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P R E F A C E

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Following the recommendation of the Board of Examiners to permit the author to resubmit his thesis, it is hereby resubmitted after incorporating the modifications suggested by the Examiners and making the necessary additions. The author wishes to thank the Examiners for their valuable suggestions.

(G.N.Keshava Murthy)

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C O N T E N T S

Introduction

Let $L^1(R)$ be the Banach space of all real valued functions f on the real line R such that

$$\|f\|_1 = \int_R |f(x)| \, dx < \infty$$

Here we identity two functions which are equal almost every where. If we define multiplication by convolution

$$(f * g)(x) = \int_R f(x-u)\, g(u) du$$

then $L^1(R)$ is a Banach algebra. If

$$\hat{f}(y) = \int_R f(t)\, e^{2\pi i t y}\, dt$$

denotes the Fourier transform of f, then by Fubini's theorem

$$\widehat{f * g}(y) = \hat{f}(y) \cdot \hat{g}(y)$$

Now let $T : L^1(R) \longrightarrow L^1(R)$ be a transformation such that

$$(1) \qquad T(f * g) = Tf * g = f * Tg \qquad f, g \in L^1(R).$$

That is, T commutes with convolution. Applying Fourier transforms, we have

$$(2) \qquad \widehat{Tf} \cdot \hat{g} = \hat{f} \cdot \widehat{Tg}$$

Then using the semisimplicity of $L^1(R)$ it is possible to find a unique function φ defined on R such that

$$(3) \qquad \widehat{Tf} = \varphi \hat{f} \qquad f \in L^1(R).$$

On the other hand, if T is a transformation for which (3) holds, then

$$Tf * g = (Tf) * g = f * (Tg) = f * Tg$$

and appealing to inverse Fourier transform it is easy to see that T satisfies (1).

Such a transformation T is called a Fourier multiplier transformation associated with φ and φ is called the multiplier function. The multiplier problem consists in asking for sufficient conditions on the function φ in order that the transformation T is bounded.

It is easy to find examples of such transformations. The inverse Fourier transform is given by the formula

$$f(x) = \int_R e^{-2\pi itx} \hat{f}(t)dt$$

(Here the integral is to be interpreted in the proper sense). Then we can write

$$(4) \qquad Tf(x) = \int_R \varphi(y) \hat{f}(y) e^{-2\pi ixy} dy$$

For suitable choices of the function φ, we obtain the partial Fourier integral, the translate, the derivative and the Riesz conjugate as special cases. Similar problems are set for Fourier series instead of Fourier integrals. In 1939, J. Marcinkiewicz proved a very important multiplier theorem on Fourier series. It gives a sufficient condition for a sequence of complex numbers $\{\lambda_n\}$ to have the property that the multiplication of the Fourier coefficients of a periodic function f by $\{\lambda_n\}$ will give a periodic function g and the mapping $f \rightarrow g$ is bounded in L^p. There are various generalizations of this result.

Thus we see that many important situations in classical Fourier analysis can be regarded as problems in multiplier theory. In addition multipliers seems to appear in many important branches such as Banach algebra, singular integrals and partial differential equations. The theory of multipliers can now be regarded as one of the fashionable fields of harmonic analysis.

Let G be a locally compact abelian group and Γ its character group. Let dx and $d\gamma$ denote the elements of the normalized Haar measures on G and Γ respectively.

If $1 \leq p \leq \infty$, $L^p(G)$ is the Lebesgue space of equivalance classes of complex valued measurable functions f on G such that

$$f_p = \left(\int_G |f(x)|^p \, dx\right)^{1/p} < \infty$$

when $p = \infty$, $\|f\|_\infty$ denotes the essential supremum of $|f|$. If f, g $L^1(G)$, the convolution is given by

$$(f * g)(t) = \int_G f(t-x)\, g(x)dx$$

$M(G)$ is the Banach space of all bounded regular complex valued measures μ on G normed by $\|\mu\| = |\mu|(G)$ = total variation of μ. If $\lambda, \mu \in M(G)$, then multiplication is defined by

$$\lambda * \mu(E) = \int_G \lambda(E-x)\, d\mu(x)$$

$M(G)$ is then a Banach algebra and $L^1(G)$ is a closed ideal in $M(G)$. If $y \in G$, then the translation operator τ_y is defined on a space of functions f on G by the formula

$$\tau_y f(x) = f(x-y) \qquad x \in G$$

A linear operator $T : L^p(G) \longrightarrow L^q(G)$ is translation invariant if

$$T\tau_x = \tau_x T$$

for all $x \in G$. The natural question that arises is whether there are such nontrivial bounded operators from $L^p(G)$ to $L^q(G)$

for various values of p and q and to obtain characterization of such operators if they exist. The work of Hormander [9] not only contains various fundamental results in this direction, but has actually given a lot of motivation for various generalizations by many authors.

Considering $G = R^n$, Hormander proved the following re result.

THEOREM A. If T is a bounded translation invariant operator from $L^p(G)$ to $L^q(G)$ then there exists a unique distribution $d \in S^1$ such that

(5) $$Tu = d*u \qquad u \in S$$

If $p < \infty$, then T is the closure of the operator $u \to d*u$. If $q < p < \infty$, the distribution d is 0 and if $p = q = \infty$ the distribution d is a bounded measure.

Here S denotes the space of C^∞-functions on R^n which decrease rapidly at infinity and S^1 is the space of tempered distributions.

A bounded translation invariant operator from $L^p(G)$ to $L^q(G)$ is termed a (p,q)-multiplier. In the case of $L^1(G) \to L^1(G)$ there are various equivalent definitions. A multiplier on $L^1(G)$ is either a continuous linear operator T which commutes with translation operators which commutes with convolutions. Notice that translation operators may be defined even though convolutions may not. Another definition is the following. A function φ defined on the character group Γ is called a multiplier for $L^1(G)$ if $\varphi \hat{f} \in L^1(G)^\wedge$ whenever $f \in L^1(G)$ where $\wedge$ denotes the Fourier transform.

When G is a locally compact abelian group and $T: L^p(G) \to L^q(G)$

is a bounded translation invariant operator then the representation (5) given in Theorem A takes the form

(6) $$Tf = \sigma * f$$

Here σ is a quasimeasure when $1 \leq p, q < \infty$ and (6) holds for all $f \in \mathcal{K}(G)$, the space of continuous functions on G with compact support. If $p = q$ and $1 < p < \infty$, the quasimeasure σ becomes a pseudomeasure and f varies over $L^1(G) \cap L^2(G) \cap L^p(G)$. If $T : L^1(G) \to L^p(G)$, then (6) is valid for all $f \in L^1(G)$ with $\sigma \in M(G)$ if $p = 1$ and $\sigma \in L^p(G)$ if $1 < p < \infty$.

The characterization of the space of multipliers on different L^p spaces were also obtained by Figa Talamanca [4], Figa-Talamanca and Gaudry [5] and Rieffel [20]. Using the idea of the tensor product these authors have characterised the multiplier space as the dual of certain Banach spaces.

In this thesis our object is to give representation theorems corresponding to the type (6) for the multiplier on several spaces and also to characterize the space of multipliers on various other spaces. Throughout we assume a multiplier to be a bounded linear operator which commutes with translations.

In Chapter I we study the properties of sums and intersection of weighted Lebesgue spaces defined on a locally compact abelian group with Haar measure dx. For our study we consider the class of weight functions introduced by P. Kree [13]. These are precisely the functions $\omega \in \Omega$, satisfying the conditions

1) ω is measurable on G, positive almost everywhere for the Haar measure dx.

2) for each $p \in (1,\infty)$ both ω^{p} and ω^{-p} are locally integrable.
Let $\omega \in \Omega$ $1 < p < \infty$. Then $L^{p,\omega}(G)$ denote the equivalence class of complex valued measurable functions on G such that $\left(\int_G |f\omega|^p dx\right)^{\frac{1}{p}} < \infty$. First we identify the dual spaces of the sums and intersections of the weighted Lebesgue spaces $L^{p,\omega}$.

For the studying of multipliers we consider a subclass Ω_0 of Ω consisting of functions ω on G which satisfy the conditions

$$\omega(x+y) \leq \omega(x)\ \omega(y) \qquad x, y \in G.$$

We also ~~in this~~ give in this chapter several properties of the weighted Lebesgue spaces where $\omega \in \Omega_0$ which we require for our future work. For $1 < p, q < \infty$ let $M(L^{p,\omega}, L^{q,\omega})$ denote the space of multipliers from $L^{p,\omega} \to L^{q,\omega}$. In Chapter II we consider the representation theorem for the elements of $M(L^{p,\omega}, L^{q,\omega})$ and also give the characterisation of $M(L^{p,\omega}, L^{q,\omega})$ ~~as~~ as ~~is~~ dual space. Here we need to assume some more conditions on the weight function $\omega \in \Omega_0$.

In Chapter III we consider the characterization of the space of multipliers for the space D_1 introduced in Chapter I which is defined for $1 < p_1, p_2 < \infty$ to be the space $D_1 = L^{p_1,\omega_1} \cap L^{p_2,\omega_2}$ where $\omega_1, \omega_2 \in \Omega_0$.

Segal algebras which are certain subalgebras of $L^1(G)$ have acquired considerable importance in recent years. Our interest here is to study multipliers on these subalgebras which we denote by $S(G)$. If $M(S)$ denotes the space of multipliers on $S(G)$ and if $M(S,L^p)$ denotes the multipliers from $S \to L^p$ for $1 \leq p < \infty$ then in the I part of the Chapter IV we have given certain abstract characterisation theorem for $M(S)$ and in the second part we have given the dual space characterization theorem for the space $M(S,L^p)$.

In Chapter V we consider two special cases of Segal algebras and study the multipliers on these spaces. The algebra $A^p_\omega(G)$ $(1 \leq p < \infty)$ which consists of functions in $L^1(G)$ whose Fourier transform is in the space $L^{p,\omega}(\hat{\Gamma})$ where ω is an even continuous function on Γ which satisfy the condition

$$\omega(x+y) \leq \omega(x)\,\omega(y)$$

We have proved that if G is a noncompact locally compact abelian group than the multipliers on $A^p_\omega(G)$ are precisely the bounded regular measures on G. Later we consider the Wiener space W(R) which consists of continuous functions f on the real line such that

$$\sum_{-\infty}^{\infty} \max_{x \in I_k} |f(x)| < \infty \quad \text{where } I_k = [k, k+1]$$
$$\text{and } k = 0, \pm 1, \pm 2, \cdots$$

This is a Banach algebra under the norm

$$\|f\|_W = \sum_{-\infty}^{\infty} \max_{x \in I_k} |f(x)| .$$

If T is a multiplier from $W \to W$ we have shown that

$$Tf = f*\mu$$

Where μ belongs to the dual of W.

In Chapter VI we have introduced a new class of functions ~~on~~ W_α which happens to be a subclass of W. <u>The class</u> W_α : Let $0 < \alpha < 1$ and let lip α denote the class of all functions f on the real line R such that

$$\sup_{x \in R} |f(x+h) - f(x)| = o(|h|^\alpha) \qquad h \to 0$$

Let W_α denote the class of all functions $f \in$ lip α, such that

$$\|f\|_{W_\alpha} = \sum \max_{x \in I_k} |f(x)| \sum \max_{x \in I_k}$$

$\|f\|_{W_\alpha} = \sum_{-\infty}^{\infty} m_k(f)<\infty$ where

$$m_k(f) = \max\left\{ \max_{x\in I_k}|f(x)|\,,\ \sup_{x,\,x+h\in I_k} \frac{|f(x+h)-f(x)|}{|h|^\alpha} \right\}$$

and $I_k = [k, k+1]$, $k = 0, \pm 1, \pm 2, \ldots$. This chapter is denoted to a systematic study of this space including the multiplier on the space .

The last chapter (Chapter VI) is a derivation from the main content of this thesis. Here we prove a theorem on the Zygmund classof functions which is analogous to a famous theorem of K.de Leuw for the Lipchitz condition.

CHAPTER 1

Weighted Spaces

Here we consider certain Lebesgue spaces defined on a locally compact abelian group G. First we consider the weights introduced by P.Kree [13] and study the properties of sums and intersections of weighted Lebesgue spaces. These spaces turn out to be Banach spaces containing the space of continuous functions on G with compact support. The results are analogous to those obtained by Liu and Wang [16] . We have identified the duals of these Banach spaces. Later we consider only a subclass of these weights which are even continuous functions ω defined on G satisfying the simple condition

$$\omega(x+y) \leq \omega(x)\,\omega(y)$$

for all x and $y \in G$.

Let G be a locally compact abelian group with Haar measure dx. Let Ω be the set of all functions ω satisfying the two conditions.

(i) ω is a measurable function on G positive almost every where for the Haar measure dx.

(ii) for each $p \in [1,\infty)$, both ω^{p} and ω^{-p} are locally integrable.

The elements of Ω are called <u>weights</u> (See P.Kree [13]).

If $\omega \in \Omega$ let $L^{p,\omega}(G)$ denote the space of all equivalance classes of complex valued functions f on G such that

$|f|\omega$ has its p^{th} power summable and has norm

$$\|f\|_{p,\omega} = \left(\int_G |f(x)|\omega(x)|^p dx\right)^{\frac{1}{p}}$$

Then $L^{p,\omega}(G)$ is a Banach space and its conjugate space is $L^{p',\omega^{-1}}(G)$ where $\frac{1}{p}+\frac{1}{p'}=1$. However if $1 < p < \infty$, then $L^{p,\omega}(G)$ is a reflexive Banach space.

<u>THEOREM 2.1. Let</u> $\omega_1, \omega_2 \in \Omega$ <u>and let</u> $1 \leq p_1, p_2 < \infty$. <u>Suppose</u> S_1 <u>is the set of all complex valued functions</u> g <u>which can be written as</u>

$$g = g_1 + g_2 \quad \text{with } (g_1, g_2) \in L^{p_1,\omega_1}(G) \times L^{p_2,\omega_2}(G)$$

<u>If we define a norm on</u> S_1 <u>by</u>

$$(1) \qquad \|g\|_{S_1} = \inf\left\{\|g_1\|_{p_1,\omega_1} + \|g_2\|_{p_2,\omega_2}\right\}$$

<u>where the infimum is taken over all such representations of</u> g, <u>then</u> S_1 <u>becomes a Banach space.</u>

<u>Proof.</u> It is easy to verify that S_1 is a vector space and that (1) defines a seminorms on S_1. We now claim it is actually a norm. To this end, let us suppose that $\|g\|_{S_1} = 0$ We have to show that $g = 0$ a.e. By definition we can choose sequence $\{g_1^{(n)}\} \subset L^{p_1,\omega_1}(G)$ and $\{g_2^{(n)}\} \subset L^{p_2,\omega_2}(G)$ such that

$$g = g_1^{(n)} + g_2^{(n)}$$

and

$$\lim_{n\to\infty} \|g_1^{(n)}\|_{p_1,\omega_1} = \lim_{n\to\infty} \|g_2^{(n)}\|_{p_2,\omega_2} = 0$$

This implies that $g_1^{(n)}$ and $g_2^{(n)}$ converge to 0 in measure. Hence $g = 0$ a.e. as desired.

We now assert that S_1 is complete in this norm. Let $g^{(n)}$ be elements in S_1 such that $\sum_{n=1}^{\infty} \|g^{(n)}\|_{S_1} < \infty$. It is enough to show that there exists $g \in S_1$ such that $g = \sum_{n=1}^{\infty} g^{(n)}$ in S_1. We can choose, for each n, elements $g_1^{(n)} \in L^{p_1,\omega_1}(G)$ and $g_2^{(n)} \in L^{p_2,\omega_2}(G)$ such that

$$g^{(n)} = g_1^{(n)} + g_2^{(n)}$$

with

(2) $$\|g_1^{(n)}\|_{p_1,\omega_1} + \|g_2^{(n)}\|_{p_2,\omega_2} < \|g^{(n)}\|_{S_1} + \frac{1}{2^n}.$$

From (2), it follows that $\sum_{n=1}^{\infty} \|g_1^{(n)}\|_{p_1,\omega_1} < \infty$ and $\sum_{n=1}^{\infty} \|g_2^{(n)}\|_{p_2,\omega_2} < \infty$. The completeness of $L^{p_i,\omega_i}(G)$ gives the existence of g_i in $L^{p_i,\omega_i}(G)$ such that

$$\lim_{n\to\infty} \|\sum_{k=1}^{n} g_i^{(k)} - g_i\|_{p_i,\omega_i} = 0$$

for $i = 1,2$. Let $g = g_1 + g_2$. Then

$$\sum_{k=1}^{n} g^{(k)} - g = \sum_{k=1}^{n} (g_1^{(k)} + g_2^{(k)}) - (g_1 + g_2)$$

so that

$$\|\sum_{k=1}^{n} g^{(k)} - g\|_{S_1} \le \|\sum_{k=1}^{n} g_1^{(k)} - g_1\|_{p_1,\omega_1} + \|\sum_{k=1}^{n} g_2^{(k)} - g_2\|_{p_2,\omega_2}$$

so that

$$\|\sum_{k=1}^{n} g^{(k)} - g\|_{S_1} \le \|\sum_{k=1}^{n} g_1^{k} - g_1\|_{p_1,\omega_1} + \|\sum_{k=1}^{n} g_2^{(k)} - g_2\|_{p_2,\omega_2}$$

from which we obtain

$$\lim_{n\to\infty} \|\sum_{1}^{n} g^{(k)} - g\|_{S_1} = 0$$

This completes the proof of Theorem 1.1.

We shall now proceed to find the conjugate space of S_1.

For any bounded linear functional T on S_1 the restrictions of T to $L^{p_i,\omega_i}(G)$ $(i = 1,2)$ are bounded linear functionals on $L^{p_i,\omega_i}(G)$. Hence there exist functions $f_i \in L^{p_i',\omega_i^{-1}}(G)$ such that

$$Tg_i = \int_G g_i(x) f_i(x)\,dx \qquad i = 1,2$$

for all $g_i \in L^{p_i,\omega_i}(G)$. If $g \in L^{p_1,\omega_1}(G) \cap L^{p_2,\omega_2}(G)$ we have

$$Tg = \int_G g(x) f_1(x)\,dx = \int_G g(x) f_2(x)\,dx$$

Since ω_i^{p} are locally integrable, $L^{p_1,\omega_1}(G) \cap L^{p_2,\omega_2}(G)$ contains characteristic functions of all sets with finite measure. This implies $f_1 = f_2$ a.e. If this common value is denoted by f, then $f \in D_2 = L^{p_1',\omega_1^{-1}} \cap L^{p_2',\omega_2^{-1}}(G)$. If $g = g_1 + g_2$ is a decomposition of g as an element of S_1, then

$$Tg = \int_G g_1(x)\, f(x)\,dx + \int_G g_2(x)\, f(x)\,dx$$

so that

(3) $$Tg = \int_G g(x)\, f(x)\,dx$$

Conversely, suppose T is defined by (3). Let $g = g_1 + g_2$ be a representation of g. Then

$$Tg = \int_G g_1(x)\, f(x)\,dx + \int_G g_2(x)\, f(x)\,dx$$

and

$$\begin{aligned} |Tg| &\le \|g_1\|_{p_1,\omega_1} \|f\|_{p_1',\omega_1^{-1}} + \|g_2\|_{p_2,\omega_2} \|f_2\|_{p_2',\omega_2^{-1}} \\ &\le \left(\|g_1\|_{p_1,\omega_1} + \|g_2\|_{p_2,\omega_2}\right) \max\left(\|f\|_{p_1',\omega_1^{-1}}, \|f\|_{p_2',\omega_2^{-1}}\right). \end{aligned}$$

Hence

$$\|T\| \le \max\left(\|f\|_{p_1',\omega_1^{-1}}, \|f\|_{p_2',\omega_2^{-1}}\right).$$

Thus if T is given by (3), then T is a bounded linear functional on S_1.

We shall now show that the equality

$$\|T\| = \max\left(\|f\|_{p_1',\omega_1^{-1}}, \|f\|_{p_2',\omega_2^{-1}}\right)$$

actually holds. This is trivial when $f = 0$ a.e. Otherwise we may suppose without loss of generality that $\|f\|_{p_1',\omega_1^{-1}} \geq \|f\|_{p_2',\omega_2^{-1}}$

Let ε be a positive number less than $\|f\|_{p_1', \omega_1^{-1}}$. Then there exists $g \in L^{p_1, \omega_1}(G) \subset S_1$ such that

$$\left|\int_G g(x) f(x)\, dx\right| \geq \|g\|_{p_1, \omega_1} \left(\|f\|_{p_1', \omega_1^{-1}} - \varepsilon\right).$$

Since $g = g + 0$ is a representation of g as an element of S_1 we have

$$\|T\| \|g\| \geq |Tg| \geq \left(\|f\|_{p_1', \omega_1^{-1}} - \varepsilon\right) \|g\|_{p_1 \omega_1} = \left(\|f\|_{p_1', \omega_1^{-1}} - \varepsilon\right) \|g\|_{S_1}$$

Hence

$$\|T\| \geq \left(\|f\|_{p_1', \omega_1^{-1}} - \varepsilon\right) = \max\left(\|f\|_{p_1', \omega_1^{-1}}, \|f\|_{p_2', \omega_2^{-1}}\right) - \varepsilon$$

Since ε is arbitrary we conclude that

$$\|T\| = \max\left(\|f\|_{p_1', \omega_1^{-1}}, \|f\|_{p_2', \omega_2^{-1}}\right).$$

Thus we have proved

<u>THEOREM 1.2.</u> <u>Let</u> $\omega_1, \omega_2 \in \Omega$ <u>and</u> $1 \leq p_1, p_2 < \infty$. <u>Then the conjugate space of</u> S_1 <u>is isometrically isomorphic to the space</u> D_2 <u>where</u>

$$D_2 = L^{p_1', \omega_1^{-1}}(G) \cap L^{p_2', \omega_2^{-1}}(G)$$

<u>with norm defined by</u>

(4) $$\|f\|_{D_2} = \max\left(\|f\|_{p_1', \omega_1^{-1}}, \|f\|_{p_2', \omega_2^{-1}}\right)$$

<u>The space</u> D_2 <u>is a Banach space with the norm given by</u> (4) <u>and</u>

<u>the bounded linear functional</u> T <u>on</u> S_1 <u>corresponding</u> $f \in D_2$ <u>is given by</u>

$$Tg = \int_G g(x) f(x)\, dx$$

Now let us denote by S_2 the set of all complex valued functions g which can be written as

$$g = g_1 + g_2 \text{ with } (g_1, g_2) \in L^{p_1', \omega_1^{-1}}(G) \times L^{p_2', \omega_2^{-1}}(G)$$

and introduce a norm in S_2 by

$$\|g\|_{S_2} = \inf \{ \|g_1\|_{p_1', \omega_1^{-1}} + \|g_2\|_{p_2', \omega_2^{-1}} \} \tag{5}$$

where the infimum is over all such decompositions of g.

We denote by D_1 the space $L^{p_1, \omega_1}(G) \cap L^{p_2, \omega_2}(G)$ with a norm

$$\|f\|_{D_1} = \max\left(\|f\|_{p_1, \omega_1}, \|f\|_{p_2, \omega_2} \right) . \tag{6}$$

Since we have not used any special property of ω's in the proofs of Theorem 1.1 and 1.2, the following result is also valid.

<u>THEOREM 1.3</u>. <u>Let</u> $\omega_1, \omega_2 \in \Omega$ <u>and let</u> $1 \le p_1', p_2' < \infty$. <u>Then the spaces</u> S_2 <u>and</u> D_1 <u>are Banach spaces and the conjugate space of</u> S_2 <u>is isometrically isomorphic to</u> D_1. <u>The norms in these spaces are respectively given by (5) and (6).</u>

We shall find the conjugate space of D_1 where p_1 and p_2 are in $[1, \infty)$.

THEOREM 2.4. Let $\omega_1, \omega_2 \in \Omega$ and let $1 \le p_1, p_2 < \infty$. Then the conjugate space of D_1 is isometrically isomorphic to S_2 and the operation of $g \in S_2$ on $f \in D_1$ is given by

(7) $$Tg = \int_G f(x)\, g(x)\, dx$$

Proof. Let $g \in S_2$ and consider the functional T defined by (7). If $g = g_1 + g_2$ is a decomposition of g as an element of S_2 then

$$|Tf| \le \left|\int_G f(x)\, g_1(x)\, dx\right| + \left|\int_G f(x)\, g_2(x)\, dx\right|$$

$$\le \|f\|_{p_1,\omega_1} \|g\|_{p_1',\omega_1^{-1}} + \|f\|_{p_2,\omega_2} \|g_2\|_{p_2',\omega_2^{-1}}$$

$$\le \|f\|_{D_1} \left(\|g_1\|_{p_1',\omega_1^{-1}} + \|g_2\|_{p_2',\omega_2^{-1}}\right)$$

This implies that $|Tf| \le \|f\|_{D_1} \|g\|_{S_2}$ so that $\|T\| \le \|g\|_{S_2}$ and T is a bounded linear functional on D_1.

Since D_1 contains the characteristic functions of all sets of finite measure, the correspondence $g \longrightarrow T$ is one to one. To complete the proof it remains to show that $\|T\| = \|g\|_{S_2}$ and the mapping is onto.

To this end, consider the Banach space $L^{p_1,\omega_1} \oplus L^{p_2,\omega_2}$ with norm

$$\|(f_1, f_2)\| = \max\left(\|f_1\|_{p_1,\omega_1}, \|f_2\|_{p_2,\omega_2}\right)$$

Now the space D_1 is embedded in this ~~ax~~ space as its diagonal by the mapping $\varphi(f) = (f,f)$ for $f \in D_1$ and φ is an isometric mapping of D_1 into $L^{p_1,\omega_1} \oplus L^{p_2,\omega_2}$. Let T be a bounded linear functional on D_1. Then $T\circ\varphi^{-1}$ is a bounded linear functional on the subspace $\varphi(D_1)$ of $L^{p_1,\omega_1}\oplus L^{p_2,\omega_2}$ and hence can be extended by Hahn Banach theorem as a bounded linear functional to the whole space without changing its norm. The conjugate space of $L^{p_1,\omega_1}\oplus L^{p_2,\omega_2}$ is $L^{p_1',\omega_1^{-1}}\oplus L^{p_2',\omega_2^{-1}}$ with the norm

$$\|(g_1, g_2)\| = \|g_1\|_{p_1',\omega_1^{-1}} + \|g_2\|_{p_2',\omega_2^{-1}},\quad g_i\in L^{p_i',\omega_i^{-1}}(G)$$

There are functions $g_1\in L^{p_1',\omega_1^{-1}}(G)$ and $g_2\in L^{p_2',\omega_2^{-1}}(G)$ such that

$$Tf = \int_G f(x)\, g_1(x)\,dx + \int f(x)\, g_2(x)\,dx$$

Define $g = g_1 + g_2$. Then $g\in S_2$ and Tf is given by (7). Since T and its extension have the same norm we have

(8) $\quad \|T\| = \|g_1\|_{p_1',\omega_1^{-1}} + \|g_2\|_{p_2',\omega_2^{-1}} \geq \|g\|_{S_2}$.

Thus we have $\|T\| = \|g\|_{S_2}$ and the proof of the theorem is completed.

Similarly we have the following

<u>THEOREM 1.5. Let</u> $\omega_1, \omega_2\in\Omega$ <u>and let</u> $1\leq p_1', p_2' < \infty$. <u>Then the conjugate space of</u> D_2 <u>is isometrically isomorphic to</u> S_1.

<u>As a consequence of all these results we have</u>

<u>THEOREM 1.6. Let</u> $\omega_1, \omega_2\in\Omega$ <u>and let</u> $1 < p_1, p_2 < \infty$. <u>Then the four spaces</u> S_1, S_2, D_1 <u>and</u> D_2 <u>are reflexive Banach spaces</u>.

We shall now consider the case when one of the p_i's is 1 and show that the corresponding space D can be thought of as a dual.

Let $\omega \in \Omega$ be a fixed function. Then $C_\omega(G)$ will denote the class of all functions h such that $h\omega \in C_0(G)$, the space of continuous functions on G which vanish at infinity.

THEOREM 1.7. *Let* $1 \leq p < \infty$ *and let* $\omega_0, \omega \in \Omega$. *If* S *denotes the set of all functions* g *which can be written as*

(9) $$g = g_1 + g_2$$

where $g_1 \in C_{\omega_0}(G)$ *and* $g_2 \in L^{p,\omega}(G)$, *then* S *becomes a Banach space with a norm given by*

(10) $$\|g\|_S = \inf\{\|g_1\|_{\infty,\omega_0} + \|g_2\|_{p,\omega}\}$$

where the infimum is taken over all decompositions of g *given by (9). The conjugate space of* S *is isometrically isomorphic to the space* D *where*

$$D = L^{1,\omega_0^{-1}}(G) \cap L^{p',\omega^{-1}}(G)$$

with norm

$$\|f\| = \max\left(\|f\|_{1,\omega_0^{-1}}, \|f\|_{p',\omega^{-1}}\right)$$

and the operation of $f \in D$ *on* $g \in S$ *is given by (3). Similar result is valid if we replace* ω_0 *and* ω *by* ω_0^{-1} *and* ω^{-1} *throughout.*

Proof. That the space S is a Banach space is proved as in Theorem 1.1. We shall here prove only that if T is a bounded linear functional on S, then T is given by (3) for some $f \in D$. The rest of the proof of this theorem follows as in Theorem 1.2. Let us now suppose that T is a bounded linear functional on S. Since $C_{\omega_0}(G) \subset S$, the restriction of T on $C_{\omega_0}(G)$ defines a bounded linear functional on $C_{\omega_0}(G)$. Hence there exists a complex measure ν on G such that ν/ω_0 is bounded satisfying

$$Tg = \int_G g(x)\, d\nu(x) \qquad g \in C_{\omega_0}(G)$$

Similarly, since $L^{p,\omega}(G) \subset S$, there is a function $f \in L^{p',\omega^{-1}}(G)$ such that

$$Tg = \int_G g(x) f(x)\, dx \qquad g \in L^{p,\omega}(G)$$

If g is a continuous function with compact support both the above formulas are valid and hence we have

$$\int_G g(x)\, d\nu(x) = \int_G g(x) f(x)\, dx$$

This implies that ν is absolutely continuous and

$$d\nu(x) = f(x)\, dx$$

Since ν/ω_0 is bounded, we have $\int_G \frac{|d\nu(x)|}{\omega_0(x)} < \infty$ hence $f \in L^{1,\omega_0^{-1}}(G)$. Hence $f \in D$, Now if g is an arbitrary function in S, let $g = g_1 + g_2$ be a decomposition of g in the form (9). Then

$$Tg = Tg_1 + Tg_2 = \int_G g_1(x)\, d\nu(x) + \int_G g_2(x)\, f(x)\,dx$$
$$= \int_G g(x)\, f(x)\,dx$$

which is representation (3). This completes the proof.

Suppose now that ω is a nonnegative function on G satisfying the inequality

(11) $\omega(x+y) \le \omega(x)\,\omega(y)$

for all $x, y \in G$. If $1/\omega$ is also bounded away from zero, then ω being locally bounded, both ω^{p} and ω^{-p} are locally integrable. Hence $\omega \in \Omega$. Here-after we shall assume that all our weight functions ω will satisfy (11).

We shall now state several results that are needed later.

<u>LEMMA 1.8.</u> <u>If</u> $\mathcal{K}(G)$ <u>denote the space of continuous functions on</u> G <u>with compact support and</u> $1 \le p < \infty$, <u>then</u>

$$\mathcal{K}(G) \subset L^{p,\omega}(G) \subset L^{p,\zeta}(G)$$

<u>Proof.</u> Trivial.

<u>LEMMA 1.9.</u> $\mathcal{K}(G)$ <u>is dense in</u> $L_1^{1,\omega}(G)$

<u>Proof.</u> This is Lemma 2 in Gaudry [7] .

<u>LEMMA 1.10.</u> $L_1^{1,\omega}(G)$ <u>has approximate identities, that is there exists</u> $\{\varphi_\alpha\}$ <u>with the following properties</u>

(i) $\varphi_\alpha \in \mathcal{K}(G)$, $\varphi_\alpha \geqslant 0$, $\|\varphi_\alpha\|_1 = 1$

(ii) $\varphi_\alpha * f \to f$ <u>in</u> $L^{1,\omega}(G)$ <u>for each</u> $f \in L_1^{1,\omega}(G)$

(iii) φ_α <u>is bounded in</u> $\bar{L}_1^{1,\omega}(G)$.

Proof. This is Lemma 2 of Gaudry [7] .

DEFINITION 1.11. Two weights defined on the same group are said to be equivalent if their quotient is bounded both above and below by a strictly positive number.

LEMMA 1.12. Every weight is equivalent to a continuous weight.

Proof. This is Prop 111. 1-3 of Spector [21] .

LEMMA 1.13. $\mathcal{K}(G)$ is dense in $L^{p,\omega}(G)$ for $1 < p < \infty$

Proof. By Lemma 1.12, we may assume that ω is continuous. Let $f \in L^{p,\omega}(G)$. Then $f\omega \in L^{p}(G)$. Since $\mathcal{K}(G)$ is dense in $L^{p}(G)$, given $\varepsilon > 0$, there exists $f_c \in \mathcal{K}(G)$ such that

$$\|f_c - f\omega\|_p < \varepsilon$$

Since ω is assumed to be a continuous function satisfying (11), it follows that $\omega(x) \neq 0$ for any $x \in G$. Now we set $g_c = f_c/\omega$. Then $g_c \in \mathcal{K}(G)$ and $\|g_c - f\|_{p,\omega} < \varepsilon$. This completes the proof.

LEMMA 1.14. Let φ_α be as in Lemma 1.10. Then for each $f \in L^{p,\omega}(G)$ we have

$$\varphi_\alpha * f \to f \text{ in } L^{p,\omega}(G)$$

for $1 < p < \infty$.

Proof. First we prove that $\varphi_\alpha * g \to g$ in $L^{p,\omega}(G)$ for each $g \in \mathcal{K}(G)$. Now

$$\varphi_\alpha * g(x) - g(x) = \int_G (g(x-y) - g(x))\, \varphi_\alpha(y)\, dy$$

Let $\omega \in \Omega_o$ and $f \in L^{p,\omega}$. We first show that the mapping $y \quad \tau_y f$ is uniformly continuous from $G \qquad L^{p,\omega}$.

By Lemma 1.13, for any $\qquad 0$, there existsx a continuous function of such that

Since g is an uniformly continuous function and ω^p is locally integrable there exists a neighbourhood of the identity V such that

Therefore we have for any $y \in V$ using theorem (3.1)

Since ω is locally bounded we have $\qquad$ for any $y \in V$.

Now we choose $\qquad$ as in Lemma 1.10 and so as to have compact supports in V. Following the arguments for the corresponding theorem for L^p as in Loomis[*] 31B we have for $h \in L^{p^1, \omega^{-1}}$ and using Holder's inequality

Now since $\qquad$ have compact supports in V and for $y \in V$ we have

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[*] L.H.Loomis, Abstract Hamonic Analysis, Von Nostrand (1953)

CHAPTER 2

2. Multipliers on Weighted Spaces

Let Ω_0 denote the subclass of Ω consisting of those even continuous functions satisfying the inequality

(1) $$\omega(x+y) \leq \omega(x)\,\omega(y)$$

for all $x,y \in G$. It then follows that

(2) $$1 \leq \omega(0) \leq \omega(x)$$

for all $x \in G$. Moreover

(3) $$\frac{1}{\omega(x)} \leq \frac{\omega(y)}{\omega(x-y)}$$

for all $x,y \in G$.

<u>THEOREM 2.1. Let</u> $\omega \in \Omega_0$. <u>If</u> $y \in G$ <u>then</u> τ_y <u>is a bounded linear operator on both the spaces</u> $L^{p,\omega}(G)$ <u>and</u> $L^{p,\omega^{-1}}(G)$. <u>Moreover</u>

(4) $$\|\tau_y f\|_{p,\omega} \leq \omega(y)\,\|f\|_{p,\omega}$$

(5) $$\|\tau_y f\|_{p,\omega^{-1}} \leq \omega(y)\,\|f\|_{p,\omega^{-1}}$$

<u>Proof.</u> It is easy to verify that is a linear operator. If $f \in L^{p,\omega}(G)$, then

$$\|\tau_y f\|^p_{p,\omega} = \int_G |f(x-y)\,\omega(x)|^p dx \leq \omega^p(x) \int_G |f(x-y)\,\omega(x-y)|^p dx$$

from which follows (4). If $f \in L^{p,\omega^{-1}}(G)$, then

$$\|\tau_y f\|^p_{p,\omega^{-1}} = \int_G \frac{|f(x-y)|^p}{\omega^p(x)} dx \leq \omega^p(y) \int_G \left|\frac{f(x-y)}{\omega(x-y)}\right|^p dx$$

which gives (5).

<u>DEFINITION 2.2.</u> Let $\omega \in \Omega_0$ and $1 < p,q < \infty$. A multipliers from $L^{p,\omega}(G)$ to $L^{q,\omega}(G)$ is a bounded linear operator from $L^{p,\omega}(G)$ to $L^{q,\omega}(G)$ which commutes with translations and let $M(L^{p,\omega}(G), L^{q,\omega}(G))$ denote the space of multipliers from $L^{p,\omega}(G)$ to $L^{q,\omega}(G)$.

<u>THEOREM 2.3.</u> *If* $T \in M(L^{p,\omega}, L^{q,\omega})$, *then*

$$T(f*g) = Tf*g = f*Tg$$

for all $f, g \in L^{1,\omega}(G) \cap L^{p,\omega}(G)$

<u>Proof.</u> First we notice that if $h \in L^{1,\omega}(G)$ and $k \in L^{p,\omega}(G)$ then $h*k \in L^{p,\omega}(G)$. For $h\omega \in L^1(G)$ and $k\omega \in L^p(G)$ so that $|(h*k)\omega| \leq |h\omega| * |k\omega|$ from which we obtain

$$\|(h*k)\|_{p,\omega} \leq \|h\|_{1,\omega} \, \|k\|_{p,\omega}.$$

Let $T \in M(L^{p,\omega}(G), L^{q,\omega}(G))$. If $f, g \in K(G)$, then $T(f*g)$ and $Tf*g$ both belong to $L^{q,\omega}(G)$. Then if $k \in L^{q',\omega^{-1}}$ and $f \in L^{p,\omega}(G)$, then

$$\begin{aligned} |\langle Tf, k\rangle| &= \left|\int Tf(x)k(x)\,dx\right| \leq \|Tf\|_{q,\omega} \, \|k\|_{q',\omega^{-1}} \\ &\leq \|T\| \, \|f\|_{p,\omega} \, \|k\|_{q',\omega^{-1}} \end{aligned}$$

Shows that $f \to \langle Tf, k\rangle$ is a bounded linear functional on $L^{p,\omega}(G)$. Since $L^{p',\omega^{-1}}(G)$ is the conjugate space of $L^{p,\omega}(G)$, there exists ℓ in $L^{p',\omega^{-1}}(G)$ such that

(6) $\quad \langle Tf, k\rangle = \langle f, \ell\rangle$ for all $f \in L^{p,\omega}(G)$.

Now if $f, g \in K(G)$ and $k \in L^{q', \omega^{-1}}(G)$ then

$$\langle Tf * g, k\rangle = \int g(y) \langle \tau_y T f, k\rangle dy$$
$$= \int g(y) \langle T \tau_y f, k\rangle dy$$
$$= \int g(y) \langle \tau_y f, l\rangle dy$$
$$= \langle f * g, l\rangle = \langle T(f*g), k\rangle$$

This implies that $Tf * g = T(f * g)$ for all $f, g \in K(G)$.

Now let $f, g \in L^{1,\omega}(G) \cap L^{p,\omega}(G)$. Then $Tf * g$ and $T(f*g)$ both belong to $L^{q,\omega}(G)$. Choose $\{f_n\}$ and $\{g_n\}$ in $K(G)$ such that $\|f - f_n\|_{p,\omega} \to 0$ and $\|g - g_n\|_{1,\omega} \to 0$

Then

$$\|T(f*g) - Tf*g\|_{q,\omega}$$
$$\leq \|T(f*g) - T(f_n * g)\|_{q,\omega} + \|T(f_n * g) - T(f_n * g_n)\|_{q,\omega}$$
$$+ \|T(f_n * g_n) - Tf_n * g\|_{q,\omega} + \|Tf_n * g - Tf * g\|_{q,\omega}$$
$$\leq \|T\| \, \|f*g - f_n * g\|_{p,\omega} + \|T\| \, \|f_n * g - f_n * g_n\|_{p,\omega}$$
$$+ \|Tf_n * (g_n - g)\|_{q,\omega} + \|T(f_n - f) * g\|_{q,\omega}$$
$$\leq \|T\| \, \|f - f_n\|_{p,\omega} \|g\|_{1,\omega} + \|T\| \, \|f_n\|_{p,\omega} \|g - g_n\|_{1,\omega}$$
$$+ \|T\| \, \|f_n\|_{p,\omega} \|g_n - g\|_{1,\omega} + \|T\| \, \|f_n - f\|_{p,\omega} \|g\|_{1,\omega}$$
$$= 2\|T\| \left\{ \|f_n - f\|_{p,\omega} \|g\|_{1,\omega} + \|f_n\|_{p,\omega} \|g - g_n\|_{1,\omega} \right\}$$

The right hand side tends to zero as $n \to \infty$ and the left hand side is independent of n. Hence $Tf*g = T(f*g)$ for all $f, g \in L^{1,\omega}(G) \cap L^{p,\omega}(G)$.

THEOREM 2.4. Let G be a locally compact abelian group and $\omega \in \Omega_0$. If $1 < p, q < \infty$ then there exists a linear isometric isomorphism of $M(L^{p,\omega}, L^{q,\omega})$ onto $M(L^{q',\omega^{-1}}, L^{p',\omega^{-1}})$

Proof. Let $T \in M(L^{p,\omega}, L^{q,\omega})$. If $f, g \in \mathcal{K}(G)$, then $Tf * g$ and $T(f*g)$ both belong to $L^{q,\omega}(G)$. For every $k \in L^{q',\omega^{-1}}(G)$, we have

$$\langle T(f*g), k\rangle = \langle Tf*g, k\rangle$$

for all $f, g \in \mathcal{K}(G)$. Now let g be a fixed element of $\mathcal{K}(G)$. Define a functional L_g on $\mathcal{K}(G)$ by the formula

$$L_g(f) = f * Tg(o).$$

Then

$$|L_g(f)| = |Tf*g(o)| \le \|Tf\|_{q,\omega} \|g\|_{q',\omega^{-1}} \le \|T\| \|f\|_{p,\omega} \|g\|_{q',\omega^{-1}}$$

so that L_g is bounded in the $L^{p,\omega}$-norm. Since $\mathcal{K}(G)$ is dense in $L^{p,\omega}(G)$ we can extend L_g to a bounded linear function on $L^{p,\omega}(G)$ without increasing its norm. Since $L^{p',\omega^{-1}}(G)$ is the conjugate space of $L^{p,\omega}(G)$, we have $Tg \in L^{p',\omega^{-1}}(G)$ and $\|Tg\|_{p',\omega^{-1}} = \|L_g\| \le \|T\| \|g\|_{q',\omega^{-1}}$.

Thus the restriction of T to $\mathcal{K}(G)$ is a bounded linear transformation from $\mathcal{K}(G)$ to $L^{p',\omega^{-1}}(G)$ which commutes with translations and hence can be extended uniquely as a multiplier from $L^{q',\omega^{-1}}(G)$ to $L^{p',\omega^{-1}}(G)$. Thus $T \in M(L^{q',\omega^{-1}}, L^{p',\omega^{-1}})$

Moreover $\|T\|_{q',p'} \leq \|T\|_{p,q}$. The opposite inequality can also be established similarly.

We need the following result of Kree ([7] , Lemma 2, p.116).

THEOREM 2.5. (Kree [7] , Lemma 2, p.116). Let $\omega \in \Omega$ then

(1) if f_0 is a complex valued measurable function on G then $\log \|f_0\|_{p,\omega^\alpha}$ is a convex function of $(\frac{1}{p}, \alpha)$ if $\alpha \in R$ and $0 \leq \frac{1}{p} \leq 1$. Thus the set of points $(\frac{1}{p}, \alpha)$ such that $f_0 \in L^{p,\omega^\alpha}(G)$ is either convex or empty.

(2) let $\theta \in [0,1]$, $p_0, p_1, q_0, q_1 \in [1,\infty]$ and $\alpha_0, \alpha_1 \in R$ If $D^\theta = L^{p_\theta, \omega^{\alpha_\theta}}(G)$ and $E^\theta = L^{q_\theta, \omega^{\alpha_\theta}}(G)$ with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

and if A is a continuous linear operator (j = 0 and 1)

$$A : D^j \longrightarrow E^j$$

then

$$\log \operatorname{Sup}\langle Af, g\rangle \quad (\text{for } \|f\|_{D^\theta} \leq 1 \quad \|g\|_{(E^\theta)^*} \leq 1)$$

is a convex function of θ.

We are now in a position to prove the following representation theorem.

THEOREM 2.6. Let $1 < p < \infty$ and $\omega \in \Omega_0$. If $T \in M(L^{p,\omega}, L^{p,\omega})$ then there exists a unique pseudomeasure σ such that

$$Tf = \sigma * f$$

for $L^{p,\omega}(G) \cap L^2(G) \cap L^1(G)$. In particular, this representation holds for all $f \in \mathcal{K}(G)$.

Proof. Let $T \in M(L^{p,\omega}, L^{p,\omega})$ and $\|T\|_{p,\omega}$ denote the operator norm of T. Then by Theorem 2.4 we also have $T \in M(L^{p',\omega^{-1}}, L^{p',\omega^{-1}})$ and $\|T\|_{p,\omega} = \|T\|_{p',\omega^{-1}}$. Theorem 2.5 when restated says that

$$\log \|T\|_{p_\theta,\, \omega^{\alpha_\theta}}$$

is a convex function of θ. We put $\alpha_0 = 1$, $\alpha_1 = -1$ and $\theta = 1/2$ with $p_0 = q_0 = p$, $p_1 = q_1 = p'$

$$\alpha_\theta = (1-\theta)\alpha_0 + \theta\alpha_1 = 1/2(1) + 1/2(-1) = 0$$

$$\frac{1}{p_\theta} = \frac{1}{q_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = 1/2\left(\frac{1}{p} + \frac{1}{p'}\right) = 1/2 .$$

Then we have

$$\log \|T\|_{2,\omega^0} \leq (1-\theta)\log \|T\|_{p,\omega} + \theta \log \|T\|_{p',\omega^{-1}}$$

which implies

$$\|T\|_{2,\omega^0} \leq \|T\|_{p,\omega} .$$

Thus for each $f \in \mathcal{K}(G)$ we have

$$\|Tf\|_2 \leq \|T\|_{p,\omega} \|f\|_2$$

Thus T when restricted to $\mathcal{K}(G)$ is a bounded linear transformation of $\mathcal{K}(G)$ into $L^2(G)$ which commutes with translations. Since $\mathcal{K}(G)$ is dense in $L^2(G)$, T can be extended as a multiplier T_1 from $L^2(G)$ to $L^2(G)$ without chaiging the norm. Then there is a pseudomeasure σ such that

$$T_1 f = \sigma * f$$

for each $f \in L^1(G) \cap L^2(G)$. From this follows our theorem.

THEOREM 2.7. Let $\omega \in \Omega_0$ and suppose that $1 < p, q < \infty$.

Then

(a) if $T \in M(L^{p,\omega}, L^{q,\omega})$ and $p > q$ then $T = 0$, the zero operator when G is noncompact locally compact abelian group.

(b) if $1 < p \leq q < \infty$ and $T \in M(L^{p,\omega}, L^{q,\omega})$, then there exists a unique quasimeasure S such that

$$Tf = S * f$$

for $f \in \mathcal{K}(G)$.

Proof. We now apply Theorem 2.5 again. We now put $\alpha_0 = 1, \alpha_1 = -1$ and $\theta = \frac{1}{2}$ with $p_0 = p, q_0 = q$ and $p_1 = q, q_1 = p'$

Then

$$\alpha_\theta = 0$$

$$\frac{1}{p_\theta} = \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q'}\right) = \frac{1}{2}\left(\frac{1}{p} + 1 - \frac{1}{q}\right)$$

and

$$\frac{1}{q_\theta} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{p'}\right) = \frac{1}{2}\left(\frac{1}{q} + 1 - \frac{1}{p}\right)$$

$$D^\theta = L^{p_\theta}(G) \qquad E^\theta = L^{q_\theta}(G).$$

As in the proof of Theorem 2.6, we find that if T is a multiplier from $L^{p,\omega}(G)$ to $L^{q,\omega}(G)$, then T is also a multiplier from $L^{p_\theta}(G)$ to $L^{q_\theta}(G)$ and the representation (*) follows from Gaudry's theorem. Now Hormander's theorem generalized by Gaudry for a locally compact noncompact abelian group says that $T = 0$ if $p_\theta > q_\theta$. This is indeed the case if $p > q$ for if $p > q$ then $\frac{1}{q} - \frac{1}{p} > 0$ and so $1 + \frac{1}{q} - \frac{1}{p} > 1 + \frac{1}{p} - \frac{1}{q}$ which in turn implies that $\frac{1}{q_\theta} > \frac{1}{p_\theta}$ which is the same as $p_\theta > q_\theta$. This completes the proof of Theorem 2.7.

We shall now give the characterization of $M(L^{p,\omega}, L^{q,\omega})$ as the dual of a certain Banach space.

THEOREM 2.8. *Let* $\omega \in \Omega_0$ *and* $1 < p < q < \infty$. *Let* $\mathfrak{a}(p,q,\omega)$ *be the space of all those functions* u *which can be represented as*

(7) $$u = \sum f_i * g_i \quad \text{a.e}$$

where $f_i \in \mathcal{K}(G)$ *and* $g_j \in L^{q',\omega^{-1}}(G)$ *and such that*

$\sum_{j=1}^{\infty} \|f_j\|_{p,\omega} \|g_j\|_{q',\omega^{-1}}$. *We define a norm on* $\mathfrak{a}(p,q,\omega)$ *by*

(8) $$\|u\| = \inf\left\{ \sum_{j=1}^{\infty} \|f_j\|_{p,\omega} \|g_j\|_{q',\omega^{-1}} \right\}$$

where the infimum is taken over all such representations of u. *Then* $\mathfrak{a}(p,q,\omega)$ *is a Banach space and if* $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, *then* $\mathfrak{a}(p,q,\omega) \subset L^{r,\omega^{-1}}(G)$.

Proof. It is easy to verify that $\mathfrak{a}(p,q,\omega)$ is a vector space and that (8) defines a seminorm on $\mathfrak{a}(p,q,\omega)$. We now claim it is actually a norm. First we notice that if $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ then $L^p(G) * L^{q'}(G) \subset L^r(G)$. Now if $f \in \mathcal{K}(G)$ and $g \in L^{q',\omega^{-1}}(G)$ we have $f \cdot \omega \in L^p(G)$ and $g/\omega \in L^{q'}(G)$ so that $f\omega * g/\omega \in L^r(G)$. Now since $\omega(x+y) \le \omega(x)\omega(y)$ and $\omega(x) = \omega(-x)$ it follows that

$$1/\omega(x) \le \frac{\omega(x-t)}{\omega(t)}$$

Now if $f \in \mathcal{K}(G)$ and $g \in L^{q',\omega^{-1}}(G)$, we have

$$f * g(x) = \int_G f(x-t)\, g(t)\, dt$$

so that

$$\frac{|f*g(x)|}{\omega(x)} \leq \int_G |f(x-t)\omega(x-t)||g(t)|\omega(t)|^{-1}\,dt$$
$$= (|f\omega| * |g/\omega|)(x)$$

from which we deduce that

$$f*g \in L^{r,\omega^{-1}}(G)$$

and

$$\|f*g\|_{r,\omega^{-1}} \leq \|f\|_{p,\omega}\,\|g\|_{q',\omega^{-1}}$$

Now if $u = \sum_{j=1}^{\infty} f_j * g_j$ is a representation of u in the given form it follows that

$$\|u\|_{r,\omega^{-1}} \leq \sum_{j=1}^{\infty} \|f_j * g_j\|_{r,\omega^{-1}} \leq \sum_{j=1}^{\infty} \|f_j\|_{p,\omega}\,\|g_j\|_{q',\omega^{-1}}$$

Thus $u \in L^{r,\omega^{-1}}(G)$ and we have $\mathfrak{a}(p,q,\omega) \subset L^{r,\omega^{-1}}(G)$

Now show that (8) defines a norm on $\mathfrak{a}(p,q,\omega)$, let us suppose that $\|u\| = 0$. Then by definition we can find elements $f_j^{(n)} \in \mathcal{K}(G)$ and $g_j^{(n)} \in L^{q',\omega^{-1}}(G)$ such that

$$g = \sum_{j=1}^{\infty} f_j^{(n)} * g_j^{(n)} \quad a.e$$

and

$$\sum_{j=1}^{\infty} \|f_j^{(n)}\|_{p,\omega}\,\|g_j^{(n)}\|_{q',\omega^{-1}} < 1/2^n$$

for n = 1,2,... . This shows that $f_j^{(n)} * g_j^{(n)}$ converges to 0 in measure for each j and hence u = a.e. This proved that (8) is a norm on $\mathfrak{a}(p,q,\omega)$.

It remains to show that $\mathfrak{a}(p,q,\omega)$ is a Banach space. To this end, let $\{u_n\}$ be a Cauchy sequence in $\mathfrak{a}(p,q,\omega)$. By the property of the Cauchy sequence it is enough to show that a subsequence of $\{u_n\}$ converges to an element of $\mathfrak{a}(p,q,\omega)$. Therefore we may assure without loss of generality, that our sequence is such that

$$\|u_{n+1} - u_n\| < 1/2^n \qquad n = 1, 2, \cdots$$

Let $\|u_1\|_N$. Then, by the definition of the norm in we can always find elements $f_j^{(k)} \in \mathcal{K}(G)$ and $g_j^{(k)} \in L^{q', \omega^{-1}}(G)$ such that

$$u_1 = \sum_{j=1}^{\infty} f_j^{(1)} * g_j^{(1)}$$

$$u_{n+1} - u_n = \sum_{j=1}^{\infty} f_j^{(n+1)} * g_j^{(n+1)}$$

with

$$\sum_{j=1}^{\infty} \|f_j^{(1)}\|_{p,\omega} \|g_j^{(1)}\|_{q',\omega^{-1}} < N+1$$

and

$$\sum_{j=1}^{\infty} \|f_j^{(n+1)}\|_{p,\omega} \cdot \|g_j^{(n+1)}\|_{q',\omega^{-1}} < \frac{1}{2^n} + \frac{1}{2^n} = 1/2^{n-1} \qquad n = 1, 2, \cdots$$

Now define

$$u = f_1^{(1)} * g_1^{(1)} + f_2^{(1)} * g_2^{(1)} + f_1^{(2)} * g_1^{(2)} + f_1^{(3)} * g_1^{(3)} + \cdots$$

Then

$$\|f_1^{(1)}\|_{p,\omega} \|g_1^{(1)}\|_{q',\omega^{-1}} + \|f_2^{(1)}\|_{p,\omega} \|g_2^{(1)}\|_{q',\omega^{-1}} + \|f_1^{(2)}\|_{p,\omega} \|g_1^{(2)}\|_{q',\omega^{-1}}$$

$$+ \|f_1^{(3)}\|_{p,\omega} \|g_1^{(3)}\|_{q',\omega^{-1}} + \cdots < N+3$$

Thus $u \in \mathfrak{a}(p,q,\omega)$. We now claim that $u_n \to u$ in $\mathfrak{a}(p,q,\omega)$. Let $\varepsilon > 0$ be given. We can find an integer N_0 such that $n > N_0$ implies $\sum_{r=n+1}^{\infty} \frac{1}{2^{r-1}} < \varepsilon$. Then, for we have

$$u - u_{n+1} = u - [(u_{n+1} - u_n) + (u_n - u_{n-1}) + \cdots + (u_2 - u_1) + u_1]$$

and

$$\| u - u_{n+1} \| \le \sum_{r=n+1}^{\infty} \left[\sum_{k=1}^{\infty} \| f_k^{(r+1)} \|_{p,\omega} \| g_k^{(r+1)} \|_{q',\omega^{-1}} \right] < \sum_{r=n+1}^{\infty} \frac{1}{2^{r-1}} < \varepsilon$$

Hence $u_n \to u$ in $\mathfrak{a}(p,q,\omega)$ and the proof is complete.

In a similar fashion we have

<u>THEOREM 2.9.</u> <u>Let</u> $\omega \in \Omega_0$ <u>and</u> let $1 < p < \infty$. <u>Then</u> $\mathfrak{a}(p,p,\omega)$ <u>is defined as in Theorem 2.8 with</u> q <u>replaced by</u> p, <u>but (7) is assumed to hold every where. Then</u> $\mathfrak{a}(p,p,\omega)$ <u>is a subspace of</u> $C_{\frac{1}{\omega}}(G)$ <u>where the norm in</u> $C_{\frac{1}{\omega}}(G)$ <u>is given by</u>

$$(9) \qquad \| h \|_{\infty,\omega^{-1}} = \sup_{x \in G} \left| \frac{h(x)}{\omega(x)} \right| \quad \text{for } h \in C_{\frac{1}{\omega}}(G)$$

<u>The topology defined by the norm (3) is stronger than the topology inherited from</u> $C_{\frac{1}{\omega}}(G)$.

<u>Proof.</u> Proof is similar to that of Theorem 2.8. We only observe that if $f \in \mathcal{K}(G)$ and $g \in L^{p',\omega^{-1}}(G)$ then $|f\omega| * |g/\omega|$ is a continuous function vanishing at infinity on G, and hence f being continuous function with compact support, f*g is a continuous functions belonging to the class $C_{\frac{1}{\omega}}(G)$. Since $\sum_{j=1}^{\infty} \| f_j \|_{p,\omega} \| g_j \|_{p',\omega^{-1}} < \infty$ we have

$$\| \sum_{j=m}^{n} f_j * g_j \|_{\infty,\omega^{-1}} \le \sum_{j=m}^{n} \| f_j \|_{p,\omega} \| g_j \|_{p',\omega^{-1}} \to 0$$

as $m,n \to \infty$. Hence $u = \sum_{j=1}^{\infty} f_j * g_j$ converges in the norm of $C_{\frac{1}{\omega}}(G)$ and the norm $\|u\|$ is stronger than the norm given by (9).

<u>THEOREM 2.10. Let G be a locally compact abelian group and $\omega \in \Omega_0$ with $\omega(0) = 1$. If $1 < p \leq q < \infty$, then the space of multipliers $M(L^{p,\omega}, L^{q,\omega})$ is isometrically isomorphic to the dual $\mathfrak{A}(p,q,\omega)^*$ of $\mathfrak{A}(p,q,\omega)$.</u>

<u>Proof.</u> Let $T \in M(L^{p,\omega}, L^{q,\omega})$ and define a linear functional t on $\mathfrak{A}(p,q,\omega)$ by

$$(10) \qquad t(u) = \sum_{i=1}^{\infty} Tf_i * g_i(0)$$

for $u = \sum_{i=1}^{\infty} f_i * g_i$ in $\mathfrak{A}(p,q,\omega)$. We claim now that is unambiguously defined. To see this, it is enough to show that if $u = \sum_{i=1}^{\infty} f_i * g_i$ in $\mathfrak{A}(p,q,\omega)$ and $\sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{q',\omega^{-1}} < \infty$ then $\sum_{i=1}^{\infty} Tf_i * g_i(0) = 0$.

We first notice that if $\varphi \in K(G)$ and T_φ is defined by

$$T_\varphi f = \varphi * f \qquad f \in L^{p,\omega}(G)$$

then $T_\varphi \in M(L^{p,\omega}, L^{q,\omega})$. To see this, let $\varphi \in K(G)$ and $f \in L^{p,\omega}(G)$. From the relation

$$|\varphi * f(x)\,\omega(x)| \leq \int_G |f(x-y)\,\omega(x-y)|\,|\varphi(y)\,\omega(y)|\,dy.$$

we obtain

$$\|(\varphi * f)\omega\|_\infty \leq \|f\omega\|_p \|\varphi\omega\|_{p'}$$

and

$$\|(\varphi * f)\omega\|_p \leq \|f\omega\|_p \|\varphi\omega\|_1$$

Then

$$\|(\varphi * f)\omega\|_q^q \leq \|(\varphi * f)\omega\|_\infty^{q-p} \|(\varphi * f)\omega\|_p^p$$

$$\leq \|\varphi\omega\|_{p'}^{q-p} \|\varphi\omega\|_1^p \|f\omega\|_p^q .$$

so that

$$\|(\varphi * f)\omega\|_q \leq \left(\|\varphi\omega\|_{p'}^{q-p} \|\varphi\omega\|_1^p\right)^{\frac{1}{q}} \|f\omega\|_p$$

Moreover $\tau_y(\varphi * f) = \varphi * \tau_y f$ for each $f \in L^{p,\omega}(G)$ which implies that $\tau_y T_\varphi f = T_\varphi \tau_y f$. Thus $T_\varphi \in M(L^{p,\omega}, L^{q,\omega})$ and

$$\|T_\varphi\| \leq \|\varphi\omega\|_1^{p/q} \|\varphi\omega\|_p^{1-p/q} .$$

We nest show that every element of $M(L^{p,\omega} L^{q,\omega})$ can be approximated boundedly in the strong operator topology by operators of the form $T_\varphi, \varphi \in K(G)$. We show that if $T \in M(L^{p,\omega}, L^{q,\omega})$ then there exists a net φ_α in $K(G)$ such that $\varphi_\alpha * f \to Tf$ in the norm of $L^{q,\omega}(G)$ for every $f \in L^{p,\omega}(G)$ and there exists constants $K_\alpha(\omega)$ which depends on ω such that

$$\|\varphi_\alpha * f\|_{q,\omega} \leq K_\alpha(\omega) \|f\|_{p,\omega} \|T\|$$

where $\lim K_\alpha(\omega)=1$, and $\{K_\alpha(\omega)\}$ is bounded. It is sufficient to show that $\varphi_\alpha * f \to Tf$ weakly in $L^{q,\omega}(G)$ and then a net of convex combinations of the φ_α's will satisfy our requirements. Let $\{h_\beta\}$ be an approximate identity in $L^{1,\omega}(G)$ with $h_\beta \in \mathcal{K}(G) * K(G)$, $\|h_\beta\|_1 \leq 1$ and h_β vanishes outside some fixed compact set for all β. Let k_δ be an approximate identity in $L^1(\Gamma)$ such that $\hat{k}_\delta \in \mathcal{K}(G)$, $\|k_\delta\|_1 = 1$. Since T commutes with convolutions by functions in (G) (Theorem 2.3), it easily follows that Th_β is continuous for all β. Now we set $\varphi_\alpha = \varphi_{(\beta,\delta)} = \hat{k}_\delta Th_\beta$ and give $\alpha = (\beta,\delta)$ the usual product ordering. Then $\varphi_\alpha \in \mathcal{K}(G)$ for each α. If $f, g \in \mathcal{K}(G)$ we have

$$\hat{k}_\delta Th_\beta * f * g(0) = \int_G \hat{k}_\delta Th_\beta(-y)\, f*g(y)\, dy$$

$$\doteq \iint_{G\,G}\int_\Gamma k_\delta(\gamma)\, Th_\beta(-y)$$

Since $\overline{\gamma}(y) = \gamma(-y)$, we have by Fubini's theorem

$$|\varphi_\alpha * f * g(0)| \leq \doteq \int_\Gamma |k_\delta(\gamma)| d\gamma \iint_{G\,G} Th_\beta(-y)\, f(y-t)\, g(t)\, \overline{\gamma}(y)\, dt\, dy$$

$$\leq \|k_\delta\|_1 \sup_{\gamma\in\Gamma} \left|\int Th_\beta(-y)\, \overline{\gamma}f * \overline{\gamma}g(y)\, dy\right|$$

$$= \sup |Th_\beta * (\overline{\gamma}f * \overline{\gamma}g)(0)|$$

$$\doteq \sup_{\gamma\in\Gamma} |T(h_\beta * \overline{\gamma}f) * \overline{\gamma}g(0)|$$

$$\leq \|T\| \sup_{\gamma\in\Gamma} \|h_\beta * \overline{\gamma}f\|_{p,\omega} \|\overline{\gamma}g\|_{q',\omega^{-1}}$$

$$\leq \|T\|\,\|h_\beta\| \quad \|f\| \quad \|g\|$$

using the relation

$$\| h_\beta \|_{1,\omega} \le K_\beta(\omega) \| h_\beta \|_1 = K_\beta(\omega)$$

where $K_\beta(\omega) = \max\{\omega(x) : x \in \operatorname{supp} h_\beta\}$, it follows that

$$| \varphi_\alpha * f * g(0) | \le \|T\| \, K_\beta(\omega) \, \|f\|_{p,\omega} \, \|g\|_{q',\omega^{-1}}$$

so that

$$\| \varphi_\alpha * f \|_{q,\omega} \le \|T\| \, K_\alpha(\omega) \, \|f\|_{p,\omega}$$

where $K_\alpha(\omega) = K_\beta(\omega)$. It is clear that $\{K_\alpha(\omega)\}$ is bounded and $\lim_\alpha K_\alpha(\omega) = 1$ since $\omega(0) = 1$. The operators T_{φ_α} satisfy

$$\| T_{\varphi_\alpha} \| \le K_\alpha(\omega) \|T\| \le K(\omega) \|T\|$$

Since each closed ball of $M(L^{p,\omega}, L^{q,\omega})$ is compact in the weak operator topology, the net T_{φ_α} has a limit point $U \in M(L^{p,\omega}, L^{q,\omega})$ (for this same topology) with $\|U\| \le \|T\|$.
We suppose that $\lim_\alpha T_{\varphi_\alpha} = U$ in the weak operator topology.
Then we have

$$\lim_\beta \lim_\delta (\hat{k}_\delta T h_\beta) * f * g(0) = Tf * g(0)$$

for $f, g \in \mathcal{K}(G)$ since $\hat{k}_\delta \to 1$ locally uniformly $\{h_\beta\}$ is an approximate identity and T commutes with convolutions by functions from $\mathcal{K}(G)$. Hence $T = U$ and our assertion is proved.

Now suppose that $\sum_{i=1}^{\infty} f_i * g_i$ is a representation of 0 as an element of $\mathfrak{a}(p,q,\omega)$ and consider the net φ_α given in the preceding paragraph. Since the series $\sum_{i=1}^{\infty} \varphi_\alpha * f_i * g_i(0)$ converges uniformly with respect to α and $\varphi_\alpha * f_i \to Tf_i$ in $L^{q,\omega}(G)$ for each i we have

$$\sum_{i=1}^{\infty} Tf_i * g_i(0) = \lim_\alpha \sum_{i=1}^{\infty} \varphi_\alpha * f_i * g_i(0)$$

Now for each α

$$\sum_{i=1}^{\infty} \varphi_\alpha * f_i * g_i(0) = \sum_{i=1}^{\infty} \int \varphi_\alpha(-y)\, f_i * g_i(y)\, dy$$

$$= \int \varphi_\alpha(-y) \underbrace{\sum_{i=1}^{\infty} f_i * g_i(y)}_{=0}\, dy$$

since $\varphi_\alpha \in K(G)$ and hence can be viewed as an element of $L^{r',\omega}(G)$ and $f_i * g_i \in L^{r,\omega^{-1}}(G)$ where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. This proves that t is well defined.

The linearity of the mapping $T \to t$, is obvious. Now we show that it is an isometry. From the relation

$$|t(u)| = |\sum_{i=1}^{\infty} Tf_i * g_i(0)| \leq \|T\| \sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{q',\omega^{-1}}$$

it follows that

$$|t(u)| \leq \|T\| \|u\|$$

and hence $\|t\| \leq \|T\|$. On the other hand

$$\|T\| = \sup\{|Tf * g(0)| : \|f\|_{p,\omega} \leq 1, \|g\|_{q',\omega^{-1}} \leq 1\}$$

$$\leq \sup\{|t(f*g)| : \|f\|_{p,\omega} \leq 1, \|g\|_{q',\omega^{-1}} \leq 1\}$$

$$\leq \|t\|$$

Therefore $\|T\| = \|t\|$.

Finally we show that the mapping $T \longrightarrow t$ is onto. Suppose $t \in \mathfrak{a}(p,q,\omega)^*$. Let $f \in \mathcal{K}(G)$ be fixed. Define

$$g \to t(f*g)$$

on the space $L^{q',\omega^{-1}}(G)$. This is a bounded linear functional on $L^{q',\omega^{-1}}$ since

$$|t(f*g)| \le \|t\| \, \|f\|_{p,\omega} \|g\|_{q',\omega^{-1}}$$

Now $L^{q,\omega}(G)$ is the conjugate space of $L^{q',\omega^{-1}}(G)$. Hence there exists a unique element, call it Tf, in $L^{q,\omega}(G)$ such that

$$Tf * g(o) = T(f*g) \quad \text{for all } g \in L^{q',\omega^{-1}}(G)$$

and $\|Tf\|_{q,\omega} \le \|t\| \|f\|_{p,\omega}$. Thus we have a continuous linear operator T defined on the dense subset $\mathcal{K}(G)$ of $L^{p,\omega}(G)$ into $L^{q,\omega}(G)$. We extend T continuously and linearly to the whole of $L^{p,\omega}(G)$ without changing the norm. We claim that this extended T belongs to $M(L^{p,\omega}, L^{q,\omega})$. Let $y \in G$. If $f \in \mathcal{K}(G) \subset L^{p,\omega}(G)$ and $g \in L^{q',\omega^{-1}}(G)$, we have

$$T(\tau_y f) * g(o) = t(\tau_y f * g) = t(f * \tau_y g)$$
$$= Tf * \tau_y g(o) = \tau_y Tf * g(o)$$

Hence $T\tau_y f = \tau_y Tf$ for all $f \in \mathcal{K}(G)$ and hence the same holds for all $f \in L^{p,\omega}(G)$. Thus $T \in M(L^{p,\omega}, L^{q,\omega})$ and our assertion is proved.

CHAPTER 3

Multipliers on $L^{p_1,\omega_1}(G) \cap L^{p_2,\omega_2}(G)$

Let $\omega_1, \omega_2 \in \Omega$ and $1 < p_1, p_2 < \infty$. We recall the space D_1 defined by

$$D_1 = L^{p_1,\omega_1}(G) \cap L^{p_2,\omega_2}(G)$$

is a Banach space under the norm given by

$$\text{(1)} \qquad \|f\|_{D_1} = \max\{\|f\|_{p_1,\omega_1}, \|f\|_{p_2,\omega_2}\}$$

for each $f \in D_1$.

DEFINITION 3.1. When $\omega_1, \omega_2 \in \Omega_0$, we define a multiplier on D_1 to be a bounded linear operator on D_1 which commutes with translations. The space of all multipliers on D_1 is denoted by $M(D_1)$.

We shall here obtain a characterization of $M(D_1)$ as the dual of a certain Banach space.

We first introduce the space $C_{\omega_1,\omega_2}(G)$. If $\omega_1, \omega_2 \in \Omega$ we denote by $C_{\omega_1,\omega_2}(G)$ the space of all functions h which can be written as

$$h = h_1 + h_2 \qquad (h_1, h_2) \in C_{\omega_1}(G) \times C_{\omega_2}(G)$$

with a definition of norm $|||\cdot|||$ given by

$$\text{(2)} \qquad |||h||| = \inf\{\|h_1\|_{\infty,\omega_1} + \|h_2\|_{\infty,\omega_2}\}$$

where the infimum is taken over all such decompositions of h, then we can prove as in earlier cases that $C_{\omega_1,\omega_2}(G)$ is a Banach space under $|||\cdot|||$.

We also recall that the space S_2 is defined to be the class of all functions g which can be represented as

$$g = g_1 + g_2 \ , \ (g_1, g_2) \in L^{p_1', \omega_1^{-1}}(G) \times L^{p_2', \omega_2^{-1}}$$

endowed with the norm

$$(3) \qquad \|g\|_{S_2} = \inf \{ \|g_1\|_{p_1', \omega_1^{-1}} + \|g_2\|_{p_2', \omega_2^{-1}} \}$$

where the infimum is taken over all such representations of g.

Now we define the space $\mathfrak{A} = \mathfrak{A}(p_1, \omega_1, p_2, \omega_2)$ to be the set of all functions u which can be represented as

$$(4) \qquad u = \sum_{k=1}^{\infty} f_k * g_k$$

where $f_k \in \mathcal{K}(G)$ and $g_k \in S_2$ with $\sum_{k=1}^{\infty} \|f_k\|_{D_1} \|g_k\|_{S_2} < \infty$

Notice that $\mathcal{K}(G)$ is a dense subset of D_1. Define a norm $u \to \|u\|$ on $\mathfrak{A}$ by

$$(5) \qquad \|u\| = \inf \left[\sum_{k=1}^{\infty} \|f_k\|_{D_1} \|g_k\|_{S_2} \right\}$$

where the infimum is taken over all such representations of u.

Then we have

THEOREM 3.2. Let $\omega_0, \omega_1 \in \Omega_0$. Then (5) defines a norm on $\mathfrak{A}$ and $\mathfrak{A}$ is complete in this norm. Furthermore $\mathfrak{A}$ is a subspace of $C_{\omega_1^{-1}, \omega_2^{-1}}(G)$ and the topology on $\mathfrak{A}$ is not weaker than the topology induced from $C_{\omega_1^{-1}, \omega_2^{-1}}(G)$.

Proof. The first part of the theorem is proved exactly as before (see the proof of Theorem 2.8). Now let $f \in \mathcal{K}(G)$ and $g \in S_2$. Suppose $g = g_1 + g_2$ is a decomposition of g

with $g_1 \in L^{p_1', \omega_1^{-1}}(G)$ and $g_2 \in L^{p_2', \omega_2^{-1}}(G)$. Since $f \in L^{p_i, \omega_i}(G)$ for i = 1,2, it follows that $f*g \in C_{\omega_1^{-1}}(G)$ and $f*g \in C_{\omega_2^{-1}}(G)$. Moreover

$$\|f*g_1\|_{\infty,\omega_1^{-1}} + \|f*g_2\|_{\infty,\omega_2^{-1}}$$
$$\leq \|f\|_{p_1,\omega_1} \|g_1\|_{p_1',\omega_1^{-1}} + \|f\|_{p_2,\omega_2} \|g_2\|_{p_2',\omega_2^{-1}}$$
$$\leq \|f\|_{D_1} \left(\|g_1\|_{p_1',\omega_1^{-1}} + \|g_2\|_{p_2',\omega_2^{-1}}\right)$$

which implies that $f*g \in C_{\omega_1^{-1},\, \omega_2^{-1}}(G)$ and

(6) $$|||f*g||| \leq \|f\|_{D_1} \|g\|_{S_2}$$

Furthermore if $u \in \mathfrak{A}$, then

$$\left|\left|\left|\sum_{k=m}^{n} f_k * g_k\right|\right|\right| \leq \sum_{k=m}^{n} \|f_k\|_{D_1} \|g_k\|_{S_2} \to 0 \text{ as } m,n \to \infty.$$

From these relations it is clear that $\mathfrak{A} \subset C_{\omega_1^{-1},\, \omega_2^{-1}}(G)$ and that the topology of $\mathfrak{A}$ is not weaker than that induced from $C_{\omega_1^{-1},\, \omega_2^{-1}}(G)$.

<u>THEOREM 3.3</u>. <u>Let</u> G <u>be a locally compact abelian group and let</u> $1 < p_1, p_2 < \infty$. <u>If</u> $\omega_1, \omega_2 \in \Omega_0$, <u>then the space of multipliers</u> $M(D_1)$ <u>is isometrically isomorphic to</u> $\mathfrak{A}^*$ <u>the conjugate space of</u> $\mathfrak{A}$.

<u>Proof</u>. For any $T \in M(D_1)$ define

$$t(u) = \sum_{k=1}^{\infty} Tf_k * g_k(0)$$

for $u = \sum_{k=1}^{\infty} f_k * g_k$ in $\mathfrak{A}$. First we show that T is well define. To this end it is sufficient to show that if

$$u = \sum_{k=1}^{\infty} f_k * g_k = 0 \text{ in } \mathfrak{A} \quad \text{and} \quad \sum_{k=1}^{\infty} \|f_k\|_{D_1} \|g_k\|_{S_2} < \infty$$

then $\sum_{k=1}^{\infty} Tf_k * g_k(0)$.

Let $\{\xi_\alpha\}$ be an approximate identity for $L^{1,\omega_1}(G)$ with $\|\xi_\alpha\|_1 = 1$ and $\{\eta_\beta\}$ an approximate identity for $L^{1,\omega_2}(G)$ with $\|\eta_\beta\|_1 = 1$. Let $\varphi_\gamma = \varphi_{(\alpha,\beta)} = \xi_\alpha + \eta_\beta - \xi_\alpha * \eta_\beta$ and give $\gamma = (\alpha, \beta)$ the usual product ordering. Now let $f \in \mathcal{K}(G)$ and consider $\varphi_\gamma * f - f$. From the relation

$$\varphi_\gamma * f = \xi_\alpha * f + \eta_\beta * f - \xi_\alpha * \eta_\beta * f.$$

it follows that

$$\|\varphi_\gamma * f - f\|_{p_1,\omega_1} \le \|\xi_\alpha * f - f\|_{p_1,\omega_1} + \|\eta_\beta * f - \xi_\alpha * \eta_\beta * f\|_{p_1,\omega_1}$$

and

$$\|\varphi_\gamma * f - f\|_{p_2,\omega_2} \le \|\eta_\beta * f - f\|_{p_2,\omega_2} + \|\xi_\alpha * f - \eta_\beta * \xi_\alpha * f\|_{p_2,\omega_2}$$

Hence

$$\|\varphi_\gamma * f - f\|_{D_1} \to 0$$

For taking the limit over the index γ , Then

$$|T(\varphi_\gamma * f_k) * g_k(0) - Tf_k * g_k(0)| = |T(\varphi_\gamma * f_k - f_k) * g_k(0)|$$
$$\le \|T\| \, \|\varphi_\gamma * f_k - f_k\|_{D_1} \|g_k\|_{S_2} \to 0$$

so that

$$\lim_\gamma T(\varphi_\gamma * f_k) * g_k(0) = Tf_k * g_k(0)$$

Since $u = \sum_{k=1}^{\infty} f_k * g_k = 0$ and the series $\sum_{k=1}^{\infty} f_k * g_k$ converges uniformly, we get

$$\sum_{k=1}^{\infty} T(\varphi_\gamma * f_k) * g_k(0) = \sum_{k=1}^{\infty} \int \tau_y T(\varphi_\gamma * f_k)(0)\, g_k(y)\,dy$$

$$= \sum_{k=1}^{\infty} \int T\tau_y (\varphi_\gamma * f_k)(0)\, g_k(y)\,dy$$

$$= \sum_{k=1}^{\infty} T(\varphi_\gamma * f_k * g_k)(0)$$

$$= T\left(\varphi_\gamma * \sum_{k=1}^{\infty} f_k * g_k\right)(0)$$

$$= 0$$

We shall now show that $\sum_{k=1}^{\infty} T(\varphi_\gamma * f_k) * g_k(0)$ converges uniformly with respect to γ.

We may suppose that the support of ξ_α is contained in a fixed compact set K_1 for each α and the support of η_β is contained in a fixed compact set K_2 for all β. Since $\{\xi_\alpha\}$ and $\{\eta_\beta\}$ are bounded respectively in $L^{1,\omega_1}(G)$ and $L^{1,\omega_2}(G)$ there exist M_1, M_2 such that $\|\xi_\alpha\|_{1,\omega_1} \le M_1$ and $\|\eta_\beta\|_{1,\omega_2} \le M_2$. Set $M = M_1 + M_2$. If $y \in G$ then from the relations

$$\|\tau_y f\|_{p_1,\omega_1} \le \omega_1(y)\, \|f\|_{p_1,\omega_1}$$

and

$$\|\tau_y f\|_{p_2,\omega_2} \le \omega_2(y)\, \|f\|_{p_2,\omega_2}$$

it follows that the translation operator τ_y on the space D_1 has a norm bounded by $\max\{\omega_1(y), \omega_2(y)\}$. Let m_1 and m_2 be the maxima of the continuous functions ω_1, ω_2 respectively on the compact set $K = K_1 + K_2$. Set $m = \max(m_1, m_2)$. Then m and M are independent of α and β and hence of γ. Now if $f \in \mathcal{K}(G)$, we have

$$\|\varphi_\gamma * f\|_{p_1,\omega_1} \le \|\xi_\alpha * f\|_{p_1,\omega_1} + \|\eta_\beta * f\|_{p_1,\omega_1} + \|\xi_\alpha * \eta_\beta * f\|_{p_1,\omega_1}$$

$$\le \|\xi_\alpha\|_{1,\omega_1}\|f\|_{p_1,\omega_1} + \|\eta_\beta * f\|_{p_1,\omega_1} + \|\xi_\alpha\|_{1,\omega_1}\|\eta_\beta * f\|_{p_1,\omega_1}$$

We shall now calculate $\|\eta_\beta * f\|_{p_1,\omega_1}$. Now, using the definition of $\eta_\beta * f$ and Minkiowsky's inequality, we get

$$\|\eta_\beta * f\|_{p_1,\omega_1} \le \int \eta_\beta(t)\,dt \left(\int |f(x-t)\,\omega_1(x)|^{p_1} dx\right)^{\frac{1}{p_1}}$$

$$\le \sup\{\|\tau_t f\|_{p_1,\omega_1} : t \in \operatorname{supp}\eta_\beta\}$$

$$\le \sup\{\|\tau_t\|\,\|f\|_{p_1,\omega_1} : t \in \operatorname{supp}\eta_\beta\}$$

$$\le m\|f\|_{p_1,\omega_1}$$

Then

$$\|\varphi_\gamma * f\|_{p_1,\omega_1} \le \|\xi_\alpha\|_{p_1,\omega_1}\|f\|_{p_1,\omega_1} + [1 + \|\xi_\alpha\|_{1,\omega_1}]\,m\|f\|_{p_1,\omega_1}$$

$$\le (M + (1+M)\,m)\,\|f\|_{p_1,\omega_1}$$

Similarly for $f \in \mathcal{K}(G)$, we also have

$$\|\varphi_\gamma * f\|_{p_2,\omega_2} \le [M + m(1+M)]\,\|f\|_{p_2,\omega_2}$$

Hence

$$\|\varphi_\gamma * f\|_{D_1} \le (M + m + mM)\|f\|_{D_1}$$

Then

$$\left|\sum_{k=1}^{\infty} T(\varphi_\gamma * f_k) * g_k(0)\right| \le \sum_{k=1}^{\infty} \|T(\varphi_\gamma * f_k)\|_{D_1} \|g_k\|_{S_2}$$

$$\le \sum_{k=1}^{\infty} \|T\| \, \|\varphi_\gamma * f_k\|_{D_1} \|g_k\|_{S_2}$$

$$\le \|T\|(M + m + mM) \sum_{k=1}^{\infty} \|f_k\|_{D_1} \|g_k\|_{S_2}$$

and the convergence of $\sum_{k=1}^{\infty} T(\varphi_\gamma * f_k) * g_k(0)$ is uniform with respect to γ. Hence

$$\sum_{k=1}^{\infty} Tf_k * g_k(0) = \lim_{\gamma} \sum_{k=1}^{\infty} T(\varphi_\gamma * f_k) * g_k(0) = 0.$$

since $T(\varphi_\gamma * f_k) * g_k(0) \longrightarrow Tf_k * g_k(0)$ for each k. Thus t is well defined. It is clearly linear. The mapping is an isometry. In fact

$$|t(\omega)| = \left|\sum_{k=1}^{\infty} Tf_k * g_k(0)\right| \le \|T\| \sum_{k=1}^{\infty} \|f_k\|_{D_1} \|g_k\|_{S_2}$$

implies

$$|t(\omega)| \le \|T\| \|u\|$$

so that $\|t\| \le \|T\|$. On the other hand

$$\|T\| = \sup\{|Tf * g(0)| : \|f\|_{D_1} \le 1;\ \|g\|_{S_2} \le 1\}$$

$$= \sup\{|t(f * g)| : \|f\|_{D_1} \le 1 \quad \|g\|_{S_2} \le 1\} \|t\|$$

To see that the mapping $T \to t$ is onto, we proceed as follows. Let $t \in \mathfrak{A}^*$. Let $f \in \mathcal{K}(G)$ be fixed. Now define a functional L on S_2 by the equation

$$L(g) = t(f * g) \qquad g \in S_2.$$

Then $|L(g)| = |t(f*g)| \le \|t\| \, \|f\|_{D_1} \|g\|_{S_2}$ which shows that L is a bounded linear functional on S_2. Since D_1 is the conjugate space of S_2 (see Theorem 1.3) there exists a unique element, call it Tf, in D_1 such that

$$Tf * g(o) = L(g) = t(f * g)$$

and $\|Tf\|_{D_1} \le \|t\| \, \|f\|_{D_1}$. Thus to each f in $\mathcal{K}(G)$, we have Tf in D_1 and the mapping T is a bounded operator from $\mathcal{K}(G)$ into D_1 when $\mathcal{K}(G)$ is considered as a subset of D_1. It is clear that the operator T is linear. Since $\mathcal{K}(G)$ is dense in D_1 we can extend T uniquely as a bounded linear operator on D_1 without increasing its norm. We claim that this extended T is a multiplier on D_1. Let $y \in G$ and let $f \in \mathcal{K}(G)$. If $g \in S_2$, then

$$\tau_y Tf * g(o) = Tf * \tau_y g(o) = t(f * \tau_y g) = t(\tau_y f * g)$$
$$= T\tau_y f * g(o)$$

holds for all $g \in S_2$. Hence

(7) $$\tau_y Tf = T\tau_y f$$

Now (7) holds for each f in $\mathcal{K}(G)$ and hence the same is valid for all $f \in D_1$. Thus $T \in M(D_1)$. This completes the proof of our theorem.

We shall now give the characterization of multipliers when one of the p_i's is 1. Let $\omega_o, \omega \in \Omega_o$ and $1 < p < \infty$. Let $D = L^{1,\omega_o}(G) \cap L^{p,\omega}(G)$ and supply a norm on D by

(8) $$\|f\|_D = \max\left(\|f\|_{1,\omega_o}, \|f\|_{p,\omega}\right)$$

Then D is a Banach space. We set

$$S = \{g : g_1 + g_2 : (g_1, g_2) \in C_{\omega_o^{-1}}(G) \times L^{p',\omega^{-1}}(G)$$

and the norm in S is defined by

(9) $\|g\|_S = \inf\{\|g_1\|_{\infty,\omega_0^{-1}} + \|g_2\|_{p',\omega^{-1}}\}$

where the infimum is taken over all such representations of g. We have proved that D can be thought of as the conjugate space of S.

We now define $\mathfrak{A}_1$ to be the set of all functions u which can be represented as

$$u = \sum_{k=1}^{\infty} f_k * g_k$$

where $f_k \in \mathcal{K}(\omega)$ and $g_k \in S$ with $\sum_{k=1}^{\infty} \|f_k\|_D \, \|g_k\|_S < \infty$

We introduce a norm on $\mathfrak{A}_1$ by

$$\|u\| = \inf\left\{\sum_{k=1}^{\infty} \|f_k\|_{D_1} \|g_k\|_S\right\}.$$

where the infimum is taken over all the admissible representations of u.

<u>THEOREM 3.4</u>. <u>Let</u> G <u>be a locally compact abelian group</u>. <u>Suppose</u> $1 < p < \infty$ <u>and</u> $\omega_0, \omega \in \Omega_0$. <u>Then the space of multipliers</u> M(D) <u>on</u> D <u>is isometrically isomorphic to</u> $\mathfrak{A}_1^*$, <u>the conjugate space of</u> $\mathfrak{A}_1$.

The proof is quite analogous to that of Theorem 3.3 above and hence we shall omit it.

CHAPTER 4

Multipliers on a Segal Algebras

There has been a considerable interest in recent years in the study of Segal algebras. A linear subspace $S(G)$ of $L^1(G)$ is called a Segal algebra if the following four conditions are satisfied

(a) $S(G)$ is dense in $L^1(G)$

(b) $S(G)$ is a Banach space under some norm $\|\cdot\|_S$ and

$$\|f\|_S \geq \|f\|_1 \qquad f \in S(G)$$

(c) Let $y \in G$ and τ_y denote the translation operators. For each $f \in S(G)$, $\tau_y f$ belongs to $S(G)$ and the mapping $y \to \tau_y f$ is continuous from G into $S(G)$.

(d) $\|\tau_y f\|_S = \|f\|_S$ for all $f \in S(G)$ and all $y \in G$.

Various properties of a Segal algebra are collected below in the form of lemmas.

LEMMA 4.1. For every $f \in S(G)$ and arbitrary $h \in L^1(G)$ the vector valued integral $\int_G h(y)\, \tau_y f dy$ exists as an element of $S(G)$ and

$$\int_G h(y) \cdot \tau_y f dy = h*f$$

Moreover

$$\|h*f\|_S \leq \|h\|_1 \cdot \|f\|_S$$

it follows immediately that if $h \in S(G)$, then

$$\|h*f\|_S \leq \|h_1\| \|f\|_S \leq \|h\|_S \|f\|_S$$

which shows that $S(G)$ is actually a Banach algebra and it is an ideal in $L^1(G)$.

LEMMA 4.2. *Let* μ *be a bounded complex valued measure on* G. *Then for any* $f \in S(G)$ *the vector valued integral*

$\int_G \tau_y f \, d\mu(y)$ *exists as an element of* $S(G)$ *and*

$$\int_G \tau_y f \, d\mu(y) = \mu * f$$

Further

$$\|\mu * f\|_S \le \|\mu\| \cdot \|f\|_S$$

Thus $S(G)$ is also an ideal in $M(G)$.

LEMMA 4.3. $S(G)$ *contains all* $f \in L^1(G)$ *such that its Fourier transform* $\hat{f}$ *has compact support.*

LEMMA 4.4. *To every compact set* $K \subset \Gamma$ *there is a constant* $C_K > 0$ *such that for every* $f \in S(G)$ *whose Fourier transform vanishes outside* K *satisfies*

$$\|f\|_S \le C_K \|f\|_1$$

LEMMA 4.5. *Given any* $f \in S(G)$ *there is for every a* $\vartheta \in S(G)$ *such that the Fourier transform* $\hat{\vartheta}$ *has compact support and* $\|\vartheta * f - f\|_S < \varepsilon$

LEMMA 4.6. *Every Segal algebra has approximate units of* L^1*-norm 1.*

The proofs of these lemmas can be found in Reiter ([18] pp.128-129 and [19] pp. 18-20, p.37).

LEMMA 4.7. *Let* $1 \le p < \infty$. *If* $f \in S(G)$ *and* $h \in L^p(G)$ *then* $f * h \in L^p(G)$ *and*

$$\|f * h\|_p \le \|f\|_1 \|h\|_p \le \|f\|_S \|h\|_p$$

DEFINITION 4.8. Let $S(G)$ be a Segal algebra on a locally compact abelian group G. A multiplier on $S(G)$ is a bounded linear operator on $S(G)$ which commutes with translations. We denote by $M(S)$ the set of all multipliers on $S(G)$.

<u>THEOREM 4.9</u>. <u>Let</u> $T \in M(S)$. <u>If</u> $f, g \in S(G)$, <u>then</u>

$$T(f * g) = Tf * g = f * Tg$$

(see Unni [22]).

<u>Proof</u>. Suppose $T \in M(S)$. Then for each $y \in G$, we have $T_y\, y = \tau_y T$. We shall now show that T commutes with convolutions. If the space of continuous linear functionals on $S(G)$ is denoted by $S(G)'$, we denote the pairing between $S(G)$ and $S(G)'$ by

$$\langle f, \varphi \rangle$$

for $f \in S(G)$ and $\varphi \in S(G')'$. Let $\|T\|$ denote the operator norm of T. Let $\varphi \in S(G)'$ be fixed. If $f \in S(G)$, the inequality

$$|\langle Tf, \varphi \rangle| \leq \|Tf\|_S \|\varphi\| \leq \|T\| \, \|f\|_S \|\varphi\|$$

shows that the mapping $f \to \langle Tf, \varphi \rangle$ is a bounded linear functional on $S(G)$ and there exists $\psi \in S(G)'$ such that

(1) $$\langle f, \psi \rangle = \langle Tf, \varphi \rangle \qquad \text{for all } f \in S(G).$$

Now it is known (see Reiter [19] , pp 52) that if $f \in L^1(G)$, $g \in S(G)$ and $\varphi \in S(G)'$, then

(2) $$\langle f * g, \varphi \rangle = \int_G f(y) \langle \tau_y g, \varphi \rangle \, dy$$

holds. Now suppose $f, g \in S(G)$. Then

$$\begin{aligned}
\langle Tf * g, \varphi \rangle &= \int_G g(y) \langle \tau_y Tf, \varphi \rangle \, dy && \text{by (2)} \\
&= \int_G g(y) \langle T \tau_y f, \varphi \rangle \, dy \\
&= \int g(y) \langle \tau_y f, \psi \rangle \, dy && \text{by (1)} \\
&= \langle f * g, \psi \rangle && \text{by (2) again} \\
&= \langle T f * g, \varphi \rangle . && \text{by (1) again}
\end{aligned}$$

Hence the relation

$$(3) \qquad \langle Tf * g, \varphi \rangle = \langle T(f * g), \varphi \rangle$$

is valid for every φ in $S(G)'$. Hahn Banach theorem now applies to show that

$$Tf * g = T(f * g)$$

for all $f, g \in S(G)$. By commutativity of the convolution product we also have

$$T(f * g) = T(g * f) = Tg * f$$

This completes the proof.

A nalogous to Theorem A stated in the introduction the following representation theorem was proved by Unni [23] .

THEOREM 4.10. *If* $T \in M(S)$, *then there exists a unique pseudomeasure* σ *such that*

$$Tf = \sigma * f$$

for all $f \in S(G)$.

We shall show that $M(S)$ is isometric and algebra isomorphic to the multiplier algebra on an abstract Banach algebra.

Let G be a locally compact abelian group and let $S(G)$ denote a Segal algebra on G. The space of bounded linear operators on $S(G)$ is denoted by $B(S)$. Then $M(S)$ is a subset of $B(S)$ consisting of those elements in $B(S)$ which commute with translations. If $T \in B(S)$, then $\|T\|_S$ will denote the operator norm of T.

If $g \in L^1(G)$, we define the operator W_g on $S(G)$ by

$$W_g(f) = g * f \qquad f \in S(G).$$

Then W_g is a linear operator commuting with translations.

From the inequality

$$\| W_g(f) \|_S \leq \| g \|_1 \; \| f \|_S$$

it follows that $W_g \in M(S)$ and $\| W_g \|_S \leq \| g \|_1$. Let

$$P = \left\{ W_g : g \in L^1(G) \right\} .$$

Then P is a linear subspace of $B(S)$. If $\check{U}(S)$ denotes the completion of P in $B(S)$ then $U(S)$ is a subspace of $M(S)$.

<u>THEOREM 4.12</u>. $U(S)$ <u>is actually a Banach algebra</u>.

<u>Proof.</u> It is easy to see that $U(S)$ is a Banach space. Let $g,h \in L^1(G)$. Then $g * h \in L^1(G)$ so that W_g, W_h, W_{g*h} all belong to P. Further, if $f \in S(G)$, then

$$W_{g*h}(f) = g * h * f = W_g \cdot W_h(f)$$

so that $W_g \circ W_h = W_{g*h}$. Hence $W_g \circ W_h \in P$. Thus P is closed under composition as multiplication. Moreover

$$\| W_{g*h} \|_S = \| W_g \circ W_h \|_S \leq \| W_g \|_S \; \| W_h \|_S .$$

It now follows that $U(S)$, being the completion of P, is itself a Banach algebra.

<u>THEOREM 4.13</u>. <u>There exists a bounded approximate identity for</u> $U(S)$.

<u>Proof.</u> Let $\{h_\alpha\}$ be an approximate identity for $S(G)$ such that h_α is bounded in L^1-norm and the Fourier transform of h_α has compact support. Then $W_{h_\alpha} \in P$ since $h_\alpha \in S(G) \subset L^1(G)$.

Now

$$\| W_{h_\alpha} * W_g - W_g \|_S \leq \| h_\alpha * g - g \|_1$$

This implies that $\lim_{\alpha} \| W_{h_\alpha} \circ W_g - W_g \| = 0$ for each $W_g \in P$. Since P is dense in $U(S)$, we also have

$$\lim_{\alpha} \| W_{h_\alpha} \circ W - W \|_S = 0$$

for each $W \in U(S)$.

<u>THEOREM 4.14</u>. *Let* $T \in M(S)$ *and* $h \in S(G)$. *Then* $W_{Th} = T \circ W_h$.

<u>Proof</u>. Now $Th \in S(G) \subset L^1(G)$ and so $W_{Th} \subset P$. Then we have

$$W_{Th}(f) = Th * f = T(h*f) = (T \circ W_h)(f)$$

for each $f \in S(G)$. Therefore

$$W_{Th} = T \circ W_h$$

<u>DEFINITION 4.15</u>. Let $A \in M(S)$ and consider the mapping defined by

$$P_A(B) = \| BA \|_S$$

for each $B \in M(S)$. Then P_A is a seminorm on $M(S)$. Now let $R(M,U)$ denote the coarest topology on $M(S)$ with respect to which each of the seminorms P_A is continuous for $A \in U(S)$ and $M(S)$ is a locally convex topological vector space with respect to the topology $R(M,U)$ (see McKennon [17], p.482).

<u>LEMMA 4.16</u>. Let r be any positive integer and let $M_r = \{ A \in M(S) : \| A \|_S \leq r \}$. If $M_r \times M_r$ <u>is given by the relativized product uniform topology</u> $R(M,U) \times R(M,U)$ <u>then the binary operation defined by</u>

$$(A,B) \longrightarrow A.B$$

<u>is continuous in</u> $R(M,U)$.

Proof. Let q and r be fixed positive numbers and B be any element of U(S). Then if E,F,G and H are in M_r such that $p_B(E-F) < q/2r$, we have

$$\begin{aligned} p_B\,(H-G)(E-F) &= \|HEB - GEB - HFB + GFB\|_S \\ &\leq \|H\|_S\,\|EB - FB\|_S + \|G\|_S\,\|EB - FB\|_S \\ &= (\|H\|_S + \|G\|_S)\;p_B(E-F) \\ &< q/2r \cdot 2r = q \end{aligned}$$

from which follows the continuity.

LEMMA 4.17. The unit ball in U(S) is dense in the unit ball in M(S) in the R(M,U) topology.

Proof. Let T be any element of M_1. Let W_{h_α} be the approximate identity given in Theorem 4.13. Then $Th_\alpha \in S$ and $W_{Th_\alpha} = T\circ W_\alpha$. By the continuity given Lemma 4.16, we have

$$\lim_\alpha W_{Th_\alpha} = \lim_\alpha T\circ W_{h_\alpha} = T\circ I = T \text{ in } R(M,U) \text{ topology}$$

Moreover

$$\|T\|_S \leq \underline{\lim}_\alpha \|W_{Th_\alpha}\|_S$$

On the other hand

$$\overline{\lim}\,\|W_{Th_\alpha}\|_S = \overline{\lim}\,\|T\circ W_{h_\alpha}\|$$

Hence $$\lim_\alpha \|W_{Th_\alpha}\|_S = \|T\|_S$$

Now $W_{Th_\alpha} \in U(S)$ and $\|W_{Th_\alpha}\|^{-1} W_{Th_\alpha} = T$ belongs to the unit ball in $U(S)$. Thus $\lim_{\alpha} \|W_{Th_\alpha}\|^{-1} W_{Th_\alpha} = T$ in the topology of $R(M,U)$ and T is the $R(M,U)$-limit of operators in the unit ball of $U(S)$.

<u>LEMMA 4.18</u>. <u>Let</u> $\{T_\alpha\}$ <u>be any</u> $R(B,U)$-<u>Cauchy net in</u> $B(S)$ <u>such that</u> $\sup_\alpha \|T_\alpha\| < \infty$. <u>Then there is an operator</u> T <u>in</u> $B(S)$ <u>such that</u> $\lim_\alpha T_\alpha = T$ <u>in both the strong operator topology and the topology</u> $R(B,U)$.

<u>Proof</u>. Let $f \in S(G)$, if $g \in L^1(G)$, then $W_g \in U(S)$ and by hypothesis $\{T_\alpha \circ W_g\}$ is a Cauchy net in $B(S)$. Since $T_\alpha \circ W_g(f) = T_\alpha(g * f)$, $T_\alpha(g * f)$ is a Cauchy net in $S(G)$ and converges to $T(g*f)$ in the Segal norm. Since $L^1(G)*S$ is dense in $S(G)$ and $\sup_\alpha \|T_\alpha\|_S < \infty$ it follows that $T_\alpha \to T$ in the strong operator topology.

Let $g \in L^1(G)$, then by hypothesis $\{T_\alpha \circ W_g\}$ is a Cauchy net in $B(S)$ and so has a limit V in the norm of $B(S)$. Then for each $f \in S(G)$, we have

$$V(f) = \lim_\alpha (T_\alpha \circ W_g) f = \lim_\alpha T_\alpha(g * f) = T(g * f)$$
$$= T \circ W_g(f)$$

Hence $V = T \circ W_g$ and $\|(T_\alpha - T) \circ W_g\|_S \to 0$ implying that $T_\alpha \to T$ in the topology of $R(B,U)$.

<u>LEMMA 4.19</u>. <u>If</u> $T \in M(S)$, <u>then</u>

$$\|T\|_S = \sup\{\|T \circ W\|_S : W \in U(S) \;\&\; \|W\|_S \le 1\}$$

<u>Proof</u>. Let $T \in M(S)$. Then if $W \in U(S)$ and $\|W\|_S = 1$ then

$$\|T \circ W\|_S \le \|T\|_S \|W\|_S = \|T\|_S$$

from which it follows that

$$\|T\|_S \geq \operatorname{Sup}\{\|T \circ W\|_S : W \in U(S),\ \|W\|_S = 1\}$$

Let $\varepsilon > 0$ be given. Choose $f \in S(G)$, such that and $\|Tf\|_S \geq \|T\|_S - \varepsilon/2$. Let $\{W_\alpha\}$ be an approximate identity for $U(S)$ (see Theorem 4.13). Then $W_\alpha \to I$ in the topology of $R(M,U)$ and therefore $T \circ W_\alpha \to T \circ I$ in the topology of $R(M,U)$. But Lemma 4.18 implies that $T \circ W_\alpha \to T \circ I$ in the strong operator topology also. Hence given $\varepsilon > 0$ there exists a γ such that

$$\|T \circ W_\gamma(f) - Tf\|_S < \varepsilon/2$$

Therefore

$$\|T \circ W_\gamma(f)\|_S \geq \|Tf\|_S - \|T \circ W_\gamma(f) - Tf\|_S \geq \|T\|_S - \varepsilon$$

Since $\|W_\gamma\|_S \leq 1$, we have

$$\begin{aligned}\|T\|_S - \varepsilon &\leq \operatorname{Sup}\{\|T \circ W_\gamma\|_S : \|W_\gamma\|_S \leq 1\} \\ &\leq \operatorname{Sup}\{\|T \circ W\|_S : W \in U(S)\ \|W_S\| \leq 1\}\end{aligned}$$

Since ε is arbitrary we obtain

$$\operatorname{Sup}\{\|T \circ W\|_S : W \in U(S) : \|W\|_S \leq 1\} \geq \|T\|_S$$

This completes the proof.

<u>THEOREM 4.20</u>. <u>Let</u> B<u>e a normed algebra with identity</u> <u>and</u> A <u>a subalgebra of</u> B <u>which is</u> $\|\cdot\|_B$ <u>complete. Suppose that the following hold.</u>

(1) <u>the unit ball</u> A_1 <u>of</u> A <u>is</u> $R(B,A)$ <u>dense in the unit ball</u> B_1 <u>of</u> B .

(ii) $\|b\|_B = \operatorname{Sup}\{\|b.a\|_B : a \in A_1\}$ <u>for each</u> $b \in B$

(iii) B_1 <u>is</u> $R(B,A)$-<u>complete</u>.

<u>Then</u> $M(A)$ <u>is isomorphic to</u> B.

This is Theorem 6 of Kelly McKennon [17] .

If we take $M(S)$ in the place of B and $U(S)$ in the place of A, we have proved that $M(S)$ and $U(S)$ satisfy the conditions of above theorem and thus we have

<u>THEOREM 4.21</u>. <u>The multiplier algebra</u> $M(S)$ <u>is isometric and algebra isomorphic to</u> $M(U,(S))$. <u>The isomorphism is given by</u> $T \to \pi_T$ <u>where</u> $\pi_T : U(S) \to U(S)$ <u>is defined by</u>

$$\pi_T(w) = T\circ w .$$

<u>DEFINITION 4.22</u>. A multiplier from $S(G)$ to $L^p(G)$ is a bounded linear operator which commutes with translations. The space of all multipliers from $S(G)$ to $L^p(G)$ is denoted by $M(S,L^p)$.

<u>THEOREM 4.23</u>. <u>Let</u> $T \in M(S,L^p)$. <u>Then if</u> $f,g \in S(G)$ <u>we have</u>

$$T(f*g) = Tf*g = f*Tg$$

<u>Proof.</u> Suppose $T \in M(S.L^p)$. For each $y \in G$, we have $T\tau_y = \tau_y T$. If $f,g \in S(G)$, then $T(f*g)$ and $Tf*g$ both belong to $L^p(G)$. Then if $k \in L^{p'}(G)$ and $f \in S(G)$ the inequality

$$|\langle Tf, k\rangle| \le \|Tf\|_p \|k\|_{p'} \le \|T\| \|f\|_S \|k\|_{p'}$$

shows that $f \to \langle Tf,k\rangle$ is a bounded linear functional on $S(G)$ and hence there exists $\varphi \in S(G)'$ such that

$$\langle Tf,k\rangle = \langle f, \varphi\rangle \quad \text{for all } f \in S(G).$$

Now if $f, g \in S(G)$ and $k \in L^{p'}(G)$, then

$$\langle Tf*g, k\rangle = \int g(y)\langle \tau_y Tf, k\rangle dy$$
$$= \int g(y)\langle T\tau_y f, k\rangle dy = \int g(y)\langle \tau_y f, \varphi\rangle dy$$
$$= \langle f*g\ \varphi\rangle = \langle T(f*g), k\rangle$$

This implies $Tf*g = T(f*g)$ for all $f, g \in S(G)$ by Hahn ~~xxx~~ Banach theorem.

We shall now obtain a characterization of $M(S, L^p)$.

Let $1 < p < \infty$. Let $\mathfrak{A}$ be the set of all functions u which can be expressed as

(4) $$u = \sum_{k=1}^{\infty} f_k * g_k$$

where $f_k \in S(G)$ and $g_k \in L^{p'}(G)$ such that

$$\sum_{k=1}^{\infty} \|f_k\|_S \|g_k\|_{p'} < \infty$$

We define a norm in $\mathfrak{A}$ by

(5) $$\|u\| = \inf \left\{\sum_{k=1}^{\infty} \|f_k\|_S \|g_k\|_{p'}\right\}$$

where the infimum is taken over all such representations of u. Then we have

<u>THEOREM 4.24</u>. <u>$\mathfrak{A}$ is a Banach space with norm given by (5). Moreover $\mathfrak{A}$ is a subspace of $L^{p'}(G)$ and the topology on $\mathfrak{A}$ is not weaker than the topology induced from $L^{p'}(G)$.</u>

<u>Proof.</u> The first part of this theorem is proved exactly as before while the second part follows from the fact that if $f \in S(G)$ and $g \in L^{p'}(G)$ then $f*g \in L^{p'}(G)$ and

$$\|f*g\|_{p'} \le \|f\|_S \|g\|_{p'}$$

THEOREM 4.25. Let G be a locally compact abelian group and let $1 < p < \infty$. The multiplier space $M(S,L^p)$ is isometrically isomorphic to the dual $\mathfrak{a}^*$ of $\mathfrak{a}$.

Proof. Let $T \in M(S,L^p)$, we define

$$t(u) = \sum_{k=1}^{\infty} Tf_k * g_k(0)$$

for $u = \sum_{k=1}^{\infty} f_k * g_k$. Since $Tf_k \in L^p(G)$ and $g_k \in L^{p'}(G)$ it is clear that $Tf_k * g_k(o)$ is properly defined. We have to show that t is well defined. To this end, it is sufficient to show that if $\sum_{k=1}^{\infty} f_k * g_k$ is a representation of 0 as an element of $\mathfrak{a}$ and $\sum_{k=1}^{\infty} \|f_k\|_S \|g_k\|_{p'} < \infty$ then $\sum_{k=1}^{\infty} Tf_k * g_k(o) = 0$. Let e_α be an approximate identity for $S(G)$ such that $\|e_\alpha\|_1 = 1$ and $\hat{e}_\alpha \in K(\Gamma)$. Then

$$|T(e_\alpha * f_k) * g_k(0) - Tf_k * g_k(0)| = |T(e_\alpha * f_k - f_k) * g_k(0)|$$
$$\leq \|T\| \, \|e_\alpha * f_k - f_k\|_S \|g_k\|_{p'}$$

so that

$$\lim_\alpha T(e_\alpha * f_k) * g_k(0) = Tf_k * g_k(0)$$

Since $u = \sum f_k * g_k = 0$ and the series $\sum_{k=1}^{\infty} f_k * g_k$ converges in $L^{p'}(G)$, and $Te_\alpha \in L^p$ we get

$$\sum_{k=1}^{\infty} T(e_\alpha * f_k) * g_k(0) = \sum_{k=1}^{\infty} Te_\alpha * f_k * g_k(0)$$

$$= Te_\alpha * \sum_{k=1}^{\infty} f_k * g_k(0)$$

$$= 0$$

We shall now show that $\sum T(e_\alpha * f_k) * g_k(0)$ converges uniformly with respect to α. This is immediate since

$$\left|\sum_{k=1}^{\infty} T(e_\alpha * f_k) * g_k(0)\right| \le \sum_{k=1}^{\infty} \|T(e_\alpha * f_k)\|_p \, \|g_k\|_{p'}$$

$$\le \sum_{k=1}^{\infty} \|T\| \, \|e_\alpha * f_k\|_S \, \|g_k\|_{p'}$$

$$\le \|T\| \sum_{k=1}^{\infty} \|e_\alpha\|_1 \, \|f_k\|_S \, \|g_k\|_{p'}$$

$$= \|T\| \sum_{k=1}^{\infty} \|f_k\|_S \, \|g_k\|_{p'}$$

Hence

$$\sum_{k=1}^{\infty} Tf_k * g_k(0) = \lim_{\alpha} \sum_{k=1}^{\infty} T(e_\alpha * f_k) * g_k(0) = 0$$

Since $T(e_\alpha * f_k) * g_k(0) \to Tf_k * g_k(0)$ for each k. Thus t is well defined. It is clearly linear. The mapping is an isometry. In fact

$$|t(u)| = \left|\sum_{k=1}^{\infty} Tf_k * g_k(0)\right| \le \|T\| \sum_{k=1}^{\infty} \|f_k\|_S \, \|g_k\|_{p'}$$

implies

$$|t(u)| \leq \|T\| \, \|u\|$$

so that $\|t\| \leq \|T\|$. On the other hand

$$\|T\| = \operatorname{Sup}\{|Tf*g(0)| : \|f\|_S \leq 1 \quad \|g\|_{p'} = 1\}$$
$$= \operatorname{Sup}\{|t(f*g)| : \|f\|_S \leq 1 \quad \|g\|_{p'} \leq 1\}$$
$$\leq \|t\|$$

To see that the mapping $T \to t$ is onto, we proceed as follows. Let $t \in \mathfrak{a}^*$. Let $f \in S(G)$ be fixed. Now define a functional L on $L^{p'}(G)$ by the equation

$$L(g) = t(f*g) \qquad g \in L^{p'}(G).$$

Then $|L(g)| = |t(f*g)| \leq \|t\| \, \|f\|_S \cdot \|g\|_{p'}$ which shows that L is a bounded linear functional on $L^{p'}(G)$. Since $1 < p < \infty$ ~~exists~~ $L^p(G)$ is the dual of space of $L^{p'}(G)$ $(1 < p' < \infty)$, there exists a unique element, call it Tf, in $L^p(G)$ such that

$$Tf*g(0) = L(g) = t(f*g) \qquad g \in L^{p'}(G)$$

and $\|Tf\|_p = \|L\| \leq \|t\| \, \|f\|_S$. Thus to each $f \in S(G)$ we have Tf in $L^p(G)$ and the mapping T is a bounded operator from $S(G)$ to $L^p(G)$. It is clear that T is linear. We now claim that T is a multiplier from $S(G)$ to $L^p(G)$. Let $y \in G$ and $f \in S(G)$. If $g \in L^{p'}(G)$, then

$$\tau_y Tf * g(0) = Tf * \tau_y g(0) = t(f * \tau_y g) = t(\tau_y f * g)$$
$$= T\tau_y f * g(0)$$

holds for all g $L^{p'}(G)$. Hence $\tau_y Tf = T\tau_y f$ for each

$f \in S(G)$ which implies that $T\tau_y = \tau_y T$, that is, $T \in M(S, L^p)$. This completes the proof.

We shall now conclude this section with a representation theorem for multipliers from $L^p(G)$ to $S(G)$.

THEOREM 4.26. *Let $S(G)$ be a Segal algebra on a locally compact abelian group G. Suppose that $1 < p < \infty$ and let $T \in M(L^p, S)$*

a) *If $p > 1$ then $M(L^p, S)$ consists of only the zero operator if G is noncompact*

b) *If $p = 1$ these exists a unique measure $\mu \in M(G)$ such that*

$$Tf = \mu * f$$

for all $f \in L^1(G)$.

Proof. If $T \in M(L^p, S)$ then $\|Tf\|_S \le \|T\| \, \|f\|_p$ for each $f \in S(G)$. Then

$$\|Tf\|_1 \le \|Tf\|_S \le \|T\| \, \|f\|_p$$

shows that $T \in M(L^p, L^1)$. If $p \neq 1$, Hormander's theorem generalized by Gaudry (see Larsen [14] p. 149) shows that $T = 0$. If $p = 1$, then there exists a unique measure $\mu \in M(G)$ such that

$$Tf = \mu * f$$

for all $f \in L^1(G)$. This completes the proof.

CHAPTER 5

Segal algebras: Particular Cases

We shall now discuss the problem of multipliers on some special cases of Segal algebras. Though Theorem 4.10 says that the multipliers on a Segal algebra are given by pseudo-measures in some special cases they reduce to measures.

The algebra $A^p(G)$ consisting of those functions $f \in L^1(G)$ whose Fourier transform $\hat{f}$ belongs to $L^p(\Gamma)$ is a Segal algebra with Segal norm $\|f\|_S = \|f\|_1 + \|\hat{f}\|_p$. The algebra $L^1(G) \cap L^p(G)$ with norm $\|f\|_S = \|f\|_1 + \|f\|_p$ is also a Segal algebra. In both these cases the multiplier space turned out to be $M(G)$ when G is noncompact. We shall consider now the algebra $A^p_\omega(G)$ and show that if G is a locally compact noncompact nondiscrete abelian group then the multipliers on $A^p_\omega(G)$ reduce to bounded measures. In fact it is pointed out by Unni [24] that there is a spectrum of Segal algebras for which multiplier space reduces to $M(G)$.

We now state a lemma which was first proved Hörmander [9] when $G = R^n$.

<u>LEMMA 5.1.</u> <u>Let</u> G <u>be a locally compact, noncompact abelian group. Then for</u> $f \in L^p(G)$, <u>we have</u>

$$\|f + \tau_y f\|_p \to 2^{\frac{1}{p}} \|f\|_p$$

as $y \to \infty$.

<u>Proof</u>. See Larsen [14] p. 78.

Let ω be a real valued even continuous function on Γ such that $\omega(\eta+\gamma) \leq \omega(\eta)\omega(\gamma)$ for all $\eta, \gamma \in \Gamma$. If $1 \leq p < \infty$, we define $A^p_\omega(G)$ to be the set of all functions f in $L^1(G)$ such that $\hat{f} \in L^{p,\omega}(\Gamma)$. We introduce a norm by

(1) $$\|f\| = \|f\|_1 + \|\hat{f}\|_{p,\omega}.$$

Then $A^p_\omega(G)$ is a Segal algebra on G (see Reiter [19] p.25) and hence is a semisimple commutative Banach algebra.

If $B(G) = \{ f \in L^1(G) : \hat{f} \in \mathcal{K}(G)$ then $B(G)$ is dense in $A^p_\omega(G)$ by Lemma 4.5.

DEFINITION 5.2. A multiplier on $A^p_\omega(G)$ is a bounded linear operator which commutes with translations and $M(A^p_\omega)$ will denote the space of multipliers on $A^p_\omega(G)$.

THEOREM 5.3. *Let G be a nondiscrete, noncompact, locally abelian group and $1 \leq p < \infty$. If $T \in M(A^p_\omega(G))$ then there exists a unique measure $\mu \in M(G)$ such that*

$$Tf = \mu * f$$

for all $f \in A^p_\omega(G)$. Further $M(A^p_\omega)$ is isometrically isomorphic to $M(G)$, the space of bounded regular Borel measures on G.

Proof. Let $T \in M(A^p_\omega)$ and $f \in B(G)$. Then there is a compact set $K \subset \Gamma$ such that f vanishes outside K. Since $Tf \in A^p_\omega(G)$, we have

(2) $$\|Tf\|_1 \leq \|Tf\| \leq \|T\| \left(\|f\|_1 + \|\hat{f}\|_{p,\omega} \right)$$

Case (1). $2 \leq p < \infty$. Using Hausdorff Young inequality we compute $\|\hat{f}\|_{p,\omega}$. Since $f \in B(G)$, we have a compact set $K \subset \Gamma$ outside of which $\hat{f}$ vanishes. Now $L^1(G) \cap L^{p1}(G)$ is a Segal algebra on G and hence $B(G) \subset L^1(G) \subset L^{p1}(G)$. Therefore for $f \in B(G)$, Therefor since $1 < p^1 < 2$ we have by Hausdorff Young inequality

$$\|\hat{f}\|_p \leq \|f\|_{p'}$$

Now

$$\int_{\Gamma} |\hat{f}|^p \omega^p \, d\gamma = \int_K |\hat{f}| \omega^p \, d\gamma$$

$$\leq ((C(K,\omega))^p \int_K |\hat{f}| \, d\gamma = (C(K,\omega))^p \int_{\Gamma} |\hat{f}|^p \, d\gamma$$

where $C(K, \omega)$ is a constant depending on the compact set K and the weight function ω. This implies

$$\|\hat{f}\|_{p,\omega} \leq C(K,\omega) \|\hat{f}\|_p \leq C(K,\omega) \|f\|_{p'}$$

Then (2) can be written as

$$(3) \qquad \|Tf\|_1 \leq \|T\| \left(\|f\|_1 + C(K,\omega) \|f\|_{p'} \right)$$

Now since $1 < p^1 \leq 2$ and G is noncompact, we have, by Lemma 5.1

$$2\|Tf\|_1 = \lim_{s \to \infty} \|Tf + \tau_s Tf\|_1 = \lim_{s \to \infty} \|T(f + \tau_s f)\|_1$$

$$\leq \lim_{s \to \infty} \|T\| \left(\|f + \tau_s f\|_1 + C(K,\omega) \|f + \tau_s f\|_{p'} \right)$$

$$= \|T\| \left(2\|f\|_1 + C(K,\omega) 2^{\frac{1}{p'}} \|f\|_{p'} \right)$$

so that

$$(4) \qquad \|Tf\|_1 \leq \|T\| \left(\|f\|_1 + C(K,\omega) 2^{\frac{1}{p'} - 1} \|f\|_{p'} \right)$$

Repeating this process n times we get

$$(5) \qquad \|Tf\|_1 \leq \|T\| \left(\|f\|_1 + C(K,\omega) 2^{n\left(\frac{1}{p'} - 1\right)} \|f\|_{p'} \right)$$

Since $p^1 > 1$, we have $\lim_{n\to\infty} 2^{n(\frac{1}{p'}-1)} = 0$. Now the left hand side of (5) is independent of n. Hence taking the limit as $n \to \infty$ on the right hand side we conclude that

(6) $$\|Tf\|_1 \leq \|T\| \, \|f\|_1$$

The inequality (6) holds for all $f \in B(G)$. Hence T when restricted to B(G) defines a linear transformation from B(G) to $L^1(G)$ which commutes with translations and which is bounded in L^1-norm. Since B(G) is dense in $L^1(G)$, we can extend T uniquely as a multiplier T_1 from $L^1(G)$ to $L^1(G)$. Hence by Theorem there exists a unique measure $\mu \in M(G)$ such that $T_1 f = \mu * f$ for $f \in L^1(G)$ and hence $Tf = \mu * f$ for all $f \in B(G)$. Moreover $\|T\| \leq \|\mu\|$. Using the fact that B(G) is dense in $A^p_\omega(G)$, we have $Tf = \mu * f$ and $\|\mu\| \leq \|T\|$. But we also have from the above that $\|\mu\| \geq \|T\|$. Hence the theorem is proved for the case $2 \leq p < \infty$.

<u>Case (ii):</u> $1 < p < 2$. Let $q = \frac{2}{p}$. Then $q > 1$. If $T \in M(A^p_\omega)$ we have the inequality (1) satisfied. Since f has compact support, we have

$$\|\hat f\|^p_{p,\omega} = \int |\hat f \omega|^p d\gamma = \int_K |\hat f \omega|^p d\gamma$$

Using Holder's inequality and the fact that ω is locally bounded, we obtain

(7) $$\|\hat f\|^p_{p,\omega} \leq C(K,\omega) \int_K |\hat f|^p d\gamma$$

$$\leq (C(K,\omega))^p \left(\int_K |\hat f|^{pq}\right)^{\frac{1}{q}} \left(\int_K 1\, d\gamma\right)^{\frac{1}{q'}}$$

$$\leq (C(K,\omega))^p \left(\int_K |\hat f|^{pq}\right)^{\frac{1}{q}}$$

where $C(K,\omega)$ is a constant depending on the weight function and the compact set K. From (7) we deduce that

(8) $$\|\hat{f}\|_{p,\omega} \leq C_1(K,\omega)\,\|\hat{f}\|_2$$

So, we have, by Plancherel's theorem

(9) $$\|\hat{f}\|_{p,\omega} \leq C_1(K,\omega)\,\|\hat{f}\|_2 = C(K,\omega)\,\|f\|_2$$

Now (2) can be written as

(10) $$\|Tf\|_1 \leq \|T\|\,(\|f\|_1 + C(K,\omega)\,\|f\|_2)$$

from which argument as in case (i) gives

(11) $$\|Tf\|_1 \leq \|T\|\,\left(\|f\|_1 + C(K,\omega)\,2^{-\frac{1}{2}}(\|f\|_2)\right)$$

Repeating the process n times we have

$$\|Tf\|_1 \leq \|T\|\,\left(\|f\|_1 + C(K,\omega)\,2^{-n/2}\,\|f\|_2\right)$$

Now letting $n \to \infty$, we obtain

The rest of the argument is as before.

In connection with the study of tauberian theorems Wiener introduced continuous functions f on $(-\infty,\infty)$ for which

$\sum_{k=-\infty}^{\infty} \max_{k\leq x\leq k+1} |f(x)|$ converges [25]. A systematic study of the space W of all such functions normed with this sum was given by Goldberg [8]. It turned out to be a Banach algebra under convolution and is an important subalgebra of the Banach algebra $L^1(-\infty,\infty)$. With a slight modification in the definition of the norm, this Wiener class becomes an example of a Segal algebra.

In [3] , Edwards observed that if T is a bounded linear operator from W to $L^1(-\infty,\infty)$ which commutes with translations, then T has a representation

(12) $$Tf = \sigma * f \qquad f \in W$$

for a suitably chosen pseudomeasure σ. We shall now investigate the representation and properties of continuous linear operators on W which commutes with translations. We shall make more precise the representation (12) of such operators.

For each $k = 0, \pm 1, \pm 2, \ldots$ let I_k denote the closed interval $[k,k+1]$ and let W denote the space of all continuous functions f on $(-\infty,\infty)$ such that

$$\sum_{k=-\infty}^{\infty} \max_{x \in I_k} |f(x)| < \infty$$

A norm on W is given by

(13) $$\|f\|_W = \sum_{k=-\infty}^{\infty} \max_{x \in I_k} |f(x)|$$

Various properties of the space W proved by Goldberg can be summarized as follows.

<u>THEOREM 5.4.</u> (a) W <u>is a linear subspace of</u> $L^1(-\infty,\infty)$ <u>and that</u> $\|f\|_1 \le \|f\|_W$ <u>for each</u> $f \in W$.

(b) W <u>is a Banach space under the norm</u> (13)

(c) <u>If</u> f <u>and</u> g <u>are any two elements of</u> W <u>then</u> $f * g$ <u>belongs to</u> W <u>and</u> $\|f * g\|_W \le 2\, \|g\|_W \; \|g\|_W$

(d) <u>The translation operator has norm</u> ≤ 2, <u>that is,</u> $\|\tau_x f\|_W \le 2\|f\|_W$, <u>the constant 2 being best possible.</u>

(e) <u>If</u> L <u>is a continuous linear functional on</u> W, <u>then there is a measure</u> μ <u>on</u> $(-\infty,\infty)$ <u>satisfying</u>

$$(14) \qquad L(f) = \int_{-\infty}^{\infty} f(t)d\mu(t)$$

and

$$(15) \qquad |\mu|(I_k) \leq \eta \qquad k = 0,1,2$$

for some $\eta > 0$. Moreover any L satisfying (14) and (15) is a bounded linear functional on W.

It follows from (3) and (4) that

$$|L(f)| = \left|\int_{-\infty}^{\infty} f(t)d\mu(t)\right| \leq \sum_{k=-\infty}^{\infty}\int_{I_k} |f(t)|\,|d\mu(t)|$$

so that

$$|L(f)| \leq \|f\|_W \cdot \sup_k |\mu|(I_k$$

and hence

$$(16) \qquad \|L\| \leq \sup_k |\mu|(I_k)$$

We denote by $\mathfrak{M}$ the set of all measure μ on $(-\infty,\infty)$ satisfying (15). The pairing between $\mathfrak{M}$ and W is denoted by

$$(17) \qquad \langle f,\mu\rangle = \int_{-\infty}^{\infty} f(t)d\mu(t)$$

We remark that every continuous function defined on the real line R having compact support belongs to W and hence W is actually dense in $L^1(-\infty,\infty)$ in the L^1-norm.

It is easy to verify that, taking the group G to be R all the axioms of a Segal algebra are satisfied in the case of W except the one which says that the translation operator has norm one. If $\alpha \in R$ and $I_{k-\alpha} = [\alpha-(k+1), \alpha-k]$ for $k = 0, \pm 1, \pm 2, \ldots$

it is clear that if $f \in W$ then

$$\sup_{\alpha \in R} \sum_{k=-\infty}^{\infty} \max_{x \in I_{k-\alpha}} |f(x)| \leq 2 \sum_{k=-\infty}^{\infty} \max_{x \in I_k} |f(x)|$$

Thus if we introduce a new norm on W by

$$\|f\|_S = \sup_{\alpha \in R} \sum_{k=-\infty}^{\infty} \max_{x \in I_{k-\alpha}} |f(x)| \tag{18}$$

then

$$\|f\|_W \leq \|f\|_S \leq 2\|f\|_W$$

(the W-norm and the S-norm are equivalent) and the translation operator now will have norm one. W is then a Segal algebra on R where the Segal norm is given by (18).

If $f \in L^1(R)$, $g \in W$ and $\mu \in \mathfrak{m}$, then we have

$$\|f * g\|_W \leq 2\|f\|_1 \|g\|_W$$

and

$$\langle f * g, \mu \rangle = \int_{-\infty}^{\infty} \langle \tau_x g, \mu \rangle f(x)\, dx$$

<u>DEFINITION 5.5</u>. A bounded linear operator on W is called a multiplier on W if it commutes with translations.

<u>THEOREM 5.6</u>. <u>Let</u> $T: W \longrightarrow W$ <u>be a bounded linear operator. Then the following are equivalent</u>

(a) $T\tau_x = \tau_x T$ for all $x \in R$

(b) $T(f*g) = Tf*g = f*Tg$ $\quad f, g \in W$.

<u>Proof</u>. Suppose (a) holds. Then if $\mu \in \mathfrak{m}$, then the inequalities

$$|\langle Tf, \mu \rangle| \leq \|Tf\|_W \|\mu\|_{\mathfrak{m}} \leq \|T\| \|f\|_W \|\mu\|_{\mathfrak{m}}$$

show that the mapping $f \longrightarrow \langle Tf, \mu\rangle$ is a bounded linear functional on W. Hence there exists a $\lambda \in m$ such that

$$\langle Tf, \mu\rangle = \langle f, \lambda\rangle$$

Now let $\mu \in m$ and suppose $f, g \in W$. Then we have

$$\langle Tf * g, \mu\rangle = \int_{-\infty}^{\infty} \langle \tau_x Tf, \mu\rangle g(x)\,dx$$

$$= \int_{-\infty}^{\infty} \langle T\tau_x f, \mu\rangle g(x)\,dx$$

$$= \int_{-\infty}^{\infty} \langle \tau_x f, \lambda\rangle g(x)\,dx$$

$$= \langle f * g, \lambda\rangle = \langle T(f*g), \mu\rangle$$

Thus $\langle Tf*g, \mu\rangle = \langle T(f*g), \mu\rangle$ for all $\mu \in m$. It then follows that $Tf*g = T(f*g)$. Since $f*g = g*f$ we have

$$Tf*g = T(f*g) = T(g*f) = f*Tg \qquad f, g \in W.$$

This proves (a) implies (b).

We shall now prove the converse. Suppose that

$$T(f*g) = Tf*g \qquad f, g \in W.$$

Let $f \in W$ and $\mu \in m$ be fixed. If $g \in W$, then

$$\langle Tf*g, \mu\rangle = \int_{-\infty}^{\infty} \langle \tau_x Tf, \mu\rangle g(x)dx$$

and

$$\langle T(f*g), \mu\rangle = \langle f*g, \lambda\rangle = \int_{-\infty}^{\infty} \langle \tau_x f, \lambda\rangle g(x)dx$$

$$\int_{-\infty}^{\infty} \langle T\tau_x f, \mu\rangle g(x)dx$$

Since $Tf*g = T(f*g)$, we have $\langle Tf*g, \mu\rangle = \langle T(f*g), \mu\rangle$

Hence

$$\int_{-\infty}^{\infty} \langle \tau_x T f, \mu \rangle g(x)\, dx = \int_{-\infty}^{\infty} \langle T \tau_x f, \mu \rangle g(x)\, dx$$

or equivalently

(19) $$\int_{-\infty}^{\infty} \langle \tau_x T f - T \tau_x f, \mu \rangle g(x)\, dx = 0$$

Since (8) holds for every $g \in W$, it follows that

$$\langle \tau_x T f - T \tau_x f, \mu \rangle = 0$$

for almost all x. Since $x \to \tau_x f$ is a continuous representation of R into W, it follows that

(20) $$\langle \tau_x T f - T \tau_x f, \mu \rangle = 0$$

for each $x \in R$ and each $\mu \in \mathfrak{M}$. This implies then that

$$T \tau_x = \tau_x T$$

This equation is valid for each $f \in W$ and hence

This completes the proof.

<u>DEFINITION 5.7.</u> For $f \in W$ and $\lambda \in \mathfrak{M}$, the convolution is defined by

$$f * \lambda (x) = \int_{-\infty}^{\infty} f(x-t)\, d\lambda (t)$$

It is clear that

$$|f * \lambda (x)| \leq 2 \|f\|_W \sup_R |\lambda|(I_R)$$

$f * \lambda$ is actually a bounded continuous function.

<u>THEOREM 5.8.</u> <u>Let</u> $T \in M(W)$. <u>Then there exists a unique</u> $\mu \in \mathfrak{M}$ <u>such that</u>

$$Tf = f * \mu \qquad f \in W$$

<u>Thus</u> $M(W,W)$ <u>is isomorphic to a proper subspace of</u> $\mathfrak{M}$ <u>and this subspace is obviously characterised by the property that</u> $f \rightarrow f*\mu$ <u>is bounded from</u> W <u>to</u> W.

<u>Proof.</u> Let $T \in M(W)$. If $f \in W$, the inequalities

$$|Tf(0)| \leq \|Tf\|_W \leq \|T\| \|f\|_W$$

imply that the mapping $f \rightarrow Tf(o)$ is a bounded linear functional on W. Hence there exists $\lambda \in \mathfrak{M}$ such that

$$Tf(o) = \int_{-\infty}^{\infty} f(t)\, d\lambda(t) = f * \tilde{\lambda}(o)$$

where $\tilde{\lambda}(E) = \lambda(-E)$. We set $\mu = \tilde{\lambda} \in \mathfrak{M}$. By the translation invariance of T, we have

$$Tf(x) = f * \mu(x)$$

Hence $Tf = f * \mu$ and the proof is completed.

CHAPTER 6

Multipliers on the space W_α

We shall now introduce a new class of functions which happens to be a subclass of the space W considered earlier and show that the structure and properties of this new class are similar to those obtained by Goldberg for the space W [8] .

Let $0 < \alpha < 1$ and let lip α denote the class of all continuous functions f on the real line R such that

$$\sup_{x \in R} |f(x+h) - f(x)| = 0 \, (|h|^\alpha) \text{ as } h \to 0.$$

For each $k = 0, \pm 1, \pm 2, \ldots$ let I_k denote the closed interval $[k, k+1]$. The space W_α then is the class of all those functions f in lip α such that

(1) $$\|f\|_{W_\alpha} = \sum_{k=-\infty}^{\infty} m_k(f)$$

is finite, where

$$m_k(f) = \max_{x \in I_k} \left\{ \max_{x \in I_k} |f(x)| , \sup_{x, x+h \in I_k} \frac{|\Delta_h f(x)|}{|h|^\alpha} \right\}$$

and

$$\Delta_h f(x) = f(x+h) - f(x).$$

It is easy to verify that (1) defines a norm on W_α and W_α is a normed linear space. If we put

$$n_k(f) = \max_{x \in I_k} |f(x)| + \sup_{x, x+h \in I_k} \frac{|\Delta_h f(x)|}{|h|^\alpha}$$

$$\|f\|_W = \sum_{k=-\infty}^{\infty} \max_{x \in I_k} |f(x)|$$

and

$$\|f\|_{\alpha} = \sum_{k=-\infty}^{\infty} \sup_{x,x+h\in I_k} \frac{|\Delta_h f(x)|}{|h|^{\alpha}}$$

it follows that

$$m_k(f) \le n_k(f) \le 2m_k(f)$$

and

$$\sum_{k=-\infty}^{\infty} n_k(f) = \|f\|_W + \|f\|_{\alpha}$$

Thus if we set

(2) $$|||f|||_{W_\alpha} = \sum_{k=-\infty}^{\infty} n_k(f)$$

we see that

$$\|f\|_{W_\alpha} \le |||f|||_{W_\alpha} \le 2\|f\|_{W_\alpha}$$

and the norms $\|\cdot\|_{W_\alpha}$ and $|||\cdot|||_{W_\alpha}$ are equivalent on W_α. Moreover $W_\alpha \subset W$ and

$$\|f\|_W \le \|f\|_{W_\alpha}$$

for each $f \in W_\alpha$.

<u>THEOREM 6.1.</u> W_α <u>is a Banach space under the norm (1).</u>

<u>Proof.</u> $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in W_α. Then, given $\varepsilon > 0$, there exists a positve integer N such that if $m,n > N$ then

(3) $$\|f_m - f_n\|_{W_\alpha} = \sum_{k=-\infty}^{\infty} m_k(f_m - f_n) < \varepsilon.$$

Hence for any fixed k, $\max_{x \in I_k} |f_m(x) - f_n(x)| \leq m_k(f_m - f_n) < \varepsilon$

and so $\max_{x \in R} |f_m(x) - f_n(x)| < \varepsilon$. Thus $\{f_n\}$ must converge uniformly to some continuous function f on $(-\infty, \infty)$. We thus have, for any k, because of the uniform convergence on I_k,

$$\max_{x \in I_k} |f(x)| = \max_{x \in I_k} \lim_{n \to \infty} |f_n(x)| = \lim_{n \to \infty} \max_{x \in I_k} |f_n(x)|$$

We claim that this function f belongs to W_α and that $f_n \to f$ in W_α . Now

$$\sup_{x,\, x+h \in I_k} \frac{|\Delta_h f(x)|}{|h|^\alpha} = \sup_{x,\, x+h \in I_k} \lim_{n \to \infty} \frac{|\Delta_h f_n(x)|}{|h|^\alpha}$$

$$= \lim_{n \to \infty} \sup_{x,\, x+h \in I_k} \frac{|\Delta_h f_n(x)|}{|h|^\alpha}$$

$$\leq \lim_{n \to \infty} m_k(f_n)$$

Thus we have

$$m_k(f) \leq \lim_{n \to \infty} m_k(f_n).$$

Using Fatou's lemma we obtain

$$\|f\|_{W_\alpha} = \sum_{k=-\infty}^{\infty} m_k(f) \leq \liminf_{n \to \infty} \sum_{k=-\infty}^{\infty} m_k(f_n)$$

$$\leq \liminf_{n \to \infty} \|f_n\|_{W_\alpha}$$

Since $\{f_n\}$ is a Cauchy sequence in W_α , it follows that the set of norms $\{\|f_n\|_{W_\alpha}\}$ is bounded and hence $f \in W_\alpha$.

To see that $f_n \to f$ in W_α ,we have for $m \geq N$

$$\max_{x \in I_k} |f_m(x) - f(x)| = \lim_{n\to\infty} \max_{x \in I_n} |f_m(x) - f_n(x)|$$

and

$$\leq \lim_{n\to\infty} m_k(f_m - f_n).$$

$$\sup_{x, x+h \in I_k} \frac{|\Delta_h f_m(x) - \Delta_h f(x)|}{|h|^\alpha} = \lim_{n\to\infty} \sup_{x,x+h\in I_k} \frac{|\Delta_h f_m(x) - \Delta_h f_n(x)|}{|h|^\alpha}.$$

$$\leq \lim_{n\to\infty} m_k(f_m - f_n)$$

Thus

$$m_k(f_m - f) \leq \lim_{n \to \infty} m_k(f_m - f_n)$$

and so

$$\|f_m - f\|_{W_\alpha} = \sum_{k=-\infty}^{\infty} m_k(f_m - f) \leq \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} m_k(f_m - f_n)$$

$$= \lim_{n\to\infty} \|f_m - f_n\| < \varepsilon$$

This completes the proof.

<u>THEOREM 6.2</u>. <u>If</u> $f \in L^1$ <u>and</u> $g \in W_\alpha$ <u>then</u> $f * g \in W_\alpha$ <u>and</u>

$$\|f*g\|_\alpha \leq 2\|f\|_1 \|g\|_{W_\alpha}$$

<u>Proof</u>. Let $u = f*g$ and for each k, we put

$$\lambda_k = \int_{I_k} |f(x)|dx \qquad \mu_k = \max_{x\in I_k} |g(x)|, \qquad \nu_k = \max_{x\in I_k} |u(x)|$$

$$\xi_k = \sup_{x, x+h \in I_k} \frac{|\Delta_h f(x)|}{|h|^\alpha} \qquad \eta_k = \sup_{x,x+h\in I_k} \frac{|\Delta_h u(x)|}{|h|^\alpha}$$

Since

$$u(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)dt$$

it follows that

(4) $$|u(x)| \le \sum_{k=-\infty}^{\infty} \int_{I_j} |f(t)||g(x-t)|\,dt \le \sum_{j=-\infty}^{\infty} \max_{t\in I_j} |g(x-t)|\,\lambda_j$$

Suppose $x \in I_k$. Then $x - I_j \subset I_k - I_j = I_{k-j+1} \cup I_{k-j}$ so that

$$\max_{t\in I_j} |g(x-t)| = \max_{t\in x-I_j} |g(t)| \le \max_{t\in I_{k-j+1}} |g(t)| + \max_{t\in I_{k-j}} |g(t)|$$
$$= \mu_{k-j+1} + \mu_{k-j}$$

Thus (4) gives

(5) $$\nu_k \le \sum_{j=-\infty}^{\infty} \lambda_j(\mu_{k-j+1} + \mu_{k-j})$$
$$\le \sum_{j=-\infty}^{\infty} \lambda_j(m_{k-j+1}(g) + m_{k-j}(g)$$

Since

$$\Delta_h u(x) = f * \Delta_h g(x)$$

the above argument applied to the function $\Delta_h g$ shows that

(6) $$\eta_k \le \sum_{j=-\infty}^{\infty} \lambda_j(\xi_{k-j+1} + \xi_{k-j})$$
$$\le \sum_{j=-\infty}^{\infty} \lambda_j(m_{k-j+1}(g) + m_{k-j}(g))$$

Thus from (4) and (5) we obtain

$$m_k(u) \le \sum_{j=-\infty}^{\infty} \lambda_j(m_{k-j+1}(g) + m_{k-j}(g))$$

so that

$$\|u\|_{W_\alpha} = \sum_{k=-\infty}^{\infty} m_k(u) \leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (m_{k-j+1}(g) + m_{k-j}(g))$$

$$\leq 2\left(\sum_{j=-\infty}^{\infty} \lambda_j\right)\left(\sum_{k=-\infty}^{\infty} m_k(g)\right)$$

$$= 2\|f\|_1 \|g\|_{W_\alpha}$$

Remark. Since $f \in W_\alpha$ implies $\|f\|_1 \leq \|f\|_{W_\alpha}$ it follows that if $N(f) = 2\|f\|_{W_\alpha}$ we have

$$N(f * g) \leq N(f)\, N(g)$$

for all $f, g \in W_\alpha$ showing that W_α is actually a Banach algebra.

THEOREM 6.3. *Every $f \in W_\alpha$ has a unique representation $f = u + v$ where $u, v \in W_\alpha$, $v(k) = 0$ for all $k = 0, \pm 1, \pm 2, \ldots$ and u is linear on each I_k.*

Proof. Let $f \in W_\alpha$. We define u by the formula

$$u(k) = f(k) \qquad k = 0, \pm 1, \pm 2, \ldots$$

and such that u is linear in each I_k. Define v now by setting $v = f - u$. Then $v(k) = 0$ for each k. In I_k, we have

$$u(x) = [(f(k+1) - f(k))](x-k) + f(k)$$

so that

$$\frac{|\Delta_h u(x)|}{|h|^\alpha} = |f(k+1) - f(k)|\, |h|^{1-\alpha} \leq |f(k+1) - f(k)| \leq 2m_k(f)$$

and

$$\max_{x \in I_k} |u(x)| = \max\{|f(x)|, |f(k+1)|\} \leq m_k(f).$$

Thus

$$m_k(u) \leq 2\, m_k(f)$$

which shows that $u \in W_\alpha$. Since $f, u \in W$ and $v = f-u$ it follows that $v \in W_\alpha$ and $v(k) = 0$ for all $k = 0, \pm 1, \pm 2, \ldots$ The representation is clearly unique.

Notation: Let D_k denote the closed triangle given by

$$\left\{ (x,h) : k \le x \le k+1, \ 0 \le h \le k+1-x \right\}$$

in the (x,h)-plane.

THEOREM 6.4. Every continuous linear functional L on W can be represented by

$$(7) \qquad L(f) = \int_{-\infty}^{\infty} f d\lambda + \int_D \frac{\Delta_h f(x)}{|h|^\alpha} d\mu(x,h)$$

where λ is a measure on $(-\infty, \infty)$ satisfying

$$(8) \qquad |\lambda|(I_k) \le M \qquad k = 0, \pm 1, \pm 2, \ldots$$

and μ is a measure $D = \cup D_k$ such that

$$(9) \qquad |\mu|(D_k) \le M \qquad k = 0, \pm 1, \pm 2, \ldots$$

for some $M > 0$.

Proof. The previous theorem asserts that W_α is a direct sum of two subspaces U and V where U is the set of all $u \in W_\alpha$ such that u is linear in each I_k and V is the set of all $v \in W_\alpha$ such that $v(k) = 0$ for all integers k. To obtain the representation of a continuous linear functional on W_α we consider continuous linear functionals on each of the subspaces U and V. If $u \in V$, we have

$$|u(k)| \le \max_{x \in I_k} |u(x)| \le m_k(u) \le |u(k)| + |u(k+1)|$$

and so

$$(10) \qquad \sum_{k=-\infty}^{\infty} |u(k)| \le \|u\|_{W_\alpha} \le 2 \sum_{k=-\infty}^{\infty} |u(k)|$$

u is completely determined by the sequence $\{u(k)\}_{k=-\infty}^{\infty}$. Moreover the inequalities (10) show that $\|\cdot\|_{W_\alpha}$ on U is equivalent to the ℓ^1 norm whose conjugate space is ℓ^∞.

Thus if L is a continuous linear functional on U there exists a bounded sequence $\{\varphi_k\}_{k=-\infty}^{\infty}$ such that

(11) $$L(u) = \sum_{k=-\infty}^{\infty} u(k)\varphi_k$$

We now define a measure ν on $(-\infty,\infty)$ concentrated on the integers with mass φ_k at the point k. That is $\nu(\{k\}) = \varphi_k$. Then (11) becomes

(12) $$L(u) = \int_{-\infty}^{\infty} u\,d\nu \qquad u \in U$$

Since $\{\varphi_k\}$ is a bounded sequence we have

$$|\nu|(I_k) = |\varphi_k| + |\varphi_{k+1}| \leq M \quad k = 0, \pm 1, \pm 2 \cdots$$

for some $M_1 > 0$.

We now look at the linear functionals on the subspace V. We do this by combining the idea of de Leeuw [15] with the method of Goldberg [8]. Given $v \in V$, let v_k be the function that agrees with v on I_k and is 0 outside of I_k. Then $v = \sum_{k=-\infty}^{\infty} v_k$. Let V_k be the space of continuous functions on I_k with W_α-norm which vanish at the end points of I_k and belong to lip α in I_k. Let Λ_k be the disjoint union $I_k \cup H_k$ where

$$H_k = \{(x,h) : k \leq x \leq k+1,\ 0 < h \leq k+1-x\}$$

Then Λ_k is a locally convex space. To each function $v_k \in V_k$ we define a function $\tilde{v}_k$ on Λ_k by

$$\tilde{v}_k(x) = v_k(x) \qquad x \in I_k$$

$$\tilde{v}_k(x,h) = \frac{\Delta_h v_k(x)}{|h|^\alpha} \qquad (x,h) \in H_k$$

Then it is easy to verify that $\|v_k\|_{W_\alpha} = \sup_{y \in \Lambda_k} |\tilde{v}_k(y)|$. Let $C_0(\Lambda_k)$ denote the Banach space of continuous functions on Λ_k which vanish at infinity with sup norm, then the mapping $j : V_k \longrightarrow C_0(\Lambda_k)$ given by $jv_k = \tilde{v}_k$ is a linear isometry of V_k with norm $\|\cdot\|_{W_\alpha}$ into $C_0(\Lambda_k)$ with sup norm on Λ_k. If φ is a continuous linear functional on V_k then it can be treated as a functional on the isometric image $j(V_k) \subset C_0(\Lambda_k)^*$. The Hahn Banach theorem provides the existence of an extension $\Phi \in C_0(\Lambda_k)^*$ such that $\|\varphi\| = \|\Phi\|$. By the Riesz tepresentation theorem there is a corresponding regular Borel measure on Λ_k with $\|\Phi\|$ = tot. var. μ and

$$\varphi(f) = \Phi(\tilde{f}) = \int_{\Lambda_k} \tilde{f}\, d\mu \qquad f \in V_k$$

Hence if L is a continuous linear functional on V its restriction to V_k is a continuous linear functional on V_k and hence there exists a measure μ_k on V_k such that

$$(13) \qquad L(v_k) = \int_{I_k} v_k\, d\mu_k + \int_{H_k} \frac{\Delta_h v_k(x)}{|h|^\alpha}\, d\mu_k(x,h)$$

for all $v_k \in V_k$. If we put $\eta_k = \mu_k|_{I_k}$ then (13) can be written as

$$(14) \qquad L(v_k) = \int_{I_k} v_k\, d\eta_k + \int_{H_k} \frac{\Delta_h v_k(x)}{|h|^\alpha}\, d\mu_k(x,h)$$

where η_k is a measure concentrated on I_k and μ_k is a measure concentrated on H_k. If $v \in V$, then $v \in \sum_{k=-\infty}^{\infty} V_k$ so that

$$L(v) = \sum_{k=-\infty}^{\infty} \int_{I_k} v_k \, d\eta_k + \sum_{k=-\infty}^{\infty} \int_{H_k} \frac{\Delta_h v(x)}{|h|^\alpha} \, d\mu_k(x,h)$$

Thus we define two measure η and μ as follows. Let η be a measure on $(-\infty,\infty)$ which agrees with η_k on I_k and has 0 mass at the integers. We define μ to be the measure on D which agrees with μ_k on H_k and assigns to I_k zero mass. Then we have

(15) $$L(v) = \int_{-\infty}^{\infty} v \, d\eta + \int_D \frac{\Delta_h f(x)}{|h|^\alpha} \, d\mu(x,h)$$

We now assert that η and μ have the following properties: there exists $M_2 > 0$ such that

$$|\eta|(I_k) \leq M_2 \qquad k = 0, \pm 1, \pm 2, \ldots$$
$$|\mu|(D_k) \leq M_2 \qquad k = 0, \pm 1, \pm 2, \ldots$$

It is sufficient to prove $|\mu|(\Lambda_k) \leq M_2$, for $k = 0, \pm 1, \pm 2, \ldots$ Suppose not. Then there exists a sequence $k_1, k_2, \ldots$ such that $|\mu|(\Lambda_{k_n}) > n$. By Riesz representation theorem, the norm of μ_{k_n} regarded as a linear functional on V_{k_n} is equal to $|\mu|(\Lambda_{k_n})$. Hence there exists $v_{k_n} \in V_{k_n}$ with $\|v_{k_n}\|_{W_\alpha} = \|\tilde{v}_{k_n}\|_{\Lambda_{k_n}} = 1$ such that

$$|L(v_{k_n})| = \left|\int \tilde{v}_{k_n} \, d\mu_{k_n}\right| > n/2$$

Now, let $v \in \sum_{n=1}^{\infty} \frac{1}{n^2} v_{k_n}$. Then $\|v\|_{W_\alpha} = \sum_{k=-\infty}^{\infty} \|v_k\|_{W_\alpha}$ $= \sum_{n=1}^{\infty} \frac{1}{n^2} \|v_{k_n}\|_{\Lambda_{k_n}} < \infty$ so that $v \in V$. But

$$L(v) = \sum_{n=1}^{\infty} \frac{1}{n^2} L(v_{k_n}) > \sum_{n=1}^{\infty} \frac{1}{n^2}\frac{n}{2} = \sum_{n=1}^{\infty} \frac{1}{2n} = \infty$$

which is a contradiction since L is a bounded linear functional on V. This proves our assertion.

If L is a continuous linear functional on W_α it follows that L restricted to V satisfies

$$L(v) = \int_{-\infty}^{\infty} v\,d\eta + \int_{\Delta} \frac{\Delta_h v(x)}{|h|^\alpha}\,d\mu(x,h) \qquad v \in V$$

and L restricted to U satisfies

$$L(u) = \int_{-\infty}^{\infty} u\,d\nu \qquad u \in U$$

Moreover there exists constants $M_1, M_2 > 0$ such that

$$|\eta|(I_k) \le M_2, \quad |\nu|(I_k) \le M_1 \quad \text{and} \quad |\mu|(D_k) \le M_2$$

for all $k = 0, \pm 1, \pm 2, \ldots$

Thus if $f \in W_\alpha$, then $f = u+v$ where $u \in U$ and $v \in V$ so that

$$L(f) = L(u)+L(v) = \int_{-\infty}^{\infty} u\,d\nu + \int_{-\infty}^{\infty} v\,d\eta + \int_D \frac{\Delta_h v(x)}{|h|^\alpha}\,d\mu(x,h)$$

and so

(16) $$L(f) = \int_{-\infty}^{\infty} f\,d\eta + \int_D \frac{\Delta_h f(x)}{|h|^\alpha}\,d\mu(x,h) + \int_{-\infty}^{\infty} u\,d\nu$$

$$- \int_{-\infty}^{\infty} u\,d\nu - \int_{-\infty}^{\infty} u\,d\eta - \int_D \frac{\Delta_h f(x)}{|h|^\alpha}\,d\mu(x,h)$$

Now from the inequalitües

$$\left|\int_{-\infty}^{\infty} u\,d\eta + \int_D \frac{\Delta_h u(x)}{|h|^\alpha}\,d\mu(x,h)\right| \le \int_{-\infty}^{\infty} |u||d\eta| + \int_D \frac{|\Delta_h u(x)|}{|h|^\alpha}\,d|\mu(x,h)|$$

$$\le \sum_{k=-\infty}^{\infty}\left(\int_{I_k} |u||d\eta| + \int_{D_k} \frac{|\Delta_h u(x)|}{|h|^\alpha}\,|d\mu(x,h)|\right)$$

$$\le \sum_{k=-\infty}^{\infty}\left(\max_{x\in I_k}|u(x)|\cdot|\eta|(I_k) + \sup_{x,\,x+h\in I_k}\frac{|\Delta_h u(x)|}{|h|^\alpha}|\mu|(D_k)\right)$$

$$\le \sum_{k=-\infty}^{\infty}\left(m_k(u)\,M_2 + m_k(u)\,M_2\right) = 2M_2\|u\|_{W_\alpha}$$

it follows that the mapping

$$u \to \int_{-\infty}^{\infty} u\,d\eta + \int_D \frac{\Delta_h u(x)}{|h|^\alpha}\,d\mu(x,h)$$

is a bounded linear functional on U so that

$$\int_{-\infty}^{\infty} u\,d\eta + \int_D \frac{\Delta_h u(x)}{|h|^\alpha}\,d\mu(x,h) = \int_{-\infty}^{\infty} u\,d\nu_1$$

where ν_1 is a measure concentrated on the integers and $|\nu_1|(I_k) \le M_1$, for all k. Thus from (16) we obtain

$$(17)\quad L(f) = \int_{-\infty}^{\infty} f\,d\eta + \int_D \frac{\Delta_h f(x)}{|h|^\alpha}\,d\mu(x,h) + \int_{-\infty}^{\infty} u\,d(\nu-\nu_1)$$

But $\nu-\nu_1$ is a measure concentrated on the integers and ϑ vanishes at the integers. Therefore

$$(18)\quad \int_{-\infty}^{\infty} \vartheta\,d(\nu-\nu_1) = 0$$

Hence, adding (17) and (18), we get

$$L(f) = \int_{-\infty}^{\infty} f\,d\eta + \int_D \frac{\Delta_h f(x)}{|h|^\alpha}\,d\mu(x,h) + \int_{-\infty}^{\infty} f\,d(\nu-\nu_1)$$

and finally we have

$$L(f) = \int_{-\infty}^{\infty} f\,d\eta + \int_D \frac{\Delta_h f(x)}{|h|^\alpha}\,d\mu(x,h)$$

where $\lambda = \eta + \nu - \nu_1$. Since

$$|\lambda|(I_k) \le |\eta|(I_k) + |\nu|(I_k) + |\nu_1|(I_k) \le M_2 + M_1 + M_1$$

We take $M = M_2 + M_1 + M_1$ and the proof is completed.

We have thus shown that if L is a continuous linear functional on W_α, then there exists a pair (λ, μ) of measures where λ is a measure on $(-\infty, \infty)$ such that $|\lambda|(I_k) \le M$ for $k = 0, \pm 1, \pm 2, \ldots$ and μ is defined on D such that $|\mu|(D_k) \le M_2$ for all k and L has a representation

$$L(f) = \int_{-\infty}^{\infty} f\,d\lambda + \int_D \frac{\Delta_h f(x)}{|h|^\alpha}\,d\mu(x,h).$$

Moreover if L satisfies the above conditions then it is easy to see that L is a bounded linear functional on W_α.

Let J denote the set of all such pairs of measures.

DEFINITION 6.5. A multiplier on W_α is a bounded linear operator on W_α which commutes with translations and let $M(W_\alpha)$ denote the set of all multipliers on W_α.

THEOREM 6.6. *If* $T \in M(W_\alpha)$, *then* $Tf * g = T(f * g)$ *for all* $f, g \in W$.

Proof. Let $T \in M(W_\alpha)$. We first notice that if $t \in R$, $f, g \in W_\alpha$ and Δ_h denotes the difference operator then we have

$$\tau_t \Delta_h = \Delta_h \tau_t$$

and

$$\Delta_h (f * g) = \Delta_h f * g = f * \Delta_h g$$

If $(\lambda, \mu) \in J$ and $f \in W_\alpha$ we use the notation

$$\langle f, \lambda \rangle = \int_{-\infty}^{\infty} f(t)\, d\lambda(t)$$

and

$$\langle\langle \Delta_h f, \mu \rangle\rangle = \int_D \frac{\Delta_h f(x)}{|h|^\alpha}\, d\mu(x,h)$$

Then

(19) $\langle f, (\lambda,\mu) \rangle = \langle f, \lambda \rangle + \langle\langle \Delta_h f, \mu \rangle\rangle$

Now if $f, g \in W_\alpha$ then

$$\langle f * g, \lambda \rangle = \int_R \langle \tau_x f, \lambda \rangle\, g(x)\, dx .$$

and

$$\begin{aligned}
\langle\langle \Delta_h f * g, \mu \rangle\rangle = \langle\langle \Delta_h f * g, \mu \rangle\rangle &= \int_D \frac{(\Delta_h f * g)(x)}{|h|^\alpha}\, d\mu(x,h) \\
&= \int_D \frac{1}{|h|^\alpha}\, d\mu(x,h) \int_R \Delta_h f(x-t)\, g(t)\, dt \\
&= \int_R g(t)\, dt \left(\int_D \frac{\Delta_h \tau_t f(x)}{|h|^\alpha}\, d\mu(x,h) \right) \\
&= \int_R g(t)\, dt \left(\int_D \frac{\tau_t \Delta_h f(x)}{|h|^\alpha}\, d\mu(x,h) \right) \\
&= \int_R \langle\langle \tau_t \Delta_h f, \mu \rangle\rangle\, g(t)\, dt
\end{aligned}$$

Thus if $f, g \in W_\alpha$ and $(\lambda,\mu) \in J$, then we have

(20)
$$\begin{aligned}
\langle f * g, (\lambda,\mu) \rangle &= \langle f * g, \lambda \rangle + \langle\langle \Delta_h f * g, \mu \rangle\rangle \\
&= \int_R \langle \tau_t f, \lambda \rangle\, g(t)\, dt + \int_R \langle\langle \tau_t \Delta_h f, \mu \rangle\rangle\, g(t)\, dt \\
&= \int_R \langle \tau_t f, (\lambda,\mu) \rangle\, g(t)\, dt
\end{aligned}$$

Let $(\lambda,\mu) \in J$ be fixed. Then the inequalities

$$|\langle Tf, (\lambda,\mu)\rangle| \leq \|Tf\|_{W_\alpha}\|(\lambda,\mu)\| \leq \|T\|\,\|f\|_{W_\alpha}\|(\lambda,\mu)\|$$

shows that the mapping $f \to \langle Tf, (\lambda,\mu)\rangle$ is bounded linear functional on W_α and hence there exists a pair $(\beta,\gamma) \in J$ such that

(21) $$\langle Tf, (\lambda,\mu)\rangle = \langle f, (\beta,\gamma)\rangle$$

for all $f \in W_\alpha$. Now let $f, g \in W_\alpha$ and $(\lambda,\mu) \in J$. Then

$$\begin{aligned}\langle Tf*g, (\lambda,\mu)\rangle &= \int_R \langle \tau_t Tf, (\lambda,\mu)\rangle g(t)\,dt \\ &= \int_R \langle T\tau_t f, (\lambda,\mu)\rangle g(t)\,dt = \int_R \langle \tau_t f, (\beta,\gamma)\rangle g(t)\,dt \\ &= \langle f*g, (\beta,\gamma)\rangle = \langle T(f*g), (\lambda,\mu)\rangle\end{aligned}$$

Thus we have $\langle Tf * g, (\lambda,\mu)\rangle = \langle T(f * g), (\lambda,\mu)\rangle$ for each $(\lambda,\mu) \in J$ and hence

$$T(f * g) = Tf * g$$

This completes the proof.

<u>DEFINITION 6.7</u>. If $f \in W_\alpha$ and $(\lambda,\mu) \in J$, we define a function $f \circ \mu$ on R by the formula

$$f\circ\mu(x) = \int_D \frac{f(x-t-h) - f(x-t)}{|h|^\alpha}\,d\mu(t,h)$$

It is clear that $f\circ\mu$ exists and

$$f\circ\mu(x) = \int_D \frac{\Delta_h \tau_t f(x)}{|h|^\alpha}\,d\mu(t,h)$$

<u>THEOREM 6.8</u>. <u>If</u> $T \in M(W_\alpha)$, <u>then there exists a pair</u> $(\lambda,\mu) \in J$ <u>such that</u>

$$Tf = f * \lambda + f\circ\mu$$

<u>where</u> $f * \lambda$ <u>denotes the convolution product defined earlier</u>.

Proof. From the boundedness of T it follows that

$$|Tf(0)| \leq \|Tf\|_{W_\alpha} \leq \|T\| \, \|f\|_{W_\alpha}$$

and hence the mapping $f \to Tf(o)$ is a bounded linear functional on W_α. Hence there exists a pair (λ_1, μ_1) such that

$$Tf(0) = \int_R f(t)\, d\lambda_1(t) + \int_D \frac{\Delta_h f(t)}{|h|}\, d\mu_1(t,h)$$

If we set $\tilde{\lambda}_1(E) = \lambda_1(-E)$ and $\tilde{\mu}_1(E \times H) = \mu_1(-E \times H)$ then

$$Tf(0) = f * \tilde{\lambda}_1(0) + f \odot \tilde{\mu}(0)$$

We take $\lambda = \tilde{\lambda}_1$ and $\mu = \tilde{\mu}_1$. Then $(\lambda, \mu) \in J$ and

$$Tf(0) = f * \lambda(0) + f \odot \mu(0)$$

The translation invariance of T then gives

$$Tf(x) = f * \lambda(x) + f * \mu(x)$$

for each $x \in R$. Thus

$$Tf = f * \lambda + f \odot \mu$$

This completes the proof.

CHAPTER 7.

A space of functions of Zygmund

Let $0 < \alpha < 2$. Let Λ_α denote the class of all continuous complex valued functions f on the real line R with period 1 such that there exists a constant K satisfying the condition

$$(1) \quad \sup_{x \in R} |f(x+t) - 2f(x) + f(x-t)| \leq K.\ |t|^\alpha \quad \text{as } t \to 0$$

We denote by λ_α the subset of Λ_α consisting of those functions f which satisfy the condition

$$(2) \quad \sup_{x \in R} \left| \frac{f(x+t) = 2f(x) + f(x-t)}{t} \right| \to 0 \quad \text{as } t \to 0$$

We set

$$\|f\|_\infty = \sup_{x \in R} |f(x)|$$

$$\|f\|_\alpha = \sup_{x, t \in R} \left| \frac{f(x+t) - 2f(x) + f(x-t)}{t^\alpha} \right|$$

and define

$$\|f\| = \max \left\{ \|f\|_\infty , \|f\|_\alpha \right\}$$

for each f in Λ_α. Then we have

THEOREM 7.1. *Λ_α is a Banach space with norm $\|\cdot\|$ and λ_α is a closed linear subspace of Λ_α.*

Proof. It is easy to verify that $\|\cdot\|$ is actually a norm on Λ_α. We shall prove only the completeness. Let $\{f_n\}$ be a Cauchy sequence in Λ_α. Then $\|f_m - f_n\| \to 0$ as $m, n \to \infty$. This implies that $\|f_m - f_n\|_\infty \to 0$ and $\|f_m - f_n\|_\alpha \to 0$ as

$m, n \to \infty$. Now for each $x \in R$, $|f_m(x) - f_n(x)| \le \| f_m - f_n \| \to 0$ so that $\{ f_n(x) \}$ is a Cauchy sequence of complex numbers and hence there exists $f(x)$ such that $f_n(x) \to f(x)$ as $n \to \infty$. We define f by setting

$$f(x) = \lim_{n \to \infty} f_n(x)$$

We claim that $f \in \Lambda_\alpha$ and $\| f_n - f \| \to 0$ as $n \to \infty$. Now

$$\sup_{x \in R} |f_n(x) - f(x)| = \sup_{x \in R} \lim_{m \to \infty} |f_n(x) - f_m(x)|$$

$$= \lim_{m \to \infty} \sup_{x \in R} |f_n(x) - f_m(x)|$$

$$\le \lim_{m \to \infty} \| f_n - f_m \|_\infty \to 0 \text{ as } n \to \infty.$$

Thus $\| f_n - f \|_\infty \to 0$ as $n \to \infty$. If $x, y \in R$, then

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

$$2 \cdot \| f - f_n \|_\infty + | f_n(x) - f_n(y) | .$$

Now given $\varepsilon > 0$, we can choose n_0 such that $\| f - f_{n_0} \| < \varepsilon/3$ and using the continuity of f_{n_0} we can find a $\delta > 0$ such that $|x - y| < \varepsilon$ implies $|f_{n_0}(x) - f_{n_0}(y)| < \varepsilon/3$. Then for this δ, we have

$$|f(x) - f(y)| \le 2 \cdot \| f - f_{n_0} \| + |f_{n_0}(x) - f_{n_0}(y)| < 2 \cdot \varepsilon/3 + \varepsilon/3 = \varepsilon$$

This shows the continuity of f. It is clear that f is of period of 1. To see that $f \in \Lambda_\alpha$, it remains to show that $\| f \|_\alpha \to \infty$. In fact

$$\sup_{x \in R} |f(x+t) - 2f(x) + f(x-t)| = \sup_{x \in R} \lim_{n \to \infty} |f_n(x+t) - 2f_n(x) + f_n(x-t)|$$

$$= \lim_{n \to \infty} \sup_{x \in R} |f_n(x+t) - 2f_n(x) + f_n(x-t)|$$

$$\lim_{n \to \infty} \| f_n \|_\alpha \cdot |t|^\alpha .$$

Since $\{f_n\}$ is a Cauchy sequence there exists $K > 0$ such that $\|f_n\|_\alpha < K$ for all n. Hence

$$\|f\|_\alpha \leq K.$$

and $f \in \Lambda_\alpha$. Now let us set $\Delta_t^2 f(x) = |f(x+t)-2f(x) + f(x-t)|$. Then

$$\sup_{x\in R} |\Delta_t^2 f_m(x) - \Delta_t^2 f(x)| = \sup_{x \in R} \lim_{n\to\infty} |\Delta_t^2 f_m(x) - \Delta_t^2 f_n(x)|$$

$$= \lim_{n\to\infty} \sup_{x\in R} |\Delta_t^2 f_m(x) - \Delta_t^2 f_n(x)|$$

$$\leq \lim_{n\to\infty} \|f_m - f_n\|_\alpha |t|^\alpha.$$

so that

$$\|f_m - f\|_\alpha \leq \lim_{n\to\infty} \|f_m - f_n\|_\alpha$$

from which it follows that $\|f_m - f\|_\alpha \to 0$ as $m \to \infty$. Thus we have proved that both $\|f_n - f\|_\infty$ and $\|f_n - f\|_\alpha \to$ as $n \to \infty$. Hence $\|f_n - f\| \to 0$ as $n \to \infty$ and Λ_α is a complete normed linear space. It is clear that if $f_n \in \lambda_\alpha$ so does f which implies that λ_α is a closed linear subspace of Λ_α.

<u>THEOREM 7.2.</u> <u>Let</u> $0 < \alpha < \alpha' < 2$. <u>Denote by</u> $\|\cdot\|$ <u>and</u> $\|\cdot\|'$ <u>the norms in</u> Λ_α <u>and</u> λ_α <u>respectively. Then</u>

i) $\Lambda_{\alpha'} \subset \lambda_\alpha$

ii) $\|f\| \leq 4^{(\alpha'-\alpha)/\alpha} \|f\|' \qquad f \in \Lambda_{\alpha'}$

<u>Proof.</u> Let $f \in \Lambda_{\alpha'}$ and let $x \in R$. Suppose $|t|^\alpha < 4$. Then

$$(*) \qquad \frac{|\Delta_t^2 f(x)|}{|t|^\alpha} = \frac{|\Delta_t^2 f(x)|}{|t|^{\alpha'}} \cdot |t|^{\alpha'-\alpha} \leq \|f\|_{\alpha'} \cdot |t|^{\alpha'-\alpha}$$

On the other hand, if $|t|^\alpha \geq 4$ then

$$\frac{|\Delta_t^2 f(x)|}{|t|^\alpha} \leq \frac{1}{4} |\Delta_t^2 f(x)| \leq \|f\|_\infty < \|f\|'$$

Since $1 < 4^{(\alpha'-\alpha)/\alpha}$, we may combine the last two results to obtain

$$\|f\|_\alpha \le 4^{(\alpha'-\alpha)/\alpha} \|f\|'$$

Since $\|f\|_\infty \le 4^{(\alpha'-\alpha)/\alpha} \|f\|'$. We conclude that

$$\|f\| \leqq 4^{(\alpha'-\alpha)/\alpha} \|f\|'$$

From the equation (*), we have

$$\frac{\Delta_t^2 f(x)}{|t|^\alpha} \le \|f\|_{\alpha'} |t|^{\alpha'-\alpha}$$

Letting $t \longrightarrow 0$, we see that $f \in \lambda_\alpha$.

Now consider the evaluation functionals φ_x defined for each $x \in R$ by

$$\varphi_x(f) = f(x) \qquad f \in C(R)$$

<u>LEMMA 7.3. For each</u> $x \in R$, φ_x <u>is bounded linear functional on</u> Λ_α <u>with</u> $\|\varphi_x\| \le 1$.

<u>Proof.</u> If $f \in \Lambda_\alpha$, then $|\varphi_x(f)| = |f(x)| \le \|f\|_\infty \le \|f\|$

Hence $\|\varphi_x\| \le 1$.

<u>LEMMA 7.4. For each pair</u> x, t <u>in</u> R, <u>we have</u>

$$\|\varphi_{x+t} - 2\varphi_x + \varphi_{x-t}\| \le |t|^\alpha$$

<u>Proof.</u> Let x, $t \in R$ be fixed. For each $f \in \Lambda_\alpha$,

$$|\varphi_{(x+t)} - 2\varphi_x + \varphi_{x-t})(f)| = |f(x+t) - 2f(x) + f(x-t)|$$
$$\le \|f\| \, |t|^\alpha$$

Hence $\|\varphi_{x+t} - 2\varphi_x + \varphi_{x-t}\| \le |t|^\alpha$.

For each F in the dual space $(\lambda_\alpha)^{**}$ of $(\lambda_\alpha)^*$, we define a function $\hat{F}$ on R by

$$\hat{F}(x) = F(\varphi_x) \qquad x \in R$$

If f is in λ_α and F_f is its image under canonical imbedding of λ_α in $(\lambda)_\alpha^{**}$, the function F_f is simply f.

LEMMA 7.5. If F is a functional in $(\lambda_\alpha)^{**}$, then the function $\hat{F} \in \Lambda_\alpha$ and $\|\hat{F}\| \leq \|F\|$

Proof. Let $F \in (\lambda_\alpha)^{**}$. Since $\varphi_x \in (\lambda)_\alpha^*$ for each $x \in R$, $\hat{F}(x)$ is well defined and

$$|\hat{F}(x)| = |F(\varphi_x)| \leq \|F\| \|\varphi_x\| \leq \|F\|$$

Hence $\|\hat{F}\|_\infty \leq \|F\|$. If $x, t \in R$, then

$$\begin{aligned} |\Delta_t^2 \hat{F}(x)| &= |F(\varphi_{x+t} - 2\varphi_x + \varphi_{x-t})| \\ &\leq \|F\| \|\varphi_{x+t} - 2\varphi_x + \varphi(x-t)\| \\ &\leq \|F\| |t|^\alpha \end{aligned}$$

Thus $\|\hat{F}\|_\alpha \leq \|F\|$ and $\|\hat{F}\| \leq \|F\|$.

We next identify the continuous linear functionals of λ_α by constructing an isomorphic imbedding of λ_α into a space of continuous functions with the sup norm. Let $U = \{ y : -1 \leq y \leq 0 \}$ and $V = \{ (x,t) : 0 \leq x \leq 1,\ 0 < t < \tfrac{1}{2} \}$. The disjoint union $U \cup V$ is denoted by W and it is locally compact topological space. Let $C_0(W)$ denote the Banach space of complex valued continuous functions on W which vanish at infinity with the norm

$$\|g\|_W = \sup_{x \in W} |g(x)| \qquad g \in C_0(W)$$

For $f \in \lambda_\alpha$ define $jf = \tilde{f}$ where

$$\tilde{f}(u) = f(u) \qquad u \in U$$

$$\tilde{f}(v) = \tilde{f}(s,t) = \frac{\Delta_t^2 f(s)}{|t|^\alpha} \qquad (s,t) \in V.$$

LEMMA 7.6. j is a linear isometry of λ_α with norm $\|\cdot\|$ into $C_0(W)$ with sup norm $\|\cdot\|_W$ on W.

Proof. It is clear that j is a linear mapping of λ_α into $C_0(W)$. If $f \in \lambda_\alpha$, f has period 1, so

$$\|f\|_\infty = \sup\{|f(u)| : u \in R\} = \sup\{|f(u)| : u \in U\}$$

and

$$\|f\|_\alpha = \sup\left\{\frac{|\Delta_t^2 f(s)|}{|t|^\alpha} : s, t \in R\right\} = \sup\left\{\frac{|\Delta_t^2 f(s)|}{|t|^\alpha} : (s,t) \in V\right\}$$

and hence $\|f\| = \|\tilde{f}\|_W$

LEMMA 7.7. For every $\varphi \in (\lambda_\alpha)^*$, there is a measure μ on W such that

$$i)\ \varphi(f) = \int_W \tilde{f}\, d\mu \qquad f \in \lambda_\alpha$$

$$ii)\ \|\varphi\| = \text{tot Var}\, \mu = |\mu|(W).$$

Proof. Let $\varphi \in (\lambda_\alpha)^*$. Treating φ as a functional on the isometric image $j(\lambda_\alpha) \subset C_0(W)$, the Hahn Banach theorem provides the existence of an extension $\Phi \in C_0(W)^*$ such that $\|\varphi\| = \|\Phi\|$. By the Riesz representation theorem there is a corresponding regular Borel measure μ on W with $\|\Phi\|$ = tot.var.μ and

$$\Phi(\tilde{f}) = \varphi(f) = \int_W \tilde{f}\, d\mu$$

Notation. Let $M(U)$ denote the space of (regular Borel) measures on U. Every $\mu \in M(U)$ determine a bounded linear functional $\varphi_\mu \in (\lambda_\alpha)^*$ by

$$\Phi_\mu(f) = \int f\,d\mu \qquad f \in \lambda_\alpha .$$

We now define two subspaces of $(\lambda_\alpha)^*$. Let

$$E_m^* = \{ \varphi_\mu \in (\lambda_\alpha)^* : \mu \in M(U) \}$$

and

$$\{ \varphi_\mu \in (\lambda_\alpha)^* : \mu \in M(U), \mu \text{ has a finite support} \}$$

Thus if $\varphi \in E_p^*$, then $\varphi = \sum_{i=1}^{n} \beta_i \varphi_{x_i}$ for some $\beta_1, \beta_2 \cdots \in C$ and some $x_1, x_2 \cdots x_n \in U$

<u>LEMMA 7.8.</u> E_m^* <u>is norm dense in</u> $(\lambda_\alpha)^*$

<u>Proof.</u> Let $\varphi \in (\lambda_\alpha)^*$. Let μ be a measure on W such that

$$\varphi(f) = \int_W \tilde{f}\,d\mu \qquad f \in \lambda_\alpha .$$

Let $\{W_n\}$ be an increasing sequence of compact sets whose union is W. That is $W = \bigcup_{n=1}^{\infty} W_n$ and

$$W_1 \subset W_2 \subset W_3 \subset \ldots .$$

For each positive integer n, define

$$\varphi_n(f) = \int_{W_n} \tilde{f}\,d\mu \qquad f \in \lambda_\alpha .$$

It now suffices to prove that

i) $\|\varphi - \varphi_n\| \to 0$ as $n \to \infty$

and ii) $\varphi_n \in E_m^*$

If $f \in \lambda_\alpha$, then

$$|(\varphi - \varphi_n)(f)| = \left| \int_{W \setminus W_n} \tilde{f}\,d\mu \right|$$

$$\leq \|\tilde{f}\|_W \cdot |\mu|(W \setminus W_n) = \|f\|\,|\mu|(W \setminus W_n)$$

Hence $\| \varphi - \varphi_n \|_\infty \leq |\mu|(W \setminus W_n)$. Since μ is countably additive and $W = \bigcup_{n=1}^{\infty} W_n$, the right hand side of the inequality tends to zero as $n \rightarrow \infty$. So $\| \varphi - \varphi_n \| \rightarrow 0$

To see (ii) , consider $f \in \lambda_\alpha$. Then

$$\varphi_n(f) = \int_{U \cap W_n} f(x) d\mu(x) + \int_{V \cap W_n} \tilde{f}(s,t) d\mu(s,t)$$

The second integral can be written as

$$\int_{V \cap W_n} \tilde{f}(s,t)\, d\mu(s,t) = \int_{V \cap W_n} \frac{\Delta_t^2 f(s)}{|t|^\alpha}\, d\mu(s,t)$$

$$= \int_{V \cap W_n} \frac{f(s+t)}{|t|^\alpha} d\mu(s,t) - 2\int_{V \cap W_n} \frac{f(s)}{|t|^\alpha} d\mu(s,t) + \int_{V \cap W_n} \frac{f(s-t)}{|t|^\alpha} d\mu(s,t)$$

Since $|t|^\alpha$ is bounded away from zero on $V \cap W_n$, there exist measures ν_1, ν_2, ν_3 such that

$$\int_{V \cap W_n} f(s+t)|t|^{-\alpha} d\mu(s,t) = \int_U f(s)\, d\nu_1(s)$$

$$\int_{V \cap W_n} f(s)|t|^{-\alpha} d\mu(s,t) = \int_U f(s)\, d\nu_2(s)$$

$$\int_{V \cap W_n} f(s-t)|t|^{-\alpha} d\mu(s,t) = \int f(s)\, d\nu_3(s)$$

Combining the preceeding equalities, we obtain

$$\varphi_n(f) = \int f(x)\, d(\mu + \nu_1 - 2\nu_2 + \nu_3)(x)$$

Thus $\varphi_n \in E_m^*$.

<u>LEMMA 7.9.</u> E_p^* <u>is norm dense in</u> $(\lambda_\alpha)^*$.

LEMMA 7.9. E_p^* is norm dense in $(\lambda_\alpha)^*$

Proof. Let $\varphi \in (\lambda_\alpha)^*$ and $\varepsilon > 0$. By Lemma 6 , we can choose a $\mu \in M(U)$ such that $\|\varphi - \varphi_\mu\| < \varepsilon/2$. Thus for every $f \in \lambda_\alpha$ we have

$$\varphi_\mu(f) = \int_U f \, d\mu .$$

Let μ denote the norm of μ as an element of $C(U)^*$ and let $S = \{g : g = f|_U , \|f\| \leq 1$ be the unit ball of λ_α. Since S is an equicontinuous family of bounded functions on U, S is conditionally compact in the topology of uniform convergence. Choose a finite set $T = \{g_1, g_2, \dots, g_n\}$ of functions in S such that the spheres $B(g_i ; \varepsilon/8\|\mu\|^*)$ cover S. Here $B(g_0, \delta) = \{g \in S . \|g - g_0\| \leq \delta\}$. The closed sphere $\Sigma_\mu = \eta \in C(U)^*$: $\|\eta\|^* \leq \|\mu\|^*$ is weak* - compact. By Krein Milman theorem, Σ_μ is the weak* - closure of the convex hull of its ~~xx~~ extreme points which are easily seen to be

$$\mathrm{Ext}(\Sigma_\mu) = \{ \varepsilon\|\mu\|^* \varphi_x : x \in U, |\varepsilon| = 1\} \text{ (See [2]).}$$

Thus measures with finite support are weak* dense in Σ_μ. The weak neighbourhood $N(\mu, g_1, g_2, \dots, g_n, \varepsilon/4)$ contains a measure $\eta \in \Sigma_\mu$ with finite support, that is,

i) $\|\eta\|^* \leq \|\mu\|^*$

ii) $\left|\int_U g_i \, d\mu - \int_U g_i \, d\eta\right| < \varepsilon/4 \qquad g_i \in T$

iii) $\eta = \sum_1^n \beta_i \varphi_{x_i}$

Denote by φ_η the functional given by $\varphi_\eta(f) = \int_U f d\eta$ $(f \in C(U))$. Now consider $g \in S$. Chose $g_i \in T$ such that $\|g - g_i\|_\infty \leq \dfrac{\varepsilon}{8\|\mu\|^*}$. Then

$$
\begin{aligned}
|(\varphi_\mu - \varphi_\eta)(g)| &= |\int_U g\,d\mu - \int_U g\,d\eta| \\
&\le |\int g\,d\mu - \int g_i\,d\mu| + |\int g_i\,d\eta - \int g_i\,d\eta| \\
&\quad + |\int g_i\,d\eta - \int g\,d\eta| \\
&\le \|g-g_i\|_\infty \|\mu\|^* + \varepsilon/4 + \|g-g_i\|_\infty \|\eta\|^* \\
&\le \varepsilon/8 + \varepsilon/4 + \varepsilon/8 = \varepsilon/2 .
\end{aligned}
$$

For any $f \in \lambda_\alpha$ with $\|f\| \le 1$, we get $g = f/_U$. Then

$$|\varphi_\mu - \varphi_\eta(f)| \le |(\varphi_\mu - \varphi_\eta)(g)| < \varepsilon/2$$

Hence $\|\varphi_\mu - \varphi_\eta\| \le \varepsilon$. Since ε is arbitrary, we conclude that E_p^s is norm dense in $(\lambda_\alpha)^*$

COROLLARY 7.10. The mapping $F \longrightarrow \hat{F}$ of $(\lambda_\alpha)^{**}$ into Λ_α onex to one.

Proof. Since the mapping is linear, it is enough to consider $F \in \lambda_\alpha^{**}$ such that $\hat{F} = 0$. If F is the zero function, then F vanishes on the set of point evaluations φ_x and hence on its closed linear span $(\lambda_\alpha)^*$. But then F is the zero functional. Thus the mapping $F \rightarrow \hat{F}$ is one to one.

LEMMA 7.11. The mapping $F \rightarrow \hat{F}$ of $(\lambda_\alpha)^{**}$ into Λ_α is onto and norm preserving

Proof. To prove the mapping is onto, let $g \in \lambda_\alpha$. We construct an F in $(\lambda_\alpha)^{**}$ such that $\hat{F} = g$. For this we convolute g with the Fegir's kernel

$$K_n(x) = \frac{2}{n+1}\left(\frac{\sin(n+1)\pi x}{\sin \pi x}\right)^2$$

so that the convolution $K_n * g$ is the nth (C,1) partial sum of the Fourier series of g and these converge uniformly to g. That is

(4) $$\lim_{n \to \infty} K_n * g(x) = g(x)$$

Moreover K_n is positive and

(5) $$\int_0^1 K_n(x)\,dx = 1$$

Then we have

$$\Delta_t^2 (K_n * g)(x) = \int_0^1 K_n(x-u)\,\Delta_t^2 g(u)\,du$$

so that

$$\| K_n * g\|_\alpha \le \int_0^1 K_n(x)dx.\, \| g\|_\alpha = \| g \|_\alpha$$

This shows that $K_n * g \in \Lambda_\alpha$. Now $K_n * g$ being a trigonometric polynomial, we have

$$\sup_{x \in R} \frac{|\Delta_t^2 K_n * g(x)|}{|t|^\alpha} = O(|t|^\alpha)$$

Since $\alpha < 2$, we have

$$\sup_{x \in R} \frac{|\Delta_t^2 K_n * g(x)|}{|t|^\alpha} \to 0 \text{ as } t \to 0$$

This shows that $K_n * g \in \lambda_\alpha$. We shall denote by F_n the functional in $(\lambda_\alpha)^{**}$ corresponding to $K_n * g$ under the canonical imbedding of λ_α in $(\lambda_\alpha)^{**}$. This means that

(7) $\quad F_n(\varphi) = \varphi(K_n * g) \qquad \varphi \in (\lambda_\alpha)^*$

Since the imbedding of (λ_α) in its second dual is an isometry, we obtain from (6)

(8) $$\|F_n\| = \|K_n * g\|_\alpha \leq \|g\|_\alpha$$

If we set $g_n = K_n * g$, we have proved that $\{g_n\}$ is a sequence of functions in λ_α such that

(i) $\sup_n \|g_n\|_\alpha < \infty$

(ii) $\lim_{n \to \infty} g_n(x)$ exists for each $x \in R$

Moreover F_n is the canonical image of g_n in $(\lambda_\alpha)^{**}$. We assert now that if $\varphi \in (\lambda_\alpha)^*$, then $\{F_n(\varphi)\}$ is a cauchy sequence of complex numbers. To prove our assertion let $M = \sup_n \|g\|_\alpha$ and let $\varepsilon > 0$ be given. Suppose $\varphi \in (\lambda_\alpha)^*$. Choose $\varphi_p \in E_p^*$ such that $\|\varphi - \varphi_p\| \leq \varepsilon/4M$. Thus $\varphi_p = \sum_1^r \beta_i \varphi_{x_i}$ for some complex numbers $\beta_1, \beta_2 \cdots \beta_r$ and some $x_1, x_2, \ldots x_r \in R$. If m and n are two positive integers, then we have

$$\begin{aligned}
|F_n(\varphi) - F_m(\varphi)| &= |\varphi(g_n - g_m)| \\
&\leq |(\varphi - \varphi_p)(g_n - g_m)| + |\varphi_p(g_n - g_m)| \\
&\leq \|\varphi - \varphi_p\| \|g_n - g_m\| + \left|\sum_{i=1}^{r} \beta_i [g(x_i) - g_m(x_i)]\right| \\
&\leq \frac{\varepsilon}{4M} \cdot 2M + \left(\sum_1^r |\beta_i|\right) \max_{1 \leq i \leq r} |g_n(x_i) - g_m(x_i)|
\end{aligned}$$

Choose an integer N such that $m, n > N$ implies

$$|g_n(x_i) - g_m(x_i)| < \frac{\varepsilon}{2(\sum_1^r |\beta_r|)} \quad \text{for } i = 1, 2, \ldots r.$$ Then if $m, n > N$, we have

$$|F_n(\varphi) - F_m(\varphi)| \leq \varepsilon/_2 + \varepsilon/_2 = \varepsilon$$

Since ε is arbitrary, $\{F_n(\varphi)\}$ is a cauchy sequence of complex numbers for each $\varphi \in (\Lambda_\alpha)^*$. Define

$$F(\varphi) = \lim_{n\to\infty} F_n(\varphi)$$

Since

$$\|F\| \le \limsup_{n\to\infty} \|f_n\| = \limsup_{n\to\infty} \|g_n\|_\alpha \le \|g\|_\alpha$$

We have $F \in (\Lambda_\alpha)^{**}$. On the other hand, since for each $x \in R$

$$\hat{F}(x) = F(\varphi_x) = \lim_{n\to\infty} F_n(\varphi_x) = \lim_{n\to\infty} g_n(x) = g(x)$$

we see that $\hat{F} = g$. We have thus proved that the mapping $F \longrightarrow \hat{F}$ is onto. From (9) it follows that $\|F\| \le \|\hat{F}\|_\alpha \le \|\hat{F}\|$. Thus to complete the proof, it remains to show that $\|\hat{F}\|_\alpha \le \|F\|$.

For each $y \in R$

$$|\hat{F}(y)| = |F(\varphi_y)| \le \|F\| \, \|\varphi_y\| \le \|F\|$$

so that

(9) $$\|\hat{F}\|_\infty \le \|F\|$$

Moreover for any $x, t \in$ in R we gave

$$|\Delta_t^2 \hat{F}(x)| = |F(\varphi_{x+t}) - 2F(\varphi_x) + F(\varphi_{x-t})|$$

$$= |F(\varphi_{x+t} - 2\varphi_x + \varphi_{x-t})|$$

$$\le \|F\| \, \|\varphi_{x+t} - 2\varphi_x + \varphi_{x-t}\| \le \|F\| \, |t|^\alpha$$

Thus

(10) $$\|\hat{F}\|_\alpha \le \|F\|$$

From (9) ~~ts~~ and (10), it follows that $\|\hat{F}\| \le \|F\|$. This completes the proof.

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