{\mu\nu} = a^2 \left(\begin{array}{c|c} \phi & -B_{;i} + S_i \\ \hline -B_{;i} + S_i & 2(\psi\gamma_{ij} - E_{;ij}) - (F_{i;j} + F_{j;i}) - h_{ij} \end{array} \right) \tag{3.14}$$

where B, ϕ, ψ, E are scalars, S, F are divergence free vectors and h is a symmetric, divergence free, traceless tensor. Hence, the decomposition can be done easily as shown below.

$$\delta g^{(s)} = a(\eta)^2 \left(\begin{array}{c|c} 2\phi & -B_{;i} \\ \hline -B_{;i} & 2(\psi\gamma_{ij} - E_{;ij}) \end{array} \right) \tag{3.15}$$

$$\delta g^{(v)} = -a(\eta)^2 \left(\begin{array}{c|c} 0 & -S_i \\ \hline -S_i & (F_{i;j} + F_{j;i}) \end{array} \right) \tag{3.16}$$

$$\delta g^{(t)} = -a(\eta)^2 \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & h_{ij} \end{array} \right) \tag{3.17}$$

Then,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} \tag{3.18}$$

Let us do some counting. In the scalar decomposition there were four functions (ϕ, ψ, B, E). In the vector decomposition, there are two divergence free vectors (S, F). Hence four more independent variables. The final tensor perturbation (h_{ij}) is a symmetric, trace-free, divergence free 3D tensor. So it has 2 free components. The total is 10 variables which then characterises a general metric perturbation. The reason for characterising the metric perturbation in this way is important for two reasons. The decomposition is unique and the decomposition operation commutes with taking derivatives. For this, consider Einstein's tensor. It is constructed from the metric and its derivatives. The above statement says that we can consider each type of perturbation and study its evolution independently of the other classes of perturbation. That is, we can consider only scalar, vector or tensor perturbations independently as they do not couple to each other. The decomposition leads to better physical realization of the perturbation theory and makes the calculation a bit easier.

3.2.2 Gauge Transformation

We will now define a general gauge transformation and see how the perturbation variables transform under the coordinate transformation. A general infinitesimal coordinate transformation can be written as the following.

$$\tilde{x}^{\alpha} := x^{\alpha} + \xi^{\alpha} \tag{3.19}$$

We will decompose the spatial part of the vector ξ^{α} in to the gradient of a scalar and a divergence less vector. Taking $\xi^{\alpha} = (\xi^0, \gamma^{ij}\xi_{;j} + \xi^i)$, where ξ^i is taken to be a divergence zero vector. The symbol (;) denotes covariant derivative. So while considering scalar transformations, we will consider only the scalar part of the coordinate transformation i.e. $\xi^{\alpha} = (\xi^0, \gamma^{ij}\xi_{;j})$. When we take vector perturbations, we will consider only the vector part of the coordinate transformation i.e. $\xi^{\alpha} = (0, \xi^i)$. To study any such transformation of the metric, we will use the lie derivative definition (3.6). Then we accommodate the change of the metric in terms of the perturbation variables. The following equations bring out this point.

- **Scalar transformations -** The coordinate transformation is given by the following:

$$\tilde{\eta} = \eta + \xi^0 \tag{3.20}$$

$$\tilde{x}^i = x^i + \gamma^{ij}\xi_{;j} \tag{3.21}$$

The the metric variables undergo the following transformations:

$$\tilde{\phi} = \phi - \frac{a'}{a}\xi^0 - \xi^{0'} \tag{3.22}$$

$$\tilde{\psi} = \psi + \frac{a'}{a} \tag{3.23}$$

$$\tilde{B} = B + \xi^0 - \xi' \tag{3.24}$$

$$\tilde{E} = E - \xi \tag{3.25}$$

- **Vector transformations -** The coordinate transformation is given by the following:

$$\tilde{\eta} = \eta \tag{3.26}$$

$$\tilde{x}^i = x^i + \xi^i \tag{3.27}$$

The the metric variables undergo the following transformations:

$$\tilde{S}_i = S_i + \xi_i' \tag{3.28}$$

$$\tilde{F}_i = F_i - \xi_i \tag{3.29}$$

The prime denotes derivative with respect to the conformal time η. It can be shown that the tensor perturbation h_{ij} is invariant under each of the two classes of transformation, the scalar and vector transformations as given above. We mention again that the quantites are scalars, vectors or tensors depending on their behaviour under three dimensional spatial diffeomorphisms.

3.2.3 Gauge Invariant Variables

From the above we can construct gauge invariant vairables for the two types of transformations. In the scalar case, we have two functions (ξ and ξ^0), which we can use to get two gauge invariant variables. The two gauge invariant variables will be denoted by Φ, Ψ. In the vector case, we have a divergence free vector which we can use to define a gauge invariant vector. As mentioned before, the tensor part is already gauge invariant under each subclass of transformations.

- For scalar transformations, the two variables are,

$$\Phi := \phi + \frac{[(B-E')a]'}{a} \tag{3.30}$$

$$\Psi := \psi - \frac{a'(B-E')}{a} \tag{3.31}$$

- For vector transformations, we have

$$\chi_\alpha := S_\alpha - F'_\alpha \tag{3.32}$$

Two popular gauges:

Even though we have described the gauge invariant formalism in the previous sections, it is however convenient to choose a suitable coordinate system to simplify and perform certain calculations. This coordinate choice corresponds to applying conditions on the gauge variant quantities. This is in analogy to the gauge fixing procedure in electromagnetic theory. In this theory, a gauge choice corresponds to choosing a constraint on the vector potentials. For the metric perturbations, we will describe gauge choices in the context of scalar perturbations i.e. we set the vector and tensor perturbations to zero. Therefore, a gauge choice corresponds to conditions imposed on the scalar variables ϕ, ψ, B, E. We have four perturbation variables and two free parameters ξ^0 and ξ that can be used to impose constraints on two of the four perturbation variables. There are two popular choices for gauges in scalar perturbations, the synchronous gauge and the conformal gauge. The table below illustrates these gauges.

Synchronous Gauge	Longitudinal Gauge or Conformal Gauge
1) $\phi = 0$, B =0 2) In this gauge, there is a further residual freedom allowed in the transformation leading to the appearance of unphysical degrees of freedom which makes the physical interpretation difficult	B = E = 0 Φ, Ψ coincide with the metric perturbations ϕ, ψ leading to easier physical interpretation. It is also easier to get the equation of motion for the gauge invariant variables. Also, all quantities are uniquely fixed and there is no residual degrees of freedom.

The longitudinal gauge is convenient in the following sense. In this coordinate system we can calculate the equations of motion ψ, ϕ and replace the variable by the gauge invariant ones, Ψ, Φ . We will use the second gauge, also called the conformal Newtonian gauge, when studying perturbations in more detail.

3.3 Einstein's Equation for Scalar Perturbations

The next step is to calculate Einstein's equation to linear order in the metric variables. We are looking for the following:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.33}$$

$$G^0_{\mu\nu} + \delta G_{\mu\nu} = 8\pi G(T^0_{\mu\nu} + \delta T_{\mu\nu}) \tag{3.34}$$

$$G^0_{\mu\nu} = 8\pi G T^0_{\mu\nu} \tag{3.35}$$

$$\Rightarrow \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \tag{3.36}$$

The left hand side is determined by the metric perturbation and the right hand side is determined from the matter perturbation. The metric perturbation part can be calculated in a straight forward way. We are interested in finding out how these perturbation transform and then rewrite the perturbed Einstein's equation completely in terms of gauge invariant variables. For this purpose we only need the background quantities and then study its lie derivative. The background FRW $G_{\mu\nu}$ is given by the following:

$$G^0{}_0 = 3a^{-2}(H^2 + K) \tag{3.37}$$

$$G^0{}_i = 0 \tag{3.38}$$

$$G^i{}_j = a^{-2}(2H' + H^2 + K)\delta^i{}_j \tag{3.39}$$

In the above set of equations, H is called the Hubble parameter which depends on time.

$$H := \frac{\dot{a}}{a} \tag{3.40}$$

Using the equation,

$$\delta G(x') - \delta G(x) = L_\xi G, \tag{3.41}$$

we can write down how the perturbations transform.

$$\delta \bar{G}^\mu{}_\nu - \delta G^\mu{}_\nu = L_\xi^{(0)} G^\mu{}_\nu \tag{3.42}$$

$$\delta \bar{G}^0{}_0 - \delta G^0{}_0 = -({}^{(0)}G^0{}_0)'\xi^0 \tag{3.43}$$

$$\delta \bar{G}^0{}_i - \delta G^0{}_i = -({}^{(0)}G^0{}_0 - \frac{1}{3}{}^{(0)}G^k{}_k)\ \xi^0{}_{|i} \tag{3.44}$$

$$\delta \bar{G}^i{}_j - \delta G^i{}_j = -({}^{(0)}G^i{}_j)'\ \xi^0 \tag{3.45}$$

By inspection, we can immediately write down the gauge invariant form of the above equations.

$$\delta \tilde{G}^0_0 = \delta G^0_0 + ({}^{(0)}G^0{}_0)'(B - E') \tag{3.46}$$

$$\delta \tilde{G}^0_i = \delta G^0_i + ({}^{(0)}G^0{}_0 - \frac{1}{3}{}^{(0)}G^k{}_k)(B - E')_{|i} \tag{3.47}$$

$$\delta \tilde{G}^i_j = \delta G^i_j + ({}^{(0)}G^i{}_j)'(B - E') \tag{3.48}$$

The $(\sim)$ tilde symbol on top the tensors in the above equations denote gauge invariant variables. The above can be checked from the known transformation properties for each of the quantities involved. We note that the energy momentum tensor is proportional to the Einstein tensor. So the above construction just carries over for the energy momentum tensor. All we need to do is to write down the perturbations for the energy momentum tensor depending on the physical system we are dealing with. This information will go into the right hand side of the perturbed Einstein's equation. Another point to note is that, if the spatial part of the perturbed energy momentum tensor is proportional to identity, then we have just one independent variable to describe scalar perturbations. This last point will be mentioned explicitly in the next section when we encounter the example of a perfect fluid. Below we have listed the full perturbed Einstein's equation which will be used to study some physical examples. We will study all examples from hence forth in the $K = 0$ or spatially flat case. This is because at present, the observational evidence seems to be clearly pointing out that the universe is flat.

$$-3H(H\Phi + \Psi') + \nabla^2\Psi = 4\pi G a^2 \delta\tilde{T}^0_0 \tag{3.49}$$

$$(H\Phi + \Psi')_{,i} = 4\pi G a^2 \delta\tilde{T}^0_i \tag{3.50}$$

$$[(2H' + H^2)\Phi + H\Phi' + \Psi'' + 2H\Psi' + \frac{1}{2}\nabla^2 D]\delta^i_j - \frac{1}{2}\gamma^{ik} D_{,kj} = -4\pi G a^2 \delta\tilde{T}^i_j \tag{3.51}$$

where $D = \Phi - \Psi$ and $H = a'/a$. Also γ^{ik} in the last equation (3.51) is the background metric in 3D as given in equation (3.9). The above equations are written in conformal time. The (′) prime symbol denotes derivatives with respect to conformal time.

The above equations are the starting point for the analysis of interest to us. We make a couple of points here. We will be concerned only with scalar perturbations for most part of the thesis. We will briefly touch upon tensor perturbations in the context of inflation later.

3.4 Matter Perturbation

We will describe as to how to specialise these equations to two systems: a perfect fluid and a scalar field. The case of the scalar field will be discussed in detail in a later chapter.

3.4.1 Perfect Fluid

The symmetry properties have to be satisfied by the energy momentum tensor as well since it enters Einstein's equation on the RHS. This implies that the energy momentum tensor is of the perfect fluid form. Then we will consider perturbations to the energy momentum tensor, write down the Einstein's equation and study some simple solutions. A more detailed approach will follow in the subsequent chapters. For the latter, we will show how a particular coordinate system can also be used to describe perturbations.

The energy momentum tensor of a perfect fluid has the form

$$T^\mu_{\ \nu} = (\rho + p)u^\mu u_\nu - p\delta^\mu_\nu, \tag{3.52}$$

where, u^μ is the fluid velocity normalised to 1. It follows that u^μ is an eigenvector of T^μ_ν with 'eigenvalue' ρ. p, ρ are the pressure and energy density respectively. The equation of state that relates the pressure and energy density will characterise the physical system we have. For example,

$$p = 0, \quad \text{is for non relativistic matter also called pressureless dust} \tag{3.53}$$

$$p = \frac{1}{3}\rho, \quad \text{is for relativistic matter also called radiation.} \tag{3.54}$$

Then we can characterise the perturbations in terms of $\delta p, \delta\rho, \delta u^i$. We get the following equations in terms of gauge invariant functions in conformal time.

$$\delta\tilde{T}^0_{\ 0} = \delta\tilde{\rho} \tag{3.55}$$

$$\delta\tilde{T}^i_{\ j} = \tilde{\delta p}\delta^i_j \tag{3.56}$$

$$\delta\tilde{T}^0_{\ i} = \delta\tilde{u}_i a^{-1}(\rho_0 + p_0) \tag{3.57}$$

We note that the spatial part of the energy momentum tensor is a multiple of identity. This has an important consequence, it reduces the number of scalar perturbation variables to just one, either Φ or Ψ. This fact can be seen from the equation (3.51).

$$[(2H' + H^2)\Phi + H\Phi' + \Psi'' + 2H\Psi' + \frac{1}{2}\nabla^2 D]\delta^i_j - \frac{1}{2}\gamma^{ik}D_{,kj} = -4\pi G a^2 \delta T^{i(gi)}_j \tag{3.58}$$

We can go to the fourier space of this equation. The term $D_{,kj}$ will then become $-k_k k_j D$. Note that the other terms in this equation are proportional to the identity matrix. Therefore we can apply the operator

$$k_i k^j - \frac{1}{3}k^2\delta^j_i \tag{3.59}$$

on this equation. The terms proportional to identity matrix drop out and we get $D = \Phi - \Psi = 0$ implying $\Phi = \Psi$. Therefore, when the spatial components of the energy momentum tensor is proportional to the diagonal matrix, we have

$$\Phi = \Psi \tag{3.60}$$

From thermodynamics of a fluid, we have the following relation,

$$\delta p = c_s^2 \delta\rho + \tau\delta S \tag{3.61}$$

where

$$c_s^2 = \frac{\dot{p}_0}{\dot{\rho}_0} \tag{3.62}$$

is the speed of sound and τ is another parameter of the system. The subscript zero denotes background quantities. We can then add the corresponding parts of the energy momentum tensor and write it in terms of δS, the entropy perturbation. Then we will have to similarly introduce the velocity of sound on the LHS of the (0-0) component of the perturbed Einstein's equation. We get the following final gauge invariant equation.

$$\Phi'' + 3H(1 + c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2H' + (1 + 3c_s^2)(H^2)]\Phi = 4\pi G a^2 \tau \delta S. \tag{3.63}$$

Setting $\delta S = 0$ is what is called *adiabatic perturbations*. We have taken $K = 0$. We will make a change of variables and write down all the corresponding equations. First we define the variables z, θ

$$\Phi := 4\pi G(\rho_0 + p_0)^{-1/2} z = (4\pi G)^{-1/2}[(H^2 - H')a^{-2}]^{1/2} z \tag{3.64}$$

$$\theta := \frac{H}{a}[\frac{2}{3}(H^2 - H')]^{-1/2} \tag{3.65}$$

Then the differential equation (3.63) becomes

$$z'' - c_s^2\nabla^2 z - \frac{\theta''}{\theta} z = 0 \tag{3.66}$$

The gauge invariant equation for density perturbation is,

$$-3H(H\Phi + \Psi') + \nabla^2\Psi = 4\pi G a^2 \delta\tilde{T}^0_0 \tag{3.67}$$

The background energy density can be written as,

$$\rho = \frac{3H^2}{8\pi G} \tag{3.68}$$

We then divide the gauge invariant density perturbation by the background energy density. But for the perfect fluid we have $\Phi = \Psi$ as described in equation (3.60). Combining the two equations, give,

$$\frac{\delta\rho^{(gi)}}{\rho_0} = 2[3H^2]^{-1}[\nabla^2\Phi - 3H\Phi' - 3H^2\Phi] \tag{3.69}$$

So we will first solve for z using equation (3.66) and then obtain Φ from equation (3.64). Finally we get the density perturbations, using equation (3.69) corresponding to adiabatic perturbations.

Solutions in simple cases, K=0:

We will work in the fourier space of all the vairables. Then we observe that we get a second order ordinary equation for z and Φ. Since the equations are linear, we get solutions for each fourier mode of all the variables. This will help in comparing the wavelength of the perturbations with respect to the Hubble radius. We will give two simple examples of dust and radiation in their respective dominant epochs [20].

For dust: Here we are considering a non relativistic pressure-less dust. Therefore, $c_s = 0$. Let $r = k\eta$. The differential equation (3.66) reduces to

$$z'' - \frac{\theta''}{\theta} z = 0 \tag{3.70}$$

Then the solution can be written as

$$u(x,\eta) = A\ \theta(\eta) + B\ \theta(\eta) \int \frac{d\eta'}{\theta^2} \tag{3.71}$$

For this dust case, we can calculate θ from its definition in equation (3.65) . Then the solution of Φ is

$$\Phi(x,\eta) = C(x) + D(x)\eta^{-5} \tag{3.72}$$

Then in the fourier space, the solution for $\delta\rho$ is,

$$r >> 1 \text{ Sub-horizon,} \qquad \frac{\delta\rho}{\rho} \sim r^2 \tag{3.73}$$

$$r \ll 1 \text{ Super-horizon,} \qquad \frac{\delta\rho}{\rho} \sim \text{constant} \tag{3.74}$$

where we have neglected terms that are decaying with η.

For radiation: In this case, $c_s^2 = 1/3$ and $r = c_s k\eta$. The following is the analysis of the differential equation (3.63) in the various limits. Again, we obtain θ from its definition and then get the differential equation for z.

$$z'' + \left(\frac{k^2}{3} - \frac{2}{\eta^2}\right) z = 0 \tag{3.75}$$

Then using the definition for r, it reduces to,

$$r^2 z'' + (r^2 - 2)z = 0 \tag{3.76}$$

The solution for the above equation can be written as

$$z(r) = A\left(\frac{\sin x}{x} - \cos x\right) + B\left(\frac{\cos x}{x} - \sin x\right) \tag{3.77}$$

After obtaining Φ, we get the following for density perturbations,

$r \gg 1$ $(Sub - Horizon)$	$r \ll 1$ $(Super - Horizon)$	
$\Phi \sim cos(r)$	$\Phi \sim \text{constant}$	(3.78)
$\frac{\delta\rho}{\rho} \sim sin(r)$	$\frac{\delta\rho}{\rho} \sim \text{constant}$	(3.79)

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Chapter 4

Detailed Physical Analysis

In the previous chapter, we described the gauge invariant formalism and applied it to some simple cases and found how the perturbations evolve in the long and short wavelength limit. Now we wish to actually study the physical universe by accounting for the various components of the universe and studying the initial conditions necessary to solve the evolution equations. For this approach we will adopt a different strategy [17].

Each component contributing to the energy density is made up of particles which are either relativistic or non-relativistic. The energy momentum tensor of each component is described statistically in terms of the four momenta of the particles and the phase space distribution function $f(\vec{x}, \vec{p}, t)$. The particles are assumed to travel along geodesics. The background functions are the equilibrium Bose-Einstein or Fermi-Dirac distribution functions at the corresponding temperature. These distribution functions are independent of position and are isotropic. But when we include perturbations, the particles are assumed to be travel along the perturbed geodesics in between consecutive collisions. Here, the perturbation variables satisfy two types of equations, the Boltzmann equation and the Einsteins equation.

The Boltzmann equation gives an evolution equation for the perturbed distribution functions in terms of the metric perturbation variables. Therefore, each Boltzmann equation will introduce a new perturbation variable. So we further need two more equations to account for the two scalar metric perturbation variables. The final two equations for the scalar perturbation variables can be obtained from Einstein's equation. This then completes the full set of equations. When we go to the fourier space of these equations, we see that the equations are ordinary differential equations. Therefore, each mode is decoupled. Finally, we need to look for initial conditions to solve these equations.

This chapter is presented as follows. First, we will setup the Boltzmann equation for each component of the universe. The photons and neutrinos constitute the radiation component of the energy density. Baryons (collectively include both protons and electrons) and dark matter constitute the matter part of the energy density. We will not consider dark energy in this thesis. We will then reduce the Boltzmann equations to obtain differential equations for the perturbation variables. The perturbation variables are defined in a convenient manner depending on whether the component is matter or radiation. These differential equations are obtained in the fourier space. Then we obtain two differential equations from the Einstein's equation. For this we need the perturbed energy momentum tensor obtanined by performing integrals over perturbed distribution functions. After all the equations are setup, we will consider a certain limit to study the initial conditions. We will see that all the variables will then depend just on one variable for the initial condition. The origin of these fluctuations and the physically important quantities like power spectrum will be discussed in the next chapter.

All physically relevant quantities are integrals over functions defined on the phase space. We wish to account for perturbations to these distribution functions coming from geometry perturbations of the spacetime. So now the distribution functions will also depend on the direction of the momentum and also on the configuration space variables. To study the evolution of these distribution functions in the presence of interaction with other components and accounting for the perturbed geometry, we will work with the Boltzmann equation. The Figure.(4.1) provides the percentage of contribution of each component to the total energy density today. This diagram would have been different when we consider earlier epochs since the energy density scales as a function of the scale factor.

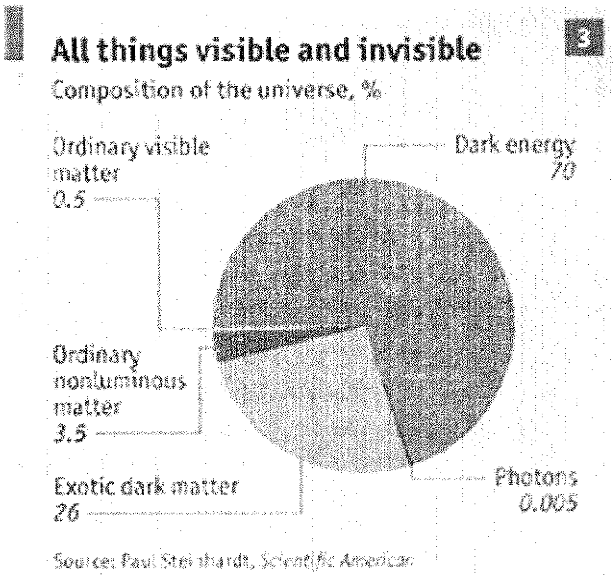

Figure 4.1: Cosmic Inventory at present [2]

Firstly, the energy momentum tensor of every statistically distributed component of the universe in the radiation epoch is given by the following:

$$T^{\mu}_{\ \nu}(\vec{x},t)|_i := \tilde{g}\int \frac{dP_1 dP_2 dP_3}{(2\pi)^3}\frac{1}{\sqrt{|g|}}\frac{P^{\mu}P_{\nu}}{P^0}f(\vec{x},\vec{p},t)|_i \tag{4.1}$$

$P^{\mu} = (P^0, P^1, P^2, P^3)$ is the four momentum of the particle and $\tilde{g}$ is the degeneracy of the phase space cell. The subscript 'i' is a label for every component we are considering. The above energy momentum tensor is the full general relativistic expression for each component. Therefore it holds in the case of the perturbed metric also. We will show that the components of the energy momentum tensor reduces to the familiar quantities even in the presence of perturbations. The metric we will consider is,

$$\left(\begin{array}{c|c} -(1+2\psi) & 0 \\ \hline 0 & a^2\delta_{ij}(1+2\phi) \end{array}\right) \tag{4.2}$$

As an example, we consider the (0-0) component of the energy momentum tensor for radiation. Now, starting from equation (4.1),

$$T^{0}_{\ 0}(\vec{x},t) := \tilde{g}\int \frac{dP_1 dP_2 dP_3}{(2\pi)^3}\frac{1}{\sqrt{|g|}}\frac{P^{0}P_{0}}{P^0}f(\vec{x},\vec{p},t) \tag{4.3}$$

The determinant is given by,

$$-(1+2\psi)[a^2(1+2\phi)]^3 = -a^6(1+2\psi+6\phi) \tag{4.4}$$

Using $P^2 = 0$, we get,

$$P^0 = p(1-\psi) \tag{4.5}$$

where

$$p^2 := g^{ij}P_iP_j \tag{4.6}$$

Similarly,

$$P^i = \frac{p^i}{a}(1-\phi) \tag{4.7}$$

Note, that the phase space integral is over momenta with lower indices. Inserting all the above expressions into equation (4.1), we get,

$$T^0_{\ 0}(\vec{x},t) = -\tilde{g}\int \frac{d^3p}{(2\pi)^3}\ E\ f(\vec{x},\vec{p},t) \tag{4.8}$$

The RHS of the above equation is equal to the negative of the energy density of radiation. So now, the perturbation to the above quantity will come only from the perturbation of the distribution function 'f'.

4.1 Boltzmann equation

As mentioned before, the Boltzmann equation [21] gives us a way to compute the evolution of the perturbation of the phase space distribution functions. The equation will relate how the distribution functions f_i of each component evolves with time taking into account the interaction with the other species. Therefore, by accounting for all the components, we will get a set of coupled differential equations. The Boltzmann equation will also incorporate metric perturbations.

$$\frac{df}{dt} = C[f] \tag{4.9}$$

where $C[f]$ accounts for interaction with the other components of the universe. Consider the LHS of the above equation. It can be expanded as the following,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial t} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\partial \hat{p}^i}{\partial t} \tag{4.10}$$

The terms

$$\frac{\partial x^i}{\partial t} \text{ and } \frac{\partial p}{\partial t} \tag{4.11}$$

will take into account the metric perturbation. This is because these two terms are calculated from the geodesic equation. The term

$$\frac{\partial f}{\partial \hat{p^i}} \frac{\partial \hat{p^i}}{\partial t} \tag{4.12}$$

can be neglected because each term in itself is a first order term, thereby making the term second order. Now we will setup and study the Boltzmann equation for each component.

4.1.1 Boltzmann Equation for Photons

The perturbation to the distribution function for radiation is most conveniently charactersied in the following way.

$$f = exp\left(\frac{p}{T(1+\Theta(x^i, \hat{p}^i, t))} - 1\right)^{-1} \tag{4.13}$$

where, $\Theta = \frac{\delta T}{T}$ is the temperature perturbation which depends on position and T is the temperature. Here we note that Θ is not a funtion of the magnitude of the three momentum because to zero order in Compton scattering, the photon's momentum changes only in direction and not in magnitude. $\hat{p}^i$ is the unit vector for the spatial part of the momentum vector. Then the distribution function can be expanded using Taylor series to first order,

$$f(p) = f^0(p) - p\frac{\partial f^0}{\partial p}\Theta \tag{4.14}$$

For photons, the most important interaction term is the Compton scattering by electrons.

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q'}) + \gamma(\vec{p'}) \tag{4.15}$$

The interaction term C[f] can now be written as,

$$\begin{aligned} C[f(p)] &= \frac{1}{p}\int\left\{\frac{d^3q}{(2\pi)^3 2E(q)}\frac{d^3q'}{(2\pi)^3 2E(q')}\frac{d^3p'}{(2\pi)^3 2E(p')}|M|^2\right. &(4.16)\\ &\times \left.(2\pi)^4\delta^4(p+q-p'-q')\ \left[f_e(\vec{q'})f(\vec{p'}) - f_e(\vec{q})f(\vec{p})\right]\right\} &(4.17)\end{aligned}$$

We can substitute for the energy, the following expressions:

$$E_\gamma(p) = p; \qquad E_e(q) = m_e + \frac{q^2}{2m_e} \tag{4.18}$$

Here we have considered the electrons to be non-relativistic. The RHS of the equation (4.10) gives the LHS of the full Boltzmann equation given below.

$$\dot{\Theta} + ik\mu\Theta + \dot{\phi} + ik\mu\psi = -n_e\sigma_T a\left(\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Theta_2\right) \tag{4.19}$$

The RHS of the full Boltzmann equation is got by computing the interaction term C[f]. The equation is written in terms of the fourier modes and we have not explicitly put any notation to say that all the terms are in the momentum space i.e. any term in the equation, say Θ, is actually $\tilde{\Theta}$. Here, one assumption is made, that the electron's velocity is in the direction of the temperature gradient i.e. v_b^i is along k^i. The equation is written in conformal time coordinate. This gives the term a multiplying the RHS. σ_T is the Thompson scattering cross section. n_e is the number density of the electrons. μ is defined as the following,

$$\mu = \frac{\vec{k}.\hat{p}}{k} \tag{4.20}$$

Then the velocity of the electrons can be written in terms of μ.

$$\vec{v_b}.\hat{p} = v_b\mu \tag{4.21}$$

In the above equation, $P_2(\mu)$ corresponds to the Legendre polynomial of degree 2 with argument μ. We also define the l^{th} moment of Θ as the following, (it appears in the above differential equation for Θ).

$$\Theta_l = \frac{1}{(-i)^l}\int_{-1}^{1} P_l(\mu)\Theta\mu\,\frac{d\mu}{2} \tag{4.22}$$

where, $P_l's$ are the Legendre polynomials. We have summed all possible particle spins and polarisation. Also in the above derivation, the scattering amplitude was taken to be a constant.

$$|M|^2 = 8\pi\sigma_T \;\; m_e^2 \tag{4.23}$$

Though the Compton scattering amplitude has an angular dependence, we neglect it since it makes the calculation easier. Also accounting for the angular dependence has a negligible contribution.

4.1.2 Boltzmann equation for Neutrinos

This follows directly from photon analysis. We just need to drop all interaction terms because neutrinos are weakly interacting and we will also not consider polarisation terms.

The temperature perturbation is written as N, the equivalent of Θ. Using the equation (4.19), we can write down,

$$\dot{N} + ik\mu N + \dot{\phi} + ik\mu\psi = 0 \tag{4.24}$$

Now we will go over to the matter sector.

4.1.3 Boltzmann Equation for Baryons

The important difference in this sector is that there are a couple of modifications to the approach we followed for radiation. Firstly, the RHS of the equation (4.10) will involve quantities that are appropriate for a massive particle. Hence, the magnitude of the three momentum will be replaced by the energy which is defined as $\sqrt{p^2 + M^2}$. Also, it is more convenient to describe perturbations through different quantities. Here we will define and study density and velocity perturbations. Baryons refer to both electrons and protons. We will assume that the Coulomb scattering which strongly couples the electron and the proton makes their bulk properties common, i.e.,

$$\delta_e = \delta_p = \delta_b \tag{4.25}$$

$$\vec{v_e} = \vec{v_p} = \vec{v_b} \tag{4.26}$$

Then the Boltzmann equation (4.9) can be written for both the electron and the proton.

$$\frac{df_e}{dt} = [e\gamma] + [ep] \tag{4.27}$$

$$\frac{df_p}{dt} = [ep] \tag{4.28}$$

The square bracket is a short form denoting the interacting terms. Then the LHS of the Boltzmann equation will be written in terms of the distribution function $f(\vec{x}, \vec{p}, t$ and its derivatives. We then perform the necessary phase space integration of the differential equations to get equations in density and velocity perturbation defined as follows. The number density is defined as the following:

$$n = \int \frac{d^3p}{(2\pi)^3} f \tag{4.29}$$

Then the perturbation in the number density is defined as the following,

$$n = n^{(0)}(1 + \delta_b) \tag{4.30}$$

Here, the number density and energy density perturbation definition coincide and is δ_b. The velocity perturbation is defined as:

$$v^i = \frac{1}{n_b}\int \frac{d^3p}{(2\pi)^3} f \frac{p\hat{p}_i}{E} \tag{4.31}$$

Then by following the procedure as in the photon case and taking moments to get all the equations, we get,

$$\dot{\delta_b} + ikv_b + 3\dot{\phi} = 0 \tag{4.32}$$

$$\dot{v_b} + \frac{\dot{a}}{a}v_b + ik\psi = \frac{-n_e\sigma_T a}{R}(v_b + 3i\Theta_1) \tag{4.33}$$

where, R is defined as the following

$$\frac{1}{R} = \frac{4\rho^0}{3\rho_b^0} \tag{4.34}$$

The RHS in equation (4.33) is due to the Coulomb scattering. The quantity R is related to the sound speed in this medium.

4.1.4 Boltzmann Equation for Cold Dark Matter (CDM)

CDM is an important component for studying structure formation. We will assume two properties of CDM which is motivated from observations. The first property is that it is non-relativistic and secondly it is non-interacting. So the RHS of the Boltzmann equation (4.9) is zero. It is more convenient to describe perturbations in terms of density perturbations and velocity perturbations. Since CDM is massive, we will take the distribution function to be a function of energy in place of the magnitude of the three momentum in the previous section (4.10). In a similar way, following the procedure used above for the baryons, and taking moments of the equation, we get equations for density and velocity perturbations. Also in the analysis, we will neglect terms of order $(p/E)^2$ and higher. We first define the number density.

$$n = \int \frac{d^3p}{(2\pi)^3} f \tag{4.35}$$

Then the velocity and density perturbations are defined in the following way as before,

$$n = n^{(0)}(1 + \delta_{DM}) \tag{4.36}$$
$$v^i = \frac{1}{n_{DM}} \int \frac{d^3p}{(2\pi)^3} f \frac{p\hat{p}_i}{E} \tag{4.37}$$
$$\tag{4.38}$$

Then using the above definitions in the Boltzmann equation gives

$$\dot{\delta_{DM}} + ikv + 3\dot{\phi} = 0 \tag{4.39}$$
$$\dot{v} + \frac{\dot{a}}{a}v + ik\psi = 0 \tag{4.40}$$

Again, we make the note that the equation is written in terms of a particular fourier mode of each variable and in conformal time η. The equations are identical to the ones describing the baryons with the difference being there are no interaction terms.

4.2 Perturbed Einstein's equation

Using the metric given in equation (4.2), we can calculate Einstein's equation to first order in the perturbation variables. The perturbed energy momentum tensor can be calculated from the definition of energy momentum tensor in equation (4.1).

The first equation is the time-time component of the Einstein's equation. We had shown earlier that the definition of (0-0) component of the energy momentum tensor as the negative of the energy density holds true even if we consider perturbations to the metric. Therefore, as an example we have for photons,

$$T^0_{\ 0}(\vec{x}, t) = -2 \int \frac{d^3p}{(2\pi)^3} p \left\{ f^0 - p\frac{\partial f^0}{\partial p}\Theta \right\} \tag{4.41}$$

After performing the integral using the background Bose-Einstein distribution in the above equation, we get,

$$T^0_{\ 0} = -\rho(1 + 4\Theta_0) \tag{4.42}$$

This will go into the RHS of Einsteins equation. A similar contribution will come from neutrinos as well. For the matter parts, we will just write it in terms of the energy densities directly. The full equation is then,

$$k^2\phi + 3\frac{\dot{a}}{a}\left(\dot{\phi} - \psi\frac{\dot{a}}{a}\right) = 4\pi G a^2[\rho_m \delta_m + 4\rho_r \Theta_{r,0}] \tag{4.43}$$

where, subscript m refers to both the baryonic and dark matter components and $r = 1, 2$ refers to photons and neutrinos respectively.

The second equation is the tracefree longitudinal part of the spatial part of the Einstein's equation. To get the longitudinal traceless part, we act on the Einstein's equation the following operator ,

$$\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij} \tag{4.44}$$

We then get the following equation after calculating the perturbed energy momentum tensor in a similar way to the previous calculation.

$$k^2(\phi + \psi) = -32\pi G a^2 \rho_r \Theta_{r,2} \tag{4.45}$$

Again, $r = 1, 2$ and stands for photons or neutrinos. The subscript m stands for matter components which are dark matter and baryons. We note that only radiation terms contribute to the the anisotropic stress part of the energy momentum tensor. The other components of Einstein's equations are redundant.
The first of the Einstein's equations gives an important insight (4.43). Consider first two terms involving ϕ.

$$k^2\phi + 3\frac{\dot{a}}{a}\dot{\phi} \tag{4.46}$$

The first term in real space will be the laplacian acting on the potential ϕ. This is the familiar term occurring in Newtonian gravitation in the Poisson equation. The second term accounts for the expansion of the universe. The second term becomes significant when the wavelength is of order Hubble radius or more. So we see that, when wavelengths are greater that the Hubble radius (H^{-1}), a general relativistic approach is necessary. This is the case in cosmology as most scales of physical interest were in the past outside the Hubble radius.

4.2.1 Summary of the perturbation equations

We will now put together all the equations, those obtained from Boltzmann equation and those from Einstein's equation.

$$\dot{\Theta} + ik\mu\Theta + \dot{\phi} + ik\mu\psi = n_e\sigma_T a\left(\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Theta_2\right) \tag{4.47}$$

$$\dot{\delta_{DM}} + ikv + 3\dot{\phi} = 0 \tag{4.48}$$

$$\dot{v} + \frac{\dot{a}}{a}v + ik\psi = 0 \tag{4.49}$$

$$\dot{\delta}_b + ikv_b + 3\dot{\phi} = 0 \tag{4.50}$$

$$\dot{v}_b + \frac{\dot{a}}{a}v_b + ik\psi = \frac{-n_e\ \sigma_T\ a}{R}(v_b + 3i\Theta_1) \tag{4.51}$$

$$\dot{N} + ik\mu N + \dot{\phi} + ik\mu\psi = 0 \tag{4.52}$$

$$k^2\phi + 3\frac{\dot{a}}{a}\left(\dot{\phi} - \psi\frac{\dot{a}}{a}\right) = 4\pi G a^2[\rho_m\delta_m + 4\rho_r\Theta_{r,0}] \tag{4.53}$$

$$k^2(\phi + \psi) = -32\pi G a^2\rho_r\Theta_{r,2} \tag{4.54}$$

So now we have setup eight equations in eight variables and would like to study their evolution. This leads us to a two important questions. What are the scales of physical interest and how to study them? What are the initial conditions for these equations? Both these questions will lead us to the study of inflation. These issues will be addressed in the following sections.

4.2.2 Importance of relevant scales

There is one important length scale in cosmology, the comoving Hubble radius $(aH)^{-1}$. The perturbations are expressed in terms of the fourier components in a sense to compare with this length scale. The Hubble radius changes with time. There is one very important puzzling problem that observations throw up. We notice that photons from causally disconnected regions have the same temperature. This is also called the horizon problem. Another way to look at it is the following.

Consider perturbations whose length scales are greater than the Hubble radius through the period of photon decoupling. Refer Fig.(4.2.2). Only length scales smaller than the Hubble radius can be affected by micro physical processes. Therefore, the perturbations on such large scales cannot be evened out to the give the isotropy of the CMB to a certain degree. We also see that length of cosmological importance i.e. for scales of $10^2 - 10^3$MPc, have only recently entered the Hubble radius, long after the decoupling of photons from matter. A way to quantify scales smaller than the Hubble radius is $k\eta << 1$. The inequality is understood as the following. We go to a sufficiently early time that η is very small and also consider wavelengths that are very large. The inequality is the statement that the comoving horizon is much smaller than the comoving wavelength.

Inflation is a plausible solution to the above problems. It is modelled in such way that scales that are causally disconnected were once small enough to be affected by physical processes. These scales then leave the Hubble radius, a time at which the initial conditions

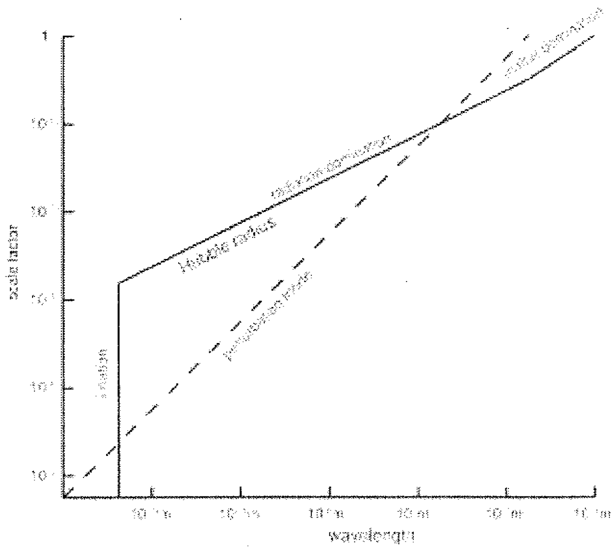

Figure 4.2: Comparison of length scales with the Hubble radius. Note that scales very early on were inside the Hubble radius. Then the scales left the Hubble radius during inflation and reentered in the radiation era - [3]

are set up. Then when they reenter the Hubble radius, they are affected by the physical processes and evolve to lead to the current structure.

4.3 Initial Conditions

We will first study the initial conditions of the differential equations provided in the summary and then study its connection to inflation [22]. Let us study the equations in the beginning of the radiation dominant epoch. In this epoch, we can simplify the equations for the following condition, $k\eta << 1$. Also during this time we can neglect higher order moments of the Θ variables. Under these approximations, we can reduce and impose constraints on the differential equations and reduce it to the dependence on one variable ϕ. Therefore, we neglect terms that are proportional to k. This says that we are dropping scales that will not be affected by the causal physics.

First we will consider the following:

$$\dot{\Theta} + ik\mu\Theta + \dot{\phi} + ik\mu\psi = n_e\sigma_T a\left(\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Theta_2\right) \tag{4.55}$$

$$\dot{\delta_{DM}} + ikv + 3\dot{\phi} = 0 \tag{4.56}$$

$$\dot{\delta_b} + ikv_b + 3\dot{\phi} = 0 \tag{4.57}$$

$$\dot{N} + ik\mu N + \dot{\phi} + ik\mu\psi = 0 \tag{4.58}$$

Neglecting higher moments and terms proportional to k, we get:

$$\dot{\Theta_0} = -\dot{\phi} \tag{4.59}$$

$$\dot{\delta_{DM}} = -3\dot{\phi} \tag{4.60}$$

$$\dot{\delta_b} = -3\dot{\phi} \tag{4.61}$$

$$\dot{N_0} = -\dot{\phi} \tag{4.62}$$

We note that all the variables are dependant only on one metric perturbation variable. Let us now consider one of the Einstein's equation,

$$k^2(\phi + \psi) = -32\pi G a^2 \rho_r \Theta_{r,2} \tag{4.63}$$

Neglecting higher moments of Θ, N, we get

$$\phi = -\psi \tag{4.64}$$

We can now consider the other Einstein's equation.

$$k^2\phi + 3\frac{\dot{a}}{a}\left(\dot{\phi} - \psi\frac{\dot{a}}{a}\right) = 4\pi G a^2[\rho_m\delta_m + 4\rho_r\Theta_{r,0}] \tag{4.65}$$

The matter terms can be left out since we are in the radiation era. Using $\phi = \psi$, we get,

$$3\frac{\dot{a}}{a}\left(\dot{\phi} - \phi\frac{\dot{a}}{a}\right) = 4\pi G a^2[4\rho_\gamma\Theta_{\gamma,0} + 4\rho_\nu\Theta_{\nu,0}] \tag{4.66}$$

where we have dropped the term proportional to k^2. During radiation, $a \sim \eta$. So the equation becomes,

$$\left(\frac{\dot{\phi}}{\eta} - \frac{\phi}{\eta^2}\right) = \frac{16}{3}\pi G a^2[\rho_\gamma\Theta_{\gamma,0} + \rho_\nu\Theta_{\nu,0}] \tag{4.67}$$

We multiply and divide by the total background energy density on the RHS. We get,

$$\left(\frac{\dot{\phi}}{\eta} - \frac{\phi}{\eta^2}\right) = \frac{16}{3}\pi G a^2\rho[\frac{\rho_\gamma}{\rho}\Theta_{\gamma,0} + \frac{\rho_\nu}{\rho}\Theta_{\nu,0}] \tag{4.68}$$

Using the background FRW equation, we get,

$$\frac{1}{\eta^2} = \frac{8}{3}\pi G a^2 \rho \tag{4.69}$$

Defining

$$f = \frac{\rho_\gamma}{\rho} \tag{4.70}$$

We then get,

$$\dot{\phi}\eta + \phi = 2(f\Theta_0 + (1-f)N_0) \tag{4.71}$$

But we can eliminate the zero moments of Θ and N from the first set of initial conditions in equation (4.62). Then we get a differential equation in ϕ.

$$\ddot{\phi}\eta + 4\dot{\phi} = 0 \tag{4.72}$$

The solution to the above equation is $\phi \sim \eta^x$ where x =0, -3. Neglecting the decaying mode, we get that ϕ is constant. Then from equation (4.71) we get,

$$\phi = 2(f\Theta_0 + (1-f)N_0) \tag{4.73}$$

We also make the assumption that at such times, the photons and neutrinos exhibit identical behaviour and

$$\Theta_0(k, \eta_i) = N_0(k, \eta_i) \tag{4.74}$$

where, η_i is some initial time. Then the initial condition on Θ_0 is,

$$\phi(k, \eta_i) = 2\Theta_0(k, \eta_i) \tag{4.75}$$

Now we will obtain the initial condition for the density perturbations for the matter components. We observe that both dark matter and baryons satisfy identical equations when considering initial conditions.

$$\dot{\delta_{DM}} = -3\dot{\phi} \tag{4.76}$$
$$\dot{\delta_b} = -3\dot{\phi} \tag{4.77}$$

But we also have,

$$\dot{\Theta_0} = -\dot{\phi} \tag{4.78}$$
$$\dot{N_0} = -\dot{\phi} \tag{4.79}$$

So we can write down,

$$\delta_{b/DM} = 3\Theta_0 + constant \tag{4.80}$$

In the next section we will describe what the consequences are for the constant to be zero or non-zero.

We will also be needing the initial condition for Θ_1 when we study origin of perturbations. Though we have neglected these moments for the above calculations, we will need the initial condition on Θ_1 when studying the spectrum of the perturbation variables that will be taken up in the next chapter. We will start from the following equations:

$$\dot{v} + \frac{\dot{a}}{a}v + ik\psi = 0 \tag{4.81}$$

$$\dot{v}_b + \frac{\dot{a}}{a}v_b + ik\psi = \frac{-n_e\,\sigma_T\,a}{R}(v_b + 3i\Theta_1) \tag{4.82}$$

The term $n_e\,\sigma_T$ in the last equation is very high during the early phase of the universe. So for this equation to remain meaningful, we have initially

$$v_{b/DM} + 3i\Theta_1 = 0 \tag{4.83}$$

So now we will need to determine the initial condition on Θ_1. This will be done by considering a gauge invariant variable in the fourier space,

$$v := ikB + \frac{\hat{k}^i T_i^0}{(\rho + P)a} \tag{4.84}$$

This variable reduces the velocity we have been using in the conformal gauge, i.e. setting $B = 0$ in the above equation. We can substitute for T_i^0 the term G_i^0 from Einstein's equation. We can also substitute for $\rho + P$ from the background equations. Then we get,

$$v = \frac{-2ik[\dot{\phi} + aH\phi]}{-4\dot{a}^2/a^2} \tag{4.85}$$

This can be simplified since we are interested in the case when ϕ = constant. We get,

$$v = \frac{ik\phi}{4(aH)} \tag{4.86}$$

Then we get the initial condition on Θ_1 using equation (4.83).

$$\Theta_1 = \frac{-k\phi}{6aH} \tag{4.87}$$

We have setup the initial conditions. There are two classes of initial conditions that will be studied in the next section. The issue of the origin of these perturbations will be studied in the next chapter.

4.3.1 Adiabatic and Isocurvature initial conditions:

We made a mention of adiabatic initial conditions in the early part of this thesis. Here we give a more explicit way to understand these. We begin with the equation for δ from above,

$$\delta = 3\Theta_0 + \text{constant} \tag{4.88}$$

The solution now falls into classes, those for which the constant is zero (adiabatic) and those for which it is non-zero (isocurvature). Isocurvature perturbations are not so important in terms of the physical quantities we are interested in and have only negligible contribution. That the constant is zero has a certain interpretation. It implies that adiabatic conditions are those for which there is a constant matter to photon ratio everywhere. The ratio is a constant both in space and time. The following equations are identical for both dark matter and baryonic matter. Consider,

$$\frac{n_{DM}}{n_\gamma} = \left(\frac{n_{DM}^{(0)}}{n_\gamma^{(0)}}\right)\left(\frac{1+\delta_{DM}}{1+3\Theta_0}\right). \tag{4.89}$$

The ratio of the background number densities is a constant because they both scale the same way. Therefore, the first fraction is a constant. This implies that the second fraction should be a constant. Writing to first order, we get

$$\delta = 3\Theta_0 \tag{4.90}$$

In the early part of the thesis, adiabatic conditions were defined as those for which $\delta S = 0$ in equation (3.63). This is an equivalent way of stating that matter and photons have a constant ratio of the number densities. This can be seen in the following way. The equations again hold for both baryon and dark matter. Consider the entropy per baryon,

$$\frac{S}{N_b} = \frac{S}{n_b a^3} = \frac{s}{n_b} \tag{4.91}$$

where 'S' is the total entropy and 's' is the entropy density.

$$\frac{s}{n_b} \sim \frac{T^3}{n_b} \sim \frac{\rho_r^{3/4}}{\rho_m} \tag{4.92}$$

since, the energy density in radiation scales T^4 and the number density of baryons is proportional to the energy density. We have not considered the perturbation of entropy per photon because it does not contribute to the entropy perturbation as both 's' and n_γ will scale the same way. Therefore we have,

$$\frac{\delta\rho_m}{\rho_m} = \frac{3}{4}\frac{\delta\rho_r}{\rho_r} \tag{4.93}$$

We note that when we use the condition that $\delta = 3\Theta_0$ and perform the phase space integrals to calculate energy density perturbations, we get the above condition. This shows that we can consider adiabatic initial conditions as either constant matter to photon number density ratio or a constant energy density perturbation as in the last equation above.

Chapter 5

Inflation

Though the standard model has explained the observations to a remarkable accuracy, there are certain problems with the model. We briefly discuss some of these problems:

- Horizon problem: We observe as isotropy in the CMB temperature to a certain degree. This leads to a problem. We see that photons that reached us from causally disconnectd regions have the same temperature.

- Flatness problem: We observe that the universe at present has curvature constant $K = 0$ to a very high accuracy i.e.
$$\frac{\rho}{\rho_{cr}} \simeq 1$$
where
$$\rho_{cr} := \frac{3H^2}{8\pi G}$$
So if we consider this ratio at early times at $T \sim 10^{15} GeV$, say, it is find tuned to the order of 10^{-57} [23]. The problem is that we need to understand as to why the initial condition is really small.

- Monopole problem: Certain Grand unified theories (GUT) predict the production of magnetic monopoles. But they are not observed.

- Cosmological constant problem [24]: Let us take that the dark energy contributes to 70% of the total energy today and that the component is described by the cosmological constant. Then the ratio of energy density of this component to its critical density is fine tuned to the order of 10^{-128} at the Planck scale.

- Matter-antimatter asymmetry: Let us assume that there were equal number of matter and anti-matter particles initially. But today we see only matter around us and no antimatter distributions. So there is some kind of asymmetry which we need to account for.

Inflation is considered a plausible model because it has been able to solve the first two of the problems [23, 25, 26]. Inflation is an accelerated state of expansion of the universe. We wish to have an inflationary model that has the following properties:

- Firstly, as mentioned before, the scale factor undergoes an accelerated expansion. This is just the statement that that the comoving hubble radius should decrease during the early phase of the universe. As the comoving hubble radius decreases, particles initially in causal contact are pushed apart to such large distances that they will not be able to communicate in the future. This is generally setup by taking H is almost a constant during inflation, i.e. inflation is characterised by an exponential expansion of the scale factor. This also says that the energy density is a constant as can be seen from the FRW equations.

- To solve for the horizon problem and the flatness problem, the scale factor should increase by a factor of atleast 10^{28} ([23]). This requires that inflation lasts for a time when the scale factor grows by at least 67 e-folds since $10^{28} \sim e^{67}$.

- Such a model has $p < -\frac{\rho}{3}$. This implies negative pressure as can be seen from the FRW equations. This is unlike any familiar matter or radiation components of the universe.

Any model that can account for these properties is a viable inflation model. The simplest model is that of a scalar field initially in a false vacuum and slowly rolling towards the true vacuum [27, 28, 29]. Refer Fig.(5). In this case, we can arrange for the potential to be greater than kinetic energy, which will be an identification of a negative pressure state. A further advantage of inflation is that it predicts a certain form of the spectrum for gravity waves when it is studied in the context of inflation. If detected, it could give a peak into physics at the scale of 10^{15} GeV[30], many orders of magnitude greater that the current particle accelerators. The description of inflation through a scalar field is described in the next section.

5.1 Scalar field Inflation

It is well accepted that inflation is driven by a scalar field. So we will need to write an energy momentum tensor for the scalar field and study its properties.

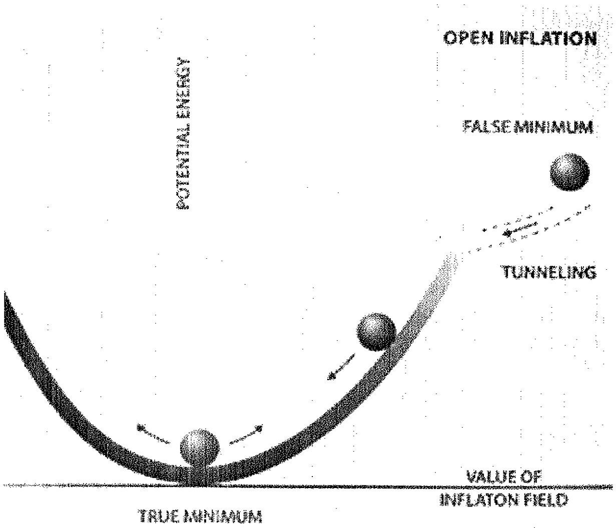

Figure 5.1: A description of the potential for the scalar field - [4]

The action for a scalar field is

$$S = \int [\frac{1}{2}\phi^{;\mu}\phi_{;\mu} - V(\phi)]\sqrt{|g|}d^4x \tag{5.1}$$

The the energy momentum tensor is given by,

$$T^{\mu}_{\ \nu} = \phi^{;\mu}\phi_{;\nu} - [\frac{1}{2}\phi^{;\mu}\phi_{;\mu} - V(\phi)]\delta^{;\mu}_{\ \nu}] \tag{5.2}$$

Immediately, we can write out the energy density and the pressure of the scalar field by calculating the (0-0) and (i-j) components of the energy momentum tensor. We get,

$$\rho = \frac{1}{2}\left(\frac{\partial\phi^0}{\partial t}\right)^2 + V(\phi^0) \tag{5.3}$$

$$P = \frac{1}{2}\left(\frac{\partial\phi^0}{\partial t}\right)^2 - V(\phi^0) \tag{5.4}$$

The equation of motion in conformal time can be written as the following,

$$\ddot{\phi}^{(0)} + 2aH\dot{\phi}^{(0)} + a^2V' = 0 \tag{5.5}$$

where V' is the derivative with respect the field. There are two quantities called slow roll parameters which describe the scalar field potential and its properties.

$$\epsilon_{sl} = \frac{d}{dt}\left(\frac{1}{H}\right) = \left(\frac{-\dot{H}}{aH^2}\right) \tag{5.6}$$

$$\delta_{sl} = \frac{1}{H}\frac{d^2\phi/dt^2}{d\phi/dt} = \frac{-1}{aH\dot{\phi}^{(0)}}[3aH\dot{\phi}^{(0)} + a^2V'] \tag{5.7}$$

Also we can calculate the perturbed equation of motion for the scalar field. Taking

$$\phi(\vec{x}, t) = \phi^{(0)}(t) + \delta\phi(\vec{x}, t) \tag{5.8}$$

We can neglect the term proportional to V'' since it is a slow varying field. Then we get the following equation,

$$\delta\ddot{\phi} + 2aH(t)\delta\dot{\phi} + k^2\delta\phi = 0 \tag{5.9}$$

We will analyse this equation after the next section.

5.2 Gravity Waves in inflation

In this section we will consider tensor perturbations of the metric in the context of inflation. The perturbed metric for tensor perturbations can be written in the following way,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_- & 0 \\ 0 & h_- & h_+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.10}$$

The spatial part of the metric is divergence free and traceless. When we plug this into Einstein's eqaution, we get identical equations for both the scalar functions h_+ and h_- in the metric,

$$\ddot{h} + \frac{2\dot{a}}{a}\dot{h} + k^2 h = 0 \tag{5.11}$$

Then we can raise these functions to operators and use our knowledge of the quantum harmonic oscillator in to work out its properties. We will be interested in a certain quantity called the power spectrum.

The power spectrum is the fourier transform of the auto-correlation function. It can be written in the following way:

$$\int dx \,\langle f(x) f(x+h)\rangle\, e^{ik.h} dh = G(k) \tag{5.12}$$

All integrals above are 3D integrals. The averaging over x is not explicitly given. We will explain the equation in the context of the discussion to gravity waves in the subsequent section. For our example, let us take the first to be f*(x). The equation can be rewritten in the following way,

$$\int dx\; dk'\; dk''\; \langle \tilde{g}*(k')\; \tilde{g}(k'')\rangle\;\; e^{ik'.x}\; e^{-ik''.x}\; e^{-ik''.h}\; dh = G(k) \tag{5.13}$$

Now all the integrals give delta functions. Now we need to input as to how to evaluate the two point function in the fourier space. For the metric perturbation variable h, for which we will calculate an expression similar to the above, we will assume that the averaging is done by describing h as a field. Then we can quantise each mode in terms of ladder operators i.e. each mode is described quantum mechanically like a quantum harmonic oscillator. We notice that the field h satisfies a free massless scalar field in an expanding background.

$$\ddot{h} + \frac{\dot{a}}{a}\dot{h} - \nabla^2 h = 0 \tag{5.14}$$

The first and third term is the free field equation of motion with the second term accounting for an expanding background. We can go over to the fourier space. Then we get the following equation:

$$\ddot{h}_{\vec{K}} + \frac{2\dot{a}}{a}\dot{h}_{\vec{K}} + k^2 h_{\vec{K}} = 0 \tag{5.15}$$

We now make a change of variables,

$$\tilde{h}_{\vec{K}} = \frac{a h_{\vec{K}}}{\sqrt{16\pi G}} \tag{5.16}$$

Then the equation of motion becomes,

$$\ddot{\tilde{h}}_{\vec{K}} + \left(k^2 - \frac{\ddot{a}}{a}\right)\tilde{h}_{\vec{K}} = 0 \tag{5.17}$$

Then we expand each mode in terms of the ladder operators,

$$\hat{\tilde{h}}_{\vec{K}}(k,\eta) = v(k,\eta)\hat{a}_{\vec{K}} + v^*(k,\eta)\hat{a}^{\dagger}_{\vec{K}} \tag{5.18}$$

Then substituting in the eq. (5.17), we get

$$\ddot{v} + \left(k^2 - \frac{\ddot{a}}{a}\right)v = 0 \tag{5.19}$$

To solve the above equation, we need the relation between a and η during inflation. For convenience, we will shift the origin of η to the end of inflation. This implies that η is negative during inflation. So we have

$$\eta \sim -\frac{1}{aH} \tag{5.20}$$

Then we get:

$$\dot{a} = a^2 H \sim \frac{-a}{\eta} \tag{5.21}$$

$$\frac{\ddot{a}}{a} \sim \frac{2}{\eta^2} \tag{5.22}$$

Then the solution to the above equation is

$$v = \frac{e^{ik\eta}}{\sqrt{2k}}\left[1 - \frac{i}{k\eta}\right] \tag{5.23}$$

Then the two point function defined by the following equation gives,

$$\langle \hat{\bar{h}}_{\vec{K}} \hat{\bar{h}}_{\vec{K'}} \rangle = |v|^2 (2\pi)^3 \delta^3(\vec{K} - \vec{K'}) \tag{5.24}$$

$$\langle \hat{h}^{\dagger}_{\vec{K}} \hat{h}_{\vec{K'}} \rangle = (2\pi)^3 P_h(k) \delta^3(\vec{K} - \vec{K'}) \tag{5.25}$$

where we have defined the power spectrum P_h as the following

$$P_h = 16\pi G \frac{|v|^2}{a^2} \tag{5.26}$$

Then for the above case, we know the solution for v. So we get

$$P_h(k) = \frac{16\pi G}{a^2} \frac{1}{2k^3\eta^2} \tag{5.27}$$

So using, equation (5.20) from above, the power spectrum can be reduced to

$$P_h(k) = \frac{8\pi G H^2}{k^3} \tag{5.28}$$

The function $P_h(k)$ is called the power spectrum of gravitational waves. It is called gravity waves because the equation of motion is the wave equation with a damping term. This leads to an important point. The function h exists only when the wavelength is greater than the hubble radius. When it enters the hubble radius, it gets damped. Therefore, only those wavelengths which are greater than the hubble radius at the time of photon decoupling, will have a signature in the CMB analysis.

Now, we can make a comment about the power spectrum of the perturbed scalar field. If we look at its equation of motion, it is identical to the equation (5.11). So we can follow the same procedure, without doing a change of variable. Therefore, the power spectrum for the scalar field is,

$$P_{\delta\phi} = \frac{H^2}{2k^3} \tag{5.29}$$

5.3 Relating scalar field perturbation to metric perturbation

We have given a brief description of the initial fluctuations in the scalar field. We would like to connect it to the spectrum of scalar perturbation variable ϕ which will then be the seed for initial conditions necessary for studying the radiation epoch. Consider the following conserved variable on super-horizon scales. This will link the two important variables [31, 32].

$$\zeta := \frac{ik_i \delta T^0_{\ i} H}{k^2(\rho + P)} - \psi \tag{5.30}$$

ζ defined above is gauge-invariant i.e. it can be written in terms of gauge invariant variables as given below:

$$\zeta = \frac{2}{3}\frac{H^{-1}\dot{\Phi} + \Phi}{1+w} + \Phi \tag{5.31}$$

This variable is also conserved when the wavelength of the fourier mode is greater than the hubble radius i.e.

$$\dot{\zeta} = 0 \tag{5.32}$$

We can evaluate the variable at horizon crossing and then post inflation and into the radiation epoch. Horizon crossing is characterised by $aH = k$. At horizon crossing during inflation, the first term in equation (5.30) is dominant, therefore the second term can be neglected.

$$\zeta|_{h.c.} = -aH\frac{\delta\phi}{\dot{\phi}_0} \tag{5.33}$$

where the input of pressure and energy density have been taken from the background energy momentum tensor. After inflation,

$$\begin{aligned} \frac{ik_i \delta T^0_{\ i} H}{k^2(\rho + P)} &= \frac{4ak\rho_r\Theta_1}{k^2(4\rho/3)} && (5.34) \\ \Rightarrow \zeta_{\text{post inflation}} &= \frac{-aH\Theta_1}{k} - \psi && (5.35) \\ &= \frac{-3}{2}\psi && (5.36) \end{aligned}$$

For the last equation, *we have used the initial condition for* Θ_1 *as mentioned in the section on initial conditions.* But

$$\psi = \frac{2}{3}aH\frac{\delta\phi}{\dot{\phi}_0} \tag{5.37}$$

from eq.(5.33). Therefore, the power spectra of the scalar perturbation $\delta\phi$ can be related to the power spectra of the metric perturbation ϕ,

$$P_\psi = P_\phi = \frac{8\pi G H^2}{9k^3\epsilon}|_{aH=k} \tag{5.38}$$

The last equation gives the primordial power spectrum of both the variables ϕ, ψ.

5.4 Summary of results and spectral indices

So after setting up the differential equations, we sought for their initial conditions. Then we wrote down the power spectrum of the various variables after quantising the variables. There are some more points to be made regarding the power spectrum.

A scale free power spectrum is one in which k^3P is a constant. Spectral indices characterise primordial deviations from the scale free spectrum. We can rewrite the power spectra in the following way

$$P_\phi = \frac{8\pi H^2}{9k^3\epsilon m_{planck}^2}|_{aH=k} = \frac{50\pi^2}{9k^3}\left(\frac{k}{H_0}\right)^{n-1}(\delta_H^2)\left(\frac{\Omega_m}{D_1(a=1)}\right)^2 \tag{5.39}$$

$$P_h = \frac{8\pi H^2}{k^3 m_{planck}^2}|_{aH=k} = A_T k^{n_T-3} \tag{5.40}$$

A_T and δ_h are amplitudes, D_1 is called the growth function. The important quantities are n, n_T. These are called the spectral indices. They characterise deviations from the scale free spectrum. $n = 1$ and $n_T = 0$ are the indices for a scale free spectrum. There is a relation between the spectral indices and the slow roll parameters. so spectral indices give an idea about the potential of the scalar inflaton. The next two equations can be got from taking the logarithmic derivative of the power spectrum with respect to log k of equation (5.40). We get,

$$n_T = -2\epsilon \tag{5.41}$$

$$n = 1 - 4\epsilon - 2\delta \tag{5.42}$$

The above two quantities are evaluated at the time of horizon crossing.

We will give an outline of how the relation between the spectral indices and the slow roll parameters is calculated.

5.4.1 Equation for n_T

Let us start from the definition of the spectral indices.

$$P_h = \frac{8\pi H^2}{k^3 m_{planck}^2}|_{aH=k} = A_T k^{n_T-3} \tag{5.43}$$

Then we have,

$$ln(P_h) = lnC + 2lnH - 3lnk = lnA_T + (n_T - 3)lnk \tag{5.44}$$

We have collected all the constant into the term C. We can now differentiate with respect to (ln k). We get,

$$\frac{d2lnH}{dlnk} - 3 = n_T - 3 \tag{5.45}$$

Since the derivatives should be evaluated at horizon crossing i.e. at $aH = k$. So we have,

$$\frac{dlnH}{dlnk} = \frac{dlnH}{d\eta}\frac{d\eta}{dlnk} \tag{5.46}$$

During inflation,

$$\eta \sim \frac{-1}{aH} \tag{5.47}$$

Therefore,

$$\frac{d\eta}{dlnk} = k\frac{d\eta}{dk} \tag{5.48}$$

$$= \frac{1}{k^2} \tag{5.49}$$

From the definition of ϵ ,

$$\epsilon = -\frac{\dot{H}}{aH^2} \tag{5.50}$$

Then we substitute the above and get,

$$\frac{dlnH}{dlnk} = \frac{dlnH}{d\eta}\frac{d\eta}{dlnk} \tag{5.51}$$

$$= \frac{dH}{Hd\eta}\frac{kd\eta}{dk} \tag{5.52}$$

$$= \frac{k}{H}\frac{1}{k^2}(-aH^2\epsilon) \tag{5.53}$$

$$= -\epsilon \tag{5.54}$$

Therefore,

$$n_T = -2\epsilon \tag{5.55}$$

5.4.2 Equation for n

Again we start from the power spectrum.

$$P_\phi = \frac{8\pi H^2}{9k^3\epsilon m_{planck}^2}|_{aH=k} = \frac{50\pi^2}{9k^3}\left(\frac{k}{H_0}\right)^{n-1}(\delta_H^2)\left(\frac{\Omega_m}{D_1(a=1)}\right)^2 \tag{5.56}$$

This can be reduced in the following way after taking the log on both sides,

$$2lnH - ln\epsilon = lnC + (n-1)lnK \tag{5.57}$$

All the constants have been collected in C above. We can now differential with respect to lnK. We get,

$$\frac{d[2lnH - ln\epsilon]}{dlnk} = n - 1 \tag{5.58}$$

The first term was evaluated when calculating the equation for n_T. So carrying that over, we get,

$$-2\epsilon \quad - \quad k\dot{\epsilon}\frac{d\eta}{dk} = n - 1 \tag{5.59}$$

$$-2\epsilon \quad - \quad \frac{\dot{\epsilon}}{\epsilon k^2} = n - 1 \tag{5.60}$$

Now,

$$\epsilon = \frac{-\dot{H}}{aH^2} \tag{5.61}$$

We can evaluate this using the background FRW equations for a scalar field and get,

$$\epsilon = \frac{4\pi G\dot{\phi}^2}{a^2H^2} := \frac{\alpha\dot{\phi}^2}{a^2H^2} \tag{5.62}$$

Also

$$\frac{d(ah)^{-1}}{d\eta} = -\frac{\dot{H}}{aH^2} + \frac{-\dot{a}}{a^2H} \tag{5.63}$$

$$= \epsilon - 1 \tag{5.64}$$

Then

$$\dot{\epsilon} = \frac{2\alpha\dot{\phi}\ddot{\phi}}{a^2H^2} + \frac{2\alpha\dot{\phi}^2(\epsilon - 1)}{aH} \tag{5.65}$$

From the definition of δ,

$$\delta = \frac{1}{H}\frac{d^2\phi/dt^2}{d\phi/dt} = \frac{\ddot{\phi}}{a\dot{\phi}H} - 1 \tag{5.66}$$

Substituting all of the above equation into the following equation gives,

$$-2\epsilon - \frac{\dot{\epsilon}}{\epsilon k^2} = n - 1 \tag{5.67}$$

$$-2\epsilon - 2(\delta + \epsilon) = n - 1 \tag{5.68}$$

Therefore, we get the equation

$$n = 1 - 4\epsilon - 2\delta \tag{5.69}$$

Chapter 6

Summary

Classical perturbation theory was applied to the FRW cosmological model under the framework of general relativity. We described the motivation for studying such a theory and then the formalism. The formalism began with the definition of perturbation. The gauge invariant approach was highlighted. This approach required the definition of gauge transformation and gauge invariance.

Then the perturbation of the geometry was studied with respect to metric perturbations. Here the decomposition of the perturbation into three classes, scalar, vector and tensor was discussed. The quantities were named depending on how the perturbation quantities behaved under 3D spatial transformations. Then the perturbation to the energy momentum tensor was discussed. Finally the perturbed Einstein's equation were written completely in terms of gauge-invariant quantities. Some simple solutions were discussed. The important class of perturbations necessary to study the types of perturbations we were interested in was the scalar perturbations

To model the actual universe, a more detailed approach was necessary. It was convenient at this point to introduce the conformal gauge and work in this particular coordinate system. The matter was described to be made up of particles that are statistically distributed according to phase space distribution function. The distribution functions in the presence of metric perturbations and interactions with other components was described by the Boltzmann equation. The equation is an evolution equation. It was studied for all the components in terms of their Fourier modes. Apart from the Boltzmann equation, we also have the perturbed Einstein' equations.

Once all the differential equations were obtained for the perturbation variables, we needed initial conditions. We took large wavelength and early time limit to study the initial conditions. We then obtained constraints among all the variables. We were then lead to a natural question as to how these perturbations originated?

The source for the origin of the fluctuations lead to the study of inflation. The power spectrum was calculated for the initial conditions by assuming that two point functions are calculated quantum mechanically.

Finally, the relation between the power spectrum of the scalar field fluctuations to that of the metric perturbations was described. We also mentioned important quantities called spectral indices that quantify the deviations of the power spectrum about the scale-free spectrum.

Issues not addressed: In this thesis we have not addressed the following points: evolution of the differential equations and relation to current matter structure, anisotropies in the CMB, alternative inflationary models [33, 34, 35, 36], quantum to classical transition of fluctuations and corrections from quantum gravity theories in the trans-planckian region.

$$\begin{tikzcd} & \text{Quantum Gravity Corrections[37, 38, 39]} & \\ \text{Inflationary Models} & \text{Future Directions} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \text{Large Scale Structure} \\ & \text{CMB Anisotropies[40]} & \end{tikzcd}$$

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