Counting Similarity Classes of Tuples of Commuting Matrices Over a Finite Field

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DECLARATION

I, Uday Bhaskar Sharma, hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

Date: **Uday Bhaskar Sharma**

List of Publications

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Others

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Contents

Notations

The following notation will be used throughout this thesis:

We denote a finite field of order q by $\mathbf{F}_q$. For any positive integer n, $M_n(\mathbf{F}_q)$ denotes the space of $n \times n$ matrices over $\mathbf{F}_q$.

For any matrix $A \in M_n(\mathbf{F}_q)$, $Z_{M_n(\mathbf{F}_q)}(A)$ denotes the centralizer algebra of A, and $Z_{GL_n(\mathbf{F}_q)}(A)$ denotes the group of units in $Z_{M_n(\mathbf{F}_q)}(A)$ (i.e., the centralizer group of A).

For a positive integer, k, let $(A_1, \ldots, A_k) \in M_n(\mathbf{F}_q)^{(k)}$. The common centralizer of $(A_1, \ldots, A_k)$ is the intersection of the centralizer algebras of the A_i's: $\bigcap_{i=1}^k Z_{M_n(\mathbf{F}_q)}(A_i)$. We denote this common centralizer algebra by $Z_{M_n(\mathbf{F}_q)}(A_1, \ldots, A_k)$, and $Z_{GL_n(\mathbf{F}_q)}(A_1, \ldots, A_k)$ denotes its group of units.

Synopsis

1 Introduction

This thesis concerns the problem of enumerating isomorphism classes of n dimensional modules over polynomial algebras $\mathbf{F}_q[x_1, \ldots, x_k]$, in k variables over a finite field of order q. This problem is the same as the classification of k-tuples of commuting $n \times n$ matrices over a finite field up to simultaneous similarity. Let $c(n, k, q)$ denote the number of isomorphism classes of n-dimensional $\mathbf{F}_q[x_1, \ldots, x_k]$-modules. In this thesis, we analyse the asymptotic behaviour of $c(n, k, q)$ as a function of k, keeping n and q fixed. We also give an explicit formula of $c(n, k, q)$ for $n \leq 4$.

This thesis is divided into four chapters. In the first chapter, we state our main results about $c(n, k, q)$. In the second chapter, we discuss the preliminaries required for understanding the thesis. In the third chapter, we discuss the asymptotic behaviour of $c(n, k, q)$ as a function of k, for a fixed n and q. In the fourth chapter, we explicitly calculate $c(n, k, q)$ for $n = 2, 3, 4$, and any $k \geq 1$.

2 Asymptotic Behaviour of $c(n, k, q)$

We study the asymptoticity of $c(n, k, q)$ as a function of k.

A lot of work has been done in understanding asymptoticity properties within $M_n(\mathbf{F}_q)$, as a function of n. Most of this work concerns the probability of a matrix over a finite field, being of a certain kind. From the theory of the rational canonical form, for a fixed n, we can show that, $c(n, 1, q)$, the number of similarity classes in $M_n(\mathbf{F}_q)$, is asymptotically q^n up to multiplication by a constant factor, as a function of q. This is also the asymptotic behaviour of $c(n, 1, q)$ as a function of n, for a fixed q (see Stong [14, Corollary 5]). For a fixed q, Neumann and Praeger [11] proved that the probability of an $n \times n$ matrix over $\mathbf{F}_q$, being non-cyclic is asymptotically q^{-3} as a fucntion of n. They also proved that the probability of a matrix in $M_n(\mathbf{F}_q)$ being non-separable is asymptotically q^{-1}.

Now we shall fix both q and n, and look at k-tuples of $n\times n$ matrices over $\mathbf{F}_q$. Let $a(n,k,q)$ denote the number of simultaneous similarity classes of k-tuples of $n\times n$ matrices over $\mathbf{F}_q$. Using Burnside's lemma, the following can be easily shown:

Lemma 2.1. *Keeping n and q fixed, $a(n,k,q)$ is asymptotically q^{n^2k} up to some constant factor, as k goes to ∞.*

The Burnside's lemma approach fails to give the asymptoticity of $c(n,k,q)$ as function of k, because the components of the k-tuples of $n\times n$-matrices over $\mathbf{F}_q$ are no longer independently chosen. Nevertheless, we are able to prove the following theorem:

Theorem 2.2. *For a fixed positive integer n, and prime power q, $c(n,k,q)$, as a function of k, is asymptotic to $q^{m(n)k}$ (up to some constant factor), where $m(n)$ is defined as*

$$m(n) = \left[\frac{n^2}{4}\right] + 1.$$

3 Explicit Calculation of $c(n,k,q)$ for $n\leq 4$

We explicitly calculate $c(n,k,q)$ for $n=2,3,4$ and prove the following result:

Theorem 3.1. *For each n in $\{2,3,4\}$, and $k\geq 1$, there exists a polynomial $Q_{n,k}(t)\in\mathbb{Z}[t]$, with non-negative integer coefficients, such that $c(n,k,q)=Q_{n,k}(q)$, for every prime power q.*

Enumeration of similarity classes of matrices over a discrete valuation ring R, with maximal ideal P, and residue field $R/P\cong\mathbf{F}_q$, has been a topic of interest in the past. Singla [13], Jambor and Plesken [9] showed that $c(n,2,q)$ is also the number of similarity classes in $M_n(R/P^2)$. Comparing the results of the study of $c(n,k,q)$ in this thesis, with the results of Avni, Onn, Prasad and Vaserstein [2], and Prasad, Singla and Spallone [12], we find that the number of similarity classes in $M_n(R/P^k)$ is $c(n,k,q)$ for $n\leq 3$, and all $k\geq 1$. The results of this chapter, along with the above mentioned results lead us to conjecture the following:

- For all positive integers n,k, there exists a polynomial $Q_{n,k}(t)$ with non-negative integer coefficients such that $c(n,k,q)=Q_{n,k}(q)$.
- $c(n,k,q)$ is the number of conjugacy classes in $M_n(R/P^k)$, for any two positive integers n and k.

Chapter 1

Introduction

1.1 Overview

Let $\mathbf{F}_q$ be a finite field of order q, and n be a positive integer. Let $M_n(\mathbf{F}_q)$ denote the algebra of $n \times n$ matrices over $\mathbf{F}_q$. For a positive integer k, consider the algebra $\mathbf{F}_q[x_1, \ldots, x_k]$ of polynomials over $\mathbf{F}_q$ in k variables. Let (V, ρ) be some n-dimensional representation of $\mathbf{F}_q[x_1, \ldots, x_k]$, i.e., ρ is a homomorphism:

$$\rho : \mathbf{F}_q[x_1, \ldots, x_k] \to M_n(\mathbf{F}_q).$$

As the variables $x_1, \ldots, x_k$ are pairwise commutative, we have for $i \neq j$:

$$\rho(x_i)\rho(x_j) = \rho(x_j)\rho(x_i).$$

Hence, a representation of $\mathbf{F}_q[x_1, \ldots, x_k]$ is determined by a k-tuple of commuting $n \times n$ matrices over $\mathbf{F}_q$. Any two representations, ρ and η, are isomorphic if there is an invertible $n \times n$ matrix X, such that for each i,

$$\eta(x_i) = X\rho(x_i)X^{-1}.$$

Thus, the problem of counting isomorphism classes of n-dimensional representations of $\mathbf{F}_q[x_1, \ldots, x_k]$, is the same as counting the number of orbits for the action of $GL_n(\mathbf{F}_q)$ on the set, $M_n(\mathbf{F}_q)^{(k)}$, of k-tuples of commuting $n \times n$ matrices over $\mathbf{F}_q$. The action is defined as follows: For $g \in GL_n(\mathbf{F}_q)$, and $(A_1, \ldots, A_k) \in M_n(\mathbf{F}_q)^{(k)}$,

$$g.(A_1, \ldots A_k) = (gA_1g^{-1}, \ldots, gA_kg^{-1}).$$

The orbits for this action are called *simultaneous similarity classes.*

1.2 The Main Results

Let $c(n, k, q)$ denote the number of simultaneous similarity classes of $M_n(\mathbf{F}_q)^{(k)}$. By the theory of the rational canonical form (explained in the end of Section 4.2), the number of similarity classes in $M_n(\mathbf{F}_q)$ is given by

$$c(n, 1, q) = \sum_{\lambda \vdash n} q^{\lambda_1},$$

where λ runs over partitions of n, and for each λ, λ_1 denotes its largest part. For a fixed n, we can discuss the asymptotic behaviour of $c(n, 1, q)$, as a function of q. As a function of q, $c(n, 1, q)$ is asymtotically cq^n, where c is some constant. If we keep q fixed, and look at $c(n, 1, q)$ as a function of n, then also it is asymptotically q^n up to multiplication by some constant. This was proved by Stong [14, Corollary 5] in 1988. In 1995, Neumann and Praeger [11] proved that the probability of an $n \times n$ matrix over $\mathbf{F}_q$ being non-cyclic, is asymptotically q^{-3} as a function of n, for a fixed q. They also proved that the probability of a matrix in $M_n(\mathbf{F}_q)$ being non-separable is asymptotically q^{-1}, as a function of n. Wall G. E. [15] proved that the probability of an $n \times n$ matrix over $\mathbf{F}_q$ being cyclic is asymptotically $(1 - q^{-5}) \prod_{r=3}^{\infty}(1 - q^{-r})$, as a function of n. In 1998, Girth [4] worked on certain probabilities for $n \times n$ upper triangular matrices and compared their asymptotic behaviour with that of corresponding probabilities for arbitrary $n \times n$ matrices over $\mathbf{F}_q$. He also did these comparisons of asymptotic behaviours as q goes to ∞, keeping n fixed. Most of the work cited above focused mainly on counting the numbers of certain kinds of matrices in $M_n(\mathbf{F}_q)$ and understanding their respective asymptotic behaviour as n goes to ∞.

Now, we shall fix n and q, and study the asymptotic behaviour of $c(n, k, q)$ as a function of k. We prove:

Theorem 1.2.1. *For a fixed positive integer n, and prime power q, $c(n, k, q)$, as a function of k, is asymptotic to $q^{m(n)k}$ (up to some constant factor), where $m(n)$ is defined as*

$$m(n) = \left[\frac{n^2}{4}\right] + 1.$$

The number $m(n)$, mentioned in Theorem 1.2.1, is the maximal dimension of any commutative subalgebra of $M_n(\mathbf{F}_q)$. This was proved by Jacobson [7] in 1944.

In this thesis, we also do an explicit calculation of $c(n, k, q)$ for $n = 2, 3$ and 4. From our calculations, we prove that:

Theorem 1.2.2. *For each n in $\{2, 3, 4\}$, and $k \geq 1$, there exists a polynomial, $Q_{n,k}(t) \in \mathbb{Z}[t]$, with non-negative integer coefficients such that, $c(n, k, q) = Q_{n,k}(q)$, for every prime power q.*

To prove this, we need the help of the generating function of $c(n, k, q)$'s in k, which we shall denote by $h_n(q, t)$. So,

$$h_n(q,t) = \sum_{k=0}^{\infty} c(n,k,q)t^k.$$

$h_n(q, t)$ has a nice property (mentioned in Lemma 1.2.3 below), which will be proved in Chapter 2 (for any finite dimensional algebra over $\mathbf{F}_q$).

Lemma 1.2.3. *$h_n(q, t)$ is a rational function.*

The most important tool we will need for proving Theorem 1.2.2 is something called a *Similarity Class Type*, which is defined below:

Definition 1.2.4. *We say that two simultaneous similarity classes of tuples of commuting matrices are of the same* **similarity class type** *(or just* **type***), if their centralizers are conjugate.*

The problem of counting of similarity classes of matrices over a discrete valuation ring R, with maximal ideal P, and residue field, $R/P \cong \mathbf{F}_q$, has been of interest to quite a few authors in the past. The results of Singla [13], Jambor and Plesken [9], show that, $c(n, 2, q)$ is also the number of simultaneous similarity classes of matrices in $M_n(R/P^2)$. From the results of Avni, Onn, Prasad and Vaserstein [2], and those of Prasad, Singla and Spallone [12], we find that, for $n \leq 3$, the number of similarity classes in $M_n(R/P^k)$ is equal to $c(n, k, q)$ for all k. Theorem 1.2.2 and the results of the papers cited above lead us to conjecture the following:

- For all positive integers n, k, there exists a polynomial $Q_{n,k}(t)$ with non-negative integer coefficients such that $c(n, k, q) = Q_{n,k}(q)$.
- $c(n, k, q)$ is the number of similarity classes in $M_n(R/P^k)$ for $k \geq 1$.

1.3 Organization of the Thesis

In Chapter 2, we discuss the preliminaries that are required for understanding the thesis. The chapter has two sections. Section 2.1 is devoted to proving Lemma 1.2.3 for any general finite dimensional algebra over $\mathbf{F}_q$. Section 2.2 is devoted to explaining the concept of *Similarity Class Types*, which was defined earlier (see Definition 1.2.4).

In Chapter 3, we prove Theorem 1.2.1 and prove a related simple result (in Section 3.2) as a corollary.

In Chapter 4, we give a detailed proof of Theorem 1.2.2. The chapter has three sections. Section 4.1 is devoted to proving Theorem 1.2.2 for the 2×2 case. Section 4.2 deals with the 3×3 case, and Section 4.3 deals with the 4×4 case.

Chapter 2

Preliminaries

2.1 Generating Functions

Let $\mathcal{A}$ be a finite dimensional algebra over $\mathbf{F}_q$, and $\mathcal{A}^*$ be the group of units of $\mathcal{A}$. For $k \geq 1$, let $\mathcal{A}^k$ denote the set of k-tuples of elements of $\mathcal{A}$. $\mathcal{A}^*$ acts on $\mathcal{A}^k$ in the following way: For $g \in \mathcal{A}^*$ and $(a_1, \ldots, a_k) \in \mathcal{A}^k$,

$$g.(a_1, \ldots, a_k) = (ga_1g^{-1}, \ldots, ga_kg^{-1}).$$

The orbits under this action are called *simultaneous similarity classes*. Let $\alpha_{\mathcal{A}}(k)$ denote the number of simultaneous similarity classes in $\mathcal{A}^k$ and $H_{\mathcal{A}}(t)$ denote the generating function in k of $\alpha_{\mathcal{A}}(k)$.

Next, we define $\mathcal{A}^{(k)}$ as the subset of $\mathcal{A}^k$ that consists of k-tuples whose entries commute with each other. Let $\beta_{\mathcal{A}}(k)$ denote the number of simultaneous similarity classes in $\mathcal{A}^{(k)}$. Let $h_{\mathcal{A}}(t)$ denote the rational function in k of $\beta_{\mathcal{A}}(k)$.

Lemma 2.1.1. *Both $H_{\mathcal{A}}(t)$ and $h_{\mathcal{A}}(t)$ are rational functions of t.*

Proof. By Burnside's lemma, $\alpha_{\mathcal{A}}(k)$ is

$$\alpha_{\mathcal{A}}(k) = \frac{1}{|\mathcal{A}^*|} \sum_{g \in \mathcal{A}^*} |Z_{\mathcal{A}}(g)|^k,$$

where, for each $g \in \mathcal{A}^*$, $Z_{\mathcal{A}}(g)$ denotes the centralizer algebra of g. Thus $H_{\mathcal{A}}(t)$ is:

$$\begin{aligned} H_{\mathcal{A}}(t) &= 1 + \sum_{k=1}^{\infty} \alpha_{\mathcal{A}}(k) t^k \\ &= 1 + \sum_{k=1}^{\infty} \left(\frac{1}{|\mathcal{A}^*|} \sum_{g \in \mathcal{A}^*} |Z_{\mathcal{A}}(g)|^k \right) t^k \\ &= 1 + \frac{1}{|\mathcal{A}^*|} \sum_{g \in \mathcal{A}^*} \left(\sum_{k=1}^{\infty} |Z_{\mathcal{A}}(g)|^k t^k \right) \\ &= 1 + \frac{1}{|\mathcal{A}^*|} \sum_{g \in \mathcal{A}^*} \frac{|Z_{\mathcal{A}}(g)| t}{1 - |Z_{\mathcal{A}}(g)| t}, \end{aligned}$$

which is a finite sum of rational functions. Therefore $H_{\mathcal{A}}(t)$ is a rational function.

But the Burnside's lemma approach fails in the case of tuples with commuting entries, since the entries, $a_1, \ldots, a_k$, cannot be chosen independently.

We need to calculate $\beta_{\mathcal{A}}(k)$. Let $(a_1, \ldots, a_k) \in \mathcal{A}^{(k)}$. Then $(a_2, \ldots, a_k) \in Z_{\mathcal{A}}(a_1)^{(k-1)}$. Hence the map,

$$(a_1, a_2, \ldots, a_k) \mapsto (a_2, \ldots, a_k),$$

induces a bijection from the set of $\mathcal{A}^*$-orbits in $\mathcal{A}^{(k)}$, which contain an element whose 1st coordinate is a_1, onto the set of $Z_{\mathcal{A}^*}(a_1)$-orbits in $Z_{\mathcal{A}}(a_1)^{(k-1)}$ for the action of simultaneous conjugation. Thus, we get, for $k \geq 1$:

$$\beta_{\mathcal{A}}(k) = \sum_{Z \subseteq \mathcal{A}} c_Z \beta_Z(k-1) \ (\beta_Z(0) = 1 \text{ for all } Z \subseteq \mathcal{A}),$$

where Z runs over subalgebras of $\mathcal{A}$, $\beta_Z(k-1)$ is the number of orbits under the action of Z^* on $Z^{(k-1)}$ by simultaneous conjugation, and c_Z is the number of elements in $\mathcal{A}$ whose centralizer is isomorphic to Z. Hence, we have

$$\begin{aligned}
h_{\mathcal{A}}(t) &= 1 + \sum_{k=1}^{\infty} \beta_{\mathcal{A}}(k) t^k \\
&= 1 + \sum_{k=1}^{\infty} \left(\sum_{Z \subseteq \mathcal{A}} c_Z \beta_Z(k-1) \right) t^k \\
&= 1 + \sum_{Z \subseteq \mathcal{A}} c_Z \left(\sum_{k=1}^{\infty} \beta_Z(k-1) t^{k-1} \right) t \\
&= 1 + \sum_{Z \subseteq \mathcal{A}} c_Z t \left(\sum_{k=0}^{\infty} \beta_Z(k) t^k \right) \qquad (2.1) \\
&= 1 + c_{\mathcal{A}} t \sum_{k=0}^{\infty} \beta_{\mathcal{A}}(k) t^k + \sum_{Z \subsetneq \mathcal{A}} c_Z t \left(\sum_{k=0}^{\infty} \beta_Z(k) t^k \right) \\
&= 1 + c_{\mathcal{A}} t . h_{\mathcal{A}}(t) + \sum_{Z \subsetneq \mathcal{A}} c_Z t . h_Z(t) \\
&\text{(Here, } h_Z(t) = \sum_{k=0}^{\infty} \beta_Z(k) t^k \text{)}
\end{aligned}$$

Therefore

$$(1 - c_{\mathcal{A}} t) h_{\mathcal{A}}(t) = 1 + \sum_{Z \subsetneq \mathcal{A}} c_Z t . h_Z(t) \qquad (2.2)$$

The above identity establishes rationality when $\mathcal{A}$ is a commutative algebra. When $\mathcal{A}$ is commutative, $\mathcal{A}^{(k)} = \mathcal{A}^k$. As $\mathcal{A}$ is commutative, $Z_{\mathcal{A}}(a) = \mathcal{A}$ for all $a \in \mathcal{A}$. Each element of $\mathcal{A}$ is a similarity class in $\mathcal{A}$. Thus, $s_{\mathcal{A}} = |\mathcal{A}|$, and $s_Z = 0$ for $Z \subsetneq \mathcal{A}$. We have, $(1 - |\mathcal{A}| t) h_{\mathcal{A}}(t) = 1$,

$$\text{hence } h_{\mathcal{A}}(t) = \frac{1}{1 - |\mathcal{A}| t},$$

which is a rational function.

If $\mathcal{A}$ is not commutative, then from identity (2.2), we are reduced to the case of algebras whose dimension is strictly less than that of $\mathcal{A}$. The rationality of $h_{\mathcal{A}}(t)$ follows by induction on the dimension of $\mathcal{A}$. When $\mathcal{A}$ is 1 dimensional, $\mathcal{A} = \mathbf{F}_q$, which is commutative. □

Now, if we replace $\mathcal{A}$ in the above lemma by $M_n(\mathbf{F}_q)$, then we have, $h_{M_n(\mathbf{F}_q)}(t) = h_n(q, t)$. Hence, by Lemma 2.1.1, $h_n(q, t)$ is a rational function. Lemma 1.2.3 is proved.

2.2 Similarity Class Types

Given $A \in M_n(\mathbf{F}_q)$ and $x \in \mathbf{F}_q^n$, define for any polynomial $f(t) \in \mathbf{F}_q[t]$, $f(t).x = f(A)x$. This endows $\mathbf{F}_q^n$ with an $\mathbf{F}_q[t]$-module structure, denoted by M^A. It is easy to check that, for matrices, A and B,

$$M^A \cong M^B \Leftrightarrow A = gBg^{-1} \text{ for some } g \in GL_n(\mathbf{F}_q).$$

We can easily see that $End_{\mathbf{F}_q[t]}(M^A)$ is $Z_{M_n(\mathbf{F}_q)}(A)$.

If A is a block diagonal matrix, $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where B and C are square matrices whose respective characteristic polynomials are coprime, then we write A as $B \oplus C$, and

$$M^{B\oplus C} = M^A \cong M^B \oplus M^C.$$

It can easily be shown that $Z_{M_n(\mathbf{F}_q)}(A)$ is isomorphic to $Z_{M_l(\mathbf{F}_q)}(B) \oplus Z_{M_{n-l}(\mathbf{F}_q)}(C)$, where l is the order of B.

Next, we have the Jordan decomposition of M^A, which will be explained in the following para.

Definition 2.2.1. *Let p be an irreducible polynomial in $\mathbf{F}_q[t]$, then the submodule*

$$M^{A_p} = \left\{ x \in M^A \,:\, p(t)^r.x = 0 \textit{ for some } r \geq 1 \right\},$$

*is called the p-**primary** part of M^A.*

Let $\mathrm{Irr}(\mathbf{F}_q[t])$ denote the set of irreducibles in $\mathbf{F}_q[t]$. Then by the primary decomposition theorem, M^A has the decomposition,

$$M^A = \bigoplus_{p \in \mathrm{Irr}(\mathbf{F}_q[t])} M^{A_p},$$

which is over a finite number of irreducibles, as M^A is finitely generated. Then by Structure Theorem of finitely generated modules over a PID (see Dummit and Foote [3]), for each p, M^{A_p} has the decomposition,

$$\frac{\mathbf{F}_q[t]}{p^{\lambda_1}} \oplus \frac{\mathbf{F}_q[t]}{p^{\lambda_2}} \oplus \cdots,$$

where $(\lambda_1 \geq \lambda_2 \geq \cdots)$ is a partition, which is denoted by λ. The primary decomposition, together with the Structure Theorem decomposition of each primary part, gives the decomposition:

$$\bigoplus_{p \in \mathrm{Irr}(\mathbf{F}_q[t])} \left(\frac{\mathbf{F}_q[t]}{p^{\lambda_1}} \oplus \frac{\mathbf{F}_q[t]}{p^{\lambda_2}} \oplus \cdots \right).$$

This decomposition is the **Jordan decomposition**.

This gives a bijection between similarity classes in $M_n(\mathbf{F}_q)$, and the set of maps from $\mathrm{Irr}(\mathbf{F}_q[t])$ to the set of partitions, Λ.

$$\{\text{Similarity classes of } M_n(\mathbf{F}_q)\} \longleftrightarrow \{\nu : \mathrm{Irr}(\mathbf{F}_q[t]) \to \Lambda\}$$

Now, for any $\nu : \mathrm{Irr}(\mathbf{F}_q[t]) \to \Lambda$, let $\mathrm{Supp}(\nu)$ denote the set of irreducible polynomials, $p(t)$, for which $\nu(p)$ is a non-empty partition. Clearly, $\mathrm{Supp}(\nu)$ is a finite set. For each partition, μ, and each $d \geq 1$, let $r_\nu(\mu, d)$ be

$$r_\nu(\mu, d) = |\{p(t) \in \mathrm{Irr}(\mathbf{F}_q[t]) : \deg(p) = d \text{ and } \nu(p) = \mu\}|.$$

This puts us in a position to define *Similarity Class Types.*

Definition 2.2.2. *Let A and B be two similarity classes in $M_n(\mathbf{F}_q)$, and let $\nu^{(A)}$ and $\nu^{(B)}$ be the maps from $\mathrm{Irr}(\mathbf{F}_q[t]) \to \Lambda$ corresponding to A and B respectively. We say that A and B are of the same* ***Similarity Class Type****, if for each partition λ, and each $d \geq 1$, $r_{\nu^{(A)}}(d, \lambda) = r_{\nu^{(B)}}(d, \lambda)$ (See Green [6]).*

We shall denote a similarity class type by

$$\lambda^{(1)}{}_{d_1}, \ldots, \lambda^{(l)}{}_{d_l},$$

where $\lambda^{(1)}, \ldots, \lambda^{(l)}$ are partitions, and $d_i \geq 1$ for $1 \leq i \leq l$, such that

$$\sum_{i=1}^{l} |\lambda^{(i)}| d_i = n.$$

For example, in $M_2(\mathbf{F}_q)$, there are four similarity class types which are described in the table below:

Type	Description of the type
$(1,1)_1$	$\lambda^{(1)} = (1,1)$, $d_1 = 1$
$(2)_1$	$\lambda^{(1)} = (2)$, $d_1 = 1$
$(1)_1(1)_1$	$\lambda^{(1)} = (1), d_1 = 1$ $\lambda^{(2)} = (1)$, $d_2 = 1$
$(1)_2$	$\lambda^{(1)} = (1)$, $d_1 = 2$

So, for all those similarity classes, $\nu : \mathrm{Irr}(\mathbf{F}_q[t]) \to \Lambda$, such that $\mathrm{Supp}(\nu) = \{f_1, \ldots, f_l\}$, where $\deg(f_i) = d_i$ and $\nu(f_i) = \lambda^{(i)}$, the similarity class type is

$$\lambda^{(1)}{}_{d_1}, \ldots, \lambda^{(l)}{}_{d_l}$$

Definition 2.2.3. *1. We say that a matrix, A, is of the* Central *type, if it is of the similarity class type*

$$(\underbrace{1, \ldots, 1}_{n\text{-ones}})_1$$

2. *And of the* Regular/Cyclic *type if it is of the class type*

$$\lambda^{(1)}{}_{d_1}, \ldots, \lambda^{(l)}{}_{d_l}$$

where for each $i = 1, \ldots, l$, the partition $\lambda^{(i)}$ has only one part.

So, now we shall define types for commuting tuples of matrices:

Definition 2.2.4. *Let $(A_1, \ldots, A_k)$ be a k-tuple and $(B_1, \ldots, B_l)$, an l-tuple of commuting matrices. We say that they are of the same **similarity class type** if their respective common centralizers,*

$$Z_{M_n(\mathbf{F}_q)}(A_1, \ldots, A_k) \text{ and } Z_{M_n(\mathbf{F}_q)}(B_1, \ldots, B_l),$$

are conjugate in $M_n(\mathbf{F}_q)$.

The above definition of types for tuples, is a precise version of Definition 1.2.4, and is consistent with the Definition 2.2.2 because, A and B are of the same type if and only if their centralizers, $Z_{M_n(\mathbf{F}_q)}(A)$ and $Z_{M_n(\mathbf{F}_q)}(B)$, are conjugate (see the definition of orbit-equivalent by Ravi S. Kulkarni [10] or the definition of z-equivalent by Rony Gouraige [5]). So, if the centralizer, $Z_{M_n(\mathbf{F}_q)}(A_1, \ldots, A_k)$, of a k-tuple, $(A_1, \ldots, A_k)$, for $k \geq 2$, is conjugate to that of some matrix, $A \in M_n(\mathbf{F}_q)$ (of some type τ), we say that the simultaneous similarity class of $(A_1, \ldots, A_k)$ is of the type, τ. Suppose, the centralizer, $Z_{M_n(\mathbf{F}_q)}(B_1, \ldots, B_l)$ of $(B_1, \ldots, B_l)$ is conjugate to $Z_{M_n(\mathbf{F}_q)}(A_1, \ldots, A_k)$, then it is conjugate to $Z_{M_n(\mathbf{F}_q)}(A)$. Hence, $(B_1, \ldots, B_l)$ too is of type τ. If for some $k > 1$, $Z_{M_n(\mathbf{F}_q)}(A_1, \ldots, A_k)$ is not conjugate to the centralizer of any matrix in $M_n(\mathbf{F}_q)$, we have a new type of similarity class.

Chapter 3

Asymptotic Behaviour of $c(n, k, q)$

3.1 Proof of Theorem 1.2.1

We defined $c(n, k, q)$ in the Introduction and stated a theorem (Theorem 1.2.1) about its asymptotic behaviour as a function of k. In this chapter we will prove that theorem, and also state and prove a related simple result.

Before going ahead, we look at the asymptoticity in k of the number of simultaneous similarity classes in $M_n(\mathbf{F}_q)^k$. Let $a(n, k, q)$ denote the number of simultaneous similarity classes in $M_n(\mathbf{F}_q)^k$. Then we have the following lemma:

Lemma 3.1.1. *For a fixed positive integer n and prime power q, $a(n, k, q)$ is asymptotically q^{n^2k} up to some constant factor, as k goes to ∞.*

Proof. We need to show that there exist positive constants m_1 and m_2 (constant with respect to k) such that $m_1 q^{n^2k} \leq a(n, k, q) \leq m_2 q^{n^2k}$. Using Burnside's Lemma, we know that $a(n, k, q)$ is equal to:

$$a(n, k, q) = \frac{1}{|GL_n(\mathbf{F}_q)|} \sum_{g \in GL_n(\mathbf{F}_q)} |Z_{M_n(\mathbf{F}_q)}(g)|^k.$$

So, in this expansion of $a(n, k, q)$, if we just consider all the g that are scalar matrices, we have $Z_{M_n(\mathbf{F}_q)}(g) = M_n(\mathbf{F}_q)$.

$$\text{Let } m_1 = \frac{q-1}{|GL_n(\mathbf{F}_q)|}.$$

then $m_1 q^{n^2k} \leq a(n, k, q)$.

Next, if g is a non-scalar matrix, then $Z_{M_n(\mathbf{F}_q)}(g) \subsetneq M_n(\mathbf{F}_q)$. We know from [1], that the maximal dimension of a proper subalgebra of $M_n(\mathbf{F}_q)$ is, $n^2 - n + 1$.

So we have

$$\begin{aligned} a(n,k,q) &= \frac{1}{|GL_n(\mathbf{F}_q)|} \sum_{g \in GL_n(\mathbf{F}_q)} |Z_{M_n(\mathbf{F}_q)}(g)|^k \\ &= \frac{1}{|GL_n(\mathbf{F}_q)|} (q-1) q^{n^2 k} + \sum_{\substack{g \in GL_n(\mathbf{F}_q) \\ g \notin \mathbf{F}_q . I}} |Z_{M_n(\mathbf{F}_q)}(g)|^k \\ &\leq \frac{1}{|GL_n(\mathbf{F}_q)|} (q-1) q^{n^2 k} + \sum_{\substack{g \in GL_n(\mathbf{F}_q) \\ g \notin \mathbf{F}_q . I}} q^{(n^2-n+1)k} \\ &= \frac{1}{|GL_n(\mathbf{F}_q)|} (q-1) q^{n^2} k \left(1 + (|GL_n(\mathbf{F}_q)| - q + 1) q^{-(n-1)k}\right). \end{aligned}$$

From this we get m_2 such that $a(n,k,q) \leq m_2 q^{n^2 k}$. $\square$

The technique used in the proof of the lemma above, using the Burnside's Lemma approach, fails to give the asymptoticity of $c(n,k,q)$ because the matrices, $A_1, ..., A_k \in M_n(\mathbf{F}_q)$ are no longer independently chosen.

To prove Theorem 1.2.1, it suffices to prove the existence of positive numbers, C_1 and C_2, such that

$$C_1 q^{m(n)k} \leq c(n,k,q) \leq C_2 q^{m(n)k}$$

for large k. For this, we need to unravel $c(n,k,q)$.

We first define the following:

Definition 3.1.2. *For $k > 0$ and any subalgebra Z of $M_n(\mathbf{F}_q)$, let $c_Z(k,q)$ denote the number of simultaneous similarity classes of k-tuples of commuting matrices in Z, under conjugation by its group of units Z^*.*

We claim:

$$c(n,k,q) = \sum_{Z \subseteq M_n(\mathbf{F}_q)} s_Z c_Z(k-1,q) \tag{3.1}$$

where Z runs over subalgebras of $M_n(\mathbf{F}_q)$, and for each Z, s_Z is the number of similarity classes in $M_n(\mathbf{F}_q)$, whose centralizer algebra is conjugate to Z; and $c_Z(0,q) = 1$ for all $Z \subseteq M_n(\mathbf{F}_q)$.

Let $(A_1, \ldots, A_k) \in M_n(\mathbf{F}_q)^{(k)}$. Let $Z = Z_{M_n(\mathbf{F}_q)}(A_1)$. Then $(A_2, \ldots, A_k) \in Z^{(k-1)}$. We know that, counting the number of simultaneous similarity classes in $M_n(\mathbf{F}_q)^{(k)}$, which contain an element whose first component is A_1, is the same as counting the orbits for the conjugation action of Z^* on $Z^{(k-1)}$. Hence we get equation (3.1).

Now, in equation (3.1), for each subalgebra Z, we can expand $c_Z(k-1, q)$ to get

$$c_Z(k-1, q) = \sum_{Z' \subseteq Z} s_{ZZ'} c_{Z'}(k-2, q)(\text{ i.e., when } k \geq 2),$$

where $s_{ZZ'}$ is the number of orbits of matrices in Z for the action of Z^* on it by conjugation, whose centralizer algebra in Z is conjugate to Z'. Proceeding this way, we get the following expansion for $c(n, k, q)$:

$$c(n, k, q) = \sum_{Z_1 \supseteq \cdots \supseteq Z_k} s_{Z_1} s_{Z_1 Z_2} \cdots s_{Z_{k-1} Z_k}, \tag{3.2}$$

where, for $1 \leq i \leq k-1$, $Z_i = Z_{M_n(\mathbf{F}_q)}(A_1, \ldots, A_i)$, for some i-tuple of commuting matrices $(A_1, \ldots, A_i)$. And $s_{Z_i Z_{i+1}}$ denotes the number of orbits of matrices in Z_i for the conjugation action of Z_i^*, whose centralizer algebra, $Z_{Z_i}(x)$, in Z_i, is conjugate to Z_{i+1}. For $Z_{i+1} \subseteq Z_i$, we say that Z_{i+1} is a **branch** of Z_i, iff $s_{Z_i Z_{i+1}} > 0$.

Here are some observations about these non-increasing sequences of subalgebras which come up in the expansion of $c(n, k, q)$. We shall state them as a lemma:

Lemma 3.1.3. *Given a non-increasing sequence of centralizer subalgebras, which appears in equation (3.2), say*

$$Z_1 \supseteq \cdots \supseteq Z_k,$$

we have the following:

1. *If for some i, Z_i is a commutative subalgebra, then*

$$Z_{i+1} = \cdots = Z_k = Z_i,$$

and for each $j \in \{i, \ldots, k-1\}$, $s_{Z_j Z_{j+1}} = q^{\dim(Z_i)}$.

2. *If for some i, Z_i is not necessarily commutative, but $Z_{i+1} = Z_i$, then $s_{Z_i Z_{i+1}} = q^{\dim(Z(Z_i))}$, where $Z(Z_i)$ is the centre of Z_i.*

Proof. For $i \geq 1$, let $(A_1, \ldots, A_i) \in M_n(\mathbf{F}_q)^{(i)}$ such that,

$$Z_{M_n(\mathbf{F}_q)}(A_1, \ldots, A_i) = Z_i.$$

Suppose Z_i is commutative. For any $x \in Z_i$, $Z_{Z_i}(x) = Z_i$. So, if we take any $A_{i+1} \in Z_i$, we have

$$Z_{i+1} = Z_{M_n(\mathbf{F}_q)}(A_1, \ldots, A_i, A_{i+1}) = Z_{Z_i}(A_{i+1}) = Z_i.$$

Therefore, $s_{Z_i Z_{i+1}} = |Z_i| = q^{\dim(Z_i)}$. Similarly, $Z_j = Z_i$ for $i+1 \leq j \leq k$. Thus, $s_{Z_j Z_{j+1}} = q^{\dim(Z_i)} \leq q^{m(n)}$ for $i \leq j \leq k-1$.

If Z_i is not necessarily commutative, but $Z_{i+1} = Z_i$, then $s_{Z_i Z_{i+1}}$ is the number of matrices in Z_i for which

$$Z_{Z_i}(A_{i+1}) = Z_i.$$

Thus $s_{Z_i Z_{i+1}}$ is the size of the centre ,$Z(Z_i)$, of Z_i. So $s_{Z_i Z_{i+1}} = q^{\dim(Z(Z_i))} \leq q^{m(n)}$ (since $Z(Z_i)$ is a commutative subalgebra of $M_n(F_q)$).

□

Finding Crude Lower and Upper Bounds for $c(n,k,q)$

First, we need to show that there exists a tuple of commuting matrices whose common centralizer is a commutative algebra of dimension, $m(n)$. Here are examples of tuples of commuting matrices whose common centralizer is a commutative subalgebra of $M_n(\mathbf{F}_q)$, of dimension $m(n)$.

Example 1. *When n is even, let $n = 2l$, where $l \geq 1$. Then $m(n) = l^2+1$. Consider the commuting tuple $(A_1, A_2, \ldots, A_{l+1})$ in which*

$$A_1 = \begin{pmatrix} 0_l & I_l \\ 0_l & 0_l \end{pmatrix} \quad \text{(0_l is the $l \times l$ 0-block)}$$

for $i \geq 2$,

$$A_i = \begin{pmatrix} 0_l & N_i \\ 0_l & 0_l \end{pmatrix},$$

where, for $i = 2, \ldots, l+1$,

$$N_i = \begin{pmatrix} 0_{(l-1)\times l} \\ e_{i-1} \end{pmatrix} \quad \text{($0_{(l-1)\times l}$ is the $(l-1)\times l$ 0-block)}$$

and e_{i-1} is the $1 \times l$ row matrix,

$$\begin{pmatrix} 0 \cdots & \underset{(i-1)th\ place}{\overset{1}{\downarrow}} & \cdots 0 \end{pmatrix}.$$

Its common centralizer algebra is

$$Z_{M_n(\mathbf{F}_q)}(A_1,\dots,A_{l+1}) = \left\{ a_0 I_n + \begin{pmatrix} 0_l & B \\ 0_l & 0_l \end{pmatrix} \mid a_0 \in \mathbf{F}_q \text{ and } B \in M_l(\mathbf{F}_q) \right\}.$$

It is commutative and is of dimension l^2+1 which is equal to $m(n)$.

Example 2. *When n is odd, let $n = 2l+1$. Then $m(n) = l(l+1)+1$. Consider the commuting tuple $(A_1, A_2, \dots, A_{l+1})$ where*

$$A_1 = \begin{pmatrix} 0_{l+1} & I_l \\ 0_{l\times(l+1)} & 0_{(l+1)\times l} \end{pmatrix},$$

and for $i = 2, \dots, l+1$,

$$A_i = \begin{pmatrix} 0_{l+1} & N_i \\ 0_{l\times(l+1)} & 0_{l\times l} \end{pmatrix},$$

where for each i, N_i is a $(l+1)\times l$-matrix of the form

$$\begin{pmatrix} 0_l \\ e_{i-1} \end{pmatrix}.$$

The common centralizer of this tuple is

$$\left\{ a_0 I_n + \begin{pmatrix} 0_{(l+1)\times(l+1)} & B \\ 0_{l\times(l+1)} & 0_{l\times l} \end{pmatrix} \mid a_0 \in \mathbf{F}_q \text{ and } B \in M_{(l+1)\times l}(\mathbf{F}_q) \right\}.$$

It is commutative and is of dimension $l(l+1)+1$, which is equal to $m(n)$.

So we can find at least one $([n/2]+1)$-tuple of commuting $n\times n$ matrices, whose common centralizer algebra is of dimension $m(n)$.

Lemma 3.1.4. *There exists $C_1 > 0$, such that $C_1 q^{m(n)k} \leq c(n,k,q)$ for large k.*

Proof. Let $l_0 = [n/2]+1$. Consider the k-tuple,

$$(A_1, A_2, \dots, A_{l_0}, A_{l_0+1}, \dots, A_k),$$

whose first l_0 entries are as in Examples 1 or 2 (depending on whether n is even or odd). We have, for each i, $Z_i = \bigcap_{j=1}^{i} Z_{M_n(\mathbf{F}_q)}(A_j)$. So Z_{l_0} is a commutative subalgebra of dimension $m(n)$ (as described in the examples). Hence, from Lemma 3.1.3, we get that, for $i = l_0 + 1, \ldots, k$, $Z_i = Z_{l_0}$. Then,

$$c(n, k, q) \geq s_{Z_1} s_{Z_1 Z_2} \cdots s_{Z_{l_0-1} Z_{l_0}} q^{m(n)(k-l_0)}.$$

Let

$$C_1 = \frac{s_{Z_1} s_{Z_1 Z_2} \cdots s_{Z_{l_0-1} Z_{l_0}}}{q^{m(n) l_0}}$$

then $c(n, k, q) \geq C_1 q^{m(n)k}$ for all large k. □

To complete the proof of the main theorem, we need this observation (Lemma 3.1.5) about the non-increasing sequences of subalgebras,

$$Z_1 \supseteq \cdots \supseteq Z_k,$$

which appear in the expansion of $c(n, k, q)$ given in equation (3.2).

Lemma 3.1.5. *$Z(Z_i) \subseteq Z(Z_{i+1})$ for $i \geq 1$, and if $Z_{i+1} \subsetneq Z_i$, then $Z(Z_i) \subsetneq Z(Z_{i+1})$*

Proof. Let $x \in Z(Z_i)$. Then, for any $y \in Z_i$ such that $Z_{Z_i}(y) = Z_{i+1}$, we have $xy = yx$. Therefore $x \in Z_{i+1}$. Now, as $x \in Z(Z_i)$, $xz = zx$ for every $z \in Z_{i+1}$. That implies, $x \in Z(Z_{i+1})$. So, $Z(Z_i) \subseteq Z(Z_{i+1})$, and thus, $\dim(Z(Z_{i+1})) \geq \dim(Z(Z_i))$.

If $Z_i \supsetneq Z_{i+1}$, then consider any $y \in Z_i$ for which $Z_{Z_i}(y) = Z_{i+1}$. Clearly, $y \in Z(Z_{i+1})$. But for $x \notin Z_{i+1}$, $yx \neq xy$. Hence, $y \notin Z(Z_i)$. Therefore $Z(Z_i) \subsetneq Z(Z_{i+1})$. Thus $\dim(Z(Z_{i+1})) > \dim(Z(Z_i))$. □

Now we are in a position to get a crude upper bound for $c(n, k, q)$. Let $k > n^2$. Let us look at any summand of $c(n, k, q)$. A summand of $c(n, k, q)$ is of the form,

$$s_{Z_1} s_{Z_1 Z_2} \cdots s_{Z_{k-1} Z_k},$$

for the non-increasing sequence of subalgebras, $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_k$. Now, let j be the number of distinct Z_i's in the non-increasing sequence. As $M_n(\mathbf{F}_q)$ is of dimension n^2, there cannot be more than n^2 distinct Z_i's in $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_k$. So $1 \leq j \leq n^2$.

We therefore rewrite $c(n, k, q)$ as

$$c(n, k, q) = \sum_{j=0}^{n^2-1} \sum_{\substack{Z_1 \supseteq \cdots \supseteq Z_k \\ j+1 \text{ distinct}}} s_{Z_1} s_{Z_1 Z_2} \cdots s_{Z_{k-1} Z_k} \tag{3.3}$$

Now, for any j between 0 and n^2-1, consider a non-increasing sequence $Z_1 \supseteq \cdots \supseteq Z_k$, in which $j+1$ of the Z_i's are distinct. Then it has a strictly decreasing subsequence

$$Z_{i_1} \supsetneq Z_{i_2} \supsetneq \cdots \supsetneq Z_{i_j} \supsetneq Z_k.$$

So, the non-increasing sequence $Z_1 \supseteq \cdots \supseteq Z_k$ looks like this:

$$Z_1 = \cdots = Z_{i_1} \supsetneq Z_{i_1+1} = \cdots = Z_{i_2} \supsetneq \cdots = Z_{i_j} \supsetneq Z_{i_j+1} = \cdots = Z_k. \tag{3.4}$$

Then, from Lemma 3.1.3, $s_{Z_1}s_{Z_1Z_2}\cdots s_{Z_{k-1}Z_k}$ is equal to:

$$s_{Z_1}q^{\dim(Z(Z_{i_1}))(i_1-1)}s_{Z_{i_1}Z_{i_2}}q^{\dim(Z(Z_{i_2}))(i_2-i_1-1)}\cdots s_{Z_{i_j}Z_k}q^{\dim(Z(Z_k))(k-i_j-1)},$$

For $1 \le u \le j-1$, we have $Z_{i_u} \supsetneq Z_{i_{u+1}}$, and $Z_{i_j} \supsetneq Z_k$. Thus, $Z_{i_u} \supsetneq Z_k$ for all $1 \le u \le j$. Hence, by Lemma 3.1.5, we have $\dim(Z(Z_{i_u})) < \dim(Z(Z_k))$ for all $1 \le u \le j$. Hence, for $1 \le u \le j$,

$$\dim(Z(Z_{i_u})) < \dim(Z(Z_k)) \le m(n).$$

Therefore,

$$\dim(Z(Z_{i_u})) \le m(n)-1$$

for $1 \le u \le j$. Hence $s_{Z_1}s_{Z_1Z_2}\cdots s_{Z_{k-1}Z_k}$ is bounded above by

$$s_{Z_{i_1}}s_{Z_{i_1}Z_{i_2}}\cdots s_{Z_{i_j}Z_k}.q^{(m(n)-1)(i_j)}.q^{m(n)(k-i_j)}.$$

Now, each of $s_{Z_{i_1}}, s_{Z_{i_1}Z_{i_2}}, \ldots, s_{Z_{i_j}Z_k}$ cannot be more than q^{n^2}. Hence,

$$\begin{aligned} s_{Z_1}s_{Z_1Z_2}\cdots s_{Z_{k-1}Z_k} &\le q^{n^2(j+1)}.q^{[(m(n)-1)i_j+m(n)(k-i_j)]} \\ &= q^{n^2(j+1)}.q^{(m(n)k-i_j)} \end{aligned}$$

Here are some observations:

- We know that, as $M_n(\mathbf{F}_q)$ is finite, it has only a finite number of distinct subalgebras. Let that number be $f(n)$. For each j, as $0 \le j \le n^2-1$, there cannot be more than $\binom{f(n)}{j+1}$ of them.
- Given a $Z_1 \supseteq \cdots \supseteq Z_k$, in which $j+1$ of the Z_i's are distinct, i.e., there is a strongly decreasing subsequence of $Z_1 \supseteq \cdots \supseteq Z_k$:

  $$Z_{i_1} \supsetneq Z_{i_2} \supsetneq \cdots \supsetneq Z_{i_j} \supsetneq Z_k,$$

  such that $Z_1 \supseteq \cdots \supseteq Z_k$, is as in expression (3.4). Given the subset, $S =$

$\{i_1,\ldots,i_j\}$, at which the descents occur, $s_{Z_1}s_{Z_1Z_2}\cdots s_{Z_kZ_{k-1}}$ is bounded above by

$$q^{n^2(j+1)}.q^{(m(n)k-\max(S))}.$$

But this S could be any size j subset of $\{1,\ldots,k-1\}$. So, $c(n,k,q)$ is bounded above by

$$\sum_{j=0}^{n^2-1}\left(\binom{f(n)}{j+1}q^{n^2(j+1)}\sum_{\substack{S\subseteq\{1,\ldots,k-1\}\\|S|=j}}q^{(m(n)k-\max(S))}\right)$$

which is equal to

$$\sum_{j=0}^{n^2-1}\left(\binom{f(n)}{j+1}q^{n^2(j+1)}\sum_{r=j}^{k-1}\sum_{\substack{S\subseteq\{1,\ldots,k-1\}\\|S|=j\\\max(S)=r}}q^{(m(n)k-r)}\right)$$

but this is equal to

$$\sum_{j=0}^{n^2-1}\left(\binom{f(n)}{j+1}q^{n^2(j+1)}\sum_{r=j}^{k-1}\binom{r-1}{j-1}q^{(m(n)k-r)}\right).$$

(Once r is chosen, the remaining $j-1$ are chosen from $1,\ldots,r-1$ in $\binom{r-1}{j-1}$ ways.)

Now, as $\binom{r-1}{j-1}\leq r^j$, we get,

$$\begin{aligned}c(n,k,q)&\leq q^{m(n)k}\sum_{j=0}^{n^2-1}\left(\binom{f(n)}{j+1}q^{n^2(j+1)}\sum_{r=j}^{k-1}r^jq^{-r}\right)\\&\leq q^{m(n)k}\sum_{j=0}^{n^2-1}\left(\binom{f(n)}{j+1}q^{n^2(j+1)}\sum_{r=0}^{\infty}r^jq^{-r}\right)\end{aligned}$$

Now, for any fixed j, we can see by any of the routine tests (either the root or ratio test) that the series,

$$\sum_{r=0}^{\infty}r^jq^{-r}\text{ converges.}$$

Now let

$$C_2 = \sum_{j=0}^{n^2-1} \left(\binom{f(n)}{j+1} q^{n^2(j+1)} \sum_{r=0}^{\infty} r^j q^{-r} \right),$$

then we have

$$c(n, k, q) \leq C_2 q^{m(n)k}.$$

So we have found positive constants C_1 and C_2 such that

$$C_1 q^{m(n)k} \leq c(n, k, q) \leq C_2 q^{m(n)k}.$$

Hence $c(n, k, q)$, as a function of k is asymptotically $q^{m(n)k}$ (up to some constant factor).

3.2 Asymptotic of Counting Tuples of Commuting Matrices

In this section, instead of looking at simultaneous similarity classes of commuting tuples, we will look at the asymptotic of counting total number of tuples of commuting matrices. Let $C(n, k, q)$ denote the total number of k-tuples of commuting $n \times n$ matrices over $\mathbf{F}_q$ i.e., the size of $M_n(\mathbf{F}_q)^{(k)}$. Then we have

$$C(n, k, q) = \sum_{Z \subseteq M_n(\mathbf{F}_q)} \frac{|GL_n(\mathbf{F}_q)|}{|Z^*|} C_Z, \tag{3.5}$$

where Z varies over conjugacy classes of subalgebras of $M_n(\mathbf{F}_q)$, Z^* is the group of units of Z, and C_Z is the total number of simultaneous similarity classes of k-tuples of commuting matrices whose common centralizer algebra is conjugate to Z.

From the previous section, we see that

$$C_Z = \sum_{\substack{Z_1 \supseteq \cdots \supseteq Z_k \\ Z_k = Z}} s_{Z_1} s_{Z_1 Z_2} \cdots s_{Z_{k-1} Z_k}.$$

We can therefore rewrite equation (3.5) as

$$C(n, k, q) = \sum_{Z_1 \supseteq \cdots \supseteq Z_k} \frac{|GL_n(\mathbf{F}_q)|}{|Z_k^*|} s_{Z_1} s_{Z_1 Z_2} \cdots s_{Z_{k-1} Z_k}. \tag{3.6}$$

Equation (3.6) is a modified version of equation (3.2).

If we consider tuples, $(A_1,\ldots,A_k)$, where the first l_0 (where $l_0 = [n/2]+1$) coordinates are as in Examples 1 and 2, then we get $|Z_k| = q^{m(n)}$ and $|Z_k^*| = (q-1)q^{\left[\frac{n^2}{4}\right]}$. So we have

$$\frac{|GL_n(\mathbf{F}_q)|}{(q-1)q^{\left[\frac{n^2}{4}\right]}} s_{Z_1}s_{Z_1Z_2}\cdots s_{Z_{l_0-1}Z_{l_0}} q^{m(n)(k-l_0)} \le C(n,k,q)$$

Thus, choose

$$D_1 = \frac{|GL_n(\mathbf{F}_q)|}{(q-1)q^{\left[\frac{n^2}{4}\right]}} s_{Z_1}s_{Z_1Z_2}\cdots s_{Z_{l_0-1}Z_{l_0}} q^{-m(n)l_0}.$$

Then we get $D_1 q^{m(n)k} \le C(n,k,q)$.

Now we can find an upper bound for $C(n,k,q)$. From equation (3.6) we have $C(n,k,q)$ equal to

$$\sum_{Z_1\supseteq\cdots\supseteq Z_k} \frac{|GL_n(\mathbf{F}_q)|}{|Z_k^*|} s_{Z_1}s_{Z_1Z_2}\cdots s_{Z_{k-1}Z_k}$$

Now, as $GL_n(\mathbf{F}_q)$ has only a finite number of subgroups, $\frac{|GL_n(\mathbf{F}_q)|}{|Z_k^*|}$ is bounded above. Let that bound be $G(q)$. So we have

$$\begin{aligned} C(n,k,q) &\le G(q) \sum_{Z_1\supseteq\cdots\supseteq Z_k} s_{Z_1}s_{Z_1Z_2}\cdots s_{Z_{k-1}Z_k} \\ &= G(q)c(n,k,q) \\ &\le G(q)C_2q^{m(n)k} \text{ (from Section 3.1)} \end{aligned}$$

So let $D_2 = G(q)C_2$, then we have $D_2 > 0$ such that $C(n,k,q) \le D_2 q^{m(n)k}$. This proves the theorem:

Theorem 3.2.1. *The total number of k-tuples of commuting $n\times n$ matrices over $\mathbf{F}_q$, $C(n,k,q)$ is asymptotic to $q^{m(n)k}$ as a function of k.*

Keeping n and q fixed, we were able to find the asymptotic behaviour of $c(n,k,q)$ and $C(n,k,q)$, as k goes to ∞. We could instead keep k and q fixed and ask what are the asymptotic behaviour of $c(n,k,q)$ and $C(n,k,q)$, as n goes to ∞. We could also keep k and n fixed and ask for the asymptotic behaviour of $c(n,k,q)$ and $C(n,k,q)$, as a function of q.

Chapter 4

Explicit Calculation of $c(n, k, q)$ for $n \leq 4$

We begin this chapter with the definition of a *branch* of a similarity class type. (Similarity Class Types were defined in Chapter 2. See definition 2.2.2).

Definition 4.0.2. *Given a matrix A of a type τ in $M_n(\mathbf{F}_q)$. We saw in the proof of Theorem 2.1.1 in Chapter 2 that, counting the number of simultaneous similarity classes of pairs of commuting matrices whose first coordinate is A, is the same as counting the orbits in $Z_{M_n(\mathbf{F}_q)}(A)$ for the conjugation by $Z_{GL_n(\mathbf{F}_q)}(A)$ on it. Each such orbit is called a* ***branch*** *of A.*

Given a matrix $A \in M_n(\mathbf{F}_q)$, whose similarity class type is τ. Let $B \in Z_{M_n(\mathbf{F}_q)}(A)$. Then, the centralizer algebra of B in $Z_{M_n(\mathbf{F}_q)}(A)$ is the common centralizer $Z_{M_n(\mathbf{F}_q)}(A, B)$, of A and B. Let ρ be the similarity class type of the commuting pair, (A, B). Then the number of branches of A, which are of type ρ is the number of matrices, B_1 for which $Z_{M_n(\mathbf{F}_q)}(A, B_1)$ is conjugate to $Z_{M_n(\mathbf{F}_q)}(A, B)$.

4.1 The 2×2 Case

We shall examine the similarity classes of k-tuples of commuting 2×2 matrices over $\mathbf{F}_q$ in this section.

In $M_2(\mathbf{F}_q)$, there are two kinds of similarity classes:

1. The Central type.

2. The Regular/Cyclic types, where $\mathbf{F}_q^2$ has a cyclic vector.

Lemma 4.1.1. *For a matrix A, of the* Central *type, the branches are given in the table below:*

Type	No. of Branches
Central	q
Regular	q^2

Proof. When A is of the Central type, $Z_{M_2(\mathbf{F}_q)}(A) = M_2(\mathbf{F}_q)$. Enumeration the similarity classes in $M_2(\mathbf{F}_q)$ leads to the table shown above. $\square$

Lemma 4.1.2. *A matrix of any of the* Regular *types has q^2* Regular *branches.*

Proof. When A is of the Regular type,

$$Z_{M_2(\mathbf{F}_q)}(A) = \{a_0 I_2 + a_1 A \mid a_0, a_1 \in \mathbf{F}_q\},$$

which is commutative. Hence, for any $B \in Z_{M_2(\mathbf{F}_q)}(A)$,

$$Z_{M_2(\mathbf{F}_q)}(A, B) = Z_{M_2(\mathbf{F}_q)}(A).$$

Thus, (A, B) is of the Regular type. Hence, each $B \in Z_{M_2(\mathbf{F}_q)}(A)$ is an orbit for the conjugation of $Z_{GL_2(\mathbf{F}_q)}(A)$ on $Z_{M_2(\mathbf{F}_q)}(A)$. Thus, A has q^2 branches of the Regular type. $\square$

Arranging the two types in the order: {Central, Regular}, we shall write down the branching matrix $\mathcal{B}_2 = [b_{ij}]$, indexed by the types, where for each i and j, b_{ij} is the number of type i branches of a tuple of type j. Here, the branching matrix is:

$$\mathcal{B}_2 = \begin{pmatrix} q & 0 \\ q^2 & q^2 \end{pmatrix}$$

We have

$$c(2, k, q) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathcal{B}_2^k \begin{pmatrix} 1 & 0 \end{pmatrix}^T.$$

As $\mathcal{B}_2$ has entries that are polynomials in q with non-negative integer coefficients, $c(2, k, q)$ is a polynomial in q with non-negative integer coefficients. Thus, Theorem 1.2.2 is proved for $n = 2$.

4.2 The 3×3 Case

In $M_3(\mathbf{F}_q)$ we have the following types of similarity classes:

1. The Central type.
2. The $(2,1)$ **nilpotent type**: $(2,1)_1$.
3. The $(2,1)$ **semi-simple type**: $(1,1)_1(1)_1$.
4. The Regular types.

We now proceed to explain the branching rules.

Lemma 4.2.1. *For a matrix A of the* Central *type, the branching rules are shown in the table below.*

Type	Number of Branches
Central	q
$(2,1)_1$	q
$(1,1)_1(1)_1$	$q^2 - q$
Regular	q^3

Proof. Since A is of Central type, $Z_{M_3(\mathbf{F}_q)}(A)$ is $M_3(\mathbf{F}_q)$. Enumeration of the similarity class types in $M_3(\mathbf{F}_q)$ gives us the table above. □

Lemma 4.2.2. *If A is a matrix of a* Regular *type, then it has q^3 branches of that same* Regular *type.*

Proof. If A is of a Regular type,

$$Z_{M_3(\mathbf{F}_q)}(A) = \{a_0 I_2 + a_1 A + a_2 A^2 \mid a_0, a_1, a_2 \in \mathbf{F}_q\},$$

which is a commutative algebra of dimension 3. Thus for any $B \in Z_{M_3(\mathbf{F}_q)}(A)$, $Z_{M_3(\mathbf{F}_q)}(A,B) = Z_{M_3(\mathbf{F}_q)}(A)$. Therefore (A,B) is Regular. The number of B for which $Z_{M_3(\mathbf{F}_q)}(A,B)$ is $Z_{M_3(\mathbf{F}_q)}(A)$, is q^3, we therefore have q^3 Regular branches. □

Lemma 4.2.3. *For matrix A of the type $(2,1)_1$, the branching rules are given in the table below.*

Type	Number of Branches
$(2,1)_1$	q^2
Regular	$q^3 + q$

Proof. A matrix A, of the type $(2,1)_1$, has the canonical form, $\begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$, where

$a \in \mathbf{F}_q$. Then,

$$Z_{M_3(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} a_0 & a_1 & b \\ 0 & a_0 & 0 \\ 0 & c & d \end{pmatrix} \mid a_0, a_1, b, c, d \in \mathbf{F}_q \right\}.$$

Let $B = \begin{pmatrix} a_0 & a_1 & b \\ 0 & a_0 & 0 \\ 0 & c & d \end{pmatrix} \in Z_{M_3(\mathbf{F}_q)}(A)$, and $X = \begin{pmatrix} x_0 & x_1 & y \\ 0 & x_0 & 0 \\ 0 & z & w \end{pmatrix} \in Z_{GL_3(\mathbf{F}_q)}(A)$ $(x_0, w \neq 0)$. Let $B' = \begin{pmatrix} a_0' & a_1' & b' \\ 0 & a_0' & 0 \\ 0 & c' & d' \end{pmatrix}$ be the conjugate, XBX^{-1} of B by X. So, we have

$$\begin{pmatrix} x_0 & x_1 & y \\ 0 & x_0 & 0 \\ 0 & z & w \end{pmatrix} \begin{pmatrix} a_0 & a_1 & b \\ 0 & a_0 & 0 \\ 0 & c & d \end{pmatrix} = \begin{pmatrix} a_0' & a_1' & b' \\ 0 & a_0' & 0 \\ 0 & c' & d' \end{pmatrix} \begin{pmatrix} x_0 & x_1 & y \\ 0 & x_0 & 0 \\ 0 & z & w \end{pmatrix}. \tag{4.1}$$

Equating the entries in the above equation gives us $a_0' = a_0$, $d' = d$ and the following equations:

$$\begin{aligned} x_0 a_1 + cy &= a_1' x_0 + b' z & (4.2) \\ x_0 b + yd &= a_0 y + b' w & (4.3) \\ a_0 z + wc &= c' x_0 + dz. & (4.4) \end{aligned}$$

We look at 2 cases: $a_0 = d$ and $a_0 \neq d$.

When $a_0 = d$,

$$B = \begin{pmatrix} a_0 & a_1 & b \\ 0 & a_0 & 0 \\ 0 & c & a_0 \end{pmatrix} \text{ and } B' = \begin{pmatrix} a_0 & a_1' & b' \\ 0 & a_0 & 0 \\ 0 & c' & a_0 \end{pmatrix}.$$

From equations (4.3) and (4.4), we have: $x_0 b = b' w$ and $wc = c' x_0$. So we have two subcases: $b = c = 0$ and $(b, c) \neq (0, 0)$.

When $b = c = 0$, equation (4.2) is reduced to $x_0 a_1 = a_1' x_0$, hence $a_1' = a_1$.

Therefore, $B = \begin{pmatrix} a_0 & a_1 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & a_0 \end{pmatrix}$ and $Z_{M_3(\mathbf{F}_q)}(A,B) = Z_{M_3(\mathbf{F}_q)}(A)$. Therefore (A,B) is of the type, $(2,1)_1$. The number of B such that

$$Z_{M_3(\mathbf{F}_q)}(A,B) = Z_{M_3(\mathbf{F}_q)}(A),$$

is $q \times q = q^2$ (as a_0 and a_1 are arbitrary). Hence, A has q^2 branches of the type, $(2,1)_1$.

When $(b,c) \neq 0$. We may first assume that $b \neq 0$. With $b \neq 0$ and $x_0 b = b'w$, we choose w such that $b' = 1$. Replacing b by $b' = 1$, we get $x_0 = w$. From this, equation (4.4) becomes $x_0 c = c' x_0$, therefore $c = c'$. Equation (4.2) reduces to $x_0 a_1 + cy = a_1' x_0 + z$. Choose z such that $a_1' x_0 = 0$. Thus, $a_1' = 0$. So B is reduced to $\begin{pmatrix} a_0 & 0 & 1 \\ 0 & a_0 & 0 \\ 0 & c & a_0 \end{pmatrix}$. Therefore

$$Z_{M_3(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & y \\ 0 & x_0 & 0 \\ 0 & cy & x_0 \end{pmatrix} \mid x_0, x_1, y \in \mathbf{F}_q \right\}.$$

But if we conjugate this by an elementary matrix (switching the 2nd and 3rd rows (resp. columns)), we get

$$Z_{M_3(\mathbf{F}_q)}(A,B) \sim \left\{ \begin{pmatrix} x_0 & y & x_1 \\ 0 & x_0 & cy \\ 0 & 0 & x_0 \end{pmatrix} \mid x_0, x_1, y \in \mathbf{F}_q \right\} = Z_{(3)_1},$$

which is the centralizer of the type, $(3)_1$, which is a Regular type. The number of B such that $Z_{M_3(\mathbf{F}_q)}(A,B) \sim Z_{(3)_1}$, is $q \times q = q^2$ (as a_0 and c are arbitrary). Thus, A has q^2 branches of this Regular type.

When $b = 0$, $c \neq 0$, in the equation $wc = c' x_0$, we choose x_0 such that $c' = 1$. Replacing c by $c' = 1$, we get $w = x_0$ and equation (4.2) becomes $x_0 a_1 + y = a_1' x_0$.

So choose a y such that $a_1' = 0$. This reduces B to $\begin{pmatrix} a_0 & 0 & 0 \\ 0 & a_0 & 0 \\ 0 & 1 & a_0 \end{pmatrix}$, and we get,

$$Z_{M_3(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & 0 \\ 0 & x_0 & 0 \\ 0 & z & x_0 \end{pmatrix} \mid x_0, x_1, z \in \mathbf{F}_q \right\},$$

which is conjugate to

$$\left\{ \begin{pmatrix} x_0 & 0 & x_1 \\ 0 & x_0 & z \\ 0 & 0 & x_0 \end{pmatrix} \mid x_0, x_1, z \in \mathbf{F}_q \right\} \sim Z_{(3)_1}$$

Hence (A,B) is of a Regular type. The number of such B, for which $Z_{M_3(\mathbf{F}_q)}(A,B)$ is as above, is q. So A has q more branches of this Regular type.

When $a_0 \neq d$: In equation (4.3) we choose y such that $b' = 0$, and in equation (4.4), we choose z such that $c' = 0$. Replacing b by $b' = 0$, and c by $c' = 0$, equation (4.2) is reduced to $x_0 a_1 = a_1' x_0$, which implies $a_1' = a_1$. Hence, $B = \begin{pmatrix} a_0 & a_1 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & d \end{pmatrix}$, and

$$Z_{M_3(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & 0 \\ 0 & x_0 & 0 \\ 0 & 0 & w \end{pmatrix} \mid x_0, x_1, w \in \mathbf{F}_q \right\},$$

which is the centralizer of a matrix of type $(2)_1(1)_1$ (which we shall denote as $Z_{(2)_1(1)_1}$), which is a Regular type. The number of B for which $Z_{M_3(\mathbf{F}_q)}(A,B) \sim Z_{(2)_1(1)_1}$, is $q(q-1)q = q^3 - q^2$ (as the entries of B: a_0, a_1 are arbitrary, and $d \neq a_0$). Thus, A has $q^3 - q^2$ branches of this Regular type. Adding up all the Regular branches, we have a total of

$$q^2 + q + q^3 - q^2 = q^3 + q \text{ Regular branches.}$$

$\square$

Lemma 4.2.4. *If A is a matrix whose similarity class is of the type $(1,1)_1(1)_1$ i.e.,* *the* $(2,1)$*-semisimple type, then it has*

- q^2 *branches of the type,* $(1,1)_1(1)_1$.
- q^3 *branches of the* Regular *types.*

Proof. A matrix A whose similarity class is of type $(1,1)_1(1)_1$, is of the form, $A' \oplus A''$, where A' is a 2×2 matrix of the Central type, and A'' is a 1×1 matrix. As $A = A' \oplus A''$, we know that $Z_{M_3(\mathbf{F}_q)}(A) = Z_{M_2(\mathbf{F}_q)}(A') \oplus Z_{M_1(\mathbf{F}_q)}(A'')$, which is $M_2(\mathbf{F}_q) \oplus M_1(\mathbf{F}_q)$. The branches of A, being in $Z_{M_2(\mathbf{F}_q)}(A') \oplus Z_{M_1(\mathbf{F}_q)}(A'')$, will be of the form,

$$B' \oplus B'',$$

where B' is a branch of A', and B'' is a branch of A''. A' has q branches of the Central type, and q^2 branches of the Regular type (Lemma 4.1.1). A'' has q branches of its own type, i.e., $(1)_1$ (as $M_1(\mathbf{F}_q) = \mathbf{F}_q$). Therefore, we have $q \times q = q^2$ branches of the type $(1,1)_1(1)_1$ and $q^2 \times q = q^3$ Regular branches. $\square$

We shall arrange the types in the order: $\{\text{Central}, (2,1)_1, (1,1)_1(1)_1, \text{Regular}\}$, and write down the branching matrix $\mathcal{B}_3 = [b_{ij}]$, indexed by the types in that order. Each entry b_{ij} of $\mathcal{B}_3$, is the number of type i branches of a type j similarity class. Then

$$\mathcal{B}_3 = \begin{pmatrix} q & 0 & 0 & 0 \\ q & q & 0 & 0 \\ q^2 - q & 0 & q^2 & 0 \\ q^3 & q^3 + q & q^3 & q^3 \end{pmatrix}$$

To make things easier, we shall interpret the branching rules in terms of *rational canonical form* (rcf) types, which we shall briefly discuss now.

The similarity class types in $M_n(\mathbf{F}_q)$ can be further classified into these rcf-types. The definition of rcf types is given below:

Definition 4.2.5. *Let $A \in M_n(\mathbf{F}_q)$. As M^A is a finitely generated $\mathbf{F}_q[t]$-module, by the Structure Theorem (see Jacobson [8]), M^A has the decomposition*

$$\frac{\mathbf{F}_q[t]}{f_1(t)} \oplus \cdots \oplus \frac{\mathbf{F}_q[t]}{f_r(t)} \tag{4.5}$$

where $f_r(t) \mid f_{r-1}(t) \mid \cdots \mid f_1(t)$. Let l_i be the degree of f_i. Then $\lambda = (l_1, \ldots, l_r)$ is a partition of n and we say that A is of ***rational canonical form*** (rcf)*-type λ. Thus, each rational canonical form of $M_n(\mathbf{F}_q)$ is a partition of n.*

Let A be a matrix with similarity class type, $\lambda^{(1)}{}_{d_1}, \ldots, \lambda^{(l)}{}_{d_l}$, where for each i, $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots)$. Then there are irreducible polynomials, $p_1(t), \ldots, p_l(t)$, with degrees, $d_1, \ldots, d_l$, respectively, such that,

$$M^A = \bigoplus_{i=1}^{l} \left(\frac{\mathbf{F}_q[t]}{p_i(t)^{\lambda_1^{(i)}}} \oplus \frac{\mathbf{F}_q[t]}{p_i(t)^{\lambda_2^{(i)}}} \oplus \cdots \right)$$

Then, in the structure theorem decomposition of M^A, as given in equation (4.5), we have (see [8])

$$f_j(t) = p_1(t)^{\lambda_j^{(1)}} p_2(t)^{\lambda_j^{(2)}} \cdots p_l(t)^{\lambda_j^{(l)}}.$$

Hence for each j, the degree of f_j is $l_j = \sum_{i=1}^{l} \lambda_j^{(i)} d_i$. Hence, $(l_1, l_2, \ldots)$ is

$$\mu = \left(\sum_{i=1}^{l} \lambda_1^{(i)} d_i, \sum_{i=1}^{l} \lambda_2^{(i)} d_i, \ldots \right).$$

This partition μ is the rcf-type of the similarity class type

$$\lambda^{(1)}{}_{d_1}, \ldots, \lambda^{(l)}{}_{d_l}.$$

Now we get back to the 3×3 case. Here, the rcf types are $(1, 1, 1)$, $(2, 1)$ and (3). We see that

1. The Central type $(1, 1, 1)_1$ is the only class type with rcf type $(1, 1, 1)$.
2. Similarity class types: $(2, 1)_1$ and $(1, 1)_1(1)_1$ are of the rcf type $(2, 1)$.
3. The Regular types are of rcf type (3)

We know that there are q^2 classes with rcf-type $(2, 1)$ in $M_3(\mathbf{F}_q)$, of which $q^2 - q$ of them are of the type $(1, 1)_1(1)_1$ and q of them are of the type $(2, 1)_1$. Hence a class of rcf type $(2, 1)$, is of type $(1, 1)_1(1)_1$ with probability $(q - 1)/q$ and is of type $(2, 1)_1$ with probability $1/q$.
So, the number of Regular branches that a matrix of rcf type $(2, 1)$, has on an average is

$$\left(\frac{q-1}{q} \times q^3 \right) + \left(\frac{1}{q} \times (q^3 + q) \right) = q^3 + 1$$

The average number of rcf type $(2, 1)$ branches of the rcf type $(2, 1)$ is

$$\frac{q-1}{q} \times q^2 + \frac{1}{q} \times q^2 = q^2.$$

So, our branching matrix is reduced to

$$\mathcal{B}_3 = \begin{pmatrix} q & 0 & 0 \\ q^2 & q^2 & 0 \\ q^3 & q^3+1 & q^3 \end{pmatrix}$$

Theorem 4.2.6. *The number of similarity classes, $c(3,k,q)$, of k-tuples of commuting matrices over $\mathbf{F}_q$ for $k \geq 2$ is given by*

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathcal{B}_3^k \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$$

Table 4.1 shows $c(3,k,q)$ calculated for $k = 1,2,3$.

k	$c(3,k,q)$
1	q^3+q^2+q
2	$q^6+q^5+2q^4+q^3+2q^2$
3	$q^9+q^8+2q^7+2q^6+3q^5+2q^4+2q^3$

Table 4.1: $c(3,k,q)$ for $k = 1,2,3$

As a consequence of Theorem 4.2.6, we get that $c(3,k,q)$ is a polynomial in q with non-negative integer coefficients for all k. This proves Theorem 1.2.2 for this case.

Before we close this chapter and move on to the 4×4 case, we shall give an explanation of how to reduce the branching matrix (whose rows and columns are indexed by the similarity class types) to a smaller matrix whose rows and columns are indexed by the rcfs, for the general $n \times n$ case.

Let $\lambda \vdash n$. For the given rcf λ, let p_τ^λ be the probability of a class of rcf type λ, being of similarity class type, τ. Let μ be another partition of n. Then, for rcf types μ and λ, the average number of rcf-type μ branches of an rcf-type λ similarity class is

$$b_{\mu\lambda} = \sum_{rcf(\tau)=\lambda} p_\tau^\lambda \left(\sum_{rcf(\gamma)=\mu} b_{\gamma\tau} \right).$$

4.3 The 4×4 Case

In the 4×4 case, we have 22 similarity class types. Table 4.2 shows the rcf types and the similarity class types of each rcf-type listed below it

$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
$(1,1,1,1)_1$	$(2,1,1)_1$ $(1,1,1)_1(1)_1$	$(2,2)_1$ $(1,1)_1(1,1)_1$ $(1,1)_2$	$(3,1)_1$ $(2,1)_1(1)_1$ $(2)_1(1,1)_1$ $(1,1)_1(1)_1(1)_1$ $(1)_2(1,1)_1$	Regular types, where $\mathbf{F}_q^4$ has a cyclic vector.

Table 4.2: rcf's and similarity class types of 4×4 matrices

Before we move ahead, we give a broader definition of Regular type.

Definition 4.3.1. *We say that a k-tuple of commuting matrices is of* Regular *type if its common centralizer algebra is a commutative algebra of dimension 4 or conjugate to that of the centralizer of a* Regular *type from $M_4(\mathbf{F}_q)$ (note that, the centralizers of* Regular *types in $M_n(\mathbf{F}_q)$ are 4-dimensional and commutative).*

We shall first state the branching rules of the Regular and the Central types and discuss the branching rules of the other types in different subsections of this section.

Lemma 4.3.2. *If A is a matrix of a* Regular *type, then it has q^4 branches of that same regular type.*

Proof. The centralizer $Z_{M_4(\mathbf{F}_q)}(A)$ of A, is the algebra of polynomials in A, which is commutative. Since the characteristic polynomial of A is of degree 4, $Z_{M_n(\mathbf{F}_q)}(A)$ is 4-dimensional. Hence, for each $B \in Z_{M_4(\mathbf{F}_q)}(A)$, (A, B) is a branch of the regular type. Therefore we have q^4 regular branches. $\square$

Lemma 4.3.3. *For A of the* Central *type, its branches are given in the table below:*

Type	No. of Branches	Type	No. of Branches
Central	q	$(3,1)_1$	q
$(2,1,1)_1$	q	$(2,1)_1(1)_1$	q^2-q
$(1,1,1)_1(1)_1$	q^2-q	$(1,1)_1(1)_1(1)_1$	$\frac{q(q-1)(q-2)}{2}$
$(2,2)_1$	q	$(1,1)_1(2)_1$	q^2-q
$(1,1)_1,(1,1)_1$	$\frac{q^2-q}{2}$	$(1,1)_1(1)_2$	$\frac{q^3-q^2}{2}$
$(1,1)_2$	$\frac{q^2-q}{2}$	Regular	q^4

Proof. As A is of Central type, $Z_{M_n(\mathbf{F}_q)}(A) = M_4(\mathbf{F}_q)$. Enumerating the similarity classes of $M_4(\mathbf{F}_q)$ gives the above table. $\square$

4.3.1 Branching Rules of the Non-Primary, Non-Regular Types.

Any non-primary similarity class type of $M_4(\mathbf{F}_q)$ is of the form

$$\lambda^{(1)}{}_{d_1} \cdots \lambda^{(l)}{}_{d_l}$$

where $l \geq 2$. Hence the centralizer algebra of matrices of such types consist of block matrices of the form

$$\begin{pmatrix} X_1 & \cdots & O \\ & \ddots & \\ O & \cdots & X_l \end{pmatrix}$$

where X_i is in the centralizer of the primary type $\lambda^{(i)}{}_{d_i}$. Therefore, the branches of such types are of the form

$$(B_1 \oplus \cdots \oplus B_l)$$

where B_i is a branch of $\lambda^{(i)}{}_{d_i}$, like we saw in Lemma 4.2.4. Thus, with the help of Lemmas 4.1.1, 4.1.2, 4.2.1 and 4.2.3, we have the following results:

Lemma 4.3.4. *For A of the type $(1,1,1)_1(1)_1$, the branching rules are given below:*

Type	Number of Branches
$(1,1,1)_1(1)_1$	q^2
$(2,1)_1(1)_1$	q^2
$(1,1)_1(1)_1(1)_1$	$q^3 - q^2$
Regular	q^4

Lemma 4.3.5. *If A is of type $(2,1)_1(1)_1$, then it has q^2 branches of the type $(2,1)_1(1)_1$ and $q^4 + q^2$ branches of the* Regular *type.*

Lemma 4.3.6. *For matrix, A, of similarity class type, $(1,1)_1(1,1)_1$, the branching* *rules are given in the table below*

Type	Number of Branches
$(1,1)_1(1,1)_1$	q^2
$(1,1)_1(2)_1$	$2q^2$
$(1,1)_1(1)_2$	$q^3 - q^2$
$(1,1)_1(1)_1(1)_1$	$q^3 - q^2$
Regular	q^4

Lemma 4.3.7. *If A is of type, $(1,1)_1(2)_1$, then it has q^3 branches of the type $(1,1)_1(2)_1$ and q^4* Regular *branches.*

Lemma 4.3.8. *If A is of the type, $(1,1)_1(1)_2$, then it has q^3 branches of the type $(1,1)_1(1)_2$ and q^4* Regular *branches.*

Lemma 4.3.9. *If A is of the type $(1,1)_1(1)_1(1)_1$, then it has q^3 branches of the type $(1,1)_1(1)_1(1)_1$ and q^4* Regular *branches.*

4.3.2 Branching Rules of the Primary Types

We have three primary types of similarity classes in the 4×4 case: $(3,1)_1$, $(2,2)_1$ and $(2,1,1)_1$. The proofs of the lemmas here will be done using the technique used in the proof of Lemma 4.2.3 i.e., reducing a matrix in $Z_{M_4(\mathbf{F}_q)}(A)$ to its simplest possible forms by conjugation by elements of $Z_{GL_4(\mathbf{F}_q)}(A)$.

Lemma 4.3.10. *If A is of the type, $(3,1)_1$, it has q^3 branches of the type $(3,1)_1$ and $q^4 + q^2$ regular branches.*

Proof. Matrix A, of type $(3,1)_1$ has the canonical form:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Any matrix $B \in Z_{M_4(\mathbf{F}_q)}(A)$ is of the form

$$B = \begin{pmatrix} a_0 & a_1 & a_2 & b \\ 0 & a_0 & a_1 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & c & d \end{pmatrix}.$$

Let $X \in Z_{GL_4(\mathbf{F}_q)}(A)$.

$$X = \begin{pmatrix} x_0 & x_1 & x_2 & y \\ 0 & x_0 & x_1 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & z & w \end{pmatrix} \; (x_0, w \neq 0).$$

Let $B' = \begin{pmatrix} a'_0 & a'_1 & a'_2 & b' \\ 0 & a'_0 & a'_1 & 0 \\ 0 & 0 & a'_0 & 0 \\ 0 & 0 & c' & d' \end{pmatrix}$ be the conjugate of B by X i.e.,

$$\begin{pmatrix} x_0 & x_1 & x_2 & y \\ 0 & x_0 & x_1 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & z & w \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & b \\ 0 & a_0 & a_1 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & c & d \end{pmatrix} = \begin{pmatrix} a'_0 & a'_1 & a'_2 & b' \\ 0 & a'_0 & a'_1 & 0 \\ 0 & 0 & a'_0 & 0 \\ 0 & 0 & c' & d' \end{pmatrix} \begin{pmatrix} x_0 & x_1 & x_2 & y \\ 0 & x_0 & x_1 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & z & w \end{pmatrix}$$

Then we have the following:

$$a'_0 = a_0,\ d' = d \text{ and } a'_1 = a_1,$$

and the following set of equations:

$$x_0 a_2 + yc = a'_2 x_0 + b'z \tag{4.6}$$
$$x_0 b + yd = b'w + a_0 y \tag{4.7}$$
$$z a_0 + wc = c' x_0 + dz \tag{4.8}$$

We will count the number of branches by looking at the following cases:

$$a_0 = d \text{ and } a_0 \neq d.$$

When $a_0 = d$: From equations 4.7 and 4.8, we get $x_0 b = b'w$ and $wc = c'x_0$. So, we look at the cases $b = c = 0$ and $(b, c) \neq (0, 0)$ separately.

<u>$b = c = 0$:</u> In this case equation (4.6) boils down to $x_0 a_2 = a'_2 x_0$, therefore

$a_2 = a_2'$. Thus B is reduced to $\begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}$. Therefore, any matrix in $Z_{M_4(\mathbf{F}_q)}(A)$ commutes with B. This means, $Z_{M_4(\mathbf{F}_q)}(A,B) = Z_{M_4(\mathbf{F}_q)}(A)$. Therefore the pair (A,B) is of the type $(3,1)_1$. We see that there are $q \times q \times q = q^3$ branches of this type.

$\underline{(b,c) \neq (0,0)}$: First we assume that $b \neq 0$. Then equation (4.7) boils down to $x_0 b = b'w$. As b is non zero, choose $x_0 = w/b$ so that $b' = 1$. Replacing b by $b' = 1$, we get $x_0 = w$. Hence, equation (4.8) boils down to $x_0 c = c' x_0$, which implies: $c' = c$. Thus, equation (4.6) becomes

$$x_0 a_2 + yc = a_2' x_0 + z.$$

Choose z such that $a_2 = 0$. Then B is reduced to

$$\begin{pmatrix} a_0 & a_1 & 0 & 1 \\ 0 & a_0 & a_1 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & c & a_0 \end{pmatrix}.$$

Therefore,

$$Z_{M_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & y \\ 0 & x_0 & x_1 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & cy & x_0 \end{pmatrix} : x_0, x_1, x_2, y \in \mathbf{F}_q \right\},$$

which is 4 dimensional and commutative (by a routine check). Hence, (A,B) is of a Regular type, and there are $q \times q^2 = q^3$ such Regular branches.

Next, when $b = 0$ and $c \neq 0$, equation (4.8) boils down to $wc = c' x_0$. Choose an appropriate w such that $c' = 1$. Replacing c by $c' = 1$, we get $x_0 = w$. Then equation (4.6) gets reduced to $a_2' x_0 = a_2 x_0 + y$. Now, we choose y such that $a_2 x_0 + y =$

0 . This gives us $a_2' = 0$. B is reduced to

$$\begin{pmatrix} a_0 & a_1 & 0 & 0 \\ 0 & a_0 & a_1 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 1 & a_0 \end{pmatrix}$$

and $Z_{M_4(\mathbf{F}_q)}(A,B)$ is

$$\left\{ \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & z & x_0 \end{pmatrix} : x_0, x_1, x_2, z \in \mathbf{F}_q \right\},$$

which is 4 dimensional and commutative. Thus (A,B) is of the Regular type and there are q^2 such branches.

When $a_0 \neq d$: As $a_0 - d \neq 0$, in equation (4.7), we choose y such that b' becomes 0, and in equation (4.8), choose z such that c' becomes 0. Therefore, equation (4.6) boils down to $x_0 a_2 = a_2' x_0$, thus giving us $a_2 = a_2'$. Hence,

$$B = \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \text{ and}$$

$$Z_{M_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & w \end{pmatrix} : x_0, x_1, x_2, w \in \mathbf{F}_q \right\},$$

which is the centralizer of a matrix of the Regular type, $(3)_1(1)_1$. So, (A,B) is of a Regular type. We see that there are $q \times (q-1) \times q^2 = q^4 - q^3$ such Regular branches. So, adding up all the Regular branches, we get the total number of regular branches of A to be

$$(q^4 - q^3) + q^3 + q^2 = q^4 + q^2.$$

$\square$

Lemma 4.3.11. *For a matrix A of similarity class type $(2,2)_1$, the branching rules are given in the table below.*

Type of branch	Number of Branches
$(2,2)_1$	q^2
Regular	q^4
New type NT1	q^2
New type NT2	$\frac{(q^3-q^2)}{2}$
New type NT3	$\frac{(q^3-q^2)}{2}$

- *The centralizer algebra of a pair of commuting matrices of type* NT1 *is:*

$$\left\{ \begin{pmatrix} x_0 & x_1 & y_1 & y_2 \\ 0 & x_0 & y_3 & y_4 \\ 0 & 0 & x_0 & x_1 \\ 0 & 0 & 0 & x_0 \end{pmatrix} : x_0, x_1, y_1, y_2, y_3, y_4 \in \mathbf{F}_q \right\}$$

and its group of units is therefore of size $q^6 - q^5$

- *The centralizer algebra of* NT2 *is:*

$$\left\{ \begin{pmatrix} p(C) & X \\ 0_2 & p(C) \end{pmatrix} : p(C) \in \mathbf{F}_q[C] \text{ and } X \in M_2(\mathbf{F}_q) \right\},$$

and its group of units is therefore of size $q^6 - q^4$. Here C is a 2×2 matrix of the type $(1)_2$.

- *The centralizer algebra of* NT3 *is*

$$\left\{ \begin{pmatrix} x_0 & 0 & y_1 & y_2 \\ 0 & x_1 & y_3 & y_4 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_1 \end{pmatrix} : x_0, x_1, y_1, y_2, y_3, y_4 \in \mathbf{F}_q \right\},$$

and its group of units is therefore of size $q^4(q-1)^2$.

Proof. Matrix A of similarity class type, $(2, 2)_1$, is of the form,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that, conjugating A by an elementary matrix, (switching the 2nd and 3rd rows(resp. columns) of A), gives us

$$\begin{pmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix and 0_2 is the 2×2 0-matrix. Thus,

$$Z_{M_4(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} C & D \\ 0_2 & C \end{pmatrix} \mid C, D \in M_2(\mathbf{F}_q) \right\}.$$

Two matrices, $B = \begin{pmatrix} C & D \\ 0_2 & C \end{pmatrix}$ and $B' = \begin{pmatrix} C' & D' \\ 0_2 & C' \end{pmatrix}$, in $Z_{M_4(\mathbf{F}_q)}(A)$, are similar if there is a matrix, $\begin{pmatrix} X & Y \\ 0_2 & X \end{pmatrix} \in Z_{GL_4(\mathbf{F}_q)}(A)$ (X is invertible) such that,

$$\begin{pmatrix} C' & D' \\ 0_2 & C' \end{pmatrix} \begin{pmatrix} X & Y \\ 0_2 & X \end{pmatrix} = \begin{pmatrix} X & Y \\ 0_2 & X \end{pmatrix} \begin{pmatrix} C & D \\ 0_2 & C \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} C'X & C'Y + D'X \\ 0_2 & C'X \end{pmatrix} = \begin{pmatrix} XC & XD + YC \\ 0_2 & XC \end{pmatrix},$$

which means that C' and C have to be similar. So, we look at 2 cases:

1. C is of Central type.

2. C is of Regular type.

When C is of Central type, $C'Y + D'X = XD + YC$ becomes $D'X = XD$. Hence, to find out which matrix in $Z_{M_4(\mathbf{F}_q)}(A)$ commutes with $B = \begin{pmatrix} C & D \\ 0_2 & C \end{pmatrix}$, we need to know when X commutes with D. For that we look at the different types of D.

When D is of the central type, then X can be any 2×2 invertible matrix.

$$\text{Hence, } Z_{M_4(\mathbf{F}_q)}\left(A, \begin{pmatrix} C & D \\ 0_2 & C \end{pmatrix}\right) = Z_{M_4(\mathbf{F}_q)}(A) \text{ itself.}$$

Therefore, (A, B) is a branch of type $(2,2)_1$. The number of such branches is $q \times q = q^2$ (C and D are arbitrary 2×2 scalar matrices).

When D is of the type $(2)_1$ i.e., $D = \begin{pmatrix} d & 1 \\ 0 & d \end{pmatrix}$, then $XD = DX$ iff $X = \begin{pmatrix} x_0 & x_1 \\ 0 & x_0 \end{pmatrix}$.

So B is $\begin{pmatrix} c & 0 & d & 1 \\ 0 & c & 0 & d \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$ $(c, d \in \mathbf{F}_q)$. Thus the centralizer group, $Z_{GL_4(\mathbf{F}_q)}(A, B)$, of B in $Z_{M_4(\mathbf{F}_q)}(A)$ is

$$\left\{ \begin{pmatrix} x_0 & x_1 & y_1 & y_2 \\ 0 & x_0 & y_3 & y_4 \\ 0 & 0 & x_0 & x_1 \\ 0 & 0 & 0 & x_0 \end{pmatrix} : x_0 \neq 0 \right\},$$

and its size is $(q-1) \times q^5 = q^6 - q^5$. But, none of the known types in $M_4(\mathbf{F}_q)$ have centralizer groups of size, $q^6 - q^5$. We thus have a new type of similarity class of pairs of commuting matrices. This is our new type NT1. There are $q \times q = q^2$ such branches (the entries, c and d of B are arbitrary).

Next, if D is of type $(1)_2$ then the matrices that commute with D are polynomials in D, i.e., $x_0 I + x_1 D$, where $x_0, x_1 \in \mathbf{F}_q$. So

$$Z_{M_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 I_2 + x_1 D & Y \\ 0_2 & x_0 I_2 + x_1 D \end{pmatrix} \mid x_0, x_1 \in \mathbf{F}_q,\, Y \in M_2(\mathbf{F}_q) \right\}.$$

An element in $Z_{M_4(\mathbf{F}_q)}(A, B)$ is invertible if and only if $(x_0, x_1) \neq (0, 0)$. Thus, the centralizer group, $Z_{GL_n(\mathbf{F}_q)}(A, B)$ has $q^4 \times (q^2 - 1) = q^6 - q^4$ matrices, which is not the size of the centralizer group of any known type in $M_4(\mathbf{F}_q)$. Thus we have $\binom{q}{2} \times q = \frac{1}{2}(q^3 - q^2)$ branches of a new type, which we shall refer to as NT2.

When D is of type $(1)_1(1)_1$ i.e., $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ where $d_1 \neq d_2$: X commutes with D iff $X = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}$. So, the common centralizer, $Z_{M_4(\mathbf{F}_q)}(A, B)$ of (A, B) is

$$\left\{ \begin{pmatrix} x_0 & 0 & y_1 & y_2 \\ 0 & x_1 & y_3 & y_4 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_1 \end{pmatrix} : x_i, y_j \in \mathbf{F}_q \right\}$$

and the size of the group, $Z_{GL_4(\mathbf{F}_q)}(A, B)$ is $(q-1)^2 \times q^4$, which is the same as that of the centralizer group of $(3,1)_1$. But $Z_{M_n(\mathbf{F}_q)}(A, B)$ is not isomorphic to the centralizer of a matrix of type $(3,1)_1$, and in $M_4(\mathbf{F}_q)$, there is no similarity class type other than $(3,1)_1$, whose centralizer group is of size $q^4(q-1)^2$. Hence we have a new type, which we shall call NT3. There are $q \times \binom{q}{2} = \frac{1}{2}(q^3 - q^2)$ branches of this new type.

Now, when C is any of the regular types of matrices:
C is of type $(2)_1$, C is of the form, $\begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix}$. Here, X commutes with C iff $X = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$, where $x \neq 0$. So we have the following equation:

$$\begin{pmatrix} a_0 & 1 & d'_1 & d'_2 \\ 0 & a_0 & d'_3 & d'_4 \\ 0 & 0 & a_0 & 1 \\ 0 & 0 & 0 & a_0 \end{pmatrix} \begin{pmatrix} x & y & z_1 & z_2 \\ 0 & x & z_3 & z_4 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{pmatrix} = \begin{pmatrix} x & y & z_1 & z_2 \\ 0 & x & z_3 & z_4 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{pmatrix} \begin{pmatrix} a_0 & 1 & d_1 & d_2 \\ 0 & a_0 & d_3 & d_4 \\ 0 & 0 & a_0 & 1 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

Then we get $d_3 = d'_3$ and the following equations:

$$d'_1 x + z_3 = x d_1 + y d_3 \tag{4.9}$$
$$d'_2 x + d'_1 y + z_4 = x d_2 + y d_4 + z_1 \tag{4.10}$$
$$d'_4 x + d_3 y = x d_4 + z_3 \tag{4.11}$$

In Equation (4.9) choose z_3 so that $d'_1 = 0$. Replacing d_1 by $d'_1 = 0$, we have $z_3 = d_3 y$. Then equation (4.11) becomes $d'_4 x = x d_4$, and therefore $d'_4 = d_4$. In equation (4.10), choose z_4 such that $d_2' = 0$. Thus, B gets reduced to $\begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & d_3 & d_4 \\ 0 & 0 & a_0 & 1 \\ 0 & 0 & 0 & a_0 \end{pmatrix}$. Thus,

$$Z_{M_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x & y & z_1 & z_2 \\ 0 & x & c_3 y & c_4 y + z_1 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{pmatrix} : x, y, z_1, z_2 \in \mathbf{F}_q \right\},$$

which is commutative and 4-dimensional. Hence, (A,B) is of a Regular type. The number of B such that (A,B) is of this Regular type, is $q \times q \times q = q^3$. Hence A has q^3 branches of this Regular type.

If C is of type $(1)_1(1)_1$, so C has the canonical form $\begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix}$, where $c \neq a_0$. So, $XC = CX$ iff $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Now we have

$$\begin{pmatrix} a_0 & 0 & d_1' & d_2' \\ 0 & c & d_3' & d_4' \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \begin{pmatrix} x & 0 & z_1 & z_2 \\ 0 & y & z_3 & z_4 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 & z_1 & z_2 \\ 0 & y & z_3 & z_4 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix} \begin{pmatrix} a_0 & 0 & d_1 & d_2 \\ 0 & c & d_3 & d_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix}.$$

This gives us $d_1' = d_1$ and $d_4' = d_4$ and the following equations:

$$cz_3 + d_3' x = yd_3 + a_0 z_3 \tag{4.12}$$
$$a_0 z_2 + d_2' y = xd_2 + z_2 c. \tag{4.13}$$

As $c \neq a_0$, we can get rid of d_2 and d_3 (in equations (4.12) and (4.13)) and reduce

B to $\begin{pmatrix} a_0 & 0 & d_1 & 0 \\ 0 & c & 0 & d_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$. Then,

$$Z_{M_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x & 0 & z_1 & 0 \\ 0 & y & 0 & z_4 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix} : x, y, z_1, z_4 \in \mathbf{F}_q \right\}.$$

If we conjugate the above algebra, by the elementary matrix (switching the 2nd and 3rd rows (resp. columns)), we get:

$$\left\{ \begin{pmatrix} x & z_1 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & z_4 \\ 0 & 0 & 0 & y \end{pmatrix} : x, y, z_1, z_4 \in \mathbf{F}_q \right\},$$

which is the centralizer algebra of a matrix of type $(2)_1(2)_1$. Hence, this branch, (A, B), is of the Regular type $(2)_1(2)_1$. The number of branches of this type is $q^2 \times \binom{q}{2} = \frac{1}{2}(q^4 - q^3)$.

When C is of type $(1)_2$, we may take C to be the companion matrix of its characteristic polynomial f (a degree 2 irreducible polynomial over $\mathbf{F}_q$). Then from the equation below,

$$\begin{pmatrix} C_f & D' \\ 0_2 & C_f \end{pmatrix} \begin{pmatrix} X & Y \\ 0_2 & X \end{pmatrix} = \begin{pmatrix} X & Y \\ 0_2 & X \end{pmatrix} \begin{pmatrix} C_f & D \\ 0_2 & C_f \end{pmatrix},$$

we have $C_f Y + D'X = XD + YC_f$ (Here X is a polynomial in C_f). We get 4 equations (by equating the 4 entries) and using the fact that the constant part of f is non-zero (since it is irreducible), we can reduce $\begin{pmatrix} C_f & D \\ 0_2 & C_f \end{pmatrix}$ to

$$B = \begin{pmatrix} C_f & \tilde{D} \\ 0_2 & C_f \end{pmatrix},$$

where $\tilde{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. Thus,

$$Z_{M_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0I_2 + x_1C_f & x_1\tilde{D} + y_0I + y_1C_f \\ 0_2 & x_0I_2 + x_1C_f \end{pmatrix} : x_0, x_1, y_0, y_1 \in \mathbf{F}_q \right\},$$

which is 4 dimensional, and is commutative (again a routine check). Therefore (A,B) is of Regular type, and there are $q^2\binom{q}{2} = \frac{q^4-q^3}{2}$ such branches. So, adding up the regular branches gives us: $\frac{q^4-q^3}{2} + \frac{q^4-q^3}{2} + q^3 = q^4$ regular types of branches. $\square$

Lemma 4.3.12. *For A of similarity class type $(2,1,1)_1$, the branching rules are given in Table 4.3.*

Type	No. of Branches	Type	No. of Branches
$(2,1,1)_1$	q^2	NT1	q
$(3,1)_1$	q^2-q	NT3	q^2
$(1,1)_1(2)_1$	q^3-q^2	New type NT4	q
$(2,1)_1(1)_1$	q^3-q^2	New type NT5	q
Regular	q^4+q^2		

Table 4.3: Branching Rules of type $(2,1,1)_1$

- *The centralizer algebra of the new type* NT4 *is of the form*

$$\left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & 0 & 0 \\ 0 & z_1 & z_2 & z_3 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \mid x_0, x_1, x_2, x_3, z_1, z_2, z_3 \in \mathbf{F}_q \right\}$$

- *The centralizer algebra of the new type* NT5 *is of the form*

$$\left\{ \begin{pmatrix} x_0 & 0 & x_1 & x_2 \\ 0 & x_0 & x_3 & x_4 \\ 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \mid x_0, x_1, x_2, x_3, x_4, y_1, y_2 \in \mathbf{F}_q \right\}$$

Proof. Matrix A of the type $(2,1,1)_1$ has the canonical form $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, Any matrix B that commutes with A, is of the form,

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & 0 & 0 \\ 0 & b_1 & b_2 & b_3 \\ 0 & c_1 & c_2 & c_3 \end{pmatrix}.$$

On conjugating B by an elementary matrix (which we shall denote by E_{243}), such that the 2nd row (column) moves to the 4th row (resp. column), the 3rd row (column) moves to the 2nd row (resp. column) and the 4th row (column) moves to the 3rd row (resp. column), we get

$$B = \begin{pmatrix} a_0 & \overrightarrow{b}^T & a_1 \\ \overrightarrow{0} & C & \overrightarrow{d} \\ 0 & \overrightarrow{0}^T & a_0 \end{pmatrix},$$

where $\overrightarrow{b}^T = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$, $\overrightarrow{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, and C is a 2×2 matrix.

Let

$$X = \begin{pmatrix} x_0 & \overrightarrow{y}^T & x_1 \\ \overrightarrow{0} & Z & \overrightarrow{w} \\ 0 & \overrightarrow{0}^T & x_0 \end{pmatrix} \in Z_{GL_4(\mathbf{F}_q)}(A).$$

Let

$$B' = \begin{pmatrix} a'_0 & \overrightarrow{b'}^T & a'_1 \\ \overrightarrow{0} & C' & \overrightarrow{d'} \\ 0 & \overrightarrow{0}^T & a'_0 \end{pmatrix},$$

be the conjugate of B by X. Then

$$\begin{pmatrix} x_0 & \overrightarrow{y}^T & x_1 \\ \overrightarrow{0} & Z & \overrightarrow{w} \\ 0 & \overrightarrow{0}^T & x_0 \end{pmatrix} \begin{pmatrix} a_0 & \overrightarrow{b}^T & a_1 \\ \overrightarrow{0} & C & \overrightarrow{d} \\ 0 & \overrightarrow{0}^T & a_0 \end{pmatrix} = \begin{pmatrix} a'_0 & \overrightarrow{b'}^T & a'_1 \\ \overrightarrow{0} & C' & \overrightarrow{d'} \\ 0 & \overrightarrow{0}^T & a'_0 \end{pmatrix} \begin{pmatrix} x_0 & \overrightarrow{y}^T & x_1 \\ \overrightarrow{0} & Z & \overrightarrow{w} \\ 0 & \overrightarrow{0}^T & x_0 \end{pmatrix}.$$

From the above equation, we get $a_0' = a_0$, and the following set of equations:

$$\begin{aligned} C'Z &= ZC && (4.14) \\ a_0 \overrightarrow{y}^T + \overrightarrow{b'}^T Z &= x_0 \overrightarrow{b}^T + \overrightarrow{y}^T C && (4.15) \\ C' \overrightarrow{w} + x_0 \overrightarrow{d'} &= Z \overrightarrow{d} + a_0 \overrightarrow{w} && (4.16) \\ \overrightarrow{b'}^T . \overrightarrow{w} + a_1' x_0 &= x_0 a_1 + \overrightarrow{y}^T . \overrightarrow{d} && (4.17) \end{aligned}$$

We replace C by C'. Then Z is a matrix that commutes with C. Hence, for each type of C, we find out Z, and with the help of equations (4.15) and (4.16), reduce B to a simpler form. To begin with there are two main cases of what C is:

- When a_0 is an eigenvalue of C. Here C could be of the types, Central ($C = a_0 I_2$), $(2)_1$ and $(1)_1(1)_1$.
- When a_0 is not an eigenvalue of C. Here C could be of the types, Central ($C = cI_2$, $c \neq a_0$), $(2)_1$, $(1)_1(1)_1$ and $(1)_2$.

When a_0 is an eigenvalue of C: We have the following subcases:

- $\overrightarrow{b} = \overrightarrow{d} = \overrightarrow{0}$
- $(\overrightarrow{b}, \overrightarrow{d}) \neq (\overrightarrow{0}, \overrightarrow{0})$

Case: $\overrightarrow{b} = \overrightarrow{d} = \overrightarrow{0}$. In this case, equation (4.17) is reduced to $x_0 a_1' = x_0 a_1$. This implies $a_1' = a_1$. Therefore, the matrix that commutes with C, and equations (4.15) and (4.16), will help determine $Z_{M_4(\mathbf{F}_q)}(A, B)$.

When C is central

$$B = \begin{pmatrix} a_0 & \overrightarrow{0}^T & a_1 \\ \overrightarrow{0} & a_0 I_2 & \overrightarrow{0} \\ 0 & \overrightarrow{0}^T & a_0 \end{pmatrix}.$$

Here, equations (4.15) and (4.16) are void. Thus, any $X \in Z_{M_4(\mathbf{F}_q)}(A)$ commutes with B. Thus $Z_{M_4(\mathbf{F}_q)}(A, B) = Z_{M_n(\mathbf{F}_q)}(A)$. Therefore (A, B) is of type $(2, 1, 1)_1$ and the number of such branches is $q \times q = q^2$.

When C is of type $(2)_1$ We have

$$C = \begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix},$$

and B is reduced to

$$\begin{pmatrix} a_0 & 0 & 0 & a_1 \\ 0 & a_0 & 1 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

Equations (4.15) and (4.16) become

$$\begin{aligned} \begin{pmatrix} a_0 y_1 & a_0 y_2 \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix} \\ \begin{pmatrix} a_0 w_1 \\ a_0 w_2 \end{pmatrix} &= \begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{aligned}$$

$\Rightarrow$ $y_1 = w_2 = 0$ and therefore,

$$Z_{M_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & 0 & y_2 & x_1 \\ 0 & z_1 & z_2 & w_1 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \mid x_0, x_1, y_2, z_1, z_2, w_1 \in \mathbf{F}_q \right\}.$$

Conjugating this matrix by elementary matrices (by switching the 3rd and 4th rows (resp. columns)), gives us

$$\left\{ \begin{pmatrix} x_0 & 0 & x_1 & y_2 \\ 0 & z_1 & w_1 & z_2 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & z_1 \end{pmatrix} \mid x_0, x_1, y_2, z_1, z_2, w_1 \in \mathbf{F}_q \right\},$$

which is the centralizer of a pair of commuting matrices of the new type, NT3. Thus the commuting pair (A,B) is of similarity class type, NT3. Thus, A has $q \times q = q^2$ branches of this new type.

When C is of type $(1)_1(1)_1$,

$$C = \begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix} \ (c \neq a_0) \text{ and } Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix}.$$

So B, in this case, is $\begin{pmatrix} a_0 & 0 & 0 & a_1 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}$. From equations (4.15) and (4.16), we have the following:

$$\begin{aligned} \begin{pmatrix} a_0 y_1 & a_0 y_2 \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix} \\ \begin{pmatrix} a_0 w_1 \\ a_0 w_2 \end{pmatrix} &= \begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{aligned}$$

which leaves us with $y_2 = w_2 = 0$ (since $a_0 \neq c$) and therefore

$$Z_{M_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & y_1 & 0 & x_1 \\ 0 & z_1 & 0 & w_1 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \mid x_0, x_1, y_1, z_1, z_2, w_1 \in \mathbf{F}_q \right\}$$

conjugating this matrix by the elementary matrix, E_{234} (which moves the 2nd row (resp. column) to the 3rd row (resp. column), 3rd row (resp. column) to the 4th row (resp. column), and the 4th row (resp. column) to the 3nd row (resp. column)), we get

$$\left\{ \begin{pmatrix} x_0 & x_1 & y_1 & 0 \\ 0 & x_0 & 0 & 0 \\ 0 & w_1 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix} \mid x_0, x_1, y_1, z_1, z_2, w_1 \in \mathbf{F}_q \right\}$$

which is the centralizer algebra of a matrix of the type $(2,1)_1(1)_1$. Hence the (A,B) is of type, $(2,1)_1(1)_1$, and we have $q \times q \times (q-1) = q^3 - q^2$ (as a_0, a_1 arbitrary and $c \neq a_0$) branches of this type.

Case: $(\overrightarrow{b}, \overrightarrow{d}) \neq (\overrightarrow{0}, \overrightarrow{0})$: In this case, we can find a suitable $\overrightarrow{y}$ or $\overrightarrow{w}$ in equation (4.17) and get rid of the entry a_1 of the matrix B. So B is:

$$\begin{pmatrix} a_0 & \overrightarrow{b} & 0 \\ \overrightarrow{0} & C & \overrightarrow{d} \\ 0 & \overrightarrow{0}^T & a_0 \end{pmatrix}$$

When $C = a_0 I$: Z is any 2×2 invertible matrix. We first assume $\overrightarrow{b} \neq \overrightarrow{0}$.

Equation (4.15) becomes

$$\overrightarrow{b'}^T Z = x_0 \overrightarrow{b}^T.$$

We may replace Z by $x_0^{-1} Z$ so that we have

$$\overrightarrow{b'}^T Z = \overrightarrow{b}^T \text{ and } Z\overrightarrow{d} = \overrightarrow{d}'.$$

Since $\overrightarrow{b} \neq \overrightarrow{0}$ and Z is invertible, we can find a suitable Z such that $\overrightarrow{b'}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Now, let $\overrightarrow{b}^T = \overrightarrow{b'}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$, then equation (4.15) gives us $Z = \begin{pmatrix} 1 & 0 \\ z_3 & z_4 \end{pmatrix}$. Hence, equation (4.16) boils down to

$$\begin{pmatrix} 1 & 0 \\ z_3 & z_4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \end{pmatrix} \tag{4.18}$$

therefore

$$\begin{pmatrix} d_1' \\ d_2' \end{pmatrix} = \begin{pmatrix} d_1 \\ z_3 d_1 + z_4 d_2 \end{pmatrix}$$

If $\overrightarrow{d} \neq \overrightarrow{0}$, with $d_1 \neq 0$, then we can find z_3 so that $z_4 d_2 + d_1 z_3 = 0$, which leaves us with $d_2' = 0$. Hence, B is reduced to

$$\begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & 0 & d_1 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix},$$

and any $X \in Z_{M_4(\mathbf{F}_q)}(A, B)$ is of the form

$$X = \begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ 0 & x_0 & 0 & d_1 y_1 \\ 0 & 0 & z_4 & w_2 \\ 0 & 0 & 0 & x_0 \end{pmatrix}$$

Conjugate this by the elementary matrix (switching the 3rd and 4th rows (resp. columns)). Then we get:

$$\begin{pmatrix} x_0 & y_1 & x_1 & y_2 \\ 0 & x_0 & d_1 y_1 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & w_2 & z_4 \end{pmatrix}.$$

Thus, $Z_{M_4(\mathbf{F}_q)}(A, B)$ is conjugate to the centralizer of a matrix of type $(3, 1)_1$. Hence (A, B) is of type $(3, 1)_1$. There are $q(q-1) = q^2 - q$ branches of this type.

Now when $\overrightarrow{d} \neq 0$ and $d_1 = 0$, then equation (4.18) becomes

$$\begin{pmatrix} d_1' \\ d_2' \end{pmatrix} = \begin{pmatrix} 0 \\ z_4 d_2 \end{pmatrix},$$

which can be reduced to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus B is reduced to

$$\begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & 1 \\ 0 & 0 & 0 & a_0 \end{pmatrix},$$

and $Z_{M_4(\mathbf{F}_q)}(A,B)$ is

$$\left\{ \begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ 0 & x_0 & 0 & y_2 \\ 0 & z_3 & x_0 & w_2 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \mid x_0, x_1, y_1, y_2, z_3, w_2 \in \mathbf{F}_q \right\}.$$

On conjugating X by the elementary matrices such that its 2nd and 3rd rows and columns are switched, we get

$$Z_{M_4(\mathbf{F}_q)}(A,B) \sim \left\{ \begin{pmatrix} x_0 & y_2 & y_1 & x_1 \\ 0 & x_0 & z_3 & w_2 \\ 0 & 0 & x_0 & y_2 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \mid x_0, x_1, y_1, y_2, z_3, w_2 \in \mathbf{F}_q \right\},$$

which is the centralizer of a pair of commuting matrices of the new type NT1. Hence (A,B) is of type NT1. So we have q branches of the new type NT1.

When $\overrightarrow{d} = \overrightarrow{0}$, then

$$B = \begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

Hence $Z_{M_4(\mathbf{F}_q)}(A,B)$ contains matrices of the form

$$X = \begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ 0 & x_0 & 0 & 0 \\ 0 & z_3 & z_4 & w_2 \\ 0 & 0 & 0 & x_0 \end{pmatrix}.$$

Thus, $Z_{M_4(\mathbf{F}_q)}(A,B)$ is 7 dimensional. As there is no known type in $M_4(\mathbf{F}_q)$ whose centralizer is 7 dimensional, we have a new type, which we call NT4. There are q branches of this type.

When $\overrightarrow{b} = \overrightarrow{0}$ and $\overrightarrow{d} \neq \overrightarrow{0}$: From equation (4.16), we can find Z such that

$Z\overrightarrow{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and our B is reduced to

$$\begin{pmatrix} a_0 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 1 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}$$

and therefore $Z_{M_4(\mathbf{F}_q)}(A,B)$ has matrices of the form

$$\begin{pmatrix} x_0 & 0 & y_2 & x_1 \\ 0 & x_0 & z_2 & w_1 \\ 0 & 0 & z_4 & w_2 \\ 0 & 0 & 0 & x_0 \end{pmatrix}$$

Hence, $Z_{M_4(\mathbf{F}_q)}(A,B)$ is 7 dimensional, but it is not conjugate to the centralizer of NT4 and therefore, the branch is of a new type, which we shall call NT5. There are q such branches.

When C is of type $(2)_1$ i.e., $C = \left(\begin{smallmatrix} a_0 & 1 \\ 0 & a_0 \end{smallmatrix}\right)$: We have $Z = \begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix}$ where $z_1 \neq 0$.

From equation (4.15), we get:

$$\overrightarrow{b'}^T \begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix} + \overrightarrow{y}^T (a_0 I_2 - C) = x_0 \overrightarrow{b}^T \tag{4.19}$$

As $a_0 I_2 - C = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, the LHS in equation (4.19) above boils down to

$$\begin{pmatrix} b'_1 z_1 & b'_1 z_2 + b'_2 z_1 - y_1 \end{pmatrix}$$

Choose y_1 so that $\overrightarrow{b}^T = \begin{pmatrix} b'_1 z_1 & 0 \end{pmatrix}$. We now have two cases: $b'_1 \neq 0$ and $b'_1 = 0$.

When $b'_1 \neq 0$, we can choose z_1 so that $\overrightarrow{b}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Replacing $\overrightarrow{b'}^T$ with

$\overrightarrow{b}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$, equation (4.15) becomes

$$\begin{pmatrix} z_1 & z_2 - y_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

which implies: $z_1 = 1$ and $y_1 = z_2$. So $\overrightarrow{y}^T = \begin{pmatrix} z_2 & y_2 \end{pmatrix}$.
Then equation (4.16) is reduced to

$$\begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \end{pmatrix}$$

which implies that we can choose w_2 appropriately so that

$$\begin{pmatrix} d_1' \\ d_2' \end{pmatrix} = \begin{pmatrix} 0 \\ d_2 \end{pmatrix}.$$

Thus B is reduced to

$$\begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & 1 & 0 \\ 0 & 0 & a_0 & d_2 \\ 0 & 0 & 0 & a_0 \end{pmatrix}, \text{ and } Z_{M_4(\mathbf{F}_q)}(A,B) \text{ is}$$

$$\left\{ \begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ 0 & x_0 & y_1 & d_2 y_2 \\ 0 & 0 & x_0 & d_2 y_1 \\ 0 & 0 & 0 & x_0 \end{pmatrix} : x_0, x_1, y_1, y_2 \in \mathbf{F}_q \right\},$$

which is 4-dimensional and commutative. This branch (A,B) is of a Regular type, and there are $q \times q = q^2$ such branches.

Now if $b_1 = 0$, then we have $\overrightarrow{b}^T = \overrightarrow{0}^T$. Then equation (4.16) becomes

$$\begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} d_1' & d_2' \end{pmatrix}$$

which gives us

$$\overrightarrow{d'} = \begin{pmatrix} z_1d_1 + z_2d_2 - w_2 \\ z_1d_2 \end{pmatrix}$$

choose w_2 such that $\overrightarrow{d'} = \begin{pmatrix} 0 \\ z_1d_2 \end{pmatrix}$.
If $d_2 \neq 0$, we can scale it to 1 and thus we have

$$B = \begin{pmatrix} a_0 & 0 & 0 & 0 \\ 0 & a_0 & 1 & 0 \\ 0 & 0 & a_0 & 1 \\ 0 & 0 & 0 & a_0 \end{pmatrix}$$

so in this case $Z_{M_4(\mathbf{F}_q)}(A, B)$ is:

$$\left\{ \begin{pmatrix} x_0 & 0 & 0 & x_1 \\ 0 & x_0 & z_2 & w_1 \\ 0 & 0 & x_0 & z_2 \\ 0 & 0 & 0 & x_0 \end{pmatrix} : x_0, x_1, w_1, z_2 \in F_q \right\}.$$

It is 4-dimensional and commutative. Therefore, this branch too is of a Regular type and the number of branches is q. So we have a total of $q^2 + q$ branches of Regular types in this case.

If $d_2 = 0$, we are back to the case $\overrightarrow{b} = \overrightarrow{d} = \overrightarrow{0}$.

When $C = \begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix}$ $(c \neq a_0)$, $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix}$. So equation (4.15) becomes

$$\begin{pmatrix} b'_1 & b'_2 \end{pmatrix} \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & a_0 - c \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$$

We get from this

$$\begin{pmatrix} z_1b'_1 & z_4b'_2 + (a_0 - c)y_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$$

As $a_0 - c \neq 0$, we can get rid of b'_2 so that $\overrightarrow{b}^T = \begin{pmatrix} z_1b'_1 & 0 \end{pmatrix}$.

If $b_1' \neq 0$, then we can reduce $\overrightarrow{b'}^T$ to $\begin{pmatrix} 1 & 0 \end{pmatrix}$. Replace $\overrightarrow{b'}^T$ by $\overrightarrow{b'}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Then, from equation (4.15), we get $\begin{pmatrix} z_1 & (a_0 - c)y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Thus $z_1 = 1$ and $y_2 = 0$. So $Z = \begin{pmatrix} 1 & 0 \\ 0 & z_4 \end{pmatrix}$.

Equation (4.16) becomes

$$\begin{pmatrix} d_1' \\ d_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z_4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a_0 - c \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

using $a_0 \neq c$, we can reduce $\overrightarrow{d'}$ to $\begin{pmatrix} d_1 \\ 0 \end{pmatrix}$. Thus

$$B = \begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & 0 & d_1 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

Then $Z_{M_4(\mathbf{F}_q)}(A, B)$ is

$$\left\{ \begin{pmatrix} x_0 & y_1 & 0 & x_1 \\ 0 & x_0 & 0 & d_1 y_1 \\ 0 & 0 & z_4 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \quad x_0, x_1, y_1, z_4 \in \mathbf{F}_q \right\},$$

which is 4-dimensional and commutative. Therefore this branch is of a Regular type. The number of such branches is $q^2(q-1) = q^3 - q^2$.

When $b_1' = 0$, then $\overrightarrow{b'}^T = \overrightarrow{0}^T$. Then equation (4.16) becomes

$$\begin{aligned} \begin{pmatrix} d_1' \\ d_2' \end{pmatrix} &= \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a_0 - c \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \begin{pmatrix} z_1 d_1 \\ z_4 d_2 + (a_0 - c) w_2 \end{pmatrix} \end{aligned}$$

As $a_0 \neq c$, we can make $z_4 d_2$ vanish by choosing w_2 appropriately. So we have

$\overrightarrow{d'} = \begin{pmatrix} z_1 d_1 \\ 0 \end{pmatrix}$. If $d_1 \neq 0$, then choose z_1 so that $\overrightarrow{d'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and B is reduced to

$$\begin{pmatrix} a_0 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 1 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

$$\text{Hence } Z_{M_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & 0 & 0 & x_1 \\ 0 & x_0 & 0 & w_1 \\ 0 & 0 & z_4 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \; x_0, x_1, w_1, z_4 \in \mathbf{F}_q \right\},$$

which is 4 dimensional and commutative. Thus the pair (A,B) is of Regular type and there are $q(q-1) = q^2 - q$ such branches. So we have a total of $(q^3 - q^2) + (q^2 - q) + (q^2 + q) = q^3 + q^2$ branches of the Regular type so far.

When a_0 is not an eigenvalue of C: In equations (4.15) and (4.16), using the fact that $C - a_0 I$ is invertible, we can reduce $\overrightarrow{b}$ and $\overrightarrow{d}$ to $\overrightarrow{0}$. Hence, equation (4.16) is reduced to $a_1' x_0 = x_0 a_1'$, thus $a_1' = a_1$. So B is reduced to

$$\begin{pmatrix} a_0 & \overrightarrow{0}^T & a_1 \\ \overrightarrow{0} & C & \overrightarrow{0} \\ 0 & \overrightarrow{0}^T & a_0 \end{pmatrix}$$

When C is of the Central type, $C = cI_2$, where $c \neq a_0$. We have

$$B = \begin{pmatrix} a_0 & \overrightarrow{0}^T & a_1 \\ \overrightarrow{0} & cI_2 & \overrightarrow{0} \\ 0 & \overrightarrow{0}^T & a_0 \end{pmatrix}$$

$$\text{and } Z_{M_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & \overrightarrow{0}^T & x_1 \\ \overrightarrow{0} & Z & \overrightarrow{0} \\ 0 & \overrightarrow{0}^T & x_0 \end{pmatrix} \mid x_0, x_1 \in \mathbf{F}_q, Z \in M_2(\mathbf{F}_q) \right\}.$$

On conjugating $Z_{M_4(\mathbf{F}_q)}(A, B)$ by E_{234}, we get

$$\left\{ \begin{pmatrix} x_0 & x_1 & \vec{0}^T \\ 0 & x_0 & \vec{0}^T \\ \vec{0} & \vec{0} & Z \end{pmatrix} \mid x_0, x_1 \in \mathbf{F}_q, Z \in M_2(\mathbf{F}_q) \right\},$$

which is the centralizer of a matrix of type, $(1,1)_1(2)_1$. Hence, (A, B) is of type, $(1,1)_1(2)_1$. A has $q \times q \times (q-1) = q^3 - q^2$ such branches.

When C is a matrix of Regular type, i.e, $C = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$, $C = \begin{pmatrix} c & 0 \\ 0 & s \end{pmatrix}$ ($c \neq a_0$, $c \neq s$ and $s \neq a_0$), and $C = C_f$, where $f(t)$ is an irreducible polynomial in $\mathbf{F}_q[t]$ of degree 2. So, B is of the form, $\begin{pmatrix} a_0 & \vec{0}^T & a_1 \\ \vec{0} & C & \vec{0} \\ 0 & \vec{0}^T & a_0 \end{pmatrix}$, where C is of any one of the above mentioned forms. Then

$$Z_{M_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & \vec{0}^T & x_1 \\ \vec{0} & Z & \vec{0} \\ 0 & \vec{0}^T & x_0 \end{pmatrix} \mid x_0, x_1 \in \mathbf{F}_q, Z \in Z_{M_2(\mathbf{F}_q)}(C) \right\},$$

which on conjugation by E_{234} becomes,

$$\left\{ \begin{pmatrix} x_0 & x_1 & \vec{0}^T \\ 0 & x_0 & \vec{0}^T \\ \vec{0} & \vec{0} & Z \end{pmatrix} \mid x_0, x_1 \in \mathbf{F}_q, Z \in Z_{M_2(\mathbf{F}_q)}(C) \right\},$$

which is the centralizer of a matrix which is of the type, $(2)_1\tau$, where τ is a Regular type in $M_2(\mathbf{F}_q)$. Hence, (A, B) is of the type $(2)_1\tau$, which is a Regular type. So A has

$$q^2(q - 1 + (q-1)(q-2)/2 + q(q-1)/2) = q^4 - q^3$$

branches of these Regular type. So, adding up the number of all the Regular branches gives

$$(q^4 - q^3) + (q^3 + q^2)$$

which is equal to $q^4 + q^2$ Regular branches and hence the Table 4.3 □

Lemma 4.3.13. *If A is of type $(1,1)_2$, then it has q^2 branches of the type $(1,1)_2$ and q^4 regular branches.*

Proof. The proof is like that of the $(1,1)_1$ case for 2×2 matrices over $\mathbf{F}_{q^2}$. $\square$

4.3.3 Branching Rules of the New Types

While finding out the branching rules for the types, $(2,1,1)_1$ and $(2,2)_1$, we got 5 new types of branches: NT1, NT2, NT3, NT4 and NT5. In this subsection, we will see the branching rules of those new types.

Lemma 4.3.14. *For a pair (A,B) of similarity class type* NT1*, the branching rules are given in the table below:*

Type of branch	No. of Branches
NT1	q^3
Regular	$q^4 - q^3$
New Type NT6	$q^4 - q^2$

The centralizer of the new type NT6 *is*

$$\left\{ \begin{pmatrix} a_0 I_2 & C \\ 0_2 & a_0 I_2 \end{pmatrix} \mid a_0 \in \mathbf{F}_q,\, C \in M_2(\mathbf{F}_q) \right\}$$

Proof. In this case,

$$Z_{M_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} a_0 I_2 + a_1 D & C \\ 0_2 & a_0 I_2 + a_1 D \end{pmatrix} \mid C \in M_2(\mathbf{F}_q),\, a_0, a_1 \in F_q \right\},$$

where $D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

To see the branching rules here, we will use a different approach from what we have been using so far. Let $M = \begin{pmatrix} a_0 I_2 + a_1 D & C \\ 0_2 & a_0 I_2 + a_1 D \end{pmatrix}$ be an invertible matrix and $X = \begin{pmatrix} x_0 I_2 + x_1 D & Y \\ 0_2 & x_0 I_2 + x_1 D \end{pmatrix}$. We have

$$MX = \begin{pmatrix} (a_0 I_2 + a_1 D)(x_0 I_2 + x_1 D) & (a_0 I_2 + a_1 D)Y + C(x_0 I_2 + x_1 D) \\ 0_2 & (a_0 I_2 + a_1 D)(x_0 I_2 + x_1 D) \end{pmatrix},$$

$$XM = \begin{pmatrix} (x_0I_2 + x_1D)(a_0I_2 + a_1D) & (x_0I_2 + x_1D)C + Y(a_0I_2 + a_1D) \\ 0_2 & (x_0I_2 + x_1D)(a_0I_2 + a_1D) \end{pmatrix}$$

So, $XM = MX$ if and only if

$$a_1DY + x_1CD = x_1DC + a_1YD,$$

which implies

$$[a_1Y - x_1C, D] = 0 \tag{4.20}$$

Thus we need to deal with 4 cases of what x_1 and Y are, in equation (4.20).

When $x_1 = 0$ and $[Y, D] = 0$: There are $qq^2 = q^3$ matrices X in this case and Equation (4.20) holds for any a_1 and any C. Thus the centralizer group, $Z_{GL_4(\mathbf{F}_q)}(A, B, X)$, of X in $Z_{GL_4(\mathbf{F}_q)}(A, B)$ is $Z_{GL_4(\mathbf{F}_q)}(A, B)$ itself.
Thus, under conjugation by $Z_{GL_4(\mathbf{F}_q)}(A, B)$:

- Orbit size of $X = 1$.
- Number of orbits is $q^3/1 = q^3$.

Thus (A, B, X) is of type NT1 and the number of branches is q^3

When $x_1 = 0$ and $[Y, D] \neq 0$: The number of X's is $q(q^4 - q^2)$. Thus, equation (4.20) boils down to $a_1[Y, D] = 0$. But $[Y, D] \neq 0$ implies $a_1 = 0$.

$$\text{Hence, } Z_{GL_4(\mathbf{F}_q)}(A, B, X) = \left\{ \begin{pmatrix} a_0I_2 & C \\ 0_2 & a_0I_2 \end{pmatrix} \mid a_0 \neq 0,\, C \in M_2(\mathbf{F}_q) \right\}.$$

The size of $Z_{GL_4(\mathbf{F}_q)}(A, B, X)$ is, $(q-1)q^4 = q^5 - q^4$. But none of the so far known types of similarity classes (the types in $M_4(\mathbf{F}_q)$ and the new types NT1, NT2, NT3, NT4 and NT5), has a 5-dimensional centralizer algebra . So we have another new type, which we shall call NT6. We get:

- Orbit size of X is $(q^6 - q^5)/(q^5 - q^4) = q$.
- Number of orbits is $q(q^4 - q^2)/q = q^4 - q^2$.

Thus (A, B, X) is of type NT6 and the number of such branches is $q^4 - q^2$.

When $x_1 \neq 0$ and $[Y, D] = 0$: The number of X's is

$$q(q-1)q^2 = q^4 - q^3.$$

Thus, Equation (4.20) boils down to $x_1[C,D] = 0$, which means that $[C,D]$ is 0. Thus, $C = b_0 I_2 + b_1 D$. So

$$Z_{GL_4(\mathbf{F}_q)}(A,B,X) = \left\{ \begin{pmatrix} a_0 I_2 + a_1 D & b_0 I_2 + b_1 D \\ 0_2 & a_0 I_2 + a_1 D \end{pmatrix} \mid a_0 \neq 0 \; b_0, b_1 \in \mathbf{F}_q \right\},$$

which is a commutative group of size $q^4 - q^3$. So $Z_{M_4(\mathbf{F}_q)}(A,B,X)$ is 4-dimensional. We get:

- Orbit size of X is $(q^6 - q^5)/(q^4 - q^3) = q^2$.
- Number of orbits is $(q^4 - q^3)/q^2 = q^2 - q$.

Thus (A,B,X) is of a Regular type and the number of such Regular branches is $q^2 - q$.

When $x_1 \neq 0$ and $[Y,D] \neq 0$: The number of $X's$ of this kind is $q(q-1)(q^4 - q^2)$. In this case, equation (4.20) remains as it is. This implies, $x_1 C - a_1 Y \in \mathbf{F}_q[D]$, and $x_1 \neq 0$ implies $C = x_1^{-1} a_1 Y + b_0 I_2 + b_1 D$. So, $Z_{GL_4(\mathbf{F}_q)}(A,B,X)$ is

$$\left\{ \begin{pmatrix} a_0 I_2 + a_1 D & x_1^{-1} a_1 Y + b_0 I_2 + b_1 D \\ 0_2 & a_0 I_2 + a_1 D \end{pmatrix} \mid a_0 \neq 0, \; a_1, b_0, b_1 \in \mathbf{F}_q \right\}.$$

It is commutative, and is of size $q^4 - q^3$. So $Z_{M_4(\mathbf{F}_q)}(A,B,X)$ is 4-dimensional. Hence, (A,B,X) is of a Regular type. We have:

- Orbit size of $X = (q^6 - q^5)/(q^4 - q^3) = q^2$.
- Number of orbits is $q(q-1)(q^4 - q^2)/q^2 = (q-1)(q^3 - q)$.

The total number of Regular branches is

$$(q^2 - q) + (q-1)(q^3 - q) = q^4 - q^3 \text{ Regular branches.}$$

□

Lemma 4.3.15. *For (A,B) of similarity class type* NT2*, the branching rules are given in the table below*

Type	Number of Branches
NT2	q^3
Regular	$q^4 - q^3$
NT6	$q^4 - q^3$

Proof. $Z_{M_4(\mathbf{F}_q)}(A,B)$ is equal to

$$\left\{ \begin{pmatrix} a_0I_2 + a_1C_f & D \\ 0_2 & a_0I_2 + a_1C_f \end{pmatrix} \mid a_0, a_1 \in \mathbf{F}_q,\, D \in M_2(\mathbf{F}_q) \right\}$$

where C_f is a 2×2 matrix, whose characteristic polynomial is a degree 2 irreducible polynomial f. A matrix in $Z_{M_4(\mathbf{F}_q)}(A,B)$ is invertible iff $(a_0,a_1) \neq (0,0)$ and hence the size of the $Z_{GL_4(\mathbf{F}_q)}(A,B)$ is $q^6 - q^4$. To prove this lemma, we will use the steps used in the proof of Lemma 4.3.14. Let

$$M = \begin{pmatrix} a_0I_2 + a_1C_f & D \\ 0_2 & a_0I_2 + a_1C_f \end{pmatrix} \in Z_{GL_4(\mathbf{F}_q)}(A,B),$$

and let $X = \begin{pmatrix} x_0I_2 + x_1C_f & Y \\ 0_2 & x_0I_2 + x_1C_f \end{pmatrix}$. Then M and X commute iff

$$[a_1Y - x_1D, C_f] = 0 \tag{4.21}$$

From equation (4.21), we have 4 cases of what x_1 and Y should be: We shall analyze the cases:

When $x_1 = 0$ and $[Y, C_f] = 0$: The number of X's is $qq^2 = q^3$. Equation (4.21) holds for any a_1 and any D. Thus, $Z_{GL_4(\mathbf{F}_q)}(A,B,X)$ is $Z_{GL_4(\mathbf{F}_q)}(A,B)$ itself. Thus there are q^3 orbits under the conjugation by $Z_{GL_4(\mathbf{F}_q)}(A,B)$. Thus the triple (A,B,X) is of type NT2. Hence we have q^3 branches of type NT2.

When $x_1 = 0$ and $[Y, C_f] \neq 0$: The number of matrices X is $q(q^4 - q^2)$. Equation (4.21) boils down to $a_1[Y, C_f] = 0$, which implies $a_1 = 0$. Thus

$$Z_{GL_4(\mathbf{F}_q)}(A,B,X) = \left\{ \begin{pmatrix} a_0I_2 & C \\ 0_2 & a_0I_2 \end{pmatrix} \mid a_0 \in \mathbf{F}_q,\, C \in M_2(\mathbf{F}_q) \right\}.$$

So (A,B,X) is of type NT6. From this we get:

- Orbit size of X is $(q^6 - q^4)/(q^5 - q^4) = q + 1$.
- Number of such orbits is $q(q^4 - q^2)/(q+1) = q^4 - q^3$.

The number of branches of type NT6 is $q^4 - q^3$.

When $x_1 \neq 0$ and $[Y, C_f] = 0$: The number of matrices X is

$$q(q-1)q^2 = q^4 - q^3.$$

From equation (4.21), $x_1[D, C_f] = 0$, which implies $[D, C_f] = 0$. Hence $D = d_0 I_2 + d_1 C_f$ and therefore

$$Z_{GL_4(\mathbf{F}_q)}(A,B,X) = \left\{ \begin{pmatrix} a_0 I_2 + a_1 C_f & d_0 I_2 + d_1 C_F \\ 0_2 & a_0 I_2 + a_1 C_f \end{pmatrix} \mid (a_0, a_1) \neq (0,0),\ d_0, d_1 \in \mathbf{F}_q \right\}$$

It is commutative and its size is $(q^2-1)q^2 = q^4 - q^2$. Hence, $Z_{M_4(\mathbf{F}_q)}(A,B,X)$ is of dimension 4. So, the triple (A,B,X) is of a Regular type. We get:

- Orbit size of $X = (q^6 - q^4)/(q^4 - q^2) = q^2$.
- Number of such orbits is $(q^4 - q^3)/q^2 = q^2 - q$.

Thus, the number of Regular branches is $q^2 - q$.

When $x_1 \neq 0$ and $[Y, C_f] \neq 0$: The number of matrices is

$$q(q-1)(q^4 - q^2).$$

Equation (4.21) gives us $D \in x_1^{-1} a_1 Y + \mathbf{F}_q[C_f]$. So, $Z_{GL_4(\mathbf{F}_q)}(A,B,X)$ is:

$$\left\{ \begin{pmatrix} a_0 I_2 + a_1 C_f & x_1^{-1} a_1 Y + d_0 I_2 + d_1 C_f \\ 0_2 & a_0 I_2 + a_1 C_f \end{pmatrix} \mid (a_0, a_1) \neq (0,0),\ d_0, d_1 \in \mathbf{F}_q \right\}.$$

It is commutative and its size is $(q^2-1)q^2 = q^4 - q^2$. So $Z_{M_4(\mathbf{F}_q)}(A,B,X)$ is of dimension 4. Thus, (A,B,X) is of a Regular type.

- Orbit size of X is $(q^6 - q^4)/(q^4 - q^2) = q^2$.
- Number of such orbits is $q(q-1)(q^4 - q^2)/q^2 = q(q-1)(q^2-1)$.

Therefore, the total number of Regular branches is

$$q(q-1)(q^2-1) + (q^2 - q) = q^4 - q^3$$

Thus we have the table mentioned in the statement. □

Lemma 4.3.16. *If A is of similarity class type* NT3*, then its branching rules are given in the table below:*

Type	Number of Branches
NT3	q^3
Regular	$q^4 - q^3$
New Type NT6	$q^4 + q^3$

Proof. The centralizer algebra of a pair (A, B) of type NT3, is

$$Z_{M_4(\mathbf{F}_q)}(A,B) = \begin{pmatrix} D(c_0,c_1) & C \\ 0 & D(c_0,c_1) \end{pmatrix},$$

where $D(c_0, c_1)$ is a 2×2 diagonal matrix with c_0 and c_1 as its diagonal entries. This $D(c_0, c_1)$ can also be written as $c_0I_2 + c_1D(0, 1)$ (replace $c_1 - c_0$ by c_1).

$$\text{Let } X = \begin{pmatrix} x_0I_2 + x_1D(0,1) & Y \\ 0_2 & x_0I_2 + x_1D(0,1) \end{pmatrix}, \text{ and}$$

$$M = \begin{pmatrix} c_0I_2 + c_1D(0,1) & C \\ 0_2 & c_0I_2 + c_1D(0,1) \end{pmatrix} \in Z_{GL_4(\mathbf{F}_q)}(A,B).$$

As M is invertible, $c_0 \neq 0$ and $c_0 + c_1 \neq 0$. So, $XM = MX$ iff

$$[c_1Y - x_1C, D(0,1)] = 0.$$

From this equation, we have four cases as to what x_1 and Y have to be, i.e.,

<u>When $x_1 = 0$ and $[Y, D(0, 1)] = 0$:</u> The number of such X's is q^3. Hence c_1 can be anything and C can be any 2×2 matrix. Hence $Z_{GL_4(\mathbf{F}_q)}(A, B, X)$ is $Z_{GL_4(\mathbf{F}_q)}(A, B)$ itself. Therefore the orbit of X is of size 1 and there are $q^3/1 = q^3$ such orbits. Hence q^3 branches of type NT3.

<u>When $x_1 = 0$ and $[Y, D(0, 1)] \neq 0$:</u> The number of such X's is $q(q^4-q^2)$. $c_1[Y, D(0, 1)] = 0$ implies $c_1 = 0$. Thus,

$$Z_{GL_4(\mathbf{F}_q)}(A,B,X) = \left\{ \begin{pmatrix} c_0I & C \\ 0_2 & c_0I \end{pmatrix} :\ c_0 \neq 0,\, C \in M_2(\mathbf{F}_q) \right\}.$$

Thus (A, B, X) is of the type NT6.

- Orbit size of X is $(q^4(q-1)^2)/(q^5-q^4) = q-1$.
- The number of such orbits is $q^3(q^2-1)/(q-1) = q^4+q^3$.

We therefore have q^4+q^3 branches of the type NT6.

When $x_1 \neq 0$ and $[Y, D(0,1)] = 0$: There are $q(q-1)q^2$ such matrices and we have $x_1[C, D(0,1)] = 0$ which implies that $C = d_0 I_2 + d_1 D(0,1)$. Hence,

$$Z_{GL_4(\mathbf{F}_q)}(A,B,X) = \left\{ \begin{pmatrix} c_0 I_2 + c_1 D(0,1) & d_0 I_2 + d_1 D(0,1) \\ 0_2 & c_0 I + c_1 D(0,1) \end{pmatrix} \mid c_0, c_1 + c_0 \neq 0 \right\}.$$

It is commutative, and is of size $q^2(q-1)^2$. So, $Z_{M_4(\mathbf{F}_q)}(A,B,X)$ is 4-dimensional. Therefore, (A,B,X) is of a Regular type.

- Orbit size of X is $(q^4(q-1)^2)/(q^2(q-1)^2) = q^2$.
- Number of such orbits is $q(q-1)q^2/q^2 = q^2-q$.

When $x_1 \neq 0$ and $[Y, D(0,1)] \neq 0$: There are $q(q-1)(q^4-q^2)$ such X and $C \in x_1^{-1}c_1 Y + \mathbf{F}_q[D(0,1)]$. Thus $C = x_1^{-1}c_1 Y + d_0 I_2 + d_1 D(0,1)$ and so the $Z_{GL_4(\mathbf{F}_q)}(A,B,X)$ consists of matrices of the form

$$\begin{pmatrix} c_0 I_2 + c_1 D(0,1) & x_1^{-1} c_1 Y + d_0 I_2 + d_1 D(0,1) \\ 0_2 & c_0 I_2 + c_1 D(0,1) \end{pmatrix}.$$

It is commutative and its size is $q^2(q-1)^2$. So, $Z_{M_4(\mathbf{F}_q)}(A,B,X)$ is 4-dimensional. Thus (A,B,X) is a Regular branch.

- Orbit size of X is $(q^4(q-1)^2/(q^2(q-1)^2) = q^2$.
- Number of such orbits is $q(q-1)(q^4-q^2)/q^2 = q(q-1)(q^2-1)$.

The total number of Regular branches is

$$q(q-1)(q^2-1) + q(q-1) = q^4 - q^3.$$

$\square$

Lemma 4.3.17. *For the commuting pair (A,B) of similarity class type* NT4 *or* NT5*, there are:*

- *q^3 branches of its own type.*
- *q^3+q^2 branches of the new type* NT6.

- q^4 Regular *type of branches.*

Proof. The proof is the same for both NT4 and NT5. So it will suffice to prove for any one of them. We shall prove it for NT4.
$Z_{M_4(\mathbf{F}_q)}(A,B)$ consists of matrices of the form

$$M = \begin{pmatrix} a_0 & b_1 & b_2 & b_3 \\ 0 & a_0 & 0 & 0 \\ 0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & a_0 \end{pmatrix}$$

which, on conjugation by elementary matrices (which switches the 2nd and 3rd rows and columns of M) becomes

$$M = \begin{pmatrix} a_0 & b_2 & b_1 & b_3 \\ 0 & c_2 & c_1 & c_3 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

We shall rewrite M as

$$\begin{pmatrix} a_0 & a_1 & b_1 & b_2 \\ 0 & b_0 & b_3 & b_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix},$$

and let M' be a conjugate of M in $Z_{M_4(\mathbf{F}_q)}(A,B)$:

$$M' = \begin{pmatrix} a'_0 & a'_1 & b'_1 & b'_2 \\ 0 & b'_0 & b'_3 & b'_4 \\ 0 & 0 & a'_0 & 0 \\ 0 & 0 & 0 & a'_0 \end{pmatrix}.$$

Then there is an invertible X such that $XM = M'X$. Let

$$X = \begin{pmatrix} x_0 & x_1 & y_1 & y_2 \\ 0 & y_0 & y_3 & y_4 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix}$$

where $x_0, y_0 \neq 0$. So we have

$$\begin{pmatrix} x_0 & x_1 & y_1 & y_2 \\ 0 & y_0 & y_3 & y_4 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & b_1 & b_2 \\ 0 & b_0 & b_3 & b_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix} = \begin{pmatrix} a'_0 & a'_1 & b'_1 & b'_2 \\ 0 & b'_0 & b'_3 & b'_4 \\ 0 & 0 & a'_0 & 0 \\ 0 & 0 & 0 & a'_0 \end{pmatrix} \begin{pmatrix} x_0 & x_1 & y_1 & y_2 \\ 0 & y_0 & y_3 & y_4 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix}.$$

From the above equation we get $a'_0 = a_0$, $b'_0 = b_0$, and the following equations:

$$\begin{aligned} a_0x_1 + a'_1y_0 &= a_1x_0 + x_1b_0 && (4.22) \\ a'_1y_3 + b'_1x_0 &= x_0b_1 + x_1b_3 && (4.23) \\ a'_1y_4 + b'_2x_0 &= x_1b_4 + b_2x_0 && (4.24) \\ b_0y_3 + b'_3x_0 &= y_0b_3 + y_3a_0 && (4.25) \\ b_0y_4 + b'_4x_0 &= y_0b_4 + y_4a_0 && (4.26) \end{aligned}$$

We have two main cases: $a_0 \neq b_0$ and $a_0 = b_0$.

If $a_0 \neq b_0$. Then, in equation (4.22), using a suitable choice of x_1, we can make $a'_1 = 0$. With a suitable choice of y_3 in equation (4.25), we can make $b'_3 = 0$. Similarly, in equation (4.26), choose a suitable y_4 so that $b'_4 = 0$. Then from equations (4.23) and (4.24), we get $b'_1 = b_1$ and $b'_2 = b_2$. So

$$M = \begin{pmatrix} a_0 & 0 & b_1 & b_2 \\ 0 & b_0 & 0 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}$$

Its centralizer, $Z_{M_4(\mathbf{F}_q)}(A, B, M)$, in $Z_{M_4(\mathbf{F}_q)}(A, B)$ is

$$\left\{ \begin{pmatrix} x_0 & 0 & y_1 & y_2 \\ 0 & y_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \mid x_0, y_0, y_1, y_2 \in \mathbf{F}_q \right\}$$

which is 4-dimensional and commutative. Therefore, this branch (A, B, M) is of a Regular type and there are $q^3(q-1) = q^4 - q^3$ such branches

If $a_0 = b_0$. Then equation (4.22) becomes $a_1' y_0 = a_1 x_0$. Here, there are two cases.

$$a_1 \neq 0 \text{ and } a_1 = 0$$

When $a_1 \neq 0$, choose y_0 such that $a_1' = 1$. So, letting $a_1 = a_1' = 1$, we have $y_0 = x_0$. Then, from equations (4.25) and (4.26) we get $b_3' = b_3$ and $b_4' = b_4$. Equation (4.23) becomes $y_3 + b_1' x_0 = x_0 b_1 + x_1 b_3$ and equation (4.24) becomes $y_4 + b_2' x_0 = x_1 b_4 + b_2 x_0$. So we can choose y_3 and y_4 appropriately so that $b_1' = b_2' = 0$ So our M reduces to

$$\begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & b_3 & b_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

Hence,

$$Z_{M_4(\mathbf{F}_q)}(A,B,M) = \left\{ \begin{pmatrix} x_0 & x_1 & y_1 & y_2 \\ 0 & x_0 & x_1 b_3 & x_1 b_4 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \mid x_0, x_1, y_1, y_2 \in \mathbf{F}_q \right\},$$

which is 4 dimensional and commutative. Thus the branch, (A,B,M), is of a Regular type. The number of such branches is q^3. So we have a total of $q^4 - q^3 + q^3 = q^4$ Regular branches.

When $a_1 = 0$, equation (4.23) becomes $b_1' x_0 = x_0 b_1 + x_1 b_3$, equation (4.24) becomes $b_2' x_0 = x_0 b_2 + x_1 b_4$, and from equations 4.25 and 4.26, we get $b_3' x_0 = y_0 b_3$ and $b_4' x_0 = y_0 b_4$. So we can divide this into two cases.

$$(b_3, b_4) = (0,0) \text{ and } (b_3, b_4) \neq (0,0)$$

When $(b_3, b_4) = (0,0)$ we have $b_1' = b_1$ and $b_2' = b_2$ and thus M reduces to

$$\begin{pmatrix} a_0 & 0 & b_1 & b_2 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

Thus, $Z_{M_4(\mathbf{F}_q)}(A,B,M)$ is the whole of $Z_{M_4(\mathbf{F}_q)}(A,B)$. Thus (A,B,M) is of the type

NT4 and we have q^3 such branches.

When $(b_3, b_4) \neq (0,0)$ and $b_3 \neq 0$. Then we can make $b_3 = 1$. Letting $b_3' = b_3 = 1$, we get $y_0 = x_0$ and therefore $b_4' = b_4$. Equation (4.23) becomes $b_1' x_0 = x_0 b_1 + x_1$, hence we can get $b_1' = 0$. Solving for x_1 by putting $b_1 = 0$, gives us $x_1 = 0$, and we get $b_2' = b_2$. Thus M is reduced to

$$\begin{pmatrix} a_0 & 0 & 0 & b_2 \\ 0 & a_0 & 1 & b_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}$$

and $Z_{M_4(\mathbf{F}_q)}(A,B,M)$ is

$$\left\{ \begin{pmatrix} x_0 I_2 & Y \\ 0_2 & x_0 I_2 \end{pmatrix} \mid x_0, \in \mathbf{F}_q,\, Y \in M_2(\mathbf{F}_q) \right\}.$$

Therefore (A,B,M) is of type NT6, and we have q^3 such branches.

If $b_3 = 0$ and $b_4 \neq 0$. Then we can make $b_4 = 1$ and by the arguments like in the above case, we can make $b_2 = 0$ and $b_1' = b_1$. So

$$M = \begin{pmatrix} a_0 & 0 & b_1 & 0 \\ 0 & a_0 & 0 & 1 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

So, $Z_{M_4(\mathbf{F}_q)}(A,B,M)$ is

$$\left\{ \begin{pmatrix} x_0 I_2 & Y \\ 0_2 & x_0 I_2 \end{pmatrix} \mid x_0, \in \mathbf{F}_q,\, Y \in M_2(\mathbf{F}_q) \right\}$$

Thus this (A,B,M) too is a branch of the new type NT6 and there are q^2 such branches. So in total we have $q^3 + q^2$ branches of the new type NT6. $\square$

Lemma 4.3.18. *For a triple (A,B,M) of similarity class type* NT6*, there are q^5 branches of the type* NT6*.*

Proof. We know that

$$Z_{M_4(\mathbf{F}_q)}(A,B,M) = \left\{ \begin{pmatrix} a_0 I_2 & C \\ 0_2 & a_0 I_2 \end{pmatrix} \mid a_0 \in \mathbf{F}_q \text{ and } C \in M_2(\mathbf{F}_q) \right\}$$

It is easy to see that this algebra is commutative. Hence, there is only one branch and it is of the type NT6 and there are q^5 of them. □

We therefore have no more new similarity class types.

4.3.4 Calculating $c(4,k,q)$

Now, that we have all the branching rules, we can form a matrix, $\mathcal{B}_4 = [b_{ij}]$, with rows and columns indexed by the types. For a given type j, b_{ij} is the number of similarity class type i branches of a tuple of similarity class type j. $\mathcal{B}_4$ is our branching matrix. Table 4.2 lists the rcfs, and under each rcf, it has a list of the types with that rcf. Let each of the new types be treated as separate rcf's. By the averaging technique mentioned in the end of Section 4.2, we can reduce $\mathcal{B}_4$ to a 11×11 matrix indexed by the 5 rcfs and the 6 new types.

rcf $(1,1,1,1)$**:** There is only one type with rcf $(1,1,1,1)$, i.e., the Central type $(1,1,1,1)_1$. It has q branches of rcf $(1,1,1,1)$, q^2 branches each of rcf types $(2,1,1)$ and $(2,2)$, q^3 branches with rcf $(3,1)$, and q^4 branches with rcf (4).

rcf (4)**:** Each Regular type is of rcf type (4), and has q^4 branches of rcf (4).

rcf $(2,1,1)$**:** An element of rcf type $(2,1,1)$ is of class type $(1,1,1)_1(1)_1$ with probability $1/q$ and of class type $(2,1,1)_1$ with probability $(q-1)/q$. So, on an average, a tuple of rcf type $(2,1,1)$ has:

- q^2 branches of rcf type $(2,1,1)$.
- q^3+q^2-q-1 branches of rcf type $(3,1)$.
- q^4+q Regular (rcf type (4)) branches.
- 1 branch each of types NT1, NT4 and NT5.
- q branches of type NT3.

rcf $(2,2)$**:** There are three similarity class types with rcf $(2,2)$. They are $(1,1)_1(1,1)_1$, $(2,2)_1$ and $(1,1)_2$. An element of rcf type $(2,2)$ is of class type $(1,1)_1(1,1)_1$ with probability $(q-1)/(2q)$, of class type $(2,2)_1$ with probability $1/q$

and is of class type $(1,1)_2$ with probability $(q-1)/(2q)$. So on an average, a tuple of rcf-type $(2,2)$ has:

- q^2 branches of rcf type $(2,2)$.
- $q^3 - q^2$ branches of rcf $(3,1)$.
- q^4 branches of rcf (4).
- q branches of the new type NT1
- $(q^2 - q)/2$ branches each of the new types NT2 and NT3.

rcf $(3,1)$**:** The similarity class types with rcf $(3,1)$ are:

- $(3,1)_1$
- $(2,1)_1(1)_1$
- $(1,1)_1(2)_1$
- $(1,1)_1(1)_2$ and
- $(1,1)_1(1)_1(1)_1$

Their probabilities are mentioned in the table below.

Class Type	Probability
$(3,1)_1$	$\frac{1}{q^2}$
$(2,1)_1(1)_1$	$\frac{q-1}{q^2}$
$(1,1)_1(2)_1$	$\frac{q-1}{q^2}$
$(1,1)_1(1)_2$	$\frac{q-1}{2q}$
$(1,1)_1(1)_1(1)_1$	$\frac{(q-1)(q-2)}{2q^2}$

All these types have branches of their own respective types and Regular branches. Hence we have on an average: q^3 branches of rcf type $(3,1)$ and $q^4 + q$ branches of rcf type (4).

So our branching matrix $\mathcal{B}_4$ is equal to

$$
\begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^3 & q^3+q^2-q-1 & q^3-q^2 & q^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^4 & q^4+q & q^4 & q^4+q & q^4 & q^4-q^3 & q^4-q^3 & q^4-q^3 & q^4 & q^4 & 0 \\
0 & 1 & q & 0 & 0 & q^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{q^2-q}{2} & 0 & 0 & 0 & q^3 & 0 & 0 & 0 & 0 \\
0 & q & \frac{q^2-q}{2} & 0 & 0 & 0 & 0 & q^3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & q^3 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^3 & 0 \\
0 & 0 & 0 & 0 & 0 & q^4-q^2 & q^4-q^3 & q^4+q^3 & q^3+q^2 & q^3+q^2 & q^5
\end{pmatrix}
$$

Let e_1 denote the 11×1 column matrix with first entry being 1 and the rest, 0. Let $\mathbf{1}'$ denote the 1×11 row matrix, whose entries are all 1's. Then we have

$$c(4, k, q) = \mathbf{1}' \mathcal{B}_4^k . e_1$$

The table below lists $c(4, k, q)$ for $k = 1, 2, 3, 4$. The calculations were done using sage.

k	$c(4, k, q)$
1	$q^4 + q^3 + 2q^2 + q$
2	$q^8 + q^7 + 3q^6 + 3q^5 + 5q^4 + 3q^3 + 3q^2$
3	$q^{12} + q^{11} + 3q^{10} + 4q^9 + 8q^8 + 8q^7 + 11q^6 + 8q^5 + 5q^4 + 2q^3$
4	$q^{16} + q^{15} + 3q^{14} + 5q^{13} + 9q^{12} + 12q^{11} + 16q^{10}$ $+17q^9 + 17q^8 + 13q^7 + 9q^6 + 4q^5 + 2q^4$

We can see that $c(4, k, q)$ is a polynomial in q with non-negative integer coefficients for $k = 1, 2, 3, 4$. But, we cannot say the same for k in general. So, we will have to use the generating function for $c(4, k, q)$:

$$h_4(q, t) = \sum_{k=0}^{\infty} c(4, k, q) t^k = \mathbf{1}'(I - t\mathcal{B}_4)^{-1} e_1.$$

In the next subsection, we will look at the expression of $h_4(q, t)$ to prove Theorem 1.2.2 in the case of $n = 4$.

4.3.5 Non-Negativity of Coefficients of $c(4,k,q)$

Now it remains to check if the coefficients of $h_4(q,t)$ are non-negative. The rational generating function $h_4(q,t)$ is:

$$h_4(q,t) = \frac{r_+(q,t) - r_-(q,t)}{(1-qt)(1-q^2t)(1-q^3t)(1-q^4t)(1-q^5t)},$$

where

$$r_+(q,t) = 1 + q^2t + 2q^2t^2 + q^3t^2 + 2q^4t^2 + q^6t^3, \text{ and}$$
$$r_-(q,t) = q^5t + q^7t^2 + q^3t^3 + 2q^7t^3 + 2q^9t^3 + q^{10}t^4.$$

We have

$$\frac{1}{(1-qt)(1-q^2t)(1-q^3t)(1-q^4t)(1-q^5t)} = \left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^jt^k\right)\right),$$

where $p_{5,k}(j)$ denotes the number of partitions of j with k parts, with the maximum part being ≤ 5. With this,

$$h_4(q,t) = (r_+(q,t) - r_-(q,t))\left[1 + \left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^jt^k\right)\right)\right].$$

Expanding this gives us

$$\begin{array}{ll}
\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^jt^k\right)\right) & -\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^{j+5}t^{k+1}\right)\right)\\
+\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^{j+2}t^{k+1}\right)\right) & -\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^{j+7}t^{k+2}\right)\right)\\
+\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} 2p_{5,k}(j)q^{j+2}t^{k+2}\right)\right) & -\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^{j+3}t^{k+3}\right)\right)\\
+\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^{j+3}t^{k+2}\right)\right) & -\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} 2p_{5,k}(j)q^{j+7}t^{k+3}\right)\right)\\
+\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} 2p_{5,k}(j)q^{j+4}t^{k+2}\right)\right) & -\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} 2p_{5,k}(j)q^{j+9}t^{k+3}\right)\right)\\
+\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^{j+6}t^{k+3}\right)\right) & -\left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{5k} p_{5,k}(j)q^{j+10}t^{k+4}\right)\right).
\end{array} \tag{4.27}$$

The coefficient, d_{jk}, of q^jt^k in equation (4.27) is

$$\begin{aligned}
d_{jk} &= (p_{5,k}(j) - p_{5,k-1}(j-5)) + (p_{5,k-1}(j-2) - p_{5,k-2}(j-7))\\
&+(2p_{5,k-2}(j-2) - p_{5,k-3}(j-3)) + (p_{5,k-2}(j-3) - 2p_{5,k-3}(j-7))\\
&+(2p_{5,k-2}(j-4) - 2p_{5,k-3}(j-9)) + (p_{5,k-3}(j-6) - p_{5,k-4}(j-10)).
\end{aligned} \tag{4.28}$$

Here are some observations which will be enough to prove that d_{jk} is non-negative.

Lemma 4.3.19. *For any $k \geq 1$, any $j : k \leq j \leq 5k$, and any l such that, $1 \leq l \leq 5$, $p_{5,k}(j) \geq p_{5,k-1}(j-l)$.*

Proof. We assume that $j - l \leq 5(k-1)$ so that $p_{5,k-1}(j-l) \neq 0$. Given a partition of $j - l$ with $k - 1$ parts with maximal part ≤ 5, we can attach the part l to this partition to get a partition of j in k parts, with maximal part ≤ 5. Hence $p_{5,k}(j) \geq p_{5,k-1}(j-l)$. $\square$

As a consequence of the above lemma, we have the following inequalities.

$$
\begin{aligned}
p_{5,k}(j) &\geq p_{5,k-1}(j-5) && (4.29)\\
p_{5,k-1}(j-2) &\geq p_{5,k-2}(j-7) && (4.30)\\
p_{5,k-2}(j-2) &\geq p_{5,k-3}(j-3) && (4.31)\\
p_{5,k-2}(j-3) &\geq p_{5,k-3}(j-7) && (4.32)\\
p_{5,k-2}(j-4) &\geq p_{5,k-3}(j-9) && (4.33)\\
p_{5,k-3}(j-6) &\geq p_{5,k-4}(j-10). && (4.34)
\end{aligned}
$$

Lemma 4.3.20. *Let $k \geq 4$. Then for j such that $j-7 \geq k-3$ we have the following:*

- *If $j - 7 = 5(k-3)$, then*

$$(p_{5,k}(j) - p_{5,k-1}(j-5)) + (p_{5,k-2}(j-3) - 2p_{5,k-3}(j-7)) \geq 0 \qquad (4.35)$$

- *If $j - 7 < 5(k-3)$ then*

$$p_{5,k-2}(j-3) - 2p_{5,k-3}(j-7) \geq 0 \qquad (4.36)$$

Proof. When $j - 7 = 5(k-3)$, given the only partition of $j - 7$ with $k - 3$ parts, we can attach two 1's to it, to get a partition of $j - 5$ in $k - 1$ parts. Hence $p_{5,k-1}(j-5) \geq p_{5,k-3}(j-7)$.

$$
\begin{aligned}
\text{Therefore } &(p_{5,k}(j) - p_{5,k-1}(j-5)) + (p_{5,k-2}(j-3) - 2p_{5,k-3}(j-7))\\
&\geq p_{5,k}(j) - 2p_{5,k-1}(j-5) + (p_{5,k-2}(j-3) - p_{5,k-3}(j-7)).
\end{aligned}
$$

Observe: $j - 7 = 5k - 15 \Rightarrow j - 5 = 5k - 13 = 5(k-1) - 8$. So any partition of $j - 5$ with $k - 1$ parts, with maximal part ≤ 5, will have atleast two parts which are strictly less than 5. So, to each of these, we can either attach a 5, or add 1 each to the two parts which are less than 5 and attach 3 as the kth part. This gives 2 partitions of j having k parts. So, $p_{5,k}(j) - 2p_{5,k-1}(j-5) \geq 0$ and therefore

$$\begin{aligned}
&(p_{5,k}(j) - p_{5,k-1}(j-5)) + (p_{5,k-2}(j-3) - 2p_{5,k-3}(j-7)) \\
&\geq p_{5,k}(j) - 2p_{5,k-1}(j-5) + (p_{5,k-2}(j-3) - p_{5,k-3}(j-7)) \\
&\geq 0 \text{ Since } (p_{5,k-2}(j-3) - p_{5,k-3}(j-7)) \geq 0 \quad \text{(from ineq. (4.32))}.
\end{aligned}$$

Hence inequality (4.35) holds.

When $j-7 < 5(k-3)$, then, for any partition of $j-7$ with $k-3$ parts with each part being atmost 5, we have atleast one part which is strictly less than 5. Given any such partition, we can either, add 1 to the part that's < 5 and attach a 3, or just attach a 4 to the existing partition, to get a partition of $j-3$ in $k-2$ parts. Hence we get two partitions of $j-3$ in $k-2$ parts. Therefore inequality (4.36) holds. □

Using Lemma 4.3.20 and inequalities (4.29) to (4.34) , we can show that the coefficient of $q^j t^k$ for each $j, k \geq 0$, is non-negative. So for each $k \geq 1$, the coefficients of $c(4,k,q)$ are the coefficients of $q^j t^k$ as j varies, which are non-negative. Therefore, the coefficients of $c(4,k,q)$ are non-negative integers.
Thus, Theorem 1.2.2 is proved for $n = 4$.

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