MULTIPLIERS OF SEGAL ALGEBRAS

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INTRODUCTION

If G is a locally compact abelian group, then there exists a nonnegative regular measure m on G, the so called Haar measure of G which is not identically zero and which is translation invariant. That is to say

$$m(E + x) = m(E)$$

for every $x \in G$ and every Borel set E in G. If m and m' are two Haar measures on G, then $m = \lambda m'$ where λ measure is a positive constant. If G is compact it is customary to normalize m so that $m(G) = 1$. If G is discrete any set consisting of a single point is assigned the measure 1.

Let G be a locally compact abelian group with character group Γ. Let dx and $d\gamma$ denote the elements of the normalized Haarm measures on G and Γ respectively. For $1 \leq p < \infty$, $L^p(G)$ is the Lebesgue space of equivalence classes of complex valued functions f on G such that

$$\| f \|_p = \left[\int_G |f(x)|^p \, dx \right]^{1/p} < \infty .$$

When $p = \infty$, $\| f \|_\infty$ denotes the essential supremum of $|f|$.

If $f, g \in L^1(G)$, the convolution is given by

$$f * g(t) = \int_G f(t-x) \, g(x) \, dx, \quad t \in G.$$

$M_{bd}(G)$ is the Banach space of all bounded regular complex-valued measures μ on G normed by

$$\|\mu\| = |\mu|(G) = \text{total variation of } \mu .$$

$M_{bd}(G)$ is then a Banach algebra and $L^1(G)$ is a closed ideal in $M_{bd}(G)$.

If $y \in G$, the translation operator τ_y is defined on a space of functions X on G by the formula

$$\tau_y f(x) = f(x-y), \quad x \in G, \quad f \in X.$$

A space A of functions on G is defined to be <u>translation-invariant</u> if, $f \in A$ implies $\tau_y f \in A$ for every $y \in G$. Let A and B be two translation invariant spaces of functions on G. A linear operator $T : A \to B$ is translation invariant if it commutes with translations, that is,

$$T \tau_x = \tau_x T \quad \text{for all } x \in G.$$

A bounded linear translation invariant operator from $A \to B$ is termed a <u>multiplier from A to B</u>. In the case of $L^1(G) \to L^1(G)$ there are various equivalent definitions. A multiplier on $L^1(G)$ is either a continuous linear operator T which commutes with translation operators or which commutes with convolutions. Notice that in some spaces translation operators may be defined even though convolutions are not. Another definition is the following: A function ϕ defined on the character group Γ is called a multiplier for $L^1(G)$ if $\phi \hat{f} \in [L^1(G)]^\wedge$ whenever $f \in L^1(G)$, where $\wedge$ denotes the Fourier transform.

A linear subspace $S(G)$ of $L^1(G)$ is called a <u>Segal Algebra</u> if the following four conditions are satisfied:

(a) $S(G)$ is dense in $L^1(G)$

(b) $S(G)$ is a Banach space under some norm $\| \ \|_s$ and

$$\| f \|_s \geq \| f \|_1 , \quad f \in S(G)$$

(c) Let $y \in G$. Then $\tau_y f \in S(G)$ for every $f \in S(G)$ and the mapping $y \longrightarrow \tau_y f$ is continuous from G into $S(G)$.

(d) $\| \tau_y f \|_s = \| f \|_s$ for all $f \in S(G)$ and all $y \in G$.

Various properties of Segal algebras are collected below in the form of lemmas.

LEMMA 0.1. <u>For every</u> $f \in S(G)$ <u>and arbitrary</u> $h \in L^1(G)$, <u>the vector-valued integral</u> $\int_G h(y) \tau_y f dy$ <u>exists as an element of</u> $S(G)$ <u>and</u>

$$\int_G h(y) \tau_y f \, dy = h * f.$$

<u>Moreover</u>

$$\| h * f \|_s \leq \| h \|_1 \| f \|_s .$$

It follows immediately that if $h \in S(G)$, then

$$\| h * f \|_s \leq \| h \|_1 \| f \|_s \leq \| h \|_s \| f \|_s$$

which shows that $S(G)$ is actually a Banach algebra and it is an ideal in $L^1(G)$.

LEMMA 0.2. <u>Let</u> μ <u>be a bounded complex valued measure on</u> G. <u>Then for any</u> $f \in S(G)$ <u>the vector-valued integral</u> $\int_G \tau_y f \, d\mu(y)$

exists as an element of $S(G)$ and

$$\int_G \tau_y f \, d\mu(y) = \mu * f$$

Further

$$\| \mu * f \|_s \leq \| \mu \| \| f \|_s .$$

Thus $S(G)$ is an ideal in $M_{bd}(G)$.

LEMMA 0.3. $S(G)$ contains all $f \in L^1(G)$ such that its Fourier transform $\hat{f}$ has compact support.

LEMMA 0.4. To every compact set $K \subset \Gamma$, there is a constant $C_K > 0$ such that every $f \in S(G)$ whose Fourier transform vanishes outside K satisfies the inequality

$$\| f \|_s \leq C_K \| f \|_1 .$$

LEMMA 0.5. Given any $f \in S(G)$, there is for every $\epsilon > 0$ a $\nu \in S(G)$ such that the Fourier transform $\hat{\nu}$ has compact support and

$$\| \nu * f - f \|_s < \epsilon .$$

LEMMA 0.6. Every Segal algebra has approximate units of L^1-norm 1.

The proof of these lemmas can be found in Reiter [20, pp.128-129] and [, pp.18-20, p.37].

If $S(G)$ is a Segal algebra on G, a multiplier on $S(G)$ is a bounded linear operator on $S(G)$ which commutes with translations. We denote by $M(S)$ the set of all multipliers on $S(G)$.

THEOREM 0.7. *Let* $T \in M(S)$. *If* $f, g \in S(G)$, *then*

$$T(f * g) = Tf * g = f * Tg$$

This is proved by Unni in [ ].

The following representation theorem for a multiplier on a Segal algebra has been proved by Unni [ ].

THEOREM 0.8. *If* $T \in M(S)$, *then there is a unique measure-measure* σ *such that*

$$Tf = \sigma * f$$

for all $f \in S(G)$.

We shall giv now give some examples of Segal algebras.

EXAMPLE. Let $1 \leq k < \infty$ and let T be the circle group. The Banach algebra $C^k(T)$ of all functions f with k continuous derivatives on T under the norm

$$\| f \|_{C^k} = \sum_{j=0}^{k} \frac{1}{\lfloor j} \| f^{(j)} \|_{\infty} ,$$

where $f^{(j)}$ denotes the jth derivative of f.

EXAMPLE 2. Let T be the circle group. Let

$$B = \left\{ f \in L^1(T) : \| f - D_N * f \|_1 \to 0 \text{ as } N \to \infty \right\}$$

where D_N is the Dirichlet kernel of order N defined by

$$D_N(t) = \sum_{j=-N}^{+N} e^{ijt} .$$

Then B is a Segal algebra with norm

$$\| f \|_B = \sup_{N \geq 1} \| D_N * f \|_1$$

EXAMPLE 3. The Banach algebra $L^A(R)$ of all functions $f \in L^2(R)$ which are absolutely continuous on the real line R with the derivative $f' \in L^2(R)$ under the norm

$$\| f \|_{L^A} = \| f \|_2 + \| f' \|_2 .$$

EXAMPLE 4. The space $W(R)$ of all continuous functions f on R for which $\sum_{k=-\infty}^{\infty} \max_{k \leq x \leq k+1} | f(x) | < \infty$, under the norm

$$\| f \|_s = \sup_{\alpha \in R} \sum_{k=-\infty}^{\infty} \max_{k \leq x \leq k+1} | f(x+\alpha) | .$$

EXAMPLE 5. The algebra $A^p(G)$ for $1 \leq p < \infty$ consisting of all those f in $L^1(G)$ such that its Fourier transform $\hat{f} \in L^p(\Gamma)$ with the norm

$$\| f \|_{A^p} = \| f \|_1 + \| \hat{f} \|_p .$$

EXAMPLE 6. Let ω be a real valued even continuous function on Γ such that

$$\omega(\eta + \gamma) \leq \omega(\eta)\, \omega(\gamma) \quad \text{for all } \eta, \gamma \in \Gamma .$$

For $1 \leq p < \infty$ we define $A^p_\omega(G)$ to be the set of all functions f in $L^1(G)$ such that $\hat{f}\omega \in L^p(\Gamma)$. We introduce a norm by

$$\| f \| = \| f \|_1 + \| \hat{f}\omega \|_p , \quad f \in A^p_\omega(G).$$

Then $A^p_\omega(G)$ is a Segal algebra (see Reiter [, pp.35]).

EXAMPLE 7. Consider a named family Ω of strictly positive functions ω on Γ which are measurable, summable with respect

to dY, together with the norm $N(\omega)$, satisfying the following conditions

1) For each $\omega \in \Omega$, $N(\omega)$ is finite and

$$0 < \int \omega dY \leq N(\omega) .$$

2) If $\omega \in \Omega$, then $1/\omega$ is locally L^{∞} on Γ.

3) For each positive number λ and each $\omega \in \Omega$, we have $\lambda\omega \in \Omega$ and

$$N(\lambda\omega) = \lambda N(\omega) .$$

4) If $\omega_1, \omega_2 \in \Omega$, then $\omega_1 + \omega_2 \in \Omega$ and

$$N(\omega_1 + \omega_2) \leq N(\omega_1) + N(\omega_2).$$

5) Ω is complete under the norm N. That is, for any sequence $\{\omega_n\}_{n=1}^{\infty} \subset \Omega$ such that $\sum_1^{\infty} N(\omega_n) < \infty$, $\omega = \sum_1^{\infty} \omega_n$ is in Ω and

$$N(\omega) \leq \sum_{n=1}^{\infty} N(\omega_n).$$

Let Ω_0 be the subset of Ω consisting of those ω such that $N(\omega) = 1$.

For a fixed p satisfying $1 < p < \infty$, we set

$$\omega' = \frac{1}{\omega^{p-1}} .$$

For each $\omega \in \Omega_0$, we consider the Banach spaces $L^p_{\omega'}(\Gamma)$ and $L^q_{\omega}(\Gamma)$ of functions measurable on Γ having the respective norms

$$\| f \|_{L^p_{\omega'}} = \left\{ \int_{\Gamma} |f|^p \omega' \, dY \right\}^{1/p}$$

and

$$\|\phi\|_{L^q_\omega} = \left\{ \int_\Gamma |\phi|^q \omega \, d\gamma \right\}^{1/q}$$

and set

$$\Lambda^p(\Gamma) = \bigcup_{\omega \in \Omega_0} L^p_\omega(\Gamma)$$

and set

$$\Theta^q(\Gamma) = \bigcap_{\omega \in \Omega_0} L^q_\omega(\Gamma)$$

where $1/p + 1/q = 1$.

Then we have the following theorem of Beurling [ ] .

THEOREM 0.11. <u>Let</u> $1 < p < \infty$ <u>and</u> $q = p/p-1$, <u>then both</u> $\Lambda^p(\Gamma)$ <u>and</u> $\Theta^q(\Gamma)$ <u>are Banach spaces if they are supplied with the norms given by</u>

$$\| f \|_{\Lambda^p} = \inf_{\omega \in \Omega_0} \| f \|_{L^p_\omega}$$

<u>and</u>

$$\| \phi \|_{\Theta^q} = \sup_{\omega \in \Omega_0} \| \phi \|_{L^q_\omega}$$

<u>respectively.</u>

Let $S^p(G) = \{ f \in L^1(G) : \hat{f} \in \Lambda^p(\Gamma) \}$. We introduce a norm on $S^p(G)$ by setting

$$\| f \|_s = \| f \|_1 + \| \hat{f} \|_{\Lambda^p(\Gamma)}, \quad f \in S^p(G).$$

Then if G is a locally compact non-discrete abelian group and $1 \leq p < \infty$, $S^p(G)$ is a Segal algebra (See Unni [ ]).

EXAMPLE 3. Let G be a locally compact abelian group with character group Γ. Let α be a locally bounded function on Γ with $\alpha(\gamma) \geq 1$ for all $\gamma \in \Gamma$. Let $S(\alpha) = \{ f \in L^1(G) : \lim \hat{f}(\gamma)\alpha(\gamma) = 0$, i.e. for every $\epsilon > 0$ there exists a compact subset K of Γ such that

$$|\hat{f}(\gamma)\alpha(\gamma)| < \epsilon, \ \gamma \notin K \} .$$

Then $S(\alpha)$ is a Segal algebra with norm

$$\| f \|_s = \sup_\gamma |\hat{f}(\gamma)\alpha(\gamma)| + \| f \|_1 .$$

In this thesis we study multipliers on some of the Segal algebras included in the examples above, particularly the algebras $A^p(G)$, $A^p_\omega(G)$, $S^p(G)$, $C^k(T)$ and $W(R)$ and characterise various spaces of multipliers. This introduction is followed by the characterisation of the space of multipliers on $A^p(G)$, when $p > 2$, answering the questions raised by Larsen [ ] and Lai [ ]. The principal results are contained in Theorems 1.13, 1.14 and 1.18 Chapter two is concerned with the Segal algebras $A^p_\omega(G)$ and $S^p(G)$ when G is a compact abelian group. We prove that there is a continuous algebra isomorphism of $M(A^p_\omega(G))$ onto $L^q(\Gamma)$ when $1 \leq p \leq 2$ and a continuous linear isomorphism of $M(A^p_\omega(G))$ onto continuous linear isomorph the dual of a certain Banach space of continuous function if $p > 2$. Analogous results are also obtained for $S^p(G)$. In Chapter three we characterise the multiplier spaces on the segal algebras $W(R)$, $C^k(T)$ and $S(\alpha)$.

Chapter four discusses bipositive and isometric isomorphisms of multiplier algebras. In particular we prove that for two groups

G_1 and G_2, a bipositive isomorphism of the multiplier spaces of the segal algebras induces a topological isomorphism between G_1 and G_2, thereby generalising the results of Gaudry [ ] and Tewari [ ]. In the case of $A_\infty^p(G)$, an isometric isomorphism of the multiplier algebras also induces a topological isomorphism of the underlying groups. In Chapter five we discuss the characterisations of the multipliers from $S(G)$ into $L^1(G)$. The sixth chapter is concerned with the problem of the restriction of a multiplier to a subset of the dual space Γ. We have extended the result of Pigno [ ] concerning the restriction of multipliers from $L^1(G)$ to $L^{p_1} \cap L^{p_2}(G)$ to the case when both the exponents p_1 and p_2 are greater than two. In the last chapter, we have answered the question raised by Larsen [ ] as to whether there exist non-zero closed translation invariant subspaces X of $L^p(G)$ satisfying $X \cap L^1(G) = \{0\}$.

CHAPTER I

MULTIPLIERS ON $A^p(G)$ ALGEBRAS

In this chapter we shall study the multipliers from $L^1(G)$ to $A^p(G)$ as also the multipliers of $A^p(G)$.

Throughout this chaper G will be a locally compact abelian group. The space $A^p(G)$ is the subset of $L^1(G)$ consisting of those functions f whose Fourier transform $\hat{f}$ belongs to $L^p(\Gamma)$. Given the norm

$$\|f\|_s = \|f\|_1 + \|\hat{f}\|_p \quad , \quad f \in A^p(G)$$

it turns out that $A^p(G)$, $1 \leq p < \infty$, is a dense ideal in $L^1(G)$ and forms a semisimple commutative Banach algebra under convolution and in fact is a Segal algebra.

We shall now introduce some notation and terminology. Let $C_c(G)$ denote the space of all continuous functions on G with compact support and let $C_o(G)$ be the space of continuous functions on G vanishing at infinity. For a fixed p satisfying $1 \leq p < \infty$, let q denote the conjugate index of p given by $1/p + 1/q = 1$.

Let $E = C_o(G) \times L^q(\Gamma)$. Then the linear space E can be made into a normed linear space by introducing either of the equivalent norms:

(1) $$\|(f, g)\|_{max} = \max(\|f\|_\infty, \|g\|_q)$$

(2) $\|(f,g)\|_{sum} = \|f\|_\infty + \|g\|_q$

where $f \in C_0(G)$, $g \in L^q(\Gamma)$.

Let H_q be the closure in E of $\{ (\check{f}, -\hat{f}) , f \in A^2(G) \}$ where $\check{f}$ is the reflexive function given by $\check{f}(x) = f(-x)$, for all $x \in G$ and $\hat{f}$ is the Fourier transform of f.

The space $C_0(G) \underset{H_q}{\vee} L^p(\Gamma)$ is the quotient space $K_q = E/H_q$ with the quotient norm, more explicitly the norm is given by

(3) $\|\{g,h\}\| = \inf \{ \|g'\|_\infty + \|h'\|_q : \{g',h'\} = \{g,h\} \mod H_q \}$

The space $C_0(G) \underset{H_q}{\wedge} L^q(\Gamma)$ is the space H_q, the norm being restriction to H_q of the maximum norm (1) on $C_0(G) \times L^q(\Gamma)$.

Let $A_p(G)$ the set of all functions u which can be expressed as

$$u = \sum_{i=1}^{\infty} f_i * g_i$$

where $f_i \in A^p(G)$, $g_i \in C_q = \{ g \in C_c(G) : \{\check{g}, \hat{g}\} \in K_q \}$ and $\sum_{i=1}^{\infty} \|f_i\|_s \|g_i\| < \infty$ where $\|g_i\| = \|\{\check{g}_i, \hat{g}_i\}\|$.

Define a norm on $A_p(G)$ by

$$|||u|||_p = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_s \|g_i\| \right\}$$

where the infimum is taken over all functions $f_i \in A^p(G)$, $g_i \in C_q$ for the representation of $u = \sum_{i=1}^{\infty} f_i * g_i$ as an element of $A_p(G)$.

The following result is then valid.

THEOREM 1.1. [ ] For $1 < p \leq 2$, the space $A_p(G)$ is a dense linear subspace of $C_0(G)$ and is a Banach space with respect to the norm $|||\ \ |||_p$ and the topology thus defined is stronger than both the top uniform topology and the topology induced from $A^p(G)$.

Let $M(L^1, A^p)$ denote the space of multipliers from $L^1(G)$ to $A^p(G)$ and $M(A^p)$ the space of multipliers of $A^p(G)$.

Concerning multipliers, the following results were proved by Lai.

THEOREM 1.2. [ ]. For $1 < p \leq 2$, the multiplier space $M(A^p)$ is isometrically isomorphic to the topological dual $[A_p(G)]^*$ of $A_p(G)$.

THEOREM 1.3. [ ]. For $1 < p \leq 2$, $M(L^1, A^p) \simeq A^p(G)$ where $\simeq$ denotes the isometric isomorphism between the two spaces.

The proof of Theorem 1.2 makes use of the following result of Liu and Rooij.

THEOREM 1.4 [ ] . *For* $1 < p \leq 2$,

$$[C_0(G) \underset{H_q}{\vee} L^q(\Gamma)]^* \cong A^p(G) .$$

Theorems 1.2, 1.3 and 1.4 were left open when $p > 2$. The question whether $A^p(G)$ is actually a dual space for $p > 2$ was raised by Larsen [ ]. He also asked for a description of $M(A^p)$ similar to Theorem 1.2 when $p > 2$ [15, p.231]. Theorem 1.3 for $p > 2$ was left open by Lai [12, p.602]. We shall prove below the results analogous to Theorems 1.2, 1.3 and 1.4 when $p > 2$.

We will often make use of the following well known results.

THEOREM 1.5 [ ]. *Let* G *be a locally compact abelian group. Suppose* T *is a multiplier from* $L^1(G)$ *into itself. Then there exists a unique measure* $\mu \in M_{bd}(G)$ *such that*

$$Tf = \mu * f$$

for each $f \in L^1(G)$

THEOREM 1.6. [ ] . *If* $\{e_\alpha\}$ *is an approximate identity for* $L^1(G)$ *with* $\|e_\alpha\| = 1$ *for all* α *and* $\hat{e}_\alpha$ *has compact support for each* α, *then* $\hat{e}_\alpha$ *converges uniformly to one on compact subsets of* Γ.

Let A be a Banach algebra. A Banach A-module is a Banach space V which is an A-module in the algebraic sense and in which the following norm inequality is satisfied:

$$\|av\| \leq \|a\| \, \|v\| , \quad a \in A, \; v \in V$$

Here $\|a\|$ denotes the norm in A and $\|av\|$ and $\|v\|$ are norms in V.

If A is a Banach algebra and V an A-module, then V is said to be an <u>essential A module</u> if AV, the linear manifold spanned by $\{av : a \in A, v \in V\}$ is dense in V.

We shall now state some results that are needed in the course of our proofs.

THEOREM 1.7. [ ] . <u>If A is a Banach algebra with bounded approximate identity $\{i_j\}$ and V is an A module then V is essential if and only if $i_j v \to v$ for every $v \in V$</u> .

THEOREM 1.8 [ ] . <u>Let A be a Banach algebra with bounded approximate identity and W is an A-module. If the A module W^* is essential, then W is also essential.</u>

It follows from Theorem 1.7 that every Segal algebra is an essential $L^1(G)$ module.

LEMMA 1.9. [ ] . <u>Let A be a normed algebra with bounded approximate identity $\{e_\alpha\}$ such that $\|e_\alpha\|_A \leq 1$ and B a normed A module such that $x * e_\alpha \to x$ for all $x \in B$, limit being taken over α. Then there is a natural isometry,</u>

$$M(A, B^*) \cong B^*$$

<u>where B^* denotes the topological dual of B.</u>

THEOREM 1.10. $[C_0(G) \underset{H_q}{\vee} L^q(\Gamma)]^* \simeq M_{bd}(G) \underset{J}{\wedge} L^p(\Gamma)$ for $1 < q < \infty$, $1/p + 1/q = 1$, where

$$J = \left\{ (\mu, h) \in M_{bd}(G) \times L^p(\Gamma) : \int_G \hat{f}\, d\mu = \int_\Gamma \hat{f} h\, d\gamma \right.$$
$$\left. \underline{\text{for all}} \quad f \in A^1(G) \right\}$$

LEMMA 1.11. [ ] . If $\mu \in M_{bd}(G)$ then $\mu \in L^1(G)$ if and only if the mapping $G \to M_{bd}(G)$ given by $s \to \tau_s \mu$ is continuous.

We now define a linear subspace $B_p(G)$ of $M_{bd}(G)$ by

$$B_p(G) = \left\{ \mu \in M_{bd}(G) : \hat{\mu} \in L^p(\Gamma) \right\}$$

If we set

$$\|\mu\|_{B_p} = \|\mu\| + \|\hat{\mu}\|_p$$

then $B_p(G)$ becomes a banach space under the norm $\| \ \|_{B_p}$. Our characterization of the space of multipliers from L^1 to A^p is given by

THEOREM 1.12. [ ] . If $p > 2$, then $M(L^1, A^p)$ is isometrically isomorphic to $B_p(G)$

REMARK. When $p > 2$, there are measures μ whose Fourier Stieltjes transform belongs to $L^p(\Gamma)$ but μ is not absolutely continuous.

PROOF OF THEOREM 1.12. Let $\{e_\alpha\}$ be an approximate identity for $L^1(G)$ with $\|e_\alpha\|_1 = 1$ for all α, $\hat{e}_\alpha$ has compact support for all α. Suppose $\mu \in B_p(G)$. If $f \in L^1(G)$, we know that $\mu * f \in L^1(G)$ and $\hat{\mu}\hat{f} \in L^p(\Gamma)$ since $\hat{\mu} \in L^p(\Gamma)$ and $\hat{f}$ is bounded. Define $T_\mu : L^1(G) \longrightarrow L^1(G)$ by

$$T_\mu(f) = \mu * f, \quad f \in L^1(G).$$

Then T_μ is a well defined linear map which is moreover bounded since

$$\|T_\mu f\|_S = \|\mu * f\|_S = \|\mu * f\|_1 + \|\hat{\mu}\hat{f}\|_p$$

$$\leq \|\mu\| \, \|f\|_1 + \|\hat{\mu}\|_p \, \|\hat{f}\|_\infty$$

$$\leq \|f\|_1 \left[\|\mu\| + \|\hat{\mu}\|_p\right] = \|f\|_1 \, \|\mu\|_{B_p}$$

Hence

$$(4) \qquad \|T_\mu\| \leq \|\mu\|_{B_p}.$$

Since T_μ is also translation invariant, it follows that $T_\mu \in M(L^1, A^p)$.

The mapping

$$\Lambda : B_p(G) \longrightarrow M(L^1, A^p)$$

given by

$$\Lambda(\mu) = T_\mu$$

is a well defined linear map.

First we show that Λ is onto. For this purpose, let $T \in M(L^1, A^p)$. The inequalities

(5) $$\|Tf\|_1 \leq \|Tf\|_S \leq \|T\|\|f\|_1$$

imply that T can be extended to a bounded linear translation invariant map of $L^1(G)$ into itself. Then by Theorem 1.4 there exists a $\mu \in M_{bd}(G)$ such that

(6) $$Tf = \mu * f \quad \text{for all} \quad f \in L^1(G) .$$

Now since $e_\alpha \in A^p(G)$, using (5) and (6) we have

$$\|\mu * e_\alpha\|_1 + \|\hat{\mu}\hat{e}_\alpha\|_p = \|Te_\alpha\|_S \leq \|T\|\|e_\alpha\|_1 = \|T\|$$

This implies that

(7) $$\|\hat{\mu}\hat{e}_\alpha\|_p \leq \|T\| \quad \text{for all } \alpha$$

Then by Alaoglu's theorem [, p.424] there exists a subnet $\{e_{\alpha_\beta}\}$ and an element $g \in L^p(\Gamma)$ such that $\hat{\mu}\hat{e}_{\alpha_\beta} \to g$ weakly in $L^p(\Gamma)$. This means that for each $\chi \in L^q(\Gamma)$ we have

(8) $$\int_\Gamma \hat{\mu}\hat{e}_{\alpha_\beta}\chi \, d\gamma \longrightarrow \int_\Gamma g\chi \, d\gamma$$

Now consider $\chi \in C_c(\Gamma)$. Since $\hat{e}_{\alpha_\beta}$ converges uniformly to one on compact subsets of Γ by Theorem 1.6, we obtain

$$\lim_\beta \int_\Gamma \hat{\mu}\hat{e}_{\alpha_\beta}\chi \, d\gamma = \int_\Gamma \hat{\mu}\chi \, d\gamma .$$

From this and (8) we get

$$\cancel{\int_\Gamma g\chi \, d\gamma = \int_\Gamma \hat{\mu}\chi \, d\gamma}$$

(9) $$\int_\Gamma g\chi\, d\gamma = \int_\Gamma \hat{\mu}\chi\, d\gamma .$$

Since (9) is valid for every continuous function χ with compact support on Γ we conclude that $\hat{\mu} = g$ a.e. on Γ. Thus $\hat{\mu} \in L^p(\Gamma)$ and so $\mu \in B_p$. Hence $T = \Phi\mu$ and Λ is onto.

Next we show that $\|T_\mu\| = \|\mu\|_{B_p}$ which will prove that Λ is an isometry. Since $\{e_\alpha\}$ is an approximate identity for $L^1(G)$, we have

$$\|\mu * f * e_\alpha - \mu * f\|_1 \longrightarrow 0 \ , \ f \in L^1(G)$$

Then if $f \in C_c(G)$, there exists a subnet $\{e_{\alpha_\beta}\}$ of $\{e_\alpha\}$ such that

$$\mu * f * e_{\alpha_\beta} \longrightarrow \mu * f \quad \text{pointwise on } G \text{ so that}$$

whithmaintin

(10) $$\lim_\beta \mu * f * e_{\alpha_\beta}(e) = \mu * f(e)$$

which is the same as

(11) $$\lim_\beta \int_G (\mu * e_{\alpha_\beta})(x)\, \check{f}(x)\, dx = \int_G \check{f}(x)\, d\mu(x) .$$

Moreover

$$\|\mu * e_{\alpha_\beta}\|_1 \le \|\mu * e_{\alpha_\beta}\|_S \le \|T\mu\|\, \|e_{\alpha_\beta}\|_1 \le \|T\| .$$

We can then find a subnet $\{e_{\alpha_{\beta_\gamma}}\}$ of $\{e_{\alpha_\beta}\}$ and a measure ν

such that $\mu * e_{\alpha_{\beta_\gamma}}$ converges weakly to ν. Hence if $f \in C_c(G)$, we have

$$(12) \qquad \lim_{\gamma} \int_G \mu * e_{\alpha_{\beta_\gamma}}(x) \tilde{f}(x)\, dx = \int_G \tilde{f}(x)\, d\nu(x)$$

From (11) and (12) we get

$$\int_G \tilde{f}(x)\, d\nu(x) = \int_G \tilde{f}(x)\, d\mu(x), \quad f \in C_c(G).$$

Since $C_c(G)$ is dense in $C_0(G)$, we have $\nu = \mu$. We have thus proved that there exists a subnet $e_{\alpha_\beta} * \mu$ of $e_\alpha * \mu$ which converges weakly to μ in $M_{bd}(G)$.

Now given $\epsilon > 0$, since $\hat{\mu} \in L^p(\Gamma)$, there exists a compact subset K of Γ such that

$$\int_{\Gamma \setminus K} |\hat{\mu}(\gamma)|^p d\gamma < \frac{\epsilon^p}{2^{2p+1}}$$

Since $\hat{e}_\alpha$ converges to one uniformly on K,

$$\int_K |\hat{\mu}(\gamma)\, \hat{e}_\alpha(\gamma) - \hat{\mu}(\gamma)|^p d\gamma \text{ converges to zero.}$$

Hence there exists a finite subset B of indices such that for all $\alpha \notin B$

$$\int_K |\hat{\mu}(\gamma)\hat{e}_\alpha(\gamma) - \hat{\mu}(\gamma)|^p d\gamma < \varepsilon^p/2^{p+1}$$

Now

$$\int_\Gamma |\hat{\mu}(\gamma)\hat{e}_\alpha(\gamma) - \hat{\mu}(\gamma)|^p d\gamma = \int_K |\hat{\mu}(\gamma)\hat{e}_\alpha(\gamma) - \hat{\mu}(\gamma)|^p d\gamma$$

$$+ \int_{\Gamma\setminus K} |\hat{\mu}(\gamma)\hat{e}_\alpha(\gamma) - \hat{\mu}(\gamma)|^p d\gamma$$

$$< \varepsilon^p/2^{p+1} + 2^p \int_{\Gamma\setminus K} |\hat{\mu}(\gamma)|^p d\gamma < \varepsilon^p/2^{p+1} + 2^p \cdot \frac{\varepsilon}{2^p}$$

$$= \varepsilon^p/2^p$$

Hence

$$\|\hat{\mu}\hat{e}_\alpha - \hat{\mu}\|_p < \varepsilon/2 \quad \text{for all } \alpha \notin B.$$

Thus

(13) $$\|\hat{\mu}\|_p \leq \|\hat{\mu}\hat{e}_\alpha\|_p + \varepsilon/2 \quad \text{for all } \alpha \notin B.$$

Given $\varepsilon > 0$, there exists $f \in C_0(G)$ such that $\|f\|_\infty \leq 1$ and

$$\|\mu\| \leq \left|\int f d\mu\right| + \varepsilon/4 .$$

Since $\int f d\mu = \lim \int f(x)\, \mu * e_{\alpha_\beta}(x)\, dx$ we can find a finite subset C of indices such that

$$\left|\int f d\mu - \int f(\mu * e_{\alpha_\beta})\, dx\right| < \varepsilon/4 \quad \text{for all } \alpha_\beta \in C .$$

Hence

$$\left|\int f d\mu\right| < \left|\int f(\mu * e_{\alpha_\beta})\, dx\right| + \varepsilon/4 \quad \text{for all } \alpha_\beta \notin C,$$

$$< \|f\|_\infty \, \|\mu * e_{\alpha_\beta}\|_1 + \epsilon/4 \quad \text{for all } \alpha_\beta \notin C,$$

$$< \|\mu * e_{\alpha_\beta}\|_1 + \epsilon/4 \quad \text{for all } \alpha_\beta \notin C.$$

We then have

$$(4) \quad \|\mu\| \leq \left|\int f d\mu\right| + \varepsilon/4 \leq \|\mu * e_{\alpha_\beta}\|_1 + \varepsilon/2 \quad \text{for all } \alpha_\beta \notin C.$$

Let $D = B \cup C$. Choose $\alpha_\beta \notin D$. Then

$$\|\mu\| + \|\hat{\mu}\|_p \leq \|\hat{\mu}\, \hat{e}_{\alpha_\beta}\|_p + \|\mu * e_{\alpha_\beta}\|_1 + \varepsilon$$

$$= \|\mu * e_{\alpha_\beta}\|_1 + \varepsilon \leq \|T_\mu\| \, \|e_{\alpha_\beta}\|_1 + \epsilon = \|T_\mu\| + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\|\mu\| + \|\hat{\mu}\|_p \leq \|T_\mu\|$.

This inequality along with the opposite inequality (4) gives

$$\|\mu\| + \|\hat{\mu}\|_p = \|T_\mu\|$$

as desired.

In the case of the Segal algebra $A^p_\omega(G)$ for any weight function ω on G given as in Example 5, if $M(A^p_\omega(G))$ denotes the multiplier algebra of this Segal algebra, then we have

THEOREM 1.13. [ ] $M(L^1, A^p_\omega(G)) \simeq A^p_\omega(G)$ for $1 \leq p < 2$ and

$$M(L^1, A^p_\omega(G)) \simeq B^p_\omega(G) \quad \text{if } p > 2$$

where $B_\omega^p(G) = \{\mu \in M_{bd}(G) : \hat{\mu}\omega \in L^p(\Gamma)$ with norm

$$\|\mu\|_{B_\omega^p} = \|\mu\| + \|\hat{\mu}\omega\|_p$$

PROOF. The proof is similar to that of Theorem 1.12.

The question whether $A^p(G)$ is a dual for $p > 2$ is answered by the following result.

THEOREM 1.14 [ ] . For $p > 2$, $A^p(G)$ is not a dual space.

PROOF. Suppose for some $p > 2$, $A^p(G)$ is the dual of a normed linear space B. Since $A^p(G)$ is an $L^1(G)$ module, B^{**} is an $L^1(G)$ module. Since $B \subset B^{**}$, B is an $L^1(G)$ module Now $A^p(G) = B^*$ is an essential $L^1(G)$ module. Hence by Theorem 1.8, B is an essential $L^1(G)$ module. By Lemma 1.9, we then conclude that

$$M(L^1, A^p) = M(L^1, B^*) \simeq B^* = A^p(G) .$$

This is a contradiction to the fact that

$$M(L^1, A^p) \simeq B^p(G) \supsetneq A^p(G) \text{ for } p>2 ,$$

This completes the proof.

We shall now obtain a representation of $M(A^p)$ analogous to that of Theorem 1.2. First we prove

LEMMA 1.15 [ ] : $K_q^* \simeq B_p(G)$ for $p>2, \frac{1}{p}+\frac{1}{q} = 1$

PROOF. By Theorem 1.10,

$$K_q^* \simeq M_{bd}(G) \underset{J}{\wedge} L^p(\Gamma)$$

Let $\{\mu, h\} \in J$ and let $f \in [C_c(\Gamma)]^\wedge$. Since $f \in L^2(G)$ by Parseval's relation we have

$$\int_G \check{f}\, d\mu = \int_\Gamma \hat{f}\hat{\mu}\, d\gamma .$$

But by the definition of J,

$$\int_G \check{f}\, d\mu = \int_\Gamma \hat{f} h\, d\gamma$$

Thus

$$(15) \quad \int_\Gamma \hat{f} h\, d\gamma = \int_\Gamma \hat{f}\hat{\mu}\, d\gamma .$$

Now (15) holds for every $\hat{f} \in C_c(\Gamma)$. Hence $\hat{\mu} = h$ a.e. on Γ and we conclude that $\hat{\mu} \in L^p(\Gamma)$ and so $\mu \in B_p(G)$. Hence

$$J = \{ \{\mu, \hat{\mu}\} : \mu \in B_p(G) \} .$$

If we equip $B_p(G)$ with the norm

$$\|\mu\| = \sup \{ \|\mu\|_{M_{bd}(G)} , \|\hat{\mu}\|_{L^p(\Gamma)} \}$$

equivalent to the one defined earlier, by definition of $M_{bd}(G) \underset{J}{\wedge} L^p(\Gamma)$, we have

$$B_p(G) \simeq M_{bd}(G) \underset{J}{\wedge} L^p(\Gamma) \simeq K_q^*$$

This completes the proof of Lemma 1.15.

LEMMA 1.26 [ ] For $\mu \in B_p(G)$, $p > 2$ and $g \in [C_c(\Gamma)]^\wedge$ we have

$$\|\mu\|_{B_p} = \sup_{\|\{\check{g}, \hat{g}\}\| \leq 1} \left| \mu * g(0) + \int_\Gamma \hat{\mu}\hat{g}\, d\gamma \right|$$

where $\{g, h\} \in K_q$, $g \in C_0(G)$, $h \in L^q(\Gamma)$.

PROOF. By Parseval's relation, we have

$$\int_\Gamma \hat{\mu}\hat{g}\, d\gamma = \int_\Gamma \hat{g} \left[\int_G \langle -x, \gamma \rangle \, d\mu(x) \right] d\gamma$$

$$= \int_G \check{g}(x)\, d\mu(x) = \mu * g(0).$$

Consider a linear functional of the form

$$t_\mu(\check{g}, \hat{h}) = \int_G \check{g}(x)\, d\mu(x) + \int_\Gamma \hat{\mu}(\gamma)\hat{h}(\gamma)\, d\gamma$$

for $g, h \in [C_c(\Gamma)]^\wedge$ and $\{\check{g}, \hat{h}\} \in C_0(G) \times L^q(\Gamma)$.

Since $(\check{g}, \hat{h}) = (\check{h}, \hat{g}) \bmod H_q$, we have

$$\|(\check{g}, \hat{h})\| = \|(\check{h}, \hat{g})\| = \inf (\|g\|_\infty + \|\hat{h}\|_q)$$

and so

$$t_\mu(\check{g}, \hat{h}) = t_\mu(\check{h}, \hat{g}) = \frac{1}{2}\left(t_\mu(\check{g}, \hat{g}) + t_\mu(\check{h}, \hat{h})\right).$$

By Lemma 1.16

$$\|\mu\|_{B_p} = \sup\{|t_\mu(\check{g},\hat{h})| : \|(\check{g},\hat{h})\| \leq 1\}$$

$$= \sup\{|t_\mu(\check{g},\hat{h})| : \|(\check{g},\hat{h})\| = \|(\check{h},\hat{g})\| \leq 1\}$$

$$= \sup\{|\tfrac{1}{2}(t_\mu(\check{g},\hat{h}) + t_\mu(\check{h},\hat{g}))| :$$

$$\|(\check{g},\hat{h})\| = \|(\check{h},\hat{g})\| = 1\}$$

$$= \sup\{|t_\mu(\widetilde{\tfrac{g+h}{2}}, \widehat{\tfrac{g+h}{2}})| : \|(\check{g},\hat{g})\| = \|(\check{h},\hat{h})\| = 1\}$$

$$\leq \sup\{|t_\mu(\check{g},\hat{g})| : \|(\check{g},\hat{g})\| \leq 1\}$$

But

$$\sup\{|t_\mu(\check{g},\hat{g})| : \|(\check{g},\hat{g})\| \leq 1\}$$

$$\leq \sup\{|t_\mu(\check{g},\hat{h})| : \|(\check{g},\hat{h})\| \leq 1\}$$

$$= \|\mu\|_{B_p}.$$

This proves Lemma 1.16.

Let $p > 2$. We define the space $C_p(G)$ to be the set of all functions u such that

$$u = \sum_{i=1}^{\infty} f_i * g_i$$

where $f_i \in A^p(G)$, $g_i \in C_q = \{g \in [C_c(\Gamma)]^\wedge : \{g, \hat{g}\} \in K_q\}$

and

$$\sum_{i=1}^{\infty} \|f_i\|_s \, \|g_i\| < \infty$$

where $\|g_i\| = \|(\check{g}_i, \hat{g}_i)\|$.

Define $|||u|||_p$ by

$$|||u|||_p = \inf \{ \sum_{i=1}^{\infty} \|f_i\|_s \, \|g_i\| \}$$

where the infimum is taken over all the representations of u as an element of $C_p(G)$.

It is easy to verify that $|||\ \ |||_{C_p}$ is actually a norm on $C_p(G)$. We now assert that $C_p(G)$ is a Banach space under the norm. To this end, let $\{u_n\} \subset C_p(G)$ be a Cauchy sequence. It suffices to show that a subsequence of $\{u_n\}$ converges to an element of $C_p(G)$. We may assume, without loss of generality, that our sequence is such that

$$|||u_{n+1} - u_n|||_p < 1/2^n \ , \ n = 1, 2, \ldots$$

Let $|||u_1|||_p = C$. By the definition of the norm in $C_p(G)$, we can always find elements $f_{1k} \in A^p(G)$ and $g_{1k} \in C_q$ such that

$$u_1 = \sum_{k=1}^{\infty} f_{1k} * g_{1k}$$

$$u_{n+1} - u_n = \sum_{k=1}^{\infty} f_{n+1,k} * g_{n+1,k}$$

with

(16) $$\sum_{k=1}^{\infty} \|f_{1k}\|_S \, \|g_{1k}\| \leq C+1$$

and

(17) $$\sum_{k=1}^{\infty} \|f_{n+1,k}\|_S \, \|g_{n+1,k}\|_S < 1/2^{n-1}, \; n = 1, 2, \ldots$$

Now define

$$u = f_{11} * g_{11} + f_{12} * g_{12} + f_{21} * g_{21} + f_{31} * g_{31} + \cdots$$

Then

$$\|f_{11}\|_S \|g_{11}\| + \|f_{12}\|_S \|g_{12}\| + \|f_{21}\|_S \|g_{21}\| + \cdots < C+3,$$

and thus $u \in C_p(G)$. We now show that u_n converges to u in $C_p(G)$. Given $\varepsilon > 0$, choose a natural number n_0 such that $\sum_{\gamma=n_0}^{\infty} 1/2^{\gamma-1} < \varepsilon$. If $n > n_0$ then

(18) $$u - u_{n+1} = u - [(u_{n+1} - u_n) + (u_n - u_{n-1}) + (u_{n-1} - u_{n-2}) + \cdots (u_2 - u_1) + u_1].$$

Now $u_1 + (u_2 - u_1) + (u_3 - u_2) + \cdots (u_{n+1} - u_n)$

$$= f_{11} * g_{11} + f_{12} * g_{12} + f_{13} * g_{13} + f_{14} * g_{14} + \cdots$$

$$+ f_{21} * g_{21} + f_{22} * g_{22} + f_{23} * g_{23} + \cdots$$

$$+ \cdots$$

$$+ f_{n+1,1} * g_{n+1,1} + f_{n+1,2} * g_{n+1,2} + \cdots$$

and

$$u = f_{11} * g_{11} + f_{12} * g_{12} + f_{21} * g_{21} + f_{31} * g_{31} + f_{22} * g_{22}$$

$$+ f_{13} * g_{13} + f_{41} * g_{41} + f_{32} * g_{32} + f_{23} * g_{23} + f_{14} * g_{14}$$

$$+ \cdots + f_{n+1,1} * g_{n+1,1} + f_{n,2} * g_{n,2} + f_{n-1,3} * g_{n-1,3}$$

$$+ \cdots f_{1,n+1} * g_{1,n+1} + \cdots$$

Thus from (28)

$$u - u_{n+1} = f_{n+2,1} * g_{n+2,1} + f_{n+2,2} * g_{n+2,2} + \cdots$$

$$+ f_{n+3,1} * g_{n+3,1} + f_{n+3,2} * g_{n+3,2} + \cdots$$

$$+ \cdots$$

Therefore

$$\|u - u_{n+1}\|_{C_p} \leq \sum_{j=1}^{\infty} \|f_{n+2,j}\|_S \|g_{n+2,j}\| + \sum_{j=1}^{\infty} \|f_{n+3,j}\|_S \|g_{n+2,j}\|$$

$$+ \cdots$$

$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \sum_{r=n+1}^{\infty} 1/2^{r-1}$$

from (17)

Hence u_n converges to u in $C_p(G)$.

THEOREM 1.17. [ ] $C_p(G)$ <u>is a dense linear subspace of</u> $C_0(G)$ <u>and is a Banach space with respect to the norm</u> $|||\ |||_p$ <u>and the topology so defined is stronger than the uniform topology.</u>

PROOF. We need only to verify only the last part of the Theorem. If $u \in C_p(G)$, then u has the representation

$$u = \sum f_i * g_i$$

where $f_i \in A^p(G)$, $g_i \in C_q$ and $\sum_{i=1}^{\infty} \|f_i\|_S \|g_i\| < \infty$

From the inequalities

$$\|\sum_{i=m}^{n} f_i * g_i\|_\infty \leq \sum_{i=m}^{n} \|f_i\|_S \|g_i\|$$

where the right hand side tends to zero as $m, n \to \infty$, we conclude that $u = \sum_{i=1}^{\infty} f_i * g_i$ is therefore a uniformly continuous function on G and the norm $|||\ |||_p$ is stronger than the uniform norm. To complete the proof of the theorem, we have to show that $C_p(G)$ is dense in $C_0(G)$. This is because of the fact that the

algebra of continuous functions on G generated by

$\{f * g : f \in A^p(G), g \in [C_c(\Gamma)]^\wedge\}$ is a self adjoint subalgebra of $C_0(G)$ and separates points on G, thus it is uniformly dense in $C_0(G)$ by the Stone Weirstrass theorem. This completes the proof of the theorem.

THEOREM 1.12 [ ] . <u>If</u> $p > 2$, <u>then the multiplier space</u> $M(A^p)$ <u>is isometrically isomorphic to the topological</u> dual $C_p^*(G)$ of $C_p(G)$.

PROOF. Suppose $T \in M(A^p)$. Define a linear functional on $C_p(G)$ by

$$\mu(u) = \sum_{i=1}^{\infty} \left(\int_G T f_i(x)\, \check{g}_i(x)\, dx + \int_\Gamma \widehat{T f_i}(\gamma)\, \hat{g}_i(\gamma)\, d\gamma \right)$$

$$= \sum_{i=1}^{\infty} \left(T f_i * g_i(0) + \int_\Gamma \widehat{T f_i}(\gamma)\, \hat{g}_i(\gamma)\, d\gamma \right)$$

for $u = \sum_{i=1}^{\infty} f_i * g_i$ in $C_p(G)$ with $f_i \in A^p(G)$, $g_i \in C_q$ and

$$\sum_{i=1}^{\infty} \|f_i\|_S \, \|g_i\| < \infty$$

To show that μ is well defined, it suffices to show that if

$$u = \sum_{i=1}^{\infty} f_i * g_i = 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \|f_i\|_S \, \|g_i\| < \infty .$$

then $\mu(u) = 0$. Let $\{e_\alpha\}$ be an approximate identity for $A^p(G)$. Let $h_\alpha = Te_\alpha$. Then

(19) $\| h_\alpha * f - Tf \|_{A^p} \to 0$

For $f \in A^p(G)$, $g \in [C_c(\Gamma)]^\wedge$, we have

(20) $$\| h_\alpha * f * g \|_\infty \leq \int_G |e_\alpha * Tf(x)| \, |\check{g}(x)| \, dx \leq \| e_\alpha \|_1 \, \| Tf \|_1 \, \| g \|_\infty$$

By assumption, the series $u = \sum_{i=1}^{\infty} f_i * g_i$ is uniformly convergent to zero on G. We therefore have

(21) $$h_\alpha * u = \sum_{i=1}^{\infty} h_\alpha * f_i * g_i = 0$$

By (19) we have

$$Tf_i * g_i(0) = \lim_\alpha h_\alpha * f_i * g_i(0)$$

Therefore

(22) $$\sum_{i=1}^{\infty} Tf_i * g_i(0) = \sum_{i=1}^{\infty} \lim_\alpha h_\alpha * f_i * g_i(0)$$

$$\leq \lim_\alpha \sum_{i=1}^{\infty} h_\alpha * f_i * g_i(0) = 0 \text{ from (21)}$$

Also

$$\int_\Gamma T\hat{f}_i(\gamma)\,\hat{g}_i(\gamma)\,d\gamma = \int_\Gamma \hat{g}_i(\gamma)\int_G \langle -x,\gamma\rangle\, Tf_i(x)\,dx$$

$$= \int_G Tf_i(x)\Big[\int_\Gamma \hat{g}_i(\gamma)\,\langle -x,\gamma\rangle\,d\gamma\Big]\,dx$$

$$= \int_G Tf_i(x)\,\check{g}_i(x)\,dx = Tf_i * g_i(0)$$

Therefore

$$\mu(u) = \sum_{i=1}^{\infty}\Big[\int_\Gamma T\hat{f}_i(\gamma)\,\hat{g}_i(\gamma)\,d\gamma + Tf_i * g_i(0)\Big]$$

$$= 2\sum_{i=1}^{\infty} Tf_i * g_i(0) = 0 \quad \text{from (22)}$$

thus proving that μ is well defined.

Since $A^p(G) \subset B_p(G)$, by Lemma 1.16

$$|\mu(u)| \leq \sum_{i=1}^{\infty}\|Tf_i\|_S\,\|g_i\| \leq \|T\|\sum_{i=1}^{\infty}\|f_i\|_S\,\|g_i\|$$

It follows that

$$|\mu(u)| \leq \|T\|\,\|u\|,$$

that is

(23) $$\|\mu\| \leq \|T\|$$

On the other hand from Lemma 1.16 we have

$$(24) \quad \|T\| = \sup_{\|f\|_S \le 1} \|Tf\|_S = \sup_{\substack{\|f\|_S \le 1 \\ \|g\|_S \le 1}} |\mu(f * g)|$$

$$\le \sup_{\|f*g\|_{C_p} \le 1} |\mu(f*g)| \le \|\mu\|.$$

From (23) and (24) we have

$$\|T\| = \|\mu\|.$$

It only remains to show that the mapping from $M(A^p)$ into $[C_p(G)]^*$ is surjective. Let $\mu \in [C_p(G)]^*$. For a fixed $f \in A^p(G)$ define the linear functional

$$t(g) = \mu(f * g) \quad \text{for } g \in [C_c(K)]^\wedge .$$

Now

$$(25) \quad |t(g)| \le \|\mu\| \|f*g\|_{C_p} \le \|\mu\| \|f\|_S \|g\|_1$$

t may be extended to an element of K_q^* and hence defines a unique $Tf \in B_p(G)$, the dual space of K_q^* by Lemma 1.15. Since

$$\|Tf\|_{B_p} = \|t\| \le \|\mu\| \|f\|_S$$

T is a bounded linear operator from $A^p(G)$ into $B_p(G)$. Also

$$\mu(f*g) = Tf*g(0) + \int_{\Gamma} \widehat{Tf}(\gamma)\,\hat{g}(\gamma)\,d\gamma\,, \quad f \in A^p(G),$$

$$g \in [C_c(\Gamma)]^\wedge .$$

Therefore if $y \in G$, we have

$$T(\tau_y f)*g(0) + \int_{\Gamma} \widehat{T\tau_y f}\,\hat{g}\,d\gamma = \mu(\tau_y f * g)$$

$$= \mu(f*\tau_y g) = \int_{\Gamma} \widehat{Tf}\,\widehat{\tau_y g}\,d\gamma + Tf*\tau_y g(0)$$

$$= \int_{\Gamma} \widehat{\tau_y Tf}(\gamma)\,\hat{g}(\gamma)\,d\gamma + \tau_y Tf*g(0)$$

that is

$$\tau_y(Tf) = T(\tau_y f)\,,$$

that is T commutes with translations. Also the map from $G \to M_{bl}(G)$ given by $y \to \tau_y(Tf)$ is continuous for every $f \in A^p(G)$. Hence by Lemma 1.11, Tf must be absolutely continuous. $Tf \in B_p(G)$ and Tf absolutely continuous implies $Tf \in A^p(G)$ for every $f \in A^p(G)$. Thus T defines a bounded linear translation-invariant map from $A^p(G)$ into itself, that is $T \in M(A^p)$. This proves that the mapping of $M(A^p)$ into $[C_p(G)]^*$ is surjective which completes the proof of the theorem.

REMARK 1.19. Theorems 1.12 and 1.14 have been obtained independently by Durham, Krogstad and Larson in [ ].

CHAPTER II

MULTIPLIERS ON THE ALGEBRAS $A^p_\omega(G)$ AND $S^p(G)$

This chapter deals with characterizations of the spaces $M(A^p_\omega(G))$ and $M(S^p(G))$.

In the case when G is a noncompact locally compact abelian group, a characterization of the multipliers on the algebras $A^p_\omega(G)$ and $S^p(G)$ have been given by Keshava Murthy and Unni in the following theorems.

THEOREM 2.1. [ ]. Let G be a nondiscrete, noncompact locally compact abelian group and $1 \leq p < \infty$. If $T \in M(A^p_\omega(G))$ then there exists a unique measure $\mu \in M_{bd}(G)$ such that

$$Tf = \mu * f$$

for all $f \in A^p_\omega(G)$. Further $M(A^p_\omega(G))$ is isometrically isomorphic to $M_{bd}(G)$.

THEOREM 2.2. [ ]. Let G be a nondiscrete, noncompact locally compact abelian group and $1 \leq p < \infty$. If $T \in M(S^p(G))$ then there exists a unique measure $\mu \in M_{bd}(G)$ such that

$$Tf = \mu * f$$

for all $f \in S^p(G)$. Further $M(S^p(G))$ is isometrically isomorphic to $M_{bd}(G)$.

In the case when G is compact abelian, we shall give

here characterizations of $M(A^p_\omega(G))$ and $M(S^p(G))$ similar to the one given by Larsen for the algebras $A^p(G)$ as the dual of Banach spaces. Throughout this section we shall assume that G is a compact abelian group.

For a semisimple commutative Banach algebra the following multiplier theorem is valid.

THEOREM 2.3. <u>If</u> A <u>is a semisimple commutative Banach algebra and</u> T <u>a multiplier on</u> A <u>then there exists a continuous function</u> φ <u>defined on the maximal ideal space</u> Δ <u>of</u> A <u>satisfying</u> $\widehat{Tx} = \varphi\hat{x}$ <u>for all</u> $x \in A$ <u>where</u> $\hat{x}$ <u>denotes the Gelfand transform of</u> x <u>, with</u> $\|\varphi\|_\infty \leq \|T\|$.

We remark that every Segal algebra is a semisimple commutative Banach algebra with maximal ideal space $\Gamma[\ ]$. Thus Theorem 2.3 is valid for all Segal algebras.

We shall now prove

THEOREM 2.4. <u>When</u> $1 \leq p \leq 2$ <u>, there exists a continuous algebra isomorphism of</u> $M(A^p_\omega(G))$ <u>onto</u> $L^\infty(\Gamma)$.

PROOF. If $T \in M(A^p_\omega(G))$ there exists a function $\hat{T} \in L^\infty(\Gamma)$ satisfying

$$\widehat{Tf} = \hat{T}\hat{f}, \quad f \in A^p_\omega(G)$$

with

$$\|\hat{T}\|_\infty \leq \|T\|.$$

The mapping $\beta : T \to \hat{T}$ is a well defined continuous linear map of $M(A^p_\omega(G))$ into $L^\infty(\Gamma)$. Also if $\hat{T}_1 = \hat{T}_2$,

$$T_1,\ T_2 \in M(A^p_\omega(G)) \qquad \text{then}$$

$$\widehat{T_1 f} = \widehat{T_2 f}\ ,\quad f \in A^p_\omega(G)$$

which implies that

$$T_1 f = T_2 f$$

that is $$T_1 = T_2$$

Thus the mapping β is one to one. We claim β is an onto map. To this end, let $\varphi \in L^\infty(\Gamma)$. Now $f \in A^p_\omega(G)$ implies $\varphi \hat{f} \omega \in L^p(\Gamma)$. Since

$$\omega(\gamma) \geq 1,\quad \gamma \in \Gamma\ ,\quad \varphi \hat{f} \in L^p(\Gamma)$$

Thus $\varphi \hat{f} \in L^p(\Gamma) \cap L^\infty(\Gamma)$. Since $1 \leq p \leq 2$, we see that $\varphi \hat{f} \in L^2(\Gamma)$. Then there exists $g \in L^2(G)$ such that $\hat{g} = \varphi \hat{f}$. Since G is compact, $g \in L^1(G)$. Moreover $\hat{g}\omega = \varphi \hat{f} \omega \in L^p(\Gamma)$ which implies that $g \in A^p_\omega(G)$.

Now

$$\|g\|_s = \|g\|_1 + \|\hat{g}\omega\|_p \leq \|g\|_q + \|\hat{g}\omega\|_p$$

$$\leq \|\hat{g}\|_p + \|\hat{g}\omega\|_p \leq 2\|\hat{g}\omega\|_p = 2\|\varphi \hat{f} \omega\|_p$$

$$\leq 2\|\varphi\|_\infty \|\hat{f}\omega\|_p \leq 2\|\varphi\|_\infty \|f\|_s$$

The mapping $T: A_\omega^p(G) \to A_\omega^p(G)$ given by $Tf = g$ is a bounded linear map which can be easily verified to be translation invariant. Therefore $T \in M(A_\omega^p(G))$ with $\hat{T} = \varphi$. Hence ρ is onto. This completes the proof.

We thus see that in the case $1 \leq p \leq 2$, there are multipliers in $M(A_\omega^p(G))$ which do not correspond to measures $\mu \in M_{bd}(G)$ since not every function in $L^\infty(\Gamma)$ is the Fourier-Stieltjes transform of a measure in $M_{bd}(G)$. In the case of $p > 2$ we now give an example of a multiplier in $M(A_\omega^p(G))$ which does not correspond to any measure in $M_{bd}(G)$.

EXAMPLE 2.5. Let $E \subset \Gamma$ be an infinite Sidon set. Let $m = p/2$, $n = \frac{m}{m-1}$. Choose r such that $0 < r < 2$ and $r \cdot n > 2$. Let φ be a bounded function on Γ satisfying

$$\varphi(\gamma) = 0, \ \gamma \notin E, \quad \sum_\gamma |\varphi(\gamma)|^2 = \infty, \quad \sum_\gamma |\varphi(\gamma)|^{rn} < \infty$$

If $f \in A_\omega^p(G)$, then

$$\sum_\gamma |\varphi(\gamma)\hat{f}(\gamma)\omega(\gamma)|^2 \leq \left(\sum_\gamma |\varphi(\gamma)|^{2p/p-2}\right)^{1-2/p} \times \left(\sum_\gamma |\hat{f}(\gamma)\omega(\gamma)|^{2p/2}\right)^{2/p}$$

$$\leq \left[\sum_\gamma |\varphi(\gamma)|^{2n}\right]^{1/n} \|\hat{f}\omega\|_p^2$$

$$\leq \|\varphi\|_\infty^{2-r} \left[\sum_\gamma |\varphi(\gamma)|^{r \cdot n}\right]^{1/n} \|\hat{f}\omega\|_p^2 < \infty$$

so that

(1) $$\sum_{\gamma} |\varphi(\gamma)\hat{f}(\gamma)\,\omega(\gamma)|^2 < K\,\|\hat{f}\omega\|_p^2$$

for some constant K which depends only on φ.

Since Fourier transformation is an isometry on L^2, we can find a $g \in L^2(G) \subset L^1(G)$ such that

(2) $$\hat{g} = \varphi\hat{f}\omega$$

Further

(3) $$\|g\|_1 \leq \|g\|_q \leq \|g\|_2 = \|\hat{g}\|_2 = \|\varphi\hat{f}\omega\|_2 < K^{1/2}\,\|\hat{f}\omega\|_p.$$

Since $g \in L^q(G)$ and $1 < q < 2$, by Hausdorff-Young theorem we have $\hat{g} \in L^p(\Gamma)$ and

(4) $$\|\hat{g}\|_p \leq \|g\|_q < K^{1/2}\,\|\hat{f}\omega\|_p$$

Analogously we prove that

(5) $$\sum_{\gamma} |\varphi(\gamma)\,\hat{f}(\gamma)|^2 \leq K\,\|\hat{f}\|_p^2 \leq K\,\|\hat{f}\omega\|_p^2$$

Then there exists $g' \in L^2(G) \subset L^1(G)$ such that

(6) $$\hat{g'} = \varphi\hat{f}$$

with

(7) $$\|g'\|_1 \leq \|g'\|_2 = \|\hat{g'}\|_2 = \|\varphi\hat{f}\|_2 \leq K^{1/2}\,\|\hat{f}\omega\|_p$$

From (3) and (6) we see that $\hat{g} = \hat{g'}\omega$. The inequalities (4) and (7) show that $g' \in A_\omega^p(G)$ and

$$\|g'\|_s = \|g'\|_1 + \|\hat{g'}\omega\|_p = \|g'\|_1 + \|\hat{g}\|_p$$
$$\leq 2K^{1/2}\|\hat{f}\omega\|_p \leq 2K^{1/2}\|f\|_s$$

The mapping $T: A_\omega^p(G) \to A_\omega^p(G)$ given by $Tf = g'$ is a multiplier on $A_\omega^p(G)$ with $\hat{T} = \varphi$. Since $\sum_\gamma |\varphi(\gamma)|^2 = \infty$, $\varphi \neq \hat{\mu}$ for any bounded regular Radon measure $\mu \in M_{bd}(G)$.

THEOREM 2.6. <u>When $p > 2$, there exists a continuous linear isomorphism of $M(A_\omega^p(G))$ onto the dual space of a Banach space of continuous functions.</u>

PROOF. We prove our theorem by showing that there exists a continuous linear isomorphism of $M(A_\omega^p(G))$ onto a dual $[R(G)]^*$ and then proving that the completion of $R(G)$ can be embedded into a space of continuous functions.

Let $A(G) = \{\hat{f} : f \in L^1(\Gamma)\}$. For each $T \in M(A_\omega^p(G))$ set

$$(8) \qquad \beta_T(\hat{f}) = \int_\Gamma \hat{T}(\gamma)\hat{f}(\gamma)\,d\gamma, \quad \hat{f} \in A(G)$$

Then β_T is a linear form on $A(G)$ and

$$|\beta_T(\hat{f})| \leq \|\hat{T}\|_\infty$$

Introduce a norm on $A(G)$ by

$$(9) \quad \|f\| = \sup\left\{ |\beta_T(f)| : T \in M(A_\omega^p(G)),\ \|T\| \leq 1 \right\}$$

It is easy to verify that (9) defines a seminorm on $A(G)$. We shall now show that $\|\cdot\|$ is actually a norm on $A(G)$. To this end, let $f \in A(G)$ such that $\|f\| = 0$. Then $\beta_T(f) = 0$ for all $T \in M(A_\omega^p(G))$. Now let $y \in G$. Since $\tau_y \in M(A_\omega^p(G))$, taking $T = \tau_y$ we have $f(-y) = \beta_T(f) = 0$, since $\hat{T}(\gamma) = \langle -y, \gamma \rangle$. This being true for every $y \in G$, we see that $f = 0$.

Our space $R(G)$ is the space $A(G)$ with the norm given by (9). The norm itself will be denoted by $\| \ \|_R$.

Now for each $T \in M(A_\omega^p(G))$, β_T is a well defined linear form on $R(G)$ and

$$|\beta_T(f)| = \left| \beta_{\frac{T}{\|T\|}}(f) \right| \|T\| \leq \|T\| \, \|f\|_R$$

by the definition of $\|f\|_R$. Thus $\beta_T \in [R(G)]^*$, the dual of $R(G)$. Consider the mapping

$$\beta : M(A_\omega^p(G)) \longrightarrow [R(G)]^*$$

given by

$$\beta(T) = \beta_T$$

Then β is well defined and linear. The inequalities

$$\|\beta(T)\| = \|\beta_T\| \leq \|T\|$$

for all $T \in M(A^p_\omega(G))$ show that β is continuous. We now claim that β is both one to one and onto. If $\beta_{T_1} = \beta_{T_2}$ then

$$\int_\Gamma \hat{T}_1(\gamma)\,\hat{f}(\gamma)\,d\gamma = \int_\Gamma \hat{T}_2(\gamma)\,\hat{f}(\gamma)\,d\gamma, \quad \hat{f} \in L^1(\Gamma).$$

Therefore $\hat{T}_1 = \hat{T}_2$ as functions of $L^\infty(\Gamma)$ which implies that β is one to one.

To prove that the mapping β is onto. Let $\lambda \in [R(G)]^*$. We denote by B(G) the set of all functions in $L^1(G)$ whose Fourier transform has compact support. If $f, g \in B(G)$ then $f * g \in A(G)$. Then

$$|\beta_T(f*g)| = \left|\int_\Gamma \hat{T}(\gamma)\,\hat{f}(\gamma)\,\hat{g}(\gamma)\,d\gamma\right| = |Tf*g(0)|$$
$$\leq \|Tf\|_1 \|g\|_\infty \leq \|T\|\,\|f\|_S \|g\|_\infty$$

so that

(10) $$\|f*g\|_R \leq \|f\|_S \|g\|_\infty$$

Also

$$\beta_T(f*g) = \int_\Gamma \hat{T}(\gamma)\,\hat{f}(\gamma)\,\omega(\gamma)\,\frac{\hat{g}(\gamma)}{\omega(\gamma)}\,d\gamma.$$

The inequality

$$|\beta_T(f*g)| \leq \|\hat{T}\|_\infty \|\hat{f}\omega\|_p \|\hat{g}/\omega\|_{p'}$$

then implies

(11) $$\|f*g\|_R \leq \|\hat{f}\omega\|_p \, \|\hat{g}/\omega\|_{p'}$$

We have already taken an element $\lambda \in [R(G)]^*$. Let $f \in B(G)$ be fixed. Define a linear form on $[B(G)]^\wedge$ by

$$F_f(\hat{g}) = \lambda(f*g), \quad \hat{g} \in [B(G)]^\wedge.$$

Then

$$|F_f(\hat{g})| = |\lambda(f*g)| \leq \|\lambda\| \, \|f*g\|_R$$

which by virtue of (11) gives

(12) $$|F_f(\hat{g})| \leq \|\lambda\| \, \|f\|_s \, \|\hat{g}/\omega\|_q$$

Now $L^{q,1/\omega}(\Gamma)$ is the Lebesgue space of measurable functions h on Γ satisfying

$$\|h\|_{q,1/\omega} = \left[\int_\Gamma \left|\frac{h(\gamma)}{\omega(\gamma)}\right|^p d\gamma \right]^{1/q} < \infty .$$

(12) implies then that F_f is a linear form on $[B(G)]^\wedge$ bounded in the norm of $L^{q,1/\omega}(\Gamma)$ and hence can be extended as a continuous linear functional to the whole space $L^{q,1/\omega}(\Gamma)$. Now $q < \infty$ and so $L^{p,\omega}(\Gamma)$ is the dual space of $L^{q,1/\omega}(\Gamma)$. Hence

there exists an $h \in L^{p,\omega}(\Gamma)$ such that

(13) $$F_f(\hat{g}) = \lambda(f * g) = \int_\Gamma \hat{g}(\gamma)\,\overline{h(\gamma)}\,d\gamma$$

for all $\hat{g} \in [B(G)]^\wedge$ and

(14) $$\|h\|_{p,\omega} = \|h\omega\|_p = \|F_f\| \leq \|\lambda\|\,\|f\|_S$$

We now define another linear form G_f on $B(G)$ by setting

$$G_f(g) = \lambda(f * g), \quad g \in B(G).$$

Then by virtue of (10) we have

$$|G_f(g)| \leq \|\lambda\|\,\|f * g\|_R \leq \|\lambda\|\,\|f\|_S\,\|g\|_\infty$$

Hence G_f can be extended to a continuous linear functional on $C_0(G)$ since $B(G)$ is dense in $C_0(G)$. Therefore there exists $\mu \in M_{bd}(G)$ such that

(15) $$G_f(g) = \lambda(f * g) = \int_G g(x)\,\overline{d\mu(x)}, \quad g \in B(G),$$

with

(16) $$\|\mu\| \leq \|\lambda\|_{R^*}\,\|f\|_S$$

Define $Tf = \mu$ for every $f \in B(G)$. Then

$$Tf * g(0) = \lambda(f * g), \quad f, g \in B(G).$$

If $y \in G$,

$$T(\tau_y f) * g(0) = \lambda(\tau_y f * g) = \lambda(f * \tau_y g)$$
$$= Tf * \tau_y g(0) = \tau_y(Tf) * g(0).$$

Thus

$$T(\tau_y f) = \tau_y(Tf), \quad y \in G.$$

Hence the mapping $G \to M_{bd}(G)$ given by $y \longrightarrow \tau_y(Tf)$ is continuous. Therefore by Lemma 1.11, Tf is absolutely continuous, that is $Tf \in L^1(G)$. Now suppose $f, g \in B(G)$. Then

$$F_f(\hat{g}) = \lambda(f * g) = G_f(g)$$

so that, by (13) and (16) we have

$$\int_\Gamma \hat{g}(\gamma)\overline{h(\gamma)}\,d\gamma = \int_G g(x)\,d\mu(x).$$

The equality

$$\int_\Gamma \hat{g}(\gamma)\overline{h(\gamma)}\,d\gamma = \int_\Gamma \hat{g}(\gamma)\overline{\hat{\mu}(\gamma)}\,d\gamma$$

holds for every $\hat{g} \in [B(G)]^\wedge$. This implies that $h(\gamma) = \hat{\mu}(\gamma)$ a.e. on Γ. Therefore $\hat{\mu}\omega \in L^p(\Gamma)$ and

(17) $\|\hat{\mu}\omega\|_p = \|h\omega\|_p \leq \|\lambda\|\,\|f\|_S$

by (14). On the other hand, by virtue of (16)

(18) $\|Tf\|_1 = \|\mu\| \leq \|\lambda\|_{R^*} \|f\|_S$

Since $Tf = \mu$, (17) and (18) imply that $Tf \in A^p_\omega(G)$ and

$$\|Tf\|_S = \|Tf\|_1 + \|(\widehat{Tf})\omega\|_p \leq \|f\|_S \|\lambda\|_{R^*} + \|f\|_S \|\lambda\|_{R^*}$$

$$= 2\|\lambda\|_{R^*} \|f\|_S$$

The mapping T which maps $B(G)$ into $A^p_\omega(G)$ can therefore be extended to a map from $A^p_\omega(G)$ into $A^p_\omega(G)$ which is bounded linear and also translation invariant, i.e. T can be extended to an element of $M(A^p_\omega(G))$, with norm $\|T\| \leq 2\|\beta\|$. We now claim that for this T, $\beta_T = \lambda$. We actually have

$$\beta_T(f * g) = \int_\Gamma \hat{T}(\gamma)\, \hat{f}(\gamma)\, \hat{g}(\gamma)\, d\gamma = \lambda(f * g), \quad f, g \in B(G).$$

Since $B(G) * B(G)$ is dense in $A(G)$ we see that

$$\beta_T(f) = \lambda(f), \quad f \in A(G)$$

Hence

$$\beta_T = \lambda$$

and the mapping β is an entomorp. To complete the proof of our

theorem, we propose to show that the completion of $R(G)$ can be embedded into space of continuous functions.

Let f be a fixed but arbitrary element of $R(G)$. Consider $T = \tau_x$ for some $x \in G$. Since $\|T\| = 1$ and $\hat{T}(\gamma) = \langle -x, \gamma \rangle$ for all $\gamma \in \Gamma$ we have

$$\|f\|_R \geq |\beta_T(f)| = \left| \int_\Gamma \langle -x, \gamma \rangle \hat{f}(\gamma)\, d\gamma \right| = |f(x)|$$

so that

$$\|f\|_\infty \leq \|f\|_R \tag{19}$$

Since f is arbitrary, (19) holds for every $f \in R(G)$.

If $\overline{R(G)}$ is the completion of $R(G)$ then $\overline{R(G)}$ can be thought of as the Cauchy sequences $\{f_n\}$ of elements of $R(G)$. From (19) it follows that every Cauchy sequence in $R(G)$ norm is also a Cauchy sequence in the essential supremum norm. Hence to each Cauchy sequence in $R(G)$ there exists a continuous function f on G such that $f_n \to f$ in the essential supremum norm. Setting $i(\{f_n\}) = f$ we have a well defined linear map from $\overline{R(G)}$ into $C(G)$ the space of continuous functions on G. The inequality (19) shows that this mapping is continuous.

To prove i is injective, it suffices to prove that if $\{f_n\} \subset R(G)$ is Cauchy and $\lim_n \|f_n\|_\infty = 0$, then

$\lim_n \|f_n\|_R = 0$. For $g \in L^1(G)$, let T_g be the element of $M(A^p_\omega(G))$ given by

$$T_g(f) = g * f, \quad f \in A^p_\omega(G).$$

Then

$$\|T_g\| \leq \|g\|_1 \quad \text{and} \quad \hat{T_g}(\gamma) = \hat{g}(\gamma), \ \gamma \in \Gamma.$$

If $g \in B(G)$, then

$$\lim_n |\beta_{T_g}(f_n)| = \lim_n \left| \int_\Gamma \hat{g}(\gamma) \hat{f_n}(\gamma)\, d\gamma \right| = \lim_n |g * f_n(0)|$$
$$\leq \lim_n \|g\|_1 \|f_n\|_\alpha = 0$$

Hence

(20) $$\lim_n \beta_{T_g}(f_n) = 0, \quad f \in B(G).$$

Let $\{\phi_\alpha\} \subset B(G)$ be an approximate identity for $A^p_\omega(G)$ satisfying $\|\phi_\alpha\|_1 \leq 1$. Then $T_{\phi_\alpha} = h_\alpha \in B(G)$ and

$$\|Tf - T_{h_\alpha}(f)\|_s = \|Tf - h_\alpha * f\|_s$$
$$= \|Tf - T\phi_\alpha * f\|_s \leq \|T\| \, \|f - \phi_\alpha * f\|_s$$
$$\to 0, \quad f \in A^p_\omega(G).$$

Therefore $Tf = \lim_\alpha T_{h_\alpha}(f)$, $f \in A^p_\omega(G)$ in the $A^p_\omega(G)$ norm. Also

$$\|T_{h_\alpha}(f)\|_S = \|h_\alpha * f\|_S = \|\phi_\alpha * Tf\|_S$$

$$\leq \|\phi_\alpha\|_1 \|Tf\|_S$$

$$\leq \|T\| \|f\|_S .$$

Hence

$$\|T_{h_\alpha}\| \leq \|T\| \quad \text{for all } \alpha.$$

We now claim that

$$\beta_{T_{h_\alpha}}(u) \longrightarrow \beta_T(u), \quad u \in R(G).$$

If $f, g \in B(G)$ then $f * g \in R(G)$ and

(21) $$\beta_{T_{h_\alpha}}(f * g) = T_{h_\alpha} f * g(0) \longrightarrow Tf * g(0) = \beta_T(f * g),$$

since $T_{h_\alpha} f \to Tf$ in the $L^1(G)$ norm. Further if $u \in R(G)$ and $\epsilon > 0$ is given we can find $f, g \in B(G)$ such that

(22) $$\|f * g - u\|_R < \epsilon / 3\|T\| .$$

Now

$$|\beta_{T_{h_\alpha}}(u) - \beta_T(u)| \leq |\beta_{T_{h_\alpha}}(u) - \beta_{T_{h_\alpha}}(f*g)| + |\beta_{T_{h_\alpha}}(f*g) - \beta_T(f*g)|$$
$$+ |\beta_T(f*g) - \beta_T(u)| .$$

$$\leq \|T_{h_\alpha}\| \, \|u - f*g\|_R + |\beta_{T_{h_\alpha}}(f*g) - \beta_T(f*g)|$$
$$+ \|T\| \, \|f*g - u\|_R$$
$$\leq 2\|T\| \, \|f*g - u\|_R + |\beta_{T_{h_\alpha}}(f*g) - \beta_T(f*g)| .$$

The first term on the right hand side can be made small by choosing f and g and then the second term is made small because of (21). We thus conclude that

$$\lim_\alpha \beta_{T_{h_\alpha}}(u) = \beta_T(u), \quad u \in R(G)$$

Hence $\beta_{T_{h_\alpha}}(f_n)$ converges to $\beta_T(f_n)$ for every n.

Therefore $\lim_n \beta_{T_{h_\alpha}}(f_n)$ converges to $\lim_n \beta_T(f_n)$.

Since $\lim_n \beta_{T_{h_\alpha}}(f_n) = 0$ for all α, $\lim_n \beta_T(f_n) = 0$ by (2

This is true for all $T \in M(A^P_\omega(G))$.

Corresponding to every $\epsilon > 0$, and every integer n there exists $T_n \in M(A^P_\omega(G))$ satisfying

$$\|T_n\| \leq 1, \qquad \|f_n\|_R \leq |\beta_T(f_n)| + \epsilon/3 ,$$

Since $\{f_n\}$ is Cauchy in $R(G)$, there exists a N such that

$$\|f_m - f_n\|_R < \epsilon/3 , \quad m, n \geq N.$$

Hence for $m \geq N$, we have

$$\|f_N\|_R \leq |\beta_{T_N}(f_N - f_m)| + |\beta_{T_N}(f_m)| + \varepsilon/3$$
$$\leq \|T_N\| \, \|f_N - f_m\| + |\beta_{T_N}(f_m)| + \varepsilon/3 .$$

Since

$$\lim_m |\beta_{T_N}(f_m)| = 0$$

we have

$$\|f_N\|_R \leq 2\varepsilon/3 .$$

Hence

$$\|f_n\|_R \leq \|f_n - f_N\|_R + \|f_N\|_R < \varepsilon, \quad n \geq N,$$

that is

$$\lim_n \|f_n\|_R = 0$$

as desired. This completes the proof of the theorem.

In the case of the algebra $S^p(G)$, for $1 \leq p < \infty$, we have the following characterizations of the multiplier space ~~$M(S^p(G$~~ $M(S^p(G))$.

THEOREM 2.7. [ ] (a) <u>When $1 \leq p \leq 2$, there exists a continuous isomorphism of</u> $M(S^p(G))$ <u>onto</u> $L^\infty(\mu)$. (b) <u>If $p \geq 2$, there exists a continuous isomorphism of</u> $M(S^p(G))$ <u>onto the dual space of a Banach space of continuous functions.</u>

PROOF. The proof of (a) is similar to that of Theorem 2.4. In the case of (b) the construction of the Banach space of continuous functions is similar to that given in Theorem 2.6 except that in the construction of $R(G)$, elements of $M(S^p(G))$ are used instead of elements of $M(A_\omega^p(G))$.

We give below an example similar to Example 2.5 of an element in $M(S^p(G)$ when $p > 2$ which is not given by convolution with a bounded Radon measure in $M_{bd}(G)$.

EXAMPLE 2.8. Let $E \subset \Gamma$ be an infinite Sidon set. Let $m = p/2$, $n = \frac{m}{m-1}$. Choose r such that $0 < r < 2$ and $rn > 2$. Choose φ as in Example 2.5. Since $\varphi \in L^\infty(\Gamma)$, if $f \in S^p(G)$, then we have $\varphi \hat{f} \in \Lambda^p(\Gamma)$ with

$$(21) \qquad \|\varphi \hat{f}\|_\Lambda \leq \|\varphi\|_\infty \|\hat{f}\|_\Lambda \leq \|\varphi\|_\infty \|f\|_S$$

Since $\hat{f} \in \Lambda^p(\Gamma)$, consider $\omega \in \Omega_0$ such that

$$\int_\Gamma \frac{|\hat{f}(\gamma)|^p}{[\omega(\gamma)]^{p-1}} < \infty .$$

Since

$$\int_\Gamma \omega \, d\gamma \leq N(\omega) = 1,$$

and Γ is discrete,

$$\omega(\gamma) \leq 1, \qquad \gamma \in \Gamma$$

Therefore

$$\int_{\Gamma} |\hat{f}(\gamma)|^p d\gamma \leq \int_{\Gamma} \frac{|\hat{f}(\gamma)|^p d\gamma}{\omega^{p-1}(\gamma)} < \infty$$

that is $\hat{f} \in L^p(\Gamma)$ with

$$\|\hat{f}\|_p \leq \|\hat{f}\|_{p,\omega'} .$$

This being true for all $\omega \in \Omega_0$ for which $\hat{f} \in L_{p,\omega'}(\Gamma)$,

$$\|\hat{f}\|_p \leq \|\hat{f}\|_\Lambda .$$

As in Example 2.5, we can prove that

$$\|\varphi \hat{f}\|_2 \leq K \|\hat{f}\|_p$$

where K is a constant depending only on φ. Then

$$\|\varphi \hat{f}\|_2 \leq K \|\hat{f}\|_\Lambda < \infty .$$

Hence there exists $g \in L^2(G) \subset L^1(G)$ satisfying $\hat{g} = \varphi \hat{f}$ with

$$(24) \quad \|g\|_1 \leq \|g\|_2 = \|\hat{g}\|_2 \leq K \|\hat{f}\|_\Lambda \leq K \|f\|_S$$

From (23) and (24), the mapping $T: S^p(G) \to S^p(G)$ given by $Tf = g$ is a multiplier on $S^p(G)$ with $\hat{T} = \varphi$. Since $\sum_\gamma |\phi(\gamma)|^2 = \infty$, $\phi \neq \hat{\mu}$ for any bounded measure $\mu \in M_{bd}(G)$.

CHAPTER III.

MULTIPLIERS ON THE SEGAL ALGEBRAS $W(R)$, $C^k(T)$, $S(\alpha)$ AND $V(T)$.

In this chapter we shall discuss the multipliers on the Wiener algebra and the algebras $C^k(T)$, $S(\alpha)$ and $W(R)$. First we consider the Wiener algebras and give a necessary and sufficient condition for a function F defined on the real line to thm be the Fourier transform of a multiplier. Next we consider the algebra $S(\alpha)$ and give a characterisation of the multipliers from $L^1(G)$ into $S(\alpha)$. If G is a compact abelian group, the space of multipliers from $L^1(G)$ to the Segal algebras $S(G)$ is characterised and particular cases are then considered. These include $C^k(T)$ and $V(T)$.

Let $W(R)$ denote the class of all continuous functions f defined on the real line R satisfying

$$\sum_{k=-\infty}^{\infty} \max_{k \le x \le k+1} |f(x)| < \infty .$$

$W(R)$ is then a Segal algebra if we define the norm on $W(R)$ by

$$\|f\|_s = \sup_{\alpha \in R} \sum_{k=-\infty}^{\infty} \max_{k \le x \le k+1} |f(x+\alpha)| .$$

We observe that if

$$\|f\|_W = \sum_{k=-\infty}^{\infty} \max_{k \le x \le k+1} |f(x)|$$

then

$$\|f\|_W \le \|f\|_S \le 2\|f\|_W$$

so that the norms $\|\cdot\|_S$ and $\|\cdot\|_W$ are equivalent.

The following theorem gives a necessary and sufficient condition for a bounded continuous function defined on the real line to be the Fourier transform of a multiplier.

THEOREM 3.2. A bounded and continuous function F defined on the real line R is the $\hat{f}$ Fourier transform of an element in $M[W(R)]$ if and only if there exists $\{f_n\} \subset W(R)$ such that

$$\hat{f}_n \to F$$

uniformly on compact subsets of R and

$$\|T_{f_n}\| \le K$$

for some constant K for all n where T_g is the multiplier corresponding to $g \in W(R)$ defined by

$$T_g(f) = g * f, \quad f \in W(R).$$

PROOF. Let $T \in M(W(R))$ be such that $\hat{T} = F$, that is

$$\widehat{Tf}(x) = F(x)\hat{f}(x), \; f \in W(R), \; x \in R.$$

Let $\{e_n\}$ be an approximate identity for $W(R)$ satisfying the conditions that each $\hat{e}_n$ has compact support and $\|e_n\|_1 = 1$ for all n. Set $f_n = Te_n$. Then

$$\hat{f}_n = F\hat{e}_n.$$

Since $\hat{e}_n \to 1$ uniformly on compact subsets of R, it follows that

$$\hat{f}_n \to F$$

Uniformly on compact subsets of R. Also

$$\|T_{f_n}(f)\|_s = \|f_n * f\|_s = \|Te_n * f\|_s = \|e_n * Tf\|_s$$

$$\leq \|e_n\|_1 \|Tf\|_s = \|Tf\|_s \leq \|T\|\|f\|_s,$$

$$f \in W(R).$$

This implies that

(1) $$\|T_{f_n}\| \leq \|T\|$$

for all n thus completing the necessary partof the theorem.

To prove the sufficiency, we proceed as follows. Let $\{f_n\}$ be a sequence of functions in $W(R)$ satisfying the condition that $\hat{f}_n \to F$ uniformly on compact subsets of R and $\cancel{\|T_{f_n}\| \leq K}$.

$\|T_{f_n}\| \leq K$ for some constant K. Let $f \in W(R)$ be such that $\hat{f}$ has compact support. Then by the definition of the sequence $\{f_n\}$,

$$\hat{f_n}\hat{f} \to F\hat{f}$$

uniformly. Since all the functions $\hat{f_n}\hat{f}$ and $F\hat{f}$ are supported by a fixed compact sets,

$$\|\hat{f_n}\hat{f} - F\hat{f}\|_1 \to 0 \quad \text{as } n \to \infty .$$

Now by considering the inverse Fourier transform we have

$$\|f_n * f - g\|_\infty \to 0 \quad \text{as } n \to \infty$$

where g is the Fourier transform of f. Then

$$\max_{k \leq x \leq k+1} |g(x+\alpha)| = \max_{k \leq x \leq k+1} \lim_{n \to \infty} |f_n * f(x+\alpha)|$$

$$= \lim_{n \to \infty} \max_{k \leq x \leq k+1} |f_n * f(x+\alpha)|$$

so that

$$\sum_{k=-\infty}^{\infty} \max_{k \leq x \leq k+1} |g(x+\alpha)| = \sum_{k=-\infty}^{\infty} \lim_{n \to \infty} \max_{k \leq x \leq k+1} |f_n * f(x+\alpha)|$$

$$\leq \liminf_{n \to \infty} \sum_{k=-\infty}^{\infty} \max_{k \leq x \leq k+1} |f_n * f(x+\alpha)|$$

(by Fatou's lemma)

$$\leq \liminf_{n\to\infty} \|f_n * f\|_S$$

$$= \liminf_{n\to\infty} \|T_{f_n}(f)\|_S \leq \liminf_{n\to\infty} \|T_{f_n}\| \|f\|_S$$

$$\leq K \|f\|_S .$$

The above inequality is true, for all $\alpha \in R$ which therefore implies that

(9) $$\|g\|_S \leq K \|f\|_S$$

Thus $g \in W(R)$. If we define $Tf = g$, we have a mapping T from $B(R)$ into $W(R)$, where $B(R) = \{ f \in W(R) : \hat{f}$ has compact support $\}$. This mapping is linear and continuous. Since $B(R)$ is dense in $W(R)$, T can be extended to $W(R)$ as a multiplier on $W(R)$. Moreover the relations

$$\widehat{Tf} = \hat{g} = F\hat{f}$$

imply that $\hat{T} = F$. This completes the proof of the theorem.

REMARK 3.2. Let $N(R) = \{ F : F$ is bounded and continuous on R, with the property that there exists a sequence $\{f_n\} \subset W(R)$ satisfying $\|T_{f_n}\|_S \leq K$ for all n and $\hat{f}_n \to F$ uniformly on compact subsets of $R \}$. Then $N(R)$ is a Banach space with the following norm

$\|F\|_N = \inf\{K$: there exists $\{f_n\} \subset W(R)$ satisfying $\|T_{f_n}\| \leq K$ for all n and $f_n^\wedge \to F$ uniformly on compact subsets of $R\}$. Equations(1) and (2) then imply that $M[W(R)]$ is isometrically isomorphic to $N(R)$.

THEOREM 3.2. can be generalised to a general segal algebra on a compact abelian group as follows.

THEOREM 3.3. [ ]. Let G be a compact abelian group with character group Γ. Let $S(G)$ be a Segal algebra on G. Let

$N(\Gamma) = \{\varphi \in L^\infty(\Gamma)$: there exists $\{f_\alpha\}_{\alpha \in A} \subset S(G)$ satisfying $\|T_{f_\alpha}\| \leq K$ for some constant K and $f_n^\wedge \to \varphi$ uniformly on compact subsets of $\Gamma\}$. $N(\Gamma)$ is then a Banach space with the norm given by: $\|\varphi\|_N = \inf\{K$: there exists $\{f_\alpha\}_{\alpha \in A} \subset S(G)$ such that $\|T_{f_\alpha}\| \leq K$ and $f_\alpha^\wedge \to \varphi$ uniformly on compact subsets of $\Gamma\}$. The space of multipliers $M[S(G)]$ is then isometrically isomorphic to $N(\Gamma)$.

PROOF. The proof follows along the same lines as in Theorem 3.2 and is hence omitted.

Let G be a locally compact nondiscrete abelian group with character group Γ. Let α be a locally bounded function on Γ with $\alpha(\gamma) \geq 1$ for all $\gamma \in \Gamma$. Let $S(\alpha)$ be defined by

$$S(\alpha) = \{ f \in L^1(G) : \lim_{\gamma} \hat{f}(\gamma)\alpha(\gamma) = 0, \text{ that is}$$

for every $\varepsilon > 0$ there exists a compact subset K of Γ such that $|\hat{f}(\gamma)\alpha(\gamma)| < \varepsilon, \gamma \notin K\}$. Then Riesemann [ ] has proved that $S(\alpha)$ is a Segal algebra with norm

$$\|f\|_S = \sup_{\gamma} |\hat{f}(\gamma)\alpha(\gamma)| + \|f\|_1$$

We have obtained a characterisation of the multipliers from $L^1(G)$ into $S(\alpha)$ as follows.

THEOREM 3.4. <u>Let</u> $B = \{\mu \in M_{bd}(G), \sup_{\gamma} |\hat{\mu}(\gamma)\alpha(\gamma)| < \infty\}$ <u>Then</u> B <u>is a Banach space with norm given by</u>

$$\|\mu\|_B = \|\mu\| + \sup_{\gamma} |\hat{\mu}(\gamma)\alpha(\gamma)| .$$

<u>The space</u> $M[L^1, S(\alpha)]$ <u>of multipliers from</u> $L^1(G)$ <u>into</u> $S(\alpha)$ <u>is then isometrically isomorphic to</u> B.

PROOF. The fact that B is a Banach space is easily verified. To prove $M[L^1, S(\alpha)] \simeq B$. Let $T \in M[L^1, S(\alpha)]$. Then $T \in (L^1, L^1)$. Therefore by Theorem 1.5, there exists a bounded Radon measure μ such that

$$Tf = \mu * f, \quad f \in L^1(G) .$$

Let $\{e_\beta\}$ be an approximate identity for $S(\alpha)$ satisfying $\|e_\beta\|_1 \leq 1$, $\hat{e}_\beta$ has compact support for all β. Then $\mu * e_\beta \in S(\alpha)$ with

$$\|\mu * e_\beta\|_S = \|\mu * e_\beta\|_1 + \sup_\gamma |\hat{\mu}(\gamma)\hat{e}_\beta(\gamma)\alpha(\gamma)| .$$

Now

$$|\hat{\mu}(\gamma)\alpha(\gamma)| = \lim_\beta |\hat{e}_\beta(\gamma)\hat{\mu}(\gamma)\alpha(\gamma)|$$

$$\leq \lim_\beta \sup_\gamma |\hat{e}_\beta(\gamma)\hat{\mu}(\gamma)\alpha(\gamma)|$$

$$\leq \lim_\beta \|\mu * e_\beta\|_S \leq \lim_\beta \|T\|\|e_\beta\|_1 = \|T\| .$$

Therefore $\mu \in B$ with

$$\sup_\gamma |\hat{\mu}(\gamma)\alpha(\gamma)| \leq \|T\| .$$

Conversely let $\mu \in B$. Then since

$$|\hat{\mu}(\gamma)\hat{e}_\beta(\gamma)\alpha(\gamma)| \leq |\hat{\mu}(\gamma)\alpha(\gamma)|\|e_\beta\|_1 = |\hat{\mu}(\gamma)\alpha(\gamma)| \leq \sup_\gamma |\hat{\mu}(\gamma)\alpha(\gamma)|$$

We have

$$\|\mu * e_\beta\|_S = \|\mu * e_\beta\|_1 + \sup_\gamma |\hat{\mu}(\gamma)\hat{e}_\beta(\gamma)\alpha(\gamma)|$$

$$\leq \|\mu\| + \sup_\gamma |\hat{\mu}(\gamma)\alpha(\gamma)| = \|\mu\|_B .$$

(4) being true for all $\{e_\beta\}$ by a result in [ ], μ corresponds to an element T_μ in $M[L^1, S(\alpha)]$ given by

$$T_\mu f = \mu * f, \qquad f \in L^1(G).$$

~~Ha Inmen(4)~~
with

$$(4) \qquad \|T_\mu\| \leq \|\mu\|_B .$$

We have thus established a mapping from $M[L^1, S(\alpha)]$ into ~~B~~ B which is linear and onto. It only remains to prove that this mapping is an isometry. If $\mu \in B$, then $\hat{\mu} \in C_0(\Gamma)$ This is because, if we take an infinite sequence $\{b_n\}$ of elements of Γ we can find an infinite subsequence $\{b_{n_k}\}$ such that

$$|\alpha(b_{n_k})| > 1/k .$$

This result is true because of a result in [ ] which says that α is bounded on Γ if and only if Γ is compact. Now $\mu \in B$ implies $\hat{\mu}$ satisfies for some constant K, the following inequality

$$|\hat{\mu}(\gamma)\alpha(\gamma)| \leq K, \quad \gamma \in \Gamma$$

Therefore

$$|\hat{\mu}(b_{n_k})|\,|\alpha(b_{n_k})| \leq K$$

that is

$$|\hat{\mu}(b_{n_k})| \leq \frac{K}{|\alpha(b_{n_k})|} \leq \frac{K}{k}$$

that is $\hat{\mu}(b_{n_k}) \to 0$ as $n_k \to \infty$. Therefore $\hat{\mu} \in C_0(\Gamma)$ which implies that given $\varepsilon > 0$ there exists a compact subset C of Γ satisfying

$$(5) \qquad |\hat{\mu}(\gamma)| < \varepsilon, \quad \gamma \notin C.$$

Since $\hat{e}_\beta \to 1$ uniformly on C, given $\varepsilon > 0$, there exists a finite subset J of the index set such that

$$|\hat{e}_\beta(\gamma)\,\hat{\mu}(\gamma)\,\alpha(\gamma) - \hat{\mu}(\gamma)\,\alpha(\gamma)| < \varepsilon/2, \quad \gamma \in K, \ \beta \notin J,$$

that is

$$|\hat{\mu}(\gamma)\,\alpha(\gamma)| < |\hat{e}_\beta(\gamma)\,\hat{\mu}(\gamma)\,\alpha(\gamma)| + \varepsilon/2, \quad \gamma \in K, \ \beta \notin J.$$

Choosing $\varepsilon < \dfrac{\|\hat{\mu}\alpha\|_\infty}{2}$, we have

$$(6) \qquad \sup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)\,\alpha(\gamma)| \le \sup_{\gamma \in \Gamma} |\hat{e}_\beta(\gamma)\,\hat{\mu}(\gamma)\,\alpha(\gamma)| + \varepsilon/2$$

Now $\mu * e_\beta$ converges weakly to μ in $M_{bd}(G)$. Therefore for a given $\varepsilon > 0$ take $f \in C_0(G)$ such that

$$(7) \qquad \|f\|_\infty \le 1, \quad \left|\int f\,d\mu\right| + \varepsilon/4 \geqslant \|\mu\|.$$

Then there exists a finite subset P of indices such that

$$(8) \qquad \left|\int f\,d\mu - \int f\,(\mu * e_\beta)\,dx\right| < \varepsilon/2$$

for all $\beta \notin P$.

Take $\Phi = L \cup P$. Then from (6), and (7) and (8) we have

$$\|\mu\| + \sup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)\alpha(\gamma)| \leq \sup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)\hat{e}_\beta(\gamma)\alpha(\gamma)| + \|\mu * e_\beta\|_1 + \varepsilon/2 + \varepsilon/2$$

that is

(9) $$\|\mu\|_B \leq \|T\mu\| + \varepsilon .$$

(9) being true for every $\varepsilon > 0$, we have

(10) $$\|\mu\|_B \leq \|T\mu\|$$

(4) and (10) combine to show that

$$\|\mu\|_B = \|T\mu\|_B$$

This proves that the mapping from $M[L^1, S(\alpha)]$ into B is an isometric isomorphism.

. Let G be a compact abelian group with character group Γ . Let $S(G)$ be a Segal algebra on G . Define $\tilde{B}$ in the following way:

$\tilde{B} = \{\mu \in M_{bd}(G)$: there exists $\{f_\alpha\}_{\alpha \in A} \subset S(G)$ satisfying $\|f_\alpha\|_S \leq K, \alpha \in A$ and $\hat{f}_\alpha \to \hat{\mu}$ pointwise on $\Gamma\}$. Then $\tilde{B}$ becomes a Banach space with norm given by

$|||\mu||| = \{\inf K :$ there exists $\{f_\alpha\}_{\alpha\in A} \subset S(G)$ satisfying $\|f_\alpha\|_S \leq K$, $\alpha \in A$ and $\hat{f}_\alpha \to \hat{\mu}$ pointwise $\}$. Then we have the following characterization of the multipliers from $L^1(G)$ into $S(G)$.

THEOREM 3.5. <u>The space of multipliers</u> $M(L^1, S)$ <u>is isometrically isomorphic to</u> $\tilde{B}$.

PROOF. Let $\{e_\alpha\}$ be an approximate identity for $L^1(G)$ consisting of functions $\{e_\alpha\}$ satisfying $\|e_\alpha\|_1 = 1$ for all α, $\hat{e}_\alpha$ has compact support for all α. Let $T \in M(L^1, S)$. Then $T \in M(L^1, L^1)$. Therefore by Theorem 1.5, there exists a bounded Radon measure μ such that

$$Tf = \mu * f, \quad f \in L^1(G).$$

Then we prove that $\mu \in \tilde{B}$. Define $h_\alpha = \mu * e_\alpha$. $\hat{h}_\alpha = \hat{\mu}\hat{e}_\alpha \to \hat{\mu}$ pointwise on Γ. Also

$$\|h_\alpha\|_S = \|\mu * e_\alpha\|_S \leq \|T\|\|e_\alpha\|_1 = \|T\|$$

for all α. This proves that $\mu \in \tilde{B}$ with

(11) $\quad \|\mu\| \leq \|T\|$

by the definition of the norm in $\tilde{B}$. Now let $\mu \in \tilde{B}$. Then

there exists a net $\{f_\alpha\}$ of functions in $S(G)$ for every $\varepsilon > 0$ satisfying

(12) $\hat{f}_\alpha(\gamma) \to \hat{\mu}(\gamma)$, $\gamma \in \Gamma$, $\|f_\alpha\|_S \leq |||\mu||| + \varepsilon$.

If $f \in B(G)$ then since $\hat{f}$ has compact support

$$\|\hat{f}_\alpha \hat{f} - \hat{\mu}\hat{f}\|_\infty \to 0$$

that is

$$\|\hat{f}_\alpha \hat{f} - \hat{\mu}\hat{f}\|_1 \to 0$$

which implies that

$$\|f_\alpha * f - \mu * f\|_\infty \to 0$$

that is, since G is compact we have

(13) $$\|f_\alpha * f - \mu * f\|_1 \to 0 .$$

Since the net of functions $\{\widehat{f_\alpha * f}\}$ and $\{\widehat{\mu * f}\}$ have compact support inside a fixed compact subset K of Γ, by Lemma 9.4, there exists a constant M such that

$$\|f_\alpha * f - \mu * f\|_S \leq M \|f_\alpha * f - \mu * f\|_1$$

for all α, that is

$$\lim_\alpha \|f_\alpha * f - \mu * f\|_S = 0 .$$

Therefore

$$\|\mu * f\|_S = \lim_\alpha \|f_\alpha * f\|_S \leq \lim_\alpha \|f_\alpha\|_S \|f\|_1$$

$$\leq (|||\mu||| + \varepsilon)\, \|f\|_1 .$$

This proves that the mapping T_μ defined on $L^1(G)$ by

$$T_\mu(f) = \mu * f$$

defines an element of $M(L^1, S)$ with

(14) $$\|T_\mu\| \leq |||\mu||| + \varepsilon .$$

The inequality (14) is true for every $\varepsilon > 0$. Thus

(15) $$\|T_\mu\| \leq |||\mu|||.$$

From (11) and (15) we then have

$$\|T_\mu\| = |||\mu|||.$$

The mapping $\mu \to T_\mu$ from $\tilde{B}$ into (L^1, S) is a linear map which is moreover an isometry. Therefore $\tilde{B}$ is isometrically isomorphic to (L^1, S).

Dunham and Goldberg in [ ] have defined $\tilde{S} = \{f \in L^1(G) :$ there exists $\{f_n\} \subset S(G)$ satisfying $\|f_n\|_S \leq K$ for some K and $\|f_n - f\|_1 \to 0$ as $n \to \infty$. They have normed it as follows.

$$\|f\|_{\tilde{S}} = \inf\{K : \text{ there exists } \{f_n\} \subset S(G)$$

satisfying $\|f_n\|_S \leq K$ for all n and $\lim_{n\to\infty} \|f_n - f\|_1 = 0\}$.

It is easily seen that $\tilde{S}$ is a closed ideal in $\tilde{B}$ and that it is isometrically embedded in $\tilde{B}$. We also have

COROLLARY 3.6. $\tilde{B} \cap L^1(G) = \tilde{S}$.

PROOF. By Corollary 2 in [ ], we have

$$(L^1, S) \cap L^1(G) = \tilde{S}.$$

This implies from Theorem 3.5 that

$$\tilde{B} \cap L^1(G) = \tilde{S}$$

COROLLARY 3.7. $S(G)$ <u>is a closed ideal in</u> $\tilde{B}$ <u>with</u>

$$\|f\|_S = |||f|||$$

PROOF. Since $S(G)$ is a closed ideal in $\tilde{S}$[ ] and $\tilde{S}$ is isometrically imbedded in $\tilde{B}$, we therefore have the required result.

THEOREM 3.8. <u>If</u> $\mu \in \tilde{B}$, $|||\mathcal{L}_y\mu - \mu||| \to 0$ <u>as</u> $y \to e$ <u>the identity of</u> G <u>if and only if</u> $\mu \in S(G)$.

PROOF. If $f \in S(G)$, since by Corollary 3.7

$$\|f\|_S = |||f|||,$$

the assertion of the theorem is true by the definition of a Segal algebra. Conversely if there exists a $\mu \in \tilde{B}$ with

$$\cancel{|||\mathcal{L}_y\mu - \mu||| \to 0 \text{ as } y \to e.}$$

$$|||\tau_y \mu - \mu||| \longrightarrow 0 \quad \text{as} \quad y \to e .$$

we have to prove that $\mu \in S(G)$. Since $\mu \in \tilde{B}$, there exists $\{f_\alpha\} \subset S(G)$ satisfying

$$\hat{f}_\alpha(\gamma) \longrightarrow \hat{\mu}(\gamma), \ \gamma \in \Gamma, \ \|f_\alpha\|_S \leq |||\mu||| + \varepsilon .$$

It is easily proved that there exists a subnet of $\{f_\alpha\}$ say $\{f_{\alpha_\beta}\}$ which converges weakly in $M_{ba}(G)$ to μ. This implies that

$$\|\mu\| \leq |||\mu||| + \varepsilon ,$$

and since this is true for every $\varepsilon > 0$, we have

$$\|\mu\| \leq |||\mu||| .$$

Therefore

$$|||\tau_y \mu - \mu||| \to 0 \quad \text{as} \quad y \to e$$

implies

$$\|\tau_y \mu - \mu\| \to 0 \quad \text{as} \quad y \to e .$$

From Lemma 1.11, μ is absolutely continuous, that is $\mu \in \tilde{B} \cap L^1(G) = \tilde{S}$. By corollary 16 in [ ], we therefore see that $\mu \in S(G)$.

Similarly using Theorem 14 [ ], we can prove the following theorem

THEOREM 3.9. If $\mu \in \tilde{B}$, $\mu \in S(G)$ <u>if and only if</u>

given $\varepsilon > 0$ there exists $\{e_\alpha\} \subset S(G)$ satisfying

$|||e_\alpha * \mu - \mu||| < \varepsilon$.

We now give two applications of Theorem 3.6 to special cases of Segal algebras. Let T denote the circle group. Let $V(T)$ be defined by

$$V(T) = \{f \in L^1(T) : \|f - D_N * f\|_1 \to 0 \text{ as } N \to \infty$$

where D_N is the Dirichlet kernel of order $N\}$. Then $V(T)$ is a Segal algebra with norm given by

$$\|f\|_S = \sup_{N \geq 1} \|D_N * f\|_1$$

In order to investigate $M[L^1, V]$ it is enough to determine the space $\tilde{B}$ for this algebra. If $\tilde{V}(T) = \{\mu \in M_{bd}(T) : \sup_{N \geq 1} \|D_N * \mu\|_1 < \infty\}$ then $\tilde{V}(T)$ is a Banach space with norm given by

$$\|\mu\|_{\tilde{V}} = \sup_{N \geq 1} \|D_N * \mu\|_1$$

We now prove the following theorem:

THEOREM 3.10. $\tilde{B}$ <u>is isometrically isomorphic to</u> $\tilde{V}(T)$.

PROOF. Let $\mu \in \tilde{B}$. Since $D_N \in L^1(T)$, $\mu * D_N \in L^1(T)$. Let $\{f_\alpha\}$ be the net of functions in $V(T)$ satisfying

$$\|f_\alpha\|_S \leq |||\mu||| + \varepsilon, \quad \hat{f}_\alpha(n) \to \hat{\mu}(n), \; n \in Z$$

for some $\varepsilon > 0$ where Z denotes the group of all integers. Then since $\hat{D}_N$ has finite support, by a proof analogous to that of Theorem 3.6, we obtain

$$\lim_{\alpha} \| f_\alpha * D_N - \mu * D_N \|_1 = 0$$

which implies that

$$\|\mu * D_N\|_1 = \lim_{\alpha} \| f_\alpha * D_N \|_1 \leq \lim_{\alpha} \sup_N \| f_\alpha * D_N \|_1$$
$$\leq \lim_{\alpha} \| f_\alpha \|_S \leq |||\mu||| + \varepsilon .$$

Therefore

$$\sup_N \|\mu * D_N\|_1 \leq |||\mu||| + \varepsilon .$$

This being true for all $\varepsilon > 0$, we have

$$\sup_N \|\mu * D_N\|_1 \leq |||\mu||| ,$$

that is $\mu \in \tilde{V}(T)$ with

(16) $$\|\mu\|_{\tilde{V}} \leq |||\mu|||$$

Conversely if $\mu \in \tilde{V}(T)$, let $\{e_n\}$ be an approximate identity for $L^1(T)$ consisting of functions satisfying $\|e_n\|_1 = 1$, and $\hat{e}_n$ has compact support for all n. Then if $f_n = \mu * e_n$, $\{f_n\}$ satisfies

$$\hat{f}_n(m) = \hat{e}_n(m)\hat{\mu}(m) \longrightarrow \hat{\mu}(m), \; m \in Z ;$$

Also

$$\|f_n\|_S = \|\mu * e_n\|_S = \sup_N \|\mu * e_n * D_N\|_1$$

$$\leq \|e_n\|_1 \sup_N \|\mu * D_N\|_1$$

$$= \|\mu\|_{\tilde{V}} .$$

Therefore xt by the definition of $\tilde{B}$, $\mu \in \tilde{B}$ with

(17) $$|||\mu||| \leq \|\mu\|_{\tilde{V}} .$$

(16) and (17) then combine to show that

$$\|\mu\|_{\tilde{V}} = |||\mu||| .$$

Thus there exists a linear, one to one, onto correspondence between $\tilde{B}$ and $V(T)$ thus proving the required result.

Regarding the Segal algebra $C^k(T)$ which is defined for an integer $k > 0$ as

$$C^k(T) = \{ f \in L^1(T) : f \text{ has } k \text{ continuous derivatives on } T \}$$

with norm

$$\|f\|_S = \sum_{j=0}^{k} \frac{1}{j!} \|f^j\|_\alpha ,$$

f^j denoting the jth differential of f, we have the following characterisation of $\tilde{B}$.

THEOREM 3.11. $\tilde{B}$ is isometrically isomorphic to the space $\tilde{C^k}(T) = \{f : f$ is k times differentiable on T, the kth derivative is a bounded function$\}$ equipped with the norm

$$\|f\|_{\tilde{C^k}} = \sum_{j=0}^{k} \frac{1}{4^j} \|f^j\|_\infty .$$

PROOF. Let $\mu \in \tilde{B}$. Then there exists $\{f_\alpha\} \subset C^k(T)$ with

(18) $$\hat{f_\alpha}(n) \to \hat{\mu}(n), \quad n \in Z, \quad \|f_\alpha\|_s \leq |||\mu||| + \varepsilon$$

for some $\varepsilon > 0$. Therefore

$$\|f_\alpha\|_\infty \leq \|f_\alpha\|_s \leq |||\mu||| + \varepsilon$$

for all α which implies that there exists a function $g \in L^\infty(T)$ and a subnet of $\{f_\alpha\}$ say $\{f_{\alpha_\beta}\}$ such that

$$f_{\alpha_\beta} \to g$$

weakly in $L^\infty(T)$. Since $e^{imt} \in L^1(T)$ for all $m \in Z$ we therefore have

(19) $$\hat{f_{\alpha_\beta}}(m) \to \hat{g}(m), \quad m \in Z$$

(18) and (19) then show that

$$\hat{g}(m) = \hat{\mu}(m), \quad m \in Z .$$

From the uniqueness theorem for Fourier Stieltjes transforms, we

then have

(20) $$g(x)\,dx = d\mu(x)\,,\ x \in \mathfrak{A}.$$

Since

$$\|f^1_{\alpha_\beta}\|_\infty \leq \|f_{\alpha_\beta}\|_S \leq |||\mu||| + \varepsilon$$

for all α_β there exists a subnet of $\{f^1_{\alpha_\beta}\}$ say $\{f^1_{\alpha_{\beta_\gamma}}\}$ and an element $h \in L^\infty(T)$ such that

$$f^1_{\alpha_{\beta_\gamma}} \to h$$

weakly in $L^\infty(T)$. This again implies that

$$\widehat{f^1_{\alpha_{\beta_\gamma}}}(m) \to \hat{h}(m)\,,\ m \in Z$$

which from (19) then shows that

$$\hat{h}(m) = \frac{1}{im}\,\hat{g}(m)\,,\quad m \in Z$$

which implies that the distributional derivative of g is h. Since $h \in L^\infty(T) \subset L^1(T)$, this implies of by a result in [3] that g is absolutely continuous, and that its derivative exists almost everywhere. Similarly proceeding, we see that h is absolutely continuous with a derivative existing almost everywhere in $L^\infty(T)$ and so on. Therefore g will have k derivatives, the kth derivative being a function in $L^\infty(T)$ which proves that $g \in C^{\infty k}(T)$. Also at each stage

$$\|g\|_\infty \leq \lim_{\alpha_\beta} \|f_{\alpha_\beta}\|_\infty\,,\ \|g^1\|_\infty \leq \lim_{\alpha_{\beta_\gamma}} \|f^1_{\alpha_{\beta_\gamma}}\|_\infty \cdots$$

Therefore there exists a subnet of $\{f_\alpha\}$ say $\{f_{\alpha_\gamma}\}$ satisfying

$$\|g\|_\infty \leq \lim_{\alpha_\gamma} \|f_{\alpha_\gamma}\|_\infty , \quad \|g^1\|_\infty \leq \lim_{\alpha_\gamma} \|f_{\alpha_\gamma}\|_\infty , \ldots$$

This implies that

$$(21) \qquad \|g\|_{\tilde{C}^k} = \sum_{j=0}^{k} \frac{1}{j!} \|g^j\|_\infty \leq \lim_{\alpha_\gamma} \sum_{j=0}^{k} \frac{\|f_{\alpha_\gamma}^j\|_\infty}{j!}$$

$$= \lim_{\alpha_\gamma} \|f_{\alpha_\gamma}\|_s \leq |||\mu||| + \varepsilon .$$

(21) being true for all $\varepsilon > 0$ we then have

$$\|g\|_{\tilde{C}^k} \leq |||\mu||| ,$$

that is from (20) we then have $\mu \in \tilde{C}^k$ with

$$(22) \qquad \|\mu\|_{\tilde{C}^k} \leq |||\mu||| .$$

Conversely if $\mu \in \tilde{C}^k(T)$ then if $\{e_n\}$ is the approximate identity considered in Theorem 3.10, set $f_n = \mu * e_n$. Then $\mu * e_n \in C^k(T)$ with

$$(23) \qquad \|f_n\|_s = \|\mu * e_n\|_s = \sum_{j=0}^{k} \frac{1}{j!} \|(\mu * e_n)^j\|_\infty = \sum_{j=0}^{k} \frac{1}{j!} \|\mu^j * e_n\|_\infty$$

$$\leq \|e_n\|_1 \sum_{j=0}^{k} \frac{1}{j!} \|\mu^j\|_\infty = \|\mu\|_{\tilde{C}^k} .$$

Also

$$\hat{f_n}(m) = \hat{\mu}(m)\,\hat{e_n}(m) \longrightarrow \hat{\mu}(m),\ m \in Z.$$

Therefore by the definition of $\tilde{B}$, $\mu \in \tilde{B}$ with

(24) $$|||\mu||| \leq \lim_n \|f_n\|_S \leq \|\mu\|_{\tilde{C^k}}$$

from (23). From (24) and (22) we then have

(25) $$\|\mu\|_{\tilde{C^k}} = |||\mu|||, \quad \mu \in \tilde{B}.$$

Therefore we have established a one to one onto correspondence between $\tilde{B}$ and $C^k(T)$ which is easily seen to be linear. By (25) we see that it is also an isometry. This completes the proof of the theorem.

We also have the following characterization of the multipliers on $C^k(T)$.

THEOREM 3.12. <u>The space of multipliers</u> $M[C^k(T)]$ <u>is isometrically isomorphic to</u> $M_{bd}(T)$, <u>the space of bounded Radon measures on</u> T.

PROOF. If $\mu \in M_{bd}(T)$, since $C^k(T)$ is a Segal algebra from Lemma 0.2, $\mu * f \in C^k(T)$ with

$$\|\mu * f\|_S \leq \|f\|_S \|\mu\|.$$

Therefore the mapping T_μ given by ~~$T_\mu f = \mu * f,\ f \in C^k(T)$~~

$$T_\mu f = \mu * f, \quad f \in C^k(T)$$

belongs to $M[C^k(T)]$ with

(26) $$\|T\| \leq \|\mu\|.$$

Conversely let $T \in M[C^k(T)]$. From Theorem 0.8, we see that there exists a unique pseudomeasure σ satisfying

$$Tf = \sigma * f, \quad f \in C^k(T)$$

For $f \in C^k(T)$, f^k is a continuous function on T and hence belongs to $L^2(T)$. Therefore $\sigma * f^k$ is a well defined function belonging to $L^2(T)$ with

$$(\sigma * f^k)^\wedge(m) = \hat{\sigma}(m)\,(f^k)^\wedge(m), \quad m \in Z$$

Also

$$(Tf)^k = (\sigma * f)^k = \sigma * f^k.$$

Therefore

(27) $$(\widehat{Tf})^k(m) = \hat{\sigma}(m)\,(f^k)^\wedge(m), \quad m \in Z,\ f \in C^k(T)$$

Let $f \in C(T)$. Consider $h(x) = f(x) - \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt$. Then

$$F(x) = \int_0^x h(t)\,dt$$

defines a continuous periodic function on T with

$$F^1(x) = h(x) = f(x) - \hat{f}(0).$$

Repeating the procedure k times, we obtain a function $g \in C^k(T)$ with

$$g^k(x) = f(x) - \hat{f}(0), \; x \in T$$

Define a mapping T' on $C(T)$ as follows

$$T'f = (Tg)^k + \hat{f}(0)\delta(0).$$

Then

$$\widehat{T'f}(m) = \widehat{(Tg)^k}(m), \; m \in 2, \; m \neq 0$$
$$= \widehat{(Tg)^k}(0) + \hat{f}(0)\delta(0), \; m = 0,$$

that is from (27) we have

(28) $$\widehat{T'f}(m) = \delta(m)(g^k)^\wedge(m) = \delta(m)\hat{f}(m), \; m \neq 0$$
$$= \delta(0)(g^k)^\wedge(0) + \delta(0)\hat{f}(0) = \delta(0)\hat{f}(0), \; m =$$

T' is then a well defined linear mapping of $C(T)$ into itself. To prove that it is continuous, we apply the closed graph theorem. Let $\{f_n\}$ converge to f in $C(T)$, and $T'f_n$ converge to g in $C(T)$. Then

(29) $$\lim_n f_n^\wedge(m) = \hat{f}(m), \; m \in 2$$

and

(30) $$\lim_n (T'f_n)^\wedge(m) = \hat{g}(m), \; m \in 2$$

But from (28) we have

(31) $$\lim_n (T' f_n)^\wedge(m) = \lim_n \sigma^\wedge(m) f_n^\wedge(m) = \sigma^\wedge(m) \hat{f}(m)$$

from (2). Thus (30) and (31) together give

$$\hat{g}(m) = \sigma^\wedge(m) \hat{f}(m) = \widehat{T'f}(m), \quad m \in Z$$

This proves that $g = T'f$ and that the mapping T' is continuous. Also since

$$(T'f)^\wedge(m) = \hat{f}(m)\, \hat{\sigma}(m), \quad m \in Z$$

T' defines a multiplier on $C(T)$. Since the multipliers on $C(T)$ correspond to measures in $M_{bd}(T)$[ ], there exists a measure $\mu \in M_{bd}(T)$ such that

(32) $$T'f = \mu * f, \quad f \in C(T)$$

with

(33) $$\|T'\| = \|\mu\|.$$

Now if $u \in C^k(T)$, there exists a $v \in C^k(T)$ such that $v^k = u - \hat{u}(0)$. Therefore

$$\begin{aligned} T'u = \mu * u &= (Tv)^k + \sigma^\wedge(0)\, \hat{u}(0) \\ &= \sigma * v^k + \sigma^\wedge(0)\, \hat{u}(0) \\ &= \sigma * [u - \hat{u}(0)] + \sigma^\wedge(0)\, \hat{u}(0) \\ &= \sigma * u, \quad u \in C^k(T). \end{aligned}$$

Therefore

$$\sigma * u = \mu * u, \quad u \in C^R(T).$$

This implies therefore that $\sigma = \mu$. Also

$$(34) \quad \|T'\| = \|\mu\| = \sup_{\substack{w \in C^R(T) \\ \|w\|_\infty \le 1}} \|T'w\|_\infty = \sup_{\substack{w \in C^R(T) \\ \|w\|_\infty \le 1}} \|Tw\|_\infty$$

$$\le \sup_{\substack{w \in C^R(T) \\ \|w\|_\infty \le 1}} \|Tw\|_S \le \sup_{\substack{w \in C^R(T) \\ \|w\|_S \le 1}} \|Tw\|_S$$

$$\le \|T\|.$$

From (34) and (26) we then have

$$\|\mu\| = \|T\|.$$

This proves the required result.

We now give an example of an element of $M[W(R)]$ which does not correspond to convolution with a bounded measure. First we prove a lemma.

LEMMA 3. 13. <u>Every element in $W(R)$ is the sum $\sum f_i$ of translates of elements with support in $(-1,1)$.</u>

Let $f \in W(R)$. For each integer n define a continuous function h_n with support contained in $(2n-1, 2n+1)$ and satisfying

$$h_n(x) = f(x), \quad x \in [2n - 1/2, 2n + 1/2]$$

$$= \text{linear outside } [2n - 1/2, 2n + 1/2].$$

Define

$$g_n(x) = f(x) - h_n(x) - h_{n+1}(x) \quad \text{for } x \in [2n+\tfrac{1}{2},\ 2n+3/2]$$
$$= 0 \quad \text{outside } [2n+1/2,\ 2n+3/2]$$

Since f, h_n, h_{n+1} are continuous so is g_n. Moreover

$$g_n(2n+1/2) = g_n(2n+3/2) = 0$$

so that the support of g_n is $[2n+1/2,\ 2n+3/2] \subset [2n,\ 2n+2)$

Thus h_n and g_n are continuous functions with support contained in open intervals of length 2 and hence can be thought of as translates of continuous functions with support contained in (-1,1). Further it is clear from the construction that

$$f(x) = \sum_{-\infty}^{\infty} (h_n(x) + g_n(x)) .$$

If

$$N_k = \max\{|f(x)| : k \le x \le k+1\}$$

then

$$\|h_n\|_\infty \le N_{2n-1} + N_{2n}$$

and

$$\|g_n\|_\infty \le 2[N_{2n} + N_{2n+1}]$$

If we set

$$f_{2n-1} = h_n, \quad n = 0, 1, 2, \ldots$$
$$f_{2n} = g_n$$

then

$$f = \sum_{j=-\infty}^{\infty} f_j = \sum_{n=-\infty}^{\infty} (h_n + g_n)$$

and

$$(35) \qquad \sum_{j=-\infty}^{\infty} \|f_j\|_\infty = \sum_{n=-\infty}^{\infty} (\|h_n\|_\infty + \|g_n\|_\infty) = 3 \sum_{k=-\infty}^{\infty} N_k = 3\|f\|_W$$

THEOREM 3.14. **There exists a function F, which is locally integrable, on the real line such that for every $f \in W(R)$, $F * f \in W(R)$ with**

$$\|F * f\|_W \leq K \|f\|_W$$

for some constant K independent of $f \in W(R)$. Also the multiplier so defined on $W(R)$ does not correspond to convolution with a bounded Radon measure on R.

PROOF. Define $F(t) = \dfrac{1}{(n+1)^2} e^{2\pi i n t}$, $n^2 < t < (n+1)^2$, $n = 0, 1, 2, \ldots$

$$= 0 \qquad \text{for } t \leq 0.$$

Then $F(t)$ is a measurable function on the real line R. However $F(t)$ is not absolutely integrable since

$$\int_{-\infty}^{\infty} |F(t)|\, dt = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \int_{n^2}^{(n+1)^2} dt = \sum_{n=1}^{\infty} \frac{(2n+1)}{n^2} = \infty$$

Therefore if we define

$$\mu(E) = \int_E F(t)\, dt$$

for all Borel subsets of R then μ defines a measure on the real line, but μ is not ~~a~~ bounded since ,

$$\int_{-\infty}^{\infty} |d\mu(t)| = \int_{-\infty}^{\infty} |F(t)|\, dt = \infty$$

Let $f \in W(R)$ be a continuous function with compact support in $[-1,1]$. Then

$$F * f(x) = \int_{-1}^{+1} F(x-t)\, f(t)\, dt$$

For $n \geq 2$, when $n^2-1 \leq x \leq n^2+1$ and $-1 \leq t \leq +1$, we have

$$n^2 - 2 \leq x - t \leq n^2 + 2$$

that is

$$(n-1)^2 < n^2 - 2 \leq x - t \leq n^2 + 2 \leq (n+1)^2 .$$

Therefore

$$|F * f(x)| = \left| \int_{-1}^{+1} F(x-t)\, f(t)\, dt \right| \leq \|f\|_\infty \int_{-1}^{+1} |F(x-t)|\, dt$$

$$\leq \|f\|_\infty \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} \right] \times 2 \leq \frac{4\|f\|_\infty}{n^2}$$

that is for $n \geq 2$ we have

$$(36) \quad \|F * f\|_{\infty\, [n^2-1,\, n^2+1]} \leq \frac{4\|f\|_\infty}{n^2} .$$

For $n \geq 2$, $n^2+1 \leq x \leq (n+1)^2-1$ and $-1 \leq t \leq +1$,

$$n^2 < x-t < (n+1)^2,$$

which implies that

$$F(x-t) = \frac{1}{(n+1)^2} e^{2\pi i n t}$$

Thus

$$F * f(x) = \frac{1}{(n+1)^2} \int_{-1}^{+1} e^{2\pi i n t} f(t)\, dt = \frac{1}{(n+1)^2} \hat{f}(n),$$

that is

$$(37) \qquad |F * f(x)| = 1/n^2 \, |\hat{f}(n)|$$

Since F has its support in $[0, 4]$ and f in $[-1, 1]$, $F*f$ is supported by $[-1, 4]$. Therefore we are only left to consider the case $-1 \leq x \leq 3$ and $-1 \leq t \leq +1$. For

$$-2 \leq x-t \leq 0, \quad F(x-t) = 0$$

$$0 \leq x-t \leq 1, \quad F(x-t) = 1$$

$$1 \leq x-t \leq 4, \quad F(x-t) = \frac{1}{4} e^{2\pi i (x-t)}$$

Therefore for $-1 \leq x \leq 3$, $-1 \leq t \leq +1$, we have

$$(38) \quad |F * f(x)| \leq \int_{-1}^{+1} \|f\|_\infty \cdot 2 = 4\|f\|_\infty$$

From (36), (37), (38) we then have

$$\|F * f\|_W = \sum_{k=-\infty}^{\infty} \max_{k \le x \le k+1} |F * f(x)| \le 4\|f\|_\infty + \sum_{n=2}^{\infty} \frac{8\|f\|_\infty}{n^2} + \sum_{n=2}^{\infty} \frac{(2n-1)\,|f(x)|}{(n+1)^2}$$

$$\le 4\|f\|_\infty + \sum_{n=2}^{\infty} \frac{8\|f\|_\infty}{n^2} + \left[\sum_{n=2}^{\infty} \frac{(2n-1)^2}{(n+1)^4}\right]^{1/2} \times \|f$$

$$\le 4\|f\|_\infty + 8\|f\|_\infty \times \sum_{n=2}^{\infty} 1/n^2 + 2\|f\|_\infty \left[\sum_{n=2}^{\infty} \frac{(2n-1)}{(n+1)}\right.$$

$$= \|f\|_\infty \left[4 + 8\sum_{n=2}^{\infty} 1/n^2 + 2\left[\sum_{n=2}^{\infty} \frac{(2n-1)^2}{(n+1)^4}\right]^{1/2}\right.$$

$$\cancel{= \|f\|_\infty}$$

Let $C = 4 + 8\sum_{n=2}^{\infty} 1/n^2 + 2\left[\sum_{n=2}^{\infty} \frac{(2n-1)^2}{(n+1)^4}\right]^{1/2}$

a constant independent of f. Then we have the following

(39) $\|f * F\|_W \le C\|f\|_\infty$.

If $g \in W(R)$, then by Lemma 3.13 there exist continuous functions g_k with compact support in $[-1, 1]$ and elements $b_k \in R$ such that

$$g = \sum_{k=-\infty}^{\infty} \tau_{b_k} g_k ,$$

and

$$\sum_{k=-\infty}^{\infty} \|g_k\|_\alpha \leq 3\|g\|_W \quad \text{from (35)}.$$

Then

$$F * g(x) = \sum_{k=-\infty}^{\infty} F * \tau_{b_k} g_k(x) .$$

Now

$$\|F * \tau_{b_k} g_k\|_W \leq \|F * \tau_{b_k} g_k\|_S = \|F * g_k\|_S \leq 2\|F*$$
$$\leq 2C\|g_k\|_\alpha \text{ from (39}$$

Therefore

$$\sum_{k=-\infty}^{\infty} \|F * \tau_{b_k} g_k\|_W \leq 2C \sum_{k=-\infty}^{\infty} \|g_k\|_\alpha \leq 2C \times 3\|g\|_W$$

which implies that

$$\|F * g\|_W \leq \sum_{k=-\infty}^{\infty} \|F * \tau_{b_k} g_k\|_W \leq 6C\|g\|_W .$$

This proves that the function F defines an element of $M[W(R)]$ which does not correspond to convolution with a bounded Radon measure on R.

CHAPTER IV

BIPOSITIVE AND ISOMETRIC ISOMORPHISMS OF MULTIPLIER ALGEBRAS

Bipositive and isometric isomorphisms of multiplier algebras were considered by Gaudry [ ]. When $1 \leq p < \infty$, he showed that if G_1 and G_2 are locally compact groups and $m_p(G_1), m_p(G_2)$ are the multiplier algebras of $L^p(G_1)$ and $L^p(G_2)$ respectively, then a bipositive ~~or~~ isometric isomorphism between $m_p(G_1)$ and $m_p(G_2)$ induces a topological isomorphism of the groups G_1 and G_2. In the case of abelian groups, the analogous results for the algebras $A^p(G), L^1 \cap L^p(G)$ and $L^1 \cap C_0(G)$ were given by Tewari [ ]. We shall prove here that the results are true for Segal algebras in general. We prove the following result.

THEOREM 4.1. [ ]. **If $S(G_1)$ and $S(G_2)$ are two Segal algebras on the locally compact abelian groups G_1 and G_2, $M[S(G_1)]$ and $M[S(G_2)]$ are their multiplier algebras. Then a bipositive isomorphism Λ of $M[S(G_1)]$ onto $M[S(G_2)]$ induces a topological isomorphism of the group G_1 onto G_2.**

Our Theorem includes the results of Tewari in the case of bipositive isomorphism of the multiplier algebras. On the other hand when G is compact, $L^2(G)$ is a Segal algebra and the following example given by Gaudry [8] shows that in the case of Segal algebras an isometric algebra isomorphism between the

multiplier algebras fails to induce a topological isomorphism between the groups.

EXAMPLE 4.2. Take $G_1 = T$, the circle group and $G_2 = T \times T$ the two dimensional torus. Then $m(L^2(G_1)) \simeq \ell^\infty(Z)$ where Z is the additive group of integers and

$$m[L^2(G_2)] \simeq \ell^\infty(Z \times Z)$$

the algebras ℓ^∞ being taken with pointwise operations and the usual sup norm. Each of Z and $Z \times Z$ is complete. Let φ be any one to one correspondence between $Z \times Z$ and Z. The mapping T_φ of $\ell^\infty(Z)$ onto $\ell^\infty(Z \times Z)$ defined by

$$T_\varphi \psi(m,n) = \psi[\varphi(m,n)]$$

is an isometric isomorphism of $\ell^\infty(Z)$ onto $\ell^\infty(Z \times Z)$. However Z and $Z \times Z$ are not algebraically isomorphic, therefore G_1 and G_2 are not isomorphic.

However in the case of the special Segal algebras $A^p_\omega(G)$ and $S^p(G)$, for $1 \leq p < \infty$, we have the following results.

THEOREM 4.3. [ ] <u>Let G_1 and G_2 be two locally compact abelian groups and ω_1, ω_2 two weight functions on Γ_1 and Γ_2 respectively. If there exists an isometric algebra isomorphism Λ between the multiplier algebras $M[A^p_{\omega_1}(G_1)]$ and</u>

$M[A^p_{\omega_2}(G_2)]$ then G_1 and G_2 are topologically isomorphic.

In the case of the algebras $S^p(G)$ defined in Example 7 of the introductory chapter, we have the following.

THEOREM 4.4. [ ] . If G_1 and G_2 are two locally compact abelian groups, then an isometric algebra isomorphism between the multiplier algebras $M[S^p(G_1)]$ and $M[S^p(G_2)]$ for $1 \leq p < \infty$, induces a topological isomorphism between the groups G_1 and G_2.

We shall now recall a few definitions and preliminary results.

By a positive multiplier on a space A of functions on G we mean a multiplier T satisfying: $f \in A$, $f \geq 0$ a.e. on G implies $Tf \geq 0$ a.e. on G. If A and B are two spaces of functions on G then a bipositive isomorphism Λ between the multiplier spaces $M(A)$ and $M(B)$ is an algebraic isomorphism which satisfies the condition that ΛT is a positive multiplier in $M(B)$ if T is a positive multiplier in $M(A)$. An isometric isomorphism Λ between $M(A)$ and $M(B)$ is an algebraic isomorphism of $M(A)$ onto $M(B)$ for which holds the equality.

$$\|\Lambda T\| = \|T\|, \quad T \in M(A)$$

Similar definitions hold if A and B are spaces of functions on the groups G_1 and G_2 respectively.

THEOREM 4.5. [ ] . If A is a semisimple commutative Banach algebra and $M(A)$ denotes the space of multipliers on A, then $M(A)$ is also a semisimple commutative Banach algebra.

THEOREM 4.6. [ ] . If G is a locally compact abelian group, T is a positive multiplier on $L^2(G)$ and σ is the measure corresponding to T, that is,

$$Tf = \sigma * f, \quad f \in L^2(G)$$

then σ reduces to a positive bounded measure in $M_{bd}(G)$.

THEOREM 4.7. [ ] If T is a norm preserving multiplier of $L^1(G)$, then there exists $a \in G$ and a complex number λ of absolute value one such that $T = \lambda \tau_a$.

THEOREM 4.8 [ ] . Let $F(G_1)$ and $F(G_2)$ be ideals of $L^1(G_1)$ and $L^1(G_2)$ respectively, which are Banach algebras in their own norm and let $M[F(G_i)]$ denote the multiplier algebras of $F(G_i)$. If Λ is a bipositive or isometric algebra isomorphism of $F(G_1)$ onto $F(G_2)$, then Λ induces a bipositive or isometric algebra isomorphism of $M[F(G_1)]$ onto $M[F(G_2)]$.

THEOREM 4.9. [ ] . If χ is a homomorphism of $L^1(G_1)$ into $M_{bd}(G_2)$, then $\widehat{\chi f} = \hat{f} \circ \alpha$ where α is a microscopic affine map of Y into Γ_1 and Y belongs to the center ring of Γ_2.

THEOREM 4.10. [ ] . Let G_1 and G_2 be two locally compact Hausdorff groups. The algebras $M_{bd}(G_i)$ are isometrically isomorphic if and only if G_1 and G_2 are topologically isomorphic.

Let

$$C_c^+(G) = \{ f \in C_c(G) : f \geq 0 \text{ on } G \}$$

$$B^+(G) = \{ f \in B(G) : f \geq 0 \text{ on } G \}$$

For $a \in G$, let ε_a denote the Dirac measure at a. We then have

LEMMA 4.11. If T is a positive multiplier on a local algebra $S(G)$ on a locally compact abelian group G there exists $\mu \in M_{bd}(G)$ such that

$$Tf = \mu * f$$

PROOF. To each multiplier T on $S(G)$ there corresponds, by Theorem 0. a unique pseudomeasure σ such that

$$Tf = \sigma * f \ , \quad f \in S(G) \ .$$

Now suppose that T is a positive multiplier. Then

$$\sigma * f \geq 0 \quad \text{a.e. on } G, \quad f \in B^+(G).$$

Let $f \in L^2(G)$ such that $f \geq 0$ a.e. on G. To each positive integer n, we can find a compact subset K_n of G with the property that

$$(1) \qquad \Big[\int_{G \setminus K_n} |f(x)|^2 dx\Big]^{1/2} < 1/2n\|\sigma\| .$$

Let $g_n(x) = f(x)\chi_{K_n}(x)$ where χ_{K_n} is the characteristic function of K_n. Let $\{e_\alpha\}$ be an approximate identity for $L^1(G)$ satisfying $e_\alpha \geq 0$ on G and $\hat{e}_\alpha$ has compact support for all α. Then $\{e_\alpha\}$ is also an approximate identity for $L^2(G)$. Hence there exists $e_n \in \{e_\alpha\}$ satisfying

$$(2) \qquad \|\sigma * e_n * f - \sigma * f\|_2 \leq 1/2n .$$

Let $h_n = g_n * e_n$.

Then

$$\|\sigma * h_n - \sigma * f\|_2 \leq \|\sigma * e_n * g_n - \sigma * e_n * f\|_2$$

$$+ \|\sigma * e_n * f - \sigma * f\|_2$$

$$\leq \|\sigma\| \, \|e_n\| \, \|g_n - f\|_2 + 1/2n \qquad \text{by (2)}$$

$$\leq \|\sigma\| \cdot \frac{1}{2n\|\sigma\|} + 1/2n = 1/n$$

Hence $\sigma * h_n$ converges to $\sigma * f$ in the $L^2(G)$ norm. We can find a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ such that

$$\sigma * h_{n_k} \to \sigma * f \quad a.e \text{ on } G.$$

Now set $h_{n_k} = e_{n_k} * g_{n_k}$. Then $h_{n_k} \in L^1(G)$ and $\hat{h}_{n_k} = \hat{e}_{n_k}\hat{g}_{n_k}$ has compact support. Moreover $h_{n_k} \geq 0$ on G so that $h_{n_k} \in B^+(G)$. Then $\sigma * h_{n_k} \geq 0$ on G for all n_k and hence $\sigma * f \geq 0$ a.e on G.

We have thus proved that if $f \in L^2(G)$ such that $f \geq 0$ a.e. on G then $\sigma * f \geq 0$ a.e. on G also. This implies that σ defines a positive multiplier on $L^2(G)$. By Theorem 4.6, therefore there exists a unique positive measure $\mu \in M_{bd}(G)$ such that

$$\sigma * f = \mu * f, \quad f \in L^2(G)$$

But $B(G) \subset L^2(G)$. Therefore

$$Tf = \sigma * f = \mu * f, \; f \in B(G)$$

Since $B(G)$ is dense in $S(G)$ we conclude that

$$Tf = \mu * f, \quad f \in S(G).$$

This completes the proof.

PROOF OF THEOREM 4.1. Let $S(G_1)$ and $S(G_2)$ be Segal algebras on the locally compact abelian groups G_1 and G_2 respectively. Suppose $M[S(G_1)]$ and $M[S(G_2)]$ denote their multiplier algebras. Suppose that Λ is a bipositive isomorphism of $M[S(G_1)]$ onto $M[S(G_2)]$. For each element $a \in G_1$ the translation operators τ_a and τ_{-a} are positive multipliers on $S(G_1)$. Since Λ is a bipositive isomorphism of $M[S(G_1)]$ onto $M[S(G_2)]$ it follows that $\Lambda\tau_a$ and $\Lambda\tau_{-a}$ are positive multipliers on $S(G_2)$. By Lemma 4.11 there exist positive bounded measures μ and ν in $M_{bd}(G_2)$ such that

$$\Lambda\tau_a(f) = \mu * f, \quad \Lambda\tau_{-a}(f) = \nu * f, \quad f \in S(G_2).$$

Since Λ is an algebraic isomorphism

$$\Lambda(\tau_a \circ \tau_{-a}) = \Lambda(\tau_{e_1}) = \tau_{e_2} = \Lambda\tau_a \circ \Lambda\tau_{-a}$$

where e_1 and e_2 denote the identities of G_1 and G_2, respectively. Therefore we have

$$\mu * \nu = \delta_{e_2}$$

We now claim that both μ and ν are measures with one point support. If possible, let b_1 and b_2 be two points in the support of μ and c a point in the support of ν. Let $\gamma \in C_c^+(G_2)$

be such that $0 \leq \chi \leq 1$. Define measures μ_1 and ν_1 by

$$\mu_1 = (\tau_{b_1}\chi)\mu + (\tau_{b_2}\chi)\mu, \quad \nu_1 = (\tau_c\chi)\nu$$

Then μ_1 and ν_1 are bounded positive measures. Moreover $\mu_1 \leq \mu$ and $\nu_1 \leq \nu$ so that

$$\mu_1 * \nu_1 \leq \mu * \nu$$

But $\mu_1 * \nu_1$ has at least two points in its support which is a contradiction. This proves our assertion that both μ and ν have one point support. Therefore there exist $a', b' \in G_2$ and positive real numbers λ_1, λ_2 such that

$$\mu = \lambda_1 \varepsilon_{\tau a'}, \quad \nu = \lambda_2 \varepsilon_{\tau b'}.$$

Since

$$\mu * \nu = \varepsilon_{\tau e_2}, \quad b' = -a', \quad \lambda_1 = 1/\lambda_2.$$

Therefore to every $a \in G_1$ we can associate a $\varphi(a) \in G_2$ and a positive real number $\lambda(a)$ such that

$$(\Lambda \tau_a)(f) = \lambda(a)\, \varepsilon_{\tau\varphi(a)} * f, \quad f \in S(G_2)$$

Since Λ is an algebraic isomorphism, it follows that φ is an algebraic isomorphism of G_1 onto G_2 and that λ is an algebraic isomorphism of G_1 onto the set of positive real numbers. Now if $\lambda(a) > 1$ for some $a \in G_1$, there exists a sequence of elements $\{a_n\} \in G_1$ such that

$$\lambda(a_n) > n^3 \text{ for all } n.$$

Consider the positive multiplier on $S(G_1)$ defined by

$$\tau = \sum_{n=1}^{\infty} \frac{1}{n^2} \tau_{a_n}$$

Then $\tau - \frac{1}{n^2}\tau_{a_n}$ is a positive multiplier on $S(G_1)$ for all n. Since Λ is bipositive, $\Lambda(\tau - \frac{1}{n^2}\tau_{a_n})$ is positive for every n. If μ denotes the positive measure in $M_{bd}(G_2)$ corresponding to $\Lambda\tau$, then we have

$$\mu \geqslant \frac{1}{n^2}\lambda(a_n)\, \xi_{\varphi(a_n)} \quad \text{for each } n$$

that is

$$\mu \geqslant n^3 \cdot \frac{1}{n^2}\xi_{\varphi(a_n)} = n\,\xi_{\varphi(a_n)} \quad \text{for each } n,$$

which is a contradiction to the fact that μ is a bounded measure. Therefore

$$\lambda(a) = 1, \quad a \in G_1$$

It remains to show that φ is a topological isomorphism. It is enough to show that φ is continuous. Then the same argument with Λ^{-1} will prove the continuity of φ^{-1}. In order to prove φ is continuous, it suffices to prove that if $a_i \to e_1$ in G_1, then $\varphi(a_i) \to e_2$ in G_2. Suppose ~~the~~ not. There exists an open neighbourhood V of e_2 and an infinite subset of $\{\varphi(a_i)\}$ such that all the elements of that subset lie outside V for all sufficiently large indices i. We shall assume

without loss of generality that $\{\varphi(a_i)\}$ belongs to the complement of V for all i.

Now $\{\varepsilon_{\varphi(a_i)}\}$ is a norm bounded net of measures in $M_{bd}(G_2)$. Therefore by Alaoglu's theorem [ ] there exists a subnet of $\{\varphi(a_i)\}$ which, without loss of generality, we assume to be $\{\varphi(a_i)\}$ itself, such that $\varepsilon_{\varphi(a_i)}$ converges to μ weakly in $M_{bd}(G_2)$ for some positive measure $\mu \in M_{bd}(G_2)$.

Take $h \in C_c^+(G_1)$. Denote the multipliers generated by h and $\varepsilon_{a_i} * h$ by W_h and $W_{\varepsilon_{a_i} * h}$ respectively. Since $a_i \to e_1$ in G_1, we have

$$\varepsilon_{a_i} * h \to h$$

in $M_{bd}(G_1)$. Now $M_{bd}(G_1)$ norm is stronger than the $M[S(G_1)]$ norm. Therefore we have

$$W_{\varepsilon_{a_i} * h} \longrightarrow W_h \quad \text{in } M[S(G_1)].$$

In other words

$$\tau_{a_i} \circ W_h \longrightarrow W_h \quad \text{in } M[S(G_1)]$$

Since $M[S(G_1)]$ and $M[S(G_2)]$ are two semi-simple commutative Banach algebras and Λ is a bipositive isomorphism between them, Λ is continuous. Hence

$$\Lambda\tau_a \circ \Lambda W_h = \Lambda(\tau_{a_i} \circ W_h) \longrightarrow \Lambda W_h$$

in $M[S(G_2)]$. ΛW_h being a positive multiplier on $S(G_2)$, we can find a corresponding positive measure $\mu_1 \in M_{bd}(G_2)$. Then

$$\varepsilon_{\varphi(a_i)} * \mu_1 * f \longrightarrow \mu_1 * f$$

in the $S(G_2)$ norm for all $f \in S(G_2)$. But the $L^1(G_2)$ norm is weaker than the $S(G_2)$ norm so that

$$\varepsilon_{\varphi(a_i)} * \mu_1 * f \longrightarrow \mu_1 * f \ , \ f \in S(G_2)$$

in the $L^1(G_2)$ norm also. Since $S(G_2)$ is dense in $L^1(G_2)$ we conclude that

(3) $$\varepsilon_{\varphi(a_i)} * \mu_1 * f \longrightarrow \mu_1 * f \ , \ f \in L^1(G_2)$$

in the $L^1(G_2)$ norm. We have also proved that

(4) $$\varepsilon_{\varphi(a_i)} \longrightarrow \mu \quad \text{weakly in } M_{bd}(G_2)$$

From (4) it follows that

$$\varepsilon_{\varphi(a_i)} * f(x) \longrightarrow \mu * f(x) \ , \ f \in C_0(G_2).$$

Thus if $f \in C_c(G_2)$, then $\mu_1 * f \in C_0(G_2)$ so that

$$\varepsilon_{\varphi(a_i)} * \mu_1 * f(x) \longrightarrow \mu_1 * \mu * f(x) \ , \ x \in G_2$$

On the other hand (3) implies

$$\varepsilon_{\varphi(a_i)} * \mu_1 * f \longrightarrow \mu_1 * f$$

in $L^1(G_2)$ so that there exists a subnet $\{\varphi(a_{i_k})\}$ of

$\{\varphi(a_i)\}$ such that

$$\xi_{\varphi(a_{i_k})} * \mu_1 * f(x) \longrightarrow \mu_1 * f(x)$$

for almost all $x \in G_2$. Thus we have

$$\mu * \mu_1 * f = \mu_1 * f \quad \text{a.e on } G_2 .$$

Since both the functions are continuous, we have

$$\mu * \mu_1 * f = \mu_1 * f , \quad f \in C_c(G_2)$$

But $C_c(G_2)$ is dense in $C_0(G_2)$. Thus we must have

$$\mu * \mu_1 = \mu_1$$

and so

$$W_\mu \circ \Lambda W_h = \Lambda W_h$$

where W_μ is the multiplier on $S(G_2)$ corresponding to μ.

This gives

$$\Lambda^{-1}(W_\mu) \circ W_h = W_h$$

and so

$$\Lambda^{-1}(W_\mu) \circ W_h(f) = W_h(f), \; f \in S(G_1)$$

Thus

$$\Lambda^{-1}(W_\mu)(h * f) = h * f , \; f \in S(G_1) .$$

If ν_1 is the positive measure corresponding to $\Lambda^{-1} W_\mu$,

we have

$$\nu_1 * h * f = h * f, \; h \in C_c^+(G_1), \; f \in S(G_1),$$

so that

$$\nu_1 = \varepsilon_{e_1}$$

and therefore

$$\Lambda^{-1}(W_\mu) = \varepsilon_{e_1}$$

which gives

$$W_\mu = \Lambda(\varepsilon_{e_1}) = \varepsilon_{e_2}.$$

This implies

$$\mu = \varepsilon_{e_2}$$

Hence

(8) $$\varepsilon_{\varphi(a_i)} \longrightarrow \varepsilon_{e_2} \text{ weakly in } M_{bd}(G_2)$$

Consider a neighbourhood W of e_2 whose closure is compact. There exists a neighbourhood U of e_2 with closure $\overline{U} \subset W$. There exists a function $f \in L^1(G_2)$ satisfying $\hat{f} = 0$ outside W and $\hat{f} = 1$ on $\overline{U}$. Since $\hat{f} \in C_0(G_2)$ by (8) we have

$$\hat{f}(\varphi(a_i)) \longrightarrow \hat{f}(e_2) = 1$$

Hence there exists an infinite subnet of $\{\varphi(a_i)\}$ say $\{\varphi(a_{i_k})\}$ inside $W \subset \overline{W}$. Since $\overline{W}$ is compact, there exists an infinite subnet of $\{\varphi(a_{i_k})\}$ say $\{\varphi(a_{i_{k_\ell}})\}$ and an $a' \in G_2$ such

that

$$\varphi(a_{i_{k_\ell}}) \longrightarrow a' .$$

Now $a' \notin V$ since $\{\varphi(a_{i_{k_\ell}})\}$ belongs to complement of V for all i_{k_ℓ}. Hence $a' \neq e_2$. Since $\varphi(a_{i_{k_\ell}}) \longrightarrow a'$

$$f(\varphi(a_{i_{k_\ell}})) \longrightarrow f(a'), \quad f \in C_0(G_2)$$

that is

$$\xi_{\varphi(a_{i_{k_\ell}})} \longrightarrow \xi_{a'} \quad \text{weakly}$$

Combining this with (5) we have $\xi_{e_2} = \xi_{a'}$ which is impossible since $a' \neq e_2$. Hence φ is continuous, thus completing the proof of the theorem.

COROLLARY 4.12. <u>A bipositive isomorphism of</u> $S(G_1)$ <u>onto</u> $S(G_2)$ <u>induces a topological isomorphism of</u> G_1 <u>onto</u> G_2.

PROOF. From Theorem 4.8, a bipositive isomorphism of $S(G_1)$ onto $S(G_2)$ induces a bipositive isomorphism of $M[S(G_1)]$ onto $M[S(G_2)]$. The conclusion then follows from Theorem 4.1.

Before we prove Theorem 4.3, we need a few lemmas.

LEMMA 4.13. <u>Let</u> G <u>be a locally compact abelian group and</u> T <u>a norm preserving multiplier of</u> $A^p_\omega(G)$ <u>for a suitable weight function</u> ω <u>defined on</u> Γ. <u>Then there exist</u> $a \in G$ <u>and a complex number</u> λ <u>of absolute value 1 such that</u> $T = \lambda \tau_a$.

PROOF. Case 1. G is compact. Let $\gamma \in \Gamma$. Then

$$\gamma * \gamma = \gamma$$

and hence

$$T(\gamma * \gamma) = T(\gamma) * \gamma = T(\gamma)$$

that is

$$T(\gamma) = \varphi(\gamma)\gamma$$

where $\varphi(\gamma)$ is a complex number. Since T is norm preserving it follows that $|\varphi(\gamma)| = 1$, $\gamma \in \Gamma$. For any $f = \sum_{i=1}^{n} a_i \gamma_i$ in $B(G)$ we ~~ha~~ then have

$$\|T(\sum_{i=1}^{n} a_i \gamma_i)\|_s = \|T(\sum_{1}^{n} a_i \gamma_i)\|_1 + [\sum_{1}^{n} |a_i \varphi(\gamma_i)\omega(\gamma_i)|^p]^{1/p}$$

$$= \|Tf\|_1 + [\sum_{i=1}^{n} |a_i \omega(\gamma_i)|^p]^{1/p}.$$

Since T is norm preserving

$$\|Tf\|_s = \|f\|_s = \|f\|_1 + [\sum_{i=1}^{n} |a_i \omega(\gamma_i)|^p]^{1/p}.$$

Therefore

$$\|Tf\|_1 = \|f\|_1, \quad f \in B(G)$$

Since $B(G)$ is dense in $L^1(G)$, there exists a unique norm preserving multiplier T' of $L^1(G)$ into itself such that

$$T'f = Tf, \quad f \in A^p_\omega(G).$$

Hence by Theorem 4.7, there exists λ and a as desired such that

$$Tf = \lambda \tau_a f, \quad f \in A^p_\omega(G)$$

Case 2. When G is noncompact locally compact abelian we have

$$M[A_\omega^p(G)] \simeq M_{bd}(G),$$

by Theorem 2.1. Therefore there exists a measure $\mu \in M_{bd}(G)$ such that

$$Tf = \mu * f, \quad f \in A_\omega^p(G)$$

with

$$\|\mu\| = 1$$

Now

$$\|\mu * f\|_S = \|\mu * f\|_1 + \|\hat{\mu}\hat{f}\omega\|_p = \|f\|_S$$
$$= \|f\|_1 + \|\hat{f}\omega\|_p$$

Therefore from (5) we have

$$\|\mu * f\|_1 = \|f\|_1, \quad f \in A_\omega^p(G).$$

Since $A_\omega^p(G)$ is dense in $L^1(G)$ an application of Theorem 4.7 yields the desired result.

LEMMA 4.14. Let G be a compact abelian group. Let $\{m_i\}_{i \in I}$ be a net of multipliers in $M[A_\omega^p(G)]$ such that $\{m_i\}_{i \in I}$ is norm bounded. Then there exists a subnet $\{m_{i_\alpha}\}$ of $\{m_i\}$

such that

$$\lim_{i_k} \int_{\Gamma} \hat{m}_{i_k}(\gamma)\, \hat{f}(\gamma)\, d\gamma = \int_{\Gamma} \hat{m}(\gamma)\, \hat{f}(\gamma)\, d\gamma, \quad f \in A(G)$$

for some multiplier $m \in M[A_\omega^p(G)]$

PROOF. Case 1. $1 \leq p \leq 2$. By Theorem 2.4 there exists a net $\{\sigma_i\}_{i \in I}$ of pseudomeasures corresponding to $\{m_i\}_{i \in I}$. Then $\{\sigma_i\}_{i \in I}$ is a norm bounded net of pseudomeasures. By Alaoglu's theorem, there exists a pseudomeasure σ and a subnet of $\{\sigma_i\}$ say $\{\sigma_{i_k}\}$ satisfying

$$\int_{\Gamma} \hat{\sigma}_{i_k}(\gamma)\, \hat{f}(\gamma)\, d\gamma \longrightarrow \int_{\Gamma} \hat{\sigma}(\gamma)\, \hat{f}(\gamma)\, d\gamma, \quad f \in A(G)$$

Let $m \in M[A_\omega^p(G)]$ correspond to σ. Then we have

$$\lim_{i_k} \int_{\Gamma} \hat{m}_{i_k}(\gamma)\, \hat{f}(\gamma)\, d\gamma = \int_{\Gamma} \hat{m}(\gamma)\, \hat{f}(\gamma)\, d\gamma, \quad f \in A(G).$$

Case 2. $2 < p < \infty$. By Theorem 2.6, $\{m_i\}$ corresponds to a norm bounded net of elements in $[R(G)]^*$. Hence there exists an element m of $[R(G)]^*$ and a subnet of $\{m_i\}$ converging to m in the weak topology of $[R(G)]^*$ that is

$$\int_{\Gamma} \hat{f}(\gamma)\, \hat{m}_{i_k}(\gamma)\, d\gamma \longrightarrow \int_{\Gamma} \hat{m}(\gamma)\, \hat{f}(\gamma)\, d\gamma, \quad f \in A(G)$$

This completes the proof of the lemma.

LEMMA 4.15. *Let G_1 and G_2 be locally compact abelian groups with character groups Γ_1 and Γ_2 respectively. For $1 \leq p < \infty$, if there exists an algebra isomorphism ψ of $M[A^p_{\omega_1}(G_1)]$ onto $M[A^p_{\omega_2}(G_2)]$, either both the groups are compact or both of them are noncompact.*

PROOF. We prove that if one of the groups say G_1 is compact, the other is also compact. Suppose G_2 is noncompact. By Theorem 2.1

$$M[A^p_{\omega_2}(G_2)] \simeq M_{bd}(G_2).$$

Thus ψ can be considered as an algebra isomorphism of $M[A^p_{\omega_1}(G_1)]$ onto $M_{bd}(G_2)$. Since every element of $L^1(G_1)$ gives an element of $M[A^p_{\omega_1}(G_1)]$, the restriction of ψ to $L^1(G_1)$ is an algebra isomorphism of $L^1(G_1)$ into $M_{bd}(G_2)$. By theorem 4.9 it follows that there exists a subset Y of Γ_2 and a piecewise affine map α of Y into Γ_1 such that for every $f \in L^1(G_1)$

$$\widehat{\psi f}(\gamma) = \begin{cases} \hat{f}(\alpha(\gamma)) & \text{if } \gamma \in Y \\ 0 & \text{if } \gamma \notin Y \end{cases}$$

Since by Theorem 0.10 every element of $M[A^p_{\omega_1}(G_1)]$ gives rise to a pseudomeasure, we identify each element of $M[A^p_{\omega_1}(G_1)]$ with the corresponding pseudomeasure on G. Thus if $\sigma \in M[A^p_{\omega_1}(G_1)]$ and $f \in B(G_1)$, we have $\sigma * f \in L^1(G_1)$ with

(6) $[\psi(\sigma * f)]^\wedge(\gamma) = (\sigma * f)^\wedge(\alpha\gamma) = \hat{\sigma}(\alpha\gamma)\hat{f}(\alpha\gamma), \gamma \in$

On the other hand

$$\psi(\sigma * f) = \psi(\sigma) * \psi(f).$$

Therefore

(7) $$[\psi(\sigma * f)]^\wedge(\gamma) = [\psi(\sigma)]^\wedge(\gamma)\,[\psi(f)]^\wedge(\gamma)$$
$$= [\psi(\sigma)]^\wedge(\gamma)\,\hat{f}(\alpha(\gamma)),\ \gamma \in Y.$$

From (6) and (7) we therefore have

(8) $$\hat{\sigma}(\alpha(\gamma))\,\hat{f}(\alpha(\gamma)) = [\psi(\sigma)]^\wedge(\gamma)\,\hat{f}(\alpha(\gamma)),\ \gamma \in Y.$$

Since (8) holds for every $f \in B(G_1)$, we have

(9) $$[\psi(\sigma)]^\wedge(\gamma) = \hat{\sigma}(\alpha(\gamma)),\ \gamma \in Y$$

Now we prove α is one to one on Y. Let $\gamma_1, \gamma_2 \in Y$ be such that $\gamma_1 \neq \gamma_2$. Choose $\mu \in M_{bd}(G_2)$ such that $\hat{\mu}(\gamma_1) \neq \hat{\mu}(\gamma_2)$. Let $\sigma \in M[A^p_{\omega_1}(G_1)]$ be such that $\psi(\sigma) = \mu$ ~~Theorem (2)~~ Then from (9) we have

$$\hat{\mu}(\gamma_1) = \hat{\sigma}(\alpha(\gamma_1)) \neq \hat{\mu}(\gamma_2) = \hat{\sigma}(\alpha(\gamma_2)).$$

Therefore

$$\hat{\sigma}(\alpha(\gamma_1)) \neq \hat{\sigma}(\alpha(\gamma_2))$$

that is α is onto one to one.

Next we show that $\alpha(Y) = \Gamma_1$. Since Γ_1 is discrete, $\alpha(Y)$ is closed in Γ_1. If $\alpha(Y) \neq \Gamma_1$ there exists $f \in B(G_1)$ such that $\hat{f} = 0$ on $\alpha(Y)$ but $\hat{f}$ is not identically zero. Since $\hat{f} \circ \alpha = 0$ we have $\psi(f) = 0$, which contradicts the one to ononess of ψ. Finally we prove that $Y = \Gamma_2$. If $Y \neq \Gamma_2$, since Y is a closed subset of Γ_2, there exists $\mu \in M_{bd}(G_2)$ such that $\mu \neq 0$, $\hat{\mu} = 0$ on Y. Choose $\sigma \in M[A^p_{\omega_1}(G_1)]$ such that $\psi(\sigma) = \mu$. By (9) we therefore have

$$\hat{\mu}(\gamma) = [\psi(\sigma)]^\wedge(\gamma) = \hat{\sigma}[\alpha(\gamma)] = 0, \gamma \in Y.$$

Since $\alpha(Y) = \Gamma_1$, $\hat{\sigma} = 0$ on Γ_1 which implies that $\sigma = 0$ and hence $\mu = 0$ which is a contradiction. This proves that $Y = \Gamma_2$.

Thus α is a piecewise affine homeomorphism of Γ_2 onto Γ_1. Since Γ_1 is discrete, Γ_2 is also discrete and hence G_2 is compact. This completes the proof of the lemma.

PROOF OF THEOREM 4.3. Since $\wedge$ is an algebraic isomorphism of $M[A^p_{\omega_1}(G_1)]$ onto $M[A^p_{\omega_2}(G_2)]$, either both G_1 and G_2 are compact or both of them are noncompact. We shall consider two cases separately.

Case 1. Suppose G_1 and G_2 are both noncompact. By Theorem 2.1

$$M[A^p_{\omega_1}(G_1)] \simeq M_{bd}(G_1)$$

$$M[A^p_{\omega_2}(G_2)] \simeq M_{bd}(G_2)$$

Since Λ is an isometric algebra isomorphism of $M[A^p_{\omega_1}(G_1)]$ onto $M[A^p_{\omega_2}(G_2)]$ it follows that

$$M_{bd}(G_1) \simeq M_{bd}(G_2)$$

From Theorem 4.10, this implies that G_1 and G_2 are topologically isomorphic.

Case 2. Suppose G_1 and G_2 are both compact. For each $a \in G_1$, $\Lambda\tau_a$ and $\Lambda\tau_{-a}$ are norm preserving multipliers on $A^p_{\omega_2}(G_2)$ since Λ is an isometric isomorphism of $M[A^p_{\omega_1}(G_1)]$ onto $M[A^p_{\omega_2}(G_2)]$. Therefore by Lemma 4.13, there exist $a', b' \in G_2$ and complex numbers λ_1, λ_2 of absolute value 1 such that

$$\Lambda\tau_a = \lambda_1\tau_{a'}, \quad \Lambda\tau_{-a} = \lambda_2\tau_{b'} .$$

Since Λ is an algebraic isomorphism, $b' = -a'$ and $\lambda_1 = 1/\lambda_2$. Also it can be easily verified that the mapping $\varphi : G_1 \to G_2$ given by $\varphi(a) = a'$ and the mapping λ which maps G_1 into the set of all complex numbers of modulus one are algebraic isomorphisms.

Since φ is an algebraic isomorphism of G_1 onto G_2, all that remains to complete the proof is to show that φ is continuous.

To show that φ is continuous, we proceed as in the proof of Theorem 4.1 to show that if $a_i \to e_1$ in G_1, then $\varphi(a_i) \to e_2$ in G_2. Suppose not. Then there exists an open neighbourhood V of e_2 such that $\{\varphi(a_i)\}$ belongs to complement of V for all i.

The net of multipliers $\{\Lambda \tau_{a_i}\}$ is a norm bounded net in $M[A^p_{\omega_2}(G_2)]$. Therefore by Lemma 4.16, there exists a multiplier m and a subnet of $\{\Lambda \tau_{a_i}\}$ which, without loss of generality we assume to be $\{\Lambda \tau_{a_i}\}$ itself, satisfying

$$(10) \qquad \int_{\Gamma_2} \widehat{\Lambda \tau_{a_i}}(\gamma)\, \hat{f}(\gamma)\, d\gamma \longrightarrow \int_{\Gamma_2} \hat{m}(\gamma)\, \hat{f}(\gamma)\, d\gamma, \quad f \in A(G_2)$$

For $h \in L^1(G_1)$, let W_h be the multiplier defined by convolution with h on $A^p_{\omega_1}(G_1)$. Since $a_i \to e_1$ in G_1, $\tau_{a_i} h \to h$ in $M_{bd}(G_1)$. Since the topology of $M_{bd}(G_1)$ is stronger than that induced by $M[A^p_{\omega_1}(G_1)]$ we have

$$W_{\tau_{a_i} h} \longrightarrow W_h$$

in $M[A^p_{\omega_1}(G_1)]$. Thus

$$\tau_{a_i} \circ W_h \longrightarrow W_h$$

in $M[A^p_{\omega_1}(G_1)]$. Λ is an isometric isomorphism of $M[A^p_{\omega_1}(G_1)]$ onto $M[A^p_{\omega_2}(G_2)]$. Therefore Λ is continuous. We then have

$$\Lambda\tau_{a_i} \circ \Lambda W_h \longrightarrow \Lambda W_h$$

in $M[A^p_{\omega_2}(G_2)]$ so that

$$\widehat{\Lambda\tau_{a_i}} \circ \widehat{\Lambda W_h} \longrightarrow \Lambda\hat{W}_h$$

in $L^\infty(\Gamma_2)$. This gives

(11) $$\lim_i \int_{\Gamma_2} \widehat{\Lambda\tau_{a_i}}(\gamma)\, \Lambda\hat{W}_h(\gamma)\, \hat{f}(\gamma)\, d\gamma = \int_{\Gamma_2} \Lambda\hat{W}_h(\gamma)\, \hat{f}(\gamma)\, d\gamma, \quad f \in A(G$$

From (10) we have

(12) $$\lim_i \int_{\Gamma_2} \widehat{\Lambda\tau_{a_i}}(\gamma)\, \widehat{\Lambda W_h}(\gamma)\, \hat{f}(\gamma)\, d\gamma = \int_{\Gamma_2} \Lambda\hat{W}_h(\gamma)\, \hat{m}(\gamma)\, \hat{f}(\gamma)\, d\gamma, \quad f \in B(G_2)$$

From (11) and (12) we therefore have

$$\int_{\Gamma_2} \Lambda\hat{W}_h(\gamma)\, \hat{f}(\gamma)\, d\gamma = \int_{\Gamma_2} \widehat{\Lambda W_h}(\gamma)\, \hat{m}(\gamma)\, \hat{f}(\gamma)\, d\gamma .$$

Since $B(G_2)$ is dense in $A(G_2)$ this implies that

(13) $$\widehat{\Lambda W_h}(\gamma)\, \hat{m}(\gamma) = \widehat{\Lambda W_h}(\gamma), \text{ a.e on } \Gamma_2 .$$

But

$$\widehat{\Lambda W_h}(\gamma)\,\hat{m}(\gamma) = \widehat{m \circ \Lambda W_h}(\gamma)$$

Thus (13) gives

$$m \circ \Lambda W_h = \Lambda W_h$$

so that

$$\Lambda^{-1} m \circ W_h = W_h$$

that is

$$\Lambda^{-1} m\,(h * f) = h * f \;, \quad f \in A^p_{\omega_1}(G_1) \;.$$

Since $A^p_{\omega_1}(G_1)$ is a Segal algebra, it is an essential $L^1(G_1)$ module and hence

$$\Lambda^{-1} m = \tau_{e_1} \;.$$

which implies

$$m = \tau_{e_2} \;.$$

(10) at then gives

$$(14) \qquad \lim_i \int_{\Gamma_2} \Lambda \hat{\tau}_{a_i}(\gamma)\,\hat{f}(\gamma)\,d\gamma = \int_{\Gamma_2} \hat{f}(\gamma)\,d\gamma \;, \quad f \in A(G_2) \;.$$

Now $\{\lambda(a_i)\}$ is a bounded sequence of complex numbers. Therefore there exists a subset of $\{\lambda(a_i)\}$ which without loss of generality we denote by $\{\lambda(a_i)\}$ itself and a complex number λ of modulus 1 such that $\{\lambda(a_i)\}$ converges to λ. Now $\{\varphi(a_i)\} \subset G_2$. Since G_2 is compact, there exists a subset of $\{\varphi(a_i)\}$ say $\{\varphi(a_{i_k})\}$ and an element $a' \in G_2$ such that $\{\varphi(a_{i_k})\}$ converges to a' in G_2. $\{\varphi(a_{i_k})\}$

belongs to complement of V for all i_k and therefore a' belongs to complement of V, that is $a' \neq e_2$. Let $f \in B(G_2)$. Then

$$(16) \quad \lambda(a_{i_k}) \int_{\Gamma_2} \widehat{\tau_{\varphi(a_{i_k})}}(\gamma)\, \hat{f}(\gamma)\, d\gamma = \lambda(a_{i_k}) \int_{\Gamma_2} \langle -\varphi(a_{i_k}), \gamma \rangle \hat{f}(\gamma)\, d\gamma$$
$$= \lambda(a_{i_k})\, f(-\varphi(a_{i_k})).$$

f is continuous and therefore we have from (15)

$$(16) \quad \lambda(a_{i_k}) \int_{\Gamma_2} \widehat{\tau_{\varphi(a_{i_k})}}(\gamma)\, \hat{f}(\gamma)\, d\gamma \longrightarrow \lambda f(-a').$$

Now

$$\lambda f(-a') = \lambda \int_{\Gamma_2} \langle -a', \gamma \rangle \hat{f}(\gamma)\, d\gamma = \lambda \int_{\Gamma_2} \hat{\tau}_{a'}(\gamma)\, \hat{f}(\gamma)\, d\gamma.$$

(16) then implies that

that is

(17)

(16) and (17) then combine to show that

$$\int_{\Gamma_2} \hat{f}(\gamma)\, d\gamma = \int_{\Gamma_2} \widehat{\lambda \varphi_{a'}}(\gamma)\, \hat{f}(\gamma)\, d\gamma, \quad f \in B(G_2).$$

The denseness of $B(G_2)$ in $A(G_2)$ then shows that

$$\widehat{\lambda \varphi_{a'}}(\gamma) = 1 \quad a.e \text{ on } \Gamma_2,$$

that is

$$\lambda = 1 \quad \text{and} \quad a' = e_2$$

which is a contradiction. Therefore ϕ is continuous. This completes the proof of the theorem.

COROLLARY 4.16. *If G_1 and G_2 are two locally compact abelian groups with character groups Γ_1 and Γ_2 and ω_1, ω_2 are two weight functions on Γ_1 and Γ_2 respectively. Then G_1 and G_2 are isomorphic as topological groups if and only if there exists an isometric or topolo algebra isomorphism between $A^p_{\omega_1}(G_1)$ and $A^p_{\omega_2}(G_2)$ for some p satisfying $1 \le p < \infty$.*

PROOF. Any isometric algebra isomorphism between $A^p_{\omega_1}(G_1)$ and $A^p_{\omega_2}(G_2)$ gives rise to an isomorphism between $M[A^p_{\omega_1}(G_1)]$ and $M[A^p_{\omega_2}(G_2)]$ by Theorem 4.8. An application of Theorem 4.3 then completes the proof of the corollary.

The proof of Theorem 4.4 is exactly similar to that of Theorem 4.3 and is hence omitted. From Theorem 4.4 and is hence we can derive the following corollary similar to corollary 4.16.

COROLLARY 4.17. *If* G_1 *and* G_2 *are two locally compact abelian groups with duals* Γ_1 *and* Γ_2 *and* $S^p(G_1)$, $S^p(G_2)$ *are two semi algebras of Beurling type for* p *satisfying* $1 \leq p < \infty$. *Then* G_1 *and* G_2 *are isomorphic as topological groups if and only if there exists an isometric algebra isomorphism between* $S^p(G_1)$ *and* $S^p(G_2)$.

REMARK 4.18. If G_1 and G_2 are noncompact, 'isometric' can be replaced by 'norm decreasing' in Theorems 4.3 and 4.4 since the multiplier algebras involved are isometrically isomorphic to $M_{bd}(G_i)$ as Banach algebras.

CHAPTER V

MULTIPLIERS FROM A SEGAL ALGEBRA INTO $L^1(G)$

Figà-Talamanca [ ] has given a characterization of the multipliers on $L^p(G)$ for a locally compact abelian group G for $1 \leq p < \infty$, as the dual space of a Banach space of continuous functions. Larsen [14] has given a similar characterization for the multiplier space of $A^p(G)$ when G is compact abelian and $p > 2$. We give here an analogous characterization of the space of multipliers from a Segal algebra $S(G)$ on a locally compact abelian group into $L^1(G)$.

Let $M(S, L^1)$ denote the space of all multipliers from $S(G)$ into $L^1(G)$. Burtham in [ ] has proved the following result.

THEOREM 5.1. *If $S_1(G)$ and $S_2(G)$ are two Segal algebras on the locally compact abelian group G and T is a multiplier from $S_1(G)$ into $S_2(G)$ then there exists a bounded continuous function φ defined on Γ satisfying*

$$\widehat{Tf}(\gamma) = \varphi(\gamma)\hat{f}(\gamma), \quad \gamma \in \Gamma, \; f \in S_1(G)$$

with

$$\|\varphi\|_\infty \leq \|T\| .$$

The following theorem of Rudin [ ] will be needed in the sequel.

THEOREM 5.2. A(G) consists precisely of the convolutions $F_1 * F_2$ with F_1 and F_2 in $L^2(G)$.

If $\varphi \in L^1(G)$ and L_φ is defined by

$$L_\varphi(f) = \varphi * f, \quad f \in S(G),$$

then L_φ is a multiplier from S(G) into $L^1(G)$. Let $T \in M(S, L^1)$. Consider the linear form on A(G) defined as follows

$$(1) \qquad \beta_T(f) = \int_\Gamma \hat{T}(\gamma)\, \hat{f}(\gamma)\, d\gamma, \quad f \in A(G),$$

where $\hat{T}$ denotes the unique bounded continuous function corresponding to T satisfying

$$\widehat{Tf}(\gamma) = \hat{T}(\gamma)\, \hat{f}(\gamma), \quad f \in S(G)$$

with

$$\|\hat{T}\|_\infty \leq \|T\|,$$

the existence of which is given by Theorem 5.1. Hence since

$$|\beta_T(f)| = \left|\int_\Gamma \hat{T}(\gamma)\, \hat{f}(\gamma)\, d\gamma\right| \leq \|T\| \|\hat{f}\|_1,$$

β_T is a well defined linear form on A(G). Remark A(G) as

follows :

(1) $$\|f\| = \sup\{|\beta_T(f)| : T \in M(S, L^1), \|T\| \leq 1\}$$

That the above defines a seminorm on A(G) can be easily verified. To prove that it defines a norm, we have to prove $\|f\| = 0$ implies $f = 0$ on G. $\|f\| = 0$ implies $\beta_T(f) = 0$, $T \in M(S, L^1)$. Taking $T = \tau_y$ for some $y \in G$ we have

$$\hat{T}(\gamma) = \langle -y, \gamma \rangle, \quad \gamma \in \Gamma, \quad \|\tau_y\| \leq 1.$$

$$\beta_T(f) = \int_\Gamma \langle -y, \gamma \rangle \hat{f}(\gamma)\, d\gamma = f(-y) = 0, \quad y \in G.$$

Hence $f = 0$ on G. With this new norm let us denote A(G) by $B_S(G)$. Denoting the norm on $B_S(G)$ by $\| \ \|_B$ we have

THEOREM 5.3. $M(S, L^1)$ <u>is isometrically isomorphic to the dual of the completion of</u> $B_S(G)$. <u>The weak operator topology on</u> $M(S, L^1)$ <u>is stronger than the weak topology on norm bounded subsets of</u> $M(S, L^1)$.

We first prove two lemmas.

LEMMA 5.4. <u>If</u> $H = \{f * g : f \in B(G), g \in C_c(G)\}$ <u>then</u> H <u>is dense in</u> $B_S(G)$.

PROOF. If $f \in B_S(G)$,

$$\|f\|_B = \sup\{|\beta_T(f)| : T \in M(S, L^1), \|T\| \leq 1\}$$

$$= \sup\{|\int_\Gamma \hat{T}(\gamma)\hat{f}(\gamma)\,d\gamma| : T \in M(S, L^1), \|T\| \leq 1\}$$

$$\leq \sup\{\|T\|\,\|\hat{f}\|_1, T \in M(S, L^1), \|T\| \leq 1\}$$

$$\leq \|\hat{f}\|_1$$

Therefore if we give another norm to A(G) as

$$\|f\|' = \|\hat{f}\|_1,$$

then

$$\|f\|_B \leq \|f\|', \quad f \in B_S(G).$$

If we therefore prove that H is dense in A(G) in $\|\cdot\|'$ we have our required result. If $f \in A(G)$, there exists $g, h \in L^2(G)$ such that $f = g * h$ by Theorem 5.2. Let $\epsilon > 0$ be given. Now $C_c(G)$ is dense in $L^2(G)$. Therefore there exists $g' \in C_c(G)$ such that

(2) $$\|g - g'\|_2 < \frac{\epsilon}{2\|h\|_2}$$

Now

$$\|g * h - g' * h\|' = \|\hat{g}\hat{h} - \hat{g}'\hat{h}\|_1 = \|(\hat{g} - \hat{g}')\hat{h}\|_1$$

$$\leq \|\hat{g} - \hat{g}'\|_2 \|\hat{h}\|_2$$

that is

(3) $\|g*h - g'*h\|' \leq \|g - g'\|_2 \|h\|_2 < \epsilon/2$

from (2). Similarly there exists $h' \in B(G)$ such that

(4) $\|\hat{h}' - \hat{h}\|_2 < \varepsilon / 2\|g'\|_2$.

Therefore we have

(5) $\|g'*h - g'*h'\|' = \|\hat{g}'\hat{h} - \hat{g}'\hat{h}'\|_1$
$= \|\hat{g}'(\hat{h} - \hat{h}')\|_1 \leq \|g'\|_2 \|\hat{h}' - \hat{h}\|_2 < \varepsilon/2$

from (4). Combining (3) and (5) we then have

$$\|g*h - g'*h'\|' \leq \|g*h - g'*h\|' + \|g*h - g'*h'\|'$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Therefore H is dense in $B_S(G)$.

LEMMA 5.5. If $T \in M(S, L^1)$, then $T = \lim_\alpha L_{\varphi_\alpha}$ in the weak operator topology where $\{\varphi_\alpha\}$ is a net of functions in B(G) with $\|\varphi_\alpha * f\|_1 \leq K\|f\|_S$ for some constant K independent of $f \in S(G)$ and independent of α .

PROOF. Let $\{h_\alpha\}$ be an approximate identity satisfying $\|h_\alpha\|_1 = 1$ and $\hat{h}_\alpha$ has compact support for all α . Set $\varphi_\alpha = Th_\alpha$.

Then $\varphi_\alpha \in B(G)$. Now

$$\lim_\alpha \|\varphi_\alpha * f - Tf\|_1 = \lim_\alpha \|Th_\alpha * f - Tf\|_1$$

$$= \lim_\alpha \|T(h_\alpha * f) - Tf\|_1 \leq \lim_\alpha \|T\| \|h_\alpha * f - f\|_S =$$

Thus for $f \in S(G)$ we have

$$\lim_\alpha \varphi_\alpha * f = Tf$$

in the $L^1(G)$ norm which shows that

$$T = \lim_\alpha L_{\varphi_\alpha}$$

in the strong operator topology which in turn implies

$$T = \lim_\alpha L_{\varphi_\alpha}$$

in the weak operator topology. Also

$$\|\varphi_\alpha * f\|_1 = \|T(h_\alpha * f)\|_1 \leq \|T\| \|h_\alpha * f\|_S$$

$$\leq \|T\| \|h_\alpha\|_1 \|f\|_S \leq \|T\| \|f\|_S ,$$

for all α and for all $f \in S(G)$. This proves the required result.

<u>PROOF OF THEOREM 5.2.</u> Given $T \in M(S, L^1)$, we have to define an element of $[B_S(G)]^*$ corresponding to it. Consider the linear form β_T defined on $[B_S(G)]$ as in (1). Since

$$|\beta_T(f)| = \|T\| \left| \beta_{\frac{T}{\|T\|}}(f) \right| \leq \|T\| \|f\|_B$$

by the definition of the norm of a function in $B_S(G)$. β_T is a continous linear functional on $B_S(G)$ with

(6) $\|\beta_T\|_{B^*} \leq \|T\|$.

If $f \in B(G)$, $g \in C_c(G)$, then $f * g \in B_S(G)$ and

$$\|f*g\|_B = \sup\{|\beta_T(f*g)| : T \in M(S, L^1), \|T\| \leq 1\}$$

$$= \sup\{|Tf * g(0)| : T \in M(S, L^1), \|T\| \leq 1\}$$

$$\leq \sup\{\|Tf\|_1 \|g\|_\alpha : T \in M(S, L^1), \|T\| \leq 1\}$$

$$\leq \sup\{\|T\| \|f\|_S \|g\|_\alpha : T \in M(S, L^1), \|T\| \leq 1\}$$

$$\leq \|f\|_S \|g\|_\alpha$$

Therefore

(7) $\|f*g\|_B \leq \|f\|_S \|g\|_\alpha$.

Now

$$\|T\| = \sup\{|Tf*g(0)| : f\in B(G),\ g\in C_c(G),\ \|f\|_S\le 1,\ \|g\|_\infty\le 1\}$$

$$= \sup\{|\beta_T(f*g)| : f\in B(G),\ g\in C_c(G),\ \|f\|_S\le 1,\ \|g\|_\infty\le 1\}$$

$$\le \sup\{\|\beta_T\|_{B^*}\|f*g\|_B : f\in B(G),\ g\in C_c(G)$$

$$\le \sup\{\|\beta_T\|_{B^*}\|f\|_S\|g\|_\infty : f\in B(G),\ g\in C_c(G),\ \|f\|_S\le 1,\ \|g\|_\infty\le 1\}$$

from (7). This implies that

(8) $\|T\| \le \|\beta_T\|_{B^*}$.

(6) and (8) together give

$$\|T\| = \|\beta_T\|_{B^*}.$$

We have therefore defined a map Λ from $M(S,L^1)$ into $[B_S(G)]^*$. This can be easily verified to be a linear map. Also Λ is an isometry. It only remains to show that Λ is onto. Given $\beta\in[B_S(G)]^*$ for a given $f\in B(G)$,

consider the linear form F_f defined on $C_c(G)$ as follows.

$$F_f(g) = \beta(f * g).$$

Then

$$|F_f(g)| = |\beta(f*g)| \leq \|\beta\|_{B^*} \|f*g\|_B \leq \|\beta\|_{B^*} \|f\|_S \|g\|_c$$

from (7). Therefore F_f defines a continuous linear functional on $C_c(G)$ endowed with the supremum norm. Since $C_c(G)$ is dense in $C_0(G)$ there corresponds a measure $Tf \in M_{bd}(G)$ satisfying

(9) $$Tf * g(0) = F_f(g) = \beta(f*g), \quad g \in C_c(G)$$

and

(10) $$\|Tf\| \leq \|\beta\|_{B^*} \|f\|_S .$$

Given $y \in G$, we have

$$\tau_y(Tf) * g(0) = Tf * \tau_y g(0) = \beta(f * \tau_y g)$$

$$= \beta(\tau_y f * g) = T(\tau_y f) * g(0),$$

$$g \in C_c(G),$$

from which we deduce that

$$\tau_y(Tf) = T(\tau_y f), \quad y \in G, \ f \in B(G).$$

The mapping $G \to M_{bd}(G)$ given by $y \to \tau_y(Tf)$ is therefore

continuous which implies by Theorem that Tf is absolutely continuous, that is $Tf \in L^1(G)$. From (10) we see that the mapping T can be extended to the whole of $S(G)$. Since T is a translation bounded linear map from $S(G)$ into $L^1(G)$, $T \in M(S, L^1)$. Also

$$\beta_T(f * g) = Tf * g(0) = \beta(f * g), \; f \in B(G), g \in C_c($$

Thus β_T and β coincide on H. Since H is dense in $B_S(G)$ by Lemma 5.4, β_T and β coincide on $B_S(G)$ that is $\beta_T = \beta$. Therefore the mapping Λ is onto. Λ then defines an isometric isomorphism between $M(S, L^1)$ and the dual of $B_S(G)$. The dual of $B_S(G)$ is the same as that of its completion. We have therefore proved the first half of the theorem.

To prove the second half of the theorem. Suppose $\{T_\alpha\} \subset M(S, L^1$ satisfies $\|T_\alpha\| \leq 1$ and T_α converges to T in the mark weak operator topology in the limit of α. Let $\{h_n\}$ be a Cauchy sequence in $B_S(G)$ and let $\varepsilon > 0$ be given. We have to prove that there exists an infinite subnet of the $\{T_\alpha\}$ say $\{T_{\alpha_\beta}\}$ such that

$$\lim_{n} |\beta_{T_{\alpha_\beta}}(h_n) - \beta_T(h_n)| < \varepsilon .$$

Let n_0 be chosen such that

(11) $\|h_{n_0} - h_n\|_B < \varepsilon/5 \quad , \quad n \geq n_0$

Let $k = f * g$ with $f \in B(G)$, $g \in C_c(G)$ be given such that

(12) $\|h_{n_0} - k\|_B < \varepsilon/5$,

Since T_α converges to T in the weak operator topology, there exists an infinite subnet of $\{T_\alpha\}$ say $\{T_{\alpha_\beta}\}$ satisfying

(13) $|T_{\alpha_\beta} f * g(0) - Tf * g(0)| < \varepsilon/5$.

Then for $n \geq n_0$ from (11), (12) and (13) we have

$$|\beta_{T_{\alpha_\beta}}(h_n) - \beta_T(h_n)| = |\beta_{T_{\alpha_\beta}}(h_n) - \beta_{T_{\alpha_\beta}}(h_{n_0})| + |\beta_{T_{\alpha_\beta}}(h_{n_0}) - \beta_{T_{\alpha_\beta}}($$

$$+ |\beta_{T_{\alpha_\beta}}(k) - \beta_T(k)| + |\beta_T(k) - \beta_T(h_{n_0})|$$

$$+ |\beta_T(h_{n_0}) - \beta_T(h_n)|$$

$$\leq \|T_{\alpha_\beta}\| \, \|h_n - h_{n_0}\|_B + \|h_{n_0} - k\|_B \, \|T_{\alpha_\beta}\| + |T_{\alpha_\beta} f * g(0) - Tf$$

$$+ \|T\| \, \|k - h_{n_0}\| + \|T\| \, \|h_{n_0} - h_n\|$$

$$\leq \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon.$$

This proves that the weak operator topology on $M(S, L^1)$ is stronger than the weak* topology on $[B_S(G)]^*$, thus completing the theorem.

THEOREM 5.6. <u>The completion</u> $\overline{B_S(G)}$ <u>of</u> $B_S(G)$ <u>can be identified with a space of continuous functions.</u>

PROOF. For each $x \in G$, $T = \tau_x \in M(S, L^1)$ with $\|\tau_x\| \leq 1$. Therefore $|\beta_{\tau_x}(h)| \leq \|h\|_B$, $h \in B_S(G)$. But

$$\beta_{\tau_x}(h) = \int_\Gamma \hat{\tau}_x(\gamma)\hat{h}(\gamma)\,d\gamma = \int_\Gamma \langle -x, \gamma\rangle \hat{h}(\gamma)\,d\gamma$$
$$= h(-x).$$

This implies that

$$|h(x)| \leq \|h\|_B, \quad h \in B_S(G), \quad x \in G$$

that is

$$\|h\|_\infty \leq \|h\|_B, \quad h \in B_S(G). \tag{14}$$

Let $\{h_n\}$ be a Cauchy sequence in $B_S(G)$. The inequality

$$\|h_n - h_m\|_\infty \leq \|h_m - h_n\|_B$$

Shows that $\{h_n\}$ is a Cauchy sequence in the supremum norm also. There exists then a continuous function h such that

$$\lim_{n\to\infty} \|h - h_n\|_\infty = 0$$

To each Cauchy sequence in $B_S(G)$, we associate the continuous function so obtained which gives us a linear map assigning a continuous function on G to each element of the completion of $B_S(G)$. To complete the proof of our theorem, it is enough if we show that

this map is one to one. To this end let $\{h_n\}$ be a Cauchy sequence in $B_S(G)$ such that

$$\lim_n \|h_n\|_\alpha = 0$$

We need to prove that

$$\lim_n \|h_n\|_B = 0$$

Let us now consider the expression

$$\beta_{L_f}(h_n) = \int_\Gamma \hat{f}(\gamma)\hat{h}_n(\gamma)\,d\gamma, \quad f \in B(G)$$

Then

$$\beta_{L_f}(h_n) = f * h_n(0)$$

and

(15) $$\lim_n |\beta_{L_f}(h_n)| = \lim_n |f * h_n(0)| \leq \lim_n \|f\|_1 \|h_n\|_\infty = 0$$

If F is the linear form on $M(S, L^1)$ defined by

$$F(T) = \lim_n \beta_T(h_n)$$

Then F is continuous with respect to the weak* topology on $M(S, L^1)$

Therefore there exists $h \in B_S(G)$ such that

(16) $$F(T) = \lim_n \beta_T(h_n) = \beta_T(h), \quad T \in M(S, L^1).$$

Now $T = \lim_{\alpha} L_{\varphi_\alpha}$ in the weak operator topology where $\{\varphi_\alpha\}$ is a net of functions in $B(G)$ by Lemma 5.5. Hence $T = \lim_{\alpha} L_{\varphi_\alpha}$ in the weak* topology by Theorem 5.3. Therefore using (15) we have

$$F(T) = \lim_{\alpha} F(L_{\varphi_\alpha}) = \lim_{\alpha} \lim_{n} \{\beta_{L_{\varphi_\alpha}}(h_n)\}$$
$$= \lim_{\alpha} \{0\} = 0 .$$

This being true for all $T \in M(S, L^1)$, it follows from (16) that

$$\beta_T(h) = 0 , \quad T \in M(S, L^1) ,$$

that is $h = 0$.

$\{h_n\}$ thus converges weakly to the zero element of the completion of $B_S(G)$. Since $\{h_n\}$ is Cauchy, we have $\lim_{n} \|h_n\|_B = 0$ and our assertion is proved.

THEOREM 5.7. $\{\tau_x : x \in G\}$ <u>is dense in</u> $M(S, L^1)$ <u>in the weak* topology.</u>

PROOF. If $h \in B_S(G)$ satisfies

$$\beta_{\tau_x}(h) = 0 , \quad x \in G ,$$

then

$$\beta_{\tau_x}(h) = h(-x) = 0 , \quad x \in G$$

Shows that $h(x) = 0$ for all $x \in G$ so that $h = 0$ on G. Now by an application of the Hahn Banach theorem, we see that $\{\beta_{\tau_x}\}$ is

weak* dense in the dual of the completion of $B_S(G)$. Therefore we have the required result.

We now give an application of Theorem 5.3 to character Segal algebras. A <u>character Segal algebra</u> is a Segal algebra $S(G)$ satisfying $f \in S(G)$, $\gamma \in \Gamma$ implies $\gamma f \in S(G)$ and

$$\|\gamma f\|_S = \|f\|_S, \quad f \in S(G), \ \gamma \in \Gamma .$$

THEOREM 5.8. <u>If $S(G)$ is a character Segal algebra on a locally compact abelian group G with dual group Γ and $T_0 \in M(S, L^1)$, $\mu \in M_{bd}(\Gamma)$ then there exists a $T_\mu \in M(S, L^1)$ with $\hat{T}_{0\mu} = \hat{T}_0 * \mu$ and</u>

$$\|T_{0\mu}\| \leq \|\mu\| \|T_0\|$$

PROOF. If $h \in B_S(G)$, $\mu \in M_{bd}(\Gamma)$, then $h\hat{\mu} \in B_S(G)$ and

$$(17) \quad |\beta_T(h\hat{\mu})| = \left|\int_\Gamma \hat{T}(\gamma)\, \hat{h} * \mu(\gamma)\, d\gamma\right| = \left|\int_\Gamma \hat{T}(\gamma) \int_\Gamma \hat{h}(\gamma - w)\, d\mu(w)\right|$$

$$= \left|\int_\Gamma \left[\int_\Gamma \hat{T}(\gamma)\, \hat{h}(\gamma - w)\, d\gamma\right] d\mu(w)\right|$$

$$= \left|\int_\Gamma \left[\int_\Gamma \hat{T}(\gamma + w)\, \hat{h}(\gamma)\, d\gamma\right] d\mu(w)\right| .$$

$$\leq \sup_{w \in \Gamma} \left| \int_{\Gamma} \hat{T}(\gamma + w) \hat{h}(\gamma)\, d\gamma \right| \int_{\Gamma} |d\mu(w)|$$

$$= \sup_{w \in \Gamma} \left| \int_{\Gamma} \hat{T}(\gamma + w) \hat{h}(\gamma)\, d\gamma \right| \|\mu\|$$

Since $S(G)$ is a character Segal algebra, it is easy to verify that T_w defined by

$$T_w f(x) = w(-x)\, T(wf)(x), \quad f \in S(G),\ x \in G$$

belongs to $M(S, L^1)$ and

$$\hat{T}_w(\gamma) = \hat{T}(\gamma + w).$$

Also

$$\|T_w\| \leq \|T\|.$$

Therefore from (17) we have

$$|\beta_T(h\hat{\mu})| \leq \sup_{w \in \Gamma} |\beta_{T_w}(h)|\, \|\mu\| \leq \sup_{w \in \Gamma} \|T_w\|\, \|h\|_B\, \|\mu\|$$

that is

(18) $$|\beta_T(h\hat{\mu})| \leq \|h\|_B\, \|\mu\|\, \|T\|.$$

(18) being true for all $T \in M(S, L^1)$, by the definition of the norm in $B_S(G)$ we see that

(19) $$\|h\hat{\mu}\|_B \leq \|h\|_B \|\mu\| .$$

Given $T_0 \in M(S, L^1)$, $\mu \in M_{bd}(\Gamma)$, define the linear form φ on $B_S(G)$ as follows

$$\varphi(h) = \beta_{T_0}(\hat{\mu}h), \quad h \in B_S(G) .$$

Then

$$|\varphi(h)| = |\beta_{T_0}(\hat{\mu}h)| \leq \|\beta_{T_0}\| \, \|\hat{\mu}h\|_B$$

From (19) we then have

$$|\varphi(h)| \leq \|T_0\| \, \|h\|_B \|\mu\|, \quad h \in B_S(G) .$$

Therefore φ defines a bounded linear functional on $B_S(G)$ with

(20) $$\|\varphi\| \leq \|T_0\| \, \|\mu\| .$$

By Theorem 5.3, there exists an element $T_{0\mu} \in M(S, L^1)$ satisfying

$$\beta_{T_{0\mu}} = \varphi, \quad \|T_{0\mu}\| = \|\varphi\| .$$

Then from (20) we have

$$\|T_{0\mu}\| \leq \|T_0\| \, \|\mu\| .$$

Moreover

$$\beta_{T_0\mu}(h) = \int_{\Gamma} \hat{T}_{0\mu}(\gamma)\,\hat{h}(\gamma)\,d\gamma = \varphi(h) = \beta_{T_0}(h\hat{\mu})$$

$$= \int_{\Gamma} \hat{T}_0(\gamma)\,\hat{h} * \mu(\gamma)\,d\gamma, \quad h \in B_S(G)$$

$$= \int_{\Gamma} \hat{T}_0 * \mu(\gamma)\,\hat{h}(\gamma)\,d\gamma, \quad h \in B_S(G)$$

Hence

$$\hat{T}_{0\mu}(\gamma) = \hat{T}_0 * \mu(\gamma), \quad \gamma \in \Gamma.$$

thus proving the required result.

CHAPTER VI

ON A THEOREM OF PIGNO

Let G be a locally compact abelian group with character group Γ. Let E be a subset of Γ. In this chapter we study the restrictions to E of the multipliers on various spaces.

The space $(L^1, L^{p_1} \cap L^{p_2}, E)$ for $1 \leq p_1, p_2 \leq \infty$ is defined to be the set of all functions φ on E satisfying the condition that for every $f \in L^1(G)$, there exists $g \in L^{p_1} \cap L^{p_2}(G)$ such that

$$\hat{g} = \varphi \hat{f} \qquad \text{a.e. on } E.$$

Then Pigno [ ] has proved the following

THEOREM 6.1. <u>For</u> $1 \leq p_1 \leq 2$, $1 \leq p_2 \leq \infty$,

$$(L^1, L^{p_1} \cap L^{p_2}, E) = (L^{p_1} \cap L^{p_2})^\wedge / E$$

<u>where</u> $(L^{p_1} \cap L^{p_2})^\wedge / E$ <u>denotes the space of restrictions to</u> E <u>of the Fourier transforms of functions in</u> $L^{p_1} \cap L^{p_2}(G)$

Here we generalize the theorem to the case when both p_1 and p_2 are greater than two. For this we need the concept of the fourier transform of a function in $L^p(G)$ as a quasi-measure as given by Gaudry in [ ].

Let K be a compact subset of G. $D_K(G)$ is the vector space of all those continuous functions u which can be represented as

$$u = \sum_{i=1}^{\infty} f_i * g_i$$

where f_i, g_i are continuous functions on G with support contained in K and $\sum_{i=1}^{\infty} \|f_i\|_\infty \|g_i\|_\infty < \infty$. If $u \in D_K(G)$ we define

$$|||u|||_{D_K} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_\infty \|g_i\|_\infty \right\}$$

where the infimum is taken over all such representations of u. Then $D_K(G)$ is a Banach space. D(G) is then defined as the internal inductive limit of the Banach spaces $D_K(G)$. This means that D(G) is the vector space $\bigcup_K D_K(G)$ and the neighbourhood base at the origin for the topology on D(G) is given by open sets of the form

$$U_r = \bigcup_K \{ u \in D_K(G) : \|u\|_{D_K} < r \} .$$

D(G) is then a locally convex topological vector space and $D(G) \subset C_c(G)$. The space of continuous linear functionals on D(G) is denoted by $\mathcal{D}(G)$ and the elements of $\mathcal{D}(G)$ are

called *quasimeasures*. Then S is a quasimeasure if and only if S is linear and the restriction of S to $D_K(G)$ is continuous in the topology of $D_K(G)$ for each compact set K. Then Gaudry has proved the following results on quasimeasures.

THEOREM 6.2. *$D(G)$ is dense in $A(G)$*

THEOREM 6.3. *If $\mu \in M_{bd}(\Gamma)$, the mapping $u \to \hat{\mu} u$ is continuous from $D_K(G)$ into $D(G)$.*

Theorem 6.3 implies that for $\mu \in M_{bd}(\Gamma)$, $u \in D_K(G)$

$$\|\hat{\mu} u\|_{D_K} \leq \|u\|_{D_K} \|\mu\| .$$

Therefore if $\mu \in M_{bd}(\Gamma)$, $S \in Q(G)$, we can define $\hat{\mu} S \in Q(G)$ by

$$\langle \hat{\mu} S, u \rangle = \langle \hat{\mu} u, S \rangle , \quad u \in D(G) .$$

Then

$$|\langle u, \hat{\mu} S \rangle| = |\langle \hat{\mu} u, S \rangle| \leq \|S\|_{Q(G)} \|\hat{\mu} u\|_{D_K}$$
$$\leq \|S\| \|\mu\| \|u\|_{D_K}$$

which implies that

$$(1) \qquad \|\hat{\mu} S\|_{Q(G)} \leq \|S\|_{Q(G)} \|\mu\| .$$

We also have

THEOREM 6.4. [ ] *Every quasimeasure with compact support is a measure.*

For $f \in L^p(G)$, $1 \leq p \leq \infty$, $\hat{f}$ is defined to be the element of $Q(M)$ satisfying

$$\langle g, \hat{f} \rangle = \langle \hat{g}, f \rangle, \quad g \in D(M)$$

If $q_1, q_2 \in Q(M)$ and E is a subset of M, q_1 is said to be equal to q_2 on E if

$$\langle \varphi, q_1 \rangle = \langle \varphi, q_2 \rangle, \quad \varphi \in D(M)$$

where φ has compact support, contained in E. For $2 < p_1, p_2 \leq \infty$, we define $(L^1, L^{p_1} \cap L^{p_2}, E)$ to be the set of all quasimeasure $q \in Q(M)$, satisfying the condition that for every $f \in L^1(G)$ there exists $g \in L^{p_1} \cap L^{p_2}(G)$ such that

$$\hat{g} = \hat{f} q \quad \text{on} \quad E.$$

Then we have the following

THEOREM 6.5. [ ] $(L^1, L^{p_1} \cap L^{p_2}, E) = (L^{p_1} \cap L^{p_2})^\wedge$ on E.

PROOF. Suppose $f \in L^{p_1} \cap L^{p_2}(G)$. If $g \in L^1(G)$ then $g * f \in L^{p_1} \cap L^{p_2}(G)$. Moreover

$$(g * f)^\wedge = \hat{g} \hat{f}$$

Then

$$\hat{f} \in (L^1, L^{p_1} \cap L^{p_2}, E)$$

To prove the converse, let $q \in (L^1, L^{p_1} \cap L^{p_2}, E)$. Let $A = \{f \in L^{p_1} \cap L^{p_2}(G) : \hat{f} = 0 \text{ on } E\}$ Then A is a linear subspace of $L^{p_1} \cap L^{p_2}(G)$. If we introduce a norm on $L^{p_1} \cap L^{p_2}(G)$ by

(2) $$\|f\| = \|f\|_{p_1} + \|f\|_{p_2} .$$

$L^{p_1} \cap L^{p_2}(G)$ become a Banach space. We claim that A is a closed subspace of this Banach space. To this end, let $\{f_n\} \subset A$ and $f \in L^{p_1} \cap L^{p_2}(G)$ such that

(3) $$\|f_n - f\| = \|f_n - f\|_{p_1} + \|f_n - f\|_{p_2} \to 0 \text{ as } n \to \infty .$$

Then we have

(4) $$\|f_n - f\|_{p_1} \longrightarrow 0 \text{ as } n \to \infty$$

and

(5) $$\|f_n - f\|_{p_2} \longrightarrow 0 \text{ as } n \to \infty .$$

(4) implies

(6) $$\lim_{n\to\infty} \int_G f_n g \, dx = \int_G f g \, dx , \quad g \in L^{q_1}(G)$$

where $1/p_1 + 1/q_1 = 1$.

Let now $\varphi \in D(\Gamma)$ be such that the support of φ is contained in E. Then

(7) $$\langle \varphi, \hat{f}_n \rangle = \int_G f_n(x)\, \check{\hat{\varphi}}(x)\, dx$$

$$\longrightarrow \int_G f(x)\, \check{\hat{\varphi}}(x)\, dx = \langle \varphi, \hat{f} \rangle$$

by virtue of (6). By hypothesis $\hat{f}_n = 0$ on E. Then (7) gives

$$\langle \varphi, \hat{f} \rangle = 0$$

We have thus proved that $\hat{f} = 0$ on E and that A is a closed linear subspace of $L^{p_1} \cap L^{p_2}(G)$.

Now consider the transformation

(8) $$T_q : L^1(G) \longrightarrow L^{p_1} \cap L^{p_2} / A$$

given by

$$T_q f = g + A$$

where g is an element of $L^{p_1} \cap L^{p_2}(G)$ such that $\hat{g} = \hat{f} q$ on E. If $\hat{g}' = \hat{f} q$ on E also, then $\hat{g} - \hat{g}' = 0$ on E and so $g - g' \in A$. T_q is thus well defined. It is clearly linear. We shall now show that T_q is continuous. For this purpose we appeal to the closed graph theorem. Suppose

$\{f_n\} \subset L^1(G)$, $\{g_n + A\} \subset L^{p_1} \cap L^{p_2} / A$ such that $f_n \to f$ and $g_n + A \to g + A$. We want to prove that $\hat{g} = \hat{f} q$ on E. Since

$$\lim_n f_n = f$$

in $L^1(G)$,

$$\lim_n \hat{f}_n q = \hat{f} q$$

in the space of quasimeasures (from (1)), that is

$$\lim_n \langle \varphi, \hat{f}_n q \rangle = \langle \varphi, \hat{f} q \rangle, \quad \varphi \in D(\Gamma).$$

Let $\varphi \in D(\Gamma)$ be such that support $\varphi \subset E$. Then

(9) $$\lim_n \langle \varphi, \hat{g}_n \rangle = \lim_n \langle \varphi, \hat{f}_n q \rangle = \langle \varphi, \hat{f} q \rangle.$$

Since $g_n + A \to g + A$ in $\dfrac{L^{p_1} \cap L^{p_2}}{A}$, there exists for each integer n, an element $h_n \in A$ such that

$$\| g_n - g + h_n \| \leq \| (g_n - g) + A \| + 1/n$$

which implies

(10) $$\| g_n - g + h_n \|_{p_1} + \| g_n - g + h_n \|_{p_2} \to 0 \text{ as } n \to \infty.$$

Then

$$\lim_n \langle (g_n - g + h_n)^\wedge, \varphi \rangle = 0$$

for each $\varphi \in D(\Gamma)$. If support $\varphi \subset E$ then

$\langle \varphi, \hat{h}_n \rangle = 0$ for all n

so that

(11) $$\lim_{n\to\infty} \langle \varphi, \hat{g}_n \rangle = \langle \varphi, \hat{g} \rangle .$$

From (11) and (9) we have

$$\langle \varphi, \hat{g} \rangle = \langle \varphi, \hat{f} q \rangle$$

for all $\varphi \in D(\mathfrak{r})$ with support $\varphi \subset E$, that is

$$\hat{g} = \hat{f} q \quad \text{on } E .$$

Hence T_q is continuous. There exists a constant K such that

(12) $$\| T_q f \| \leq K \| f \|_1 , \quad f \in L^1(\mathfrak{G}) .$$

Let $\{e_\alpha\}$ be an approximate identity in $L^1(\mathfrak{G})$ such that $\|e_\alpha\|_1 = 1$ for all α and $\hat{e}_\alpha$ has compact support for all α. For each $\varepsilon > 0$, there exists $h_\alpha \in L^{p_1} \cap L^{p_2}(\mathfrak{G})$ such that $T_q e_\alpha = h_\alpha + A$ and

(13) $$\| h_\alpha \|_{p_1} + \| h_\alpha \|_{p_2} < \| T_q e_\alpha \| + \varepsilon \leq K \| e_\alpha \|_1 + \varepsilon \leq K + \varepsilon .$$

Hence there exists a subset of $\{h_\alpha\}$ which without loss of generality we denote by $\{h_\alpha\}$ itself and a function $h \in L^{p_1}(\mathfrak{G})$ such that h_α converges weakly to h in $L^{p_1}(\mathfrak{G})$. Now there

exists a subset of $\{h_\alpha\}$ say $\{h_{\alpha_\beta}\}$ and a function g in $L^{p_2}(G)$ satisfying

$$\lim_{\alpha_\beta} h_{\alpha_\beta} = g$$

weakly in $L^{p_2}(G)$.

Therefore if $\varphi \in L^{q_1} \cap L^{q_2}(G)$ where $1/p_i + 1/q_i = 1$, $i = 1, 2$ considering φ as an element of $L^{q_1}(G)$ we have

$$(13) \qquad \lim_{\alpha_\beta} \int_G h_{\alpha_\beta} \varphi \, dx = \int_G h \varphi \, dx .$$

Also considering φ as an element of $L^{q_2}(G)$ we have

$$(14) \qquad \lim_{\alpha_\beta} \int_G h_{\alpha_\beta} \varphi \, dx = \int_G g \varphi \, dx .$$

Hence from (13) and (14) we have

$$(15) \qquad \int_G h \varphi \, dx = \int_G g \varphi \, dx , \quad \varphi \in L^{q_1} \cap L^{q_2}(G),$$

that is

$$h = g \quad \text{a.e on } G.$$

Hence $h \in L^{p_1} \cap L^{p_2}(G)$. To complete the proof we have to show that $\hat{h} = q$ on E. Let $\varphi \in D(\Gamma)$ be such that support of $\varphi = K \subset E$. There exists $f \in L^1(G)$ such that

$\hat{f} = 1$ on K. Then

$$\langle \varphi, \hat{e}_\alpha q \rangle = \langle \hat{f}\varphi, \hat{e}_\alpha q \rangle = \langle \varphi, \hat{f}\hat{e}_\alpha q \rangle$$

$$= \langle \varphi, \widehat{f * e_\alpha}\, q \rangle .$$

Since

$$\lim_\alpha e_\alpha * f = f$$

in the $L^1(G)$ norm,

$$\lim_\alpha \widehat{e_\alpha * f}\, q = q$$

in $Q(G)$ by virtue of inequality (1). We therefore have

$$\lim_\alpha \langle \varphi, \hat{e}_\alpha q \rangle = \lim_\alpha \langle \varphi, \widehat{e_\alpha * f}\, q \rangle$$

$$= \langle \varphi, \hat{f} q \rangle$$

$$= \langle \hat{f}\varphi, q \rangle = \langle \varphi, q \rangle .$$

Since support $\varphi \subset E$ and $\hat{h}_\alpha = \hat{e}_\alpha q$ on E, we have

(16) $$\lim_\alpha \langle \varphi, \hat{h}_\alpha \rangle = \lim_\alpha \langle \varphi, \hat{e}_\alpha q \rangle = \langle \varphi, q \rangle .$$

Since h_α converges weakly to h in $L^{p_1}(G)$,

(17) $$\lim_\alpha \langle \varphi, \hat{h}_\alpha \rangle = \langle \varphi, \hat{h} \rangle, \quad \varphi \in D(\Gamma) .$$

From (16) and (17) we get

$$\langle \varphi, \hat{h} \rangle = \langle \varphi, q \rangle$$

for all $\varphi \in D(\Gamma)$ with support $\varphi \subset E$, that is

$$\hat{h} = q \text{ on } E .$$

This completes the proof of the theorem.

If q is a quasimeasure on Γ, we now derive a necessary and sufficient condition for q to be the Fourier transform of a function in $L^p(G)$ for some $p > 2$. If $\{e_\alpha\}$ is the approximate identity considered in the proof of Theorem 6.5, we see that $\hat{e}_\alpha q$ has compact support for all α. This implies that $\hat{e}_\alpha q$ is a pseudomeasure for every α by Theorem 6.4. Hence there exists $\{h_\alpha\} \subset L^\infty(G)$ satisfying

$$\hat{h}_\alpha = \hat{e}_\alpha q$$

for each α. The following theorem then gives a criterion for a quasimeasure in $\mathcal{S}(\Gamma)$ to be the Fourier transform of an element of $L^p(G)$.

THEOREM 6.6. <u>Given a quasimeasure</u> q <u>on</u> Γ, <u>there exists</u> $h \in L^p(G)$ <u>for some</u> $p > 2$ <u>satisfying</u> $\hat{h} = q$ <u>if and only if</u> $\{h_\alpha\} \subset L^p(G)$ <u>for each</u> α <u>with</u>

$$\|h_\alpha\|_p \leq K$$

for all α.

PROOF. Suppose there exists $h \in L^p(G)$ such that $\hat{h} = q$. Then $h * e_\alpha \in L^p(G)$ with

$$\hat{e_\alpha}\hat{h} = \hat{e_\alpha} q = \hat{h_\alpha} .$$

Therefore

$$\langle \varphi, \hat{e_\alpha}\hat{h} \rangle = \langle \varphi, \hat{h_\alpha} \rangle, \quad \varphi \in D(\Gamma)$$

that is

$$\int_G h_\alpha(x) \check{\hat{\varphi}}(x)\, dx = \int_G e_\alpha * h(x) \check{\hat{\varphi}}(x)\, dx, \quad \varphi \in D(\Gamma)$$

By Theorem 6.2 since $D(\Gamma)$ is dense in $A(\Gamma)$, $[D(\Gamma)]^\wedge$ is dense in $L^1(G)$ and so

$$h_\alpha(x) = e_\alpha * h \quad \text{a.e on } G.$$

Therefore $\{h_\alpha\} \in L^p(G)$ for each α with

$$\|h_\alpha\|_p = \|e_\alpha * h\|_p \leq \|e_\alpha\|_1 \|h\|_p \leq \|h\|_p .$$

Conversely if $h_\alpha \in L^p(G)$ with $\|h_\alpha\|_p \leq K$ for all α then there exists a subnet say $\{h_{\alpha_\beta}\}$ and a $h \in L^p(G)$ such that h_{α_β} converges to h weakly. Hence $\langle \hat{h_{\alpha_\beta}}, \varphi \rangle$ converges to $\langle \hat{h}, \varphi \rangle$ for all $\varphi \in D(\Gamma)$. By arguments similar to that used in the Proof of Theorem 6.5, we can show that

$$\lim_{\alpha} \langle \varphi, e_\alpha^\wedge q \rangle = \langle \varphi, q \rangle, \quad \varphi \in D(\Gamma)$$

But $e_\alpha^\wedge q = h_\alpha^\wedge$. Hence

$$\lim_{\alpha} \langle h_\alpha^\wedge, \varphi \rangle = \langle q, \varphi \rangle, \quad \varphi \in D(\Gamma)$$

that is

$$h^\wedge = q.$$

This completes the proof of the theorem.

For $1 \le p < \infty$, we define (L^1, A^p, E) to be the set of all functions φ defined on the subset E of Γ satisfying the condition that for every $f \in L^1(G)$, there exists $g \in A^p(G)$ such that

$$\hat{g} = \varphi \hat{f} \quad \text{a.e on } E,$$

We then have the following

THEOREM 6.6. $(L^1, A^p, E) = (A^p)^\wedge|_E$ <u>for</u> $1 \le p < 2$

<u>and</u> $(L^1, A^p, E) = (B_p)^\wedge|_E$ for $2 < p < \infty$

<u>where</u> $(A^p)^\wedge|_E$ <u>and</u> $(B_p)^\wedge|_E$ <u>denote the restrictions to</u> E <u>of the Fourier transforms and the Fourier-Stieltjes transforms of functions in</u> $A^p(G)$ <u>and measures in</u> $B_p(G)$ <u>respectively.</u>

PROOF. The proof is similar to that of Theorem 6.5 and is hence omitted.

Similarly if ω is a weight function defined on Γ for $1 \leq p < \infty$, we define (L^1, A^p_ω, E) to be the set of functions φ defined on Γ such that for every $f \in L^1(G)$ there exists $g \in A^p_\omega(G)$ such that

$$\hat{g} = \varphi \hat{f} \quad \text{a.e on } E$$

Then we have the following

THEOREM 6.7. $(L^1, A^p_\omega, E) = (A^p_\omega)^\wedge/_E$ for $1 \leq p \leq 2$

and $(L^1, B^p_\omega, E) = (B^p_\omega)^\wedge/_E$ for $p > 2$

where $(A^p_\omega)^\wedge/_E$ and $(B^p_\omega)^\wedge/_E$ denote the restrictions to E of the Fourier transforms and the Fourier-Stieltjes transforms of the functions in $A^p_\omega(G)$ and the measures in $B^p_\omega(G)$ respectively.

CHAPTER VII

ON TRANSLATION INVARIANT SUBSPACES

Larson has asked the following question in [ ] . If G is a noncompact group and $1 < p < 2$, do there exist nonzero closed translation invariant subspaces X of $L^p(G)$ such that $X \cap L^1(G) = \{0\}$. In this chapter we give an answer to this question.

Let G be a noncompact locally compact abelian group with dual Γ . If E is a measurable subset of Γ and $1 \leq p < \infty$, E is a set of uniqueness for $L^p(G)$, if there exists no non-zero element $g \in L^1(\Gamma)$ such that $g(\gamma) = 0$ for almost all γ in $\Gamma \setminus E$ the complement of E in Γ and $\hat{g} \in L^p(G)$. The existence of such a set has been proved by Figa-Talamanca and Gaudry in [ ] .

THEOREM 7.1. <u>If E is a measurable subset of Γ with positive Haar measure, then given $\varepsilon > 0$, there exists a subset F of E such that $m(F) > m(E) - \varepsilon$ where m denotes the Haar measure and F is a set of uniqueness of $L^p(G)$ for all p lying in $1 \leq p < 2$.</u>

Let $X_E^p = \{ f \in L^p(G) : \check{f} = 0 \}$ outside E .

there

$$\check{f}(\gamma) = \hat{f}(-\gamma) \quad , \gamma \in \Gamma .$$

Then Larsen has proved the following theorem.

THEOREM 7.2.[ ] . Let G be a noncompact locally compact abelian group and E a measurable subset of Γ . If $1 \leq p \leq r \leq 2$ and $X_E^r \cap L^p(G) = \{0\}$, then E is a set of uniqueness for $L^p(G)$.

THEOREM 7.3. [ ] . Let G be a noncompact locally compact abelian group. Let $1 \leq p < 2$ and E be a measurable subset of Γ with finite Haar measure. If E is a set of uniqueness of $L^p(G)$ and $p < r \leq 2$, then $X_E^r \cap L^p(G) = \{0\}$.

THEOREM 7.4. [ ] . If G is a noncompact locally compact abelian group and $1 < p < q < 2$, then there exists a subset E of Γ of finite positive Haar measure such that E is a set of uniqueness for $L^p(G)$, but not a set of uniqueness for $L^q(G)$.

We then have

THEOREM 7.5. If G is a noncompact locally compact abelian group and $1 < p < 2$, there exists a nonzero closed translation invariant subspace X of $L^p(G)$ such that $X \cap L^1(G) = \{0\}$

PROOF. First we prove the existence of a measurable subset E of Γ with finite positive ~~subset E of with finite positive~~ Haar measure such that E is a set of uniqueness for $L^1(G)$ but not a set of uniqueness for $L^p(G)$. Suppose there does not

existsuch a set. Choose q satisfying $1 < q < p < 2$. By Theorem 7.4 there exists a measurable subset E' of Γ with finite positive Haar measure such that E' is a set of uniqueness for $L^q(G)$ but not a set of uniqueness for $L^p(G)$. From Theorem 7.3, since every set of uniqueness for $L^q(G)$ Fmrm Emmrm is a set of uniqueness for $L^1(G)$, E' is a set of uniqueness for $L^1(G)$. By hypothesis, E' is a set of uniqueness for $L^p(G)$, thus contradicting the definition of E'. Hence there exists at least one measurable subset E of Γ of finite positive Haar measure such that E is a set of uniqueness for $L^1(G)$ but not a set of uniqueness for $L^p(G)$.

$X = X_E^p$ is then a closed translation invariant subspace of $L^p(G)$ which is nonzero, since if $X_E^p = \{0\}$ by Theorem 7.2, E will be a set of uniqueness for $L^p(G)$. Since E is a set of uniqueness for $L^1(G)$, by Theorem 7.3,

$$X \cap L^1(G) = X_E^p \cap L^1(G) = \{0\}$$

This proves the required result.

THEOREM 7.6. <u>If G is a noncompact locally compact abelian group and $1 \leq q < 2 \leq p$ there exists a non-zero closed translation invariant subspace X of $L^p(G)$ satisfying</u>

$$X \cap L^q(G) = \{0\}.$$

PROOF. From Theorem 7.1, we know that there exists a measurable subset of E of Γ, the character group of G, such that E is of finite positive Haar measure and it is a set of uniqueness for $L^q(G)$. Since the only sets of uniqueness for $L^p(G)$ for $p \geq 2$ are the sets of Haar measure zero, E is not a set of uniqueness for $L^p(G)$.

For $f \in L^p(G)$, as in chapter six we can define $\hat{f}$ to be the quasimeasure on Γ satisfying

$$\langle \varphi, \hat{f} \rangle = \langle \tilde{\hat{\varphi}}, f \rangle, \quad \varphi \in D(\Gamma)$$

Define $X_E^p = \{ f \in L^p(G) : \hat{f} = 0 \text{ on } \Gamma \setminus E \}$

Suppose $X_E^p = \{0\}$. Let $g \in L^1(\Gamma)$ be such that

$g(\gamma) = 0$ for almost all $\gamma \in \Gamma \setminus E$ and $\hat{g} \in L^p(G)$.

Define $f(x) = \check{g}(x) = \hat{g}(-x), x \in G$. Then $f \in L^p(G)$ and

$$\hat{f} = g = 0 \text{ on } \Gamma \setminus E .$$

Therefore $f \in X_E^p$ which implies that $f = 0$ and hence that $g = 0$. This contradicts the fact that E is not a set of of uniqueness for $L^p(G)$. Using arguments similar to that used in the proof of Theorem 6.5, we can show that X is closed. Therefore X is a closed nonzero linear subspace of $L^p(G)$ which is moreover translation invariant since if $f \in X$, and $a \in G$,

$$\langle \varphi, \widehat{\tau_a f} \rangle = \langle \langle a, \gamma \rangle \varphi, \hat{f} \rangle, \quad \varphi \in D(\Gamma)$$

which implies that

$$\langle \varphi, \widehat{\tau_a f} \rangle = 0, \quad \varphi \in D(\Gamma)$$

with support $\varphi \subset \Gamma \setminus E$.

To complete the proof of the theorem it only remains to verify that $X \cap L^q(G) = \{0\}$. Suppose not. Let $f \in X \cap L^q(G)$ Then $f \in L^p(G) \cap L^q(G)$. Since $q < 2 \leq p$, $f \in L^2(G)$. Therefore

$$f \in L^2(G), \quad \hat{f} = 0 \quad \text{a.e on } \Gamma \setminus E,$$

that is $f \in X_E^2 \cap L^q(G)$. Since E is a set of uniqueness for $L^q(G)$, from Theorem 7.3,

$$X_E^2 \cap L^q(G) = \{0\}$$

This implies that $f = 0$. Therefore

$$X \cap L^q(G) = \{0\}.$$

The proof of the theorem is now complete.

If $S(G)$ is a Segal algebra on a noncompact locally compact abelian group G and $M[S(G)]$ is its multiplier algebra, then to every element of $M[S(G)]$ there corresponds to a bounded continuous function on Γ. Let $\mathfrak{M}[S(G)]$ denote the space $\{\hat{T} : T \in M[S(G)]$. Then we have the following

THEOREM 7.7. $\mathfrak{M}[S(G)] \cap C_0(\Gamma) \neq C_0(\Gamma)$

PROOF. Let F be a compact subset of Γ with positive Haar measure. By Theorem 7.1 there exists a measurable subset E of F with finite positive Haar measure, such that E is a set of uniqueness for $L^1(G)$. If χ_E denotes the characteristic function of E, then χ_E does not belong to $\mathfrak{M}[S(G)]$. If we assume that it does, then we will arrive at a contradiction in the following way. For every $f \in S(G)$ there exists $g \in S(G)$ such that $\hat{g} = \chi_E \hat{f}$. Now choose $f \in S(G)$, $\hat{f} = 1$ on E and $\hat{f} = 0$ outside F. Then there exists $g \in S(G)$ such that $\hat{f}\chi_E = \chi_E = \hat{g}$. Now $\hat{g}$ has compact support and therefore it belongs to $L^1(\Gamma)$. Also $\hat{g} = 0$ a.e. outside E and $(\hat{g})^\vee = g \in L^1(G)$. Since E is a set of uniqueness for $L^1(G)$, we then have $g = 0$ which is impossible since $\hat{g} = \chi_E \neq 0$ on Γ. Therefore $\chi_E \notin \mathfrak{M}[S(G)]$.

Suppose $g * \chi_E \in \mathfrak{M}(S(G))$ for every $g \in L^1(\Gamma)$. Then we have a mapping Λ from $L^1(\Gamma)$ into $M[S(G)]$ given by $\Lambda(g) = T_g$ where $T_g \in M[S(G)]$ satisfies

$$\hat{T_g} = g * \chi_E .$$

To prove Λ is continuous, we apply the closedgr graph theorem.

Let $\lim_n \|g_n - g\|_1 = 0$ and $\lim_n \|T_{g_n} - T\| = 0$ in $M[S(G)]$. To prove $T = T_g$. We have

$$\|\hat{T} - \hat{T}_g\|_\infty \leq \|\hat{T} - \hat{T}_{g_n}\|_\infty + \|\hat{T}_{g_n} - \hat{T}_g\|_\infty$$

$$\leq \|T - T_{g_n}\| + \|g_n * \chi_E - g * \chi_E\|_\infty$$

$$\leq \|T - T_{g_n}\| + \|\chi_E\|_1 \|g_n - g\|_\infty$$

The right hand side tends to zero as $n \to \infty$. Therefore
~~lmt~~
we have

$$\hat{T} = \hat{T}_g \quad \text{a.e. on } \Gamma,$$

that is $T = T_g$. Therefore there exists a constant K such that

$$\|T_g\| \leq K\|g\|_1, \quad g \in L^1(M).$$

Let $\{e_\alpha\}$ be an approximate identity in $L^1(M)$ with $\|e_\alpha\|_1 \leq 1$, $\hat{e}_\alpha$ has compact support for all α. Then

$$\|T_{e_\alpha}\| \leq K\|e_\alpha\|_1 \quad \text{for all } \alpha,$$

that is

$$\|T_{e_\alpha}\| \leq K$$

that is

$$\|T_{e_\alpha} f\|_S \leq K$$

for all $f \in S(G)$ with

$$\|f\|_S \leq 1 .$$

Choose $f \in S(G)$ such that $\hat{f} = 1$ on E and $\hat{f} = 0$ outside F. We then have

$$\|T_{e_\alpha} f\|_1 \leq \|T_{e_\alpha} f\|_S \leq K \|f\|_S$$

for all α. There exists a subnet of $\{e_\alpha\}$ say $\{e_{\alpha_\beta}\}$ and a measure $\mu \in M_{bd}(G)$ satisfying

$$T_{e_{\alpha_\beta}} f \to \mu$$

weakly in $M_{bd}(G)$. If $g \in B(G)$ we have

$$\lim_{\alpha_\beta} \int_G T_{e_{\alpha_\beta}} f(x)\, g(x)\, dx = \int_G g(x)\, d\mu(x)$$

that is

$$\lim_{\alpha_\beta} \int_\Gamma \widehat{T_{e_{\alpha_\beta}}}(\gamma)\, \hat{f}(\gamma)\, \hat{g}(\gamma)\, d\gamma = \int_\Gamma \hat{g}(\gamma)\, \hat{\mu}(\gamma)\, d\gamma ,$$

that is

$$(1) \quad \lim_{\alpha_\beta} \int_\Gamma e_{\alpha_\beta} * \chi_E(\gamma)\, \hat{f}(\gamma)\, \hat{g}(\gamma)\, d\gamma = \int_\Gamma \hat{g}(\gamma)\, \hat{\mu}(\gamma)\, d\gamma$$

Since

$$\lim_\alpha \|e_\alpha * \chi_E - \chi_E\|_1 = 0 .$$

we have

$$(2) \qquad \lim_{\alpha_\beta} \int_\Gamma e_{\alpha_\beta} * \chi_E(\gamma)\hat{f}(\gamma)\hat{g}(\gamma)\,d\gamma = \int_\Gamma \chi_E(\gamma)\hat{f}(\gamma)\hat{g}(\gamma)\,d\gamma$$

(1) and (2) together tell us that

$$\int_\Gamma \hat{g}(\gamma)\hat{\mu}(\gamma)\,d\gamma = \int_\Gamma \chi_E(\gamma)\hat{f}(\gamma)\hat{g}(\gamma)\,d\gamma, \quad g \in B(G).$$

This implies that

$$\hat{\mu}(\gamma) = \chi_E(\gamma) \quad \text{a.e on } \Gamma.$$

Now $\hat{\mu} \in \mathfrak{M}[S(G)]$ and therefore $\chi_E \in \mathfrak{M}[S(G)]$ which contradicts that which we have already proved. Therefore there exists at least one $g \in L^1(\Gamma)$ such that $g * \chi_E \notin \mathfrak{M}[S(G)]$ Since $g * \chi_E \in C_0(G)$, we therefore see that

$$\mathfrak{M}[S(G)] \cap C_0(G) \neq C_0(G).$$

In the case of compact groups, Theorem 7.7 is not valid since $L^2(G)$ is a Segal algebra for which $\mathfrak{M}[S(G)] = L^\infty(\Gamma)$. This will therefore satisfy:

$$\mathfrak{M}(S(G)) \cap C_0(G) = C_0(G).$$

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