

STUDIES IN GENERALIZED CLIFFORD ALGEBRAS,
GENERALIZED CLIFFORD GROUPS AND
THEIR PHYSICAL APPLICATIONS



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PREFACE

This thesis is based upon the work done by me during the period 1971-1975 on Generalized Clifford Algebras and their applications under the guidance of Professor Alladi Ramakrishnan, Director, MATSCIENCE, The Institute of Mathematical Sciences, Madras.

I am extremely grateful to Professor Alladi Ramakrishnan for his constant encouragement, guidance and collaboration during the course of this work. It is a pleasure to record gratefully the benefit derived from useful discussions and collaboration with Professor N.R. Ranganathan, who initiated my research in certain group theoretical aspects of Generalized Clifford algebras and their applications. I am thankful to Professor R. Vasudevan for introducing to me the subject of Weyl's rule and Wigner distribution function in quantum mechanics and useful collaboration in the application of G.C.A's to them. I wish to thank Professor K.H. Mariwalla for useful discussions and bringing to my notice often many useful references on the subject of projective representations of groups. I like to thank Mr. M.N.V. Dutt for collaboration in a piece of work on Dirac's positive energy wave equation. With pleasure I thank Professor V. Radhakrishnan, Professor T.E. Santhanam, Dr.A.R. Tekumalla and Dr.G.N. Keshava Murthy for many useful discussions.

MATSCIENCE

Particularly I like to thank very much Dr.A.P.Tekumalla for bringing to my notice the book of Morris Newman on Integral Matrices which helped me in my investigations on projective representations of Abelian groups. I wish to thank Mr.Krishnaswami Alladi for introducing to me certain number theoretic concepts which have been of great use. I take great pleasure in thanking all my colleagues who have helped me in many ways.

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INTRODUCTION

This thesis presents some recent developments in the study of Generalized Clifford Algebras (G.C.A's), their associated structures and their physical applications. These investigations are extensions of the research programme initiated by Alladi Ramakrishnan with the development of L-Matrix Theory dealing with the Grammer of Dirac Matrices and their generalizations. Alladi Ramakrishnan and his collaborators have carried out extensive analysis of G.C.A's and their applications to physical problems. Results of these investigations are found in the book 'L-Matrix Theory or Grammer of Dirac Matrices' by Alladi Ramakrishnan¹.

The new developments presented in this thesis include:

- i. Introduction of concepts 'commutation matrices' and 'product transforms'² and the formulation of Alladi Ramakrishnan's 'tenon and mortice coupling method' of representation of G.C.A's in terms of them.
- ii. Generalisation of a 'matrix decomposition theorem' due to Alladi Ramakrishnan and discussion of his 'new approach to matrix theory' in relation to the work of Hermann Weyl, Wigner and Schwinger.
- iii. Discussion of canonical transformations in quantum mechanics using C and D matrices of G.C.A.
- iv. Formulation of new group structures called 'Generalized Clifford Groups' (G.C.G's) and discussion of their significance in the study of the problem of Bloch electrons in homogeneous magnetic field^{3,4,5}.

- v. A complete, simple and explicit solution to the problem of projective representations of finite abelian groups.
- vi. Proposal of a negative energy relativistic wave equation as a counterpart of Dirac's positive energy relativistic wave equation⁶.
- vii. Discussion of 'commuting quaternion algebras' of Clifford and L-Matrix Theory and 'Kasevskii's approach⁸ to Clifford algebras and its generalisation to G.C.A's.

CHAPTER 1 presents a summary of main results of Alladi Ramakrishnan and his collaborators in L-Matrix theory which are essential to the investigations in later Chapters. New concepts 'Commutation Matrices' and 'Product Transforms' are introduced and Alladi Ramakrishnan's 'Ternion and Mortice Coupling Method' of representations of Clifford algebras is reformulated in terms of product transforms.

CHAPTER 2, studies a 'Matrix Decomposition Theorem' due to Alladi Ramakrishnan and its generalizations. Significance of these theorems in relation to quantum kinematics is discussed and canonical transformations of conjugate variables, especially affine canonical transformations, are studied using the basic generating elements of G.C.A.

CHAPTER 3 and 4 study some group theoretical aspects of G.C.A. by formulating new group structures called Generalised Clifford Groups (G.C.G's). All the irreducible and inequivalent representations are explicitly constructed and studied in detail.

CHAPTER 5. determines explicitly all the inequivalent irreducible representations of all G.C.A's associated with the finite Abelian group $\mathbb{Z}_{m_1} \otimes \dots \otimes \mathbb{Z}_{m_N}$. This provides a complete and simple solution to the problem of explicit determination of all the inequivalent irreducible projective representations of finite Abelian groups.

CHAPTER 6 studies the problem of Bloch electrons in homogeneous magnetic field in the light of above investigations using the isomorphism of G.C.O's and Magnetic translation groups and arrive at a new and simple version of so-called magnetic Bloch function convenient for playing the role of Bloch function in study solid state phenomena in presence of an external homogeneous magnetic field.

CHAPTER 7 proposes and studies a negative energy relativistic wave equation as a counter-part to Dirac's positive energy relativistic wave equation.

CHAPTERS 8 and 9 are mathematical appendices, the first dealing with Clifford's 'Commuting quaternion algebras' viewed in terms of L-Matrix Theory, and second dealing with Rosevskii's approach to Clifford algebra and its generalisation to G.C.A's.

CHAPTER 3.FUNDAMENTALS OF L-MATRIX THEORY

L-matrix theory was initiated by Alladi Ramakrishnan with the objective of understanding the mathematical procedure of obtaining Dirac matrices from the basic Pauli set. This led to a detailed study by him and his collaborators, of the mathematical structure - 'too fundamental to be unnoticed, too consistent to be ignored and much too pretty to be without consequence' - as a common basis of understanding various branches of theoretical physics. L-matrix theory studies Clifford Algebra, its generalizations and their physical applications using explicit matrix representations.

Briefly the course of major developments in L-matrix theory can be sketched as follows. Alladi Ramakrishnan developed in a series of definitive papers the view point of Dirac Hamiltonian as a member of a hierarchy of matrices analysing helicity and energy as numbers of a hierarchy of eigenvalues. He developed further the connection of these hierarchy of matrices - L-matrices - to quaternions, propagators and Cartan spinors. In collaboration with Vasudevan, Ranganathan, Santhanam and Chandrasekharan, he extended this approach to Generalized Clifford algebras and this led to the development of representations of Kerner Algebra, para-Fermi rings, certain polynomial algebras and Unitary groups from the elements of G.C.A.

^{*}Alladi Ramakrishnan, in 'A New Approach to Quantum Numbers in Elementary Particle Physics', 'The Structure of Matter', Proceedings of the Rutherford Centennial Symposium 1971, University of Canterbury, Christchurch, New Zealand, p.156.

With the realisation of a shell structure in L-Matrices he studied weak interaction Hamiltonian in this approach. Following the mathematical logic behind the symmetries associated with the roots of the Unit matrix as a guide to physical thought he interpreted the internal quantum numbers of quarks in terms of roots of unity with an exciting generalization of Gell-Mann-Nishijima relation. Leaving these interesting aspects to the reference of his book 'L-Matrix Theory or Grammar of Dirac Matrices' let us consider in this chapter the mathematical foundations of L-Matrix theory essential for our investigations in latter chapters.

Clifford Algebra originated in the paper of H.E.Clifford 'Applications of Grassmann's Extensive Algebra' (*Am. J. Math.* 2, 350-358, 1878) in which he generalized Hamilton's 'Quaternion algebra'¹⁰ with three generators to higher dimensions. He called the resulting algebra with n generators as 'n-way geometric algebra', due to its relation to Grassmann's algebra of polysectors in n-dimensional Euclidean space¹⁰. Clifford algebra became the basis for spin representations of orthogonal groups. The spin representation of the special orthogonal group by means of Clifford algebra was discovered by Lipschitz¹¹ and then forgotten. E.Cartan¹² gave the theory of spinors and Cartan's theory was developed by Brauer and Weyl¹³. The first application of spinors to physics was by Pauli¹⁴ who introduced his famous spin matrices. The fully relativistic theory

of the electron spin was discovered by Dirac¹⁵ who by his famous linear relativistic wave equation showed the connection between spinors and the Lorentz group. Freudenthal¹⁶ rediscovered the results of Lipschitz and developed them using the theory of characters. Mathematical aspects and physical applications of spinors were later developed by van der Waerden¹⁷, Fierz¹⁸, Bargmann and Wigner¹⁹ and others²⁰.

Clifford algebra and its generalizations can be studied algebraically from the point of view of projective representations of finite Abelian groups. The problem of finding the projective (ray) representations of finite groups was stated and 'general' methods of finding the irreducible representations were given by Schur in a series of papers in 1904-11²¹. In mathematical literature Generalized Clifford Algebras (G.C.A's) were introduced and studied in detail by Norinaga and Nono²² while linearizing n-th order partial differential equations for $n > 2$. Yamazaki²³ introduced G.C.A's in dealing with projective representations and ring extensions of finite groups and Morris²⁴ studied these algebras in explicit detail. Popovici and Turtoi²⁵ studied these algebra in relation to generalizations of spinor structures. Nono²² and Morris²⁴ determined explicitly the projective representations of $G = \mathbb{Z}_{n_1} \otimes \dots \otimes \mathbb{Z}_{n_k}$ (n copies), for some factor systems. But so far no theory exists providing an explicit construction technique for determination of all the projective representations of an arbitrary finite Abelian group $G = \mathbb{Z}_{n_1} \otimes \dots \otimes \mathbb{Z}_{n_k}$. (For a detailed discussion of the subject of projective representations of finite groups cf. Morris²⁴). Recent achievement in this direction is due to Backhouse and

Bradley²⁶, who have determined explicitly the formula for the dimension of representation associated with a given factor system for G. This thesis solves this problem completely in a simple manner using the techniques of L-Matrix theory.

In physics literature H. Weyl²⁷ discussed the projective representations of Abelian groups and interpreted the fundamental laws of quantum kinematics in terms of them. Schwinger²⁸ developed his approach more elaborately in relation to measurement algebra in quantum kinematics and emphasized the significance of these structures. But in the work of both Weyl and Schwinger these structures in finite dimensions occur as only intermediate steps in a limiting process. Only after the development of L-Matrix theory by Alladi Ramakrishnan and his collaborators, the physical significance of these structures in finite dimensions have been realised and developed extensively. This thesis presents further applications of G.C.A's to the problem of Bloch electrons in magnetic field. Through out this thesis we shall use mainly the techniques of L-Matrix theory and so first we shall summarise briefly the main results of L-Matrix theory essential to the investigations in later chapters.

1. L-Matrices and generation method

A linear combination of the three Pauli matrices

$$L_3^{(1)} = \sum_{i=1}^3 x_i \sigma_i = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \quad (1.1)$$

is recognised to be a member of hierarchy of matrices, called L-matrices, with the property

$$\begin{aligned} L_n^{(2)} &= \sum_{i=1}^n x_i L_{(n,i)}^{(2)} \\ \left(L_n^{(2)}\right)^2 &= \left(\sum_{i=1}^n x_i^2\right) I \end{aligned} \quad (1.2)$$

LEM.

$$\begin{aligned} (a) \quad L_{(n,i)}^{(2)} L_{(n,j)}^{(2)} &= -L_{(n,j)}^{(2)} L_{(n,i)}^{(2)} ; \quad i \neq j \\ (b) \quad \left(L_{(n,i)}^{(2)}\right)^2 &= I ; \quad i = 1, 2, \dots, n \end{aligned} \quad (1.3)$$

First let $n = 2r+1$. Operation method due to Alladi Ramakrishnan obtains $L_{2r+1}^{(2)}$ from $L_{2r-1}^{(2)}$ as follows. In $L_3^{(2)}$ any one of the parameters x_1, x_2, x_3 is replaced by $L_{2r-1}^{(2)}$ and the other two parameters are replaced by $x_{2r}I$ and $x_{2r+1}I$, where I is the identity matrix of the same dimension as $L_{2r-1}^{(2)}$.

$$L_{2r+1}^{(2)} = \begin{pmatrix} x_{2r+1}I & L_{2r-1}^{(2)} + ix_{2r}I \\ L_{2r-1}^{(2)} + ix_{2r}I & -x_{2r+1}I \end{pmatrix} = \sigma \left(L_{2r-1}^{(2)} \right) \quad (1.4)$$

Taking $L_1^{(2)} = x_1$, this relation provides explicit representation of $L_3^{(2)}, L_5^{(2)}, \dots$. When $n = 2r$, any one of the parameters in $L_{2r-1}^{(2)}$ is put equal to zero. It is seen that at each step of going from $L_{2r-1}^{(2)}$ to $L_{2r+1}^{(2)}$ the dimension of L is doubled and hence

$$\dim(L_{2r+1}^{(2)}) = 2^r \quad (1.5)$$

Also

$$\dim(L_{2r}^{(2)}) = 2^r \quad (1.6)$$

or

$$\dim_{\mathbb{R}} (L_n^{(2)}) = 2^{\lceil n/2 \rceil} \quad (1.7)$$

where $\lceil n/2 \rceil$ denotes the largest integer part of $n/2$.

The irreducibility of the above representation is seen as follows. The matrices $(L_{(n,i)}^{(2)}, i = 1, 2, \dots, n)$ called 'generator matrices' are seen to be traceless by the construction formula (1.4). This is also inferred from (1.8) as follows.

$$\text{Tr} (L_{(n,i)}^{(2)} L_{(n,j)}^{(2)} L_{(n,i)}^{(2)} - 1) = -\text{Tr} (L_{(n,j)}^{(2)}) ; i \neq j \quad (1.8)$$

or

$$\text{Tr} (L_{(n,j)}^{(2)}) = 0 ; j = 1, 2, \dots, n.$$

Now consider the case $n = 2r$. If we form all possible products of all possible powers of the L 's the 2^{2r} matrices obtained are given by

extended $\left\{ \prod_{i=1}^{2r} (L_{(2r,i)}^{(2)})^{k_i} \mid k_i = 0, 1, \dots, r, i = 1, 2, \dots, n \right\}$. Consider one of the terms $(L_{(2r,1)}^{(2)} L_{(2r,2)}^{(2)} \dots L_{(2r,r)}^{(2)})$. If r is even

$$l_1 < l_2 < \dots < l_r$$

then this term anticommutes with any of the $\left\{ L_{(2r,i)}^{(2)} \mid r = 1, 2, \dots, n \right\}$

and hence by the argument of (1.8) its trace is zero. If r is odd

then any $L_{(2r,j)}^{(2)}$ which is not contained as a factor in the product

$\prod_{j=1}^r L_{(2r,l_j)}^{(2)}$ anticommutes with it. Hence all the numbers of the set $\left\{ \prod_{i=1}^{2r} (L_{(2r,i)}^{(2)})^{k_i} \mid k_i = 0, 1, \dots, r, i = 1, 2, \dots, n \right\}$ are traceless matrices except the identity matrix corresponding to the case all $k_i = 0$. Consider the equation

$$A = \sum_{(k_1, k_2, \dots, k_{2r} = 0, 1)} a_{k_1 k_2 \dots k_{2r}} (L_{(2r,1)}^{(2)})^{k_1} \dots (L_{(2r,2r)}^{(2)})^{k_{2r}} = 0 \quad (1.9)$$

it follows

$$\text{Tr} A = 0 \quad (1.10)$$

and hence $a_{00\ldots0} = 0$. Multiplying A on the left by

$$(L_{(2Y,2Y)}^{(2)})^{k_{2Y}} \cdots (L_{(2Y,1)}^{(2)})^{k_1}$$

$$(L_{(2Y,2Y)}^{(2)})^{k_{2Y}} \cdots (L_{(2Y,1)}^{(2)})^{k_1} A \equiv a_{k_1 k_2 \dots k_{2Y}} I + (\dots) = 0 \quad (1.11)$$

where (\dots) is again a sum of traceless terms. So from

$$\text{Tr} \left\{ (L_{(2Y,2Y)}^{(2)})^{k_{2Y}} \cdots (L_{(2Y,1)}^{(2)})^{k_1} A \right\} = 0 \quad (1.12)$$

it follows

$$a_{k_1 k_2 \dots k_{2Y}} = 0 \quad (1.13)$$

for all k_1, \dots, k_{2Y} . This means that all the 2^{2Y} elements $\left\{ \prod_{i=1}^{2Y} (L_{(2Y,i)}^{(2)})^{k_i} \mid k_i = 0, 1, \dots, i = 1, 2, \dots, 2Y \right\}$ are linearly independent. Since their dimension as given by (1.6) is 2^Y , this representation is irreducible since the condition for irreducibility of a D dimensional representation of a set of elements is that the set of matrices generated by taking all possible products of all possible powers of them should contain D^2 linearly independent matrices.

Considering the case of $n = 2r+1$, the product

$$(L_{(2Y+1)}^{(2)}) (L_{(2Y+2)}^{(2)}) \cdots (L_{(2Y+2Y+1)}^{(2)}) = 0 \quad (1.14)$$

commutes with all $L_{(2Y+1)}^{(2)}$

$$\left[\begin{array}{c} (2) \\ \text{(2r+1,i)} \end{array} \right] Q = Q \left[\begin{array}{c} (2) \\ \text{(2r+1,i)} \end{array} \right] ; i=1,2,\dots,n . \quad (1.16)$$

Hence by Schur's lemma - the total matrix algebra of dimension of

$$Q \sim I \quad (1.16)$$

where \sim means proportionality. Or

$$\left[\begin{array}{c} (2) \\ \text{(2r+2r+1)} \end{array} \right] \sim \left[\begin{array}{c} (2) \\ \text{(2r+1)} \end{array} \right] \left[\begin{array}{c} (2) \\ \text{(2r+1,2)} \end{array} \right] \dots \left[\begin{array}{c} (2) \\ \text{(2r+1,2r)} \end{array} \right] \quad (1.17)$$

which is obtained by multiplying both sides of (1.14) by $L_{(2r+2r+1)}^{(2)}$

and using (1.20) and (1.3), (b). Hence in the set of all possible products of all possible powers of $L_{(2r+1,i)}^{(2)}$'s out of 2^{2r+1}

elements there are only 2^{2r} linearly independent elements. Hence 2^r dimensional representation of $L_{(2r+1,i)}^{(2)}$'s is irreducible.

2. Properties of Clifford Algebra $C_n^{(2)}$.

The n elements obeying (1.3), $\left\{ L_{(n,i)}^{(2)} \mid i=1,2,\dots,n \right\}$ generate a set of 2^n elements taking all possible products of all possible powers of them. In the set $\left\{ \prod_{i=1}^n (L_{(n,i)}^{(2)})^{k_i} \mid k_i = 0,1,\dots; i = 1,2,\dots,n \right\}$, product of any two elements is defined by

$$\left(\prod_{i=1}^n (L_{(n,i)}^{(2)})^{k_i} \right) \left(\prod_{j=1}^n (L_{(n,j)}^{(2)})^{l_j} \right) = \left\{ (-1)^{\sum_{i>j} k_i l_j} \right\} \left(\prod_{i=1}^n (L_{(n,i)}^{(2)})^{(k_i + l_i)_2} \right)$$

(1.18)

where $(k_i + l_i)_2 \text{ mod. } 2 = k_i + l_i$. Hence the set of elements

$$\left\{ \prod_{i=1}^n (L_{(n,i)}^{(2)})^{k_i} \mid k_i = 0, 1 ; i = 1, 2, \dots, n \right\} \quad \text{form the basis for an algebra called Clifford Algebra } C_n^{(2)}. \quad \text{The basis of}$$

algebra called Clifford algebra, and denoted by $C_{2^k}^{(2)}$. The basis of $C_{2^k}^{(2)}$ contains a set of 2^{2^k} linearly independent matrices when represented by 2^k dimensional matrices as shown above and hence $C_{2^k}^{(2)} = \mathbb{H}_{2^k}(\mathbb{K})$ = the total matrix algebra of dimension 2^k over a field \mathbb{K} containing i . The algebra $C_{2^{k+1}}^{(2)}$ can be shown to be a direct sum of two $C_{2^k}^{(2)}$'s, and hence $C_{2^{k+1}}^{(2)} = \mathbb{H}_{2^k} \oplus \mathbb{H}_{2^k}$. (for more details cf. 20).

2. Generalized Clifford Algebra $C_n^{(m)}$ and their properties.

Generalization of L-matrix hierarchy (1.2) to include matrices obeying

$$(a) \quad L_n^{(m)} = \sum_{i=1}^n x_i L_{(n,i)}^{(m)} \quad (1.19)$$

$$(b) \quad \left(L_n^{(m)} \right)^m = \left(\sum_{i=1}^n x_i^{(m)} \right) I$$

was considered by Alladi Ramakrishnan, R. Vasudevan, K.R. Ranganathan, T.S. Santhanam and P.S. Chandrasekharan as an extension of the problem for $n = 2$, and they showed that ϵ -operation method can be generalized to yield the irreducible representations of (1.19). These matrices (1.19) are associated with the so-called ϵ -commutation relations

$$(a) \quad L_{(n,i)}^{(m)} L_{(n,j)}^{(m)} = \omega^{(m)} L_{(n,j)}^{(m)} L_{(n,i)}^{(m)} ; \quad i, j = 1, 2, \dots, n. \quad (1.20)$$

$$(b) \quad \left(L_{(n,i)}^{(m)} \right)^m = I ; \quad i = 1, 2, \dots, n.$$

where $\omega(n) = \exp(2\pi i/n)$. If we call (1.2) as Clifford conditions then these are generalized Clifford conditions since for $n = 2$, (1.20) is the same as (1.2). From (1.20), (1.19) follows due to the fact

Then

$$\sum_{\substack{\text{sum over all} \\ \text{permutations of} \\ i_1, i_2, \dots, i_m}} L_{(n,i_1)}^{(m)} L_{(n,i_2)}^{(m)} \cdots L_{(n,i_m)}^{(m)} = m! \sum_{i_1, i_2, \dots, i_m=1,2,\dots,n} \delta_{i_1 i_2 \cdots i_m}$$

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1 & \text{if } i_1 = i_2 = \cdots = i_m \\ 0 & \text{otherwise} \end{cases}$$

(1.21)

Here we assume that we substitute the values of the indices i_1, i_2, \dots, i_m in the symmetric sum containing $m!$ terms. Then if $i_1 = i_2 = \cdots = i_m = l$ then

$$m! (L_{(n,i)}^{(m)})^m = m! I \quad \text{or} \quad (L_{(n,i)}^{(m)})^m = I ; \quad i=1, 2, \dots, n.$$

(1.22)

By taking all possible products of all possible powers of the n generators $\{L_{(n,i)}^{(m)} \mid i=1, \dots, m\}$ we get a set of m^n elements $\left\{ \prod_{i=1}^{m^n} (L_{(n,i)}^{(m)})^{k_i} \mid k_i = 0, 1, \dots, m-1 \right\}$. Let us consider first the case $m=2$. Let us denote by $g(k_1, k_2, \dots, k_{2^n})$ the element $\left\{ \prod_{i=1}^{2^n} (L_{(2^n,i)}^{(m)})^{k_i} \right\}$. Obviously

$$g(0, 0, \dots, 0) = I \quad (1.23)$$

Now let us determine whether there are any relationships among the m^n elements $\{g(k_1, k_2, \dots, k_n) \mid \sum_{i=1}^n k_i \leq m-1\}$ of the type $g(k_1, k_2, \dots, k_n) \sim g(l_1, l_2, \dots, l_n)$ for some values of $\{k_i\}$ and $\{l_i\}$. Such a relationship for a pair of sets of values $\{k_i\}$ and $\{l_i\}$ would imply the relationship $g(k_1, \dots, k_n) g(l_1, \dots, l_n)^{-1} \sim I$. Or from the help of (1.22) it is seen to imply that $g(k_1 - l_1, \dots, k_n - l_n) \sim I$. Due to Schur's lemma this would happen if there exists any element $g(\gamma_1, \gamma_2, \dots, \gamma_n)$, with at least one $\gamma_i > 0$, which commutes with all generating elements $\{L_{(n,i)}^{(m)} \mid i=1, \dots, n\}$.

The condition for this is seen to be

$$L_{(n,i)}^{(m)} g(\gamma_1, \gamma_2, \dots, \gamma_n) L_{(n,i)}^{(m)-1} = g(\gamma_1, \gamma_2, \dots, \gamma_n) \quad \forall i=1 \dots n. \quad (1.24)$$

This leads to the condition

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ -1 & -1 & 0 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & 1_m \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ mod. } m \quad (1.25)$$

or $T(\underline{\gamma}) = (\underline{0}) \text{ mod. } m.$

When $N=2V$, the matrix T is invertible since $|\det T|=1$ and hence $(\underline{\gamma})$ has integer solutions as

$$(\underline{\gamma}) = (\underline{0}) \text{ mod. } m. \quad (1.26)$$

showing that the element $g(\gamma_1, \gamma_2, \dots, \gamma_{2V})$ commutes with all the generators i.e. $0 \cdot \text{mod. } m = \gamma_i$. This shows that really there are no relationships of the type sought for. Thus for any $\underline{\gamma} \in (\underline{\gamma}_1, \underline{\gamma}_2, \dots, \underline{\gamma}_{2V})$ if $\gamma_i \neq 0$, $\forall i=1 \dots 2V$ there exists some element $L_{(2V, 1)}^{(m)}$ with the property

$$L_{(2V, 1)}^{(m)} g(\gamma_1, \dots, \gamma_{2V}) L_{(2V, 1)}^{(m)}^{-1} = \omega(m)^s g(\gamma_1, \dots, \gamma_{2V}) \quad (1.27)$$

Hence taking trace of both sides with $s \cdot \text{mod. } m > 0$

$$\text{Tr}(g(\gamma_1, \dots, \gamma_{2V})) = \omega(m)^s \text{Tr}(g(\gamma_1, \dots, \gamma_{2V})). \quad s \cdot \text{mod. } m > 0$$

$$\text{Tr}(g(\gamma_1, \dots, \gamma_{2V})) = 0 \quad (1.28)$$

Only $\text{Tr}(g(0, 0, \dots, 0)) \neq 0$ As before consider the equation

$$A = \sum_{0 \leq k_i \leq m-1} a_{k_1, k_2, \dots, k_{2V}} g(k_1, \dots, k_{2V}) = 0 \quad (1.29)$$

and multiplying it on both sides from left by

$$g(k_1, \dots, k_{2V})^{-1} \sim g(m-k_1, \dots, m-k_{2V}) \quad (1.30)$$

it follows by taking trace of both sides

$$a_{k_1, \dots, k_{2V}} = 0 \quad \forall 0 \leq k_1, \dots, k_{2V} \leq m-1. \quad (1.31)$$

This proves the linear independence of the m^{2V} elements

$$\{g(k_1, \dots, k_{2V}) \mid 0 \leq k_i \leq m-1\}$$

When $N=2V+1$, $\det T=0$ and hence there exists a relationship of the type proposed. Since $\text{rank } T = 2V$, there is only one such relationship possible. One can easily convince himself that this relationship is given by a noted by Morris²⁴,

$$g(\underbrace{m-1, \dots, m-n}_{\nu}, \underbrace{n, \dots, n}_{\nu}) \sim I \quad (1.32)$$

This means that one of the generators $\{L_{(2\nu+1, i)}^{(m)} | i=1 \dots 2\nu+1\}$ is expressible as a product of powers of others. Thus the set of all possible products of all possible powers of $\{L_{(2\nu+1, i)}^{(m)} | i=1 \dots 2\nu+1\}$ splits into a direct sum of n sets each containing $m^{\frac{n}{2}}$ linearly independent elements. Thus in both cases of $N=2\nu$ and $N=2\nu+1$ there are only $m^{\frac{n}{2}}$ linearly independent matrices showing that in general the irreducible dimension is given by

$$\dim L_{(n, i)}^{(m)} = m^{\left[\frac{n}{2}\right]} \quad (1.33)$$

Incidentally this also establishes that there cannot exist more than $2\nu+1$ mutually ω -commuting matrices of dimension in the sense of eqn. (1.30a).

The set of all possible products of possible powers of $\{L_{(n, i)}^{(m)} | i=1 \dots n\}$, namely $\left\{ \prod_{i=1}^n (L_{(n, i)}^{(m)})^{R_i} \mid 0 \leq R_i \leq m-1; i=1 \dots n \right\}$ is easily seen to be the basis of an algebra, the product of basis elements being given by

$$\left\{ \prod_{i=1}^n L_{(n, i)}^{(m)} R_i \right\} \left\{ \prod_{i=1}^n L_{(n, i)}^{(m)} l_i \right\} = \omega(m)^{\delta} \left\{ \prod_{i=1}^n L_{(n, i)}^{(m) (R_i + l_i)} \right\}$$

$$\delta \text{ mod. } m = -(k_1 \dots k_n) \begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix} \quad (1.34)$$

and

$$(k_i + l_i) \text{-mod. } m = k_i + l_i \quad \forall i=1 \dots n. \quad (1.35)$$

This is the so-called Generalized Clifford algebra, one of many such types of structures which we shall study in latter chapters. All these algebras can be introduced from different points of view as projective representations of finite Abelian groups (Weyl, Yamaki, Morris)

Linearization of certain partial differential equations (Morinaga and Nono) or generalization of spinor structure (Popovici and Turtoi). Now we shall describe the generalized σ -operation method due to Alladi Ramakrishnan and his collaborators for finding the irreducible representations of this $G_2 C_3 A_6$.

Let *

$$a) \quad L_{(3,1)}^{(m)} = C(m) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$b) \quad L_{(3,2)}^{(m)} = B(m) = \begin{pmatrix} 1 & & & & 0 \\ & \omega(m) & & & \\ & & \ddots & & \\ 0 & & & \omega(m)^{m-1} & \end{pmatrix}$$

$$c) \quad L_{(3,3)}^{(m)} = \zeta C(m)^{m-1} B(m), \quad \zeta = \begin{cases} \omega(m)^{\frac{1}{2}} & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd.} \end{cases} \quad (1.36)$$

It is verified that $\{L_{(3,i)}^{(m)} | i=1,2,3\}$ obey the σ -commutation relations

$$L_{(3,i)}^{(m)} L_{(3,j)}^{(m)} = \omega(m) L_{(3,j)}^{(m)} L_{(3,i)}^{(m)} \quad ; i, j = 1, 2, 3. \quad (1.37)$$

$$\left(L_{(3,i)}^{(m)} \right)^m = 1 \quad \forall i = 1, 2, 3. \quad (1.37)$$

ζ is introduced in $L_{(3,3)}^{(m)}$ when m is even since

$$(C(m)^{m-1} B(m))^m = \omega(m)^{\frac{1}{2} m(m-1)} = -I \quad (1.38)$$

As in the case of $C_m^{(2)}$, we consider first

* Henceforth wherever $C(m)$ and $B(m)$ occur they will refer to these matrices, and when there is no chance of confusion the index m will be omitted.

$$L_3^{(m)} = \sum_{i=1}^3 x_i L_{(3,i)}^{(m)} = \begin{pmatrix} x_2 & x_1 & 0 & \dots & \omega(m)^{m-1} x_3 \\ x_3 & \omega(m) x_2 & x_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \quad (1.39)$$

Now replace x_1 by $L_3^{(m)}$ itself and x_2 and x_3 by $x_4 I$ and $x_5 I$ respectively where I is the identity matrix of dimension $m = \dim(L_3^{(m)})$. This gives now

$$\begin{aligned} L_5^{(m)} &= L_{(3,1)}^{(m)} \otimes \left(\sum_{i=1}^3 x_i L_{(3,i)}^{(m)} \right) + x_4 L_{(3,2)}^{(m)} \otimes I + x_5 L_{(3,3)}^{(m)} \otimes I \\ &= \sum_{i=1}^5 x_i L_{(5,i)}^{(m)} \end{aligned} \quad (1.40)$$

Similarly

$$L_7^{(m)} = L_{(3,1)}^{(m)} \otimes \left(\sum_{i=1}^5 x_i L_{(5,i)}^{(m)} \right) + x_6 L_{(3,2)}^{(m)} \otimes I + x_7 L_{(3,3)}^{(m)} \otimes I \quad (1.41)$$

with $\dim I = \dim(L_5^{(m)}) = m^2$. In general

$$\begin{aligned} L_{2v+1}^{(m)} &= \sigma \left(L_{2v-1}^{(m)} \right) = \sum_{i=1}^{2v+1} x_i L_{(2v+1,i)}^{(m)} \\ &= L_{(3,1)}^{(m)} \otimes \left(\sum_{i=1}^{2v-1} x_i L_{(2v-1,i)}^{(m)} \right) + x_{2v} L_{(3,2)}^{(m)} \otimes I + x_{2v+1} L_{(3,3)}^{(m)} \otimes I \end{aligned} \quad (1.42)$$

with $\dim I = \dim (L_{2v+1}^{(m)})^{v-1} = m^v$. Explicitly writing

$$(a) L_{(2v+1, i)}^{(m)} = L_{(3, 1)}^{(m)} \otimes L_{(2v-1, i)}^{(m)} ; i = 1, 2, \dots, 2v-1$$

$$(b) L_{(2v+1, 2v)}^{(m)} = L_{(3, 2)}^{(m)} \otimes I, \quad (1.43)$$

$$(c) L_{(2v+1, 2v+1)}^{(m)} = L_{(3, 3)}^{(m)} \otimes I, \text{ with } \dim I = m^{v-1}$$

For $m = 2$, (1.43) gives Dirac's procedure of obtaining the anti-commuting elements (for further details cf. Alladi Ramakrishnan¹).

The above procedure gives as shown before,

$$\dim (L_m^{(m)}) = m^{[m/2]} \quad (1.44)$$

showing the irreducibility of the above representation. Since $C_{2v}^{(m)}$ contains as shown before m^{2v} linearly independent elements as basis

$$C_{2v}^{(m)} \cong M_{m^v}(K) \quad (1.45)$$

where $M_{m^v}(K)$ is the total matrix algebra of dimension m^v over a field K (containing $\zeta = \omega(m)^{1/2}$ when m is even). As

earlier shown the set of m^{2v+1} all possible products of all possible powers of $L_{(2v+1, i)}^{(m)}$, i.e. $\left\{ \prod_{i=1}^{2v+1} (L_{(2v+1, i)}^{(m)})^{k_i} \mid 0 \leq k_i \leq m-1, i=1, 2, \dots, 2v+1 \right\}$

does not contain completely linearly independent elements due to the relation (1.32). This makes this set split into a direct sum of m

sets each containing $\binom{m}{2v}$ linearly independent terms. This is seen by writing $L_{(2v+1, 2v+1)}^{(m)}$ in terms of the other $2v$ elements using (1.32). Hence we get

$$L_{(2v+1)}^{(m)} \cong M_{m^v}^{(k)} \oplus \dots \oplus M_{m^v}^{(k)} \quad (\text{m copies})$$

(for more details see^{22, 23, 24,}) Rosevskii²⁵ has given a complete geometric treatment of Clifford algebra and in Chapter IX we shall attempt a generalization of this approach to Generalized Clifford algebra using a generalization of determinants due to Ranganathan²⁶.

4. Comutation Matrices and Product Transforms

Recently Alladi Ramakrishnan and I²⁷ introduced the concepts of comutation matrices and product transforms in connection with the α -comutation relations. Consider the relation

$$L_{(n,i)}^{(m)} L_{(n,j)}^{(m)} = \omega(n) t_{ij} L_{(n,j)}^{(m)} L_{(n,i)}^{(m)} ; i, j = 1, 2, \dots, n. \\ t_{ij} \in \mathbb{Z}, \quad t_{ij} = -t_{ji} \quad (1.47)$$

Henceforth let us drop the indices n and m and simply denote

$L_{(n,i)}^{(m)} = L_i$ when there is no chance of confusion among y . Since the integers t_{ij} satisfy $t_{ji} = -t_{ij}$ they define an antisymmetric integer matrix as

$$\Gamma = (t_{ij}) \quad (1.48)$$

called comutation matrix associated with the system of equations (1.47)

Now let us define a product transform on L_i 's as

$$L_i^* = \prod_{j=1}^n L_j^{u_{ij}} \quad (1.49)$$

where (U_{ij}) are integers, positive or negative. They define an integer matrix

$$U = (U_{ij}) \quad (1.50)$$

Now determining the commutation relations among L_i^* 's we get

$$L_i^* L_j^* = \left(\prod_{k=1}^n L_k^{U_{ik}} \right) \left(\prod_{l=1}^m L_l^{U_{jl}} \right)$$

$$= \omega(m) (UT\tilde{U})_{ij} \left(\prod_{l=1}^m L_l^{U_{jl}} \right) \left(\prod_{k=1}^n L_k^{U_{ik}} \right)$$

$$= \omega(m) t_{ij}^* L_j^* L_i^*$$

$$i, j = 1, 2, \dots, n.$$

where

$$\left(t_{ij}^* \right) = T^* = UT\tilde{U}. \quad (1.51)$$

Thus the commutation matrix $T^* = (t_{ij}^*)$ associated with the new set of matrices is given by

$$T^* = UT\tilde{U}. \quad (1.52)$$

If $|\det U| = 1$ then there exists a unimodular matrix V such that

$$V = U^{-1}. \quad (1.53)$$

and

$$T = VT^* \tilde{V} \quad (1.54)$$

This implies that there exists a product transform on L_i^* ,

$$L_i' = \prod_{j=1}^m L_j^{*v_{ij}} ; \quad i = 1, 2, \dots, n. \quad (1.55)$$

where

$$(v_{ij}) = V \quad (1.56)$$

such that L_i' 's have the same commutation matrix T . Actually explicitly writing

$$\begin{aligned} L_i' &= \prod_{j=1}^m \left(\prod_{k=1}^n L_k^{u_{jk}} \right)^{v_{ij}} \\ &\sim \prod_{k=1}^n \left(L_k^{\left(\sum_{j=1}^m v_{ij} u_{jk} \right)} \right) = \prod_{k=1}^n L_k^{\delta_{ik}} \\ &= L_i ; \quad \text{ie } L_i' = L_i ; \quad i = 1, 2, \dots, n. \end{aligned} \quad (1.57)$$

The V defines in a sense an inverse transform to U -transform.

Now the commutation matrix associated with (1.20) is given by

$$T = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 1 & \dots & 1 & 1 \\ \vdots & & & & & \\ -1 & -1 & -1 & \dots & 0 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{bmatrix} \quad (1.58)$$

5. Sonon and Mortice coupling method in terms of Product transform

Following α -operation method Alladi Ramakrishnan formulated another equivalent method called Sonon and mortice coupling method for the representation of Clifford algebras. Here we reformulate this method in terms of product transforms introduced above Consider the commutation relations of the type

$$L_{2i-1}^* L_{2i}^* = \omega(m) L_{2i}^* L_{2i-1}^* ; i=1, 2, \dots, v. \quad (1.59)$$

$$L_k^* L_l^* = L_l^* L_k^* \text{ otherwise } ; l, k=1, 2, \dots, 2v.$$

Relabelling these as

$$L_{2i-1}^* = H_1^i \quad i=1, 2, \dots, v \quad (1.60)$$

$$L_{2i}^* = H_2^i$$

we have

$$H_1^i H_2^i = \omega(m) H_2^i H_1^i ; \quad i=1, 2, \dots, v. \quad (1.61)$$

$$H_k^i H_l^j = H_l^j H_k^i ; \quad l \neq j ; \quad l, k=1, 2.$$

The set of $2v$ matrices $\{H_k^i | i=1, 2; k=1, 2\}$ form a set of v pairs, the members of each pair α -commuting among themselves but members of different pairs commuting with each other. To each pair $\{H_k^i | k=1, 2\}$ a third one can be attached

$$H_3^i = (H_1^i)^{-1} H_2^i \quad (1.62)$$

such that the three form a complete set of mutually α -commuting matrices. Looking at the problem of the irreducible dimensionality of representation the following is clear. If we consider only two pairs then since the second pair commutes with the first one if the first one is represented irreducibly by $n \times n$ matrices the members of the

second pair are to be multiples of identity matrices by Schur's lemma, but this contradicts the requirement that the second pair is also mutually commuting pair whose irreducible dimension is $m \times m$. Hence the irreducible dimension has to be at least m^2 . By taking all possible products of all possible powers of the members of the two pairs we get m^{ν} linearly independent elements whose linear independence can be easily proved by arguments similar to the one given above. Thus the irreducible dimension is m^{ν} . In general if we take ν pairs the set of all possible products of all powers of them, consists of $m^{2\nu}$ linearly independent elements thus requiring that the irreducible dimension has to be m^{ν} . It is very easy to construct these representations as follows

$$H_1^i = \underbrace{I \otimes \dots \otimes I}_{\nu-1} \otimes C(m) \otimes \underbrace{I \otimes \dots \otimes I}_{i-1}$$

$$H_2^i = \underbrace{I \otimes \dots \otimes I}_{\nu-1} \otimes B(m) \otimes \underbrace{I \otimes \dots \otimes I}_{i-1}$$

where

$$C(m)B(m) = \omega(m)B(m)C(m)$$

$$\dim. C(m) = \dim. B(m) = \dim. I = m. \quad (1.63)$$

A linear combination of the three matrices of the same set

$$H^i = \sum_{j=1}^3 x_j H_j^i \quad (1.64)$$

has been called 'generalised Helicity Matrices' since in the case $n = 2$, $\nu = 2$, they are called Helicity matrices in quantum mechanics.

Now in terms of these $\{H_k^i | i = 1, 2, \dots, \nu; k = 1, 2\}$ form the set of matrices

$$L_1 = H_1^v$$

$$L_2 = H_2^v$$

$$L_3 = (H_1^v)^{-1} H_2^v H_1^{v-1}$$

$$L_4 = (H_1^v)^{-1} H_2^v H_2^{v-1}$$

$$\vdots \quad L_{2r-1} = (H_1^v)^{-1} H_2^v (H_1^{v-1})^{-1} \dots H_1^{v-(r-1)}$$

$$L_{2r} = (H_1^v)^{-1} H_2^v (H_1^{v-1})^{-1} \dots H_2^{v-(r-1)}$$

$$\gamma = 1, 2, \dots, v.$$

(1.65)

These are seen to obey the ordered α -commutation relations

$$L_i L_j = \omega(m) L_j L_i ; i, j = 1, 2, \dots, 2v. \quad (1.66)$$

$i < j$ procedure

This is the first part of tenon and mortice coupling method. The matrices obtained by this procedure are same as those obtained by co-operation method. Converse of this procedure of obtaining H -matrices from L_i 's has also been considered in detail by Alladi Ramakrishnan¹ and for $m=2$, this procedure of obtaining H -matrices from L_i 's has been considered by Clifford himself in 1878. We make a comparison of both in Chapter VIII. Now we shall reformulate the above method by using the product transforms. As already noted in (1.63) the commutation matrix associated with the system of ordered α -commutation relations is given by

$$T = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 1 & \dots & 1 & 1 \\ -1 & -1 & 0 & \dots & 1 & 1 \\ \vdots & & & & & \\ -1 & -1 & -1 & \dots & 0 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{bmatrix} \quad (1.67)$$

The \mathbb{N} -matrices defined by (1.50 - 1.61) interpreted in terms of \mathbb{L} -matrices have the commutation matrix

$$T^* = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 0 & 1 & 0 \\ & 0 & 1 & & \\ & -1 & 0 & & \\ 0 & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix} \quad (1.68)$$

and one solution for V connecting T and T^* ~~consequently~~ by

$$T = VT^*V^{-1}; \quad |\det V| = 1, \quad v_{ij} \in \mathbb{Z}. \quad (1.69)$$

is given by, for $n = 2V$,

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1 & 1 \\ \vdots & & & & & & & & \\ 1 & 0 & -1 & 1 & \dots & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & \dots & -1 & 1 & -1 & 1 \end{bmatrix} \quad (1.70)$$

Hence as shown in IV, the matrices associated with T can be constructed from those associated with T^* by using V as the kernel of the product transformation as

$$L_i = \prod_{j=1}^{2V} L_j^{*v_{ij}} \quad ; \quad i = 1, 2, \dots, 2V. \quad (1.71)$$

and this reproduces the formula (1.65) as

$$\begin{aligned} L_{2R-1} &= L_{2v-1}^{*-1} L_{2v} L_{2v-3} \cdots L_{2(v-k)+1}^* \\ L_{2k} &= L_{2v-1}^{*-1} L_{2v} L_{2v-3} \cdots L_{2(v-k)+2}^* \end{aligned} \quad k = 1 \dots v. \quad (1.72)$$

One has to remember that after getting these L_i 's they have to be sometimes normed suitably to obey the condition $L_i^{*v} = I, \forall i$. However this even this situation will arise. The monomials of V in (1.60) does not matter as we shall see in later chapters. All V 's lead to equivalent representations in the above case. This will become clear in Chapter V. Generalization of the above method to obtain all the irreducible representations of algebras generated by elements with an arbitrary commutation matrix T will be considered in Chapter V leading to the complete and explicit solution of the problem of explicitly determining all the projective representations of any finite Abelian group.

Schur's Representation Theorem

The following theorem of matrix decomposition is due to Alfred Schur. In fact we shall study this theorem and an extension of it due to Albert Büchner and myself in the next Chapter.

Theorem Any real matrix M can be expressed uniquely as

$$M = \sum_{k,l=0}^{m-1} Q_{kl} B^k C^l \quad (1.73)$$

where $a_{kl} = (A)$ is determined by the relation, $A = S^{-1} R$. Here

$$R = \begin{bmatrix} M_{00} & M_{01} & \cdots & M_{0,m-1} \\ M_{11} & M_{12} & & M_{1,m-1} \\ \vdots & & & \vdots \\ M_{m-1,m-1} & M_{m-1,0} & & M_{m-1,m-2} \end{bmatrix}$$

and

S^{-1} being the inverse of Sylvester matrix S

$$S = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{m-1} \\ \vdots & & & & \\ 1 & \omega^m & \omega^{2(m-1)} & \dots & \omega^{(m-1)^2} \end{bmatrix} \quad S^{-1} = \frac{1}{N} S^+$$

$$(S^{-1})_{ij} = \frac{1}{N} \omega^{-ij}$$

$$(S)_{ij} = \omega^{ij}; i, j = 0, 1, \dots, m-1.$$

$$\omega = \omega(n)$$

Proof of this follows by noting first that M can be written as a linear combination

$$M = M_0 I + M_1 C(m) + \dots + M_{r-1} C(m)^{r-1} + \dots + M_{m-1} C(m)^{m-1}$$

where

$$M_i = \begin{bmatrix} M_{0,i} & & & & 0 \\ M_{1,(i+1)} & & & & \\ 0 & & M_{(m-1), (i+(m-1))} & & \\ & & & \text{of order } m \times m & \end{bmatrix} \quad (i+k) \bmod m = i+k$$

$$i = 0, 1, \dots, m-1 \quad (1.77)$$

and any diagonal matrix D can be written as

$$D = d_0 I + d_1 B(m) + \dots + d_{m-1} B(m)^{m-1} \quad (1.78)$$

$$\text{with } d_i = \sum_{j=0}^{m-1} (S^{-1})_{ij} D_{jj}; i = 0, 1, \dots, m-1.$$

Putting together we arrive at the theorem

Let us put the formula for a_{kl} 's in the following form for use later

$$a_{kl} = \frac{1}{m} \text{Tr} \left(M C(m)^{-l} B(m)^{-k} \right) \quad (1.79)$$

or

$$a_{kl} = \frac{\omega(m)^{kl}}{m} \text{Tr} \left(M B(m)^{-k} C(m)^{-l} \right) \quad (1.80)$$

Significance of this theorem arises from the fact that it expresses any matrix (operator) in an orthonormal unitary matrix (operator) 28 basis. Weyl's rule and Wigner distribution function used in quantum mechanics are derived from this fact only.²¹ Some details of application of these matrices to understand the algebraic structure of quantum kinematics have been treated explicitly by Schwinger²² and we shall consider those in Chapter XI.

Summary of Important points.

Any Generalized Clifford Algebra(G.C.A.) has as its basis the set of all possible products of all possible powers of n generators obeying the commutation relations

$$L_i L_j = \omega(m)^{t_{ij}} L_j L_i ; t_{ij} \in \mathbb{Z}, t_{ji} = -t_{ij}$$

$$L_i^{m_i} = I ; m_i \in \mathbb{Z}, i, j = 1, 2, \dots, n.$$

When $t_{ij}=1$ ($\forall i < j$) and $\forall m_i=m$ this algebra becomes the usual G.C.A.
 $C_n^{(m)}$. The antisymmetric integer matrix T associated with $\{L_i | i=1..n\}$

is called a 'commutation matrix'. If a 'product transform' is defined

$$L_i^* = \prod_{j=1}^n L_j^{u_{ij}} \quad i = 1, \dots, n. \quad u_{ij} \in \mathbb{Z}.$$

The commutation matrix associated with $\{L_i^*\}$ say T^* is related to T by $T^* = V T \tilde{V}$. If $|\det V| = 1$ then $V^{-1} = V$ relates T and T^* by $T = V T^* \tilde{V}$. This gives $L_i = \prod_{j=1}^n L_j^{* u_{ij}}$. Thus if any irreducible representation of $\{L_i^*\}$ is known an irreducible representation of $\{L_i\}$ can be obtained if T^* and V are also explicitly known. Irreducibility of the representation of $\{L_i^*\}$ implies the irreducibility of the representation thus obtained, since the mapping -product transform- is invertible. The commutation relations and hence T essentially determine the matrices upto indeterminacy of normalization factors which are to be determined to fit the requirements $L_i^{m_i} = I$. This product transform provides a powerful tool for representation theory of G.C.A.'s to any antisymmetric integer matrix T . There is a unique

skew normal form $T^* = V \tilde{Y} U$, with $|\det U| = 1$, of the simple type

$$T^* = \sum_{i=1}^s \oplus \begin{bmatrix} 0 & t_i \\ -t_i & 0 \end{bmatrix} \oplus 0_{n-2s} \quad s \left(\leq \left[\frac{n}{2} \right] \right) = \text{rank of } T$$

for which the corresponding set $\{L_i^* | i=1 \dots n\}$ having the commutation relations

$$L_i^* L_j^* = \omega(n) t_{ij}^* L_j^* L_i^* ; i, j = 1 \dots n ; (t_{ij}^*) = T^*$$

can be represented easily irreducibly leading to the determination of a n irreducible representation of $\{L_i | i=1 \dots n\}$ through the product transform, $L_i = \prod_{j=1}^n L_j^* v_{ij}$, due to the possibility of explicit determination of a corresponding V and U (Cf. Morris Newman²²) we shall take up this matter further in Chapter V.

CHAPTER 2

MAIN FEATURES OF A MATRIX DECOMPOSITION THEOREM(i) New approach to Matrix theory and its uses.

This new approach to Matrix theory due to Aladi Ramakrishnan¹ consists in viewing an $m \times m$ matrix as a linear combination of the m^2 matrices $\{B^R C^l | R, l = 0, \dots, m-1\}$ generated by taking all possible products of all possible powers of B and C . For some purposes this approach is useful as is demonstrated in the following problems.

(a) To find the number of inequivalent irreducible representations of

a) $L_1 L_2 = \omega(m)^l L_2 L_1$

b) $L_1^m = L_2^m = 1 \quad (l, m) = d \quad (2.1)$

Let $\frac{l}{d} = l'$ and $\frac{m}{d} = m' ; (l', m') = 1 \quad (2.2)$

Then $\omega(m)^l = \exp(2\pi i \frac{l}{m}) = \exp(2\pi i l'/m') = \omega(m')^{l'} \quad (2.3)$

Now consider the relations

$$L_1^l L_2^l = \omega(m')^{l'} L_2^l L_1^l \quad (2.4)$$

$$L_1^m = L_2^m = 1 \quad (l', m') = 1.$$

These relations are same as (1.20) for $m = m', n = 2$ except that

$\omega(n')$ has been replaced by $\omega(m')^{l'}$ which is also a primitive n' th root. Hence the irreducible representations of (2.4) are given by the same matrices as in (1.26) except that $\omega(n')$ has been replaced by $\omega(m')^{l'}$. Hence L_1, L_2 are represented by n' -dimensional C and B matrices with $\omega(m') \rightarrow \omega(m')^{l'}$ or,

$$L'_1 = C(m'), \quad L'_2 = B(m')^{l'} \quad (2.5)$$

In the relations (2.1) the first determines the structure of the matrix upto the uncertainty of a phase factor and the second is a normalisation condition which shows that the matrices obeying both can always be multiplied by n th root of unity. Since the first relation determines the irreducible dimension its equivalence with (2.4a) shows that

$$L_1 \sim L'_1 \quad (2.6)$$

$$L_2 \sim L'_2$$

The possibilities are given completely by

$$\{L_1 = \omega(m)^{k_1} L'_1, L_2 = \omega(m)^{l_1} L'_2 \mid k_1, l_1 = 0, 1, \dots, m-1\} \quad (2.7)$$

Thus there are m^2 possibilities. But all of those are not inequivalent. If two representations say,

$$R_1 : \{L_1 = \omega(m)^{k_1} L'_1; L_2 = \omega(m)^{l_1} L'_2\} \quad (2.8)$$

$$R_2 : \{L_1 = \omega(m)^{k_2} L'_1; L_2 = \omega(m)^{l_2} L'_2\}$$

are equivalent then there exists a nonsingular $m' \times m'$ matrix S such that

$$\begin{aligned}\omega(m)^{k_1} L_1' S &= S \omega(m)^{k_2} L_1' \\ \omega(m)^{l_1} L_2' S &= S \omega(m)^{l_2} L_2'\end{aligned}\quad (2.10)$$

or

$$\begin{aligned}L_1' S &= \omega(m)^{k_2 - k_1} S L_1' = \omega(m)^{h_1} S L_1' \\ L_2' S &= \omega(m)^{l_2 - l_1} S L_2' = \omega(m)^{h_2} S L_2'\end{aligned}\quad (2.11)$$

Now use the theorem (1.13) to write

$$S = \sum_{k,r=0}^{m'-1} a_{kr} L_1'^k L_2'^r \quad (2.12)$$

so

$$L_1' \left\{ \sum_{k,r=0}^{m'-1} a_{kr} L_1'^k L_2'^r \right\} = \omega(m)^{h_1} \left\{ \sum_{k,r=0}^{m'-1} a_{kr} L_1'^k L_2'^r \right\} L_1'$$

$$L_2' \left\{ \sum_{k,r=0}^{m'-1} a_{kr} L_1'^k L_2'^r \right\} = \omega(m)^{h_2} \left\{ \sum_{k,r=0}^{m'-1} a_{kr} L_1'^k L_2'^r \right\} L_2'. \quad (2.13)$$

Multiplying the first by $L_1'^{m'-1}$ and second by $L_2'^{m'-1}$ from left we get

$$\begin{aligned}\left\{ \sum_{k,r=0}^{m'-1} a_{kr} L_1'^k L_2'^r \right\} &= \omega(m)^{h_1} \left\{ \sum_{k,r=0}^{m'-1} a_{kr} \omega(m')^{-r} L_1'^k L_2'^r \right\} \\ \left\{ \sum_{k,r=0}^{m'-1} a_{kr} L_1'^k L_2'^r \right\} &= \omega(m)^{h_2} \left\{ \sum_{k,r=0}^{m'-1} a_{kr} \omega(m')^{-k} L_1'^k L_2'^r \right\}\end{aligned}\quad (2.14)$$

By the uniqueness of the expansion of S it follows

$$\begin{aligned} a_{kr} &= \omega(m)^{h_1} a_{kr} \omega(m)^{-l'r} = \omega(m)^{h_2} a_{kr} \omega(m)^{kl'} \\ &= \omega(m)^{h_1 - lr} a_{kr} = \omega(m)^{h_2 + lr} a_{kr} : \forall a_{kr} \neq 0. \end{aligned}$$
(2.15)

or

$$h_1 - lr = 0 \pmod{m}$$

$$h_2 + lr = 0 \pmod{m} \quad \forall k, r \text{ for which } a_{kr} \neq 0 \text{ in (2.12).}$$
(2.16)

If there is only one $a_{kr} \neq 0$ then immediately the solution for h_1 and h_2 follows

$$h_1 = lr \quad (2.17)$$

$$h_2 = -lr \text{ or } l(m-r)$$

But if more than one $a_{kr} \neq 0$ then no solution exists for h_1 , h_2 or stating it conversely for a given (h_1, h_2) the S can only be a multiple of any one of the members of the set

$\{L_1^{rk} L_2^{rl} \mid k, l = 0, 1, \dots, m'-1\}$ and cannot be a sum of more than one term. This can be established by the following argument also. Let

S be $\left(\sum_{k,l=0}^{m'-1} a_{kr} L_1^{rk} L_2^{rl} \right)$. For S to be a sum of more than one term member of $\{L_1^{rk} L_2^{rl} \mid k, l = 0, 1, \dots, m'-1\}$, those terms must

have the same commutation relation with L_1' and L_2' . But if $L_1^{rk_1} L_2^{rl_1}$ and $L_1^{rk_2} L_2^{rl_2}$, $(k_1, l_1) \neq (k_2, l_2)$ both have same commutation relation with L_1' and L_2' . $(L_1^{rk_1} L_2^{rl_1})^{-1} (L_1^{rk_2} L_2^{rl_2})$ must commute with L_1' and L_2' or by Schur's lemma $(L_1^{rk_1} L_2^{rl_1})^{-1} (L_1^{rk_2} L_2^{rl_2}) \sim I$

$\text{if } k_1 = k_2, l_1 = l_2 \text{ contrary to the initial assumption that}$
 $(k_1, l_1) \neq (k_2, l_2)$. Thus R can not be a sum of more than one
 number of $\{L'_1 L'_2 | 0 \leq k, l \leq m'-1\}$. Thus all the m'^2
 elements $\{L'_1 L'_2 | 0 \leq k, l \leq m'-1\}$ have m^2 different commutation
 relations with L'_1 and L'_2 . From (2.17) it is seen that for
 different (r, k) values h_1 and h_2 are different and as (r, k)
 take values $(0, 1, \dots, m'-1)$ and $w(m)^{h_1}, w(m)^{h_2}$ each takes all
 values of the m' -th roots of unity. Thus all the m^2 different
 pairs given by

$$\{w(m)^{h_1} L'_1, w(m)^{h_2} L'_2 | h_1, h_2 \neq l\} \quad (2.18)$$

are equivalent. So R_1 and R_2 are equivalent iff

$$k_1 - k_2, l_1 - l_2 \neq l \quad (2.19)$$

or

$$\begin{aligned} w(m)^{k_1 - k_2} &= w(m')^T \\ w(m)^{l_1 - l_2} &= w(m')^R \end{aligned} \quad (2.20)$$

for some value of $r, k = 0, 1, \dots, m'-1$. Thus if we start with a representation,

$$R_1: \{L_1 = w(m)^{k_1} L'_1, L_2 = w(m)^{l_1} L'_2\} \quad (2.21)$$

then the representations given by

$$\{L_1 = w(m)^{k_1 + lr} L'_1, L_2 = w(m)^{l_1 + rk} L'_2 | r, k = 0, 1, \dots, m'-1\} \quad (2.22)$$

are all equivalent. Thus by starting with a particular set of

values (k_i, l_i) out of (2.7) all those m'^2 can be classified into a single representation; another set of values for (k_i, l_i) is chosen and all those equivalent to it are classified and in this way one arrives at $(m^2/m'^2) = d^2$ inequivalent representations. The above procedure leads to classification of representations as

$$R_i : \left\{ L_i = \omega(m)^{\delta_i} L'_i, L_2 = \omega(m)^{k_i} L'_2 \right\}; i=1, 2, \dots, d^2. \quad (2.29)$$

where δ_i and k_i take values

$$\delta_i, k_i = 0, 1, 2, \dots, d-1. \quad (2.30)$$

Proof. $\{\omega(m)^{\delta_i}, \omega(m)^{k_i}\}$ should be chosen from the quotient set

$$\frac{\{1, \omega(m), \dots, \omega(m)^{m'-1}\}}{\{1, \omega(m)^l, \dots, \omega(m)^{l(m'-1)}\}} \quad (2.35)$$

Now upto rearrangement,

$$\{1, \omega(m')^l, \dots, \omega(m')^{l(m'-1)}\} \equiv \{1, \omega(m'), \dots, \omega(m')^{m'-1}\} \quad (2.36)$$

since

$$\omega(m')^{l'k} = \omega(m')^r \quad ; \quad 0 \leq r \leq m'-1 \quad (2.37)$$

has a unique solution for r by Euler-Format theorem

$$k = r \ell^{\phi(m')-1} \pmod{m'} \quad (2.38)$$

writing

$$\{1, \omega(m'), \dots, \omega(m')^{m'-1}\} = \{1, \omega(m)^d, \dots, \omega(m)^{m-d}\} \quad (2.39)$$

it follows

$$(2.25) \equiv \{1, \omega(m), \dots, \omega(m)^{d-1}\} \quad (2.30)$$

Thus we have found $d^2 = \left(\frac{m}{m'}\right)^2$ inequivalent representations of (2.1)

Corresponding to one representation of (2.4). Now yet we do not know whether these d^2 exhaust all possible representations. Suppose there exists another representation inequivalent to those $\{L_1, L_2\}$ which we denote by $\{L_1', L_2'\}$. The relations (2.1) imply the following

$$\begin{aligned} L_1^{m'} L_2 &= \omega(m)^{lm'} L_2 L_1^{m'} \quad \text{or} \quad L_1^{m'} L_2 = L_2 L_1^{m'} \\ L_1 L_2^{m'} &= \omega(m)^{lm'} L_2^{m'} L_1 \quad \text{or} \quad L_1 L_2^{m'} = L_2^{m'} L_1 \end{aligned} \quad (2.31)$$

or $L_1^{m'}$ and $L_2^{m'}$ ~~must~~ commute with both L_1 and L_2 and hence with all possible products of all possible powers of them. Thus by Schur's lemma

$$L_1^{m'} \sim 1 \quad L_2^{m'} \sim 1 \quad (2.32)$$

Let

$$L_1^{m'} = \xi I, \quad L_2^{m'} = \eta I. \quad (2.33)$$

Then 1 b) gives

$$(L_1^{m'})^d = \xi^d I = I \quad (2.34)$$

$$(L_2^{m'})^d = \eta^d I = I$$

$$\xi = \exp(2\pi i k/d) = \omega(d)^k \quad k, k = 0, 1, \dots, d-1.$$

$$\eta = \exp(2\pi i t/d) = \omega(d)^t$$

(2.35)

Thus the following holds

$$(\lambda L_1)^{m'} = 1, \quad (\mu L_2)^{m'} = 1. \quad (2.36)$$

where

$$\lambda^{m'} \omega(d)^k = 1; \quad \mu^{m'} \omega(d)^t = 1 \quad (2.37)$$

or

$$\begin{aligned} \lambda^{m'} &= \omega(d)^{d-k} \quad \text{if } \lambda = \omega(m)^{d-k} \\ \mu^{m'} &= \omega(d)^{d-t} \quad \text{if } \mu = \omega(m)^{d-t} \quad ; k, t = 0, 1, \dots, d-1. \end{aligned} \quad (2.38)$$

Thus for any representation of L_1, L_2 , there exist some value of k and t such that

$$\begin{aligned} (\omega(m)^{d-k} L_1)^{m'} &= 1 & k, t < d. \\ (\omega(m)^{d-t} L_2)^{m'} &= 1 \end{aligned} \quad (2.39)$$

Thus $\{\omega(m)^{d-k} L_1, \omega(m)^{d-t} L_2\}$ obey (2.4). As Mono² and

Norris³ have shown the relations (2.1) have only one inequivalent irreducible representation. Thus there exists some S , such that

for L_1'' and L_2'' for some k and t

$$\begin{aligned} \omega(m)^{d-k} L_1'' S &= S L_1' \\ \omega(m)^{d-t} L_2'' S &= S L_2' \end{aligned} \quad (2.40)$$

or

$$\begin{aligned} S^{-1} L_1'' S &= \omega(m)^{m+k-d} L_1' = \omega(m)^r \omega(m')^s L_1' \\ S^{-1} L_2'' S &= \omega(m)^{m+t-d} L_2' = \omega(m)^u \omega(m')^v L_2' \end{aligned}$$

$$s, u < m'$$

$$r, u < d.$$

$$(2.41)$$

since

$$\begin{aligned} m - (d - k) &= s'd + r & r < d \\ m - (d - t) &= v'd + u & u < d \end{aligned} \quad (2.42)$$

has solutions and then set

$$\begin{aligned} s \bmod m' &= s' \\ v \bmod m' &= v' \end{aligned} \quad (2.43)$$

By the arguments given above there exists a U such that

$$\begin{aligned} \omega(m')^s L_1' &= UL_1 U^{-1} \\ \omega(m')^v L_2' &= UL_2 U^{-1} \end{aligned} \quad (2.44)$$

so

$$\begin{aligned} (SU)^{-1} L_1'' SU &= \omega(m)^s L_1' \\ (SU)^{-1} L_2'' SU &= \omega(m)^v L_2' ; \quad 0 \leq r, u \leq d-1 \end{aligned} \quad (2.45)$$

Thus it follows that any representation $\{L_1'', L_2''\}$ is equivalent to any one of the d^2 representations $\{\omega(m)^{s_i} L_1', \omega(m)^{t_i} L_2' | 0 \leq s_i, t_i \leq d-1\}$ where $\{L_1', L_2'\}$ is the unique (upto equivalence) representation of (2.4). Here we have assumed the result of Nono² and Morris³ for proving that (2.1) has $d^2 = (l, m)^2 = m^2/D^2$ inequivalent representations of same dimension $D = \frac{m}{(l, m)} = m'$. Later in Chapter XIII - V we shall prove this result a priori in a more general way. The result that the relations

$$\begin{aligned} L_1 L_2 &= \omega(m) L_2 L_1 \\ L_1^m &= L_2^m = 1. \end{aligned} \quad (2.46)$$

have only one irreducible representation is known to physicists from the work of Hermann Weyl⁴ who used these representations to show that the quantum operators for position and momentum are given by the Schrodinger representation

$$\begin{aligned} \text{position } q &\rightarrow \hat{q} \times = \hat{q} \\ \text{momentum } p &\rightarrow -i\hbar \frac{d}{dq} = \hat{p}. \end{aligned} \quad (2.47)$$

\hbar = Planck's constant. Thus the uniqueness of representation of (2.46) implies the uniqueness of Schrodinger representation quantum kinematics.

Usually Weyl's form of canonical commutation rule is written as

$$U_\alpha V_\beta = e^{i\hbar \alpha \hat{p}} V_\beta U_\alpha \quad (2.48)$$

where

$$U_\alpha = e^{i\alpha \hat{p}}, \quad V_\beta = e^{i\beta \hat{q}} \quad [\hat{q}, \hat{p}] = i\hbar 1. \quad (2.49)$$

$$(U_\alpha \Psi)(q) = \Psi(q + \hbar \alpha); \quad (V_\beta \Psi)(q) = e^{i\beta q} \Psi(q) \quad (2.50)$$

Von Neumann⁵ proved the uniqueness of Schrodinger representation (2.46, 2.48) upto unitary equivalence and since then many other proofs have been given. For more general discussions on these considerations reference can be made to⁶.

b) Understanding of Weyl's rule and Wigner distribution function.

Following the statement of Weyl⁴ that we can understand the relations existing in Hilbert space by analogy with or as limiting cases of those existing in spaces of a finite number of dimensions, here we shall understand the mathematical aspects of Weyl correspondence from the matrix decomposition theorem.

Now let $M = 2v+1$ and let us take our generator matrices C and B as follows

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad B = \begin{pmatrix} w(2v+1)^{-v} & & & & \\ & w(2v+1)^{-v+1} & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & w(2v+1)^v \end{pmatrix}$$

(2.51)

Let us choose the basis elements as

$$\mathcal{B} = \left\{ w(2v+1)^{\frac{1}{2}kl} B^k C^l \mid k, l = -v, -v+1, \dots, 0, 1, \dots, v \right\}$$

Then any $(2v+1) \times (2v+1)$ matrix is written uniquely as

$$M = \sum_{k=-v}^{v} a_{kl} w(2v+1)^{\frac{1}{2}kl} B^k C^l$$

where $a_{kl} = \frac{1}{2v+1} \text{tr} \left(M w(2v+1)^{-\frac{1}{2}kl} C^{-l} B^{-k} \right)$

Let $X \equiv \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_v \end{pmatrix}$ be any $(2v+1)$ dimensional vector. The effect of

C^l and B^k on it are given by

$$(C^l X)_j = x_{j+l}, (B^k X)_j = \omega(2\nu+1)^{jk} x_j; i, j = -\nu, \dots, +\nu. \quad (2.54)$$

In analogy to this finite dimensional case consider the function space (square integrable) where now $\Psi(q)$ is regarded a q th coordinate of the ∞ (nonenumerably) dimensional vector Ψ . Now following Weyl consider the relations.

$$\begin{aligned} (C^l \Psi)(q) &= \Psi(q + \xi l); (B^k \Psi)(q) = \int_q^{q+k} \Psi(\tau) \\ &\equiv \Psi(q + \sigma) \end{aligned} \quad (2.55)$$

which are analogues of (2.54) for $\nu \rightarrow \infty$ in a continuous way. When $\nu \rightarrow \infty$ continuously l and k are also continuous. So C^l and B^k form one parameter continuous groups as l, k vary and hence by Stone's theorem they can be represented exponentially

$$C^l \rightarrow e^{i\sigma \hat{P}}; B^k \rightarrow e^{i\tau \hat{Q}} \quad (2.56)$$

Following Weyl, by returning to the infinitesimal transformations from finite transformations \hat{P} and \hat{Q} are identified as Schrödinger representation of momentum and position operators namely $-i\frac{d}{dq}$ and $q \times$ respectively (taking $\hbar = 1$). With this identification (2.56) in the case of infinitesimal (non-denumerably) vector space, the analogue of (2.55) becomes for any operator in this space $+\infty$

$$M = \iint_{-\infty}^{+\infty} a(\tau, \sigma) e^{i\frac{\sigma}{2}\tau} e^{i\tau \hat{Q}} e^{i\sigma \hat{P}} d\sigma d\tau \quad (2.57)$$

or using the fact

$$e^{\frac{i}{2}\sigma\tau} e^{i\tau\hat{q}} e^{i\sigma\hat{p}} = e^{i(\tau\hat{q} + \sigma\hat{p})} \quad (2.58)$$

then

$$[\hat{q}, \hat{p}] = i \cdot 1. \quad (2.59)$$

we have

$$M = \iint_{-\infty}^{+\infty} a(\tau, \sigma) e^{i\tau\hat{q} + i\sigma\hat{p}} d\sigma d\tau \quad (2.60)$$

where

$$a(\tau, \sigma) = \frac{1}{2\pi} \text{Tr} \left\{ M e^{-\frac{i}{2}\sigma\tau} e^{i\sigma\hat{p}} e^{i\tau\hat{q}} \right\} = \frac{1}{2\pi} \text{Tr} \left\{ M e^{\frac{i}{2}\sigma\tau} e^{i\tau\hat{q}} e^{i\sigma\hat{p}} \right\} \quad (2.61)$$

Now we shall use carefully the language of Dirac delta function to evaluate this trace in position representation of the wave functions. In position representation, $e^{i\tau\hat{q}}$ is diagonal since

$$e^{i\tau\hat{q}} \delta(q - q') = e^{i\tau q'} \delta(q - q') \quad (2.62)$$

Hence

$$\langle q'' | e^{i\tau\hat{q}} | q' \rangle = e^{i\tau q'} \delta(q' - q'') \quad (2.63)$$

We have

$$\langle q'' | e^{i\sigma\hat{p}} | q' \rangle = \delta(q'' - q' - \sigma) = \langle q'' | q' + \sigma \rangle \quad (2.64)$$

It is usual to write this as $\langle q'' - \frac{1}{2}\sigma | q' + \frac{1}{2}\sigma \rangle$ So we shall write (2.61) as

$$\begin{aligned} a(\tau, \bar{\tau}) &= \frac{1}{2\pi} \int dq' \langle q' | M e^{-\frac{1}{2}i\sigma\tau} e^{-i\sigma\hat{p}} e^{-i\tau\hat{q}} | q' \rangle \\ &= \frac{1}{2\pi} \int dq' \int dq'' \langle q'' | M | q'' \rangle \langle q'' | e^{\frac{1}{2}i\sigma\tau} e^{i\sigma\hat{p}} e^{i\tau\hat{q}} | q' \rangle \\ &= \frac{1}{2\pi} \int dq' \langle q' - \frac{1}{2}\sigma | M | q' + \frac{1}{2}\sigma \rangle e^{-i\tau q'} \end{aligned} \quad (2.65)$$

This same relation can be obtained directly by taking the limit of relation (1.74) for a_{kl}

$$a_{kl} = \sum_r (S^{-1})_{kr} R_{rl} \quad (2.66)$$

or

$$\begin{aligned} a_{kl} &= \frac{1}{2N+1} \sum_{r=-N}^N w^{-kr} w^{-\frac{1}{2}kl} R_{r,r+l} \\ &= \frac{1}{2N+1} \sum_{r=-N}^N w^{-k(r+\ell/2)} R_{r,r+l} = \frac{1}{2N+1} \sum_{r'=-N+\frac{\ell}{2}}^{N+\frac{\ell}{2}} w^{-kr'} R_{r'-\frac{\ell}{2}, r'+\frac{\ell}{2}} \end{aligned} \quad (2.67)$$

This becomes

$$a(\tau, \bar{\tau}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq' e^{-i\tau q'} \langle q' - \frac{\sigma}{2} | M | q' + \frac{\sigma}{2} \rangle \quad (2.68)$$

as in (2.65). Thus we have shown that the expansion (2.60) of operators (Hilbert-Schmidt operators to be precise) on a Hilbert space of single variable can be understood as a limiting case of Alladi Ramakrishnan theorem (2.53) for finite dimensions.

In classical mechanics each observable attribute of a physical system is represented by a function on the phase space of the system. Weyl⁴ suggested the following method of obtaining quantum mechanical operators corresponding to classically observable functions on phase space. In (2.50) if we regard \hat{q} and \hat{p} as classical canonical variables q and p then $M(q, p)$ is a function in phase space and $a(\tau, \sigma)$ are Fourier Coefficients of $M(q, p)$ defined by

$$a(\tau, \sigma) = \iint_{-\infty}^{+\infty} M(q, p) e^{-i\tau q - i\sigma p} dq dp \quad (2.50)$$

Then Weyl suggested to regard (2.50), with the replacement $q \rightarrow \hat{q}$, $p \rightarrow \hat{p}$, as the quantum mechanical operator corresponding to the observable represented classically by the phase space function. S.C.T.Pool⁷ has given a well defined meaning to this expression (2.50) using basic results of measure theory, the theory of Hilbert spaces and harmonic analysis.

Now taking the hermitian conjugate of (2.50)

$$\begin{aligned} M^+ &= \sum_{k,l=-\nu}^{+\nu} a_{kl}^* \omega(2\nu+1) \frac{-\frac{1}{2}kl}{C} \frac{-l-k}{B} \\ &= \sum_{k,l=-\nu}^{+\nu} a_{kl}^* \omega(2\nu+1) \frac{-\frac{1}{2}kl}{\omega(2\nu+1)} \frac{kl}{B} \frac{-k-l}{C} \\ &= \sum_{k,l=-\nu}^{+\nu} a_{kl}^* \omega(2\nu+1) \frac{\frac{1}{2}kl}{B} \frac{-k-l}{C} \end{aligned} \quad (2.50)$$

If $M^* = M$ then this shows

$$a_{kl}^* = a_{-k, -l} \quad (2.70)$$

Let $A = (a_{kl})$ be a real matrix ($k, l = -v \dots +v$) then define the double Fourier transforms

$$\tilde{a}_{\sigma, \tau} = \sum_{k, l=-v}^{+v} a_{kl} \omega(2v+1)^{\sigma k + \tau l} \quad (2.71)$$

We have

$$\tilde{a}_{-\sigma, -\tau} = \sum_{k, l=-v}^{+v} a_{kl} \omega(2v+1)^{-\sigma k - \tau l} = \tilde{a}_{\sigma, \tau}^* \quad (2.72)$$

Hence the matrix

$$M = \sum_{\sigma, \tau=-v}^{+v} \tilde{a}_{\sigma, \tau} \omega(2v+1)^{\frac{1}{2}\sigma\tau} B^\sigma C^\tau \quad (2.73)$$

is hermitian. It is this property of this expansion that to every real matrix A there is associated a unique Hermitian matrix

$$M = \sum_{\sigma, \tau=-v}^{+v} \sum_{k, l=-v}^{+v} a_{kl} \omega(2v+1)^{-\sigma k - \tau l + \frac{1}{2}\sigma\tau} B^\sigma C^\tau$$

$$(2.74)$$

which is the basis of Weyl correspondence. The elements of the real matrix A associated with a Hermitian matrix M , i.e., the inverse

transform $\Pi \Rightarrow A$ is given by the formula

$$\begin{aligned}
 a(p, q') &= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} dq' e^{-i\tau q'} \langle q' - \frac{\sigma}{2} | M | q' + \frac{\sigma}{2} \rangle \right\} d\tau d\sigma e^{i\tau q + i\sigma p} \\
 &= \int d\sigma \int dq' \int d\tau \langle q' - \frac{\sigma}{2} | M | q' + \frac{\sigma}{2} \rangle e^{i\sigma p} e^{i\tau(q - q')} \\
 &= \int d\sigma \int dq' \langle q' - \frac{\sigma}{2} | M | q' + \frac{\sigma}{2} \rangle e^{i\sigma p} \delta(q - q') \\
 &= \int d\sigma e^{i\sigma p} \langle q - \frac{\sigma}{2} | M | q + \frac{\sigma}{2} \rangle
 \end{aligned} \tag{2.75}$$

This expresses the inverse of Weyl transform - $a(p, q')$ is the classical observable corresponding to the quantum mechanical operator Π and if Π is Hermitian A is real. This formula can also be understood as the limiting case of inverse formula of (2.74). We will not treat this point any more in detail. The Wigner quasi-probability distribution function $s_w(q, p)$ is obtained by taking the Weyl transform of $M = |\psi\rangle\langle\psi|$, the density matrix by substituting in (2.75) $M = |\psi\rangle\langle\psi|$ we get the familiar form of the Wigner function

$$s_w(q, p) = \int d\sigma e^{i\sigma p} \psi^*(q - \frac{\sigma}{2}) \psi(q + \frac{\sigma}{2}) \tag{2.76}$$

(For very detailed consideration of these refer⁸).

Schringer⁹ developed further the approach of Weyl to consider the algebraic structure of quantum kinematics as a limiting case of

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the algebraic structure manifested in finite dimension by the n matrices(operators) C and B with a-commutation relation $CB = \omega BC$. He shows that the set of m^2 matrices $\{ \chi_{kl} = m^{-\frac{1}{2}} e^{i[\pi k l / m]} B^k C^l \}_{0 \leq k, l \leq m-1}$ form a complete orthonormal operator basis and therefore together supply the foundation for a full description of a physical system possessing m states. Interpreting in terms of measurement algebra he says that the properties of C and B exhibit the maximum degree of incompatibility and thus are C and B form a complementary pair of operators. He feels that though the algebraic properties of C and B have been known long since the work of Hermann Weyl, there has been a lacking of appreciation of these operators as generators of a completed operator basis for any n and of their optimum incompatibility as summarized in the attribute of complementarity. He also has stressed that an a priori classification of all possible types of physical degrees of freedom emerges from these considerations. While leaving further interesting details to the reference of Schwinger⁹ let us pass on to a generalization of matrix decomposition theorem due to Alladi Ramakrishnan and myself, which is an explicit version of Schwinger's suggestion of commutative factorization of the unitary operator basis, provided by G.C.A., for classification of several degrees of freedom of a physical system with finite number of states.

II. Extended versions of matrix decomposition theorem.

THEOREM Any $n \times n$ matrix can be expanded as

$$M = \sum_{\substack{0 \leq k_i, l_i \leq m_i - 1 \\ i=1 \dots r}} a_{k_1 l_1 \dots k_r l_r} B(m_1)^{k_1} C(m_1)^{l_1} \otimes \dots \otimes B(m_r)^{k_r} C(m_r)^{l_r} \quad (2.77)$$

where $B(m_i), C(m_i)$ are B and C matrices of dimension m_i .

obeying $C(m_i)B(m_i) = \omega(m_i) B(m_i)C(m_i)$, $C(m_i)^{m_i} = B(m_i)^{m_i} = I$ $\forall i=1$
 and $m = \prod_{i=1}^r m_i$. If A is the matrix of the coefficients
 $(a_{k_1 l_1 \dots k_r l_r})$ in which the element $a_{k_1 l_1 \dots k_r l_r}$ occurs in
 the position of the element $d_{k_1 l_1 \dots k_r l_r}$ in a matrix

$D_1 \otimes \dots \otimes D_r$, D_i being any matrix of dimension m_i with
 elements $(d_{k_i l_i} | 0 \leq k_i, l_i \leq m_i - 1)$, then it is given by the prescrip-
 tion

$$A = \mathcal{S}^{-1} R \quad (2.72)$$

where

$$\mathcal{S}^{-1} = S_1^{-1} \otimes \dots \otimes S_r^{-1}$$

$$(S_i^{-1})_{kl} = \frac{1}{m_i} \omega(m_i)^{-kl} \quad 0 \leq k, l \leq m_i - 1 \\ \forall i = 1, \dots, r.$$

and R is the rearranged matrix of N , the rearrangement being done
 in r stages, first rearranging N as an $m_1 \times m_1$ partitioned matrix,
 then rearranging each of the constituent matrices as a partitioned
 matrix of order $m_2 \times m_2$ and so on till finally the parti-
 tioning stops at the r th stage. Proof of this proposition follows
 easily repeating the arguments for the case of $r=1$ dealt with in
 the last chapter. First the matrix N can be decomposed into a
 linear sum of matrices as

$$N = \sum_{\substack{0 \leq l_i \leq m_i - 1 \\ \forall i = 1, \dots, r}} K_{l_1 l_2 \dots l_r} C(m_1)^{l_1} \otimes \dots \otimes C(m_r)^{l_r} \quad (2.73)$$

where K_{l_i} are unique diagonal matrices. Then each of these

diagonal matrices can be written as a linear sum

$$K_{l_1 l_2 \dots l_r} = \sum_{\substack{0 \leq k_i \leq m_i - 1 \\ \gamma_i = 1 \dots r}} a_{k_1 l_1 \dots k_r l_r} B(m_1)^{k_1} \otimes \dots \otimes B(m_r)^{k_r} \quad (2.80)$$

This follows the theorem and the relation (2.79) prescribes the rearrangement operation $M \rightarrow R$. R contains the elements of $(K_{l_1 l_2 \dots l_r})$ as columns with lexicographic ordering from left to right.

The decomposition of the type (2.77) stops at the unique decomposition $m = \prod_{i=1}^r p_i^{d_i}$ where p_i 's are distinct primes.

Schrodinger calls the unique quantity $\sum_{i=1}^r d_i = \Omega(m)$ as the number of freedom for a system possessing m states. Following the same procedure as we did for the case of $r=1$, we can go to the limit of each $m_i \rightarrow \infty$ in (2.77). In this also as before one has to make the necessary in-consequential changes of labelling as

$-v_i \leq k_i, l_i \leq +v_i$ with $m_i = 2v_i + 1; \forall i=1 \dots r$ and multiplying each $B(2v_i + 1)^{k_i} C(2v_i + 1)^{l_i}$ by $\omega(2v_i + 1)^{\frac{1}{2}k_i l_i}$. Then following the

same steps and associating the correspondence

$$(m_i)^{k_i} \xrightarrow{e^{i\pi_i \hat{p}_i}}, B(m_i)^{k_i} \xrightarrow{e^{i\tau_i \hat{q}_i}} \quad \forall i=1 \dots r. \quad (2.81)$$

the continuous analogue of (2.77) becomes for any operator M involving r variables

$$M(p_i, q_i) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{\tau} \prod_{i=1}^r d\sigma_i d\hat{p}_i e^{i(\sum_{i=1}^r \tau_i \hat{q}_i + \sum \sigma_i \hat{p}_i)} \quad (2.22)$$

with

$$a(\tau_i, \sigma_i) = \frac{1}{(2\pi)^r} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{\tau} \prod_{j=1}^r dq_j e^{-i\left(\sum_{j=1}^r \tau_j q_j\right)} \langle q_j - \frac{\sigma_j}{2} - \alpha_r - \frac{\tau_j}{2} | M | q_j + \frac{\sigma_j}{2} \rangle \quad (2.23)$$

where $\langle \dots | M | \dots \rangle'$ are the matrix elements of M in position representation. This becomes analogously the basis for the Weyl rule for several degrees of freedom. Since the arguments and steps are straight forward we omit them.

(iii) Canonical transformations and C and B matrices

Let us consider the so-called canonical transformations of a pair of conjugate variables obeying the commutation relation

$[q, p] = i1$. These transformations $q \rightarrow q'(q, p)$, $p \rightarrow p'(q, p)$ are said to be canonical if they leave invariant the commutator i.e.

$[q, p] = [q', p'] = i1$. As seen above any function of q, p can be expressed in a basis generated by $\{e^{i(\sigma q + \tau p)} | -\infty \leq \sigma, \tau \leq \infty\}$. The Heisenberg commutation relation $[q, p] = i1$ take the Weyl form as $e^{i\mu \hat{q}} e^{i\nu \hat{p}} = e^{-i\mu\nu} e^{i\nu \hat{p}} e^{i\mu \hat{q}}$. The transformation

$q \rightarrow q'(q, p)$, $p \rightarrow p'(q, p)$ would also preserve the Weyl commutation relation

for them $e^{i\mu\hat{q}'} e^{i\nu\hat{p}'} = e^{-i\mu\nu} e^{i\nu\hat{q}'} e^{i\mu\hat{p}'}$. The unique (upto equivalence) of the representation of unitary operators obeying Heisenberg commutation relations implies the existence of a unitary transformation S such that $S e^{i\mu\hat{q}'} S^{-1} = e^{i\mu\hat{q}}$, $S e^{i\nu\hat{p}'} S^{-1} = e^{i\nu\hat{p}}$; $\forall \mu, \nu$.

This can be seen in the finite case also. C and B matrices, as shown before, have only one representation upto equivalence if required to obey $CB = \omega BC$, $C^m = B^m = I$. Thus any other matrix representations

which are in turn expressible as $\left(\sum_{R,L=0}^{m-1} a_{RL} B^R C^L \right)$ should be equivalent to C and B and thus all the canonical transformations of C and B , $B \rightarrow F(C, B)$, $C \rightarrow G(C, B)$ obeying $FG = \omega GF$ should be connected to C and B by unitary equivalence when $m \rightarrow \infty$

This fact implies the existence of unitary equivalence between any (hermitian) representations of the conjugate pair (q, p) obeying $[q, p] = i1$. These are very well known facts since the birth of quantum mechanics. We shall use the fact of correspondence

$B^\mu \rightarrow e^{i\mu\hat{q}}$, $C^\nu \rightarrow e^{i\nu\hat{p}}$ to derive the explicit form of the similarity transformations in case of affine canonical transformations

in which $q = k\hat{p} + l\hat{q}$, $p = m\hat{p} + n\hat{q}$ with the condition $| \begin{matrix} n & m \\ l & k \end{matrix} | = 1$ so that $[f, g] = i1$. Hence we should find the

similarity transformation S , such that

$$S e^{i\mu\hat{q}} S^{-1} = e^{i\mu(n\hat{q} + m\hat{p})}$$

$$S e^{i\nu\hat{p}} S^{-1} = e^{i\nu(k\hat{q} + l\hat{p})}$$

(2.84)

Splitting the expressions on the right hand side

$$\begin{aligned} e^{i\mu(n\hat{q} + m\hat{p})} &= e^{(i\mu^2 nm/2)} e^{i\mu n \hat{q}} e^{i\mu m \hat{p}} \\ e^{i\nu(l\hat{q} + k\hat{p})} &= e^{(i\nu^2 lk/2)} e^{i\nu l \hat{q}} e^{i\nu k \hat{p}} \end{aligned} \quad (2.85)$$

Now we can try this problem as the limiting case of finding the similarity transformation in

$$\begin{aligned} SB^\mu S^{-1} &= \omega(N)^{\frac{1}{2}\mu^2 nm} B^{\mu n} C^{\mu m} \\ SC^\nu S^{-1} &= \omega(N)^{\frac{1}{2}\nu^2 lk} B^{\nu l} C^{\nu k} \end{aligned} \quad (2.86)$$

Assume N to be even then the factors $\omega(N)^{\frac{1}{2}\mu^2 nm}, \omega(N)^{\frac{1}{2}\nu^2 lk}$ are necessitated to have the condition $B'^N = C'^N = I$ satisfied. So we have to determine the S such that

$$\begin{aligned} SB^\mu &= \omega(N)^{\frac{1}{2}\mu^2 nm} B^{\mu n} C^{\mu m} S \\ SC^\nu &= \omega(N)^{\frac{1}{2}\nu^2 lk} B^{\nu l} C^{\nu k} S \end{aligned} \quad (2.87)$$

$\forall \mu, \nu = 0, 1, \dots, N-1$ * Writing these equations in terms of matrix elements

$$\sum_{\lambda} S_{\alpha\lambda}(C')_{\lambda\beta} = \omega(N)^{\frac{1}{2}\nu^2 lk} \sum_{\lambda} (B^{\nu l} C^{\nu k})_{\alpha\lambda} S_{\lambda\beta} \quad (2.88)$$

$$\sum_{\lambda} S_{\alpha\lambda}(B^\mu)_{\lambda\beta} = \omega(N)^{\frac{1}{2}\mu^2 nm} \sum_{\lambda} (B^{\mu n} C^{\mu m})_{\alpha\lambda} S_{\lambda\beta} \quad (2.89)$$

Substituting

$$\begin{aligned}(B^{\mu})_{\lambda\beta} &= \omega(N)^{\mu\beta} \delta_{\lambda\beta} \\ (C^{\nu})_{\lambda\beta} &= \delta_{\lambda, \beta-\nu}\end{aligned}\quad (2.90)$$

and summing over λ on both sides to arrive at coupled equations for $S_{\alpha\beta}$ as

$$\begin{aligned}S_{\alpha, \beta-\nu} &= \omega(N)^{\nu\lambda(\alpha + \nu k/m)} S_{\alpha + \nu k, \beta} \\ S_{\alpha, \beta} \omega(N)^{\mu\beta} &= \omega(N)^{\mu n(\alpha + \mu m/2)} S_{\alpha + \mu m, \beta}\end{aligned}\quad (2.91)$$

Solving for these equations we obtain

$$S_{\alpha, \beta} = \omega(N)^{\frac{i}{2}(\frac{k}{m}\alpha^2 + \frac{n}{m}\beta^2 - 2\alpha\beta)} \quad (2.92)$$

Considering the limiting case of $N \rightarrow \infty$ by comparing

$C^k B^l = \omega(N)^k B^l C^k$ and $e^{ik\hat{p}} e^{il\hat{q}} = e^{ikl} e^{il\hat{q}} e^{ik\hat{p}}$ we have to replace $\omega(N) \rightarrow e^{i\omega t}$. The n, d, β are to be interpreted as continuous and k, n, m are also any real numbers. Then

$$S(\alpha, \beta) = e^{\frac{i}{2}(\frac{k}{m}\alpha^2 + \frac{n}{m}\beta^2 - 2\alpha\beta)} \quad (2.93)$$

Replacing α and β by q' and q to be more suggestive, we get

$$S(q', q) = e^{\frac{i}{2}(\frac{k}{m}q'^2 + \frac{n}{m}q^2 - 2q'q)} \quad (2.94)$$

which are the well-known unitary integral transformations which carry the transformation of operators $\hat{q} \rightarrow \hat{q}' = \gamma_B \hat{q} + m \left(-i \frac{d}{dq} \right)$ and correspondingly $\left(i \frac{d}{dq} \right) \rightarrow \ell \hat{q} + k \left(-i \frac{d}{dq} \right)$ such that $\begin{vmatrix} n & m \\ \ell & k \end{vmatrix} = 1$. The Fourier transform corresponds the case $m = -1$, $\ell = 1$, $k, n = 0$. Then

$$S(q', q) = \ell^{i q' q} \quad (2.95)$$

Thus we have achieved our aim of showing that finite dimensional C and B matrices can be used as a powerful tool in dealing with canonical transformations. Also there has been an attempt by Santanam and Tekumalla¹¹ to construct a quantum mechanics in finite dimension using C and B matrices.

I wish to express my gratitude to Professor H. H. Stone for useful discussions on Weyl's rule and bringing to my notice relevant references.

Summary of important points.

In this chapter we have shown that the set of elements L_1, L_2 obeying $L_1 L_2 = \omega(m)^l L_2 L_1, L_1^m = L_2^m = I$ with $(l, m) = d$ has $d^2 = (l, m)^2$ inequivalent irreducible representations of same dimension (m/d) and all these representations are specified by $\{\omega(m)^{\delta_1} C, \omega(m)^{\delta_2} B \mid 0 \leq \delta_1, \delta_2 \leq d-1\}$ where L_1 and L_2 are any one set of $\left(\frac{m}{d}\right) \times \left(\frac{m}{d}\right)$ matrices obeying $CB = \omega(m)^l BC$, $m' = \frac{m}{d}$, $l' = \frac{l}{d}$.

Extension of Alladi Ramakrishnan's theorem determining explicitly representation

$$M = \sum_{k,l=0}^{m-1} a_{kl} B^k C^l$$

has been considered and the procedure of determining explicitly the coefficients in the expansion

$$M = \sum_{\substack{0 \leq k_i, l_i \leq m_i - 1 \\ i=1, \dots, r}} a_{k_1, l_1, \dots, k_r, l_r} B(m_1)^{k_1} C(m_1)^{l_1} \otimes \dots \otimes B(m_r)^{k_r} C(m_r)^{l_r}$$

$; \quad m = \dim M = \prod_{i=1}^r m_i$

is given.

Weyl's rule, Wigner distribution function and canonical transformations of conjugate pairs of operators are also discussed.

CHAPTER 3GENERALIZED CLIFFORD GROUPS - I

In this chapter we shall formulate and study a group structure associated with the basis of the generalized Clifford Algebra $C_n^{(m)}$ generated by the relations

$$L_i L_j = \omega(m) L_j L_i ; L_i^m = 1 ; \quad i, j = 1, \dots, n. \quad (3.1)$$

This chapter is based mainly on the paper of Ranganathan and myself.¹

1) Dirac groups associated with ordinary Clifford Algebra

The basis of Clifford algebra $C_n^{(2)}$ generated by the relations (3.1) corresponding to the case $m=2$, namely

$$\begin{aligned} L_i L_j &= -L_j L_i ; \quad i, j = 1, 2, \dots, n. \\ L_i^2 &= 1. \quad i = 1, 2, \dots, n. \end{aligned} \quad (3.2)$$

consists of 2^n elements given by

$$\left\{ \prod_{i=1}^n L_i^{k_i} \mid k_i = 0, 1 \right\} \quad (3.3)$$

It is easy to see that product of any two elements of this set is ± 1 times another element of the set. So this set of elements do not form a group. But it is immediately seen that if we add to this set the set of elements (3.3) multiplied by -1 the set of 2^{n+1} elements given by

$$\mathbb{D}_n^2 \equiv \left\{ \prod_{i=0}^n L_i^{k_i} \mid k_i = 0, 1; \quad L_0 = -1 \right\} \quad (3.4)$$

form a group. This group has been called a Dirac group. This has been studied in the context of L-Matrix theory by Alladi Ramakrishnan and RaghavaCharyulu². This group was studied earlier by Jordan and Wigner³, Pauli⁴, Losonc⁵, Case⁶, and others. DeVries and van Zanten⁷ studied the Dirac matrix group corresponding to $n = 4$ in relation to Fierz transformations in theory of weak interactions in elementary particle physics. (cf. also Nahm⁸) The generalization of this group structure for $n > 2$ associated with a generalized Clifford algebra, $C_n^{(n)}$, has been noted very briefly with out detailed study of their properties and representations by Deepak et. al.⁹ In this paper they have also noted the properties of what we call product transforms associated with commutation matrices but its essential power as a tool of representation theory of generalized Clifford algebras has not been realized. Here in this chapter we study the properties and representations of these group structures in all details.

The number of conjugate classes in the group \mathcal{B}_n^2 is given by

$$2^{2V} + 1 \quad (3.5)$$

when $n = 2V$, and

$$2^{2V+1} + 2 \quad (3.6)$$

when $n = 2V+1$.

Proof. When $n = 2V$. The elements $+1$ and -1 are obviously self-conjugate elements. Among the other elements consider one element say

$$g(k_1, \dots, k_{2V}) = \prod_{i=1}^{2V} L_i^{k_i} \quad (3.7)$$

If it is a product of odd number of elements $\{L_i | i=1 \dots n\}$ from the set of all elements say L_j not contained in the product anti-commute with it. Hence g and $L_j g L_j^{-1} = -g$ are in the same class. If g contains an even number of factors L_i then it anticommutes with all its factor elements L_i . Thus in this case also g and $-g$ are in the same class. Hence the set of elements

$$\{g, -g | g \neq \pm 1\}$$

contains

$$\frac{1}{2}(2^{2v+1} - 2) = 2^{2v} - 1$$

classes. Including the two self-conjugate elements ± 1 , the total number of classes in D_{2v} is given by

$$2 + (2^{2v} - 1) = 2^{2v} + 1$$

(3.8)

When $n = 2v+1$, the self conjugate elements are given by $\{+1, -1, +\eta, -\eta\}$ where $\eta = 2v+1$

$$\eta = \prod_{i=1}^{2v+1} L_i$$

(3.9)

Since η commutes with all L_i and hence all products of L_i 's. The other elements $(2^{2v+2} - 4)$ in number are grouped into $\frac{1}{2}(2^{2v+2} - 4) = 2^{2v+1} - 2$ classes since by the same argument as above each element g of this set is equivalent to $-g$ so total number of classes in this case is

$$(2^{2v+1} - 2) + 4 = 2^{2v+1} + 2 \quad (3.10)$$

By Burnside's theorem the total number of classes is equal to the total number of irreducible inequivalent representations. Hence total number of representations of D_n^2 are given by

$$\left\{ \begin{array}{l} 2^{2v} + 1 \text{ when } n = 2v \\ 2^{2v+1} + 2 \text{ when } n = 2v + 1 \end{array} \right\} = \frac{n^2}{2} \quad (3.11)$$

Now let us construct all these representations. When d_i 's are dimension of the representation then by Burnside's theorem

$$\sum_{i=1}^{N_n^2} d_i^2 = \text{total number of elements in } D_n^2 = |D_n^2| \quad (3.12)$$

Hence it follows from (3.11) and (3.12) that D_{2v}^2 has 2^{2v} one dimensional representations and one 2^v dimensional representation and D_{2v+1}^2 has 2^{2v+1} one dimensional representation and 2^{2v} dimensional representations. The 2^n one dimensional representation for both $n = 2v$ and $2v+1$, are given by the representations of the Abelian group

$$G \cong \mathbb{Z}_2 \otimes \cdots \otimes \mathbb{Z}_2 \quad (\text{n copies}) \quad (3.13)$$

which is isomorphic to D_n^2

$$D_n^2 / \mathbb{Z}_2 \cong \mathbb{Z}_2 \otimes \cdots \otimes \mathbb{Z}_2 \quad (\text{n copies}) \quad (3.14)$$

Also it is seen easily that \mathbb{Z}_2 is the centre of D_n^2 and thus D_n^2 is the central extension of $\mathbb{Z}_2 \otimes \dots \otimes \mathbb{Z}_2$ with \mathbb{Z}_2 as kernel of extension. Thus all the 2^n dimensional representations of D_n^2 are constructed by taking each $L_i = \pm 1$ independent of each other.

Higher dimensional representations are constructed as follows. Let $n = 2^\nu$. Then relation (1.4) or (1.44) a, b with $n = 2$ provide 2^ν dimensional representations of the 2^ν anticommuting generators. From (3.11) and (3.12) these cannot exist any other representation. Let $n = 2^\nu + 1$. Then relation (1.4) or (1.44) a, b, c with $n = 2$ provide a 2^ν dimensional representation of the $2^\nu + 1$ anticommuting generators. Let us now put

$$L_i' = -L_i \quad i=1, 2, \dots, 2^\nu + 1 \quad (3.15)$$

obviously $\{L_i' | i=1, \dots, 2^\nu + 1\}$ also provide an irreducible representation of these $2^\nu + 1$ elements. Let us now prove that this representation is inequivalent to $\{L_i | i=1, 2, \dots, 2^\nu + 1\}$. If there two were equivalent then there exists a nonsingular matrix S such that

$$L_i' = S L_i S^{-1} = -L_i \quad \forall i=1, 2, \dots, 2^\nu + 1. \quad (3.16)$$

$$\text{or } S L_i = -L_i S \quad \forall i=1, 2, \dots, 2^\nu + 1.$$

But as has been proved earlier in Chapter 1 when $2^\nu + 1$ L_i 's are represented by 2^ν dimensional matrices there is a relation

$$L_1 \dots L_{2^\nu + 1} \sim I \quad (3.17)$$

and hence (3.16) would imply

$$S L_1 \dots L_{2^\nu + 1} = -L_1 \dots L_{2^\nu + 1} S \quad \text{or} \quad S I = -I S. \quad (3.18)$$

which is absurd. Hence these two representations

$$\{L_i = L_{(2v+1, i)}^{(2)} \mid i=1, \dots, 2v+1\} \text{ and } \{L_i = -L_{(2v+1, i)}^{(2)} \mid i=1, \dots, 2v+1\}$$

are inequivalent. Thus the two irreducible representations have been found and according to (3.16, 3.17) there is no more representation. The above facts are known as Pauli's theorem⁽¹⁰⁾ that $C_{2v}^{(2)}$ has only one irreducible representation of dimension > 1 and $C_{2v+1}^{(2)}$ has two such representations of same dimension > 1 .

(ii) Generalization of Dirac group or Generalized Clifford group associated with Generalized Clifford algebras.

Considering the algebra C_n^m generated by (3.1) the basis of this algebra is given by m^n elements

$$\left\{ \prod_{i=1}^n L_i^{k_i} \mid 0 \leq k_i \leq m-1 \right\} \quad (3.19)$$

It is easy to see that product of any two elements of this set is in general only a scalar multiple of another element of the set as given by (1.36), so that this set of elements (3.19) do not form a group. But as was done in the case of C_n^2 it is easy to form a group. The following set of m^{n+1} elements

$$G_{1m}^m = \left\{ \prod_{i=0}^n L_i^{k_i} \mid 0 \leq k_i \leq m-1; L_0 = \omega(m) \right\} \quad (3.20)$$

form a group which we shall call a generalized clifford group (3.6.6). It can be called also a Generalized Dirac Group. The choice of '3.6.6' will be made clear in the last section of this Chapter.

In this Chapter we shall study in detail $G_{\text{c.c.g.}}$, G_n^m only for the case of $m = \text{prime number}$. Let us denote the group G_n^m corresponding to m being a prime number by $G_n^{p_m}$.

iii) Properties of G.C.G. $G_n^{p_m}$

Let us denote

$$g(k_0, k_1, \dots, k_n) = \prod_{i=0}^n L_i^{k_i} \quad (3.21)$$

Then product is defined by

$$\begin{aligned} g(k_0, k_1, \dots, k_n)g(j_0, j_1, \dots, j_n) \\ = g((k+j)_0, (k+j)_1, \dots, (k+j)_n) \end{aligned} \quad (3.22)$$

where

$$(k+j)_i \bmod m = k_i + j_i, \forall i = 1, \dots, n. \quad (3.23)$$

and

$$(k+j+l)_0 \bmod m = k_0 + j_0 + l_0 \quad (3.24)$$

$$l_0 \bmod m = (m-1) \left[(j_1 j_2 \dots j_n) \begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & & & \ddots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \right]$$

$$= (m-1) \sum_{x < y} j_x k_y. \quad (3.25)$$

The inverse of an element is given by

$$g(k_0, k_1, \dots, k_n)^{-1} = g(k'_0, k'_1, \dots, k'_n) \quad (3.26)$$

where

$$k'_i = m - k_i \quad \forall i = 1, \dots, n. \quad (3.27)$$

$$k'_0 \bmod m = m - k_0 + K \quad (3.22)$$

$$\begin{aligned} K \bmod m &= (m-1) \left[(k'_1, k'_2, \dots, k'_n) \begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} k'_1 \\ k'_2 \\ \vdots \\ k'_n \end{pmatrix} \right] \\ &= (m-1) \sum_{x < y} k'_x k'_y \end{aligned} \quad (3.23)$$

Now let us determine the numbers of conjugate classes in it. It follows from (3.22 - 3.24) that

$$\begin{aligned} g(j_0, j_1, \dots, j_m) g(k_0, k_1, \dots, k_n) g(j_0, j_1, \dots, j_m)^{-1} \\ = g(k'_0, k'_1, \dots, k'_n) \end{aligned} \quad (3.24)$$

where

$$\begin{aligned} k'_0 \bmod m &= k_0 + l_0 \\ l_0 \bmod m &= (j_1, j_2, \dots, j_m) \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & 0 & 1 & \dots & 1 \\ \vdots & & & & \vdots \\ -1 & -1 & -1 & \dots & 0 \\ -1 & -1 & -1 & \dots & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \sum_{x < y} (j_x k_y - j_y k_x) \end{aligned} \quad (3.25)$$

First let $n = 2V$. For a fixed $(k_0, k_1, \dots, k_{2V})$ let k be the greatest common division of all k_i , $i = 1, \dots, 2V$. Then by the theory of linear diophantine equations, the equation

$$k = \sum_{x=1}^{2V} k_x \left\{ \sum_{y=1}^{x-1} j'_y - \sum_{y=x+1}^{2V} j'_y \right\} = \sum_{x=1}^{2V} k_x j''_x \quad (3.26)$$

has a solution for $(j''_x \mid x=1 \dots 2V)$ admitting them to have

negative values also. Since

$$\begin{pmatrix} j_1'' \\ j_2'' \\ \vdots \\ j_{2v}'' \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 \\ 1 & 0 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} j_1' \\ j_2' \\ \vdots \\ j_{2v}' \end{pmatrix}$$

(3.33)

$(j_i'; i=1, 2, \dots, 2v)$ have solution given by

$$\begin{pmatrix} j_1' \\ j_2' \\ \vdots \\ j_{2v}' \end{pmatrix} = \begin{pmatrix} 0 & -1 & +1 & \cdots & -1 \\ +1 & 0 & -1 & \cdots & +1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & +1 & -1 & \cdots & -1 \\ +1 & -1 & +1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} j_1'' \\ j_2'' \\ \vdots \\ j_{2v}'' \end{pmatrix}$$

(3.34)

(j_1', \dots, j_{2v}') can now be reduced to modulo m . The resulting set of values $\{(j_1, \dots, j_{2v}) \mid j_i \bmod m = j_i'; \forall i\}$ provide a solution to the equation

$$k \cdot \text{mod. } m = \sum_{x=1}^{2v} k_x \left\{ \sum_{y=1}^{x-1} j_y - \sum_{y=x+1}^{2v} j_y \right\} \quad (3.35)$$

or

$$\begin{aligned} g(j_0, j_1, \dots, j_{2v}) g(k_0, k_1, \dots, k_{2v}) g(j_0, j_1, \dots, j_{2v})^{-1} \\ = g(k'_0, k_1, \dots, k_{2v}) \end{aligned} \quad (3.36)$$

with

$$k'_0 \bmod m = k_0 + k \quad (3.37)$$

$$k \cdot \text{mod. } m = \sum_{x=1}^{2v} R_x \left\{ \sum_{y=1}^{x-1} j_y - \sum_{y=x+1}^{2v} j_y \right\} \quad (3.38)$$

Hence

$$\begin{aligned} & |g(j_0, j_1, \dots, j_{2v})|^r g(k_0, k_1, \dots, k_{2v}) (g(k_0, j_1, \dots, j_{2v}))^{-1} \\ & = g(k(r)', k_1, \dots, k_{2v}) \end{aligned} \quad (3.39)$$

with

$$k(r)' \text{ mod. } m = k_0 + k(r) \quad (3.40)$$

$$k(r) \cdot \text{mod. } m = rk \quad (3.41)$$

when m is prime $k(r) \text{ mod. } m$ takes all values $0, 1, 2, \dots, m-1$ as r takes values $0, 1, 2, \dots, m-1$. Thus this shows that all the elements of the set

$$\{ g(k_0, k_1, \dots, k_{2v}) \mid k_0 = 0, 1, \dots, m-1 \} \quad (3.42)$$

for fixed (k_1, \dots, k_{2v}) are members of a class when at least one of the R_i 's is nonzero. In case $R_i = 0, \forall i = 1 \dots 2v$ then it is obvious that the element $g(k_0, 0, \dots, 0)$ is a self conjugate element or it is a class in itself. Thus there are m self conjugate elements corresponding to $k_0 = 0, 1, \dots, m-1$ in $g(k_0, 0, \dots, 0)$. The set of remaining $m^{2v+1}-m$ elements is divided into

$$(m^{2v+1} - m)/m = m^{2v} - 1. \quad (3.43)$$

classes each containing n elements of the form (3.42). The total number of classes in G_{2V}^m is

$$N_{2V}^{(m)} = m^{2V} - 1 + m \quad (3.44)$$

Now let us consider the case of G_{2V+1}^m . For an element $g(k_0, k_1, \dots, k_n)$ to be self conjugate the condition is from (3.31) (3.45):

$$0 \cdot \text{mod } m = \sum_{x=1}^n k_x \left\{ \sum_{y=1}^{x-1} j_y - \sum_{y=x+1}^n j_y \right\} \quad (3.45)$$

for all values of $\{(j_1, \dots, j_n) \mid 0 \leq j_i \leq m-1; \forall i\}$. In the case of $n = 2V$, only the set $\{k_i = 0 \mid \forall i\}$ satisfies this condition.

But when $n = 2V+1$, the following set of n^2 elements

$$\left\{ g(\underbrace{k_0, m-r, \dots, m-r}_V, \underbrace{r, \dots, r}_V, m-r) \mid 0 \leq k_0 \leq m-1, 0 \leq r \leq m-1 \right\} \quad (3.46)$$

satisfy this condition. But in this the elements with $r=0$ are $\sim I$ as is directly seen. Denoting

$$\eta = g(0, \underbrace{m-1, \dots, m-1}_V, \underbrace{1, \dots, 1}_V, m-1) \quad (3.47)$$

since it commutes with all $g \in G_{2V+1}^m$, it is $\sim I$ by Schur's lemma. It is seen that we can write

$$\begin{aligned} L_{2V+1} &= \left[g(0, m-1, \dots, m-1, 1, \dots, 1, 0)^{-1} \eta \right]^{m-1} \\ &= \eta^{m-1} \left[g(0, m-1, \dots, m-1, 1, \dots, 1, 0)^{-1} \right]^{m-1} \end{aligned} \quad (3.48)$$

Hence in the product $\left\{ \prod_{i=0}^{2V+1} L_i^{k_i} \right\}$, L_{2V+1} can be replaced as
as a product of the $\{L_i | i=1 \dots 2V\}$ as

$$g(k_0, k_1, \dots, k_{2V+1}) = g(k_0, k_1, \dots, k_{2V}, 0) \left\{ \eta^{m-1} [g(0, m-1 \dots m-1, 1, \dots, 1, 0)^{-1}]^{m-1} \right\} \\ = g(k'_0, k'_1, \dots, k'_{2V}, 0) \quad (3.49)$$

Thus in the set of group elements $\in G_{2V+1}^{,m}$

$$\{ g(k_0, k_1, \dots, k_{2V+1}) | 0 \leq k_i \leq m-1 ; i=1, \dots, 2V+1 \} \quad (3.50)$$

each element is a product powers of only $\omega(m)$ and the L_i 's and hence each element other than self-conjugate elements gave rise to a class of m elements which are multiples of it, by

$\{\omega(m)^l | 0 \leq l \leq m\}$ as per the arguments presented in the case of G_{2V}^m . Thus remembering that $G_{2V+1}^{,m}$ has m^2 self-conjugate elements (3.46) the total number of class in $G_{2V+1}^{,m}$ is given by

$$N_{2V+1}^{,m} = \left(\frac{m^{2V+2} - m^2}{m} \right) + m^{2V} = m^{2V+1} + m(m-1) \quad (3.51)$$

iv) Representations of S_n^m

By (3.44) G_{2V}^m has $m^{2V} + (m-1)$ conjugate classes and hence it must have so many irreducible inequivalent representation. It can be realised that a scheme of m^2 \times 1-dimensional representations and $(m-1)$, m^2 -dimensional representations fits the requirement that

$$|G| = \text{Order of the group} = \sum_{i=1}^N d_i^2 \quad (3.52)$$

where d_i is the dimension of the i th representation and N is the total number of representations. In our case we have

$$\sum_1^{m^{2v}} 1^2 + \sum_1^{m-1} (m^v)^2 = m^{2v} + (m-1)m^{2v} = m^{2v+1} = |G'_{2v}| \quad (3.53)$$

a) One dimensional representations

These representations come from the homomorphism of the generality relations (3.1) with

$$L_i L_j = L_j L_i ; i, j = 1 \dots 2v. \quad (3.54)$$

$$\sum_i L_i^m = 1 \quad \forall i = 1 \dots n$$

or in other words, these are due to the homomorphism of G'_{2v} to $\mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m$ ($2v$ copies). The kernel of homomorphism is the normal subgroup $\{1, \omega(m), \dots \omega(m)^{2v}\} \subseteq \mathbb{Z}_m$ or

$$G'_{2v}/\mathbb{Z}_m \cong \mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m \quad (\text{2v - copies}) \quad (3.55)$$

Thus to generate the one dimensional representations simply set

$$L_i = \omega(m)^{k_{ri}} \quad ; i = 1 \dots 2v$$

$$0 \leq k_{ri} \leq m-1. \quad (3.56)$$

where T labels the representation. Thus $T = 1, \dots, m^{2v}$; m^{2v} representations arise from the fact each of the L_i 's can take m values independent of others.

b) Higher dimensional representations

We have seen in (3.53) that there should be $(n-1)$, n^{ν} - dimensional representations. These are recognised to arise from the $(n-1)$ isomorphic G.C.A's whose generating relations are

$$\begin{aligned} L_i L_j &= \omega(n)^l L_j L_i ; \quad i, j = 1, \dots, 2\nu. \\ L_i^{m^{\nu}} &= 1 ; \quad \forall i = 1, \dots, 2\nu. \\ l &= 1, \dots, m-1. \end{aligned} \tag{3.57}$$

Since n is prime

$$(\ell, m) = 1. \quad \forall \ell = 1, \dots, m-1. \tag{3.58}$$

Let us denote the G.C.A's generated by (3.57) by $\{C_{2\nu}^{(m)}(\ell) \mid \ell = 1 \dots m-1\}$. All the $\{\omega(n)^{\ell} \mid \ell = 1 \dots m-1\}$ are primitive m th roots and hence all $C_{2\nu}^{(m)}(\ell)$ are isomorphic. Hence for the group $G_{2\nu}^{(m)}$ representations of all these algebras provide a representation. Let us consider one $C_{2\nu}^{(m)}(\ell)$. We shall use the product-transform method in Chapter 1 for construction of irreducible representations. The commutation matrix T associated with the system of relations (3.57) is

$$T_{\ell} = \begin{pmatrix} 0 & \ell & \ell & \dots & \ell & \ell \\ -\ell & 0 & \ell & \dots & \ell & \ell \\ \vdots & & & & & \\ -\ell & -\ell & -\ell & \dots & 0 & \ell \\ -\ell & -\ell & -\ell & \dots & -\ell & 0 \end{pmatrix} = \ell \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 1 & \dots & 1 & 1 \\ \vdots & & & & & \\ -1 & -1 & -1 & \dots & 0 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{pmatrix} \tag{3.59}$$

where ℓ is dimension of irreducible representation of Group 3, where $\omega(n)^{\ell}$ is a primitive m th root for all $\ell = 1, \dots, m-1$.

$$d_1 = m \cdot \nu \cdot \ell = m \cdot 2\nu \cdot \ell \tag{3.60}$$

So the skew-normal form of \tilde{g} is

$$T_{\tilde{g}}^* = \begin{pmatrix} 0 & l & & & 0 \\ -l & 0 & & & \\ & 0 & l & & \\ & & -l & 0 & \\ 0 & & & 0 & l \\ & & & & -l & 0 \end{pmatrix} = l \begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & 0 & 1 & & \\ & & -1 & 0 & \\ 0 & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix} \quad (3.59)$$

and hence the skew-normal form of the relations (3.57) is

$$\begin{aligned} L_{2i-1}^* L_{2i}^* &= \omega(m)^l L_{2i}^* L_{2i-1}^* ; \quad i=1, \dots, v. \\ L_i^* L_j^* &= L_j^* L_i^* \quad \text{otherwise.} \end{aligned} \quad (3.60)$$

Following the procedure in (1.64) it is easy to construct

$$\begin{aligned} L_{2i-1}^* &= \underbrace{| \otimes | \otimes \dots \otimes |}_{v-i} \underbrace{| \otimes C_i \otimes |}_{(i-i+1)} \underbrace{\otimes \dots \otimes |}_{i-1} \\ L_{2i}^* &= \underbrace{| \otimes | \otimes \dots \otimes |}_{v-i} \underbrace{| \otimes B_i \otimes |}_{(i-i+1)} \underbrace{\otimes \dots \otimes |}_{i-1} \end{aligned} \quad (3.61)$$

with

$$C_i B_i = \omega(m)^l B_i C_i ; \quad i=1, 2, \dots, v \quad (3.62)$$

From (3.62) it follows that

$$\begin{aligned} \det C_i B_i &= \omega(m)^l d_i \det B_i C_i = \det C_i B_i \cdot \omega(m)^{l d_i} \\ \text{or } \omega(m)^{l d_i} &= 1. \end{aligned} \quad (3.63)$$

where d_i = dimension of irreducible representation of C_i and B_i
Since $\omega(m)^l$ is a primitive n th root for all $l=1, \dots, m-1$.

$$d_i = m , \quad \forall i=1, \dots, 2v. \quad (3.64)$$

From (1.36) it is easy to construct the irreducible representation of C_i and B_i as

$$C_i = C^k, \quad B_i = B^T \quad (3.66)$$

with

$$l(\text{mod } m) = kr. \quad (3.66)$$

without loss of generality we can take $k=1, r=l$ and hence

$$\begin{aligned} L_{2i-1}^* &= \underbrace{1 \otimes \dots \otimes 1}_{r-i} \otimes C \otimes \underbrace{1 \otimes \dots \otimes 1}_{l-1} \\ L_{2i}^* &= \underbrace{1 \otimes \dots \otimes 1}_{r-i} \otimes B \otimes \underbrace{1 \otimes \dots \otimes 1}_{l-1} \end{aligned} \quad (3.67)$$

Now since T_l and T_l^* are identical with T and T^* given in (1.68) and (1.69) respectively except for a multiplicative factor l . The U and V matrices are also the same. Hence the solution for V in

$$T_V = VT_l^* \tilde{V} \quad (3.68)$$

is given by (1.70)

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1 \\ \vdots & & & & & & & \\ 1 & 0 & -1 & 1 & \dots & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 & \dots & -1 & 1 & -1 \end{bmatrix} \quad (3.69)$$

Hence by the product transform method outlined in Chapter 1.

are represented by

$$\begin{aligned} L_i &= \prod_{j=1}^{2N} L_j^{*} V_{ij} \\ &= \begin{cases} L_{2i-1}^{*-1} L_{2N}^{*-1} L_{2N-1}^{*-1} \dots L_{2(N-k)+1}^{*-1} & \text{if } i = 2k-1 \\ L_{2i-1}^{*-1} L_{2N}^{*-1} L_{2N-1}^{*-1} \dots L_{2(N-k)+2}^{*-1} & \text{if } i = 2k \end{cases} \\ &\quad k=1 \dots N. \end{aligned} \quad (3.70)$$

Explicitly writing

$$\begin{aligned}
 L_{2k+1} &= \underbrace{C^{m-1} B^l \otimes C^{m-1} B^l \otimes \cdots \otimes C^{m-1} B^l}_{k-1} \otimes C \otimes \underbrace{I \otimes \cdots \otimes I}_{n-k} \\
 L_{2k} &= \underbrace{C^{m-1} B^l \otimes C^{m-1} B^l \otimes \cdots \otimes C^{m-1} B^l}_{k-1} \otimes C^{m-1} B^l \otimes I \otimes \cdots \otimes I \\
 k &= 1, 2, \dots, n.
 \end{aligned}
 \tag{3.71}$$

These also satisfy the condition as can be seen easily

$$L_i^{(m)} = I \quad \forall i = 1, 2, \dots, 2n.
 \tag{3.72}$$

Thus the dimension of these representations is m^n . Corresponding to each value of $l = 1, \dots, m-1$, we get the representations of all the relations (3.57), each of them being of dimension m^n . These are obviously inequivalent since they have different algebraic relations and equivalence transformations cannot alter algebraic relations among the set of given matrices. Thus we have obtained all the $(n-1)$, m^n dimensional representations of $G_{2n}^{(m)}$ in terms of the generators obeying (3.57).

iii) Representations of $G_{2n+1}^{(m)}$

We can guess from (3.51) that $G_{2n+1}^{(m)}$ should have m^{2n+1} one dimensional representations and $n(n-1)$, m^n dimensional representations since this scheme fits the requirement (3.52) as

$$\sum_1^{2n+1} 1^2 + \sum_1^{m(m-1)} (m^n)^2 = m^{2n+1} + m(m-1)m^{2n} = m^{2n+2} = |G_{2n+1}^{(m)}|$$

a) One-dimensional representations

as in the case of G_{2v}^m those arising from the representations of $\mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m$ ($v+1$ copies) since G_{2v+1}^m is isomorphic to it. Since this

$$L_i = \omega(m)^{k_{ri}} \quad i=1, 2, \dots, 2v+1$$

$$0 \leq k_{ri} \leq m-1 \quad (3.70)$$

$$r=1, \dots, m^{2v+1}$$

where r labels the representations. Since each L_i can take a value independent of others there are m^{2v+1} representations.

b) Higher dimension representations

Again we will use the project transfer method. Now these higher dimensional representations arise from the (real) isomorphisms $G.C.A's$

$$C_{2v+1}^{(m)(l)}$$

$$L_i L_j = \omega(m)^l L_j L_i \quad ; \quad i, j = 1, 2, \dots, 2v+1. \quad (3.71)$$

$$L_i^{mv} = 1. \quad \forall i = 1, \dots, 2v+1.$$

$$L_i^{(m)} = 1. \quad l = 1, 2, \dots, m-1.$$

Consider one $C_{2v+1}^{(m)(l)}$. The companion matrix T_l associated with (3.71) is

$$T_l = l \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ \vdots & & & & \\ -1 & -1 & -1 & \cdots & 0 \end{bmatrix} \quad (3.72)$$

and its above normal form is

$$T_l^* = l \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 0 & 1 & \\ & -1 & 0 & & \\ 0 & & -1 & 0 & \\ & & & \ddots & 0 \end{bmatrix} \quad (3.73)$$

and the solution for V is

$$T_l = VT_l^* \tilde{V} \quad (3.78)$$

is given by

$$V = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 1 & \dots & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & \dots & -1 & 1 & -1 & 1 \end{bmatrix} \quad (3.79)$$

Corresponding to T_l^* the skew normal form of the relation (3.75) is

$$\begin{aligned} a) \quad L_{2i-1}^* L_i^* &= \omega(m)^l L_i^* L_{2i-1}^* \quad i=1, 2, \dots, v. \\ L_i^* L_j^* &= L_j^* L_i^* \text{ otherwise} \\ b) \quad L_i^* L_{2v+1}^* &= L_{2v+1}^* L_i^* \quad i=1, 2, \dots, 2v. \end{aligned} \quad (3.80)$$

The relations (3.80a) are the same as those occurring in the case of G_{2v}^m and hence by the arguments given above their representation is given by the same m dimensional matrices as in (3.71). All these L_i^* , L^* -matrices are seen to generate a set of m^{2v} linearly independent matrices by following the arguments of Chapter I. So L_{2v+1}^* has to be represented by a scalar which can be taken as I_3 without loss of generality. Hence corresponding to the product transform with V in (3.79) the representations of L_i 's are given by

$$L_i = \prod_{j=1}^{2v+1} L_j^* v_{ij} \quad (3.81)$$

$$= \begin{cases} L_{2v-1}^{*-1} L_{2v}^* L_{2v-3}^{*-1} \dots L_{2(v-k)+1}^* & \text{if } i = 2k-1 \\ L_{2v-1}^{*-1} L_{2v}^* L_{2v-3}^{*-1} \dots L_{2(v-k)+2}^* & \text{if } i = 2k \\ L_{2v+1}^* L_{2v-1}^{*-1} L_{2v}^{*-1} L_{2v-3}^{*-1} \dots L_2^* L_1^{*-1} & \text{for } i = 2v+1 \end{cases} \quad h = 1, 2, \dots, v$$

Since these provide a representation of (3.75) for a value of ℓ .

Borris has shown that in the case of $C_{2V+1}^{(m)}$ there are n inequivalent representations of same dimension m^V which are explicitly given by

$$\left\{ \omega(m)^{\ell r} L_i \mid i=1, \dots, 2V+1; r=0, 1, \dots, m-1 \right\} \quad (3.80)$$

where L_i 's are those constructed in (3.81) explicitly given.

Proof. Consider two representations are equivalent. Then

$$\begin{aligned} &\left\{ \omega(m)^{\ell r} L_i \mid i=1, \dots, 2V+1 \right\} \\ &\left\{ \omega(m)^{\ell(r+1)} L_i \mid i=1, \dots, 2V+1 \right\} \end{aligned} \quad (3.81)$$

These two will be equivalent only if there exists a nonsingular

m^V -dimensional matrix S such that

$$S \omega(m)^{\ell r} L_i = \omega(m)^{\ell(r+1)} L_i S \quad \forall i=1, \dots, 2V+1. \quad (3.82)$$

or there exists a matrix S such that

$$S L_i = \omega(m)^{\ell} L_i S \quad \forall i=1, \dots, 2V+1 \quad (3.83)$$

This implies that the set of $2V+2$ matrices $\{L_1, L_2, \dots, L_{2V+1}, S\}$

obey mutual commuting relations of the type (3.87) and we can put

$S = L_{2V+2}$. But as shown in section (3.11) these $2V+2$

elements must have only one irreducible representation of dimension

m^{V+1} and cannot have a m^V -dimensional representation as

the validity of (3.80) would imply. Hence there cannot exist a

matrix S satisfying (3.83). Thus the two representations (3.80)

are inequivalent. Since it is true for all $r=0, 1, \dots, m-1$ in (3.82)

the n representations in (3.80) are all inequivalent. Since ℓr

reduced modulo n takes all values $0, 1, \dots, n-1$ as τ takes values $0, 1, \dots, n-1$, the n inequivalent representations (3.86) can be written as

$$\{ \omega(n)^{\delta} L_i \mid i = 1, 2, \dots, 2n+1; \delta = 0, 1, \dots, n-1 \} \quad (3.86)$$

The above considerations hold for the case of all values of $\lambda = 1, 2, \dots, n-1$ and each algebra $C_{2n+1}^{(n)}(\lambda)$ gives rise to n inequivalent representations as given by (3.86). Obviously representations of different algebras $C_{2n+1}^{(n)}(\lambda)$ are inequivalent. Thus totally we get $n(n-1)$ inequivalent irreducible representations of dimension n^{λ} each, as is required. Incidentally these considerations prove that $C_{2n+1}^{(n)}(\lambda)$ has only n inequivalent irreducible representations of dimension n^{λ} for each λ when n is prime.

Let us denote by $\Gamma^{(\lambda)}$ the representation of $G_{2n}^{(n)}$ arising from $C_{2n}^{(n)}(\lambda)$ and by $\Gamma^{(\lambda, \beta)}$ the representation of $G_{2n+1}^{(n)}$ arising from $C_{2n+1}^{(n)}(\lambda)$ and corresponding to the phase factor $\omega(n)^{\beta}$ as in (3.86). Let L_0 denote a one dimensional representation of $G_n^{(n)}$ where $\beta \in (0_1, 0_2, \dots, 0_n)$ $0 \leq 0_i \leq n-1$, $\forall i = 1, \dots, n$ which corresponds to the choice

$$L_i = \omega(n)^{\beta_i} \quad \forall i = 1, \dots, n. \quad (3.87)$$

Here we like to make an important note about the construction of representations. In considering the group $G_n^{(n)}$ as an abstract group, the element L_0 should be interpreted as

$$L_0 = L_i \cdot L_j \cdot L_i^{-1} \cdot L_j^{-1}; \quad \forall i < j = 1 \dots n. \quad (3.88)$$

and hence if for example L_i 's correspond to the generators of $C_n^{(n)}(\lambda)$ then $L_0 = \omega(n)^{\lambda}$. In case of 1-dimensional representations $L_0 = 1$ always.

iv) Character Tables

We shall illustrate the above considerations by a simple example G_2^3 . Order of the group $= n^{m+1} = 3^{3+1} = 27$. The elements are given by $\{ \omega^{k_0} L_1^{k_1} L_2^{k_2} \mid k_0, k_1, k_2 = 0, 1, 2 \}$. Total number of classes are $m^n + (m-1) = 3^2 + (3-1) = 11$. These are given by

$$\begin{aligned}
 C_1 &= \{ \lambda^k L_1 \mid k = 0, 1, 2 \} & C_5 &= \{ \lambda^k L_1 L_2 \mid k = 0, 1, 2 \} \\
 C_2 &= \{ \lambda^k L_1^2 \mid k = 0, 1, 2 \} & C_6 &= \{ \lambda^k L_1^2 L_2 \mid k = 0, 1, 2 \} \\
 C_3 &= \{ \lambda^k L_2 \mid k = 0, 1, 2 \} & C_7 &= \{ \lambda^k L_2^2 \mid k = 0, 1, 2 \} \\
 C_4 &= \{ \lambda^k L_2^2 \mid k = 0, 1, 2 \} & C_8 &= \{ \lambda^k L_1^2 L_2^2 \mid k = 0, 1, 2 \} \\
 C_9 &= \{ 1 \} & C_{10} &= \{ \lambda \} & C_{11} &= \{ \lambda^2 \}
 \end{aligned} \tag{3.89}$$

where λ stands for the primitive 3rd root of unity. Number of irreducible inequivalent representations are 11, 2, 3-dimensional and 9 one dimensional.

The 9 one dimensional representations are $\Gamma_0^{(0,0)}, \Gamma_0^{(0,1)}$, $\Gamma_0^{(0,2)}, \Gamma_0^{(1,0)}, \Gamma_1^{(1,1)}, \Gamma_0^{(1,2)}, \Gamma_0^{(2,0)}, \Gamma_0^{(2,1)}, \Gamma_0^{(3,2)}$ in the notation introduced above. The two 3-dimensional representations are obtained by setting

$$\Gamma^{(1)} : L_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \quad L_0 = \omega(3) \tag{3.90}$$

$$\Gamma^{(2)} : L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 0 & 0 & 0 \end{pmatrix} \quad L_0 = \omega(3)^2 \tag{3.91}$$

v) Direct product representations and Clebsch-Gordan series.

With the help of the character table it is easy to obtain the Clebsch-Gordan series. We give here briefly the results

i) $n = 2^v$.

(a) Let $\Gamma^{(\tau_1)}$ and $\Gamma^{(\tau_2)}$ be two faithful m^v -dimensional representations. Let $a_{\tau}^{[\tau_1, \tau_2]}$ denote the number of times the representation $\Gamma^{(\tau)}$ one dimensional or higher dimensional, occurs in the direct product representation $\Gamma^{(\tau_1)} \otimes \Gamma^{(\tau_2)}$. The n

(i) if $0 \bmod n = r_1 + r_2$

$$a_{\tau}^{[\tau_1, \tau_2]} = \begin{cases} 1 & \text{for all the } m^v \text{ one-dimensional representations,} \\ 0 & \text{for all other higher dimensional representations.} \end{cases} \quad (3.93)$$

(ii) if $0 \bmod n \neq r_1 + r_2$

$$a_{\tau}^{[\tau_1, \tau_2]} = \begin{cases} m^v & \text{for } r \bmod n = r_1 + r_2 \text{ corresponding to } \Gamma^{(\tau)} \\ 0 & \text{for all other representations.} \end{cases}$$

(b) If $\Gamma_0^{(\alpha)}$ and $\Gamma_0^{(\beta)}$ are two one dimensional representations then $\Gamma_0^{(\alpha)} \otimes \Gamma_0^{(\beta)}$ is given a one dimensional representation given by $\Gamma_0^{(\alpha+\beta)}$ where $(\alpha+\beta)_i \bmod m = \alpha_i + \beta_i ; \forall i = 1 \dots 2^v$.

(c) If $\Gamma^{(\tau)}$ is an m^v -dimensional representation and $\Gamma_0^{(\beta)}$ a one dimensional representation then $\Gamma^{(\tau)} \otimes \Gamma_0^{(\beta)}$ is a faithful representation equivalent to $\Gamma^{(\tau)}$ itself.

Proof. $\Gamma^{(\tau)}$ corresponds to choice of $L_0 = \omega(m)^{\tau}$ and $\Gamma_0^{(\beta)}$ corresponds to the one dimensional representation generated by the

choice $L_i = \omega(m) \beta_i ; \forall i = 1 \dots 2v$. The direct product

$\Gamma^{(r)} \otimes \Gamma_0^{(k)}$ corresponds to the choice of generators $\{ \omega(m)^{\beta_i} L_i | i=1 \dots 2v \}$
 $L_0 = \omega(m)^r \}$. This representation is equivalent to
 $\Gamma^{(r)}$ itself if we can find a non-singular matrix M such that

$$L_i M = M \omega(m)^{\beta_i} L_i ; \forall i = 1 \dots 2v \quad (3.94)$$

Taking

$$M = \prod_{i=1}^{2v} L_i^{k_i} \quad (3.95)$$

we have the condition as

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 1 & \dots & 1 & 1 \\ \vdots & & & & & \\ -1 & -1 & -1 & \dots & 0 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_{2v} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{2v} \end{pmatrix} \text{ mod. } m \quad (3.96)$$

The inverse transformation always exists giving a solution for k_i 's
namely

$$\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_{2v} \end{pmatrix} \text{ mod. } m = \begin{pmatrix} 0 & -1 & +1 & \dots & \dots & -1 \\ +1 & 0 & -1 & \dots & \dots & +1 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ -1 & +1 & -1 & \dots & \dots & -1 \\ +1 & -1 & +1 & \dots & \dots & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{2v} \end{pmatrix} \quad (3.97)$$

This proves the statement (c)

2) $n = 2v+1$

$$[r_1, l_1; r_2, k]$$

a) Let $a_{[r, t]}$ denote the number of times the

representation $\Gamma^{(\tau, t)}$
 $\Gamma^{(r, l)} \otimes \Gamma^{(r_2, k)}$

occurs in the direct product representa-

(i) If $0 \bmod n = r_1 + r_2$ then $\Gamma^{(r_1, l)} \otimes \Gamma^{(r_2, k)}$ contains
 all the n^2 one dimensional representations which have the ele-
 ment $g(n-1, n-1), \dots, (n-1, 1, \dots, 1, n-1)$ represented by $a(n)^k$, $n \bmod n =$
 $l+k$

(ii) If $0 \bmod n \neq r_1 + r_2$

$$\alpha_{[\tau, t]}^{[\tau_1, l; \tau_2, k]} = \begin{cases} n^2 & \text{for } r \bmod n = r_1 + r_2 \\ t \bmod n = l+k & \\ 0 & \text{for other representations} \end{cases} \quad (3.98)$$

(b) The direct product of two one dimensional representations $\Gamma_0^{(\alpha)}$ and $\Gamma_0^{(\beta)}$ is again a one dimensional representation $\Gamma_0^{(\alpha+\beta)}$ with $(\alpha+\beta)_1 \bmod n = \alpha_1 + \beta_1$

(c) The direct product representation $\Gamma^{(r, l)} \otimes \Gamma_0^{(\beta)}$ is equivalent to a $\Gamma^{(r, t)}$ where t is uniquely determined by l and

Proof. The representation $\Gamma^{(r, l)} \otimes \Gamma_0^{(\beta)}$ corresponds to to the choice of generators $\{\omega(m)^{l+\beta_i} L_i \mid i=1, \dots, 2v+1, L_0 = \omega(m)^r\}$ and this is equivalent to a $\Gamma^{(r, t)}$ if there is a nonsingular matrix M such that

$$\omega(m)^t L_j \cdot M = M \omega(m)^{l+\beta_j} L_j \quad \forall j = 1, \dots, 2v+1. \quad (3.99)$$

Taking M as $\prod_{i=1}^{2v+1} L_i^{n_i}$ the condition for (3.99) is

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 1 & \dots & 1 & 1 \\ \vdots & & & & & \\ -1 & -1 & -1 & \dots & 0 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_{2v+1} \end{pmatrix} = \begin{pmatrix} \beta_1 + l - t \\ \beta_2 + l - t \\ \vdots \\ \beta_{2v+1} + l - t \end{pmatrix} \text{ mod. } m \quad (3.100)$$

This does not admit a solution for k_i 's when β_i 's and l are fixed and t is arbitrary. Hence let us suppose $k_{2v+1} = 0$. Then we have two conditions

$$-(k_1 + k_2 + \dots + k_{2v}) = \beta_{2v+1} + l - t \text{ mod. } m \quad (3.101)$$

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 1 & \dots & 1 & 1 \\ \vdots & & & & & \\ -1 & -1 & -1 & \dots & 0 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_{2v} \end{pmatrix} = \begin{pmatrix} \beta_1 + l - t \\ \beta_2 + l - t \\ \vdots \\ \beta_{2v} + l - t \end{pmatrix} \text{ mod. } m. \quad (3.102)$$

From (3.102) we find

$$\begin{aligned} -(k_1 + k_2 + \dots + k_{2v}) &= (-\beta_1 + \beta_2 - \dots + \beta_{2v}) \\ &= (\beta_{2v+1} + l - t) \text{ by (3.101)} \end{aligned} \quad (3.103)$$

or

$$t = (\beta_1 - \beta_2 + \beta_3 - \dots - \beta_{2v} + \beta_{2v+1} + l) \quad (3.104)$$

Now with this value of t substituted in (3.102) solution for k_i 's can be obtained uniquely as given in (3.97). Hence this proves (c).

vi) Generalized Clifford group $C_n^{(m)}$ and the group of linear transformations leaving invariant $\sum_{i=1}^n (\chi^i)^m$ for $n > 2$

It is well known (soft Doerner⁽¹¹⁾) that in the case of $C_n^{(2)}$ -ordinary Clifford algebra, if we define

$$S_{jk\kappa} = \frac{1}{4} L_j L_k ; j, k = 1 \dots n. \quad (3.105)$$

then they satisfy

$$[S_{jk\kappa}, S_{rl\tau}] = \delta_{kr} S_{jl} + \delta_{jl} S_{kr} - \delta_{kl} S_{jr} - \delta_{jr} S_{kl} \\ j, k, r, l = 1 \dots n. \quad (3.106)$$

which is the structure of the infinitesimal ring of the $(n+1)$ -dimensional rotation group. Thus the connection of Clifford algebra with the rotation group which forms the basis of the so-called spin representations of orthogonal groups is well known. Let us describe the situation as follows. Let the L-matrix

$$L(x) = \sum_{i=1}^n x_i L_i \quad (3.107)$$

be associated with the n -dimensional vector x with coordinates $(x_1 \dots x_n)$. Then after a rotation of the space, let the coordinates of $x = x'$ be $(x'_i | i=1 \dots n)$ in the new frame. Then let the corresponding L-matrix be

$$L' = \sum_{i=1}^n x'_i L_i = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} x_j \right) L_i \quad (3.108)$$

where the old and new coordinates are related by the orthogonal transformation A

$$x'_i = \sum_{j=1}^n a_{ij} x_j ; AA^T = I ; \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x'_i^2 \quad (3.109)$$

we have

$$L'^2 = L^2 = \left(\sum_{i=1}^n x_i'^2 \right) I = \left(\sum_{i=1}^n x_i'^2 \right) I \quad (3.10)$$

Since

$$L' = \sum_{i=1}^n x_i' L_i = \sum_{j=1}^n x_j \left(\sum_{i=1}^n a_{ij} L_i \right) = \sum_{j=1}^n x_j L_j' \quad (3.11)$$

L_j' 's also satisfy

$$L_j' L_k' = -L_k' L_j' ; \quad k, j = 1 \dots n.$$

$$L_j'^2 = I ; \quad \forall j = 1 \dots n. \quad (3.12)$$

But since L_j 's have only one representation in case of $n = 2v$ and only two in case of $n = 2v+1$ by Pauli's theorem the following equations must have solution for S

$$a) \quad L' = S^{-1} L S \quad \text{if } n = 2v. \quad (3.13)$$

$$b) \quad L' = \pm S^{-1} L S \quad \text{if } n = 2v+1.$$

Thus the group of matrices S induced by rotations A of the space form a group homomorphic to the rotation group which has been called a Clifford group (cf. Kahan (8), H. Freudenthal and H. de Vries (12)) Dirac group D_n is a subgroup of this Clifford group, and corresponds to only permutations and reflections of the basis vectors. Now in the case of generalized Clifford algebra there exists a similar connection between the group of linear transformations leaving invariant the expression

$$\sum_{i=1}^n x_i^m \quad (3.14)$$

and the group G_n^m which we have called a generalized Clifford group. This is the reason for calling G_n^m a G.C.G. rather than Generalized Dirac Group. Now we shall describe this relationship in detail. As Nono⁽¹³⁾ has observed the set of all linear transformations leaving invariant the expression $\sum_{i=1}^n x_i^m$ for $n > 2$ is a finite group of order $n^n n!$ and these are given by the transformations

$$\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = \begin{pmatrix} \omega(m) \lambda_1 & & & \\ & \omega(m) \lambda_2 & & \\ & & \ddots & \\ & & & \omega(m) \lambda_n \end{pmatrix} \begin{pmatrix} P \\ \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad (3.115)$$

or $(x') = SP(x)$

where P is any $n \times n$ permutation matrix and $0 \leq \lambda_i \leq m-1$
 $\forall i = 1, \dots, n$. Now consider the subgroup of the above transformations with $P \equiv I$ = identity matrix. Consider analogous to (3.107) the matrix

$$L(x) = \sum_{i=1}^n x_i L_i \quad (3.116)$$

where

$$L_i L_j = \omega(m)^l L_j L_i \quad (3.117)$$

$$(l, m) = 1$$

then

$$L(x)^m = \left(\sum_{i=1}^n x_i^m \right) I \quad (3.118)$$

The transformation $(x) \rightarrow (x') = S(x)$ is represented by the transformation

$$\begin{aligned} L(x) &= \sum_{i=1}^n x_i L_i \rightarrow L'(x) = \sum_{i=1}^n x'_i L_i \\ &= \sum_{i=1}^n x_i L_i' = \sum_{i=1}^n x_i (\omega^{k_i} L_i) \end{aligned} \quad (3.119)$$

Here also there are two cases corresponding to whether n is even or odd. When $n = 2v$ similar to (3.113a)

$$L'(x) = M^{-1} L(x) M \quad (3.120)$$

where $M = \prod_{i=0}^{2v} L_i^{k_i}$ in which $\{k_i | i=0, \dots, 2v\}$ are fixed

uniquely by S as discussed in (v.1.c) and k_0 can take all m values $0, 1, \dots, m-1$. In G_n^m all these elements form a class.

When $n = 2v+1$, analogous to the case (3.113.b) we have

$$L'(x) = K M^{-1} L(x) M \quad (3.121)$$

where

$$K = \omega(m)^t \quad t \bmod m = \sum_{k=1}^{2v+1} (-1)^{k+1} \lambda_k \quad (3.122)$$

and

$$M = \prod_{i=0}^{2v} L_i^{k_i} \quad (3.123)$$

with $\{k_i | i=0, \dots, 2v\}$ being fixed uniquely by S as discussed in (v.2c) and $0 \leq k_0 \leq m-1$. This set of elements K form a class in G.C.G. Thus just like Clifford group provides double valued representations of the orthogonal group leaving invariant the quadratic expression $\sum_{i=1}^n x_i^2$, G.C.G. provides m -valued representation of the group proper (excluding permutations) transformations leaving

invariant the expression $\sum_{i=1}^n x_i^n$ for $n > 2$.

Summary of important points.

We call the group of elements $G_n^m = \left\{ \prod_{i=0}^n L_i^{k_i} \mid 0 \leq k_i \leq m-1 \right\}$ as a Generalized Clifford group (G.C.G.) if the generators $\{L_i \mid i=1..n\}$ obey

$$\underbrace{L_i L_j}_{i < j} = L_0 L_j L_i ; \forall i, j = 1..n ; L_i^m = 1 ; L_0 L_j = L_j L_0 \quad \forall i = 0, 1, .. n. \quad \forall j = 1..n.$$

When n is assumed to be a prime number G_n^m has the following properties.

Total number of elements $= m^{n+1}$, Total number of conjugate classes and hence the total number of irreducible inequivalent representations is $m^{2v} + (m-1)$ when $n = 2v$ and $m^{2v+1} + m(m-1)$ when $n = 2v+1$. All the representations are obtained by assuming the various permitted values of $L_0 = \exp(2\pi i l/m)$; $l=0, 1, \dots, m-1$. When $l=0$, m^n one dimensional representations of G_n^m are generated for both the cases

$n = 2v$ and $n = 2v+1$ by the representations of $\mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m$ (n copies) where \mathbb{Z}_m is the cyclic group of order m . For each value of $l = 1, 2, \dots, m-1$, there is one irreducible representation of dimension m^v when $n = 2v$. In the case of $n = 2v+1$ each value of $l = 1, 2, \dots, m-1$ corresponds to a generalized Clifford algebra having m inequivalent irreducible representations of same dimension m^v . This chapter determines explicitly all the inequivalent irreducible representations and also studies direct product representations.

CHAPTER 4GENERALIZED CLIFFORD GROUPS - XI

In this chapter we shall study in detail the properties and representations of G.C.G. G_n^m when m is any integer. This chapter is based mainly on the paper of Ranganathan and myself¹.

(2) Properties of G.C.G. G_n^m when m is not a prime number.

As defined in the previous chapter G_n^m is a group of n^{n+1} elements

$$G_n^m = \left\{ \prod_{i=0}^n L_i^{k_i} \mid 0 \leq k_i \leq m-1 \right\} \quad (4.1)$$

where L_i 's obey the commutation relations

$$\begin{aligned} L_i L_j L_i^{-1} L_j^{-1} &= L_0; \quad i, j = 1, \dots, n. \\ i < j \quad \text{or } L_i L_j &= L_0 L_j L_i \\ L_0 L_i &= L_i L_0; \quad i = 1, \dots, n; \quad L_i^m = 1; \quad i = 0, 1, \dots, n \end{aligned} \quad (4.2)$$

As before denoting by

$$g(k_0, k_1, \dots, k_n) = \prod_{i=0}^n L_i^{k_i} \quad (4.3)$$

the product and inverse of elements are given by the same formula

(3.22-25). Let us now count the number of classes in this group.

First let $n = 2^v$. In this case there exists no relation of linear dependence among the elements $\{g(0, k_1, \dots, k_2) \mid 0 \leq k_i \leq m-1\}$. Let $D \equiv \{1 = d_1 < d_2 < \dots < d_{\tau(m)} = m\}$ be the set of all divisors of m .

in ascending order. $T(n)$ denotes the number of divisor of n .

An element $g_j = g(0, j_1 \dots j_{2v})$ obeys

$$g_j^{d_j} = \begin{cases} +1 & \text{if } d_j \text{ is odd} \\ -1 & \text{if } d_j \text{ is even} \end{cases} \quad (4.4)$$

where $d_j \in D$, and

$$\sum_k d_j = 0 \pmod{n} \quad \forall k = 1, 2, \dots, 2v \quad (4.5)$$

Proof. Let j_k be such that $\sum_k j_k d_j = 0 \pmod{n}$

Then

$$g_j^{d_j} = \left\{ \prod_{k=1}^{2v} L_k^{j_k d_j} \right\}^{d_j} = L_0^{\frac{1}{2}(d_j-1) d_j K} \quad (4.6)$$

where

$$K = (j_1 \dots j_{2v}) \left(\begin{array}{cccc|c} 0 & -1 & -1 & \dots & -1 \\ 0 & 0 & -1 & \dots & -1 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & -1 \\ 0 & 0 & 0 & \dots & 0 \end{array} \right) \left(\begin{array}{c} j_1 \\ j_2 \\ \vdots \\ j_{2v} \end{array} \right) \quad (4.7)$$

Since $L_0^m = 1$, $L_0^{K d_j} = 1$. Hence if $d_j = 2s+1$

$$g_j^{d_j} = L_0^{\frac{1}{2} 2s(2s+1) K} = L_0^{K s d_j} = +1 \quad (4.8)$$

If $d_j = 2s$ then

$$g_j^{d_j} = L_0^{\frac{1}{2} K (2s-1) 2s} = L_0^{\frac{1}{2} K (2s-1) d_j} = \pm 1. \quad (4.9)$$

For a given element $g(0, j_1 \dots j_{2v})$ let us choose d_j to be the minimum $d \in D$ satisfying (4.4) and (4.5). It is easy to see that for any element $g(0, j_1 \dots j_{2v})$ such a $d_j \in D$ exists. Then

we have

$$g_j^{d_j} = \pm 1 \rightarrow g_k (g_j^{d_j}) g_k^{-1} = (g_k g_j g_k^{-1})^{d_j} = \pm 1 \quad (4.10)$$

or

$$g_k g_j g_k^{-1} = L_0^{r_k q_j} g_j \quad \text{for any } k \quad (4.11)$$

where

$$q_j = \frac{m}{d_j} \quad ; 0 \leq r_k \leq d_j - 1. \quad (4.12)$$

when we vary $g_k \in G_{(2)}^m$ overall elements then we obtain a set $\{L_0^{r_k q_j} g_j\}$ which are in a class. If it happens that d_j is nonprime then there is a possibility that $\{L_0^{r_k q_j}\}$ may not contain all the d_j -th roots of unity which arises when $(r_k, d_j) \neq 1 \forall k > 1$, for all r_k . In that case let l_j be the greatest common divisor of $\{(r_k, d_j) | \forall k\}$. Then the set will contain only all $\{\frac{d_j}{l_j}\}-$ th roots of unity and this would imply that $g_j^{(d_j/l_j)}$ commutes with all $g_k \in G_{(2)}^m$. By Schur's lemma this means that $g_j^{(d_j/l_j)} \sim I$ which is in contradiction to the fact that d_j was the minimum divisor of m satisfying this condition. Hence the set $\{L_0^{r_k q_j} g_j\}$ contains d_j distinct elements corresponding to $0 \leq r_k \leq d_j - 1$. This shows that the set of m elements

$$\{g(j_0, j_1, \dots, j_{2^n}) | 0 \leq j_i \leq m-1\} \quad (4.13)$$

are partitioned in $\frac{m}{d_j} = q_j$ classes each containing d_j elements.

Explicitly the classes are given by

$$\left\{ \left[g(j_0' + r d_j, j_1, \dots, j_{2v}) \mid r=0,1,\dots,d_j-1 \right] \mid j_0' = 0, 1, \dots, q_j - 1 \right\} \quad (4.13)$$

Let N_j be the total number of elements with $j_0 = 0$ obeying the relation (4.4) with d_j as the minimum for a $d_j \in D$. Then the total number of classes is G_{2v}^m

$$N_{2v}^m = \sum_{j=1}^{T(m)} N_j \left(\frac{m}{d_j} \right) \quad (4.14)$$

Obviously $N_1 = 1$, $N_{T(m)} = \left\{ m - \sum_{j=1}^{T(m)-1} N_j \right\}$. For others

$$N_j = d_j^{2v} - \sum_{s < j} N_s e(s) \quad (4.15)$$

where

$$e(s) = \begin{cases} 1 & \text{if } d_s \mid d_j \\ 0 & \text{if } d_s \nmid d_j \end{cases} \quad (4.16)$$

These follow from the observation that the condition

$$j_k d_j = 0 \pmod{m} \quad \forall k = 1, \dots, 2v \quad (4.17)$$

is satisfied whenever $0 \leq \frac{j_k}{q_j} \leq d_j - 1$, $\forall k = 1 \dots 2v$ and thus there are d_j^{2v} solutions for this condition. Substituting (4.15) in (4.14), we have

$$N_{2v}^m = \sum_{j=1}^{T(m)} \left(\frac{m}{d_j} \right) \left\{ d_j^{2v} - \sum_{s < j} N_s e(s) \right\} \quad (4.18)$$

In our paper¹ It was conjectured that

$$N_{2V}^m = \sum_{l=1}^m (l, m)^{2V} = \sum_{l=1}^m d_l^{2V} \quad (4.19)$$

d_l' is the greatest common division of l and m . This implies the identity

$$\sum_{j=1}^{T(m)} \left(\frac{m}{d_j} \right) \left\{ d_j^{2V} - \sum_{s < j} N_s e(s) \right\} = \sum_{l=1}^m (l, m)^{2V} \quad (4.20)$$

This identity was first proved recently by Krishnaswamy Alladi².

Following his proof we shall reformulate it using a matrix approach to the underlying number theoretic problem.

Let us denote N_j by N_d where $d|m$. The suffix j in d_j is unnecessary since all d_j 's are distinct. Then the definition (4.15) can be written as

$$\sum_{d|d} N_d = d^{2V} \quad (4.21)$$

where summation is over all $d'|d$.

Möbius inversion formula tell us that if an arithmetic function f_n is defined by

$$\sum_{d|n} f_d = g_n \quad (4.22)$$

then f_n is uniquely given by

$$f_n = \sum_{d|n} g\left(\frac{n}{d}\right) \mu(d) \equiv \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) \quad (4.23)$$

¹I wish to thank Professor Paul Erdős for very encouraging and useful discussion on this identity. Also I wish to thank Professor Hirschhorn University of New South Wales, School of Mathematics, Australia for a similar proof of the identity³. The proof of Mr. Krishnaswami Alladi applies to more general context and consequently he has used this identity in other number theoretic problems².

where $\mu(n)$ is the Möbius function

$$\mu(n) = \begin{cases} 1 & \text{if } n=1 \\ (-1)^r & \text{if } n = \prod_{i=1}^r p_i, p_i \text{'s are distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$
(4.24)

Applied to (4.21) this gives

$$N_d = \sum_{d'|d} \left(\frac{d}{d'}\right)^{2V} \mu(d') = \sum_{d'|d} \mu\left(\frac{d}{d'}\right) d'^{2V}$$
(4.25)

Let us now introduce matrices Δ and S such that

$$\Delta_{dd'} = \begin{cases} 1 & \text{if } d'|d \\ 0 & \text{if } d \neq d' \quad 1 \leq d', d \leq \infty \end{cases}$$
(4.26)

$$S_{ij} = i \delta_{ij} \quad 1 \leq i, j \leq \infty$$
(4.27)

Associated to any arithmetic function A_n let us introduce a matrix A such that

$$A_{dd'} = \begin{cases} A\left(\frac{d}{d'}\right) & \text{if } d'|d \\ 0 & \text{if } d \neq d' \end{cases}$$
(4.28)

Then (4.23) can be written as

$$\Delta F = G$$
(4.29)

where F, G are column vectors with f_n and g_n as elements
(4.29) implies

$$F = \mu G$$
(4.30)

where μ is the matrix associated with Möbius function $\mu(n)$ defined by (4.24). From (4.29) and (4.30) it follows

$$\Delta^{-1} = \mu$$
(4.31)

Euler's function $\phi(n)$ is defined as the number of ~~integers less than~~
n which are coprime to n, i.e.

$$\sum \phi(n) = \sum_{\substack{(d,n)=1 \\ d < n}} 1 \quad (4.32)$$

It obeys the identity

$$\sum_{d|n} \phi(d) = n \quad (4.33)$$

or using Möbius inversion

$$\phi(n) = \sum_{d|n} \frac{n}{d} \mu(d) \quad (4.34)$$

Let an $l|n$. Then

$$\begin{aligned} \phi\left(\frac{n}{l}\right) &= \sum_{d|\frac{n}{l}} \frac{n}{dl} \mu(d) = \sum_{d'|\frac{l}{2}} \frac{n}{d'} \mu\left(\frac{d'}{2}\right) ; \quad d' = ld \\ &= \sum_{d'|\frac{n}{l}} \sum_{d|\frac{l}{2}} n \Delta_{nd'} \frac{1}{d'} \mu\left(\frac{d'}{2}\right) \end{aligned} \quad (4.35)$$

or writing in matrix form

$$\underline{\Phi} = S \Delta S^{-1} \mu \quad (4.36)$$

where $\underline{\Phi}$ is the matrix associated with Euler $\phi(n)$. From (4.32) it is clear that

$$\Delta H = D \quad (1 \leq d \leq \infty) \quad (4.37)$$

where H is the column vector with elements H_d and D is the column vector with elements $D_d = d^{2V}$ ($1 \leq d \leq \infty$) (4.36) gives

$$\underline{\Phi} \Delta = S \Delta S^{-1} \mu \quad (4.38)$$

Hence

$$\underline{\Phi} \Delta H = S \Delta S^{-1} H \quad (4.39)$$

$$\underline{\Phi} D = S \Delta S^{-1} H \quad (4.40)$$

writing in terms of elements

$$\sum_{d|n} \phi\left(\frac{n}{d}\right) D_d = \sum_{d|n} \frac{n}{d} N_d$$

$$\sum_{d|n} \phi\left(\frac{n}{d}\right) d^{2v} = \sum_{d|n} \frac{n}{d} N_d$$
(4.41)

We have from the definition of $\phi(n)$

$$\phi\left(\frac{n}{d}\right) = \sum_{\substack{d' \\ d' < \frac{d}{2}}} 1 = \sum_{\substack{(d', n) = 1 \\ d' < \frac{d}{2}}} 1$$
(4.42)

Hence (4.41) gives

$$\sum_{d|n} \phi\left(\frac{n}{d}\right) d^{2v} = \sum_{l=1}^n (l, n)^{2v} = \sum_{d|n} \frac{n}{d} N_d$$
(4.43)

With $n = m$ this proves the identity (4.30). Incidentally the general nature of the identity (4.41) has to be noticed. Any two sequences \mathbf{N}, \mathbf{D} related by (4.37) will obey (4.41) and the identity (4.30) is a special case of this. Thus the total number of classes in G_{2v+1}^m is

$$\sum_{l=1}^m (l, m)^{2v}.$$

In the case of G_{2v+1}^m the set of m^{2v+2} group elements $\{g(k_0, k_1, \dots, k_{2v+1}) \mid \forall i, 0 \leq k_i \leq m-1\}$ can be written as a sum of m subsets

$$\left\{ \left(\prod_{i=0}^{2v} L_i^{k_i} \mid 0 \leq k_i \leq m-1 \right), \left(\prod_{i=0}^{2v} L_i^{k_i} L_{2v+1}^{m-1} \mid 0 \leq k_i \leq m-1 \right), \dots, \right.$$

$$\left. \left(\prod_{i=0}^{2v} L_i^{k_i} L_{2v+1}^{m-1} \mid 0 \leq k_i \leq m-1 \right) \right\}$$

$$\equiv \left\{ \left(g(k_0, k_1, \dots, k_{2v}) \mid 0 \leq k_i \leq m-1 \right), \left(g(k_0, k_1, \dots, k_{2v}, 1) \mid 0 \leq k_i \leq m-1 \right), \right. \\ \left. \dots, \left(g(k_0, k_1, \dots, k_{2v}, m-1) \mid 0 \leq k_i \leq m-1 \right) \right\} \quad (4.44)$$

Thus we can prove that there should be m^m such sets $1 \leq l \leq m, (l, m)$

Now as we have seen earlier (ref. chapter 1.) there exists a relationship among the $2v+1$ generators, L_i namely

$$\gamma = g(0, \underbrace{m-1, m-1, \dots, m-1}_v, \underbrace{1, 1, \dots, 1}_v, m-1) \sim I \quad (4.45)$$

and hence L_{2v+1} can be replaced as a product of powers of other $2v$

L_i 's so that each subset of (4.44) given in brackets contains only product of powers of $2v, L_i$'s. Or the set G_{2v+1}^m splits into a sum of m subsets each isomorphic to G_{2v}^m . Hence the total number of classes in G_{2v+1}^m is m times the number of classes in G_{2v}^m . Thus denoting by N_n^m the number of classes in G_n^m , we have

$$N_{2v}^m = \sum_{l=1}^m (l, m)^{2v} \quad (4.46)$$

$$N_{2v+1}^m = m \sum_{l=1}^m (l, m)^{2v} \quad (4.47)$$

(ii) Representations of G_{2v}^m has m^{2v+1} elements and $\sum_{l=1}^m (l, m)^{2v}$ conjugate classes.

It is seen that

$$m^{2v+1} = \sum_{l=1}^m (l, m)^{2v} \left[\left(\frac{m}{(l, m)} \right)^v \right]^{2v} = \sum_{\text{m times}} m^{2v} = m^{2v+1} \quad (4.48)$$

Thus we can guess that there should be for each $1 \leq l \leq m$, $(l, m)^{2v}$ representations of dimension $\left(\frac{m}{(l, m)} \right)^v$. As done for G_n^m we should look for these representations in the n inequivalent isomorphic algebras $C_m^{\infty}(l)$, $1 \leq l \leq m$ generated by

$$\underset{i < j}{L_i L_j} = \omega(m)^l L_j L_i ; i, j = 1, 2, \dots, 2v. \quad (4.49)$$

$$L_i^m = 1 ; \forall i = 1, 2, \dots, 2v ; 1 \leq l \leq m$$

respectively.

a) One dimensional representations

These correspond to the case $l = m$ or $l = 0$. Then (4.49) becomes

$$\begin{aligned} L_i L_j &= L_j L_i ; i, j = 1, 2, \dots, 2v \\ L_i^m &= 1 ; \forall i = 1, \dots, 2v \end{aligned} \quad (4.50)$$

These being the generating relations of the Abelian group $G \cong \mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m$ ($2v$ -copies) as before these representations arise from the homomorphism of G_{2v}^m to $\mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m$ ($2v$ copies)

$$G_{2v}^m / \mathbb{Z}_m \cong \mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m \quad (2v\text{-copies}) \quad (4.51)$$

Thus all those representations totally $(l, m)^{2V} = (m, m) = m^{2V}$ one dimensional representations arise by putting

$$L_i = \omega(m)^{T_{ki}} ; \quad 0 \leq T_{ki} \leq m-1 ; \quad i=1 \dots 2V. \quad (4.52)$$

where k labels the representations $k=1 \dots m^{2V}$. $L_0 = 1$ in all these representations.

b) Higher dimensional representations

Consider now (4.40) for $\lambda > 0$. For each $1 \leq l \leq m-1$ find the representations. Let

$$(l, m) = p_l , \quad q_l = \frac{m}{p_l} , \quad r_l = \frac{l}{p_l} , \quad (q_l, r_l) = 1 \quad (4.53)$$

Then (4.40) becomes

$$\begin{aligned} L_i L_j &= \omega(q_l)^{r_l} L_j L_i ; \quad i, j = 1, \dots, 2V. \\ L_i^m &= 1. \quad \forall i = 1 \dots 2V. \end{aligned} \quad (4.54)$$

When $V=1$ the problem reduces to the one considered in Chapter II-a). There we showed that two elements L_1, L_2 obeying

$L_1 L_2 = \omega(m)^l L_2 L_1, L_1^m = L_2^m = 1, (l, m) = d$ have d^2 inequivalent irreducible representations of dimension $m' = \left(\frac{m}{d}\right)$ which are given by $\left\{ \omega(m)^{\frac{si}{d}} L'_1, \omega(m)^{\frac{ti}{d}} L'_2 \mid i=1 \dots d^2; 0 \leq s_i, t_i \leq d-1; \forall i \right\}$

L'_1 and L'_2 being the unique (upto equivalence) representations of

$$L'_1 L'_2 = \omega(m')^{l'} L'_2 L'_1, \quad L'_1^{m'} = L'_2^{m'} = 1; \quad m' = \frac{m}{d}, \quad l' = \frac{l}{d} \quad (V, m') = 1.$$

Uniqueness of L'_1, L'_2 was assumed in proving this result. Now due to the above group theoretical considerations the uniqueness of L'_1, L'_2

stands proved as follows. Considering the group $G_2^{m'}$ for which 99
 L'_1, L'_2 provide a representation corresponding to ℓ' such that
 $(i', m') = 1$, there should be only $(v, m')^2 = 1$ representation of
dimension m' . Thus for L'_1, L'_2 there cannot be any other repre-
sentation.

Now corresponding to (4.53) consider the relations

$$\begin{aligned} L'_i L'_j &= \omega(\varphi_\ell)^{\tau_\ell} L'_j L'_i \quad ; i, j = 1, 2, \dots, 2v. \\ L_i^{\varphi_\ell} &= 1 \quad ; \forall i = 1 \dots 2v \quad (\varphi_\ell, \tau_\ell) = 1. \end{aligned} \quad (4.55)$$

These relations would provide a representation for $\widehat{G}_{2v}^{\varphi_\ell}$ and for
these there should be only $(\varphi_\ell, \tau_\ell)^{2v} = 1$ representation. Thus
(4.55) should have only one representation. Based on this fact
following the same type of arguments as in Chapter II a, we can prove
that (4.54) have p_ℓ^{2v} representations of dimension φ_ℓ^{2v} which
are given by $\left\{ (\omega(m)^{\beta_{ij}} L'_i \mid i = 1 \dots 2v) \mid j = 1 \dots p_\ell^{2v}, 0 \leq \beta_{ij} \leq k-1 \right\}$
 $(L'_i \mid i = 1 \dots 2v)$ being the unique (upto equivalence) representations
of (4.55). Here we shall follow a simpler reasoning to arrive at
this result. Rewrite (4.54) as

$$a) L_i L_j = \omega(\varphi_\ell)^{\tau_\ell} L_j L_i \quad ; i, j = 1 \dots 2v. \quad (4.56)$$

$$b) L_i^m = (L_i^{\varphi_\ell})^{p_\ell} = 1 \quad ; \forall i = 1 \dots 2v$$

From a) it follows that $L_i^{\varphi_\ell} \nmid 1, \forall i = 1 \dots 2v$, commute with each
other so that they are $\sim I$. By b) it follows that $L_i^{\varphi_\ell}$ can

take any of the p_e , p_e^{th} roots of unity. So

$$L_i^{q_e} = \omega(p_e)^{t_e} I \quad ; \quad 0 \leq t_e \leq p_e - 1. \quad (4.57)$$

Hence with $L_i' = \xi L_i$ where $L_i'^{q_e} = 1$, $L_i' L_j' = \omega(q_e)^{t_e} L_j' L_i'$
 $i, j = 1 \dots 2V$; $\xi^{q_e} = \omega(p_e)^{t_e}$; $0 \leq t_e \leq p_e - 1$ or $\xi = \omega(p_e)^{t_e/q_e} = \omega(m)^{t_e}$
 $0 \leq t_e \leq p_e - 1$. Thus

$$L_i' = \omega(m)^{t_e} L_i' \quad ; \quad \forall i = 1 \dots 2V \quad (4.58)$$

$$0 \leq t_e \leq p_e - 1$$

L_i' 's having only one representation, there are not any other representations possible for L_i .

For the representations of L_i 's obeying the relations (4.58) we can use the method of product transforms detailed in Chapters I and III. The commutation matrix T associated with (4.55) is

$$T = r_l \begin{bmatrix} 0 & +1 & +1 & \dots & +1 \\ -1 & 0 & +1 & \dots & +1 \\ \vdots & & & & \\ -1 & -1 & -1 & \dots & +1 \\ -1 & -1 & -1 & \dots & 0 \end{bmatrix} \quad (4.59)$$

and hence its canonical form is

$$T^* = r_l \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (4.60)$$

associated with the relations

$$\begin{aligned} L'_{2i-1}^* L'_{2i}^* &= \omega(q_{\ell})^r L'_{2i}^* L'_{2i-1}^* ; i = 1 \dots v \\ L'_{i}^* L'_{j}^* &= L'_{j}^* L'_{i}^* \text{ otherwise.} \end{aligned} \quad (4.61)$$

Following the same arguments as in the representation of (3.57) we get the representation of L_i' 's as

$$L_i' = \prod_{j=1}^{2v} L_j'^* v_{ij} \quad (4.62)$$

using the same V in (3.68) or explicitly using the formula (3.71)

$$\begin{aligned} L'_{2k-1} &= \underbrace{(C^{q_{\ell}-1} B^{\ell} \otimes \dots \otimes C^{q_{\ell}-1} B^{\ell})}_{k-1} \underbrace{C^{q_{\ell}-1} B^{\ell} \otimes I \otimes \dots \otimes I}_{v-k} \\ L'_{2k} &= \underbrace{(C^{q_{\ell}-1} B^{\ell} \otimes \dots \otimes C^{q_{\ell}-1} B^{\ell})}_{k} \underbrace{C^{q_{\ell}-1} B^{\ell} \otimes I \otimes \dots \otimes I}_{v-k} \end{aligned} \quad (4.63)$$

$k = 1 \dots v.$

Summarising the representations of L_i' 's obeying (4.61) are given by

$$\left\{ (\omega(m)^{\delta_{ji}} L_i' \mid i = 1 \dots 2v) \mid j = 1 \dots p_{\ell}^{2v}; 0 \leq \delta_{ji} \leq p_{\ell}^{-1} \right\} \quad (4.64)$$

where j labels the representations and L_i' 's are given by (4.63). Thus there are totally $p_{\ell}^{2v} = (l, m)^{2v}$ inequivalent representations of size dimension $q_{\ell}^v = [m/(l, m)]^v$. L_0 has to be taken as $\omega(m)^l = \omega(q_{\ell})^r$ corresponding to (4.54). Thus corresponding to each value of $1 \leq l \leq m-1$, there are $(l, m)^{2v}$ inequivalent irreducible representations of dimension $[m/(l, m)]^v$ respectively. Thus we have got the all the representations of G_{2v}^m .

In this case we found we have $m \sum_{l=1}^m (l, m)^{2V}$ classes.
The number of group elements is m^{2V+2} . Let us write now

$$\begin{aligned} m^{2V+2} &= \sum_{l=1}^m (l, m)^{2V+1} \left(\frac{m}{(l, m)} \right)^{2V+1} \\ &= \sum_{l=1}^m (l, m)^{2V+1} \left(\frac{m}{(l, m)} \right) \left[\left(\frac{m}{(l, m)} \right)^V \right]^2 \\ &= \sum_{l=1}^m m (l, m)^{2V} \left[\left(\frac{m}{(l, m)} \right)^V \right]^2 \end{aligned} \quad (4.65)$$

This tells us that there should be $m(l, m)^{2V} = \left\{ (l, m)^{2V+1} \frac{m}{(l, m)} \right\}$ representations of dimension $\left(\frac{m}{(l, m)} \right)^V$ corresponding to each l , $1 \leq l \leq m$.

a) One dimensional representations.

These correspond to the case of $l=m$ or 0. Corresponding to this there should be m^{2V+1} representations. As in earlier cases it is obvious that these arise due to the homomorphism of

G_{2V+1}^m to $\mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m$ ($2V+1$ copies). All these representations are obtained by putting $L_i = \omega(m)^{r_{ki}}$, $0 \leq r_{ki} \leq m-1$; $i=1 \dots 2V+1$ where k is the representation index $k=1 \dots m^{2V+1}$. $L_0 = 1$ in all these cases.

b) Higher dimensional representations

As usual consider the relations

$$L_i L_j = \omega(m)^l L_j L_i ; i, j = 1, \dots, 2v+1.$$

$$\underset{i < j}{\text{L}_i^m} = 1 ; \forall i = 1, \dots, 2v+1.$$

(4.66)

for $1 \leq l \leq m-1$. If $(l, m) = p_l$, then letting $q_l = \frac{m}{p_l}$, $r_l = \frac{l}{p_l}$ to have

$$L_i L_j = \omega(q_l)^{r_l} L_j L_i ; i, j = 1, \dots, 2v+1.$$

$$L_i^m = (L_i^{q_l})^{p_l} = 1 ; \forall i = 1, \dots, 2v+1. (q_l, q_l) = 1$$

Letting L'_i 's obey

$$L'_i L'_j = \omega(q_l)^{r_l} L'_j L'_i ; i, j = 1, \dots, 2v+1$$

(4.68)

$$(L'_i)^{q_l} = 1 . \forall i = 1, \dots, 2v+1.$$

L'_i 's obeying (4.68) are given the representation

$$\left\{ (L'_i = \omega(m)^{\gamma_{ri}} L'_i) \mid i = 1, \dots, 2v+1 ; 0 \leq \gamma_{ri} \leq p_l^{-1} \right\} \quad (4.69)$$

following the same arguments leading to the similar result (4.68) in the case of $2v$, L'_i 's. Now the difference between the two cases G_{2v}^m and G_{2v+1}^m lies in the fact that L'_i 's have been more than one representations. While dealing with G_{2v+1}^m it was shown that the relations (3.75) exactly identical to (4.68) except for the replacement $\gamma_l = l$, $q_l = m$, have m inequivalent irreducible

representations given by

$$\left\{ \omega(m) L_i^{''} \mid i=1 \dots 2v+1, \lambda = 0, 1, \dots, m-1 \right\} \quad (4.70)$$

where $L_i^{''}$'s are given by (cf. 3.78 and 3.81)

$$L_i^{''} = \prod_{j=1}^{2v+1} L_j^* v_{ij} \quad (4.71)$$

with

$$V = \begin{bmatrix} 0000 \dots 00100 \\ 0000 \dots 00000 \\ \vdots \\ 0010 \dots -11-110 \\ 0001 \dots -11-110 \\ 10-11 \dots -11-110 \\ 0111 \dots -11-110 \\ -1111 \dots -11-111 \end{bmatrix} \quad (4.72)$$

L_j^* satisfying

$$\begin{aligned} a) \quad & L_{2i-1}^* L_{2i}^* = \omega(m)^l L_{2i}^* L_{2i-1}^* \quad i=1 \dots v \\ b) \quad & L_i^* L_j^* = L_j^* L_i^* \text{ otherwise } ; i, j = 1 \dots 2v+1. \end{aligned} \quad (4.73)$$

and having representations

$$\begin{aligned} L_{2i-1}^* &= \overbrace{I \otimes \dots \otimes \overset{(v-i)}{\underset{\downarrow}{B}} \otimes I \otimes \dots \otimes I}^{i-1} \\ L_{2i}^* &= I \otimes \dots \otimes \overset{l}{\underset{\downarrow}{B}} \otimes I \otimes \dots \otimes I \\ L_{2v+1}^* &= I \otimes \dots \otimes I \otimes \dots \otimes I \equiv \text{identity} \end{aligned} \quad i=1 \dots v. \quad (4.74)$$

Now just taking $l = \gamma_\ell$, $m = q_\ell$ we get for $L_i^{''}$'s, q_ℓ representations

$$\left\{ \omega(q_\ell) L_i^{''} \mid i=1 \dots 2v+1 ; \lambda = 0, 1, \dots, q_\ell-1 \right\} \quad (4.75)$$

where L_i 's are given by (4.71) in which the C and B matrices are \mathcal{V}_l -dimensional matrices and l is to be replaced by T_l . By considering the group $G_{2v+1}^{\mathcal{V}_l}$ for which (4.68) would generate representations corresponding to T_l , there should be only $(\gamma_l, \mathcal{V}_l)^{2v} \mathcal{V}_l = \mathcal{V}_l^{2v}$ representations. This shows that (4.68) there cannot be more than \mathcal{V}_l representations which are given by (4.75). Now going back to (4.69) corresponding to each of the \mathcal{V}_l representations of L_i there are p_l^{2v+1} representations and hence thus totally there are $\mathcal{V}_l p_l^{2v+1} = m p_l^{2v}$ representations for the relations (4.68). Thus summarising all the $p_l^{2v} m$ representations of (4.68) are given by

$$\left\{ L_i = \omega(m) \begin{matrix} s_{\gamma_l} + p_l \cdot s \\ L_i'' \end{matrix} \mid \begin{array}{l} i=1 \dots 2v+1, \gamma=1 \dots p_l^{2v+1} \\ 0 \leq s_{\gamma_l} \leq p_l - 1, 0 \leq s \leq \mathcal{V}_l - 1 \end{array} \right\}$$

L_i'' being given by (4.71) and $(l, m) = p_l, \mathcal{V}_l = \frac{m}{p_l}, \gamma_l = \frac{l}{p_l}$ Considering each value of $l = 1 \dots m-1$ all the corresponding relations of the type (4.68) respectively provide all the higher dimensional representations of the generators of G_{2v+1}^m . $L_0 = \omega(m)$ corresponding to (4.68).

(iv) Examples.

We shall illustrate the above considerations by means of two examples.

- (a) G_2^4 : The total number of elements N is $4^{3+1} = 64$ and the number of classes $= \sum_{l=1}^4 (l, 4)^2 = 1^2 + 2^2 + 1^2 + 4^2 = 22$. There

exist $(4,4)^2 = 16$ one dimensional representations corresponding to $l=4$, which arise from the representations of $\mathbb{Z}_4 \otimes \mathbb{Z}_4$.

These are given by taking $L_1 = \omega(4)^\gamma = i^\gamma$, $L_2 = i^\lambda$ ($\gamma, \lambda = 0, 1, 2, 3$), $L_0 = 1$. Corresponding to the roots $\omega(4)^1$ and $\omega(4)^3$ which are primitive there are four dimensional representations for each which are given by taking the generators as

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = C(4); \quad L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega(4) & 0 & 0 \\ 0 & 0 & \omega(4)^2 & 0 \\ 0 & 0 & 0 & \omega(4)^3 \end{pmatrix} = B(4)$$

$$L_0 = \omega(4).$$
(4.77)

and

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = C(4); \quad L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega(4)^3 & 0 & 0 \\ 0 & 0 & \omega(4)^1 & 0 \\ 0 & 0 & 0 & \omega(4)^3 \end{pmatrix} = B(4)$$

$$L_0 = \omega(4)^3.$$
(4.78)

respectively. Corresponding to the value of $l=2$, there are $(2,4)^2 = 4$, $\frac{m}{(l,m)} = \frac{4}{2} = 2$ -dimensional representations, these are now given according to the formula (4.74) by

$$\left\{ L_i = \omega(4)^{\sum_j s_{ji}} L'_i \mid i=1, 2, s_{ji}=0, 1 \right\}$$
(4.79)

where L'_1, L'_2 obey

$$L'_1 L'_2 = \omega(4)^{2\sqrt{L'_2 L'_1}} = -L'_2 L'_1$$
(4.80)

or

$$L'_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; L'_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.81)$$

(b) G_3^4 : Total number of elements $= 4^{3+1} = 256$ and number of classes $= 4 \times \sum_{l=1}^4 (l, 4)^2 = 88$. There are $4 \times (4 \times 4)^2 = 64$ one-dimensional representations given by the representations of

$\mathbb{Z}_4 \otimes \mathbb{Z}_4 \otimes \mathbb{Z}_4$. Corresponding to each of the primitive roots $\omega(4)$, and $\omega(4)^3$ there are 4 inequivalent representations of dimension 4. Those are given respectively by , according to formula (4.76)

$$\left\{ L_i = \omega(4)^b L_i' \mid i=1, 2, 3, 0 \leq b \leq 3, L_0 = \omega(4)^3 \right\} \quad (4.82)$$

and

$$\left\{ L_i = \omega(4)^b L_i'' \mid i=1, 2, 3, 0 \leq b \leq 3, L_0 = \omega(4)^3 \right\} \quad (4.83)$$

where

$$L'_1 = C(4), \quad L'_2 = B(4), \quad L'_3 = C(4)^{-1} B(4) \quad (4.84)$$

and

$$L_1'' = C(4), \quad L_2'' = B(4)^3, \quad L_3'' = C(4)^{-1} B(4)^3 \quad (4.85)$$

There are $4 \cdot (2, 4)^2 = 16$, two dimensional representations which are given by, according to formula (4.76) ,

$$\left\{ L_i = \omega(4)^{s_i} L_i' \mid i=1, 2, 3; s_i = 0, 1 \right\} \quad (4.86)$$

and

$$\left\{ L_i = -\omega(4)^{s_i} L_i' \mid i=1, 2, 3; s_i = 0, 1 \right\}$$

where L_i' 's are

$$L_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = C(2); L_2' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B(2); L_3' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iC(2)B(2)^{-1}$$
(4.87)

(iv) Direct product representations

Here we shall make some general observations about direct product representations. Considering $G_{2\nu}^m$ if $\Gamma^{(l)}$ and $\Gamma^{(t)}$ correspond to two faithful representations corresponding to roots $\omega(m)^l$ and $\omega(m)^t$ respectively then the direct product representation $\Gamma^{(l)} \otimes \Gamma^{(t)}$ has generators as $L_i^{(l+t)} = L_i^{(l)} \otimes L_i^{(t)}$ and hence the commutation relations are

$$\underset{i < j}{\prod} L_i^{(l+t)} L_j^{(l+t)} = \omega(m)^{\sum_{i < j} l+t} L_j^{(l+t)} L_i^{(l+t)} ; i, j = 1 \dots 2\nu.$$
(4.88)

Let $r \bmod m = (l+t)$. (4.88) shows that $\Gamma^{(l)} \otimes \Gamma^{(t)}$ contains only $\Gamma^{(r)}$. Since we have

$$\dim \Gamma^{(l)} \otimes \Gamma^{(t)} = \left(\frac{m}{(l,m)}\right)^\nu \times \left(\frac{m}{(t,m)}\right)^\nu$$

$$\dim \Gamma^{(r)} = \left(\frac{m}{(r,m)}\right)^\nu$$
(4.89)

the number of times $\Gamma^{(r)}$ is contained is

$$\alpha(r; l, t) = \frac{m^{2\nu}}{(l,m)^\nu (t,m)^\nu} \Bigg/ \frac{m^\nu}{(r,m)^\nu}$$

$$= \frac{m^\nu (r,m)^\nu}{(l,m)^\nu (t,m)^\nu} = \left(\frac{m(r,m)}{(l,m)(t,m)} \right)^\nu$$
(4.90)

But there are $(r, m)^{2v}$ inequivalent representations corresponding to $\omega(m)^r$. Which of these occur in the $\alpha(r; l, t)$ representations is interesting but difficult to analyse. In the case of G_{2v+1}^m the same result (4.90) holds as is seen easily.

(v) A different approach to the representation problem of G_n^m .

In a direct approach to the representation problem of G_n^m , by counting the number of classes and guessing at the representations we have seen that the analysis of class structure is a difficult affair, involving tricky number theoretic considerations. Avoiding this and having a unified approach to both the cases $n = 2v$ and $n = 2v+1$ is possible utilising the generalized matrix decomposition theorems developed in Chapter II. From these we know that any d^v dimensional matrix can be represented as a linear sum of d^{2v} linearly independent matrices given by the set

$$\left\{ C \begin{smallmatrix} k_1 & l_1 \\ B & \otimes \dots \otimes C \begin{smallmatrix} k_v & l_v \\ B & \end{smallmatrix} \mid 0 \leq k_i, l_i \leq d-1 \right\} \quad \text{where } C \text{ and } B \text{ are}$$

d -dimensional matrices obeying $CB = \omega(d) BC ; C^d = B^d = I$. The method of product transforms gives the irreducible representations of $L_i L_j = \omega(d)^r L_j L_i ; L_i^d = I \forall i, j = 1 \dots n$ as some

$$(C \begin{smallmatrix} k_1 & l_1 \\ B & \otimes \dots \otimes C \begin{smallmatrix} k_v & l_v \\ B & \end{smallmatrix}) ; v = \left[\frac{n}{2} \right], CB = \omega(d) BC, C^d = B^d = I$$

If we denote by $\{L_i \mid i = 1 \dots n\}$ one representation got by this process then due to the normalization condition $L_i^m = 1, \forall i$, the m^n different sets $\{\omega(m)^{l_{ri}} L_i \mid i = 1 \dots n; l_{ri} = 0, 1 \dots m-1; r = 1 \dots m^n\}$ are also representations. But all of these may or may not be inequivalent.

If two of these representations are equivalent then there should be a non-singular matrix S of dimension d^n connecting the two representations by equivalence transform which can be written as linear combination of the d^{2v} matrices $\left\{ \prod_{i=1}^v (\mathbb{Z}_{m_i} \otimes C^{k_i} B^{l_i}) \right\}$. But since each of these d^{2v} basis matrices induces different phase factors on similarity transformation on $\{\mathbb{Z}_{m_i}\}$ two representations will be equivalent if and only if the set of phase factor differences is a member of the class of d^{2v} sets of phase factors generated by these. Hence out of the m^n possible representations there are (m^n/d^{2v}) inequivalent representations. When $n = 2v$ there are $(m/d)^{2v}$ representations and when $n = 2v+1$ there are $m(m/d)^{2v}$ representations as was found earlier. The phase factors corresponding to different inequivalent representations can be found explicitly by taking the quotient set of the set of all m^n sets of phase factors by the set of ~~m^n~~ d^{2v} sets of phase factors corresponding to equivalent representations. The completeness of all these representations i.e. non-existence of any other representation, can be then proved by considering the group G_n^m and showing that the relation $|G_n^m| = \sum d_i^2$ is satisfied exactly. In the next chapter we shall explain in detail this procedure and use it to obtain all the inequivalent projective representations of the finite abelian group $\mathbb{Z}_{m_1} \otimes \dots \otimes \mathbb{Z}_{m_n}$.

(vi) On certain limiting cases of G_n^m

When $m \rightarrow \infty$ in the representations of the group G_n^m we get relations of the type

$$\underset{i < j}{\sum} L_i L_j = L_0 L_j L_i ; i, j = 1 \dots n.$$

$$\lim_{m \rightarrow \infty} L_i^m = 1 \quad \forall i = 1 \dots n \quad (4.91)$$

$$\lim_{m \rightarrow \infty} L_0^m = 1.$$

Thus L_0 can take all values $\exp(2\pi i \xi)$, with rational $\xi \in [0, 1)$

Consider a particular value of $\xi = \frac{l}{p}$; $(l, p) = 1$ corresponding to this case we have

$$\underset{i < j}{\sum} L_i L_j = \omega(p)^l L_j L_i ; i, j = 1 \dots n \quad (4.92)$$

$$\lim_{m \rightarrow \infty} L_i^m = 1 ; \quad \forall i = 1 \dots n$$

so the matrices obeying

$$\underset{i < j}{\sum} L_i L_j = \omega(p)^l L_j L_i ; i, j = 1 \dots n. \quad (4.93)$$

$$L_i^m = 1 ; \quad p | m \quad \forall i = 1 \dots n.$$

are represented by p^v -dimensional matrices and there are $\left(\frac{m}{p}\right)^{2v}$ or $m \left(\frac{m}{p}\right)^{2v}$ inequivalent representations for $n = 2v$ and $n = 2v+1$ respectively which are given by

$$\left\{ \omega(m)^{\delta_{ri}} L_i \mid i = 1 \dots 2v, 0 \leq \delta_{ri} \leq \left(\frac{m}{p}-1\right); r = 1 \dots \left(\frac{m}{p}\right)^{2v} \right\} \quad (4.94)$$

and

$$\left\{ \omega(m)^{\delta_{ri} + \frac{mt}{p}} L_i \mid i = 1 \dots 2v+1, 0 \leq \delta_{ri} \leq \left(\frac{m}{p}-1\right); r = 1 \dots \left(\frac{m}{p}\right)^{2v+1}; 0 \leq t \leq p-1 \right\} \quad (4.95)$$

where L_i 's are any representation of

$$L_i L_j = \omega(p)^l L_j L_i ; i, j = 1 \dots n.$$

$$L_i^p = 1 ; \forall i = 1 \dots n \quad (4.96)$$

Since (4.92) is the limiting case of (4.93) as $m \rightarrow \infty$ according to (4.94) and (4.95) there are infinite representations. In this case

as $\lim_{m \rightarrow \infty} \omega(m)^{\delta_{ij}} \quad (0 \leq \delta_{ij} \leq \frac{m}{p}-1)$ become quasi-continuous and take on all values $\exp(2\pi i \eta_j), \quad (0 \leq \eta_j < \frac{1}{p})$. Thus the representations of (4.92) are given by

$$\left\{ \exp(2\pi i \eta_i) L_i \mid i = 1, \dots, n ; 0 \leq \eta_i < \frac{1}{p} \right\} \quad (4.97)$$

and

$$\left\{ \exp 2\pi i \left(\eta_i + \frac{t}{p} \right) L_i \mid i = 1 \dots n+1 ; 0 \leq \eta_i < \frac{1}{p} ; 0 \leq t \leq p-1 \right\} \quad (4.98)$$

for $n = 2N$ and $n = 2N+1$ respectively. Thus representations of

$\lim_{m \rightarrow \infty} G_n^m$ are given by (4.97) and (4.98) corresponding to each value of $(0 \leq \xi = \frac{t}{p} < 1)$ for L_0 .

Next let us consider the relations

$$\underset{i < j}{L_i L_j} = \exp(2\pi i \xi) L_j L_i ; i, j = 1 \dots n. \quad (4.99)$$

where ξ is any irrational $< 1, > 0$.

We have

$$L_i L_j^m = \exp(2\pi i m \xi) L_j^m L_i \quad (4.100)$$

$\forall i = 1 \dots n \text{ for any } j.$

But since $\eta \pmod{1} = m\xi$ is always irrational for any $m \leq \infty$, no finite m exists such that $L_j^m = 1$; $\forall j$. Thus the group of elements generated by products of $\{L_0 = \exp(2\pi i \xi), L_1, L_2, \dots, L_n\}$ is free infinite discrete group G and if H is any normal subgroup of it then $|H| = \infty$ as well as $|G/H| = \infty$. Such groups have been called non-type I groups and representation theory of such groups is not well-developed. Using the analogy of the relations (4.100) with the usual relations of the type (4.9c) we can develop formally a representation taking the basic C and B matrices, out of which L_i 's are built by direct products, to be now

$$C_\infty = \begin{pmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad B_\infty = \begin{pmatrix} 1 & \exp 2\pi i \xi & 0 \\ 0 & \exp 4\pi i \xi & 0 \\ 0 & 0 & \ddots \end{pmatrix}$$

(4.101)

It is seen evidently that these are representations only in the sense of a limiting case of $C_\infty = \lim_{m \rightarrow \infty} C_m$ and $B_\infty = \lim_{m \rightarrow \infty} B_m$ where C_m and B_m are m dimensional matrices where m is denominator of a rational approximation to ξ .

(viii) More general α -commutation relations and associated group structures.

So far we considered ordered α -commutation relations with

$\underset{i < j}{L_i L_j} = \omega L_j L_i ; i, j = 1 \dots n$. When we have general α -commutation relations of the type

$$L_i L_j = \omega(m) t_{ij}^{m_i} L_j L_i ; \quad i, j = 1 \dots n. \quad (4.102)$$

and corresponding normalization conditions as

$$L_i^{m_i} = 1 ; \quad \forall i = 1 \dots n. \quad (4.103)$$

The consistency of the two require

$$\omega(m) t_{ij}^{m_i} = \omega(m) t_{ij}^{m_j} = 1 ; \quad \forall i, j = 1 \dots n. \quad (4.104)$$

For all these relations to be satisfied the $\{t_{ij}, m_i, m_j, m | i, j = 1 \dots n\}$ should be restricted as follows

$$\forall i, j \quad \frac{t_{ij}}{m} = \frac{\gamma_{ij}}{K_{ij}}, (\gamma_{ij}, K_{ij}) = 1 \quad K_{ij} | l_{ij}, l_{ij} = (m_i, m_j)$$

$$m = (\text{l.c.m. of all } K_{ij})$$

Such type of such relationships as (4.102-103) generate a Generalised

Clifford algebra with a basis $\left\{ \prod_{i=1}^n L_i^{k_i} \mid 0 \leq k_i \leq m_i - 1 \right\}_{i=1 \dots n}$

$\prod_{i=1}^n m_i$ elements. These basic elements are seen to form a 'ray group' i.e. groups with multiplication property

$$xy = \xi(x, y)z \quad ; \quad x, y, z \in G \quad (4.105)$$

By completing the basis, adding more elements, one can readily form a vector group which obeys the multiplication rule $xy = z; x, y, z \in G$.

$\text{if } \xi(x, y) = 1 \quad \forall x, y \in G$. Such a group structure G would contain

$(m \prod_{i=1}^n m_i)$ elements which are given by

$$\left\{ \prod_{i=0}^n L_i^{k_i} \mid 0 \leq k_i \leq m_i - 1 ; 0 \leq k_0 \leq m - 1. L_0 = \omega(m) \right\}_{i=1 \dots n}$$

Denoting an element $\left\{ \prod_{i=0}^n L_i^{k_i} \right\}$ by $g(k_0, k_1, \dots, k_n)$ the product of two such elements is given by

$$\begin{aligned} g(k_0, k_1, \dots, k_n) g(j_0, j_1, \dots, j_n) \\ = g((k+j)_0, (k+j)_1, \dots, (k+j)_n) \end{aligned} \quad (4.106)$$

where

$$(k+j)_i \bmod m_i = k_i + j_i \quad \forall i = 1 \dots n.$$

$$(k+j)_0 \bmod m = k_0 + j_0 + t_0 \quad (4.107)$$

with

$$t_0 \bmod m = (k_1, \dots, k_n) \begin{pmatrix} 0 & t_{12} & t_{13} & \dots & t_{1n} \\ 0 & 0 & t_{23} & \dots & t_{2n} \\ \vdots & \vdots & 0 & \dots & t_{n,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \\ \vdots \\ j_n \end{pmatrix} \quad (4.108)$$

From this we have

$$\begin{aligned} g(j_0, j_1, \dots, j_n) g(k_0, k_1, \dots, k_n) g(j_0, j_1, \dots, j_n)^{-1} \\ = g(K, k_1, \dots, k_n) \end{aligned} \quad (4.109)$$

where

$$K \bmod m = k_0 + S$$

$$S \bmod m = (k_1, \dots, k_n) \begin{pmatrix} 0 & t_{12} & t_{13} & \dots & t_{1n} \\ -t_{12} & 0 & t_{23} & \dots & t_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -t_{1n} & -t_{2n} & -t_{3n} & \dots & 0 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \\ \vdots \\ j_n \end{pmatrix} \quad (4.110)$$

Now compared to the case of $m_i = m, \forall i = 1 \dots n$, the counting of classes becomes still more difficult and a direct analysis is seldom

possible. But there are very interesting number theoretic problems arising here. In the case of G_{2V}^m we found that an element $g(k_0, k_1, \dots, k_{2V})$ obeying $g^p \sim I$ with p being minimum gives rise to $(\frac{m}{p})$ classes and if N_p is the number of such elements for a given $p|m$ then there is an interesting identity $\sum_{p|m} N_p \frac{m}{p}$

$$= \sum_{l=1}^m (l, m)^{2V} * \text{Actually } N_p \text{ being the number of } 2V \text{-tuples } (k_1 \dots k_{2V}), k_i \leq m, \forall i=1 \dots 2V \text{ obeying } k_i p = 0 \pmod{m}, \forall i=1 \dots n$$

with p as minimum it is the so called Jordan's totient function

$$J_{2V}(p) = \text{no. of } 2V\text{-tuples such that the greatest common divisor of } (k_1 \dots k_{2V}, m) = \frac{m}{p} \text{ which is a generalization of the Euler function } \phi(p) = \text{no. of } 1 \leq k \leq m \text{ with } (k, m) = \frac{m}{p}.$$

In the above case when m_i 's are different for $i=1 \dots 2V$ as similar analysis of class structure should yield a generalization of Jordan's function of the type $J_{2V}^*(p) = \text{no. of } 2V\text{-tuples } (k_1 \dots k_{2V}); 1 \leq k_i \leq m_i, \forall i \text{ such that } k_i p = 0 \pmod{m_i}, \forall i$ and a corresponding identity should result by equating the number of classes and number of representations (found otherwise, since even if a formula is obtained for number of classes usually it is a sum over certain set of integers which is again a difficult affair).

In the case of the group G generated by (4.102-103) the commutation matrix is

$$T = \begin{bmatrix} 0 & t_{12} & t_{13} & \cdots & t_{1n} \\ -t_{12} & 0 & t_{23} & \cdots & t_{2n} \\ \vdots & & & & \\ -t_{1n} & -t_{2n} & -t_{3n} & \cdots & 0 \end{bmatrix} \quad (4.111)$$

and let its skew canonical form be

$$T^* = \begin{bmatrix} 0 & t_1 & & \\ -t_1 & 0 & t_2 & \\ & 0 & t_3 & \\ 0 & & -t_3 & 0 \end{bmatrix} \quad (4.112)$$

As in the case of G_n^m all the representations of the G_r must

arise from the m values for $L_0 = \omega(m)^l$; $0 \leq l \leq m-1$ in

$$L_i L_j = L_0 L_j L_i \quad i, j = 1 \dots n \quad \text{or these arise from relations}$$

associated with commutation matrices $\{L T \mid l=0, 1, \dots, m-1\}$.

Corresponding to $l=0$, the one dimensional representations arise

which are obtained by putting $L_0 = 1$ and $L_i = \omega(m_i)^{l_i}$; $0 \leq l_i \leq m_i-1$.
Thus there are $\prod_{i=1}^n m_i$ one dimensional representations. As ex-

plained in the next Chapter corresponding any $1 \leq l \leq m-1$,

there are $\left(\prod_{i=1}^n m_i\right) / \left(\prod_{i=1}^s \frac{m}{(m, l t_i)}\right)^2$ representations of dimension

$\left(\prod_{i=1}^s \frac{m}{(m, l t_i)}\right)$. Thus there are totally

$$N_R = \sum_{l=0}^{m-1} \left\{ \left(\prod_{i=1}^n m_i\right) / \left(\prod_{i=1}^s \frac{m}{(m, l t_i)}\right)^2 \right\} \equiv \left(\prod_{i=1}^n m_i\right)^{m-1} \sum_{l=0}^{m-1} \left(\prod_{i=1}^s \frac{m}{(m, l t_i)}\right)^2$$

and hence so many classes. When $m_i = m$; $\forall i = 1 \dots n$, we have

$s = v$, $t_i = 1 \forall i = 1 \dots v$ for both $n = 2v$ and $n = 2v+1$ and $N_R = \sum_{l=0}^{m-1} (m, l)^{2v}$

$$\text{for } n = 2v \quad N_R = m \sum_{l=0}^{m-1} (m, l)^{2v} \quad \text{for } n = 2v+1.$$

In these types of structures the case of $n = 3$ has become important in the study of the problem of Bloch electrons in homogeneous magnetic field and this will be considered in detail in Chapter VI.

viii) Some More Results Theory.

Continuing the matrix version of the Möbius inversion (4.29) let us observe the following interesting number theoretic relations. Generally if an arithmetic function $f(n)$ defined through the Dirichlet product

$$\sum_{d|n} g\left(\frac{n}{d}\right) f(d) = h(n)$$

when $g(n)$ and $h(n)$ are known, we can write it in matrix form as

$$G(F) = (H)$$

where G is the matrix associated to function $g(n)$ by

$$G_{nd} = \begin{cases} g\left(\frac{n}{d}\right) & \text{if } d|n \\ 0 & \text{if } d \nmid n. \end{cases}$$

and (F) and (H) are the column vectors with elements as $f(n)$ and $h(n)$ respectively. Noting that G is a lower triangular matrix with $G_{mm} = g(1)$, the condition for invertibility of G becomes $g(1) \neq 0$. Hence if $g(1) \neq 0$ then we can find G^{-1} and write

$$(F) = G^{-1}(H)$$

Remarkably G^{-1} is also a matrix of the same type as G namely,

$$G^{-1}_{nd} = \begin{cases} \neq 0 & \text{if } d|n \\ 0 & \text{if } d \nmid n. \end{cases} \text{ Thus we can write}$$

$$f(n) = \sum_{d|n} g^{-1}\left(\frac{n}{d}\right) h(d)$$

where $g^{-1}\left(\frac{n}{d}\right) = G^{-1}_{nd}; \forall d|n$. Without going into the details of proof

let me assert that

$$g^{-1}(n) = \sum_{s=1}^{n-1} \frac{(-1)^s}{g(1)^{s+1}} \left\{ \sum_{*} \frac{s!}{\prod_{i=1}^r (s_i)!} \left\{ \prod_{i=1}^r g(n_i)^{s_i} \right\} \right\}$$

where * denotes summation over all s_i such that $\sum_{i=1}^r s_i = s$
and $\prod_{i=1}^r n_i^{s_i} = n$ with distinct n_i .

Let us also observe the following dual to the Möbius inversion formula. Analogous to the matrix relation

$$\mathbf{G}(\mathbf{F}) = (\mathbf{H})$$

leading to the Dirichlet product of arithmetic functions let us write

$$\mathbf{\tilde{F}} \mathbf{G} = \mathbf{\tilde{H}}$$

where $\mathbf{\tilde{F}}$ is a row vector and $\mathbf{\tilde{H}}$ is also. Written in terms of matrix elements this means a product

$$\sum_{d=1}^{\infty} g(d) f(dn) = h(n)$$

Then as before if $g(1) \neq 0$ it follows that there is an inversion for f in terms of h and g namely

$$f(n) = \sum_{d=1}^{\infty} g^{-1}(d) h(dn)$$

If ∞ is trouble some one can have equally well,

if

$$\left[\frac{M}{n} \right]$$

$$\sum_{d=1}^{\left[\frac{M}{n} \right]} g(d) f(dn) = h(n) ; \quad n=1, \dots, M$$

$$d=1$$

then

$$\left[\frac{M}{n} \right]$$

$$\sum_{d=1}^{\left[\frac{M}{n} \right]} g^{-1}(d) h(dn) = f(n) ; \quad n=1, \dots, M.$$

Finding g^{-1} from G through involving lengthy computation is possible since G has a triangular structure and hence the elements of G^{-1} upto any desired row can be obtained considering only those elements in G upto that particular row.

When $g(1)=0$ let us call the arithmetic function as singular which does not have an inverse in the above sense. But following the development of pseudo-inverses or generalized inverses for singular matrices considered by Moore, Penrose and C.R.Rao ⁽⁴⁾ - one should be able to arrive at pseudo-inverses of arithmetic functions and corresponding inversion formulas for Dirichlet type products of arithmetic functions. Further developments in this direction will be published elsewhere.

Summary of important points.

The G.C.G., \mathbb{G}_n^m has the following properties when m is any integer. Total number of elements $= m^{n+1}$. Total number of conjugate classes $= \sum_{v=1}^m (\ell, m)^{2v}$ for $n = 2v$, and

$$\sum_{\ell=1}^m (\ell, m)^{2v} \quad \text{for } n = 2v+1, \text{ where } (\ell, m) = \text{g.c.d of } \ell \text{ and } m.$$

Just as in the case of G_n^m , for prime n , all the representations arise due to the various permitted values of $\frac{L_0}{k}$ given by $\left\{ \exp\left(\frac{2\pi i l}{m}\right) \mid 0 \leq l \leq m-1 \right\}$. Corresponding to $l=0$ are the m^n one dimensional representations, same as those of $\mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m$ (n copies) for both $n=2v$ and $n=2v+1$. When $l \neq 0$, there are $m(l, m)^{2v}$ representations of same dimension $m^v/(l, m)^v$ for $n=2v+1$ and for $n=2v$ there are $(l, m)^{2v}$ representations of same dimension $m^v/(l, m)^v$. All these inequivalent irreducible representations have been explicitly constructed in this chapter and direct product representations are studied. The limiting case of $m \rightarrow \infty$ in G_n^m is studied and also G.C.G - type of group structures associated to the generating relations

$$L_i L_j = b^{tij} L_j L_i; L_0 L_j = L_j L_0; L_i^{m_i} = 1; i, j = 1 \dots n.$$

are analysed, where L_0 and b^{tij} are such that $L_0^{tij m_i} = L_0^{tij m_j} = 1$. Thus

$$L_0^{tij} = \exp\left\{2\pi i \frac{b^{tij}}{(m_i, m_j)}\right\} = \exp\left(2\pi i \frac{r_{ij}}{K_{ij}}\right), (r_{ij}, K_{ij}) = 1, \forall i, j = 1 \dots n.$$

letting

$$m = \text{lcm}\{K_{ij}\}; L_0^{tij m} = 1 \forall i, j. \quad \text{This group has } \prod_{i=1}^n m_i \text{ one dimensional}$$

representations same as those of $\mathbb{Z}_m \otimes \dots \otimes \mathbb{Z}_m$ and higher dimensional representations arise corresponding to all the $(n-1)$ values of

$$L_0 = \exp\left(2\pi i \frac{l}{m}\right); \quad 0 \leq l \leq m-1. \quad \text{For a particular value of } l \text{ there are}$$

$$\left(\prod_{i=1}^n m_i \right) / \left(\prod_{i=1}^n \frac{m}{(m, lt_i)} \right)^2 \quad \text{representations of dimension} \quad \left(\prod_{i=1}^n \frac{m}{(m, lt_i)} \right)$$

where t_i 's are elements of the skew normal form of $T = (t_{ij})$.

CHAPTER 5ON PROJECTIVE REPRESENTATIONS OF ABELIAN GROUPS

Here we develop a simple procedure for explicitly determining all the irreducible inequivalent projective representations of finite abelian groups using the concept of 'product transform' introduced by us recently¹. This procedure is a generalization of 'tenon-and mortise coupling method' of representation of Clifford algebra, studied by Alladi Ramakrishnan² during the development, by him and his collaborators, of L-Matrix Theory dealing with properties and applications of Clifford algebra and its generalizations.

Projective representations of groups were first studied by Schur^{3,4} in a series of definitive papers and general methods of representations were developed. But surprisingly even for Abelian finite finite groups there does not exist any explicit procedure of determining all the irreducible inequivalent projective representations. Except in a few special cases of groups of small order and simple factor sets application of little group technique of Wigner⁵ or the same so called induced representation technique of Frobenius⁶, Mackey⁷ and Clifford⁸, becomes too complicated to handle. Hence special simpler methods to deal with particular group structures are of interest. In this connection Generalized Clifford algebras were introduced and studied in different fashions by Noriaga and Nono⁹, Kenmotsu¹⁰, Morris¹¹ and Popovici and Turtoi¹². By now it is clear from their work that linear representations of all G.C.A's associated with a finite Abelian group $\mathbb{Z}_{m_1} \otimes \dots \otimes \mathbb{Z}_{m_n}$ provide all the projective representations of G. But so far only certain special G.C.A's have been studied explicitly since the work of Herman Weyl¹³.

who first used basic relations of a G.C.A in interpreting the fundamental laws of quantum mechanics as ray (projective) representations of Abelian groups and proving the uniqueness of Schrodinger representation of position and momentum operators in quantum mechanics. He has solved the problem of projective representations of continuous Abelian groups in terms of canonically conjugate pairs of operators obeying Heisenberg commutation relations¹³. But in the case of finite Abelian groups till recently there is no explicit procedure. The basic principle of the following method is essentially the same as that used by Moyl¹³ in the case of continuous groups, but a great difference arises due to the finite nature of the group bringing in difficulties in details. Only recently Backhouse and Bradley¹⁴ have determined the dimensions of representations for arbitrary choice of factor systems. Our procedure determines all the representations explicitly in terms of simple matrices.

(i) Projective representations of finite Abelian groups and Generalized Clifford algebras.

First let us recall some elements of Schur's theory of projective representations. A representation D of a group G is called a projective representation when

$$\begin{aligned} D(g_i)D(g_j) &= \zeta(g_i, g_j) D(g_i g_j), \quad \forall g_i, g_j \in G \\ D(g_0) &= I \end{aligned} \tag{5.1}$$

where g_0 is the identity element of G . $\zeta(g_i, g_j)$'s are elements of the field of complex numbers $\mathbb{C}^* = \mathbb{C} - \{0\}$, for us. When $\zeta(g_i, g_j) = 1, \forall g_i, g_j \in G$, D is the ordinary or linear representation of G . Associativity of G requires

$$\varsigma(g_i, g_j) \varsigma(g_i g_j, g_k) = \varsigma(g_i, g_j g_k) \varsigma(g_j, g_k) \\ + g_i, g_j, g_k \in G \quad (5.3)$$

Also

$$\varsigma(g_0, g_j) = \varsigma(g_j, g_0) = \varsigma(g_0, g_0) = 1 \quad \forall g_j \in G$$

$$\varsigma(g_j, g_j^{-1}) = \varsigma(g_j^{-1}, g_j) \quad (5.3)$$

The set $\Sigma = \{\varsigma(g_i, g_j) \mid \forall g_i, g_j \in G\}$ is called a factor set of G in C^* and two factor sets ς , and η are equivalent if there exists a mapping $\mu : G \rightarrow C^*$ such that $\mu(g_0) = 1$ and

$$\eta(g_i, g_j) = \frac{\mu(g_i) \mu(g_j)}{\mu(g_i g_j)} \varsigma(g_i, g_j); \quad \forall g_i, g_j \in G \quad (5.4)$$

Let $M(G, C^*)$ denote the set of all factor sets of G in C^* and it is an Abelian group with the product $\varsigma \circ \eta(g_i, g_j) = \varsigma(g_i, g_j) \eta(g_i, g_j)$

A factor set $\varsigma \in M(G, C^*)$ is said to be normalized if $\varsigma(g_i, g_i^{-1}) = 1 \quad \forall g_i \in G$, in which case $D(g_i)^{-1} = D(g_i^{-1})$. Any factor

set $\varsigma \in M(G, C^*)$ can be normalized by an equivalence transformation (5.4) with $\mu(g_i) = \frac{1}{\sqrt{\varsigma(g_i, g_i^{-1})}} \quad \forall g_i \in G$. It is known that for any $\varsigma \in M(G, C^*)$ there is a projective representation.

The relation of equivalence of two factor sets given by (5.4) is easily verified to be an equivalence relation on $M(G, C^*)$. Let

$\{\varsigma\}$ denote the equivalence class on which contains $\varsigma \in M(G, C^*)$.

The set of all equivalence classes denoted by $H^2(G, C^*)$ is called a finite Abelian group with the product $\{\varsigma\}\{\eta\} = \{\varsigma \circ \eta\}, \forall \varsigma, \eta \in M(G, C^*)$ and is called the Schur multiplier of G in C^* . A group C^* , called a representation group, can be constructed by central extension of G with $H^2(G, C^*)$ as kernel of the extension such that

$$G^*/H^2(G, C^*) \cong G, H^2(G, C^*) \subseteq Z(G) \text{ and every projective representation}$$

of \mathbb{G} is obtained as a linear representation of \mathbb{G}^0 . (For a detailed account of the subject cf. Morris¹⁵).

Let us consider a finite Abelian group $G \cong \mathbb{Z}_{m_1} \otimes \cdots \otimes \mathbb{Z}_{m_n}$ where \mathbb{Z}_{m_i} is a cyclic group of order m_i . Any element $g_i \in G$ can be written as $\prod_{i=1}^n e_i^{k_i}$, $1 \leq k_i \leq m_i$, $\forall i = 1 \dots n$ where e_i 's are generators of \mathbb{G} obeying

$$\begin{aligned} e_i^{m_i} &= 1 & \forall i = 1 \dots n \\ e_i e_j &= e_j e_i & \forall i, j = 1 \dots n \end{aligned} \quad (5.5)$$

Let ς be a factor set. Define another associated set ω_ς by

$$\omega_\varsigma(g_i, g_j) = \frac{\varsigma(g_i, g_j)}{\varsigma(g_j, g_i)}, \quad \forall g_i, g_j \in G \quad (5.6)$$

It is easy to see that $\omega_\varsigma \equiv \omega_\eta$ if ς and η are equivalent.

Thus ω_ς denotes the ω -factor set for the entire class

$\{\varsigma\} \in H^2(G, \mathbb{C}^*)$. For Abelian groups the projective representations D can be characterized in terms of ω -factor sets as

$$\begin{aligned} D(g_i)D(g_j) &= \varsigma(g_i, g_j) D(g_i g_j) = \varsigma(g_i, g_j) D(g_j g_i) \\ &= \varsigma(g_i, g_j) \varsigma(g_j, g_i)^{-1} D(g_j) D(g_i) \\ &= \omega_k(g_i, g_j) D(g_j) D(g_i) \quad \forall g_i, g_j \in G \end{aligned} \quad k = 1, 2, \dots, |H^2(G, \mathbb{C}^*)| \quad (5.7)$$

where $|H^2(G, \mathbb{C}^*)|$ denotes the order of $H^2(G, \mathbb{C}^*)$. It follows from (5.5) and (5.6) that

$$\begin{aligned} \omega(g_i, g_j) \omega(g_i g_j, g_k) &= \omega(g_i, g_j g_k) \omega(g_j, g_k) \\ &\quad \forall g_i, g_j, g_k \in G \end{aligned} \quad (5.8)$$

and

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$$\omega(g_0, g_i) = \omega(g_i, g_0) = \omega(g_i, g_i^{-1}) = \omega(g_i^{-1}, g_i) = 1 \quad \forall g_i \in G \quad (5.9)$$

$$\omega(g_i, g_j) = \omega(g_j, g_i)^{-1}; \forall g_i, g_j \in G \quad (5.10)$$

(5.9)-(5.10) give

$$\begin{aligned} \omega(g_i, g_j g_k) &= \omega(g_i, g_j) \omega(g_i, g_k); \forall g_i, g_j, g_k \in G \\ \omega(g_i, g_i^k) &= 1 \quad \forall g_i \in G \end{aligned} \quad (5.11)$$

Hence the set $\{\omega_k(e_i, e_j) | i, j = 1 \dots n\}$ completely specifies the set $\{\omega(g_i, g_j) | \forall g_i, g_j \in G\}$. Since a representation D_ζ of G for the factor set ζ obeys (5.7) as a first stage of determining D_ζ , we can consider the problem of determination of a representation L_{ω_ζ} such that

$$L(g_i)L(g_j) = \omega_\zeta(g_i, g_j) L(g_j)L(g_i); \forall g_i, g_j \in G \quad (5.12)$$

Denoting $L(e_i) = L_i$, $\forall i = 1 \dots n$, it is enough to determine first L_i 's since $L(g_i)$, $\forall g_i \in G$ can be constructed from these as the following procedure will make it clear. These L_i 's obey from (5.12)

$$L_i L_j = \omega_{ij} L_j L_i; i, j = 1 \dots n \quad (5.13)$$

denoting $\omega_{ij} = \omega_\zeta(e_i, e_j)$. From (5.1) it follows that we can write

$$D(g_i g_j) = \zeta(g_i, g_j)^{-1} D(g_i) D(g_j) \quad (5.14)$$

and hence

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$$D\left(\prod_{i=1}^n e_i^{k_i}\right) = \varsigma(e_i, \prod_{l=2}^n e_l^{k_l-1} e_i^{k_l})^{-1} D(e_i) D\left(\prod_{i=2}^n e_i^{k_i-1} e_i^{k_i}\right) \quad (5.15)$$

and so on, finally leading to

$$D\left(\prod_{i=1}^n e_i^{k_i}\right) = \left[\prod_{s=1}^n \left\{ \prod_{r_s=1}^{k_s} \varsigma\left(e_s, \prod_{l=s+1}^n e_l^{k_l-r_s} e_i^{k_i}\right)^{-1} \right\} \right] \times \left(\prod_{i=1}^n D(e_i)^{k_i} \right) \quad (5.16)$$

Thus

$$D(e_i^{m_i}) = \left\{ \prod_{k_i=1}^{m_i} \varsigma(e_i, e_i^{k_i})^{-1} \right\} D(e_i)^{m_i} = I, \text{ by (5.1)} \quad (5.17)$$

Defining

$$K_i = \left(\prod_{k_i=1}^{m_i-1} \varsigma(e_i, e_i^{k_i})^{-1} \right)^{-1} = \left(\prod_{k_i=1}^{m_i-1} \varsigma(e_i, e_i^{k_i}) \right) \quad (5.18)$$

we have

$$D(e_i)^{m_i} = K_i \cdot I \quad (5.19)$$

and K_i 's are uniquely fixed by ς . A recent theorem due to Bachhouse and Bradley¹⁴ is of importance for further considerations. They have shown that if D_1 and D_2 are two representations of G

with the same factor set ζ , then there exist a unitary transformation U and a linear character χ such that

$$U^{-1} D_1(g_i) U = \chi(g_i) D_2(g_i) ; \forall g_i \in G \quad (5.10)$$

$$\chi(g_i)\chi(g_j) = \chi(g_i g_j) ; \forall g_i, g_j \in G \quad (5.11)$$

χ is thus a one dimensional linear representation of G , and is generated by

$$\chi(e_i) = \exp(2\pi i l_i/m_i) \quad ; 0 \leq l_i \leq m_i \quad (5.12)$$

There are only $\left(\prod_{i=1}^n m_i\right)$ possible choices of χ and hence by the above theorem all the inequivalent (linearly) projective representations of G for a given factor set ζ are contained in the set of

$$\left(\prod_{i=1}^n m_i\right)^n \text{ representations given by } \left\{ \chi_\gamma(g_i) D(g_i) \mid \forall g_i \in G; \gamma = 1 \dots \prod_{i=1}^n m_i \right\} \quad \text{where } \left\{ D(g_i) \mid \forall g_i \in G \right\}$$

representations of G for the given factor set ζ . All these representations may not be inequivalent and we have to choose them by some method which we shall consider later. Hence without loss of generality we can take $D(e_i) = L_i$, $\forall i = 1 \dots n$ which generates a representation of G for the factor set ζ by (5.10). From (5.10) L_i 's satisfy

$$L_i L_j = w_{ij} L_j L_i ; \forall i, j = 1 \dots n \quad (5.13)$$

$$L_i^{m_i} = K_i I ; \forall i = 1 \dots n$$

This shows that $w_{ij}^{m_i}$ cannot take arbitrary values. They must satisfy

$$w_{ij}^{m_i} = w_{ij}^{m_j} = 1 \quad (5.24)$$

or

$$w_{ij} = \exp(2\pi i \gamma_{ij}/k_{ij}) \quad , \quad (\gamma_{ij}, k_{ij}) = 1 \\ k_{ij} | l_{ij} \quad , \quad l_{ij} = (m_i, m_j) \quad ; \quad \forall i, j = 1 \dots n \quad (5.25)$$

These show that since w_{ij} can take l_{ij} values given by

$\{\exp(2\pi i \lambda_{ij}/l_{ij}) \mid 1 \leq \lambda_{ij} \leq l_{ij}\}$ there are totally $\prod_{i,j=1, (i < j)}^n l_{ij}$ n -factor sets and hence so many are the equivalence classes of factor sets in $M(G, \mathbb{C}^*)$, i.e. order of $H^2(G, \mathbb{C}^*)$ given by

$$|H^2(G, \mathbb{C}^*)| = \left(\prod_{\substack{i,j=1 \\ (i < j)}}^n l_{ij} \right) = \left\{ \prod_{\substack{i,j=1 \\ (i < j)}}^n (m_i, m_j) \right\} \quad (5.26)$$

Defining a matrix W by $W_{ij} = w_{ij} ; \forall i, j = 1 \dots n$ it is a Hermitian matrix, $W^+ = W$, and all the $\left(\prod_{\substack{i,j=1 \\ (i < j)}}^n (m_i, m_j) \right)$ matrices W form the group $\cong H^2(G, \mathbb{C}^*)$ under the Hadamard product $(W_1 \circ W_2)_{ij} = (W_{1,ij} W_{2,ij})$

$\forall i, j = 1 \dots n$. We see that corresponding to fix a factor set ς , the representation can be specified in terms of representations of L_i 's obeying (5.25) and all the inequivalent representations are contained in a set of $\left(\prod_{i=1}^n m_i \right)$ representations which arise from taking different values for $\{\chi(\ell_i)\}$ in $\{L_i = \chi(\ell_i) \tilde{L}_i \mid \forall i = 1 \dots n\}$ where $\{\tilde{L}_i\}$ is any one representation of (5.23), and $\chi(\ell_i) = \exp(2\pi i \ell_i/m_i)$

$1 \leq i \leq m$, taking two sets of values for $\{\chi(e_i)\}$, say $\{\chi_1\}, \{\chi_2\}$ the two representations would be equivalent if there exists a matrix S such that

$$S \chi_1(e_i) \tilde{L}_i S^{-1} = \chi_2(e_i) \tilde{L}_i, \quad \forall i=1\dots n \quad (5.27)$$

or

$$S \tilde{L}_i S^{-1} \tilde{L}_i^{-1} = \chi_2(e_i) \chi_1(e_i)^{-1} = \Psi(e_i) I \quad (5.28)$$

This implies that two representations with $\{\chi_1(e_i)\}$ and $\{\chi_2(e_i)\}$ as sets of coefficients will be equivalent iff the set $\{\Psi(e_i)\} =$

$\{\chi_2(e_i) \chi_1(e_i)^{-1}\}$ is a member of the set of sets

$$\Psi \equiv \{(S \tilde{L}_i S^{-1} \tilde{L}_i^{-1}) \mid i=1\dots n\} \quad \forall S \in M_D(\mathbb{C}), \exists S \tilde{L}_i S^{-1} \tilde{L}_i^{-1} \quad \forall i=1\dots n$$

where $M_D(\mathbb{C})$ is the total matrix algebra over \mathbb{C} of dimension $D =$ the dimension of representation of \tilde{L}_i 's. So if this

set Ψ is determined then the sets $\{\chi(e_i)\}$ belonging to inequivalent representations can be identified with members of different classes under the equivalence relation that $\{\chi_1(e_i)\}$ is equivalent to $\{\chi_2(e_i)\}$ if $\{\chi_1(e_i) \chi_2(e_i)^{-1}\} \in \Psi$. Then the number of inequivalent representations is given by $(\prod_{i=1}^n m_i) / |\Psi|$, where $|\Psi|$ is the number of elements in Ψ .

Thus our primary task is to determine a representation of \tilde{L}_i obeying (5.28) and the corresponding set Ψ . In (5.28) the second relation $\tilde{L}_i^{m_i} = K_i I, i=1\dots n$ is a normalization condition for and for a given factor set ζ, K_i are fixed uniquely and

hence without loss of generality instead of (5.29) we can consider the problem of representation of

$$L_i L_j = \omega_{ij} L_j L_i ; i, j = 1 \dots n$$

$$L_i^{m_i} = 1 ; i = 1 \dots n \quad (5.30)$$

$$\omega_{ij}^{m_i} = \omega_{ij} ; i, j = 1 \dots n.$$

It is seen easily that the set $\left\{ \prod_{i=1}^n L_i^{k_i} \mid 0 \leq k_i \leq m_i - 1 \right\}_{i=1}^n$ forms

the basis for an algebra. The product of two elements of the basis is given by

$$\left\{ \prod_{i=1}^n L_i^{k_i} \right\} \left\{ \prod_{i=1}^n L_i^{t_i} \right\} = \left\{ \prod_{j=1}^n \left(\prod_{i=0}^{n-j-1} \omega_{n-i, j}^{k_{n-i} + t_j} \right) \right\} \times \left\{ \prod_{i=1}^n L_i^{(k_i + t_i)} \right\} \quad (5.30)$$

where $(k_i + t_i) \bmod m_i = k_i + t_i ; i = 1 \dots n$. This algebra has been called a 'Generalised Clifford Algebra'. When $m_i = 2, \forall i = 1 \dots n$ and $\omega_{ij} = -1 ; \forall i, j = 1 \dots n$ this algebra is seen to be the Clifford algebra $C_n^{(2)}$ generated by n mutually anticommuting elements. So far representation theory of G.C.A.'s only with $m_i = m, \forall i = 1 \dots n$ and $\omega_{ij} = \exp(2\pi i l/m), \forall i, j = 1 \dots n, 0 \leq l \leq \frac{m-1}{2}$ have been considered in detail (2, 9, 11, 16, 17, 18).

ii) Representation theory.

Let us rewrite the equations (5.29) as

$$L_i L_j = \omega(m)^{t_{ij}} L_j L_i ; i, j = 1 \dots n.$$

$$t_{ij} = -t_{ji}, L_i^{m_i} = 1 ; i = 1 \dots n \quad (5.31)$$

where

$$\omega(m) = \exp(2\pi i/m)$$

$$\omega_{ij} = \exp(2\pi i \tau_{ij}/k_{ij}) ; (\tau_{ij}, k_{ij}) = 1$$

$$k_{ij}|l_{ij} ; l_{ij} = (m_i, m_j)$$

$$m = \text{l.c.m} \{ k_{ij} \mid \forall i, j = 1, \dots, n \}$$

(5.33)

Then defining an antisymmetric integer matrix T by

$$T_{ij} = t_{ij} ; i, j = 1, \dots, n \quad (5.34)$$

The product (5.30) becomes

$$\left(\prod_{i=1}^n L_i^{k_i} \right) \left(\prod_{i=1}^n L_i^{t_i} \right) = \omega(m) S(k, t) \left(\prod_{i=1}^n L_i^{(k_i + t_i)} \right)$$

(5.34)

where

$$(k_i + t_i) \bmod m_i = k_i + t_i ; i = 1, \dots, n.$$

(5.35)

and

$$S(k, t) \bmod m = (k_1, k_2, \dots, k_n) \begin{pmatrix} 0 & t_{12} & t_{13} & \dots & t_{1n} \\ 0 & 0 & t_{23} & \dots & t_{2n} \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & t_{n-1, n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}$$

(5.36)

It follows from (5.34) that

$$\left(\prod_{i=1}^n L_i^{(k_i + t_i)} \right) = \omega(m) -S(t, k) \left(\prod_{i=1}^n L_i^{t_i} \right) \left(\prod_{i=1}^n L_i^{k_i} \right)$$

(5.37)

Substituting this in the right hand side of (5.36)

$$\left(\prod_{i=1}^n L_i^{k_i} \right) \left(\prod_{i=1}^n L_i^{t_i} \right) = w(n) S[k, t] \left(\prod_{i=1}^n L_i^{t_i} \right) \left(\prod_{i=1}^n L_i^{k_i} \right)$$

(5.36)

where

$$\begin{aligned} S[k, t] &= S(k, t) - S(t, k) \\ &= (k_1, k_2, \dots, k_n) \begin{bmatrix} & & & t_1 \\ & \vdots & & t_2 \\ & & \ddots & \\ & & & t_n \end{bmatrix} = k^T t. \end{aligned} \quad (5.37)$$

Let us call the antisymmetric integer matrix T , the commutation matrix associated with the set $\{L_i\}$.

Let $U = (u_{ij})$; $i, j = 1 \dots n$ be an integer matrix. Then let the following be the definition of a 'product transform' from the set $\{L_i\}$ to $\{L_i^*\}_n$

$$L_i^* = \prod_{k=1}^n L_k^{u_{ik}} ; \quad i = 1 \dots n \quad (5.40)$$

Any L_k^{-r} may be rewritten as $L_k^{sm_k - r}$, such that $0 \leq sm_k - r \leq m_k$. Using (5.36) the commutation relations of L_i^* 's are

$$L_i^* L_j^* = w(n) S[u_i, u_j] L_j^* L_i^* ; \quad i, j = 1 \dots n. \quad (5.41)$$

where u_m denotes the s -th row of U . Denoting $S[u_i, u_j] = t_{ij}^*$

it is seen that $t_{ij}^* = -t_{ji}^*$ and $T^* = \{L_i^*\}$ is the commutation matrix associated with the set $\{L_i^*\}$. We have

$$T^* = UT\tilde{U} \quad (5.42)$$

Ans 12 If $|\det U| = 1$, there exists the integer matrix $U^{-1} = V = (v_{ij})$ such that

$$T = VT^*\tilde{V} \quad (5.43)$$

This means that the set $\{L'_i\}$ defined by

$$L'_i = \prod_{j=1}^n L_j^{*v_{ij}} ; \quad i = 1 \dots n \quad (5.44)$$

has the same commutation matrix T as the set $\{L_i\}$. Actually substituting for $L_j^{*v_{ij}}$ in (5.44) from (5.43) it can be seen that

$$L'_i = c_i L_i \quad ; \quad i = 1 \dots n \quad (5.45)$$

where c_i 's are some n th roots of unity. This implies that if $\{L_i\}$ are irreducible so are $\{L'_i\}$ and vice versa. Because if $\{L_i^*\}$ become reducible while $\{L_i\}$ are not initially the relation (5.45) would imply $\{L_i\}$ are reducible contradictory to initial condition. Only when $\det U = 0$, $\{L_i^*\}$ become reducible starting with irreducible $\{L_i\}$ and then inverse transformation $\{L_i^*\} \rightarrow \{L_i\}$ is impossible. We are not interested in this case. Thus we notice that if an irreducible representation of $\{L_i^*\}$ is known then an irreducible representation of $\{L_i\}$ can be found readily if T^* and T are related by the relation (5.43) and with $|\det V| = 1$ and if V can be explicitly determined. Now a basic theorem on antisymmetric integer matrices comes to our help. We follow

the treatment of Morris Newman. according to this theorem for any $n \times n$ antisymmetric integer matrix T , there is an unique skew normal form given by

$$UT\tilde{U} = T^* = \sum_{i=1}^s \oplus \begin{pmatrix} 0 & h_i \\ -h_i & 0 \end{pmatrix} \oplus 0_{n-2s} \quad (5.46)$$

where $h_i | h_{i+1}, \dots, 1 \leq i \leq s-1$, $s = \text{rank of } T$, and 0_{n-2s} is a null matrix of order $n-2s$. The matrix U is an integer matrix, $|\det U| = 1$ and a U can always be explicitly constructed to satisfy (5.46) (cf. Appendix B).

A set of matrices $\{L_i^*\}$ having T^* as their commutation matrix obey the relations

$$\begin{aligned} L_{2i-1}^* L_{2i}^* &= \omega(m)^{h_i} L_i^* L_{2i-1}^* ; i=1 \dots s \\ L_k^* L_l^* &= L_l^* L_k^* \text{ otherwise; } k, l = 1 \dots n \end{aligned} \quad (5.47)$$

Let us put

$$\left. \begin{aligned} L_{2i-1}^* &= I_1 \otimes I_2 \otimes \dots \otimes I_{i-1} \otimes C_i \otimes I_{i+1} \otimes \dots \otimes I_s \\ L_{2i}^* &= I_1 \otimes I_2 \otimes \dots \otimes I_{i-1} \otimes B_i \otimes I_{i+1} \otimes \dots \otimes I_s \\ L_k^* &= I_1 \otimes \dots \otimes I_s \quad k = 2s+1, \dots, n \end{aligned} \right\} \quad (5.48)$$

where

$$y_i = (h_i, m)$$

$$z_i = m/y_i, \quad x_i = h_i/y_i ; \quad \frac{h_i}{m} = \frac{x_i}{z_i} ; \quad (x_i, z_i) = 1$$

(5.49)

Note: pages 136 and 137 have been interchanged.

next page is 136.

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Taking trace on both sides it follows immediately that the coefficient of the identity matrix $a_{00\dots 0} = 0$. Multiplying on both sides by $\left(\prod_{k=1}^n L_j^{*ik}\right)^{-1}$ and taking trace it follows $a_{j_1 j_2 \dots j_n} = 0$ since in the product on the left hand side $a_{j_1 j_2 \dots j_n}$ becomes the coefficient of the identity matrix. Thus all these matrices, D^2 totally $\left\{ \prod_{k=1}^n L_j^{*ik} = C_1^{j_1} B_1^{j_2} \otimes \dots \otimes C_s^{j_{2s}} B_s^{j_{2s}} \mid 0 \leq j_{2i-1}, j_{2i} \leq z_i - 1; i = 1 \dots s \right\}$ are linearly independent and form a basis for $M_D(\mathbb{C})$, the full matrix algebra of dimension D over \mathbb{C} . Hence this representation of \mathcal{D} is irreducible. Explicit versions of such matrix decomposition theorems have been studied by Alain Connes, Alain Valette, and Alain Ramakrishnan and myself.

Let $U^{-1} = V$, which is also a unimodular integer matrix, be constructed to satisfy

$$T = VT^* \tilde{V} \quad (5.52)$$

though U and hence V are not unique it does not matter for us since we need only one representation to start with to construct all the others which differ only in phase factors as we have seen already. Thus we form the representation

$$\begin{aligned} L'_i &= \prod_{j=1}^n L_j^{*ij} \\ &= C_1^{v_{i1}} B_1^{v_{i2}} \otimes C_2^{v_{i3}} B_2^{v_{i4}} \otimes \dots \otimes C_s^{v_{i,2s-1}} B_s^{v_{i,2s}} \end{aligned} \quad (5.53)$$

where (v_{i1}, \dots, v_{in}) is the i th row of V . As seen earlier

$$C_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} = \text{cyclic matrix of dimension } Z_i$$

$$B_i = \begin{pmatrix} 1 & & & & 0 \\ & \omega(z_i)^{\chi_i} & & & \\ & & \ddots & & \\ & & & \omega(z_i)^{\chi_i(Z_i-1)} & \\ 0 & & & & \end{pmatrix}$$

$$C_i^{Z_i} = B_i^{Z_i} = I_i ; \quad C_i B_i = \omega(z_i)^{\chi_i} B_i C_i ; \quad \forall i = 1 \dots n.$$

I_i = Identity matrix of dimension Z_i

$i = 1 \dots s$.

Hence

$$\dim(L_i^*) = \prod_{i=1}^s Z_i = \prod_{j=1}^s \frac{m}{(h_j, m)} = D. \quad (5.50)$$

It is easily verified that $\{L_i^*\}$ given by (5.48) faithfully obey the relation (5.47).

All possible products of all possible powers of $\{L_i^*\}$ are given by the set of $\prod_{i=1}^s Z_i^2 = D^2$ elements

$$\left\{ \prod_{k=1}^n L_k^{* \gamma_k} \mid 0 \leq \gamma_{2i-1}, \gamma_{2i} \leq Z_i - 1 ; i = 1 \dots s ; \gamma_k = 0 \quad \forall k = 2s+1 \dots n \right\}$$

In this set except the identity matrix all are traceless. Consider the equation

$$\sum_{\substack{(\text{sum over}) \\ \text{all } \gamma_k \text{'s}}} a_{\gamma_1 \gamma_2 \dots \gamma_n} \left(\prod_{k=1}^n L_k^{* \gamma_k} \right) = 0 \quad (5.51)$$

$\{L'_i\}$ obey faithfully the required relations

$$L'_i L'_j = \omega(m)^{t_{ij}} L'_j L'_i ; i, j = 1 \dots n. \quad (5.54)$$

Since $\omega(m)^{t_{ij}} = \omega_{ij}; i, j = 1 \dots n$ are such that $\omega_{ij}^{m_i} = 1, \forall j = 1 \dots n, L'_i$ commutes with all $L'_j; j = 1 \dots n$ and hence by Schur's lemma

$$L'_i^{m_i} = P_i I, \forall i = 1 \dots n. \quad (5.55)$$

where P_i 's can be calculated from (5.53). So now define

$$\begin{aligned} L_i &= (P_i^{-\frac{1}{m_i}}) L'_i ; \forall i = 1 \dots n \\ &= \omega(m)^{\left\{ \frac{1}{2}(m_i - 1) \sum_{j=1}^s h_j v_{i,2j-1} v_{i,2j} \right\}} \prod_{j=1}^s \otimes c_j^{v_{i,2j-1}} c_j^{v_{i,2j}} \end{aligned} \quad (5.56)$$

which obey the required relations

$$\begin{aligned} L_i L_j &= \omega(m)^{t_{ij}} L_j L_i ; i, j = 1 \dots n \\ L_i^{m_i} &= I; i = 1 \dots n. \end{aligned} \quad (5.57)$$

The irreducibility of $\{L_i\}$ follows from the irreducibility of $\{L_i^*\}$.

Consider now the set of $\prod_{i=1}^n m_i$ representations differing in phase factors only given by $\Phi \equiv \left\{ (\chi_i^{(r)} L_i) \mid \chi_i^{(r)} = \omega(m_i)^{l_i} \right.$

$0 \leq l_i \leq m_i - 1, i = 1 \dots n, r = 1 \dots \prod_{i=1}^n m_i \}$ all of which satisfy (5.55). We

found earlier that all the inequivalent representations of the generators of G for the factor set Σ , associated with the m -factor set

generated by $\{\omega_{ij}\}$, are contained in this set Φ and two representations with the sets of coefficients $\{x_i^{(1)}\}$ and $\{x_i^{(2)}\}$ were found to be equivalent iff the set $\{\psi_i^{(2)} = x_i^{(1)} x_i^{(2)-1}\}$ is a member of the set of sets of coefficients.

$$\Psi \equiv \left\{ \left(SL_i S^{-1} L_i^{-1} \mid i=1 \dots n \right) \mid \forall S \in M_D(\mathbb{C}), \exists \begin{array}{c} SL_i S^{-1} L_i^{-1} \sim I \\ \forall i=1 \dots n \end{array} \right\}$$

Now any $S \in M_D(\mathbb{C})$ can be written as a linear sum of the D^2 matrices $\{S_{(j)} \equiv S_{j_1 \dots j_D} = \prod_{i=1}^D \otimes C_i^{j_{2i-1}} B_i^{j_{2i}} \mid 0 \leq j_{2i-1}, j_{2i} \leq z_i^{-1}\}$

We observe that

$$S_{(j)} L_i \sim L_i S_{(j)} \text{ or } S_{(j)} L_i S_{(j)}^{-1} L_i^{-1} \sim I$$

$$S_{(j)}^{-1} L_i = (S_{(j)} L_i S_{(j)}^{-1} L_i^{-1})^{-1} L_i S_{(j)}^{-1} \text{ or } S_{(j)}^{-1} L_i S_{(j)} L_i^{-1}$$

$$S_{(j)}^{-1} S_{(j')} L_i S_{(j)}^{-1} S_{(j')} L_i^{-1} = (S_{(j)} L_i S_{(j)}^{-1} L_i^{-1})(S_{(j')}^{-1} L_i S_{(j')} L_i^{-1})^{-1} = (S_{(j)} L_i S_{(j)}^{-1} L_i^{-1})^{-1}$$

$$V = \{(S_{(j)} L_i S_{(j)}^{-1} L_i^{-1})(S_{(j')} L_i S_{(j')}^{-1} L_i^{-1})^{-1} \mid \dots\}$$

(5.58)

These imply that if

$$S_{(j)} L_i S_{(j)}^{-1} L_i^{-1} = S_{(j')} L_i S_{(j')}^{-1} L_i^{-1}; \quad \forall i=1 \dots n$$

then

$$S_{(j')}^{-1} S_{(j)} L_i S_{(j)}^{-1} S_{(j')} L_i^{-1} = I; \quad \forall i=1 \dots n$$

or by Schur's lemma $S_{(j)}^{-1} S_{(j')} \sim I$

L.C.

$$S_{(j')} \sim S_{(j)}$$

This means that

$$(S_{(j)} L_i S_{(j)}^{-1} L_i^{-1}) \neq (S_{(j')} L_i S_{(j')}^{-1} L_i^{-1}) \quad (5.59)$$

when $(j) \neq (j')$ or when any $S \in M_{\mathbb{D}}$ satisfies

$$S L_i S^{-1} L_i^{-1} = \psi_i I \quad \text{and} \quad S L_i = \psi_i L_i S; \forall i=1\dots n \quad (5.60)$$

S can not be a sum of $S_{(j)}$'s, but has to be a multiple of one of the \mathbb{D}^2 , $S_{(j)}$'s. Thus the set Ψ contains only \mathbb{D}^2 distinct elements and are given by

$$\Psi = \left\{ (S_{(j)} L_i S_{(j)}^{-1} L_i^{-1} \mid i=1\dots n) \mid (j) \equiv (j_1, j_2, \dots, j_{2s}) \right. \\ \left. 0 \leq j_{2i-1}, j_{2i} \leq z_i - 1; \forall i=1\dots s \right\}$$

$$\cong \left\{ (\omega(m) S^* [\underline{j}, \underline{v_i}] \mid i=1\dots n) \mid (j) \equiv (j_1, j_2, \dots, j_{2s}, 0, \dots, 0) \right. \\ \left. 0 \leq j_{2i-1}, j_{2i} \leq z_i - 1; \right. \\ \left. i=1\dots s \right\} \quad (5.61)$$

where

$$S^* [\underline{j}, \underline{v_i}] \bmod m = (j_1, j_2, \dots, j_{2s}, 0, \dots, 0) \begin{bmatrix} T^* \\ \vdots \\ \vdots \end{bmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix} \quad (5.62)$$

Defining the equivalence relation that two coefficient sets $\{x_i^{(1)}\}$ and $\{x_i^{(2)}\}$ are equivalent if $\{x_i^{(1)}, x_i^{(2)-1}\} \in \Psi$, the entire set of all coefficient sets $\{(x_i^{(r)} \mid x_i^{(r)} = w(m)^{l_i}; 0 \leq l_i \leq m_i - 1 \mid r=1 \dots n\} \mid \prod_{i=1}^n m_i\}$ is partitioned into $(\prod_{i=1}^n m_i)/D^2$ equivalence classes. Thus factorising χ as

$$\chi = \left[\{x_i^{*(1)}\} \Psi, \{x_i^{*(2)}\} \Psi, \dots, \{x_i^{*(N)}\} \Psi \right]; N = \left(\prod_{i=1}^n m_i \right) / D^2. \quad (5.63)$$

all the inequivalent representations are given by the set

$$\Phi^* = \left[(x_i^{*(r)} \mid i=1 \dots n) \mid r=1 \dots N = \left(\prod_{i=1}^n m_i \right) / D^2 \right] \quad (5.64)$$

Thus the total number of irreducible inequivalent projective representations of G for the factor set ζ is

$$N = \left(\prod_{i=1}^n m_i \right) / D^2 \quad (5.65)$$

where D is the dimension of the irreducible representations, same for all the representations. D is given by

$$D = \prod_{i=1}^n z_i = \prod_{i=1}^n \frac{m}{(h_i, m)}. \quad (5.66)$$

which is the formula obtained by Backhouse and Ben Bradley. (14)

Finally in writing the representation of G in terms of L_i 's one has to normalize them to satisfy according to (5.23)

$$L_i^{m_i} = K_i I; i=1 \dots n \text{ if } L_i \text{ is obtained in } \\ (5.56) \text{ will have to be multiplied by } (K_i)^{-1/m_i} \text{ respectively.} \quad (5.67)$$

This completes the task of finding the projective representations of G for the given factor set ζ since once the representations of generators $\{L_i\}$ are found they have to be just substituted in the formula (5.16) to obtain the representations of other group elements.

Let us now consider the Schur representation group G^* for G constructed by central extension of G with the Schur multiplier $H^2(G, C^*)$ as the kernel of extension. This group can be described by the following generating relations

$$\begin{aligned} L_i L_j L_i^{-1} L_j^{-1} &= L_{ij}; \quad i, j = 1 \dots n \\ L_{ij}^{m_i} &= L_{ij}^{m_j} = 1; \quad i, j = 1 \dots n \\ L_j L_k &= L_k L_{jk}; \quad \forall i, j, k = 1 \dots n \\ L_i^{m_i} &= 1; \quad i = 1 \dots n. \end{aligned} \quad (5.68)$$

Choosing the factor set ζ such that $K_i = 1, \forall i = 1 \dots n$ as can be achieved by redefining ζ as

$$\zeta' \quad (5.69)$$

The corresponding n -factor sets are the same. The group G^* generated by (5.68) has the elements

$$\left\{ \prod_{\substack{i,j \\ (i < j)}}^n L_{ij}^{\lambda_{ij}} L_k^{\mu_k} \mid \begin{array}{l} 0 \leq \lambda_{ij} \leq (m_i, m_j) - 1 \\ 0 \leq \mu_k \leq m_k - 1; \quad \forall i, j, k = 1 \dots n \end{array} \right\}$$

Thus the order of G^* is

$$|G^*| = \left(\prod_{\substack{i,j \\ i < j}}^n (m_i, m_j) \right) \left(\prod_{i=1}^n m_i \right) = |H^2(G, \mathbb{C}^*)| |G|$$

(5.70)

It is easy to see that the normal subgroup $H \equiv \left\{ \prod_{\substack{i,j \\ i < j}}^n L_{ij}^{\lambda_{ij}} \mid 0 \leq \lambda_{ij} \leq (m_i, m_j) \right\}$
 $\cong H^2(G, \mathbb{C}^*)$ and $H \subseteq Z(G^*)$. Thus G^* is the central extension of G with $H^2(G, \mathbb{C}^*)$ as kernel of extension i.e.

$G^*/H^2(G, \mathbb{C}^*) \cong G$. The relations $L_{ij}^{m_i} = L_{ij}^{m_j} = 1$ and

$L_{ij} L_k = L_k L_{ij}, \forall i, j, k = 1 \dots n$ show that L_{ij} 's are scalars and

L_{ij} has to be an (m_i, m_j) -th root of unity where $m_{ij} = (m_i, m_j)$. Thus

all the representations of (5.66) are obtained by the possible

$\prod_{i < j=1}^n (m_i, m_j)$ choices for the set $\{L_{ij} \mid i, j = 1 \dots n\}$. We

found that if a particular choice is made as

$$\left\{ L_{ij} = \exp(2\pi i t_{ij}/m) \mid i, j = 1 \dots n \right\}$$

$$= \left(\prod_{i=1}^n m_i \right) / \left(\prod_{i=1}^n \frac{m}{(m, h_i)} \right)^2$$

there are $\left(\prod_{i=1}^n m_i \right) / D^2$

inequivalent irreducible representations of the same dimension D where $\{h_i, s\}$ etc. are uniquely fixed by (t_{ij}) . Then summing the squares of the dimensions of all the representations of G^* , corresponding to each choice of $\{L_{ij}\}$ we get

$\left(\prod_{i=1}^n m_i \right) \left(\prod_{i < j=1}^n (m_i, m_j) \right) = |G^*|$ as is required by Burnside's theorem, showing that no more representations can occur. From this one can find the number of conjugate classes in G^* as

$$\sum_{\text{overall choices of } \{L_{ij}\}} \left(\prod_{i=1}^n m_i \right) / \left(\prod_{i=1}^n \frac{m}{(h_i, m)} \right)^2 = \left(\prod_{i=1}^n m_i \right) \sum_{\text{overall choices of } \{L_{ij}\}} \prod_{i=1}^n (m, h_i)^2 / m^2.$$

As is evidently seen this is a very complicated expression, impossible to obtain directly by counting the classes in G^* .

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APPENDIX (cf. Morris Newman³⁰)

(i) How to obtain an $n \times n$ integer matrix D_n with a given row of n integers (d_1, \dots, d_n) as its first row and determinant $= \delta_n =$ the greatest common divisor of (d_1, \dots, d_n) ?

The theorem is that it is always possible to do so. We proceed by induction on n . For $n = 1$ the theorem is trivial for $n = 1$. Let $n = 2$. Then two integers ρ, σ may be determined so that $\rho d_1 - \sigma d_2 = \delta_2$ and thus we may choose

$$D_2 = \begin{bmatrix} d_1 & d_2 \\ \sigma & \rho \end{bmatrix}$$

Now suppose the theorem true for $n-1, \geq 3$ and let D_{n-1} be an $(n-1) \times (n-1)$ integer matrix with the first row $(d_1, d_2, \dots, d_{n-1})$ and determinant $= \delta_{n-1} = (d_1, d_2, \dots, d_{n-1})$. Since

$$\delta_n = ((d_1, d_2, \dots, d_{n-1}), d_n) = (\delta_{n-1}, d_n)$$

we can find integers ρ, σ such that $\rho \delta_{n-1} - \sigma d_n = \delta_n$ put

$$D_n = \begin{bmatrix} D_{n-1} & d_n \\ \frac{d_1 \sigma}{\delta_{n-1}} & \frac{d_2 \sigma}{\delta_{n-1}} - \dots - \frac{d_{n-1} \sigma}{\delta_{n-1}} & \rho \end{bmatrix}$$

Then $\det D_n = \delta_n$

How to obtain a unimodular integer matrix U , such that $T = UT^*U$ where T is antisymmetric integer matrix and T^* is its unique skew normal form.

Since $T \neq 0$ ^a non-zero element may be brought to the first row by a permutation of the rows and a corresponding permutation of the columns. Therefore we may assume without loss of generality that the first row of T contains a non-zero element which cannot occur in the (1,1) position of course. Since T is antisymmetric. Define

$$(t_{12}, t_{13}, \dots, t_{1n}) = \delta$$

and let x_2, x_3, \dots, x_n be integers such that

$$\sum_{j=2}^n a_{1j} x_j = \delta$$

Since $\delta \neq 0$ certainly $(x_2, x_3, \dots, x_n) = 1$. We may determine a unimodular matrix D_{n-1} with its first column as $(x_2, x_3, \dots, x_n)^T$. Then let $D_n^{(1)} = I_1 \oplus D_{n-1}$. put

$$T^{(1)} = D_n^{(1)T} T D_n^{(1)} = \begin{bmatrix} 0 & t_{12}^{(1)} & t_{13}^{(1)} & \cdots \\ -t_{12}^{(1)} & 0 & \cdots & \cdots \\ -t_{13}^{(1)} & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then a brief computation shows that $t_{12}^{(1)} = \delta$ and that $t_{12}^{(1)} | t_{1j}^{(1)}$ $3 \leq j \leq n$. Now add $-t_{1j}^{(1)}/t_{12}^{(1)}$ times row 2 to row j and then add $-t_{1j}^{(1)}/t_{12}^{(1)}$ times column 2 to column j , $3 \leq j \leq n$. The result is that $T^{(1)}$ gets transformed to a matrix say, $T^{(2)}$ whose first row is $(0, t_{12}^{(1)}, 0, 0, \dots, 0)$. The procedure is now repeated with the submatrix obtained by striking out the first row and column until a triple diagonal matrix, say $T^{(3)}$, is reached so that

$$T^{(3)} = \begin{bmatrix} 0 & t_{12}^{(3)} & 0 & \cdots & \\ -t_{12}^{(3)} & 0 & t_{23}^{(3)} & \cdots & \\ 0 & -t_{23}^{(3)} & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & \cdots & \end{bmatrix}$$

where $t_{12}^{(3)} = t_{12}^{(4)} \neq 0$. There are now two possibilities either $t_{12}^{(3)}$ divides every element of $T^{(3)}$ or it does not. If not, add rows $2, 3, \dots, n$ to row 1 and column $2, 3, \dots, n$ to column 1. The first row of the matrix obtained is

$$(0, t_{12}^{(3)} - t_{23}^{(3)}, t_{23}^{(3)} - t_{34}^{(3)}, \dots, t_{n-2,n-1}^{(3)}, -t_{n-1,n}^{(3)}, t_{n-1,n}^{(3)})$$

New greatest common divisor γ of the first row elements is

$$\gamma = (t_{12}^{(3)}, t_{23}^{(3)}, \dots, t_{n-1,n}^{(3)})$$

and γ contains fewer prime divisors than $t_{12}^{(3)}$. As in the first part of the procedure $t_{12}^{(3)}$ may now be replaced by γ and the entire process repeated until a triple diagonal matrix is obtained having γ in the $(1,2)$ position. Thus in a finite number of steps a triple diagonal matrix, $T^{(4)}$, congruent to T may be obtained in which $(1,2)$ divides every element of the matrix i.e. $t_{12}^{(4)} | t_{ij}^{(4)}$ $(1 \leq i, j \leq n)$. Add $t_{23}^{(4)} / t_{12}^{(4)}$ times row 1 to row 3, and then add $t_{23}^{(4)} / t_{12}^{(4)}$ times column 1 to column 3. Then T becomes congruent to a matrix of the form

$$\begin{pmatrix} 0 & t_1 \\ -t_1 & 0 \end{pmatrix} \oplus E$$

where t_1 divide every element of \mathbb{E} . The process is now repeated with \mathbb{E} until it becomes congruent to a matrix $\begin{pmatrix} 0 & t_2 \\ -t_1 & 0 \end{pmatrix} \oplus E'$ carrying on this one finally arrives at a skew normal form

$$\sum_{i=1}^s \oplus \begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix} \oplus O_{n-2s} ; s \leq \left[\frac{n}{2} \right]$$

and by keeping track of all the operations performed on \mathbb{T} , one can construct the matrix U , which is the product of all these operations preserving the order, such that $UT\tilde{\mathcal{V}} = T^* \tilde{\mathcal{V}}$ can be shown to be unique and t_i 's are nothing but invariant factors of \mathbb{T} . But U is not unique as is seen easily and $|\det U| = 1$ follows from the observation that at each step of the above process the transformation matrices applied to \mathbb{T} on both sides are unimodular. Since that t_i 's are invariant factors they can be specified as $t_i = h_i/k_{i-1}$ where h_i is the greatest common divisor of the $2^{i-1} \times 2^{i-1}$ minors of \mathbb{T} and k_i is the greatest common divisor of $2^i \times 2^i$ minors of \mathbb{T} .

Summary of important points.

All the inequivalent irreducible projective representations of $G \cong \mathbb{Z}_{m_1} \otimes \dots \otimes \mathbb{Z}_{m_n}$ are obtained as follows. Let e_i 's denote the generators of G obeying $e_i e_j = e_j e_i ; \forall i, j = 1 \dots n$ $e_i^{m_i} = 1 ; i = 1 \dots n$. Then for a given factor let ζ such that

$$\zeta(g_i, g_j) \zeta(g_j, g_k) = \zeta(g_i, g_j g_k) \zeta(g_j, g_k) \quad \forall g_i, g_j, g_k \in G.$$

the projective representations are given by

$$\mathcal{D}\left(\prod_{i=1}^n e_i^{k_i}\right) = \left[\prod_{s=1}^m \left\{ \prod_{j_s=1}^{R_s} \zeta(e_s, \prod_{i=s+1}^n e_s^{R_s - r_s} e_i^{k_i}) \right\} \right] \left(\prod_{i=1}^n L_i^{k_i} \right)$$

where

$$L_i = \chi_i^{*(r)} \left(\prod_{k=1}^m \zeta(e_i, e_i^{k_i}) \right)^{-\frac{1}{m_i}} \exp \left\{ \frac{2\pi i}{m} \left[\frac{1}{2} (m_i - 1) \sum_{j=1}^s h_j v_{i,2j-1} v_{i,2j} \right] \right\}$$

$$\times \left\{ \prod_{j=1}^s \otimes C_j^{v_{i,2j-1}} B_j^{v_{i,2j}} \right\}$$

$$C_j = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad B_j = \begin{pmatrix} 1 & \exp(2\pi i x_j/z_j) & 0 & & & \\ 0 & \ddots & & & & \\ 0 & & \exp(2\pi i x_j(z_j-1)/z_j) & & & \end{pmatrix}$$

$$x_j = h_j / (h_j, m), z_j = m / (h_j, m) = \dim C_j = \dim B_j$$

$$j = 1, \dots, s.$$

$(v_{ij}) = V$ is a unimodular matrix such that

$$T = VT^* \tilde{V}$$

$$T^* = \sum_{j=1}^s \oplus \begin{pmatrix} 0 & h_j \\ -h_j & 0 \end{pmatrix} \oplus 0_{n-2s}$$

$T = (t_{ij})$ is the antisymmetric integer matrix defined by through the relations

$$\zeta(e_i, e_j) \zeta(e_j, e_i)^{-1} = \omega_{ij}$$

if $\omega_{ij} = \exp(2\pi i t_{ij}/k_{ij})$ with $(T_{ij}, K_{ij}) = 1$

then let $\omega_{ij} = \exp(2\pi i t_{ij}/m)$ where $m = l \cdot c \cdot m(C K_{ij} | i, j = 1 \dots n)$

Thus $\dim D = 1 = \left(m^{\frac{1}{2}}/\frac{1}{m}\right)$. There are $\left(\prod_{i=1}^n m_i\right)/D^2$ sets of values for the set $(x_i^{*(r)} | i = 1 \dots n)$ and each set of values, labelled by $r = 1 \dots \left(\prod_{i=1}^n m_i\right)/D^2$, gives rise to different (inequivalent) representations. These values of the sets $(x_i^{*(r)} | i = 1 \dots n)$ are obtained by the relation

$$\left\{ (x_i^{*(r)} | i = 1 \dots n) \mid r = 1 \dots \left(\prod_{i=1}^n m_i\right)/D^2 \right\} * \left\{ \left(\exp\left(\frac{2\pi i}{m} s^*[\underline{j}, v_i]\right) \right)_{i=1 \dots n} \mid (\underline{j}) = (j_1, j_2, \dots, j_{2s}, 0, \dots, 0) \right\}$$

$$= \left\{ \left(x_i^{(k)} \mid x_i^{(k)} = \exp(2\pi i l_i/m_i); 0 \leq l_i \leq m_i - 1, i = 1 \dots n \right) \mid k = 1 \dots \frac{n}{\prod_{i=1}^n m_i} \right\}$$

$$s^*[\underline{j}, v_i] \bmod m = (j_1, j_2, \dots, j_{2s}, 0, \dots, 0) \begin{bmatrix} & & \\ & T^* & \\ & & \end{bmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix}$$

$$0 \leq j_{2i-1}, j_{2i} \leq z_i^{-1}; \forall i = 1 \dots s$$

and the $\{ \cdot \}^*$ means $\{ A, B, \dots \}^* \{ A', B', \dots \} = \{ A \circ A', A \circ B', \dots, B \circ A', B \circ B', \dots \}$

where the product \circ of two sets A, B is defined by $A \circ B = (A_i B_j | i=1 \dots n)$

$$\text{Thus there are } \left(\prod_{i=1}^n m_i \right) / D^2 = \left(\prod_{i=1}^n m_i \right) \left(\prod_{j=1}^s (m, h_j)^2 \right) / m^{2s}$$

inequivalent representations of dimension D for the given factor

set S . This chapter also studies the Schur representation group G^* of G , which is a central extension of G with the scalar multiplier group $H^2(G, \mathbb{C}^*) \subset Z(G^*)$ as the kernel of extension.

CHAPTER 6THEORY OF BLOCH ELECTRONS IN HOMOGENEOUS MAGNETIC FIELD AND
GENERALIZED CLIFFORD GROUPS

Aim of this chapter is to present a simple version of the theory of Bloch electrons in homogeneous magnetic field using Generalized Clifford groups. This is an extension of our earlier work in this direction¹.

The knowledge of the behaviour of Bloch electrons in a magnetic field is fundamental to the study of various phenomena in solids under the influence of external magnetic field such as galvanomagnetic, thermomagnetic, magnetic acoustic and magneto-optical effects, micro-wave absorption in magnetic field and de Haas-van Alphen effect etc. which supply very important informations about the electronic structure of solids. Almost all relevant effects except a few like magnetic break-down can be well understood in terms of semi-classical theory².

Attempts at quantum theory of the problem started with Landau³ and Peierls⁴ and further investigations by Luttinger⁵, Kohn⁶, Blount⁷, Wigner and Freedkin⁸, Roth⁹ and Zibermann¹⁰ confirmed Onsager's¹⁰ quantization rule and close connection of the quantum theory to the electron orbits of semiclassical theory was established. But since this relation is established by using non-convergent expansions in powers of field strength it becomes useless at high field strengths. In extremely high magnetic fields

the basic electronic structure of the solid a will strongly depend on the field strength independent study of the problem of Bloch electron in magnetic field was initiated by Harper¹¹, Brown¹², Zuk¹³, Fischbeck¹⁴, Opochouskii and Tom¹⁵ and others elucidated the basic symmetry of the problem and new approaches to the solution of the Schrödinger equation were studied. So far some general properties of eigenvalues and eigenfunctions of the Hamiltonian have been stated but unfortunately, as Fischbeck¹⁴ observes, till now all these results have been of no practical interest. Very strange features of the problem such as the dependence of energies and wavefunctions on certain spasmodically changing quantities associated with monotonically varying field strengths remain unexplained (Cf. Fischbeck¹⁴ for an excellent review of the present state of affairs and extensive bibliography)

The group theoretical approach to the problem^{11,12,13} brings in algebraic structures which we have called Generalised Clifford groups in earlier chapters. We shall use this isomorphism of G.C.G. with magnetic translation group (M.T.G) - the symmetry group of the Hamiltonian of the Bloch electron in homogeneous magnetic field - to derive in a very simple manner more complete and improved results, than those of Brown¹², Zuk¹³, Fischbeck¹⁴ and others on the properties of eigenfunctions and eigenvalues of the Hamiltonian. We make an attempt, unlike others, to study both the cases of rational and irrational magnetic fields at the same time, though as it has been shown already, mathematically accurate statements can be made only in the rational case. Hence our results for the irrational case are of

only of explorative and tentative nature due to the inherent mathematical difficulties involved. Our main aim has been to derive the Magnetic Bloch Theorem in the simplest form so that by analogy with the role of the ordinary (zero field) Bloch theorem methods of energy band structure calculations may be developed. First let us recall some relevant facts of the theory of electrons in free lattice potential.

(1) Electrons in periodic potential and Bloch functions

The Schrödinger equation of an electron in a periodic potential is given by

$$\mathcal{H}_0 \psi(\underline{r}) = E \psi(\underline{r})$$

$$\mathcal{H}_0 = \left\{ \frac{\mathbf{p}^2}{2m} + V(\underline{r}) \right\} \quad \mathbf{p} = -i\hbar \nabla$$

$$V(\underline{r} + \underline{R}) = V(\underline{r}), \quad \forall \underline{R} = n_1 \underline{a}_1 + n_2 \underline{a}_2 + n_3 \underline{a}_3 \quad (6.1)$$

where m is the mass of the electron, $(\underline{a}_1, \underline{a}_2, \underline{a}_3)$ are primitive vectors of a unit cell of the lattice and n_1, n_2, n_3 are integers.

The group of lattice translation operators

$$T = \left\{ T(\underline{R}) = \exp\left\{ \frac{i}{\hbar} \underline{R} \cdot \underline{p} \right\} \mid \underline{R} \in L \right\}$$

$$T(\underline{R}) \psi(\underline{r}) = \psi(\underline{r} + \underline{R}), \quad L = \left\{ \sum_{i=1}^3 n_i \underline{a}_i \mid n_1, n_2, n_3 \in \mathbb{Z} \right\} \quad (6.2)$$

Commute with the Hamiltonian \mathcal{H}_0 i.e.

$$[T(\underline{R}) \mathcal{H}_0 - \mathcal{H}_0 T(\underline{R})] \psi(\underline{r}) = 0; \quad \forall T(\underline{R}) \in T \quad (6.3)$$

Hence by the well-known Wigner's theorem on applications of group

theory to quantum mechanics, eigenfunctions of \hat{H}_0 can be chosen simultaneously as also basis functions of irreducible representations of the group T. The group T is Abelian i.e.

$$T(\underline{R}_1)T(\underline{R}_2) = T(\underline{R}_2)T(\underline{R}_1) = T(\underline{R}_1 + \underline{R}_2) \quad (6.4)$$

$\forall T(\underline{R}_1), T(\underline{R}_2) \in T.$

and hence its irreducible representations are one dimensional, given by

$$\Gamma_{\underline{k}} : \left\{ \Gamma_{\underline{k}}(\underline{R}) = \exp(i\underline{k} \cdot \underline{R}) \mid \forall \underline{R} \in L \right\} \quad (6.5)$$

where $\underline{k} = y_1 \underline{k}_1 + y_2 \underline{k}_2 + y_3 \underline{k}_3$ labels the representations, $\underline{k}_1, \underline{k}_2, \underline{k}_3$ being primitive vectors of reciprocal space, defined by

$$\underline{k}_1 = \frac{2\pi \underline{a}_2 \wedge \underline{a}_3}{\Omega}, \quad \underline{k}_2 = \frac{2\pi \underline{a}_3 \wedge \underline{a}_1}{\Omega}, \quad \underline{k}_3 = \frac{2\pi \underline{a}_1 \wedge \underline{a}_2}{\Omega}$$

$$\Omega = (\underline{a}_1 \wedge \underline{a}_2) \cdot \underline{a}_3 \quad (6.6)$$

and y_1, y_2, y_3 are any real numbers. Let L' denote the set of all vectors of the reciprocal lattice $L' \equiv \left\{ G = \sum_{i=1}^3 m_i \underline{k}_i \mid m_i \in \mathbb{Z} \right\}$.

Then

$$\Gamma_{\underline{k}} \equiv \Gamma_{\underline{k} + \underline{G}}, \quad \forall \underline{G} \in L' \quad (6.7)$$

Corresponding to the representation $\Gamma_{\underline{k}}$ the basis functions can be obtained, using the technique of projection operators, which gives the particular functions of a representation of a group G by the formula

$$f_j = \xi \sum_{g \in G} D(g)_{jj}^* g v \equiv P_j v \quad j=1, 2, \dots, m = \dim D.$$

where $D(g)_{jj}$ is the (jj) th element of the matrix $D(g)$,

v is an arbitrary function of the space on which G acts, and ξ is a normalisation constant. Applying this to the present case we get

$$\begin{aligned}\Psi_{\underline{k}}(t) &= \xi \sum_{R \in L} U_{\underline{k}}(R) T(R)v(t) \\ &= \xi \sum_{R} \exp(-i\underline{k} \cdot R) v(t+R) \\ &= \xi \exp(i\underline{k} \cdot t) \sum_{R} \exp(-i\underline{k} \cdot [t+R]) v(t+R) \\ &= \xi \exp(i\underline{k} \cdot t) u_{\underline{k}}(t)\end{aligned}\tag{6.8}$$

ξ being a normalisation constant such that $\int |\Psi_{\underline{k}}(t)|^2 dt = 1$. $v(t)$ is an arbitrary function and $U_{\underline{k}}(t) = \sum_{R} \exp(-i\underline{k} \cdot [t+R]) v(t+R)$ (over all space) is a periodic function with the same period as $v(t)$

$$U_{\underline{k}}(t+R) = U_{\underline{k}}(t), \quad \forall R \in L.\tag{6.9}$$

Also it is seen easily that

$$\Psi_{\underline{k}+G}(t) = \Psi_{\underline{k}}(t); \quad \forall G \in L'\tag{6.10}$$

Γ being the symmetry group of H_0 , solutions of H_0 are
of the same form as $\Psi_{k\zeta}(\tau)$ which is a product of $\exp(i\zeta \cdot \tau)$
and a periodic function of τ with the same period as $V(\tau)$.
This is known as Bloch theorem. Substituting $\Psi_{k\zeta}(\tau)$ in the
Schrodinger equation (6.1) we get an equation for $U_{n\zeta}(\tau)$,

$$\left\{ \frac{1}{2m} \left(\hat{p} + \hbar \frac{\partial}{\partial \tau} \right)^2 + V(\tau) \right\} U_{n\zeta}(\tau) = E_n(k) U_{n\zeta}(\tau) \quad (6.11)$$

$$\Psi_{n\zeta}(\tau) = N \exp(i k \cdot \tau) U_{n\zeta}(\tau)$$

(6.12)

not restricted under the group Γ of lattice translation and (6.12)
is, i.e., the wave symmetry condition for $\Psi_{n\zeta}$ is satisfied automatically
where $E_n(k)$ is the eigenvalue and $U_{n\zeta}(\tau)$ is the corresponding
eigenfunction. n is called a band index and the full eigenfunction is
called a Bloch functions is given by (6.12). k is called crystal
momentum of electron and due to (6.10) it follows

$$E_n(k + G) = E_n(k) ; \forall G \in L \quad (6.13)$$

Thus in knowing $E_n(k)$, k need to specified only over a con-
veniently chosen (to reflect the symmetry of the crystal) unit cell
of the reciprocal space, called Brillouin Zone. Equations (6.12) and
(6.13) constitute the main theory we seek to generalize to the
case of presence of an external homogeneous magnetic field.

(ii) Bloch electrons in homogeneous magnetic field.

The time independent Schrodinger equation for Bloch electron
in an external homogeneous magnetic field \vec{B} is given by, (dis-
regarding spin)

$$\begin{aligned}\mathcal{H}_m \Psi(\underline{r}) &= E \Psi(\underline{r}) \\ \mathcal{H}_m &= \frac{1}{2m} \left\{ \underline{p} + \frac{e}{c} \underline{A} \right\}^2 + V(\underline{r}) \\ V(\underline{r} + \underline{R}) &= V(\underline{r}) ; \forall \underline{R} \in L, e > 0 \\ \underline{B} &= \nabla \wedge \underline{A}\end{aligned}\tag{6.14}$$

without loss of generality the vector potential \underline{A} can be chosen in symmetric gauge as

$$\underline{A} = \frac{1}{2} \underline{B} \wedge \underline{r}\tag{6.15}$$

Though \underline{B} is constant since $\underline{A}(\underline{r} + \underline{R}) \neq \underline{A}(\underline{r})$, \mathcal{H}_m is not invariant under the group T of lattice translation operators (6.2). The new symmetry operators for \mathcal{H}_m , called magnetic translation operators were first studied group theoretically by Brown and then by Zuk¹³ Fizchbeck¹⁴ and others. The origin of these operators can be traced to the work of Peierls¹⁵. First it is noticed that, for the gauge (6.15),

$$\left[\left(\underline{p} + \frac{e}{c} \underline{A} \right), \left(\underline{p} - \frac{e}{c} \underline{A} \right) \right] = 0\tag{6.16}$$

Then it follows that

$$\left[\frac{i}{\hbar} \underline{R} \cdot \left(\underline{p} - \frac{e}{c} \underline{A} \right), \mathcal{H}_m \right] = 0, \forall \underline{R} \in L.\tag{6.17}$$

The following set of relations are easy to verify

- $$\left(\underline{p} + \frac{e}{c} \underline{A} \right) \wedge \left(\underline{p} + \frac{e}{c} \underline{A} \right) = -i \frac{e \hbar}{c} \underline{B} = -i \hbar^2 \underline{\beta}$$
- $$\left(\underline{p} - \frac{e}{c} \underline{A} \right) \wedge \left(\underline{p} - \frac{e}{c} \underline{A} \right) = i \frac{e \hbar}{c} \underline{B} = i \hbar^2 \underline{\beta}$$
- $$\left[\underline{R} \cdot \left(\underline{p} - \frac{e}{c} \underline{A} \right), \underline{R}' \cdot \left(\underline{p} - \frac{e}{c} \underline{A} \right) \right] = i \hbar^2 \underline{\beta} \cdot (\underline{R} \wedge \underline{R}')$$

Also, in the gauge (6.18),

$$[\underline{R} \cdot \underline{P}, \underline{R}' \cdot \underline{A}] = -\frac{i}{2} \hbar^2 \beta \cdot (\underline{R} \wedge \underline{R}') \quad (6.18)$$

$$\text{i.e. } [\underline{R} \cdot \underline{P}, \underline{R} \cdot \underline{A}] = 0$$

and hence

$$\exp\left\{\frac{i}{\hbar} \underline{R} \cdot \underline{P}\right\} \exp\left\{-\frac{i}{\hbar} \frac{e}{c} \underline{R} \cdot \underline{A}\right\} = \exp\left\{\frac{i}{\hbar} \underline{R} \cdot \left(\underline{P} - \frac{e}{c} \underline{A}\right)\right\}$$

due to

$$e^A e^B = e^{A+B}; \text{ if } [A, B] = 0 \quad (6.20)$$

The set of operators $\{\tau(\underline{R}) = \exp\left\{\frac{i}{\hbar} \underline{R} \cdot \left(\underline{P} - \frac{e}{c} \underline{A}\right)\right\} \mid \forall \underline{R} \in L\}$

commuting with \mathcal{H}_m are called magnetic translation operators.

Due to (6.18) C we have

$$\begin{aligned} \tau(\underline{R})\tau(\underline{R}') &= \exp\left\{-\frac{i}{2} \beta \cdot (\underline{R} \wedge \underline{R}')\right\} \tau(\underline{R} + \underline{R}') \\ &= \exp\left\{-i \beta \cdot (\underline{R} \wedge \underline{R}')\right\} \tau(\underline{R}')\tau(\underline{R}) \\ &\quad \forall \underline{R}, \underline{R}' \in L. \end{aligned} \quad (6.21)$$

This shows that the set $\{\tau(\underline{R}) \mid \forall \underline{R} \in L\}$ does not form a group and in exact analogy with the formation of G.C.G. from basic elements of G.C.A., one has to form a group only by adding more elements to the set which are multiples of $\tau(\underline{R})$ by phase factors of the type

$\exp\left\{-\frac{i}{2} \beta \cdot (\underline{R} \wedge \underline{R}')\right\}$. such a group $\{\exp(i\phi) \tau(\underline{R})\}$ has been

called a Magnetic translation group by Brown, Zek, Fischbeck and others

We shall follow a different presentation of the group using generators. For this let

$$\tau_i \equiv \tau(\underline{\alpha}_i), \quad i=1, 2, 3 \quad (6.22)$$

$$\therefore \tau_i \tau_j = \exp \left\{ -i \frac{1}{2} \beta_k^3 \right\} \tau_j \tau_i ; \quad (i, j, k = 1, 2, 3) \\ \text{(in cyclic order.)} \quad (6.23)$$

Then

$$\tau_1^{n_1} \tau_2^{n_2} \tau_3^{n_3} = \exp \left\{ -i \frac{1}{2} \underline{\alpha} \cdot (n_1 n_2 \beta_3 - n_1 n_3 \beta_2 + n_2 n_3 \beta_1) \right\} T(\underline{\alpha}) \\ \forall \underline{\alpha} = n_1 \underline{\alpha}_1 + n_2 \underline{\alpha}_2 + n_3 \underline{\alpha}_3 \in L. \quad (6.24)$$

Thus any $T(\underline{\alpha})$ is a multiple of a phase factor and product of powers of $\{\tau_i | i=1, 2, 3\}$. Thus we shall consider the group generated by τ_i 's for the purpose of obtaining the symmetry among adopted functions. There is no loss of generality of this in choosing the smallest lattice vector in the direction of the magnetic field \underline{B} to be $\underline{\alpha}_3$ and write $\underline{B} = \frac{2\pi}{L} \eta \underline{\alpha}_3$. Then (6.23) get much simplified. Also let us first consider η to be a rational number $= \frac{s}{N}$ with $(s, N) = 1$. Then the field \underline{B} is said to be rational. To make the group finite for convenience one can impose boundary conditions on the solutions consistent with the relations (6.23) just like imposition of Born-van Karman conditions in the free Bloch electron case i.e. $\psi(\underline{k}) = \psi(\underline{k} + N \underline{R})$, $\forall \underline{R} \in L$ where N is a very large

integer $\rightarrow \infty$. This condition makes the crystal momentum to vary quasicontinuously through rational values $K = y_1 K_1 + y_2 K_2 + y_3 K_3$, $y_1, y_2, y_3 \in \mathbb{Q}$.

Assuming the field to be rational i.e. $\beta = \frac{2\pi}{N} \eta \alpha_3$ and imposing the boundary conditions $T(R)^M = 1, \forall T(R) \in T, 2N|M, M \sim 0$, the group $\mathcal{T} \equiv \left\{ \exp\left(-2\pi i \frac{m\beta}{2N}\right) T_1^{n_1} T_2^{n_2} T_3^{n_3} \mid 0 \leq m \leq 2N-1, 0 \leq n_i \leq M-1 \right\}_{\forall i=1,2,3}$ is a finite group of order $2NM^3$. As an abstract group this is isomorphic to the group \tilde{G} generated by the relations

$$\begin{aligned} L_1 L_2 &= L_0^2 L_2 L_1 \\ L_3 L_1 &= L_1 L_3, \quad L_2 L_3 = L_3 L_2, \quad L_0^{2N} = 1, \\ L_0 L_i &= L_i L_0; i=1,2,3 \\ \left(\prod_{i=0}^{2N-1} L_i^{k_i} \right)^M &= 1, \quad \forall k_i = 0, 1, \dots, M-1; \quad 2N|M. \end{aligned} \tag{6.25}$$

which is one of the types of groups we have called Generalized Clifford groups. The correspondence is $T_i = L_i$, $i=1,2,3$. From their representation theory developed in earlier chapters we know, that all the representations are obtained by faithfully representing the N sets of relations (6.25) arising by varying

$L_1 L_2 L_1^{-1} L_2^{-1} = L_0^2 = \sqrt[N]{1}$ over all its N possible values $\left\{ \exp\left(2\pi i \frac{l}{N}\right) \mid 0 \leq l \leq N-1 \right\}$. In each of these N cases the representations are of the same dimension which depends on the value of L_0^2 chosen and differ only in phase factors. One can easily count the different representations arising from different phase factors. If we choose a particular value of L_0^2 , say $\exp\left\{2\pi i \frac{l}{N}\right\}$ then L_0 has two values

$\pm Q = \text{set of rational numbers (real)}$.

$\pm \exp\left(\frac{i\pi l}{N}\right)$. L_3 commutes with both L_1 and L_2 and hence all group elements. Thus in irreducible representations it has to be a scalar obeying $L_3^M = 1$ or L_3 has M values $\{\exp\left(2\pi i \frac{k_3}{M}\right) | 0 \leq k_3 \leq M-1\}$. Since $L_1^M = L_2^M = 1$ to each of them can be attached M phase factors $\{\exp\left(2\pi i \frac{k_1}{M}\right) | 0 \leq k_1 \leq M-1\}$ and $\{\exp\left(2\pi i \frac{k_2}{M}\right) | 0 \leq k_2 \leq M-1\}$. Thus there are totally M^3 possible different representations of same dimension with only difference of phase factors. But among these, as we have often seen in earlier chapters, two representations would be equivalent if one differs from the other only in the phase factors of L_1 and L_2 by any powers of $L_1 L_2 L_1^{-1} L_2^{-1} = \exp\left(2\pi i \frac{l}{N}\right) = \exp\left(2\pi i \frac{l'}{N'}\right)$ with $(l', N') = 1$ when $\exp\left(2\pi i \frac{l}{N}\right) = \exp\left(2\pi i \frac{l'}{N'}\right)$, $(l, N') = 1$. The dimension of representation is N' , and out of the M choices for the phase factors of L_1 and L_2 the above condition of equivalence implies that phase factors of each L_1 and L_2 can take only (M/N') values $\{\exp\left(2\pi i \frac{k_i}{M}\right) | 0 \leq k_i \leq \frac{M}{N'}, -1\}$ corresponding to inequivalent representations. Since $N' | N, 2N | M, N' | M$. Thus totally there are $2M\left(\frac{M}{N'}\right)^2 = 2\frac{M^3}{N'^2}$ inequivalent irreducible representations for (6.35) corresponding to a choice of $L_1 L_2 L_1^{-1} L_2^{-1} = \exp\left(2\pi i \frac{l}{N}\right) = \exp\left(2\pi i \frac{l'}{N'}\right), (l, N') = 1$. All representations of \tilde{G} are obtained by varying $l = 0, 1, \dots, N-1$. The set of magnetic translation operators T faithfully correspond to relations

(6.26) with a choice $L_1 L_2 L_1^{-1} L_2^{-1} = \exp(-2\pi i s/N)$; ($s \in N$) and

$L_0 = +\exp\left(-i\frac{\pi s}{N}\right)$. Corresponding to this choice there should be $\binom{M^3}{N^2}$ representations of dimension N which are specified by the choices

$$\begin{aligned} L_1 &= \exp\left(\frac{2\pi i k_1}{M}\right) L_1^*, \quad 0 \leq k_1 \leq \frac{M-1}{N} \\ L_2 &= \exp\left(\frac{2\pi i k_2}{M}\right) L_2^*, \quad 0 \leq k_2 \leq \frac{M-1}{N} \\ L_3 &= \exp\left(\frac{2\pi i k_3}{M}\right) I \quad 0 \leq k_3 \leq M-1 \\ L_0 &= +\exp\left(-i\frac{\pi s}{N}\right) \end{aligned} \quad (6.26)$$

where L_1^* , L_2^* are any two irreducible $M \times M$ matrices obeying

$$L_1^* L_2^* = \exp\left(-2\pi i \frac{s}{N}\right) L_2^* L_1^* \quad (6.27)$$

Let us choose well known matrices

$$L_1^* = B_N^s = \begin{pmatrix} 1 & & & & & & & 0 \\ & \omega(N)^s & & & & & & \\ & & \omega(N)^{2s} & & & & & \\ & & & \ddots & & & & \\ 0 & & & & \ddots & & & s(N-1) \\ & & & & & \ddots & & \omega(N) \end{pmatrix} \quad (6.28)$$

with $\omega(N) = \exp\left(\frac{2\pi i}{N}\right)$

and

$$L_2^* = C_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad (6.29)$$

$$(B_N^s)^N = C_N^N = I.$$

Then from (6.34)

$$\begin{aligned}
 \tilde{\tau}_K(R) &= \exp\left(\frac{2\pi i n_1 n_2 s}{2N}\right) \tau_1^{n_1} \tau_2^{n_2} \tau_3^{n_3} \\
 &= \exp\left(\frac{i\pi n_1 n_2 s}{N}\right) \exp\left(2\pi i\left[n_1 \frac{k_1}{M} + n_2 \frac{k_2}{M} + n_3 \frac{k_3}{M}\right]\right) B_N^{s n_1} C_N^{s n_2} \\
 &= \exp\left(i\pi n_1 n_2 \frac{s}{N}\right) \exp\left(i \frac{K \cdot R}{N}\right) \left(\begin{array}{cccccc} 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & 0 & 0 & \exp\left(i \frac{K \cdot R}{N}\right) \end{array} \right)
 \end{aligned}$$

(6.36)

where $\underline{K} = \frac{1}{M}(k_1 \underline{K}_1 + k_2 \underline{K}_2 + k_3 \underline{K}_3)$

or

$$\begin{aligned}
 (\tilde{\tau}_K(R))_{jl} &= \exp\left(i\pi n_1 n_2 \frac{s}{N}\right) \exp\left(i \frac{K \cdot R}{N}\right) \exp\left(i j \frac{s}{N} K \cdot \underline{R}\right) \delta_{j, (l-n_2)} \\
 &= \exp\left(i\pi n_1 n_2 \frac{s}{N}\right) \exp\left(i \left[\underline{K} + i \frac{s}{N} \underline{K}_1\right] \cdot \underline{R}\right) \delta_{j, (l-n_2)}
 \end{aligned}$$

$$j, l = 0, 1, \dots, N-1; \quad (l-n_2) \bmod N = l-n_2$$

$$\forall \underline{R} = n_1 \underline{a}_1 + n_2 \underline{a}_2 + n_3 \underline{a}_3 \in L.$$

\underline{K} is label the representations and if we now take the limit

$M \rightarrow \infty$, \underline{K} varies quasicontinuously in a setzone of the Brillouin Zone, Magnetic Brillouin Zone - containing the set of all reciprocal space vectors $\{ \underline{K} = k_1 \underline{K}_1 + k_2 \underline{K}_2 + k_3 \underline{K}_3 \mid 0 \leq k_1 \leq \frac{1}{N}, 0 \leq k_2 \leq \frac{1}{N}, 0 \leq k_3 \leq 1 \}$

Shape of the magnetic Brillouin zone can also be taken conveniently.

Thus two representations $\Gamma_{\underline{K}}$ and $\Gamma_{\underline{K}'}$ are equivalent if

$$\underline{K}' = \underline{K} + \underline{G}_m \quad \text{where} \quad \underline{G}_m \in L'_m = \left\{ \frac{1}{N} (m_1 \underline{K}_1 + m_2 \underline{K}_2) + m_3 \underline{K}_3 \mid m_i \in \mathbb{Z}, i=1,2,3 \right\}$$

Correspondingly the generalization of (6.12) is

$$E_n(\underline{K} + \underline{G}_m) = E_n(\underline{K}) \quad \forall \underline{G}_m \in L'_m \quad (6.32)$$

where n is the band index. Let us now construct the symmetry adapted functions corresponding to the representation $\Gamma_{\underline{K}}$. Before proceeding further to find the partner functions for $\Gamma_{\underline{K}}$ of our problem let us notice that for other representations of the group T ,

considered as isomorphic to \tilde{G} , the projection operator

$$P_j = \left\{ \xi \sum_{g \in G} D(g)^* g_j g \right\}^* \quad \text{vanishes identically} \quad \forall j. \quad \text{This happens}$$

as follows. In T each $T(R)$ occurs $2N$ times as multiples of

the $2N$, $2N$ th roots of unity due to the phase factors

$\left\{ \exp\left(-2\pi i \frac{m^2}{2N}\right) \mid 0 \leq m \leq 2N-1 \right\}$. If a representation corresponds to a choice of $\underline{L}_1 \underline{L}_2 \underline{L}_1^{-1} \underline{L}_2^{-1} = L_0^2 = \exp\left(-\frac{2\pi i l}{N}\right)$ with $l \neq 0$ or even

$$L_0 = -\exp\left(-\frac{2\pi i l}{2N}\right) \quad \text{and} \quad l=0, \quad \text{for each } T(R) \text{ in the}$$

summation in P_j coefficients add up to 0. Hence it follows that

$$P_j = 0, \quad \forall j.$$

Using the projection operation, the zeroth function is given by

$$\Psi_{\underline{K},0}(\underline{r}) = \xi \sum_{\substack{0 \leq m \leq 2N-1 \\ \underline{R} \in L}} \left\{ \exp\left(-\frac{2\pi i m}{2N}\right) (\tau_{\underline{K}}(\underline{R}))_{00} \right\}^* \exp\left(-\frac{2\pi i m}{2N}\right) \tau(\underline{R}) v(\underline{r}) \quad (6.32)$$

On summation over m this gives

$$\begin{aligned} \Psi_{\underline{K},0}(\underline{r}) &= 2N\xi \sum_{\substack{\underline{R} \in L \\ \underline{R}}} \left\{ \tau_{\underline{K}}(\underline{R})_{00} \right\}^* \tau(\underline{R}) v(\underline{r}) \\ &= 2N\xi \sum_{\substack{\underline{R} \in L \\ \underline{R}}} \exp\{-i\pi n_1 n_2 s\} \exp\{-i(\underline{K} \cdot \underline{R})\} \times \\ &\quad \exp\left\{ \frac{i}{\hbar} \underline{R} \cdot \left(\underline{p} - \frac{e}{c} \underline{A} \right) \right\} v(\underline{r}) \end{aligned} \quad (6.32a)$$

where $L \equiv \{n_1 \underline{a}_1 + n_2 N \underline{a}_2 + n_3 \underline{a}_3 \mid n_1, n_2, n_3 \in \mathbb{Z}\}$

Now using (6.19) $\Psi_{\underline{K},0}(\underline{r})$ becomes apart from normalization

$$\Psi_{\underline{K},0}(\underline{r}) = \sum_{\substack{\underline{R} \in L}} \exp\{-i\pi n_1 n_2 s\} \exp\left\{ -i \left(\underline{K} + \frac{e}{c\hbar} \underline{A} \right) \cdot \underline{R} \right\} v(\underline{r} + \underline{R}) \quad (6.34)$$

where $v(\underline{r})$ is an arbitrary function. This function has a remarkable property analogous to the Bloch functions (6.12). We shall prove that it can be written as

$$\Psi_{\underline{K},0}(\underline{r}) = \exp(i\underline{K} \cdot \underline{r}) u_{\underline{K},0}^*(\underline{r}) \quad (6.35)$$

where

$$\tau(\underline{R}') \psi_{K,0}^*(\underline{x}) = \exp \{-i\pi n_1 n_2 s\} \psi_{K,0}^*(\underline{x})$$

$$\forall \underline{R}' = n'_1 \underline{a}_1 + n'_2 N \underline{a}_2 + n'_3 \underline{a}_3 \in \tilde{L}$$
(6.36)

Proof.

$$\begin{aligned} \psi_{K,0}(\underline{x}) &= \sum_{\underline{R} \in \tilde{L}} \exp \{-i\pi n_1 n_2 s\} \exp \left\{ -i \left(K \cdot \underline{R} + \frac{e}{2\hbar c} B \wedge \underline{x} \cdot \underline{R} \right) \right\} V(\underline{x} + \underline{R}) \\ &= \exp(iK \cdot \underline{x}) \sum_{\underline{R} \in \tilde{L}} \exp \{-i\pi n_1 n_2 s\} \exp \left\{ -iK \cdot (\underline{x} + \underline{R}) \right\} \\ &\quad \times \exp \left\{ -i \frac{e}{2\hbar c} B \wedge \underline{x} \cdot \underline{R} \right\} V(\underline{x} + \underline{R}) \\ &= \exp(iK \cdot \underline{x}) \psi_{K,0}^*(\underline{x}) \end{aligned}$$

$$\psi_{K,0}^*(\underline{x} + \underline{R}') = \sum_{\underline{R} \in \tilde{L}} \exp \{-i\pi n_1 n_2 s\} \exp \left\{ -iK \cdot (\underline{x} + \underline{R}' + \underline{R}) \right\} \times$$

$$\exp \left\{ -i \frac{e}{2\hbar c} B \wedge (\underline{x} + \underline{R}') \cdot \underline{R} \right\} V(\underline{x} + \underline{R}' + \underline{R})$$

where $\underline{R}' = n'_1 \underline{a}_1 + n'_2 N \underline{a}_2 + n'_3 \underline{a}_3$

Let $\underline{R}'' = \underline{R}' + \underline{R}$. Substituting $\underline{R} = \underline{R}'' - \underline{R}'$ and $n_i'' = n_i + n'_i$ for $i=1,2,3$

$$\begin{aligned} \psi_{K,0}^*(\underline{x} + \underline{R}') &= \sum_{\underline{R}'' \in \tilde{L}} \exp \left\{ -i\pi s(n'' n''_2 - n'_1 n'_2 - n_1 n'_2 - n'_1 n_2) \right\} \\ &\quad \exp \left\{ -iK \cdot (\underline{x} + \underline{R}'') \right\} \exp \left\{ -i \frac{e}{2\hbar c} B \wedge (\underline{x} + \underline{R}') \cdot (\underline{R}'' - \underline{R}') \right\} \\ &\quad \times V(\underline{x} + \underline{R}'') \end{aligned}$$

$$\begin{aligned}
 &= \exp \left\{ \frac{ie}{2\hbar c} \underline{B} \wedge \underline{T} \cdot \underline{R}' \right\} \sum_{\underline{R}'' \in \tilde{\mathcal{L}}} \exp \left\{ -i\pi s(n_1^m n_2^m - n_1^m n_2' - n_1^m n_2' - n_1' n_2) \right\} \times \\
 &\quad \exp \left\{ -\frac{ie}{2\hbar c} \underline{B} \wedge \underline{R}' \cdot \underline{R} \right\} \exp \left\{ -i\underline{K} \cdot (\underline{T} + \underline{R}'') \right\} \exp \left\{ -\frac{ie}{2\hbar c} \underline{B} \wedge \underline{T} \cdot \underline{R}'' \right\} \times \\
 &\quad \mathcal{U}(\underline{T} + \underline{R}'') \\
 &= \exp \left\{ \frac{ie}{2\hbar c} \underline{B} \wedge \underline{T} \cdot \underline{R}' \right\} \sum_{\underline{R}'' \in \tilde{\mathcal{L}}} \exp \left\{ -i\pi s n_1^m n_2^m \right\} \exp \left\{ -i\pi s (-n_1^m n_2' - \right. \\
 &\quad \left. - n_1^m n_2' - n_1' n_2) \right\} \exp \left\{ -i\pi s (n_1^m n_2' - n_2^m n_1') \right\} \times \\
 &\quad \exp \left\{ -i\underline{K} \cdot (\underline{T} + \underline{R}'') \right\} \exp \left\{ -\frac{ie}{2\hbar c} \underline{B} \wedge \underline{T} \cdot \underline{R}'' \right\} \mathcal{U}(\underline{T} + \underline{R}'') \\
 &= \exp \left\{ \frac{ie}{2\hbar c} \underline{B} \wedge \underline{T} \cdot \underline{R}' \right\} \exp \left\{ i\pi s n_1^m n_2^m \right\} \sum_{\underline{R}'' \in \tilde{\mathcal{L}}} \exp \left\{ -i\pi s n_1^m n_2^m \right\} \times \\
 &\quad \exp \left\{ -i\underline{K} \cdot (\underline{T} + \underline{R}'') \right\} \exp \left\{ -\frac{ie}{2\hbar c} \underline{B} \wedge \underline{T} \cdot \underline{R}'' \right\} \times \\
 &\quad \mathcal{U}(\underline{T} + \underline{R}'')
 \end{aligned}$$

$$\begin{aligned}
 \exp \left\{ -\frac{i}{\hbar} \underline{R}' \cdot \frac{eA}{c} \right\} u_{\underline{K},0}^*(\underline{T} + \underline{R}') &= \exp \left\{ \frac{i}{\hbar} \underline{R}' \cdot \left(\underline{P} - \frac{e}{c} \underline{A} \right) \right\} u_{\underline{K},0}^*(\underline{T}) \\
 &= T(\underline{R}') u_{\underline{K},0}^*(\underline{T}) = \exp(-i\pi s n_1^m n_2^m) u_{\underline{K},0}^*(\underline{T}) \\
 &\quad \forall \underline{R}' \in \tilde{\mathcal{L}}
 \end{aligned}$$

Once we have found the zeroth partner function the other partner functions can be obtained by applying to it the operator

$\{\exp(-i\tilde{K} \cdot \tilde{a}_2) T(\tilde{a}_2)\}$ successively since this operator is represented by cyclic matrix C_N implying that the corresponding operator will change the j th function to $j+1$ th function. The inverse operation will change j th function to $j-1$ th function one can get all the partner function directly from the operation (6.32). We shall follow the procedure to obtain a clearer interpretation of the functions. Hence using this we get

$$\begin{aligned}\Psi_{\tilde{K},j}(\tilde{x}) &= \exp(i\tilde{K} \cdot j\tilde{a}_2) T(-j\tilde{a}_2) \Psi_{\tilde{K},0}(\tilde{x}) \\ &= \exp\left\{i\tilde{K} \cdot \tilde{x}\right\} \exp\left\{\frac{i e}{\hbar c} j\tilde{a}_2 \cdot \tilde{A}\right\} U_{\tilde{K},0}^*(\tilde{x} - j\tilde{a}_2) \\ &= \exp\left\{i\tilde{K} \cdot \tilde{x}\right\} U_{\tilde{K},j}^*(\tilde{x}) \quad ; \quad j = 0, 1, \dots, N-1\end{aligned}\quad (6.37)$$

$$\Psi_{\tilde{K},j+N}(\tilde{x}) = \Psi_{\tilde{K},j}(\tilde{x}) \quad ; \quad \forall j = 0, 1, \dots, N-1. \quad (6.38)$$

Also one can write

$$\begin{aligned}U_{\tilde{K},j}^*(\tilde{x}) &= \sum \exp\left\{-i\pi n_1(n_2 N + j)\frac{\Delta}{N}\right\} \exp\left\{-i\tilde{K} \cdot (\tilde{x} + \tilde{R})\right\} \times \\ &\quad \tilde{R} = (n_1 \tilde{a}_1 + (n_2 N + j) \tilde{a}_2 + n_3 \tilde{a}_3) \exp\left\{-\frac{i e}{\hbar c} \tilde{A} \cdot \tilde{R}\right\} V(\tilde{x} + \tilde{R}) \\ &\quad \forall j = 0, 1, \dots, N-1\end{aligned}\quad (6.39)$$

Analogous to the property (6.36) we have

$$\begin{aligned}T(\tilde{R}') U_{\tilde{K},j}^*(\tilde{x}) &= \exp\left\{i\pi \frac{8}{N} n_1'(n_2' N + j)\right\} U_{\tilde{K},j}^*(\tilde{x}) \\ &\quad \forall \tilde{R}' \in \tilde{L}.\end{aligned}\quad (6.40)$$

Proof is exactly same as that of (6.36).

Summarising, we have found the Magnetic Bloch functions - symmetry adapted functions for Bloch electron in homogeneous magnetic field - to be given by the formula -

$$\Psi_{\underline{k},j}(\underline{x}) = \xi \exp \{ i \underline{k} \cdot \underline{x} \} u^*_{\underline{k},j}(\underline{x}) \quad ; \quad j=0, 1, \dots, N-1$$

$$u^*_{\underline{k},j}(\underline{x}) = \sum_{\underline{R}} \exp \left\{ -i \frac{\pi s m_1}{N} (n_2 N + j) \right\} \exp \left\{ -i \frac{\beta}{2} \underline{x} \cdot \underline{R} \right\} \times \\ \underline{R} = (n_1 \underline{a}_1 + (n_2 N + j) \underline{a}_2 + n_3 \underline{a}_3) \\ \exp \left\{ -i \underline{k} \cdot (\underline{x} + \underline{R}) \right\} v(\underline{x} + \underline{R}) \quad (6.41)$$

$$u^*_{\underline{k},j}(\underline{x} + \underline{R}) = \exp \left\{ i \frac{\beta}{2} \underline{x} \cdot \underline{R} \right\} \exp \left\{ i \frac{\pi s}{N} m_1 (n_2 N + j) \right\} u^*_{\underline{k},j}(\underline{x})$$

$$u^*_{\underline{k},j+rN}(\underline{x}) = u^*_{\underline{k},j}(\underline{x}) \quad \forall r \in \mathbb{Z}.$$

where $\beta = \frac{e}{\hbar c} B = \frac{2\pi s}{\Omega N} \alpha_3$ and \underline{k} varies in the Magnetic Brillouin Zone which is $\frac{1}{N^2}$ times the usual Brillouin Zone and in which the points in the $(\underline{k}_1, \underline{k}_2)$ plane differ by only $\frac{1}{N}(k_1 \underline{a}_1 + k_2 \underline{a}_2)$, $0 \leq k_i, k_i < 1$. Also as proved earlier the energy band function obeys,

$$E_n(\underline{k} + \underline{G}_m) = E_n(\underline{k}) ; + \underline{G}_m = \frac{1}{N}(m_1 \underline{a}_1 + m_2 \underline{a}_2 + m_3 \underline{a}_3) \\ m_1, m_2, m_3 \in \mathbb{Z}$$

In the limit $\beta = 0$, we can choose $s=0, N=1$, and there is only one function corresponding to $j=0$ (6.41) shows that this is exactly the Bloch function.

Now substituting (6.41) in the Schrodinger equation (6.14) the equation for $u_{n,\underline{k},j}^*(\underline{x})$ is easily seen to be, exactly similar to (6.11),

$$\left[\frac{1}{2m} \left\{ (\underline{p} + \frac{e}{c} \underline{A}) + \frac{e}{c} \underline{A} \right\}^2 + V(\underline{x}) \right] u_{n,\underline{k},j}^*(\underline{x}) = E_n(\underline{k}) u_{n,\underline{k},j}^*(\underline{x})$$

$j = 0, 1, \dots, N-1.$

j is only a degenerate index and is not involved in energy value.

Finally let us have a closer look at the functions (6.41). They can be written in the following fashion bringing out their remarkable similarity with the usual Bloch functions

$$\Psi_{\underline{k},j}(\underline{x}) = \sum_{\underline{R} \in \underline{L}} \exp \left\{ - \frac{i e}{\hbar c} \Phi(\underline{x} - j\underline{a}_2, n_1 \underline{a}_1, n_2 N \underline{a}_2, n_3 \underline{a}_3, \underline{x} + \underline{R}) \right\} \exp \left\{ - i \underline{k} \cdot (\underline{x} + \underline{R}) \right\} V(\underline{x} + \underline{R})$$

where $\Phi(\underline{x} - j\underline{a}_2, n_1 \underline{a}_1, n_2 N \underline{a}_2, n_3 \underline{a}_3, \underline{x} + \underline{R})$ is the flux of the magnetic field through the polygon connected by the vectors $(\underline{x} - j\underline{a}_2, n_1 \underline{a}_1, n_2 N \underline{a}_2, n_3 \underline{a}_3, \underline{x} + \underline{R})$. Thus we see that except for the change in the range of values of j and introduction of a phase function these magnetic Bloch functions have not been realized in this remarkable version. Thus we believe that the establishment of this closest connection between the Bloch functions in field free case and in presence of magnetic field makes the above form the most suitable as starting point for study of solid state phenomena in presence of an external homogeneous magnetic field. We shall return to this point in the conclusion.

(iii) Free electrons in homogeneous magnetic field - Landau levels:

The case of free electrons in homogeneous magnetic field corresponds to the limit $V(\underline{x}) = 0$. Then the magnetic translation

group becomes continuous with \underline{R} in $\tau(\underline{R})$ varying over the entire space. Letting $\underline{\beta} = \frac{e}{\hbar c} \underline{B}$ to be in the \underline{a}_3 direction as before the commutation relations between the magnetic translation operators become

$$\tau(\underline{x})\tau(\underline{x}') = \exp\{-i(\underline{x}\wedge\underline{x}')\cdot\underline{\beta}\}\tau(\underline{x}')\tau(\underline{x}) \quad (6.42)$$

$\forall \underline{x}, \underline{x}'$

where we have replaced \underline{R} by \underline{x} to denote their continuous nature. Thus for any $\underline{\beta}$ the phase factors $\exp\{-i(\underline{x}\wedge\underline{x}')\cdot\underline{\beta}\}$ vary continuously taking all values on the unit circle in complex plane. If we have relations of the type

$$UV = \exp(2\pi i \eta) VU \quad (6.43)$$

then taking determinants on both sides

$$\det U \det V = \exp(2\pi i \eta d) \det V \det U \quad (6.44)$$

d being the dimension of the representation. If η is such that $\exp(2\pi i \eta d) \neq 1$ for any $d < \infty$ then U and V must have infinite dimensions. Hence in (6.42) also, since the phase factors $\exp(-i(\underline{x}\wedge\underline{x}')\cdot\underline{\beta})$ assume all values when \underline{x} and \underline{x}' vary, for many values of \underline{x} and \underline{x}' this condition happens and hence dimension of representation of the magnetic translation group is ∞ . Thus the parameter $N \rightarrow \infty$ (6.41), due to n_1, n_3 being continuous and $N = \infty$ the summation over n_1, n_3 is to be replaced by integration over them and n_2 has to be removed. Also since $N = \infty$, the Magnetic Brillouin zone shrinks to the origin in the $(\underline{a}_1, \underline{a}_2)$ plane and hence the different representations are specified by only the 3rd component for \underline{K} . This third component k_3 of \underline{K} can take all values from $-\infty$ to $+\infty$ due to the continuous

nature of the group. This can be seen from the analogous case for discrete group in equation (6.7) considering an one parameter continuous Abelian group the representations are given by (6.5)

$$\Gamma_k : \{ \Gamma_k = \{ \exp(i k x) \mid -\infty \leq x \leq \infty \} \}$$

$$\Gamma_{k+g} : \{ \Gamma_{k+g} = \exp(i(k+g)x) \mid -\infty \leq x \leq \infty \}$$

It is obvious $\Gamma_{k+g} \neq \Gamma_k$ for any value of $g > 0$ since there is no solution to the condition $\exp(igx) = 1, \forall x, g > 0$. Hence using (6.41), and taking the primitive vector a_1, a_2, a_3 as unit orthogonal vectors,

$$\begin{aligned} \Psi_{k_3, j}(x) &= \xi \exp(i k_3 x_3) \times \\ &\quad \iint_{-\infty}^{+\infty} dx'_1 dx'_3 \exp\left(\frac{i}{2}\beta x_1\right) \exp\left(\frac{i}{2}\beta x_2 x'_1 + \frac{i}{2}\beta x'_2 x'_1\right) \\ &\quad \exp\{-ik_3(x_3+x'_3)\} V(x_1+x'_1, x_2-j, x_3+x'_3) \\ &= \xi \exp\left(i k_3 x_3 - \frac{1}{2}\beta x_2 x_1 + i j \beta x_1\right) \times \\ &\quad \iint_{-\infty}^{+\infty} dx'_1 dx'_3 \exp\left\{\frac{i}{2}\beta x_2(x_1+x'_1)\right\} \times \exp\left\{\frac{1}{2}\beta\delta(x_1+x'_1)\right\} \\ &\quad \exp(-ik_3(x_3+x'_3)) V(x_1+x'_1, x_2-j, x_3+x'_3). \end{aligned} \tag{6.45}$$

The integrations over x'_1, x'_3 wipe out the x_1, x_3 dependence leading to

$$\Psi_{k_3, j}(x) = \xi \exp(i k_3 x_3 + i j \beta x_1) \exp(-\frac{1}{2}\beta x_2 x_1) \phi(x_2-j) \tag{6.46}$$

where $\phi(x_1 - j)$ is an arbitrary function of $(x_1 - j)$ only. j th function was obtained in (6.37) by the operation of $\exp(iK_j \alpha_2)$ $\tau(-j\alpha_2)$ where j is integer. But in this case j can assume all values since $\tau(-j\alpha_2)$ is a symmetry operator for all $-\infty \leq j \leq \infty$. Hence j is a continuous index. Since j is only a label for the rows of the matrix representation it is a degenerate index with respect to eigenvalue. Writing $j\beta = k$, we get the familiar London solution

$$\Psi_{k_3, k_1}(\vec{r}) = \xi \exp\left\{i(k_1 x_1 + k_3 x_3)\right\} \exp\left\{-\frac{i}{2} \beta x_1 x_2\right\} \phi(x_1 - k/\beta) \quad (6.47)$$

where $\phi(x_1 - k/\beta)$ is determined by requiring ψ to be the solution of the Schrodinger equation for free electron in homogeneous magnetic field

$$\frac{1}{2m} \left(\hat{p} + \frac{e}{c} \frac{1}{2} \vec{\beta} \wedge \vec{r} \right)^2 \psi(\vec{r}) = E \psi(\vec{r}) \quad (6.48)$$

The results are well known London levels

$$E_{n, k_z} = \left(n + \frac{1}{2}\right) \hbar \omega + \frac{\hbar^2 k_z^2}{2m} ; \quad \omega = \frac{e |B|}{mc} \quad (6.49)$$

Thus we have demonstrated the correct behaviour of the magnetic Bloch functions (6.41) in both the limiting cases of $B=0$ and $V(\vec{r})=0$.

iv. Irrational Magnetic Fields.

The magnetic field is said to be irrational if

$$\beta = \frac{e}{hc} B = \frac{d\pi}{l} \eta \alpha_3 \quad (6.50)$$

and η is irrational. Let us use formally the relation (6.41) to construct the symmetry adapted functions in this case, simply replacing $\frac{\lambda}{N}$ by η . Also where ever N occurs explicitly it has to be put equal to ∞ . This is seen from noting that the irreducible representations of generators T_i 's have to be of infinite dimension due to their commutation relation $T_1 T_2 = \exp(-2i\pi i\eta) T_2 T_1$. This makes the summation over n_2 disappear. Only here one gets into serious doubts. We shall explore the implications of this assumption. Certainly N cannot be made a finite integer $< \infty$, however large one may think of it. Then

$$\begin{aligned}\Psi_{\underline{k},j}(\underline{x}) &= \exp\left\{i\underline{k} \cdot \underline{x}\right\} u_{\underline{k},j}^*(\underline{x}) \\ u_{\underline{k},j}^*(\underline{x}) &= \exp\left(\frac{ni}{2} \beta \wedge \underline{x} \cdot \underline{a}_2\right) \times \\ &\quad \sum \exp\left\{i\pi\eta n_{ij}\right\} \exp\left(-\frac{i}{2} \beta \wedge \underline{x} \cdot \underline{R}\right) \times \\ &\quad R = n_1 \underline{a}_1 - j \underline{a}_2 + n_3 \underline{a}_3 \\ &\quad \exp(-i\underline{k} \cdot (\underline{x} + \underline{R})) v(\underline{x} + \underline{R})\end{aligned}$$

$$j = 0, \pm 1, \pm 2, \dots, \pm \infty$$
(6.51)

j takes only discrete values since the symmetry group T is still discrete and in (6.37) only for few discrete m_j values we get degenerate functions. Thus degeneracy of each level is now countable infinity. Only in determining the possible values of \underline{k} one gets into difficulty. The third component of \underline{k} can assume all values between 0 and \underline{k}_3 and two representations with \underline{k} and \underline{k}' would be equivalent if $\underline{k} - \underline{k}' = m_3 \underline{k}_3, m_3 \in \mathbb{Z}$. This follows from the fact

that the set of magnetic translation operators in the third direction $\{\tau(n_3 \alpha_3)\}$ form a discrete Abelian group and commute with all the operators of the entire group T . The above functions correspond to the representations, in analogy with (6.30)

$$\tau_{\underline{k}}(\underline{R}) = \exp(i\pi n_1 n_2 \eta) \exp(i\underline{k} \cdot \underline{R}) B_{\infty, \eta}^{n_1} C_{\infty}^{n_2} \quad (6.52)$$

with

$$B_{\infty, \eta} = \begin{pmatrix} \exp(i\eta \underline{k}_1 \cdot \underline{R}) & & & \\ & \exp(i2\eta \underline{k}_1 \cdot \underline{R}) & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$C_{\infty} = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \quad (6.53)$$

both of ∞ dimensions and C_{∞} is thought of as $\lim_{N \rightarrow \infty} C_N$.

Then $C_{\infty} B_{\infty, \eta} = \exp(i\eta \underline{k}_1 \cdot \underline{R}) B_{\infty, \eta} C_{\infty}$. One can think of the algebra of $\infty \times \infty$ dimensional discrete indexed matrices to be spanned by a

set of matrices $\{C_{\infty}^k B(l) \mid \begin{array}{l} k=0, \pm 1, \dots, \infty \\ 0 \leq l < 1 \end{array}\}$ where

$$C_{\infty}^{-k} = (C_{\infty}^{-1})^k = \begin{pmatrix} 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}^k \quad C_{\infty}^{+k} = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}^k$$

$$B(l) = \begin{pmatrix} \exp(2\pi i l) & & & \\ & \exp(4\pi i l) & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad (6.54)$$

Then any $\infty \times \infty$ matrix $S = \int_0^1 dl \sum_{k=-\infty}^{+\infty} a_k(l) C_\alpha^k B(l)$. Finding all possible values of $S B_{\alpha\eta} S^{-1} B_{\alpha\eta}^{-1}$ and $S C_\alpha S^{-1} C_\alpha^{-1}$ varying S over such matrices which make these quantities scalars (i.e., restricting S to the set $\{C_\alpha^k B(l) \mid k= \pm 1 \dots \pm \infty \}$ as proved in IIIrd Chapter in a similar situation) we find that two sets $\{\exp(i\tilde{k} \cdot \underline{\alpha}_1) B_{\alpha\eta}, \exp(i\tilde{k} \cdot \underline{\alpha}_2) C_\alpha\}$ and $\{\exp(i\tilde{k}' \cdot \underline{\alpha}_1) B_{\alpha\eta}, \exp(i\tilde{k}' \cdot \underline{\alpha}_2) C_\alpha\}$ will be equivalent if $\tilde{k}' = \tilde{k} + (m + \mu)\tilde{k}_1 + \lambda \tilde{k}_2$ where λ is any real number and μ is any linear sum of elements of the set $\{\xi \pmod{1} = m\eta \mid m \in \mathbb{Z}\}$ with integer coefficients. Thus we should expect in this case

$$E_n(\tilde{k} + (m + \mu)\tilde{k}_1 + \lambda \tilde{k}_2 + m_3 \tilde{k}_3) = E_n(\tilde{k}) \quad (6.55)$$

where m, m_3 are any integers, λ is any real number and μ is any linear sum of the elements of the set $\{\xi \pmod{1} = m\eta \mid m \in \mathbb{Z}\}$ with integer coefficients. This shows that \tilde{k} does not contain the second component just like in the free electron in homogeneous magnetic field. Thus it is seen that when the field becomes irrational the magnetic Brillouin Zone assumes very odd shape described by (6.55).

Iv. Conclusion.

The important message of this chapter is that in presence of an external homogeneous magnetic field \tilde{B} the group theoretical problem requires the eigenfunctions to be of the following form which

we shall call Magnetic Bloch function

$$\Psi_{K,j}(\tau) = \xi \exp(iK \cdot \tau) \sum \exp \left\{ -i [K \cdot (\tau + R) + \frac{e}{\hbar c} \Phi(\tau, R)] \right\} V(\tau + R)$$

$$R = (n_1 a_1 + (m_N - j) a_2 + n_3 a_3)$$

Method of calculation of approximate energy band function $E_n(K)$ for many crystals are mainly based on choosing trial functions to satisfy the boundary conditions implied by the symmetry of the Hamiltonian. So usually one constructs function which are to be of the Bloch type. Hence Bloch functions are important starting points for understanding many phenomena. So far in the case of solids under a homogeneous magnetic field, the magnetic Bloch functions has not been in a form bringing out the similarity with the usual Bloch function in a striking fashion so that its usage as a substitute for Bloch functions in dealing with problems in presence of magnetic field can be developed extensively. For example in the tight binding approximation method of calculation of energy bands in solids the arbitrary function $V(\tau)$ in the Bloch sum is taken to be one of the core orbitals. Then calculating the average value of the Hamiltonian with this trial function one gets a picture of the band spreading of the core energy levels as the effect of the crystalline field. Though each type of approximation is valid only under certain circumstances one has to be satisfied with the best of what one can get out of these since an exact solution of the Schrodinger equation is impossible. Hence in replacing tight binding method in presence of magnetic field one can take $V(\tau)$ to be a core orbital of the individual atom in presence of that magnetic field and calculate the band energies. Similarly one can perhaps try to adapt other methods also with the striking similarity between the usual Bloch functions and magnetic

Bloch functions providing a guide to such developments. Further developments in these directions are in progress and successful results would be published elsewhere. Due to lack of space I would conclude this chapter omitting the discussions on certain other phenomena such as notion of free electron wave packets in homogeneous magnetic fields etc. which I had planned earlier to include.

Summary of important points.

According to Bloch theorem the wave functions of an electron in a periodic potential $V(\underline{x})$, obeying the Schrödinger equation

$$\left\{ \frac{\underline{p}^2}{2m} + V(\underline{x}) \right\} \psi(\underline{x}) = E \psi(\underline{x}), \quad V(\underline{x} + \underline{R}) = V(\underline{x}); \quad \forall \underline{R} \in L.$$

Can be represented as

$$\psi_{\underline{k}}(\underline{x}) = \xi \exp(i \underline{k} \cdot \underline{x}) u_{\underline{k}}(\underline{x})$$

where $u_{\underline{k}}(\underline{x})$ is a periodic function

$$u_{\underline{k}}(\underline{x} + \underline{R}) = u_{\underline{k}}(\underline{x})$$

This property of the eigenfunctions arises from the fact that the Hamiltonian \mathcal{H}_0 is invariant under the Lattice translation group of operators

$$T \equiv \{ T(\underline{R}) = \exp\left(\frac{i}{\hbar} \underline{R} \cdot \underline{P}\right) \mid \forall \underline{R} \in L \}$$

when there is an external magnetic field the so called magnetic translation group given by

$$\tau \equiv \{ \tau(\underline{R}) = \exp\left\{ \frac{i}{\hbar} \underline{R} \cdot (\underline{P} - \frac{e}{c} \underline{A}) \right\} \mid \forall \underline{R} \in L \}$$

becomes isomorphic to what we have called Generalized Clifford group. Using this fact we construct the representations using the theory developed in earlier chapters and by the use of standard projection operator techniques the symmetry adapted functions are derived in the following form, called magnetic Bloch functions

$$\Psi_{\underline{k},j}(\underline{r}) = \xi \exp(i\underline{k} \cdot \underline{r}) \sum \exp \left\{ -i[\underline{k} \cdot (\underline{r} + \underline{B}) + \frac{e}{\hbar c} \Phi(\underline{r}, \underline{B})] \right\} \chi_{\underline{R} = [n_1 \underline{\alpha}_1 + (n_2 N - j) \underline{\alpha}_2 + n_3 \underline{\alpha}_3]} V(\underline{r} + \underline{R})$$

$$j = 0, 1, \dots, N-1 \quad \text{when } \underline{B} = \frac{\hbar c}{e} \frac{2\pi}{N} \frac{j}{N} \underline{\alpha}_3$$

$\Phi(\underline{r}, \underline{B})$ = flux of the field \underline{B} through the polygon of vectors $(\underline{r}, -j \underline{\alpha}_2, n_1 \underline{\alpha}_1, n_2 N \underline{\alpha}_2, n_3 \underline{\alpha}_3, -(\underline{r} + \underline{B}))$

$V(\underline{r})$ is an arbitrary function of \underline{r}

$$\underline{k} = k_1 \underline{\alpha}_1 + k_2 \underline{\alpha}_2 + k_3 \underline{\alpha}_3, \quad 0 \leq k_i \leq \frac{1}{N}, \quad 0 \leq k_3 \leq 1.$$

Rational magnetic field is $\underline{B} = \frac{\hbar c}{e} \frac{2\pi}{N} \frac{j}{N} \underline{\alpha}_3$ and irrational field

$\underline{B} = \frac{\hbar c}{e} \frac{2\pi}{N} m \underline{\alpha}_3$ are discussed in a unified fashion and the shapes of the Brillouin zones for both cases are derived. The form

of the magnetic Bloch functions given above has not been noticed so far and its striking similarity to the usual Bloch functions suggests a generalisation of the methods of energy band structure calculations by simple appropriate replacement of the role of Bloch function in them. The results for irrational magnetic field case are explorative and tentative due to certain probably correct assumptions made in view of the inherent mathematical difficulties.

One may start now a simple form of valence electrons, assumed to be related to an integer value positive or negative, and calculate the dispersion of the eigenvalues of the matrix

$$L = \sum_i x_i L_i$$

where L_i are good quantum numbers of the electrons in the displacement relations

$$L_{ij} = L_i - L_j = \delta_{ij} d, \quad d = 1/3, 1/4, \dots$$

where $i, j = 1, \dots, n$ and the dimension is measured by the relation

$$L = L_1 + L_2 + \dots + L_n$$

This chapter is based on a much paper written by myself and published in the year 1966.

2.1. ENERGY BANDS AND THE BLOCH FUNCTIONS

In some work reported a problem seems to have been overlooked concerning the position of negative energy levels. Allowing only positive values for the energy within the band calculation very easily disappears in Fermi's theory, equation of state, result and spectrum

CHAPTER 2.A NEGATIVE ENERGY RELATIVISTIC WAVE EQUATION

We shall demonstrate that by a simple change in the structure of the internal variables of Dirac's positive energy relativistic wave equation², we can arrive at a negative energy relativistic wave equation admitting only negative energy solutions. To achieve this we shall use a simple fact of Clifford algebra, studied in relation to an eigen value problem of Alladi Ramakrishnan³. The eigenvalue of an eigenvector of the matrix:

$$L = \sum_{i=1}^4 x_i L_i \quad (7.1)$$

where L_i 's are 4×4 matrices generators of $C_4^{(2)}$, obeying the anticommutation relations

$$L_i L_j + L_j L_i = 2 \delta_{ij} I ; \quad i, j = 1, 2, 3, 4. \quad (7.2)$$

changes sign from + to - when the eigenvector is transformed by the matrix

$$L_5 = L_1 L_2 L_3 L_4 \quad (7.3)$$

This chapter is based on a recent paper written by me in collaboration with Dutt³.

(1) Dirac's positive energy relativistic wave-equation

In 1971 Dirac² proposed a positive energy relativistic wave equation for particles of non-zero rest mass, allowing only positive values for the energy unlike the usual relativistic wave equations, originating in Dirac's famous equation of 1928⁴, which are symmetrical

between positive and negative energies. As has been pointed out by Dirac, the new equation formally very similar to the old equation for the electron but the physical significance is very different. In particular he has shown that the new equation gives integral values for the spin.

The new wave equation for a particle with unit rest mass reads (taking $\hbar = c = 1$)

$$\left(\frac{\partial}{\partial x_0} + \sum_{\gamma=1}^3 \alpha_\gamma \frac{\partial}{\partial x_\gamma} + \beta \right) (\Psi) \psi = 0 \quad (7.4)$$

where

$$\alpha_\gamma \alpha_\ell + \alpha_\ell \alpha_\gamma = 2 \delta_{\gamma \ell}; \quad \gamma, \ell = 1, 2, 3. \quad (7.5)$$

$$\alpha_\gamma \beta + \beta \alpha_\gamma = 0; \quad \gamma = 1, 2, 3.$$

$$\beta^2 = -I \quad (7.6)$$

α_γ 's are real symmetric,

$$\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (7.7)$$

$$(\Psi) = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \quad (7.8)$$

$(q_1, q_3 = p_1), (q_2, p_2 = q_4)$ are two pairs of dynamical variables describing internal degrees of freedom involving two harmonic oscillators and having the commutation relations

$$[q_a, q_b]_- = q_a q_b - q_b q_a = i \beta_{ab} \quad (7.9)$$

$$a, b = 1, 2, 3, 4.$$

and the wave function Ψ is a one component function of two commuting q_μ 's, say q_1 and q_2 , as well as of the four x^μ . Thus $(q)\Psi$ is a column matrix with four elements $(q_1\Psi, q_2\Psi, q_3\Psi, q_4\Psi)$ and the 4×4 matrices α_Y, β are to be multiplied into this column matrix in the usual way.

Putting $\delta^\mu \equiv \partial/\partial x_\mu$ and $a_0 = 1$ the wave equation (7.4) can be written concisely as

$$\left(\sum_{\mu=0}^3 \alpha_\mu \delta^\mu + \beta \right) (q)\Psi = 0 \quad (7.10)$$

or putting $i\delta^\mu \equiv p^\mu$ these become

$$(p_0 - \sum_{Y=1}^3 \alpha_Y p_Y + i\beta)(q)\Psi = 0 \quad (7.11)$$

(metric used here is $(1, -1, -1, -1)$) with a particular choice of α_Y 's as

$$\alpha_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (7.12)$$

These equations become

$$\begin{pmatrix} p_3 & 0 & -p_1 - i & p_2 \\ 0 & p_3 & p_2 & p_1 - i \\ -p_1 + i & p_2 & -p_3 & 0 \\ p_2 & p_1 + i & 0 & -p_3 \end{pmatrix} \begin{pmatrix} q_1\Psi \\ q_2\Psi \\ q_3\Psi \\ q_4\Psi \end{pmatrix} = p_0 \begin{pmatrix} q_1\Psi \\ q_2\Psi \\ q_3\Psi \\ q_4\Psi \end{pmatrix}$$

$$[D](q)\Psi = p_0(q)\Psi. \quad (7.13)$$

Taking a solution corresponding to an eigenstate of momentum and energy we can consider p_0, p_1, p_2, p_3 in (7.13) as real numbers

obeying the equation $\sum_{\mu=0}^3 p^\mu p_\mu = 1$ which is the relativistic condition naturally arising out of a necessary condition on the solution ψ that it must satisfy the de Broglie equation $(\sum_{\mu=0}^3 p_\mu \partial^\mu + 1)\psi = 0$ for all values of its internal variables (cf. Dirac¹ for further details). Hence $p_0 = \pm \sqrt{1 + p_1^2 + p_2^2 + p_3^2} = \pm |p_0|$. Taking $p_0 = \pm |p_0|$ putting $\Psi(q, x) = \phi(q) \exp(-i \sum_{\mu=0}^3 p^\mu x_\mu)$, we get

$$(q) \phi(q) = u_1^+ \xi_1 + u_2^+ \xi_2 \quad (7.14)$$

where u_1^+ and u_2^+ are two independent eigenvectors of $[D]$ both corresponding to the eigenvalue $\pm |p_0|$ and ξ_1, ξ_2 are arbitrary functions of q_1 and q_2 only. It is to be remembered that the matrix $[D]$ satisfies $[D]^2 = |p_0|^2 I$ and has corresponding to each eigenvalue $\pm |p_0|$ and $-|p_0|$ two linearly independent eigenvectors, since $[D]$ is hermitian and hence diagonalizable. Taking

$$u_1^+ = \begin{pmatrix} |p_0| + p_3 \\ 0 \\ -p_1 + i \\ p_2 \end{pmatrix}, \quad u_2^+ = \begin{pmatrix} 0 \\ |p_0| + p_3 \\ p_2 \\ p_1 + i \end{pmatrix} \quad (7.15)$$

we got a set of equations, taking explicitly $q_{13} = -i \frac{\partial}{\partial q_1}$, $q_{14} = -i \frac{\partial}{\partial q_2}$, $q_{11}\phi = (|p_0| + p_3)\xi_1$, $q_{12}\phi = (|p_0| + p_3)\xi_2$

$$-i \frac{\partial \phi}{\partial q_{11}} = (-p_1 + i)\xi_1 + p_2 \xi_2, \quad -i \frac{\partial \phi}{\partial q_{12}} = p_2 \xi_1 + (p_1 + i)\xi_2 \quad (7.16)$$

Substituting $\xi_1 = q_{11}\phi / (|p_0| + p_3)$, $\xi_2 = q_{12}\phi / (|p_0| + p_3)$ in the second set and integrating, one readily gets Dirac's solution

$$\phi(q) = K \exp \left\{ -\frac{1}{2} \left[q_1^2 + q_2^2 + i p_1 (q_1^2 - q_2^2) - 2i p_2 q_1 q_2 \right] \times (|p_0| + p_3)^{-1} \right\} \quad (7.17)$$

and hence

$$\Psi_+ = K \exp \left\{ -\frac{1}{2} \left[q_1^2 + q_2^2 + i p_1 (q_1^2 - q_2^2) - 2i p_2 q_1 q_2 \right] \times (|p_0| + p_3)^{-1} \right\} \exp \left\{ -i \sum_{\mu=0}^3 p^\mu x_\mu \right\} \quad (7.18)$$

Corresponding to the negative energy eigen value $-|p_0|$ we can proceed similarly by taking U_1^- and U_2^- instead of U_1^+ and U_2^+ in (7.14). It is seen that U_1^- and U_2^- are given by simply changing from $+|p_0|$ to $-|p_0|$ in (7.15) leading thus to the negative energy solution

$$\Psi_- = K \exp \left\{ -\frac{1}{2} \left[q_1^2 + q_2^2 + i p_1 (q_1^2 - q_2^2) - 2i p_2 q_1 q_2 \right] / (-|p_0| + p_3) \right\} \exp \left\{ -i \sum_{\mu=0}^3 p^\mu x_\mu \right\} \quad (7.19)$$

when energy is positive $(|p_0| + p_3) > 0$ always and Ψ_+ is normalizable and hence physically permissible. But when energy is $-ve$, $(-|p_0| + p_3) < 0$ always making $\Psi_- \sim \exp \left\{ \frac{1}{2} (q_1^2 + q_2^2) \right\}$ which is not normalizable and not physically permissible.

(ii) A negative energy wave equation

Now we shall derive the negative counterpart of the above equation which will admit only negative energy solution as physically

permissible. Following (7.3) let us define

$$\Gamma = \alpha_1 \alpha_2 \alpha_3 \beta = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (7.20)$$

Then $\Gamma^2 = -I$ and $\Gamma [D] \Gamma^{-1} = -[D]$ where $[D] = \left(\sum_{\gamma=1}^3 \alpha_\gamma p_\gamma - i\beta \right)$ and the wave equation (7.11) is written as

$$[D](q) \psi = p_0(q) \psi \quad (7.21)$$

Hence

$$\Gamma [D] \Gamma^{-1} \Gamma(q) \psi = +\Gamma p_0(q) \psi$$

or

$$[D](\Gamma(q)) \psi = -p_0(\Gamma(q)) \psi \quad (7.22)$$

This shows that if $(q)\psi$ satisfies (7.21) corresponding to an energy eigenvalue p_0 , then $(\Gamma(q))\psi$ satisfies (7.22) corresponding to an energy eigenvalue $-p_0$. Thus by a transformation $(q) \rightarrow (\Gamma(q)) = (k)$ we arrive at an equation similar to (7.21) or (7.4) but with only negative energy solution being normalizable and hence physically permissible. This negative energy relativistic wave equation would then read

$$(p_0 - \sum_{\gamma=1}^3 \alpha_\gamma p_\gamma + i\beta)(k)\psi = 0 \quad (7.23)$$

or which is same as

$$\left\{ \frac{\partial}{\partial x_0} + \sum_{\gamma=1}^3 \alpha_\gamma \frac{\partial}{\partial x_\gamma} + \beta \right\} (k)\psi = 0 \quad (7.24)$$

where

$$(k) \equiv (\Gamma(q)) = \begin{pmatrix} -q_4 \\ q_3 \\ -q_2 \\ q_1 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} \quad (7.25)$$

Now k 's obey

$$[k_a, k_b]_- = -i\beta_{ab} \quad ; \quad a, b = 1, 2, 3, 4. \quad (7.26)$$

with the same matrix β . Hence (7.24) would describe a particle with only negative energy states, when the internal variables obey the commutation relations (7.26) instead of (7.9). Perhaps this may be physically interpreted that the internal oscillators are flowing backward in time. Thus the wave equation

$$\left\{ \frac{\partial}{\partial x_0} + \sum_{\gamma=1}^3 \alpha_\gamma \frac{\partial}{\partial x_\gamma} + \beta \right\} (\psi) \psi = 0 \quad (7.27)$$

with $[\psi_a, \psi_b]_- = i\varepsilon \beta_{ab}$; $\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

would describe a particle with only positive energy states if $\varepsilon = +1$ (Dirac) and would describe a particle with if only negative energy states if $\varepsilon = -1$. The solutions of the negative energy equation are given by, exactly similar to (7.18) and (7.19) except for the interchange their roles,

$$\psi_- = K \exp \left\{ \frac{i}{2} (q_1^2 + q_2^2 + i[q_1^2 - q_2^2] - 2ip_2 q_1 q_2) / (-ip_0 + p_3) \right\} \times \exp \left\{ -i \sum_{\mu=0}^3 p^\mu x_\mu \right\} \quad (7.28)$$

$$\psi_+ = K \exp \left\{ \frac{i}{2} [q_1^2 + q_2^2 + i(q_1^2 - q_2^2) - 2ip_2 q_1 q_2] / (ip_0 + p_3) \right\} \times \exp \left\{ -i \sum_{\mu=0}^3 p^\mu x_\mu \right\} \quad (7.29)$$

Clearly it is seen that only ψ_- is normalizable and hence physically permissible. Starting with the equation (7.21) with q 's obeying now

instead of (7.9),

$$[q_V a, q_V b]_- = -i \beta_{ab} ; \quad a, b = 1, 2, 3, 4. \quad (7.30)$$

Using an explicit representation

$$q_V 3 = i \frac{\partial}{\partial V_1} ; \quad q_V 4 = i \frac{\partial}{\partial V_2} \quad (7.31)$$

the solutions (7.28) and (7.29) can be obtained in exactly similar manner to the derivation of (7.18) and (7.19) and hence we do not repeat this procedure of solution of the negative energy equation here.

(iii) Relativistic invariance and spin

We shall follow Dirac in discussing the relativistic invariance of the equation (7.31). It is seen that both cases $\epsilon = \pm 1$ can be treated simultaneously in the same manner as follows. Equation (7.10) can be written as

$$\left(\sum_{\mu=0}^3 \gamma_\mu \delta^\mu - 1 \right) (q) \psi = 0 \quad (7.32)$$

where $\gamma_\mu = \beta \alpha_\mu$ and $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2 \gamma_{\mu\nu}$; $\mu, \nu = 0, 1, 2, 3$. (7.32) is not manifestly relativistic since the four γ 's cannot be regarded as a 4-vector as γ_0 is skew while $\gamma_1, \gamma_2, \gamma_3$ are symmetrical. Hence let us consider the original form (7.10) in the satisfying

$$\alpha_\mu \beta \alpha_\nu + \alpha_\nu \beta \alpha_\mu = 2 \beta \gamma_{\mu\nu} ; \quad \mu, \nu = 0, 1, 2, 3. \quad (7.33)$$

Applying an infinitesimal Lorentz Transformation

$$x_\mu^* = x_\mu + \sum_{\nu=0}^3 a_\mu^\nu x_\nu \quad (7.34)$$

$$\delta^\mu = \delta^\mu + \sum_{\nu=0}^3 a_\nu^\mu \delta^\nu ; \quad \mu = 0, 1, 2, 3.$$

the $\{a_{\mu\nu}\}$ being infinitesimal coefficients such that $a_{\mu\nu} = -a_{\nu\mu}$

The equation (7.10) then gives to first order

$$\left\{ \sum_{\mu=0}^3 a_{\mu} (\partial^{\mu*} - \sum_{\nu=0}^3 a_{\mu\nu}^* \partial^{\nu*}) + \beta \right\} (\varphi) \psi = 0 \quad (7.35)$$

or

$$\left\{ \sum_{\mu=0}^3 (a_{\mu} + \sum_{\nu=0}^3 a_{\mu\nu}^* \partial^{\nu*}) \partial^{\mu*} + \beta \right\} (\varphi) \psi = 0 \quad (7.36)$$

Defining

$$N = \frac{1}{4} \sum_{\sigma=0}^3 a_{\sigma}^* d_{\rho} \beta d_{\sigma} \quad (7.37)$$

$$\tilde{N} = N$$

we have

$$d_{\mu} \beta N - N \beta d_{\mu} = - \sum_{\sigma=0}^3 a_{\mu}^* d_{\sigma} \quad (7.38)$$

Thus multiplying (7.36) by $(1 - N \beta)$ on the left we get

$$\left\{ \sum_{\mu=0}^3 d_{\mu} + (1 - N \beta) \partial^{\mu*} + \beta + N \right\} (\varphi) \psi = 0 \quad (7.39)$$

$$\text{or } \left(\sum_{\mu=0}^3 a_{\mu} \partial^{\mu*} + \beta \right) (1 - N \beta) (\varphi) \psi = 0$$

$$\left(\sum_{\mu=0}^3 a_{\mu} \partial^{\mu*} + \beta \right) (\varphi^*) \psi = 0 \quad (7.40)$$

where

$$(\varphi^*) = (1 - N \beta) (\varphi) \quad (7.41)$$

Thus the wave equation takes the same form in the new system of coordinates with (φ^*) replacing (φ) . The four φ^* 's are linear functions of the four φ 's with real coefficients (on account of a 's, β having only real elements). The new φ^* 's satisfy the

commutation relations

$$\epsilon = \pm 1$$

$$[\psi_a^*, \psi_b^*]_- = i\epsilon \beta_{ab}; \quad a, b = 1, 2, 3, 4. \quad (7.42)$$

to the first order, thus having the same properties as the ψ 's. This shows that the form of the wave equation is unchanged by an infinitesimal, and thus also finite Lorentz transformations not involving reflections.

There is a unitary transformation connecting ψ 's and ψ^* 's

Letting

$$W = (\tilde{\psi}) N(\psi) = \sum_{a,b=1}^4 \psi_a N_{ab} \psi_b \quad (7.43)$$

we have

$$W\psi_a - \psi_a W = -2i\epsilon \sum_{b,c=1}^4 \beta_{ab} N_{bc} \psi_c; \quad a = 1, 2, 3, 4. \quad (7.44)$$

or

$$W(\psi) - (\psi)W = -2i\epsilon \beta N(\psi)$$

an equation in which every term is a column matrix. Then to the first order (7.41) becomes

$$(\psi^*) = (1 - \frac{1}{2} i\epsilon W)(\psi) (1 + \frac{1}{2} i\epsilon W) \quad (7.45)$$

Having seen that the negative energy equation (7.27) with $\epsilon = -1$ is exactly similar to the positive energy equation (7.27) with $\epsilon = +1$ except for the change of ϵ from $+1 \rightarrow -1$, with regard to the relativistic covariance we shall, without repeating the contents of Dirac's paper, state that the spin operators are given by

$$S_{\rho\sigma} = \epsilon \left\{ \left(-\frac{1}{4} (\tilde{\psi}) d\rho^\beta d_\sigma (\psi) + \frac{1}{2} i g_{\rho\sigma} \right) \right\} \quad (7.47)$$

Explicitly with the choice of d^β in (7.12) we have

$$\begin{aligned} S_{01} &= \frac{1}{4} (\psi_1 \tilde{\psi}_2^2 - \psi_3^2 + \psi_4^2) & S_{12} &= \frac{1}{2} (\psi_2 \psi_3 - \psi_1 \psi_4) \\ S_{02} &= \frac{1}{2} (\psi_3 \psi_4 - \psi_1 \psi_2) & S_{23} &= \frac{1}{2} (\psi_1 \psi_2 + \psi_3 \psi_4) \\ S_{03} &= \frac{1}{2} (\psi_1 \psi_3 + \psi_4 \psi_2) & S_{31} &= \frac{1}{4} (\psi_1^2 - \psi_2^2 + \psi_3^2 - \psi_4^2) \end{aligned} \quad (7.48)$$

giving

$$S_{23}^2 + S_{31}^2 + S_{12}^2 = \frac{1}{16} (\gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2) - \frac{1}{4} \quad (7.49)$$

Defining the magnitude S of spin according to quantum mechanics by

$$S(S+1) = S_{23}^2 + S_{31}^2 + S_{12}^2$$

we find

$$S = \frac{1}{4} (\gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2) - \frac{1}{2} \quad (7.50)$$

The eigenvalues of $\frac{1}{2}(\gamma_1^2 + \gamma_3^2)$ and $\frac{1}{2}(\gamma_2^2 + \gamma_4^2)$ are $m + \frac{1}{2}$, and $m' + \frac{1}{2}$, with m and m' non-negative integers. Thus eigenvalues of S are $\frac{1}{2}(m+m')$ which are always integers. These results apply to both the case $\epsilon = \pm 1$, thus showing that the case $\epsilon = -1$ would perfectly well describe a particle with only negative energy states and integer spin. In both cases value of the spin S depends on the wave function and thus on the momentum of the particle. The quantities $S_{\rho\sigma}$ ($\rho, \sigma = 0, 1, 2, 3$) provide a representation of the infinitesimal operators of the Lorentz group. They are associated mathematically with four more quantities

$$S_{\mu\bar{\nu}} = -S_{\bar{\nu}\mu} = \frac{1}{4}\epsilon(\tilde{q})\delta_{\mu\bar{\nu}}(q); \mu = 0, 1, 2, 3. \quad (7.51)$$

The ten quantities $S_{ab} = -S_{ba}$ ($a, b = 0, 1, 2, 3, 5$) then provide a representation of the $3 + 2$ de Sitter group as discovered by Dirac⁴ in 1963 for the case of $\epsilon = +1$. We observe that in both cases of $\epsilon = \pm 1$, this is true as can be verified directly

$$[S_{ab}, S_{ac}] = S_{bc} \text{ for } a = 0, 1, 2, 3$$

$$= -S_{bc} \text{ for } a = 4, 5$$

(7.53)

which are the commutation relations of the infinitesimal operators of the $3 + 2$ de Sitter group, the group of rotations of five real variables x_0, x_1, x_2, x_3, x_5 which leave the quadratic form $x_1^2 + x_2^2 + x_3^2 - x_0^2 - x_5^2$ invariant.

I am very grateful to Professor Dirac for his comment on the equation (7.24) that it would correctly describe a particle with only negative energy states and would be the counterpart to his positive energy equation (7.4). But he feels that the equation (7.24) would not have physical application⁵. Of course, one can not predict the failure of a theory simply because it seems improbable. If a time comes when nature reveals the existence of a particle with only positive energy states described exactly by Dirac's equation should one not search for its probable counterpart?

Summary of important points

Dirac's new relativistic wave equation for a particle of unit mass ($\hbar = c = 1$) is

$$\left(\frac{\partial}{\partial x_0} + \sum_{r=1}^3 \alpha_r \frac{\partial}{\partial x_r} + \beta \right) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \Psi(v, q_2, x_\mu) = 0$$

$\mu = 0, 1, 2, 3$

with

$$[q_a, q_b] = i \beta_{ab} ; \quad a, b = 1, 2, 3, 4$$

and

$$\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} ; \quad \alpha_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ; \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} ; \quad \alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This allows only positive energy states as physically permissible.

By changing the commutation relations of internal variables
 φ_b as

$$[\varphi_a, \varphi_b] = -i\beta_{ab}; \quad a, b = 1, 2, 3, 4.$$

We observe that the same equation describes particles with only negative energy states.

CHAPTER 8ON CLIFFORD'S GENERALIZING QUATERNION ALGEBRAS

In his fundamental paper 'Applications of Grassmann's Extensive algebra' (American Journal of Mathematics Pure and Applied Vol.1, pp.360-388 (1870), H.E.Clifford¹ shows that his $\frac{2\mu+1}{2\mu+1}$ -way geometric algebra² - now called the Clifford algebra $C_{2\mu+1}^{(2)}$ is a compound of $\frac{2\mu}{2\mu}$ quaternion algebras, the units of which are commutative with one another. In the development of \mathbb{L} -matrix such commuting structures, called Generalized Helicity matrices have been built by Alladi Ramakrishnan and his collaborators³. Extended version of matrix decomposition theorems due to Alladi Ramakrishnan and myself considered in Chapter II are also associated with such structures. Here we shall see that these are generalizations of Clifford result.

Hamilton's quaternions apart from being of great mathematical importance as a generalisation of the field of complex numbers, have often been considered to be of importance to physics. It was recognised early that special relativity can be elegantly written in quaternion notation⁴. But it never gained any popularity due to the greater convenience of using tensors. With the invention of spin of electron and Dirac equation there was again a renewed attempt to introduce the quaternions⁵ but spinor formalism took over. Possibility of replacing complex numbers by quaternions in quantum mechanics has been studied by several authors⁶ and also there have been attempts to describe elementary particles by means of quaternions⁷.

There seems to be no where any reference to Clifford's commuting quaternion algebras. A careful study of his original paper gives a clue to the concept of tensor product and his commuting structures are most easily understood in terms of tensor products in the modern mathematical language.

According to Clifford if $Q_l^v(q)$ and $Q_j^v(q')$ are two different types of quaternions associated with the Clifford algebra $C_{n+1}^{(2)}$ then

$$Q_l^v(q) Q_j^v(q') = Q_j^v(q') Q_l^v(q) \text{ for } l \neq j, q = v \neq v'$$

$$Q_l^v(q) Q_j^v(q') \neq Q_j^v(q') Q_l^v(q) \text{ if } q \neq v' \quad (2.1)$$

where q and q' are 4-tuples (q_0, q_1, q_2, q_3) and (q'_0, q'_1, q'_2, q'_3) respectively.

The irreducible representations of a quaternion in terms of Pauli matrices are given by

$$\begin{aligned} Q(q) &= q_0 e + q_1 \underline{i} + q_2 \underline{j} + q_3 \underline{k} \\ &= \begin{pmatrix} q_0 - iq_3 & -iq_1 - q_2 \\ -iq_1 + q_2 & q_0 + iq_3 \end{pmatrix} = q_0 I + \sum_{j=1}^3 q_j (-i\sigma_j) \quad (2.2) \end{aligned}$$

with the units (e, i, j, k) obeying

$$\underline{i}^2 = \underline{j}^2 = \underline{k}^2 = -1; \quad e^2 = +1$$

$$ei = \underline{i}e, \quad \underline{j}e = ej, \quad ek = \underline{k}e. \quad \underline{i}\underline{j} = \underline{k}, \quad \underline{j}\underline{k} = \underline{i}, \quad \underline{k}\underline{i} = \underline{j} \quad (2.3)$$

Clifford algebra $C_n^{(2)}$ has generating relations

$$\begin{aligned} k_i k_j &= -k_j k_i \quad i, j = 1, \dots, n \\ k_i^2 &= 1 \quad \forall i = 1, \dots, n. \end{aligned} \quad (2.4)$$

Quaternion algebra is same as the Clifford algebra $C_2^{(2)}$ except for a difference in phase factors of the generators. $C_2^{(2)}$ has two generators ℓ_1, ℓ_2 which can be identified as $\ell_1 = i\hat{i}$, $\ell_2 = ij$, $\ell_3 = ik = -ijk$. Clifford constructed algebraic structures obeying (3.1) starting from the Clifford algebra $C_{2\nu+1}^{(2)}$.

Following Alladi Renekrichnan and his collaborators, we shall consider the construction of mutually commuting subalgebras of the Generalized Clifford algebra $C_{2\nu}^{(m)}$ which is generated by the relations

$$\ell_i \ell_j = \omega(m) \ell_j \ell_i ; i, j = 1, \dots, 2\nu. \quad (3.2)$$

$$\ell_i^m = 1 ; i = 1, \dots, \nu.$$

The entire basis of the algebra $C_{2\nu}^{(m)}$ is $\left\{ \prod_{i=1}^{2\nu} \ell_i^{k_i} \mid 0 \leq k_i \leq m-1 \right\}$
Now define iteratively

$$\ell_{2i-1}^* = H_1^i = (H_3^{\nu})^{m-1} (H_3^{\nu-1})^{m-1} \dots (H_3^{i+1})^{m-1} L_{2(\nu-i)+1} \quad (3.3)$$

$$\ell_{2i}^* = H_2^i = (H_3^{\nu})^{m-1} (H_3^{\nu-1})^{m-1} \dots (H_3^{i+1})^{m-1} L_{2(\nu-i)+2} \quad (3.4)$$

$$H_1^i H_2^i = \omega(m) H_2^i H_1^i ; i = 1, \dots, \nu ; [H_3^i = \zeta (H_1^i)(H_2^i)] ; \zeta = \begin{cases} \omega^{\frac{k}{2}} & \text{if } k \text{ even} \\ 1 & \text{if } k \text{ odd} \end{cases} \quad (3.5)$$

Each pair $\{H_1^i, H_2^i\}$ generates a subalgebra with a basis

$\{H_1^i k_1, H_2^i k_2 \mid 0 \leq k_1, k_2 \leq m-1\}$ and elements of one subalgebra commutes with the elements of another. If we substitute the explicit matrix representations of ℓ_i 's given in Chapter I, we see that

$$H_1^i = \underbrace{I \otimes \dots \otimes I}_{\nu-i} \otimes C \otimes \dots \otimes I \quad (3.6)$$

$$H_2^i = \underbrace{I \otimes \dots \otimes I}_{\nu-i} \otimes B \otimes \dots \otimes I \quad (3.7)$$

and hence the subalgebras generated by $\{H_1^i, H_2^i\}$ for $i=1 \dots v$
 each isomorphic to $C_2^{(m)}$. When the direct product representation
 is used it is easy to see that any element of $C_{2v}^{(m)}$ is a linear
 combination of matrices which are direct products of basic matrices
 or using (8.2) one sees that any element of $C_{2v}^{(m)}$ is a linear
 combination of a set of basic matrices $\{\prod_{i=1}^v H_1^{k_i} H_2^{l_i}\}$. Any such
 element $\sum_{k_i, l_i=0}^{m-1} a_{k_1 l_1 \dots k_v l_v} H_1^{k_1} H_2^{l_1} \dots H_1^{k_v} H_2^{l_v}$ can be written, due
 to the commutativity of different pairs $\{H_1^i, H_2^j\}; i, j = 1 \dots v$, as

$$\sum_{k_i, l_i} \left\{ \sum_{l \geq 1} H_1^{k_1 l_1} H_2^{k_2 l_2} \dots a_{k_1 l_1, k_2 l_2 \dots k_v l_v} \right\} \prod_{i \geq 1} H_1^{k_i} H_2^{l_i}$$

This means that elements of $C_{2v}^{(m)}$ can be considered as elements of $C_{2v-2}^{(m)}$ with coefficients as elements of $C_2^{(m)}$. Again elements of $C_{2v-2}^{(m)}$ can be considered as elements of $C_{2v-4}^{(m)}$ with co-efficients as elements in $C_2^{(m)}$ and so on. Thus the algebra $C_{2v}^{(m)}$ can be considered as a 'compound' of v commuting subalgebras all isomorphic to $C_2^{(m)}$. When $m=2$, $C_2^{(2)}$ is the quaternion algebra and hence Clifford's proposition immediately follows that $C_{2v}^{(2)}$ is a 'compound' of v commuting quaternion algebras. In terms of matrix representation this means that any $m^v \times m^v$ matrix may be regarded as a compound of v , m -dimensional matrices if first we can regard it as a $m \times m$ matrix where its elements are $m^{v-1} \times m^{v-1}$ matrices which are again $m \times m$ matrices of $m^{v-2} \times m^{v-2}$ matrices and so on. These are the well known results of partitioning of matrices. This is the basic principle behind the generalised matrix decomposition theorem considered in Chapter XI.

More generally one can consider any $N \times N$ matrix algebra as a compound of many matrix algebras of different dimensions if N is a *composite integer*. This is achieved by representing the two basic generators of the basis of $C_2^{(N)}$, say $C(N), B(N)$ obeying $C(N)B(N) = \omega(N)B(N)C(N)$ as direct product of smaller dimensional C, B matrices. We shall consider such a commutative factorization of the basis, as Schwinger⁸ calls it, corresponding to the factorization of N as $\prod_{i=1}^r p_i^{\alpha_i}$ where p_i 's are distinct primes, and $\alpha_i > 0, \forall i=1 \dots r$. We can put

$$C(N) = C(p_1^{\alpha_1})^{\beta_1} \otimes \dots \otimes C(p_r^{\alpha_r})^{\beta_r} \quad (8.9)$$

$$B(N) = B(p_1^{\alpha_1}) \otimes \dots \otimes B(p_r^{\alpha_r})$$

where $C(p_i^{\alpha_i}) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \dots & -1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$ of dimension $p_i^{\alpha_i}$, $B(p_i^{\alpha_i}) = \begin{pmatrix} 1 & w(p_i^{\alpha_i}) & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & w(p_i^{\alpha_i})^{p_i^{\alpha_i}-1} \end{pmatrix}$

$$w(p_i^{\alpha_i}) = \exp\left(2\pi i / p_i^{\alpha_i}\right)$$

$$C(p_i^{\alpha_i})B(p_i^{\alpha_i}) = w(p_i^{\alpha_i}) B(p_i^{\alpha_i}) C(p_i^{\alpha_i}) ; \forall i=1 \dots r \quad (8.10)$$

$$B(p_i^{\alpha_i})^{p_i^{\alpha_i}} = C(p_i^{\alpha_i})^{p_i^{\alpha_i}} = 1.$$

If p_i 's are chosen such that

$$N \sum_{i=1}^r \frac{p_i^{\alpha_i}}{p_i^{\alpha_i}} = 1 \text{ mod. } N. \quad (8.11)$$

then $C(N), B(N)$ obey

$$C(N)B(N) = \omega(N)B(N)C(N) \quad (8.12)$$

Since

$$\prod_{i=1}^r \omega(p_i^{\alpha_i})^{\beta_i} = \exp \left(2\pi i \sum_{i=1}^r \frac{\beta_i}{p_i^{\alpha_i}} \right) = \exp \left(\frac{2\pi i}{N} \right) = \omega(N)$$

due to (8.11) (8.13)

In general one can choose

$$C(N) = \prod_{i=1}^r \otimes C(p_i^{\alpha_i})^{\mu_i} B(p_i^{\alpha_i})^{\lambda_i}$$

$$B(N) = \prod_{i=1}^r \otimes C(p_i^{\alpha_i})^{\nu_i} B(p_i^{\alpha_i})^{\delta_i}$$

where

$$N \sum_{i=1}^r \frac{1}{p_i^{\alpha_i}} \begin{vmatrix} \mu_i & \lambda_i \\ \nu_i & \delta_i \end{vmatrix} = 1 \bmod N. \quad (8.15)$$

Since prime number decomposition of an integer is unique and the maximum possible starting from this one can easily show that if $N = n_1 n_2 \dots n_k$ where n_i 's are not necessarily powers of primes, $C(N)$, $B(N)$ can be expressed as tensor direct product of powers of $C(n_i)$'s and $B(n_i)$'s $i=1 \dots k$. These results leads to the fact of partitioning that $N \times N$ matrix algebra can be regarded as a spin compound of matrix algebras of dimensions n_1, n_2, \dots, n_k if $N = n_1 n_2 \dots n_k$. For the corresponding G.C.A. $C_{\frac{N}{2}}^{(N)}$ this implies that $C_{\frac{N}{2}}^{(N)}$ can be regarded as a spin compound of commuting algebras isomorphic to $(C_{\frac{n_1}{2}}, \dots, C_{\frac{n_k}{2}}^{(n_k)})$

Of all these results the original result of Clifford that $C_{\frac{N}{2}}^{(2)}$ is a compound of N -commuting quaternion algebras is perhaps of some interest for physics especially for attempts⁵ to generalise quantum mechanics by enlarging the underlying number field from

complex numbers to quaternions. The principal conceptual difficulty realised in such generalizations is in the theory of composite systems where the ordinary tensor product fails due to the non-commutativity of the quaternions. This prevents the formation of a composite system in such a way that all the observables associated with one of the systems commute with all the observables of the other system. Now in the light of the fact that a Clifford algebra can be viewed as a compound of commuting quaternion algebras, perhaps it is worthwhile to investigate the possibility of using commuting quaternions in the description of independent systems and forming composite systems by usual tensor product. It should be noted that all these commuting quaternion algebras are not totally independent - there is a common thread - they can be built out of elements of a Clifford algebra.

Summary of important points.

Expressing any matrix of dimension $N = \prod_{i=1}^r n_i$ in a basis given by $\left\{ \prod_{i=1}^r \otimes C(n_i)^{k_i} B(n_i)^{l_i} \mid 0 \leq k_i, l_i \leq n_i - 1 \right\}$ it can be observed that the G.C.A. $C_2^{(N)} \cong M_N$ the total $N \times N$ matrix algebra can be thought of as a 'compound' of r G.C.A's $\left\{ C_2^{(n_i)} \mid i = 1 \dots r \right\}$. In terms of matrices this means that any $N \times N$ matrix can be thought of as a compound of matrices of order n_1, n_2, \dots, n_r where $N = \prod_{i=1}^r n_i$; ie first $N \times N$ the matrix can be thought of as a $n_1 \times n_1$ matrix (Partitioning) with each of its elements as an $n_2 \times n_2$ matrix, with each of its elements as an $n_3 \times n_3$ matrix, and so on. When $N = 2^r$

$$C_2^{(2^r)} \cong M_{2^r} \cong C_{2^r}^{(2)}$$

and hence the Clifford algebra $C_{2^r}^{(2)}$

generated by $2r$ anticommuting elements is a compound of r Clifford algebras $C_2^{(2)}$ which are Hamilton's quaternion algebras. This in 1878, in his fundamental paper on what we now call as Clifford algebra, W.K.Clifford proved that the Clifford algebra $C_{2r}^{(2)}$ is a compound of r commuting quaternion algebras !

(1) In the same year, the German mathematician and physicist Hermann Grassmann proposed an algebraic expansion of vectors, no dot and cross product of the elements and also the concept of exterior products of vectors and the elements are called "exterior forms". These basic ideas were developed by the German mathematician and physicist Sophus Lie in 1886. He introduced the concept of Lie-algebra which became the foundation of the theory of Lie groups. He studied and worked² this extended algebraic representation of Lie groups involving vector composition and multiplication. He could have chosen a suitable element of vector or matrix rather than the complex numbers. In contrast, when we follow the other, a representation theory of Lie groups using different base elements, albeit representation was the same, the mathematical formulation of the representation theory was so cumbersome. He tried to prove in this paper that all of Lie's findings³ could be easily deduced from the theory of Lie groups. He did not succeed in this, but he did prove that the theory of Lie groups is a generalization of the theory of Lie algebras. He also showed that the theory of Lie groups is a generalization of the theory of Lie algebras. The group theory of Lie groups is a generalization of the theory of Lie algebras.

CHAPTER 9.ON GENERALIZATION OF CERTAIN GEOMETRICAL ASPECTS OF CLIFFORDALGEBRA TO GENERALIZED CLIFFORD ALGEBRA

(1) Since the time of Clifford it is well known that a Clifford algebra² $C_n^{(2)}$ can be regarded as algebra of aggregates of scalar, vector and polyvectors in the n-dimensional Euclidean space. This is what made Clifford call his algebra as geometric algebra¹. Since then there have been many investigations by Lipschitz², Cartan³, Brauer and Weyl⁴, Freudenthal⁵ and others on the geometrical aspects of Clifford algebra which became the foundation of the theory of spinors. Popovici and Turtoi⁶ have considered certain generalization of spinor structures using generalized Clifford algebra. Rosevskii⁷ has given a beautiful account of theory of spinors using entirely geometrical approach to Clifford algebra and he has also given a representation theory of Clifford algebras different from others. Alladi Ramakrishnan and his collaborators⁸ have considered generalization of his representation theory, but it is incomplete. In this chapter we shall first record an 'algebraic' (since I have failed to see any 'geometry' behind it!) generalization of the geometric approach to Clifford algebra, to a generalization of Rosevskii's theory of representation of Clifford algebras for Generalized Clifford algebras. The generators of Clifford algebra $C_n^{(2)}$, $\{L_i | i = 1 \dots n\}$ obey

$$L_i L_j + L_j L_i = 2 \delta_{ij}; \forall i, j = 1 \dots n. \quad (9.1)$$

Associated with a vector $\underline{a} = (a^1, a^2, \dots, a^n)$ in n-dimensional complex Euclidean space R_n^+ , is an element of $C_n^{(2)}$

$$\underline{a} = \sum_{i=1}^n a^i L_i \quad (9.2)$$

Then the norm of the vector \underline{a} , $\|\underline{a}\|$ is defined by

$$\|\underline{a}\|^2 = \sum_{i=1}^n (a^i)^2 = a \cdot a \quad (9.3)$$

Thus the element $\sum_{i=1}^n a^i L_i \in C_n^{(2)}$ is regarded as a vector in R_n^+ . The scalar product of two vector becomes

$$(\underline{a}, \underline{b}) = \sum_{i=1}^n a^i b^i = \frac{1}{2} (ab + ba) = \frac{1}{2} \{a, b\} \quad (9.4)$$

where $a = \sum_{i=1}^n a^i L_i$ and $b = \sum_{i=1}^n b^i L_i$. Two orthogonal vectors have $(\underline{a}, \underline{b}) = \frac{1}{2} \{a, b\} = 0$ and hence the basic relations (9.1) denote the orthogonality of the basis $\{L_i | i=1 \dots n\}$.

A totally antisymmetric tensor of rank k is called a simple k -vector if it is obtained as a skew-product of k vectors

$\underline{p}_1 \dots \underline{p}_k$ as

$$a^{i_1, i_2, \dots, i_k} = \frac{1}{k!} \begin{vmatrix} p_1^{i_1} & p_1^{i_2} & \dots & p_1^{i_k} \\ p_2^{i_1} & p_2^{i_2} & \dots & p_2^{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ p_k^{i_1} & p_k^{i_2} & \dots & p_k^{i_k} \end{vmatrix} \quad (9.5)$$

symbolically the k -vector a is written as

$$a = P[\underline{p}_1, \dots, \underline{p}_k] \quad (9.6)$$

To this is associated the element

$$a = \sum_{i_1=1}^n a^{i_1 i_2 \dots i_k} L_{i_1} L_{i_2} \dots L_{i_k} \quad (0.7)$$

$(1 \leq j \leq k)$

where $L_{i_1} L_{i_2} \dots L_{i_k} = L_{i_1 i_2 \dots i_k}$ represents a basic k -vector. A linear combination of a scalar a^0 , vector $\{a^i\}$, bivector $\{a^{i_1 i_2}\}$... an n -vector $\{a^{i_1 i_2 \dots i_n}\}$ is called an aggregate and is written as

$$A = a^0 I + \sum_{i=1}^n a^i L_i + \sum_{i_1, i_2=1}^n a^{i_1 i_2} L_{i_1 i_2} + \dots + \sum_{i_1, i_2, \dots, i_n=1}^n a^{i_1 i_2 \dots i_n} L_{i_1 i_2 \dots i_n} \quad (0.8)$$

The product of two basic polyvectors being again a polyvector products of two aggregates is well defined and is again another aggregate. These aggregates are thus elements of an algebra and this algebra is called Clifford algebra. (For more details cf. Bascovskii⁷).

If we choose $\{L_i \mid i=1 \dots n\}$ to obey generalised Clifford condition

$$\begin{aligned} L_i L_j &= \omega(m) L_j L_i; \quad \omega(m) = \exp(2\pi i/m) \\ i < j &\quad i, j = 1 \dots n. \\ L_i^m &= 1 \quad \forall i = 1 \dots n. \end{aligned} \quad (0.9)$$

instead of (0.1) to the corresponding element of (0.7) namely

$a = \sum_{i=1}^n a^i L_i$ can be associated a norm $\|a\|_m$ such that

$$(\|a\|_m)^m = a^m = \sum_{i=1}^n (a^i)^m \quad (0.10)$$

Q2

$$\|\alpha\|_m = \left(\sum_{i=1}^n (\alpha^i)^m \right)^{\frac{1}{m}}$$

The basis of \mathcal{C} a generalized Clifford algebra (G.C.A.)

$C_n^{(m)}$ generated by (9.9) is given by $\left\{ \prod_{k=1}^m L_k^{i_k} \mid 0 \leq i_k \leq m-1; \forall k=1 \dots n \right\}$
and elements of it are aggregates of the type

$$A \equiv a^0 + \sum_{i=1}^n a^i L_i + \sum_{i_1, i_2=1}^n a^{i_1 i_2} L_{i_1} L_{i_2} + \sum_{i_1, i_2, i_3=1}^n a^{i_1 i_2 i_3} L_{i_1} L_{i_2} L_{i_3} + \dots$$

$$\dots + \sum_{i_1, i_2, \dots, i_{n(m-1)}=1}^n a^{i_1 i_2 \dots i_{n(m-1)}} L_{i_1} L_{i_2} \dots L_{i_{n(m-1)}}$$

(9.11)

in which maximum ($n-1$) indices can take the same value. In the case of $C_n^{(2)}$ the polyvector $\alpha = P[p_1, \dots, p_n]$ can be represented as

$$\alpha = \begin{pmatrix} \sum_i p_1^i L_i & \sum_i p_2^i L_i & \dots & \sum_i p_n^i L_i \\ \vdots & & & \\ \sum_i p_1^i L_i & \sum_i p_2^i L_i & \dots & \sum_i p_n^i L_i \end{pmatrix}$$

(9.12)

which is the exterior product or Grassmann product of the vectors (p_1, \dots, p_n) .

Now using a generalisation of the determinant due to Ranganathan⁹ the basic components of the aggregate A (8.11) can be written in a similar form. If we define following Ranganathan

$$\det_{\lambda} A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ | & | & & | \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}_{\lambda} = \sum_{i=1}^n \lambda^{i-1} a_{1i} A_{1i}(\lambda) \quad (8.13)$$

where $A_{1i}(\lambda)$ is the usual cofactor of a_{1i} with the difference that λ replaces (-1) in its evaluation when $\lambda = -1$ it becomes the usual determinant. For example

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}_{\lambda} = a(ek + \lambda fh) + \lambda b(dk + \lambda fg) + \lambda^2 c(ah + \lambda eg) \\ = aek + \lambda afh + \lambda bdk + \lambda^2 bfg + \lambda^2 cdh + \lambda^3 ceg \quad (8.14)$$

Then the quantity

$$W[p_1, \dots, p_k] = \begin{vmatrix} \sum_i p_1^i L_i & \sum_i p_2^i L_i & \cdots & \sum_i p_k^i L_i \\ | & | & & | \\ \sum_i p_1^i L_i & \sum_i p_2^i L_i & \cdots & \sum_i p_k^i L_i \end{vmatrix}_{w(m)} \quad (8.15)$$

generalises (8.12). This becomes

$$W[p_1, \dots, p_k] = \sum_{i_1, i_2, \dots, i_k=1}^n a^{i_1 i_2 \dots i_k} W[L_{i_1}, L_{i_2}, \dots, L_{i_k}] \quad (8.16)$$

where

$$\alpha^{i_1 i_2 \dots i_k} = \begin{vmatrix} p_1^{i_1} & p_1^{i_2} & \dots & p_1^{i_k} \\ \vdots & \vdots & & \vdots \\ p_k^{i_1} & p_k^{i_2} & \dots & p_k^{i_k} \end{vmatrix}_{\omega(m)} \times (N_{i_1 i_2 \dots i_k})^{-1}$$

$$N_{i_1 i_2 \dots i_k} L_i L_{i_2} \dots L_{i_k} = \begin{vmatrix} L_{i_1} L_{i_2} \dots L_{i_k} \\ \vdots \\ L_{i_1} L_{i_2} \dots L_{i_k} \end{vmatrix}_{\omega(m)} = W [L_{i_1} L_{i_2} \dots L_{i_k}]$$

(9.17)

The aggregate A can be rewritten as

$$A = a^0 + \sum_{i_1, i_2=1}^n a^{i_1 i_2} W [L_{i_1} L_{i_2}] + \dots + \sum_{i_1, i_2, \dots, i_{n-1}=1}^n a^{i_1 i_2 \dots i_{n-1}} W [L_{i_1} \dots L_{i_{n-1}}]$$

(9.18)

in which an index can get repeated only upto $(n-1)$ times. Product of two quantities of the type (9.18) is well defined and is again a quantity of the same type. Thus this demonstrates an 'algebraic' generalisation of the geometrical aspect of Clifford algebra. May be these quantities (9.18) can also be considered as 'geometric' in a space with the $\|\cdot\|_m$ norm. But unlike the case of Euclidean space with $n = 3$, there does not exist continuous group of $\|\cdot\|_m$ norm preserving transformations of $\alpha = \sum_{i=1}^n a^i L_i$. The group

of $\| \cdot \|_m$ norm preserving transformations are trivial diagonal and permutation transformations as established by Horinaga and Nono¹⁰ and we have already discussed them in Chapter III.

(ii) Now we shall give the correct generalized version of Rosevskii's approach to representation of G.C.A. generated by (9.9). To represent $C_{2N}^{(m)}$ we start with the basis of $C_v^{(m)}$. Any element of $C_v^{(m)}$ is written as

$$\Lambda_v = \sum_{\substack{i_1, i_2, \dots, i_v \\ i_1 + i_2 + \dots + i_v = 0}}^{m-1} \lambda_{i_1, i_2, \dots, i_v} L_1^{i_1} L_2^{i_2} \dots L_v^{i_v} \quad (9.10)$$

Associate with the set of m^v basis elements $\left\{ \prod_{k=1}^v L_k^{i_k} \right\}$ linearly independent vectors of an m^v -dimensional vector space. Thus Λ_v is associated with an m^v -dimensional vector

$$\Lambda_v = \sum_{\substack{i_1, i_2, \dots, i_v \\ i_1 + i_2 + \dots + i_v = 0}}^{m-1} \lambda_{i_1, i_2, \dots, i_v} \underline{u}_{i_1, i_2, \dots, i_v} \quad (9.11)$$

where $\left\{ \underline{u}_{i_1, i_2, \dots, i_v} \right\}$ spans the m^v -dimensional space. Now define a set of $2N$ operators (matrices) L_1, L_2, \dots, L_{2N} by the following relations

$$a) \quad \underline{\Lambda}_v L'_{v-i+1} = \underline{L}_i \underline{\Lambda}_v ; \quad i=1 \dots v$$

$$b) \quad \zeta \underline{L}_i \underline{\Lambda}_v^* = \underline{L}_{v+i} \underline{\Lambda}_v ; \quad i=1 \dots v \quad (9.12)$$

$$\zeta = \begin{cases} \omega(m)^{\frac{1}{2}} & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$$

where

$$\Lambda_v^* = \sum_{\substack{i_1, i_2, \dots, i_v=0 \\ i_1+i_2+\dots+i_v=m}}^{m-1} w(m) \lambda_{i_1, i_2, \dots, i_v} L_1^{i_1} L_2^{i_2} \dots L_v^{i_v}$$

$$(k \cdot \text{mod } m = i_1 + i_2 + \dots + i_v)$$

(9.22)

Then the $2v$ operators (matrices) obey the generating relations
of $C_{2v}^{(m)}$

$$\begin{aligned} L_i L_j &= w(m) L_j L_i; \quad i, j = 1, 2, \dots, 2v \\ L_i^m &= 1 \quad ; \quad i = 1 \dots 2v \end{aligned} \quad (9.23)$$

Proof.

$$\text{For } i, j \leq v : L_i L_j \Lambda_v = \Lambda_v L'_{v-j+1} L'_{v-i+1}$$

$$L_j L_i \Lambda_v = \Lambda_v L'_{v-j+1} L'_{v-i+1}$$

If $i < j$

$$L'_{v-j+1} L'_{v-i+1} = w(m) L'_{v-i+1} L'_{v-j+1}$$

$$\therefore L_i L_j = w(m) L_j L_i$$

For $i, j \geq v$

$$L_i L_j \Lambda_v = \zeta L'_{i-v} (\zeta L'_{j-v} \Lambda_v^*)^*$$

$$L_j L_i \Lambda_v = \zeta L'_{j-v} (\zeta L'_{i-v} \Lambda_v^*)^*$$

$$\text{If } i < j \quad L'_{i-v} L'_{j-v} = \omega(m) L'_{j-v} L'_{i-v}$$

$$\therefore \underset{i < j}{L_i L_j} = \omega(m) L_j L_i$$

For $i \leq v, j \geq v$

$$L_i L_j \underset{\sim}{\Lambda}_v = (\zeta L'_{j-v} \Lambda_v^*) L'_{v-i+1}$$

$$L'_j L_i \underset{\sim}{\Lambda}_v = \zeta L'_{j-v} (\Lambda_v L'_{v-i+1})^*$$

$$= \omega(m)^{m-1} \zeta L'_{j-v} \Lambda_v^* L'_{v-i+1}$$

$$\therefore \underset{i < j}{L_i L_j} = \omega(m) L_j L_i$$

Thus in all cases

$$L_i L_j = \omega(m) L_j L_i ; \begin{matrix} i < j \\ \forall i, j = 1 \dots 2v \end{matrix}$$

Since the generators of $\binom{m}{j}, \{L'_i | i=1 \dots v\}_{\sim}^m$ obey $L'_i{}^m = 1 ; \forall i=1 \dots v$

all those $\{L_i | i=1 \dots 2v\}$ obey automatically $L_i{}^m = 1, \forall i = 1 \dots 2v$.

due to the construction (o.2)

Summary of important points

An incomplete attempt has been made to obtain a generalization of the geometrical interpretation of Clifford algebra to suit Generalized Clifford Algebra. Basovskii's theory of representation of Clifford algebras has been generalized to obtain the representation of G.C.A., which is an improvement of an earlier version by Alladi Ramakrishnan and his collaborators.

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