SPLINES IN HILBERT SPACES

THESIS

SUBMITTED TO

THE UNIVERSITY OF MADRAS

FOR

THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

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APRIL 1973

ACKNOWLEDGEMENTS

This thesis is based on the work done during the years 1969-1973 under the guidance of Professor K.R.Unni, MATSCIENCE, The Institute of Mathematical Sciences, Madras. It is my pleasant privilege to acknowledge my deep gratitude to Professor K.R.Unni for his inspiring guidance and constant encouragement and for proposing a C.S.I.R. scheme under which I had been awarded a junior research fellowship. I am also thankful to Professor Alladi Ramakrishnan, Director of the Institute of Mathematical Sciences, for providing excellent facilities for research.

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CONTENTS

INTRODUCTION

Polynomial splines, as defined by I.J.Schoenberg ([18]), consist of pieces of polynomials joined together at certain partition points of a closed interval of the real line. In recent years, the concept of 'spline' has been generalized in various directions. Schoenberg ([19]) himself initiated the departure by introducing the notion of trigonometric splines. Ahlberg, Nilson and Walsh ([1] , [2]) studied the properties of splines associated with certain differential operators. In 1964, Schoenberg ([20]) established a <u>minimal</u> property of the polynomial spline interpolating prescribed values at the partition points. Motivated by this optimal property of the polynomial spline, Marc Atteia ([5] , [6]) set the theory of splines in a Hilbert space framework.

Let X and Y be two real Hilbert spaces. Suppose that T is a continuous linear transformation of X onto Y and Φ is a prescribed set of elements of X, called the constraint set. For various choices of X, Y, T and Φ Atteia ([7] , [8] , [9]), Anselone and Laurent ([4]), Laurent ([16]), and Jerome and Schumaker ([15]) defined a generalized 'interpolating spline' σ as an element of X satisfying

$$\|T\sigma\|_Y = \min\{\|T\phi\|_Y : \phi \in \Phi\}.$$

The concept of a generalized 'smoothing spline' ([7]) also originated from an extremal property of the polynomial spline given in ([21]).

In this thesis, we study the properties of splines in a Hilbert space under weaker assumptions than those used by earlier authors. Our assumptions, though simpler, are quite sufficient to establish the existence of a minimal element for different types of constraint sets, the minimal element being the image of a set of splines. We prove the existence of, not just one (as was generally supposed) but, two distinct classes of splines. Various interesting and new results for both interpolating and smoothing splines are obtained. In many cases (for example see [4] , [7] , [8] , [9]) the existence theorems also imply the uniqueness of the spline. But in our case, the situation is more general and the problem of uniqueness is to be distinguished from that of existence. The significance of certain compact, convex sets is brought out. An ordered class of constraint sets giving rise to an ordered class of splines is also constructed.

Chapter I deals with the reduction of the constraint sets appearing in [4], [7], [8], [9], [15] and [16] to one of the following forms: (1) the translate of a closed subspace, (2) the union over a closed and convex set of translates of a closed subspace, (3) the set of elements whose orthogonal projections on a closed subspace is a singleton and (4) the set of elements whose orthogonal projections on a closed subspace belong to a closed and convex set.

In the second chapter, the notion of a generalized interpolating spline is introduced. The existence of two classes of such splines and the uniqueness criteria are discussed. Various properties of the two classes are investigated.

In chapter III, we have proved the existence of a minimal element in a constraint set derived from compact, convex sets of a Hilbert space, the minimal element being the image of a set of interpolating splines. Interpolating splines belonging to a variety of constraint sets are characterised and the problem of existence is studied in dual forms.

Chapter IV is concerned with the study of smoothing splines. We show that every smoothing spline is also an interpolating spline and obtain two classes of smoothing splines coinciding with the two classes of interpolating splines. The explicit relationship is given connecting the two classes of splines through certain operators associated with interpolating and smoothing splines.

A further generalisation of the concept of interpolating spline is given in the last chapter. An analogue of the existence theorem related to compact, convex sets is proved.

CHAPTER I

REDUCTION OF CONSTRAINT SETS TO CONVENIENT FORMS

1.1. Introduction:

For the study of splines in Hilbert spaces, various constraint sets were used by different authors. In this chapter, we show that most of these constraint sets can be brought to one of the following forms: (1) the translate of a closed subspace, (2) the union over a closed and convex set of translates of a closed subspace, (3) the set of elements whose orthogonal projection on a closed subspace is a singleton and (4) the set of elements whose projections on a closed subspace belong to a closed and convex subset of the subspace. This, in fact, is also a motivation for investigating splines under weaker assumptions in the later chapters.

1.2. Definition of a spline:

We shall first recall the definition of a spline in some special cases.

a) Polynomial spline: For each positive integer n, let $C^n[a,b]$ denote the class of all functions defined on the closed interval $[a,b]$ having continuous n^{th} derivatives and let Π_n denote the class of polynomials of degree not exceeding n. If $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a,b]$, then the polynomial spline of degree n, as defined by I.J.Schoenberg in 1946 ([18])

is such that $s(x) \in \Pi_n$ in each of the intervals (x_i, x_{i+1}), $0 \le i \le n$ and $s(x) \in C^{n-1}[a,b]$.

b) <u>Splines in $H^q[a,b]$</u> : Let us consider the space $H^q = H^q[a,b]$ consisting of these real valued functions defined on $[a,b]$ such that its $(q-1)^{st}$ derivative is absolutely continuous and its q^{th} derivative is square integrable on $[a,b]$. $H^q[a,b]$ becomes a Hilbert space if we define the inner product by

$$\langle f, g \rangle_{H^q} = \sum_{j=0}^{q} \int_a^b f^{(j)}(t)\, g^{(j)}(t)\, dt \,, \quad f, g \in H^q[a,b].$$

We denote by $L_2 = L_2[a,b]$ the Lebesgue space of square integrable functions with the usual inner product

$$\langle f, g \rangle_{L_2} = \int_a^b f(t)\, g(t)\, dt \qquad f, g \in L_2[a,b]$$

Then the corresponding norms are given by

$$\|f\|^2_{H^q} = \langle f, f \rangle_{H^q} \quad \text{and} \quad \|f\|^2_{L_2} = \langle f, f \rangle_{L_2}$$

If $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $r = (r_1, \dots, r_n) \in \mathbb{R}^n$, we define the <u>constraint set</u> Φ by

$$\Phi = \{ f \in H^q : f(x_i) = r_i \;,\; 1 \le i \le n \}$$

where the x_i's are the partition points of $[a, b]$. For $n \geq q$, Schoenberg ([20], [21]) proved the existence and uniqueness of a polynomial spline σ of degree 2q-1 satisfying

$$\|\sigma^{(q)}\|_{L_2} = \min_{f \in \Phi} \|f^{(q)}\|_{L_2} \tag{1.2.1}$$

and of a polynomial spline s of degree $2q-1$ satisfying

$$\|s^{(q)}\|^2_{L_2} + \rho \|s_x - r\|^2_{\mathbb{R}^n} = \min_{f \in H^q} \left\{ \|f^{(q)}\|^2_{L_2} + \rho \|f_x - r\|^2_{\mathbb{R}^n} \right\}, \rho > 0 \tag{1.2.2}$$

where $f_x = (f(x_1), \ldots, f(x_n))$ and $s_x = (s(x_1), \ldots, s(x_n))$.

σ and s are respectively called the interpolating spline and the smoothing spline.

c) $\mathcal{L}g$ -spline: Consider the linear differential operator $\mathcal{L}$ of the form

$$\mathcal{L} = \sum_{j=0}^{q} a_j \left(\frac{d}{dx}\right)^j, \quad a_q(x) \neq 0 \text{ on } [a,b],\ a_j \in C^j[a,b], \quad 0 \leq j \leq q.$$

Denote by $\mathcal{H}^q = \mathcal{H}^q[a,b]$ the Hilbert space of real-valued functions $f \in C^{q-1}[a,b]$ such that $f^{(q-1)}$ is absolutely continuous and $\mathcal{L}f \in L_2[a,b]$ with the inner product

$$\langle f, g\rangle_{H^q} = \sum_{j=0}^{q-1} f^{(j)}(a)\, g^{(j)}(a) + \int_a^b \mathcal{L}f\mathcal{L}g \; .$$

Let Λ be a finite sequence of linearly independent continuous linear functionals on $\mathcal{H}^q$. The $\mathcal{L}g$-spline of Jerome and Schumaker ([15]) is an element s of $\mathcal{H}^q$ solving the following minimisation problem

$$\|\mathcal{L}s\|_{L_2} = \min_{f \in U(r)} \|\mathcal{L}f\|_{L_2}$$

where

$$U(r) = \{ f \in \mathcal{H}^q \; : \; \lambda_i f = r_i, \; 1 \le i \le n \}, \; r = (r_1, r_2, \dots r_n) \in \mathbb{R}$$

d) Splines in an abstract Hilbert space: The notion of a generalized spline in abstract Hilbert spaces was motivated by the minimal properties given in (1.2.1) and (1.2.2). If T is a continuous linear transformation of X onto Y, where X and Y are real Hilbert spaces, an interpolating spline for a given constraint set Φ relative to T is an element $\sigma \in X$ satisfying the minimal property

$$\|T\sigma\|_Y = \min_{\phi \in \Phi} \|T\phi\|_Y \; .$$

Various types of constraint sets Φ were considered by Atteia ([7] , [8] , [9]), Anselone and Laurent [4] ,

Laurent ([16]).

Let us now consider a third Hilbert space Z and a continuous linear transformation T^{1} of X onto Z. If z is any element of Z, the smoothing spline, as defined in [4] , [7] and [9] is an element s of X satisfying

$$\|Ts\|_Y^2 + \rho \|T^1 s - z\|_Z^2 = \min_{x \in X} \left(\|Tx\|_Y^2 + \rho \|T^1 x - z\|_Z^2 \right), \quad \rho > 0$$

1.3 Constraint sets:

We shall collect below the different constraint sets used by Atteia ([7] , [8] , [9]), Anselone and Laurent ([4]), Laurent ([16]) and Jerome and Schumaker ([15]).

Let ℓ_2 be the Hilbert space of square summable sequences of real numbers. If $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are in ℓ_2, then the inner product is given by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$

Let X, Y and Z be three real Hilbert spaces, T and T^{1} continuous linear transformations of X onto Y and Z respectively. Consider two sets of linearly independent continuous linear functionals on X represented by $k_i \in X \ (1 \leq i \leq n)$ and $\ell_j \in X \ (j = 1, 2, \ldots)$

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\alpha = (\alpha_1, \ldots \alpha_n)$ and $\beta = (\beta_1, \ldots \beta_n)$ with $\alpha_i \leq \beta_i \ (1 \leq i \leq n)$ be prescribed elements of $\mathbb{R}^n$, $\rho = (\rho_1, \rho_2, \ldots)$ an element of the Hilbert space ℓ_2, z an arbitrary element of Z and Γ a closed and convex subset of Z. The constraint sets considered by Atteia ([7], [8], [9]) are the following

$$\Phi_\gamma = \{\phi \in X : \langle k_i, \phi \rangle_X = \gamma_i, \ 1 \leq i \leq n\} \tag{1.3.1}$$

$$\Phi_\rho = \{\phi \in X : \langle \ell_i, \phi \rangle_X = \rho_i, \ i = 1, 2, \ldots\} \tag{1.3.2}$$

$$\Phi_z = \{\phi \in X : T'\phi = z\} \tag{1.3.3}$$

$$\Phi_{\alpha\beta} = \{\phi \in X : \alpha_i \leq \langle k_i, \phi \rangle_X \leq \beta_i, \ 1 \leq i \leq n\} \tag{1.3.4}$$

$$\Phi_\Gamma = \{\phi \in X : T'\phi \in \Gamma\} \tag{1.3.5}$$

The constraint sets of the form

$$\Phi_{-\infty\gamma} = \{\phi \in X : \langle k_i, \phi \rangle_X \leq \gamma_i, \ 1 \leq i \leq n\} \tag{1.3.6}$$

were considered by P.J.Laurent ([16]). Jerome and Schumaker [15] have used the constraint sets

$$\mathcal{U}(\gamma) = \{ f \in \mathcal{H}^q : \lambda_i f = \gamma_i, \ 1 \le i \le n \} \qquad (1.3.7)$$

and

$$\mathcal{U}(\underline{\gamma}, \bar{\gamma}) = \{ f \in \mathcal{H}^q : \underline{\gamma_i} \le \lambda_i f \le \bar{\gamma}_i, 1 \le i \le n \} \qquad (1.3.8)$$

1.4. The reduction of the constraint sets to convenient forms:

The constraint sets stated above can be represented in one of the four forms mentioned in the introduction of this chapter. We first observe that the sets (1.3.7) and (1.3.8) can be brought into the form (1.3.1) and (1.3.3) using the Riesz representation theorem. It is therefore sufficient to consider the sets (1.3.1) to (1.3.6).

Let us consider (1.3.1). Let K be the subspace spanned by $\{k_i\}_{i=1}^n$ and $\{h_i\}_1^n$ a base of K such that $\langle k_i, h_j \rangle = \delta_{ij}$. Then $\Phi_\gamma = h_\gamma + K^\perp$ where $h_\gamma = \sum_{i=1}^n \gamma_i h_i$ and $K^\perp$ is the orthogonal complement of K in X (see [7]). Thus Φ_γ is the translate of a closed subspace. Also $\Phi_\gamma = \{ \phi \in X : P_K \phi = h_\gamma \}$, P_K being the projection operator mapping X onto K, or in other words, Φ_γ is the set of elements of X whose orthogonal projection on K is precisely h_γ. The set (1.3.2) can be represented analogously.

To consider the set of the type (1.3.3), we denote by $N(T')$ the kernel of (T') and by $K(T')$ its cokernel. Then

$$\Phi_z = \{ \phi \in X : T'\phi = z \} = k_z + N(T')$$
$$= \{ \phi \in X : P_{K(T')}\phi = k_z \}$$

where k_z is the unique element of $K(T')$ such that $T'k_z = z$ and $P_{K(T')}$ is the projection operator mapping X onto $K(T')$. Thus Φ_z can also be represented in the forms (1) and (3) mentioned in the introduction.

Now consider $\Phi_{\alpha\beta}$, the constraint set given by (1.3.4). It is easy to see that

$$\Phi_{\alpha\beta} = \bigcup_{\gamma \in [\alpha,\beta]} (h_\gamma + K^\perp) \qquad \text{where } h_\gamma = \sum_{i=1}^{n} \gamma_i k_i$$

Let $C = \{ h_\gamma : \gamma \in [\alpha,\beta] \}$. Since $\gamma \longrightarrow h_\gamma$ is linear and continuous, C is a compact and convex subset of X. Then

$$\Phi_{\alpha\beta} = \bigcup_{h_\gamma \in C} (h_\gamma + K^\perp) = \{ \phi \in X : P_K\phi \in C \}$$

Thus $\Phi_{\alpha\beta}$ is the union over a compact, convex subset of translates of a subspace, or equivalently, the set of elements in X whose orthogonal projections on K belong to the compact convex subset C of K.

Now consider the constraint set (1.3.5). Φ_Γ reduces to

$$\Phi_\Gamma = \bigcup_{\gamma\in\Gamma} \{\phi \in X : T'\phi = \gamma\}$$
$$= \bigcup_{\gamma\in\Gamma} (k_\gamma + N(T')), \quad k_\gamma \in K(T')$$
$$\text{and } T'(k_\gamma) = \gamma$$

$$= \bigcup_{k_\gamma\in D} (k_\gamma + N(T')) \quad \text{where } D = \{k_\gamma : \gamma\in\Gamma\}$$
$$= \{\phi\in X : P_{K(T')}\phi \in D\}$$

If Γ is compact and convex, then D is also compact and convex. If Γ is just closed and convex, then so is D. In either case, Φ_Γ is the union over a closed and convex set of translates of $N(T')$ and has an equivalent representation in terms of the projection operator $P_{K(T')}$.

The set $\Phi_{-\infty\gamma}$ has the representation

$$\Phi_{-\infty\gamma} = \bigcup_{\gamma\in[-\infty,\gamma]} (h_\gamma + K^\perp) \quad \text{where } h_\gamma = \sum_{i=1}^{n} \gamma_i h_i$$

$$= \bigcup_{h_\gamma\in E} (h_\gamma + K^\perp) \quad \text{where } E = \{h_\gamma : \gamma \in (-\infty, \gamma]\}$$

Now E is closed and convex. Thus $\Phi_{-\alpha r}$ has been reduced to the form (2) mentioned in the introduction. Further,

$$\Phi_{-\alpha r} = \{\phi \in X : P_K \phi \in E\}$$

and hence is the set of elements of X whose projections on K belong to the closed and convex subset E of K.

CHAPTER II

INTERPOLATING SPLINES WHEN THE CONSTRAINT SET IS THE TRANSLATE OF A CLOSED SUBSPACE

2.1. Introduction:

We have observed in the previous chapter that the constraint sets can be brought to one of the convenient forms. We now define an interpolating spline when the constraint set is the translate of a closed subspace and study the problem of its existence and uniqueness. The existence of two classes of interpolating splines under a simple condition is established and their properties are investigated. A sequence of sets is constructed such that the existence of an interpolating spline in one set implies the existence of an interpolating spline in the succeeding set.

The following notation will be adopted throughout this chapter: If H is a real Hilbert space, $\|\cdot\|_H$, $\langle , \rangle_H$ and θ_H denote the norm, inner product and zero element respectively in H. If M is a closed subspace of H, then $M^{\perp}$ denotes its orthogonal complement in H and P_M the projection operator which takes H onto M. If T is a transformation of H onto another Hilbert space H', then $N(T)$ and $K(T)$ denote the kernel and co-kernel respectively of T.

2.2. Definition and existence of interpolating splines:

If M is a closed subspace of the Hilbert space H and $h \in H$, we denote by $\Phi(h; M)$ the translate of M by h, i.e.,

$$\Phi(h; M) = h + M$$

DEFINITION 2.2.1. Let T be a continuous linear transformation of a Hilbert space H onto a Hilbert space H'. Suppose that M is a closed subspace of H and $m \in M$. If there exists an element $s \in \Phi(m; M^{\perp})$ satisfying

$$\|Ts\|_{H'} = \min\{\|T\phi\|_{H'} : \phi \in \Phi(m; M^{\perp})\}$$

then s is called an interpolating spline of $\Phi(m; M^{\perp})$ relative to T.

We now have the following

THEOREM 2.2.2. Let X be a real Hilbert space and suppose that Y and Z are two Hilbert spaces isomorphic to two closed subspaces A and B respectively of X, the isomorphisms being denoted by I_A and I_B. Let $T = I_A P_A$ and $\tau = I_B P_B$. If

$$A^{\perp} + B^{\perp} \text{ is closed} \tag{2.2.1}$$

then for each $a \in A$ and $b \in B$, there exist two sets S_a and Σ_b of interpolating splines satisfying

$$\| \tau s_a \|_Z = \min \{ \| \tau \phi \|_Z \,;\, \phi \in \Phi(a\,; A^{\perp}) \} \quad \underline{\text{for all}}\ s_a \in S_a$$

and

$$\| T \sigma_b \|_Y = \min \{ \| T \phi \| : \phi \in \Phi(b\,; B^{\perp}) \} \quad \underline{\text{for all}}\ \sigma_b \in \Sigma_b .$$

In order to prove theorem 2.2.2 we need the following

LEMMA 2.2.3. ([9] p.195). *If G is a closed and convex subset of a real Hilbert space H and T is a continuous linear transformation of H onto a real Hilbert space H′, then T(G) is closed if and only if G + N (T) is closed.*

PROOF OF THEOREM 2.2.2. Under the hypotheses of theorem 2.2.2, it is clear that $N(T) = A^{\perp}$ and $N(\tau) = B^{\perp}$. Since $A^{\perp} + B^{\perp}$ is closed, taking $G = B^{\perp}$ in lemma 2.2.3, we see that $TB^{\perp}$ is a closed subspace of Y. Similarly $\tau A^{\perp}$ is a closed subspace of Z. Hence if $a \in A$ and $b \in B$, we have

$$\| \tau a - P_{\tau A^{\perp}}(\tau a) \|_Z = \min_{x^{\perp} \in A^{\perp}} \| \tau a - \tau(x^{\perp}) \|_Z$$

and

$$\| Tb - P_{TB^\perp}(Tb) \|_Y = \min_{x^\perp \in B^\perp} \| Tb - T(x^\perp) \|_Y$$

The required sets S_a and Σ_b are then given by

$$S_a = a - (A^\perp)_a$$

and

$$\Sigma_b = b - (B^\perp)_b$$

where

$$(A^\perp)_a = \{ x^\perp \in A^\perp ; \ \tau(x^\perp) = P_{\tau A^\perp}(\tau a) \}$$

and

$$(B^\perp)_b = \{ x^\perp \in B^\perp ; \ T(x^\perp) = P_{TB^\perp}(Tb) \}$$

Then, for each $s_a \in S_a$ and $\sigma_b \in \Sigma_b$, we have

$$\| \tau s_a \|_Z = \min \{ \| \tau \phi \|_Z : \phi \in \Phi(a ; A^\perp) \}$$

and

$$\| T \sigma_b \|_Y = \min \{ \| T \phi \|_Y : \phi \in \Phi(b ; B^\perp) \}$$

This completes the proof of the theorem.

It will be assumed throughout this chapter that X, A, B, Y, Z, T and τ satisfy the conditions of theorem 2.2.2.

REMARK 2.2.4. We notice that the condition (2.2.1) is satisfied if either $A^{\perp}$ or $B^{\perp}$ is of finite dimension (see [14]). Hence it follows from theorem 2.2.2, that interpolating splines relative to a continuous linear transformation whose kernel is finite-dimensional always exist.

The following theorem gives a necessary and sufficient condition for an interpolating spline to be unique.

THEOREM 2.2.5. <u>A necessary and sufficient condition for</u> S_a <u>and</u> Σ_b <u>to reduce to a single element each is that</u> $A^{\perp} \cap B^{\perp} = \{\theta_x\}$

PROOF. We shall prove that S_a reduces to a single element if and only if $A^{\perp} \cap B^{\perp} = \{\theta_x\}$. The proof for the case Σ_b is similar. Assume that $A^{\perp} \cap B^{\perp} = \{\theta_x\}$ We shall show that S_a reduces to a single element. It is enough to show that $(A^{\perp})_a$ reduces to a singleton. If not, suppose there exists $a_1^{\perp}$, $a_2^{\perp}$ in $(A^{\perp})_a$ with $a_1^{\perp} \neq a_2^{\perp}$ Then $\tau a_1^{\perp} = \tau a_2^{\perp}$ which implies that $a_1^{\perp} - a_2^{\perp} \in B^{\perp} = N(\tau)$. On the other hand $a_1^{\perp} - a_2^{\perp} \in A^{\perp}$. Thus $A^{\perp} \cap B^{\perp} \neq \{\theta_x\}$ which gives a contradiction.

Now suppose that S_a reduces to a single element. Then $(A^{\perp})_a$ consists of only one element $x_0^{\perp} \in A^{\perp}$ and

$\tau(x_0^\perp) = P_{\tau A^\perp}(\tau a)$. If $A^\perp \cap B^\perp \neq \{\theta_x\}$ then there exists a non-zero element $x_1^\perp \in A^\perp \cap B^\perp$. Now $\tau(x_0^\perp + x_1^\perp) = \tau(x_0^\perp) + \tau(x_1^\perp) = \tau(x_0^\perp)$. Thus $x_0^\perp + x_1^\perp$ belongs to $(A^\perp)_a$ and is different from $x_0^\perp$. This is impossible since $(A^\perp)_a$ consists of a single element. Hence $A^\perp \cap B^\perp = \{\theta_x\}$

2.3. Properties of interpolating splines:

We now define the two classes S and Σ of interpolating splines, as promised in the introduction, by setting

$$S = \bigcup_{a \in A} S_a \tag{2.3.1}$$

and

$$\Sigma = \bigcup_{b \in B} \Sigma_b \tag{2.3.2}$$

THEOREM 2.3.1. *The class S of interpolating splines has the following properties*

(i) $\tau S = (\tau A^\perp)^\perp$

(ii) $S = (\tau^* \tau A^\perp)^\perp$

(iii) $\tau^* \tau S = A \cap B$

where τ^* *is the adjoint operator of* τ.

PROOF. We first notice that the adjoint operator τ^* of τ exists and is a linear continuous, one-to-one map of Z onto B.

(i) Let $s \in S$. Then there exists $a \in A$ such that $s \in S_a$. By the definition of S_a, it follows that $\tau s \in (\tau A^{\perp})^{\perp}$. This proves that $\tau S \subset (\tau A^{\perp})^{\perp}$. To prove the converse, we proceed as follows: Let $z \in (\tau A^{\perp})^{\perp}$. Since there exists $x \in X$ such that $z = \tau x$, we have $\langle \tau x, \tau a^{\perp} \rangle = 0$ for all $a^{\perp} \in A^{\perp}$. Putting $x = a_x + a_x^{\perp}$ with $a_x \in A$ and $a_x^{\perp} \in A^{\perp}$, it is seen that $\langle \tau a_x + \tau a_x^{\perp}, \tau a^{\perp} \rangle_Z = 0$ for all $a^{\perp} \in A^{\perp}$. This implies that $(-\tau a_x^{\perp})$ is the projection of τa_x on $\tau A^{\perp}$ so that $z = \tau x \in \tau S_{a_x} \in \tau S$ Hence $(\tau A^{\perp})^{\perp} \subset \tau S$.

We shall now prove (ii). Let $s \in S$. Since $\tau S = (\tau A^{\perp})^{\perp}$ (by (i)), we have

$$\langle \tau s, \tau a^{\perp} \rangle_Z = 0 \quad \text{for all } a^{\perp} \in A^{\perp}$$

or equivalently,

$$\langle s, \tau^* \tau a^{\perp} \rangle_X = 0 \quad \text{for all } a^{\perp} \in A^{\perp}$$

Hence $s \in (\tau^* \tau A^{\perp})^{\perp}$. This proves $S \subset (\tau^* \tau A^{\perp})^{\perp}$

If $x \in (\tau^* \tau A^{\perp})^{\perp}$, then $\langle x, \tau^* \tau a^{\perp} \rangle_X = 0$ for all $a^{\perp} \in A^{\perp}$, so that $\langle \tau x, \tau a^{\perp} \rangle_Z = 0$ for all

$a^\perp \in A^\perp$, from which it follows that $x \in S$. Hence $(\tau^*\tau A^\perp)^\perp \subset S$.

(iii) If $s \in S$, then $\tau s \in (\tau A^\perp)^\perp$ and we have

$$\langle \tau s, \tau a^\perp \rangle = 0 \quad \text{for all } a^\perp \in A^\perp$$

which implies that

$$\langle \tau^*\tau s, a^\perp \rangle_X = 0 \quad \text{for all } a^\perp \in A^\perp$$

so that $\tau^*\tau S \subset A$. On the other hand, τ^* maps Z onto B. Thus $\tau^*\tau S \subset A \cap B$. Conversely, let $x \in A \cap B$, τ^* being a continuous, linear, one to one map of Z onto B, we can find a unique $\gamma \in Z$ such that $\tau^*\gamma = x$. Let $\gamma = \tau x_0$. Then $\tau^*\tau x_0 \in A$ and so

$$\langle \tau^*\tau x_0, a^\perp \rangle = 0 \quad \text{for all } a^\perp \in A^\perp$$

which is equivalent to

$$\langle \tau x_0, \tau a^\perp \rangle = 0 \quad \text{for all } a^\perp \in A^\perp$$

Thus $x_0 \in S$ and hence $A \cap B \subset \tau^*\tau S$

COROLLARY 2.3.2. 1) S <u>is a closed subspace of</u> X <u>and</u> $B^\perp \subset S$.

2) ***If*** $s \in S$, ***then there exists*** $a \in A$ ***such that*** $s \in S_a$ ***and***

$$\| \tau\phi - \tau s \|_Z^2 = \| \tau\phi \|_Z^2 - \| \tau s \|_Z^2 \quad \underline{\text{for all}} \ \phi \in \Phi(a; A^\perp).$$

This is an analogue of "the first integral relation". (see [3])

3) τS ***is a closed subspace of*** Z ***and hence if*** $x = a_x + a_x^\perp \in X$, $a_x \in A$, $a_x^\perp \in A^\perp$ ***then there exists*** $S_x \subset S$ ***satisfying***

$$\| \tau x - \tau s_x \|_Z = \min_{s \in S} \| \tau x - \tau s \|_Z \quad \underline{\text{for all}} \ s_x \in S_x$$

<u>where</u>

$$S_x = a_x - (A^\perp)_{a_x}$$

Properties analogous to those given in theorem 2.3.1 and corollary 2.3.2 also hold for the class Σ . Since the proof is similar, we shall state the results and omit the proof.

THEOREM 2.3.3. ***The class*** Σ ***of interpolating splines has the following properties***

(i) $T\Sigma = (TB^\perp)^\perp$

(ii) $\Sigma = (T^*TB^\perp)^\perp$

(iii) $T^*T\Sigma = B \cap A$

where T^* **is the adjoint operator of** T.

COROLLARY 2.3.4. 1) Σ **is a closed subspace of** X and $A^{\perp} \subset \Sigma$

2) **If** $\sigma \in \Sigma$, **then there exists** $b \in B$ **such that** $\sigma \in \Sigma_b$ **and**

$$\| T\phi - T\sigma \|_Y^2 = \| T\phi \|_Y^2 - \| T\sigma \|_Y^2 \quad \underline{\text{for all}}\ \phi \in \Phi(b; B^{\perp})$$

3) $T\Sigma$ **is a closed subspace of** Y **and hence if** $x = b_x + b_x^{\perp}$, $b_x \in B$, $b_x^{\perp} \in B^{\perp}$, **then there exists** $\Sigma_x \subset \Sigma$ **satisfying**

$$\| Tx - T\sigma_x \|_Y = \min_{\sigma \in \Sigma} \| Tx - T\sigma \|_Y \quad \underline{\text{for all}}\ \sigma_x \in \Sigma_x$$

where

$$\Sigma_x = b_x - (B^{\perp})_{b_x}$$

It is an immediate consequence of the property (iii) given in theorems 2.3.1 and 2.3.3 that the two classes S and Σ are connected by the relation

$$\boxed{\tau^* \tau S = T^* T \Sigma} \qquad (2.3.3)$$

2.4. Dual Property : Extension of Joly's result:

We shall now extend the result of Joly ([7] p.77) to our case in the following form

THEOREM 2.4.1. *Let* S *be defined by* (2.3.1) *If* S_k *is the projection of* S *on* $K(\tau)$ *and* a_2 *is an element of* $A \cap K(\tau)$, *let* $W : X \longrightarrow Z$ *be defined by* $Wx = \tau^{*-1} x_2$ *where* $x = x_1 + x_2$, $x_1 \in N(\tau)$, $x_2 \in K(\tau)$ *then*

$$\| Wa_2 \|_Z = \min \{ \| W\phi \|_Z : \phi \in \Phi(a_2 ; S_k^\perp) \}$$

and

$$\| W\phi - Wa_2 \|_Z = \min_{a \in A \cap B} \| W\phi - Wa \|_Z \quad \text{for all } \phi \in \Phi(a_2 ; S_k^\perp)$$

PROOF. If $s^\perp \in S_k^\perp$, then we have

$$\langle Wa_2 , Ws^\perp \rangle_Z = \langle \tau^{*-1} a_2 , (\tau^{*-1} s^\perp)_2 \rangle_Z$$

where $(s^\perp)_2$ is the projection of $s^\perp$ on $K(\tau)$. Since $\tau^* \tau S = \tau^* \tau S_k = A \cap B$ there exists an interpolating spline s' whose projection $(s')_2$ on $K(\tau)$ is such that

$$\langle Wa_2 , Ws^{\perp} \rangle_Z = \langle \tau(s')_2 , \tau^{*-1}(s^{\perp})_2 \rangle_Z = \langle (s')_2 , (s^{\perp})_2 \rangle = 0$$

For $\phi \in \Phi(a_2 ; S_k^{\perp})$, we have

$$\| W\phi - Wa_2 \|_Z^2 = \| W\phi \|_Z^2 - \| Wa_2 \|_Z^2$$

Hence

$$\| Wa_2 \|_Z = \min \{ \| W\phi \|_Z : \phi \in \Phi(a_2 ; S_k^{+}) \} \qquad (2.4.1)$$

If $a_2 , \alpha_2 \in A \cap B$, then $a_2 - \alpha_2 \in A \cap B$. Thus from (2.4.1), we have

$$\| W\phi - W(a_2 - \alpha_2) \|_Z^2 = \| W\phi \|_Z^2 - \| W(a_2 - \alpha_2) \|_Z^2$$

$$\text{for all } \phi \in \Phi(a_2 - \alpha_2 ; S_k^{+})$$

Since $\Phi(a_2 - \alpha_2 ; S_k^{\perp}) = \Phi(a_2 ; S_k^{+}) - \alpha_2$

the above equation gives

$$\| W\phi - Wa_2 \|_Z \le \| W\phi - W\alpha_2 \|_Z \quad \text{for all } \phi \in \Phi(a_2 ; S_k^{+})$$

This inequality holds for any $\alpha_2 \in A \cap B$. Hence

$$\|W\phi - Wa_2\|_Z = \min_{a \in A \cap B} \|W\phi - Wa\|_Z \quad \text{for all } \phi \in \Phi(a_2; S_k^\perp)$$

An analogue of theorem 2.4.1 holds for the class Σ of interpolating splines.

<u>2.5. Representations for the sets $\Phi(a;A^\perp)$ and $\Phi(b;B^\perp)$:</u>

Let $\{e_i\}_{i \in I}$ be an orthonormal basis for the closed subspace A of X. The index set I is finite or infinite according as the dimension of A is finite or infinite. Any element $a \in A$ has the representation $a = \sum_{i \in I} \langle a, e_i \rangle e_i$. The set $\Phi(a; A^\perp)$ can now be represented in either of the following two equivalent forms:

$$\Phi(a; A^\perp) = \{\phi \in X / \langle e_i, \phi \rangle_X = \langle e_i, a \rangle_X, \ i \in I\}$$

and

$$\Phi(a; A^\perp) = \{\phi \in X \mid T\phi = y\} \quad \text{where } y = Ta$$

Similar representations can be obtained for the set $\Phi\,(b;\, B^{\perp})$

2.6. A sequence of splines:

We shall now construct a sequence of constraint sets each of which contains an interpolating spline, We shall assume in this section that $A^{\perp} \cap B^{\perp} = \{\theta_x\}$ is satisfied by the closed subspaces A and B so that S_a and Σ_b reduce to single elements. From corollary 2.3.2, since $B^{\perp} \subset S$, $S^{\perp} \cap B^{\perp} = \{\theta_x\}$ and $S^{\perp}$ and $B^{\perp}$ being orthogonal subspaces, $S^{\perp} + B^{\perp}$ is closed. Thus, from theorem 2.2.2, (replacing $A^{\perp}$ by $S^{\perp}$) there exists a unique interpolating spline $s_a^{\perp}$ of $\Phi\,(s_a\,;\, S^{\perp})$ relative to τ and a class $S_1 = \{s_a' \in X : s_a \in S\}$ of interpolating splines. Proceeding in this manner we can construct a sequence of sets $\Phi(s_n;\, S_n^{\perp})$, $(n=1,2,\cdots)$, $s_n \in S_n$ such that the existence of the interpolating spline in one set implies the existence of the interpolating spline in the succeeding set. It is to be noted that each S_n is a class of interpolating splines relative to τ . Further, for a positive integer n , we have

1) The spline of $\Phi\,(s_n,\, S_n^{\perp})$ is the unique element of minimal norm in $\Phi\,(s_{n+1} \cdot S_{n+1}^{\perp})$. This implies, in particular, that $\|s_n\|_X$ is a monotonic increasing function.

2) $B^{\perp} \subset \bigcap\limits_{n=1}^{\infty} S_n$

3) $\tau^* \tau S_{n+1} = S_n \cap B$

4) $\Phi(\mathcal{S}_n ; S_n^{\perp}) \cap S_n = \mathcal{S}_n$; $\Phi(\mathcal{S}_n ; S_n^{\perp}) \cap S_n^{+} = \mathcal{N}$

where $\mathcal{N}$ is the empty set

5) $S_{n+1} = [\tau^* \tau (S_n^{\perp})]^{\perp}$

6) At the n^{th} stage

$$S_n^{\perp} = (\tau^* \tau)^n A^{\perp}$$

where $(\tau^*\tau)^n = (\tau^*\tau)(\tau^*\tau) \cdots (\tau^*\tau)$

applied n times.

2.7. Remarks:

1) The relation (2.3.3) gives the link between the classes S and Σ of interpolating splines. We also have

$$(TS)^{\perp} \subset T\Sigma$$

and

$$(\tau\Sigma)^{\perp} \subset \tau S$$

2) If $A^{\perp} \subset B$ or $B^{\perp} \subset A$, then $A^{\perp} + B^{\perp}$ is closed and $A^{\perp} \cap B^{\perp} = \{\theta_x\}$ and the corresponding interpolating splines exist and are unique.

3) If $A^{\perp} \cap B^{\perp} = \{\theta_x\}$, then A and S have

the same dimension and B and Σ are of the same dimension.

4) Since $B^{\perp} \subset S$ and $A^{\perp} \subset \Sigma$, any element of $A^{\perp} \cap B^{\perp}$ can be considered either as a spline relative to T or as a spline relative to τ

5) The existence and uniqueness of interpolating splines depend only on the kernels of the transformation under consideration. Hence if an interpolating spline relative to a continuous linear transformation T exists, then interpolating splines relative to any continuous linear transformation with the same kernel as that of T also exist.

6) The condition $A^{\perp} \cap B^{\perp} = \{\theta_x\}$ which is required for the uniqueness of the interpolating spline together with the finite dimensionality of $B^{\perp}$ has been extensively used by Atteia ([7] , [8]) and Anselone and Laurent ([4]) for the existence of the splines belonging to the particular constraint sets considered by them. In [9] Atteia has used this condition along with our existence requirement, namely $A^{\perp} + B^{\perp}$ is closed, to prove the existence of the spline. On the other hand, Jerome and Schumaker ([15]) used the finite dimensionality of the kernel of the transformation for the existence of the $\mathcal{L}g$-spline. We have studied interpolating splines under the simple condition that $A^{\perp} + B^{\perp}$ be closed, which is weaker than the conditions of Atteia and Anselone and Laurent

and more general than the condition used by Jerome and Schumaker.

2.8. Some special cases:

We shall now study the $\mathcal{L}g$-splines defined by Jerome and Schumaker ([15]). We have introduced $\mathcal{L}g$-splines in section 2(c) of chapter I. Recall that $\mathcal{L}$ is a linear differential operator on the Hilbert space $\mathcal{H}^q[a,b]$, $\Lambda = \{\lambda_i\}_1^n$ is a set of linearly independent, continuous linear functionals on $\mathcal{H}^q$ and $r = (r_1, r_2 \cdots r_n)$ is a prescribed element of $\mathbb{R}^n$. An $\mathcal{L}g$-spline interpolating r with respect to Λ (see [15]) is an element $s \in \mathcal{H}^q$ solving the following minimisation problem

$$\left.\begin{array}{l} \|\mathcal{L}s\|_{L_2} = \min\{\|\mathcal{L}\phi\|_{L_2} : \phi = U(r)\} \\ \text{where} \quad U(r) = \{\phi \in \mathcal{H}^q \mid \lambda_i\phi = r_i \ , 1 \le i \le n\} \end{array}\right\} \quad (2.8.1)$$

By the Riesz representation theorem, there exist a set $\{m_i\}_1^n$, of linearly independent elements in $\mathcal{H}^q$ such that for $f \in \mathcal{H}^q$,

$$\lambda_i f = \langle f, m_i \rangle_{\mathcal{H}^q} \qquad 1 \le i \le n$$

The set $\{m_i\}_1^n$ spans a closed subspace M of $\mathcal{H}^q$. Let $\{k_i\}_1^n$ be a base of M such that $\langle m_i, k_j\rangle_{\mathcal{H}^q} = \delta_{ij}$

Then the constraint set Φ_r takes the form

$$\Phi_r = \{ f \in \mathcal{H}^q / \langle m_i, f\rangle_{\mathcal{H}^q} = r_i \quad 1 \le i \le n\}$$

$$= \sum_{i=1}^{n} r_i \cdot k_i + M^{\perp}$$

The differential operator $\mathcal{L}$ is a bounded linear transformation from $\mathcal{H}^q$ onto $L_2[a,b]$. Its kernel $\mathcal{U}$ is spanned by functions $\{u_i\}_1^q$ in $C^q[a,b]$

Now the extremal problem (2.8.1) can be studied in the framework introduced in this chapter. Consider the transformation $\mathfrak{m}$ of $\mathcal{H}^q$ onto $\mathbb{R}^n$ defined by

$$\mathfrak{m}(f) = (\langle m_1, f\rangle_{\mathcal{H}^q} \ldots \langle m_n, f\rangle_{\mathcal{H}^q}) \quad f \in \mathcal{H}^q$$

$\mathfrak{m}$ is continuous and linear.

$M^{\perp} + \mathcal{U}$ is closed since $\mathcal{U}$ is of finite dimension. Thus from theorem 2.2.2, there exist two sets S_r and $\Sigma_{\mathcal{U}^{\perp}}$ of interpolating splines satisfying

$$\| \mathcal{L} s \|_{L_2} = \min \{ \| \mathcal{L}\phi \|_{L_2} : \phi \in \Phi(\Sigma r_i k_i ; M^{\perp}) \}$$

for all $s \in S_r$

and

$$\| \mathfrak{m}\sigma \|_{\mathbb{R}^n} = \min \{ \| \mathfrak{m}\phi \|_{\mathbb{R}^n} : \phi \in \Phi(\mathcal{U}^{\perp} : \mathcal{U}) \}$$

for all $\sigma \in \Sigma_{\mathcal{U}^{\perp}}$

The set S_r is in fact the set of $\mathcal{L}g$-splines interpolating r with respect to Λ. Thus theorem 2.1 and corollary 2.2 of [15] can be deduced from our theorem 2.3.1. If $S_{\mathcal{L}}$ and Σ_{m} denote the class of interpolating splines relative to $\mathcal{L}$ and m respectively, then the class $S_{\mathcal{L}}$ is precisely the class of $\mathcal{L}g$-splines and all the results obtained in this chapter also hold for the two classes $S_{\mathcal{L}}$ and Σ_{m}.

2) We will now consider the constraint set mentioned in sections 3 and 4 of chapter I. Under the conditions that (1) dim $N(T) = q$, (2) $n \geqslant q$ and (3) $K^{\perp} \cap N(T) = \{\theta_x\}$, Atteia ([7]) and Anselone and Laurent ([4]) established the existence of a unique element $\sigma \in \Phi_r$ satisfying

$$\|T\sigma\|_Y = \min\{\|T\phi\|_Y : \phi \in \Phi_r\}$$

They defined σ as the interpolating spline of Φ_r relative to T. By virtue of the fact that Φ_r reduces to the translate of a closed subspace (see section 4 of chapter I), the constraint sets in [4] and [7] reduce to the form studied in this chapter. The assumption (1) implies that $N(T) + K^{\perp}$ is closed. Thus the existence of interpolating splines minimising the norm of $T\Phi_r$ is a consequence of our theorem 2.2.2. The additional assumption (3) implies that the interpolating spline of Φ_{γ} relative to T is unique (see theorem 2.2.5). Hence the results of Atteia in chapter VII (sections 1,2 and 4) of [7] and those of Anselone and Laurent [4] (propositions 2.1 and 6.1) follow either as special cases or can be deduced from our theorems 2.3.1 and 2.4.1.

CHAPTER III

MORE RESULTS ON INTERPOLATING SPLINES

3.1. Introduction:

The purpose of this chapter is to extend the results obtained by us in the previous chapter when the constraint sets are the union of translates of a closed subspace. If C is an arbitrary subset of a Hilbert space X and M is a given closed subspace of X, then we set

$$\Phi(C;M) = \bigcup_{x \in C} \Phi(x;M)$$

In particular, we shall deal with the two cases when C is compact and convex and when C is closed and convex.

We shall use the notation and terminology as in the previous chapter. A and B are two closed subspaces of a real Hilbert space X which are isomorphic to Y and Z respectively with corresponding isomorphisms I_A and I_B. The operators T and τ are defined as before by $T = I_A P_A$ and $\tau = I_B P_B$. We further assume that $A^{\perp} + B^{\perp}$ is closed in X.

3.2. Interpolating splines when C is compact and convex:

Let C be a compact, convex subset of A. We shall now consider the problem of finding the minimal element when the constraint set is of the form $\Phi(C; A^{\perp})$. This includes, in particular, the problems considered by Atteia ([8]) and Jerome and Schumaker ([15] , p.45). It may be mentioned that similar problems with different assumptions have been studied by Daniel and Schumaker ([11] , p.17) and Atteia ([9] , p.195). The main result here can be stated as follows:

THEOREM 3.2.1. *If C is a compact, convex subset of* A, *then there exists* $a_0 \in C$ *such that if* S_{a_0} *denotes the set of interpolating splines belonging to* $\Phi(a_0; A^{\perp})$, *then*

$$\| \tau s \|_z = \min \{ \| \tau \phi \|_z : \phi \in \Phi(C; A^{\perp}) \} \tag{3.2.1}$$

for all s *in the set* $\Delta = [(S_{a_0} + B^{\perp}) \cap \Phi(C; A^{\perp})]$

and $\tau(\Delta) = z_{a_0}$ *is the unique element of* Z *satisfying* (3.2.1). *Moreover, there exists a subset* $\Sigma_{b_0} \subset \Sigma$ *of splines such that*

$$\| T \sigma_{b_0} \|_Y \leq \| T s \|_Y \quad \text{for all } s \in \Delta, \text{ for all } \sigma_{b_0} \in \Sigma_{b_0}$$

Further, Δ reduces to a single element if and only if $T(\Delta)$ reduces to a singleton and $A^{\perp} \cap B^{\perp} = \{\theta_X\}$

PROOF. We have already seen in theorem 2.3.1 that corresponding to each $a \in C$, we have the constraint set $\Phi(a; A^{\perp})$ and a set of interpolating splines $S_a \subset \Phi(a; A^{\perp})$ satisfying the relation

$$\|\tau s_a\|_Z = \min\{\|\tau\phi\|_Z : \phi \in \Phi(a; A^{\perp})\} \text{ for all } s_a \in S_a$$

We set $\gamma_a = \tau(S_a)$ and define a continuous function f on C by setting $f(a) = \|\gamma_a\|_Z$ for $a \in C$. The continuity of f follows from the fact that, if $a_1, a_2 \in C$, then

$$|f(a_1) - f(a_2)| \leq \|\tau a_1 - \tau a_2\|_Z + \|P_{\tau A^{\perp}}(\tau a_1) - P_{\tau A^{\perp}}(\tau a_2)\|_Z$$

$$\leq 2\|\tau a_1 - \tau a_2\|_Z$$

$$\leq 2\|\tau\| \, \|a_1 - a_2\|_X$$

Since f is continuous on the compact set C, it attains its minimum value so that there exists $a_0 \in C$ satisfying

$$\|z_{a_0}\|_Z = \min_{a \in C} \|z_a\|_Z$$

But

$$\|z_a\|_Z = \min \{ \|\tau\phi\|_Z : \phi \in \Phi(a; A^{\perp}) \}$$

Thus, we obtain

$$\|z_{a_0}\|_Z = \min \{ \|\tau\phi\|_Z : \phi \in \Phi(C; A^{\perp}) \} \text{ for all } s_{a_0} \in S_{a_0}$$

or equivalently, we have

$$\|\tau s_{a_0}\|_Z = \min \{ \|\tau\phi\|_Z : \phi \in \Phi(C; A^{\perp}) \} \text{ for all } s_{a_0} \in S_{a_0} \tag{3.2.2}$$

It is easy to see that every element of the set $\triangle = [(S_{a_0} + B^{\perp}) \cap \Phi(C; A^{\perp})]$ also satisfies (3.2.2) and by virtue of corollary 2.3.2, we see that each element of Δ is an interpolating spline. Moreover, $\tau(\Delta) = \gamma_{a_0}$ and γ_{a_0} is the unique element of minimal norm in $\tau(\Phi(C; A^\perp))$ since Z is a real Hilbert space and C is convex.

If $\Phi_0 = \{\phi \in X : \tau\phi = \gamma_{a_0}\}$, then $\Delta \subset \Phi_0$

Further,

$$\begin{aligned} \Phi_0 &= k_0 + N(\tau) \quad \text{where } k_0 \in K(\tau) \text{ and } \tau k_0 = \gamma_{a_0} \\ &= b_0 + B^\perp, \qquad b_0 = k_0 \in B \\ &= \Phi(b_0; B^\perp) \end{aligned}$$

Since $A^\perp + B^\perp$ is closed, there exists a set Σ_{b_0} of interpolating splines relative to T such that

$$\|T\sigma_{b_0}\|_Y = \min\{\|T\phi\|_Y : \phi \in \Phi(b_0; B^\perp)\} \text{ for all } \sigma_{b_0} \in \Sigma_{b_0}$$

In particular,

$$\|T\sigma_{b_0}\|_Y \leq \|T\delta\|_Y \text{ for all } \sigma_{b_0} \in \Sigma_{b_0}, \text{ for all } \delta \in \Delta.$$

Now Δ consists of a single element if and only if

$$P_A(\Delta) = \{a_0\} \quad \text{and} \quad A^{\perp} \cap B^{\perp} = \{\theta_X\}$$

Since $T = I_A P_A$, $P_A(\Delta) = \{a_0\} \iff T(\Delta) = \{y_0\}$

where $y_0 = Ta_0$

COROLLARY 3.2.2. If $\Phi(C; A^{\perp})$ and Δ are as in theorem 3.2.1, then

$$\|\tau\phi - \tau\delta\|_Z^2 \leq \|\tau\phi\|_Z^2 - \|\tau\delta\|_Z^2 \quad \text{for all } \phi \in \Phi(C; A^{\perp})$$

$$\text{for all } \delta \in \Delta$$

PROOF.

$$\|\tau\phi - \tau\delta\|_Z^2 = \|\tau\phi\|_Z^2 - \|\tau\delta\|_Z^2 + 2\langle \tau\delta, \tau\delta - \tau\phi\rangle_Z$$

and from theorem 3.2.1, $\quad \langle \tau\delta, \tau\delta - \tau\phi\rangle_Z \leq 0$

for all $\phi \in \Phi(C; A^{\perp})$, for all $\delta \in \Delta$

3.3. Some remarks on theorem 3.2.1:

1) Since a closed and convex subset of a Hilbert space contains a unique element of minimal norm, the standard procedure adopted when looking for a minimal element in a set is to prove that the set is closed and convex. However, in order to establish the existence of

a minimal element in $\tau\Phi(C\,;A^{\perp})$, it has not been found necessary to prove that $\tau\Phi(C\,;A^{\perp})$ is closed. In that sense, the proof of theorem 3.2.1 is a departure from standard techniques. The criterion leading to the characterisation of the minimal element can be used since it does not require that $\tau\Phi(C\,;A^{\perp})$ should be closed. (see [12] , p.99).

2) If we impose the additional condition that $A^{\perp}\cap B^{\perp}=\{\theta_X\}$, then corresponding to each $\Phi(a;A^{\perp})\subset\Phi(C\,;A^{\perp})$, there exists a unique spline $s_a\in\Phi(a;A^{\perp})$. Then Δ reduces to $\Delta=[(s_{a_0}+B^{\perp})\cap\Phi(C;A^{\perp})]$

Set $\bar{\Delta}=\{s_a : a\in C\}$.

Now Δ and $\bar{\Delta}$ are closed and convex subsets of X and hence we have the following best approximation property: __Given__ $x\in X$, __there exists__ $\bar{s}_x\in\bar{\Delta}$ __and__ $s_x\in\Delta$ __such that__

$$\|x-\bar{s}_x\|_X = \min\{\|x-s_a\|_X : s_a\in\bar{\Delta}\}$$

__and__

$$\|x-s_x\|_X = \min\{\|x-s\|_X : s\in\Delta\}$$

3) From theorem 2.3.1, we have $Z=\tau S\oplus\tau A^{\perp}$

It is also interesting to note that the best approximation in $\tau\Phi(C\,;A^{\perp})$ to any element in τS is the image of a set of splines. More precisely, we have

THEOREM 3.3.1. If $\gamma \in \tau S$ then there exists $\gamma_c \in \tau\Phi(C; A^\perp)$ such that

$$\| \gamma - \gamma_c \|_Z = \min \{ \| \gamma - \tau\phi \|_Z : \phi \in \Phi(C; A^\perp) \}$$

and γ_c is the image of a set of interpolating splines.

Theorem 3.3.1 is a direct consequence of theorem 3.2.1.

4) If S_{a_1}, $S_{a_2} \subset S$, by theorem 3.3.1, there exist two sets Δ_1 and Δ_2 of splines in $\Phi(C; A^\perp)$ such that

$$\| \tau s_{a_1} - \tau(\Delta_1) \|_Z = \min \{ \| \tau s_{a_1} - \tau\phi \|_Z : \phi \in \Phi(C : A^\perp) \}$$
for all $s_{a_1} \in S_{a_1}$

and

$$\| \tau s_{a_2} - \tau(\Delta_2) \|_Z = \min \{ \| \tau s_{a_2} - \tau\phi \| : \phi \in \Phi(C; A^\perp) \}$$
for all $s_{a_2} \in S_{a_2}$

The sets S_{a_1}, S_{a_2}, Δ_1 and Δ_2 of splines are related in the following manner

$$\| \tau(\Delta_1) - \tau(\Delta_2) \|_2 \leq \| \tau(S_{a_1}) - \tau(S_{a_2}) \|_2 .$$

3.4. The different representations for $\Phi\ (C ; A^{\perp})$:

The set $\{e_i\}_{i \in I}$ is an orthonormal basis for A and C is a compact, convex subset of A. Hence for each $i \in I$, we can find two elements $c_i^{(1)}$ and $c_i^{(2)} \in C$ such that

$$\langle e_i , c_i^{(1)} \rangle_X \leq \langle e_i , a \rangle_X \leq \langle e_i , c_i^{(2)} \rangle_X$$

for all $a \in C$. Then we have the following representation for the set $\Phi\ (C; A^{\perp})$:

$$\Phi(C ; A^{\perp}) = \left\{ \phi \in X : \langle e_i , c_i^{(1)} \rangle_X \leq \langle e_i , \phi \rangle_X \leq \langle e_i , c_i^{(2)} \rangle_X , i \in I \right\} \tag{3.4.1}$$

Let $\Gamma = T\ (C)$. Then Γ is compact and convex. Now $\Phi\ (C;A^{\perp})$ can also be represented as

$$\Phi(C ; A^{\perp}) = \{ \phi \in X : T\phi \in \Gamma \} \tag{3.4.2}$$

and

$$\Phi(C ; A^{\perp}) = \{ \phi \in X : P_A \phi \in C \} \tag{3.4.3}$$

The representations of $\Phi\ (C;A^{\perp})$ in the three equivalent forms (3.4.1), (3.4.2) and (3.4.3) allows for flexibility in the study of the interpolating splines belonging to the set Δ . The first representation is useful in characterising the splines of Δ (see remark 3.7.1), the second in studying the case when Δ contains just one element (see theorem 3.6.1) and the third in studying the minimisation problem of theorem 3.2.1 in a dual form (see section 3.5).

3.5. Different formulations of the minimal problem of theorem 3.2.1:

For any subset E of X, we define a function χ_E on X as follows

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E \end{cases}$$

and set

$$p(x) = \frac{1}{2}\|\tau x\|_Z^2 + \chi_C(P_A x)$$

If g is a convex function defined on X, its dual g^* is defined by

$$g^*(v) = \sup\{[\langle u, v\rangle_X - g(u)] : u \in X\}$$

Now set $\alpha(x) = \frac{1}{2}\|\tau x\|_Z^2$, $\beta(x) = \chi_C(P_A x)$

and $q(x) = -[\alpha^*(x) + \beta^*(x)]$. Using techniques similar to those in [9] , we can prove that

$$q(x) = \begin{cases} -\infty & \text{if } x \notin A \cap B \\ -\frac{1}{2}\|\tau^{*-1}x\|_2^2 - \max\limits_{c \in C} \langle x, c \rangle & \text{if } x \in A \cap B \end{cases}$$

and

$$\sup_{v \in X} q(v) = \inf_{\omega \in X} p(\omega)$$

Thus the extremal problem of theorem 3.2.1 can be viewed in any one of the following three equivalent forms:

1) Minimise the norm of $\tau\phi$ for $\phi \in \Phi\ (C;A^{\perp})$
2) Minimise $p(x) = \frac{1}{2}\|\tau x\|_2^2 + \chi_C(P_A x)$ for $x \in X$
3) Maximise $q(x) = -[\alpha^*(x) + \beta^*(x)]$ for $x \in A \cap B$.

3.6. A uniqueness theorem:

The last part of theorem 3.2.1 gives a necessary and sufficient condition for z_{a_0} to be the image of a unique spline. Since $\tau^*\tau S = T^* T\Sigma$ (from 2.3.3), there exists a unique $y_0 \in T\Sigma$ such that $\tau^* z_{a_0} = T^* y_0$. Now y_0 is the image of a set of interpolating splines relative to T, i.e., there exists a set $\Sigma_{y_0} \subset \Sigma$ of interpolating splines such that $T(\Sigma_{y_0} + A^{\perp}) = y_0$. We shall now give in terms of y_0 a condition which is both necessary and sufficient for z_{a_0} to be the image of a unique spline.

THEOREM 3.6.1. <u>If $A^{\perp} \cap B^{\perp} = \{\theta_X\}$, then a necessary and sufficient condition for Δ to reduce to a single element is that the hyperplane of support of Γ with the equation</u>

$$\langle y_0 , y \rangle_Y = \min \{ \langle y_0 , \gamma \rangle_Y : \gamma \in \Gamma \}$$

<u>meet Γ at a unique point.</u>

The proof of theorem 3.6.1 runs along the same lines as the proof of a similar theorem in [9] (p.199) and hence we omit it.

<u>3.7. A special case of the set $\Phi(C;A^{\perp})$:</u>

We shall now study in more detail the minimisation problem of theorem 3.2.1 for a particular choice of the constraint set $\Phi(C;A^{\perp})$. Suppose that $\{1,2,--,2N\}$ is the index set I so that the dimension of A is 2N and the constraint set $\Phi(C;A^{\perp})$ is such that

$$\langle e_i , c_i^{(1)} \rangle_X = \langle e_{i+N} , c_i^{(1)} \rangle_X$$

and

$$\langle e_i , c_i^{(2)} \rangle_X = \langle e_{i+N} , c_i^{(2)} \rangle_X .$$

Set $k_i = e_i , (1 \le i \le N)$ and $\ell_i = e_{i+N} , (1 \le i \le N)$. The set $\{ k_i - \ell_i \}_1^N$ is a linearly independent set in A and spans a closed subspace, say $A_{k-\ell}$ of A.

Consider the set

$$\Phi_{k\ell} = \{\phi \in X \mid \eta_i \le \langle k_i, \phi\rangle_X = \langle \ell_i, \phi\rangle_X \le \xi_i , \quad 1 \le i \le N\}$$

where

$$\langle c_i^{(1)}, e_i\rangle_X = \eta_i = \langle c_i^{(1)}, e_{i+N}\rangle_X \qquad 1 \le i \le N$$

and

$$\langle c_i^{(2)}, e_i\rangle_X = \xi_i = \langle c_i^{(2)}, e_{i+N}\rangle_X \quad , \quad 1 \le i \le N$$

Now $\Phi_{k\ell} = \Phi(C; A^\perp) \cap A_{k-\ell}^{\perp}$. It is, however, a particular case of a set of the form $\Phi\ (C; A^\perp)$ itself and hence there exists a minimal element in $\tau\Phi_{k\ell}$ which is the image of set of interpolating splines (see theorem 3.2.1). Let $\tilde{z}$ be the unique element of $\tau\Phi_{k\ell}$ such that

$$\|\tilde{z}\|_Z = \min\{\|\tau\phi\|_Z : \phi \in \Phi_{k\ell}\} \qquad (3.7.1)$$

and

$$\tilde{\Delta} = \{ s \in X \mid \tau s = \tilde{z} \} \qquad (3.7.2)$$

We shall now characterise the set $\tilde{\Delta}$ of interpolating splines. First we observe that

$$\langle \tau\phi - \tilde{z}, \tilde{z}\rangle_Z \geq 0 \qquad \text{for all } \phi \in \Phi_{k\ell}$$

That is,

$$\langle \tau^*\tilde{z}, \phi - s\rangle_x \geq 0, \text{ for all } \phi \in \Phi_{kl},$$
$$\text{for all } s \in \tilde{\Delta} \qquad (3.7.3)$$

Now by theorem 2.3.1, $\tau^*\tilde{z}$ belongs to $A \cap B$ so that

$$\tau^*\tilde{z} = \sum_{i=1}^{N} \lambda_i k_i + \sum_{i=1}^{N} \mu_i l_i$$

with

$$P_B^{\perp}\left(\sum \lambda_i k_i + \sum_{i=1}^{N} \mu_i l_i\right) = \theta_x$$

Let

$$\Phi_\gamma = \{\phi \in X \,/\, \langle k_i, \phi\rangle_X = \langle l_i, \phi\rangle_X = \gamma_i, \ 1 \leq i \leq N\}$$

Then

$$\Phi_{kl} = \bigcup_{\gamma \in [\eta, \xi]} \Phi_\gamma \quad \text{and} \quad \Phi_\gamma = \sum_{i=1}^{N} [\gamma_i(k_i + l_i)] + A^{\perp}$$

(3.7.3) then gives

$$\left\langle \sum_{i=1}^{N} \lambda_i k_i + \sum_{i=1}^{N} \mu_i l_i, \sum_{i=1}^{N} \gamma_i(k_i + l_i) - \sum_{i=1}^{N} \rho_i(k_i + l_i)\right\rangle \geq 0$$

where

$$\sum_{i=1}^{N} \rho_i (k_i + l_i) \in P_A(\bar{\Delta}) \quad \text{and} \quad \eta_i \le \gamma_i \le \xi_i$$

From this we infer that

$$\sum_{i=1}^{N} (\lambda_i + \mu_i)(\gamma_i - \rho_i) \geqslant 0$$

Suppose that for an index $i_0 : \eta_{i_0} < \rho_{i_0} < \xi_{i_0}$
We can find numbers γ_i' and γ_i'' such that
$\eta_i \le \gamma_i' \le \xi_i$, $\eta_i \le \gamma_i'' \le \xi_i$ and
$\gamma_i' = \gamma_i'' = \rho_i$ with $\rho_{i_0} - \gamma_{i_0}' < 0$
and $\rho_{i_0} - \gamma_{i_0}'' > 0$. Then $\lambda_{i_0} + \mu_{i_0} = 0$
On the other hand, if $\rho_{i_1} = \eta_{i_1}$ for some index $i_1 \in I$
then $\lambda_{i_1} + \mu_{i_1} \geqslant 0$. Similarly for an index i_2 such
that $\rho_{i_2} = \xi_{i_2}$, $\lambda_{i_2} + \mu_{i_2} \le 0$

Thus there exists a unique set of coefficients
$\lambda_i^{(1)}, \mu_i^{(1)}, \lambda_i^{(2)}, \mu_i^{(2)}, \lambda_i^{(3)}, \mu_i^{(3)}$ at least
some of which are non-zero such that $\lambda_i^{(1)} + \mu_i^{(1)} \geqslant 0$, $i \in I_1$
$\lambda_i^{(2)} + \mu_2^{(2)} \le 0$, $i \in I_2$, $\lambda_i^{(3)} + \mu_i^{(3)} = 0$ $i \in I_3$
and the set $\tilde{\Delta}$ of splines given by (3.7.2) is characterised as follows:

$$\left.\begin{aligned}
\tau^{*}\tilde{\gamma} &= \sum_{i\in I_1}(\lambda_i^{(1)}k_i+\mu_i^{(1)}\ell_i)+\sum_{i\in I_2}(\lambda_i^{(2)}k_i+\mu_i^{(2)}\ell_i)\\
&\quad+\sum_{i\in I_3}(\lambda_i^{(3)}k_i+\mu_i^{(3)}\ell_i);\\
&\textbf{with}\\
&\lambda_i^{(1)}+\mu_i^{(1)}\geqslant 0,\quad \lambda_i^{(2)}+\mu_i^{(2)}\leq 0\quad \lambda_i^{(3)}+\mu_i^{(3)}=0\\
&\textbf{and}\\
&P_{B}^{\perp}\Big[\sum_{i\in I_1}(\lambda_i^{(1)}k_i+\mu_i^{(1)}\ell_i)+\sum_{i\in I_2}(\lambda_i^{(2)}k_i+\mu_i^{(2)}\ell_i)\\
&\quad+\sum_{i\in I_3}\lambda_3^{(3)}k_i+\mu_i^{(3)}\ell_i)\Big]=\Theta_X;\\
&\textbf{where}\\
&I_1=\{i\mid 1\leq i\leq N \text{ and } \langle k_i,\delta\rangle_X=\langle \ell_i,\delta\rangle_X=\eta_i \text{ for all } \delta\in\tilde{\Delta}\}\\
&I_2=\{i\mid 1\leq i\leq N \text{ and } \langle k_i,\delta\rangle_X=\langle \ell_i,\delta\rangle=\eta_i \text{ for all } \delta\in\tilde{\Delta}\}\\
&I_3=\{i/1\leq i\leq N \text{ and } \eta_i<\langle k_i,\delta\rangle_X=\langle \ell_i,\delta\rangle<\xi_i \text{ for all } \delta\in\tilde{\Delta}\}
\end{aligned}\right\}(3.7.4)$$

REMARK 3.7.1. The interpolating splines of $\Delta = [(S_{a_0} + B^{\perp}) \cap \Phi(C; A^{\perp})]$ can be characterised in a similar manner. In fact, we have

$$\left.\begin{aligned}
&\tau^* \gamma_{a_0} = \sum_{i \in I'} \lambda_i e_i - \sum_{i \in I''} \mu_i e_i, \quad \lambda_i \geqslant 0 \quad \mu_i \geqslant 0 \\
&P_{B^\perp}\Big[\sum_{i \in I'} \lambda_i e_i - \sum_{i \in I''} \mu_i e_i\Big] = \theta_X \\
&I' = \{ i \in I \,/\, \langle e_i, s\rangle_X = \langle e_i, c_i^{(1)}\rangle_X \text{ for all } s \in \Delta \} \\
&I'' = \{ i \in I \,/\, \langle e_i, s\rangle_X = \langle e_i \; c_i^{(2)}\rangle_X \\
&\qquad\qquad \text{for all } s \in \Delta \}
\end{aligned}\right\} (3.7.5)$$

A characterisation of the type (3.7.5) has been obtained by Atteia in [8] when the index set $I = \{1, 2, \dots, N\}$.

We shall now study the minimal problem (3.7.1) in a dual form. We first define

$$\Psi(x,\lambda) = \frac{1}{2}\|Tx\|_Z^2 - \sum_{i=1}^{N} m_i^{(1)}[\langle k_i, x\rangle_X - \eta_i]$$

$$- \sum_{i=1}^{N} n_i^{(1)}[\langle \ell_i, x\rangle_X - \eta_i] + \sum_{i=1}^{N} m_i^{(2)}[\zeta_i - \langle k_i, x\rangle_X]$$

$$+ \sum_{i=1}^{N} n_i^{(2)}[\zeta_i - \langle \ell_i, x\rangle_X] + \sum_{i=1}^{N} m_i^{(3)}[\langle k_i, x\rangle_X - \langle \ell_i, x\rangle_X] \quad (3.7.6)$$

where

$$\lambda = (m_1^{(1)}, \dots m_N^{(1)}, n_1^{(1)} \dots n_N^{(1)}, m_1^{(2)} \dots m_N^{(2)}, n_1^{(2)} \dots n_N^{(2)}, m_1^{(3)} \dots m_N^{(3)})$$

and set

$$\lambda^k = (m_1^{(1)*} \dots m_N^{(1)*}, n_1^{(1)*} \dots n_N^{(1)*}, m_1^{(2)*} \dots m_N^{(2)*} n_1^{(2)*} \dots n_N^{(2)*}, m_1^{(3)*} \dots m_N^{(3)*})$$

with

$$m_i^{(1)*} = \begin{cases} \mu_i^{(1)} & \text{if } i \in I_1 \\ 0 & \text{if } i \notin I_1 \end{cases}, \quad m_i^{(2)*} = \begin{cases} \lambda_i^{(2)} & \text{if } i \in I_2 \\ 0 & \text{if } i \notin I_2 \end{cases};$$

$$m_i^{(3)*} = \begin{cases} -\lambda_i^{(3)} & \text{if } i \in I_3 \\ 0 & \text{if } i \notin I_3 \end{cases}, \quad n_i^{(1)*} = \begin{cases} \mu_i^{(1)} & \text{if } i \in I_1 \\ 0 & \text{if } i \notin I_1 \end{cases};$$

$$n_i^{(2)*} = \begin{cases} \mu_i^{(2)} & \text{if } i \in I_2 \\ 0 & \text{if } i \notin I_2 \end{cases}.$$

and

$$\Lambda = \{ \lambda / \lambda = (m_1^{(1)} \cdots m_N^{(1)}, n_1^{(1)} \cdots n_N^{(1)}, m_1^{(2)} \cdots m_N^{(2)}$$

$$n_1^{(2)} \cdots n_N^{(2)}; m_1^{(3)} \cdots m_N^{(3)}), m_i^{(1)} + n_i^{(1)} \geq 0,$$

$$m_i^{(2)} + n_i^{(2)} \leq 0, \quad m_i^{(3)} \text{ arbitrary } 1 \leq i \leq N \}$$

Now the problem of minimising $\tau\phi$ for $\phi \in \Phi_{k\ell}$ is equivalent to finding the value of $\Psi(s, \lambda^k)$ for $s \in \tilde{\Delta}$ Using the techniques of Atteia in [8] we prove the following

THEOREM 3.7.2.

$$\inf_{x \in X} \sup_{\lambda \in \Lambda} \Psi(x, \lambda) = \sup_{\lambda \in \Lambda} \inf \Psi(x, \lambda) = \Psi(s, \lambda^*)$$

$$\underline{\text{for all}}\ s \in \tilde{\Delta}.$$

We will first prove the

LEMMA 3.7.3.

$$\Psi(s, \lambda) \leq \Psi(s, \lambda^*) \leq \Psi(x, \lambda^*)\ \underline{\text{forall}}\ x \in X\ \underline{\text{forall}}\ s \in \tilde{\Delta}$$

$$\text{and}\ \underline{\text{forall}}\ \lambda \in \Lambda$$

PROOF. Suppose $s \in \tilde{\Delta}$ Then (3.7.6) reduces to

$$\Psi(s,\lambda) = \frac{1}{2}\|\tau s\|_Z^2 - \sum_{i=1}^{N}(m_i^{(1)}+n_i^{(1)})[\langle k_i, s\rangle_X - \eta_i]$$

$$+ \sum_{i=1}^{N}(m_i^{(2)}+n_i^{(2)})[\xi_i - \langle k_i, s\rangle_X]$$

$$= \frac{1}{2}\|\tau s\|_Z^2 - \sum(m_i^{(1)}+n_i^{(1)})[\langle k_i, s\rangle_X - \eta_i]$$

$$+\sum_{i\notin I_2}(m_i^{(2)}+n_i^{(2)})[\xi_i - \langle k_i, s\rangle_X]$$

which shows that if $\lambda\in\Lambda$, then

$$\Psi(s,\lambda) \leq \frac{1}{2}\|\tau s\|_Z^2 = \Psi(s,\lambda^*)$$

Now

$$\|\tau x - \tau s\|_Z^2 = \|\tau x\|_Z^2 - \|\tau s\|_Z^2 + \langle \tau s, \tau s - \tau x\rangle_Z$$

Hence

$$\frac{1}{2}\|\tau s\|^2 \leq \frac{1}{2}\|\tau x\|_Z^2 + \langle \tau s, \tau s - \tau x\rangle_Z$$

But

$$\langle \tau s, \tau s - \tau x\rangle_Z = \langle \tau^*\tau s, s - x\rangle_X$$

$$= \Big\langle \sum_{i\in I_1}(\lambda_i^{(1)}k_i + \mu_i^{(1)}\ell_i) + \sum_{i\in I_2}\lambda_i^{(3)}(k_i - \ell_i)$$

$$+ \sum_{i\in I_3}\lambda_i^{(3)}(k_i - \ell_i)\ ,\ s - x\Big\rangle_X$$

or equivalently,

$$\langle \tau s, \tau s - \tau x\rangle_X = \sum_{i\in I_1} [(\lambda_i^{(1)} + \mu_i^{(1)})\eta_i - \lambda_i^{(1)}\langle k_i, x\rangle_X - \mu_i^{(1)}\langle l_i, x\rangle_X]$$

$$+ \sum [(\lambda_i^{(2)} + \mu_i^{(2)})\xi_i - \lambda_i^{(2)}\langle k_i, x\rangle_X - \mu_i^{(2)}\langle l_i, x\rangle_X]$$

$$- \sum_{i\in I_3} \lambda_i^{(3)} [\langle k_i, x\rangle_X - \langle l_i, x\rangle_X]$$

Now

$$\Psi(x, \lambda^*) = \frac{1}{2}\|\tau x\|_Z^2 - \sum_{i\in I_1} \lambda_i^{(1)} [\langle k_i, x\rangle_X - \eta_i]$$

$$- \sum_{i\in I_1} \mu_i^{(1)} [\langle l_i, x\rangle_X - \eta_i] + \sum_{i\in I_2} \lambda_i^{(2)} (\xi_i - \langle k_i, x\rangle_X]$$

$$+ \sum_{i\in I_2} \mu_i^{(2)} [\xi_i - \langle l_i, x\rangle_X - \sum_{i\in I_3} \lambda_i^{(3)} [\langle k_i, x\rangle_X - \langle l_i, x\rangle]$$

Hence

$$\Psi(s, \lambda^*) = \frac{1}{2}\|\tau s\|_Z^2 \le \tfrac{1}{2}\|\tau x\|_Z^2 + \langle \tau s, \tau s - \tau x\rangle_Z$$

$$= \Psi(x, \lambda^*)$$

which implies that

$$\Psi(s, \lambda^*) \le \Psi(x, \lambda^*)$$

This completes the proof of the lemma.

PROOF OF THEOREM 3.7.2. The function $\overline{\Psi}(x, \lambda)$ is defined by equation (3.7.6). We shall consider separately two cases according as x is or is not an element of $\overline{\Phi}_{k\ell}$

Case (i): Suppose $x \notin \overline{\Phi}_{k\ell}$. Then there exists an integer i_0 such that $\langle k_{i_0}, x\rangle_X = \langle \ell_{i_0} \; x\rangle_X$ with $\langle k_{i_0}, x\rangle_X - \eta_{i_0} < 0$ and $\xi_{i_0} - \langle k_i, x\rangle_X > 0$

For the value of λ such that $m_i^{(2)} = n_i^{(2)} = m_i^{(3)} \; (1 \le i \le N)$ and $m_i^{(1)} = n_i^{(1)} = 0$ for $i \neq i_0$

It follows that

$$\overline{\Psi}(x, \lambda) = \frac{1}{2} \| \tau x \|_Z^2 + (m_{i_0}^{(1)} + n_{i_0}^{(1)}) \gamma_{i_0} \qquad \gamma_{i_0} > 0$$

Then

$$\sup_{\lambda \in \Lambda} \overline{\Psi}(x, \lambda) = +\infty \quad \text{if } x \notin \Phi_{k\ell}$$

Case (ii): $x \in \Phi_{k\ell}$ In this case we have

$$\langle k_i, x\rangle_X = \langle \ell_i, x\rangle_X \quad \langle k_i, x\rangle - \eta_i \geqslant 0 \quad \xi_i - \langle k_i, x\rangle_X \geqslant 0$$

so that

$$\overline{\Psi}(x ; \lambda) = \frac{1}{2} \| \tau x \|_Z^2 - \sum_{i=1}^{N} m_i^{(1)} + n_i^{(1)} [\langle k_i, x\rangle_X - \eta_i]$$

$$+ \sum_{i=1}^{N} (m_i^{(2)} + n_i^{(2)}) [\xi_i - \langle k_i, x\rangle_X]$$

which implies that

$$\sup_{\lambda \in \Lambda} \Psi(x, \lambda) = \frac{1}{2} \|\tau x\|_Z^2 \quad \text{for } x \in \Phi_{kl}$$

Thus

$$\sup_{\lambda \in \Lambda} \Psi(x, \lambda) = \begin{cases} +\infty & \text{if } x \notin \Phi_{kl} \\ \frac{1}{2} \|\tau x\|_Z^2 & \text{if } x \in \Phi_{kl} \end{cases}$$

It follows that

$$\inf_{x \in X} \sup_{\lambda \in \Lambda} \Psi(x, \lambda) = \frac{1}{2} \|\tau x\|_Z^2 = \Psi(s, \lambda^*) \text{ forall } s \in \tilde{\Delta}$$

This proves one part of the assertion.

To prove the second part, consider the Frechet differential of $\Psi(x, \lambda)$:

$$\frac{\partial}{\partial x} \Psi(x, \lambda) = \tau^* \tau x - \sum_{i=1}^{N} (m_i^{(1)} + m_i^{(2)} - m_i^{(3)}) k_i - \sum_{i=1}^{N} (n_i^{(1)} + n_i^{(2)} + m_i^{(3)}) l_i$$

Denote by

$$\Lambda_B = \left\{ \lambda \in \Lambda : P_{B^\perp}\left[\sum_{i=1}^{N} (m_i^{(1)} + m_i^{(2)} + m_i^{(3)}) k_i + \sum_{i=1}^{N} (n_i^{(1)} + n_i^{(2)} + m_i^{(3)}) \ell_i \right] = \theta_X \right\}.$$

If $\lambda \in \Lambda_B$, then there exists a unique $\hat{x}_B(\lambda) \in B$ such that

$$\tau^* \tau \hat{x}_B(\lambda) = \sum_{i=1}^{N} (m_i^{(1)} + m_i^{(2)} - m_i^{(3)}) k_i + \sum_{i=1}^{N} (n_i^{(1)} + n_i^{(2)} + m_i^{(3)}) b_i$$

and so we have

$$\inf \Psi(x, \lambda) = \Psi(\hat{x}_B(\lambda), \lambda) = \Psi(\hat{x}(\lambda), \lambda)$$

for all $\hat{x}(\lambda)$ such that $\hat{x}(\lambda) - \hat{x}_B(\lambda) \in B^\perp$

Suppose now that $\lambda \notin \Lambda_B$. Then

$$P_{B^\perp}\left[\sum_{i=1}^{N} (m_i^{(1)} + m_i^{(2)} - m_i^{(3)}) k_i + \sum_{i=1}^{N} (n_i^{(1)} + n_i^{(2)} + m_i^{(3)}) \ell_i \right] = \tilde{x}_{B^\perp} \neq \theta_X$$

and

$$\Psi(\rho\tilde{x}_{B^\perp},\lambda) = \frac{1}{2}\|\tau(\rho\tilde{x}_{B^\perp})\|^2$$

$$-\sum_{i=1}^{N}(m_i^{(1)}+m_i^{(2)}+m_i^{(3)})\left\langle k_i,\ \rho P_{B^\perp}\Big[\sum_{i=1}^{N}(m_i^{(1)}+m_i^{(2)}-m_i^{(3)})k_i + \sum_{i=1}^{N}(n_i^{(1)}+n_i^{(2)}+m_i^{(3)})l_i\Big]\right\rangle_X$$

$$-\sum_{i=1}^{N}(n_i^{(1)}+n_i^{(2)}+m_i^{(3)})\left\langle l_i,\ \rho P_{B^\perp}\Big[\sum_{i=1}^{N}(m_i^{(1)}+m_i^{(2)}-m_i^{(3)})k_i + \sum_{i=1}^{N}(n_i^{(1)}+n_i^{(2)}+m_i^{(3)})l_i\Big]\right\rangle_X$$

$$+\sum_{i=1}^{N}(m_i^{(1)}+n_i^{(1)})\eta_i + \sum_{i=1}^{N}(m_i^{(2)}+n_i^{(2)})\xi_i$$

which gives, since $\tau(\rho\tilde{x}_{B^\perp}) = \theta_Z$,

$$\Psi(\rho\tilde{x}_{B^\perp},\lambda) = -\rho\left\langle\sum_{i=1}^{N}(m_i^{(1)}+m_i^{(2)}-m_i^{(3)})k_i,\ P_{B^\perp}\Big[\sum_{i=1}^{N}(m_i^{(1)}+m_i^{(2)}-m_i^{(3)})k_i\Big]\right\rangle_X$$

$$-\rho\left\langle\sum_{i=1}^{N}(n_i^{(1)}+n_i^{(2)}+m_i^{(3)})l_i,\ P_{B^\perp}\Big[\sum_{i=1}^{N}(n_i^{(1)}+n_i^{(2)}+m_i^{(3)})l_i\Big]\right\rangle_X$$

$$-2\rho\left\langle P_{B^\perp}\Big[\sum_{i=1}^{N}(m_i^{(1)}+m_i^{(2)}+m_i^{(3)})l_i\Big],\ P_B^\perp\Big[\sum_{i=1}^{N}(n_i^{(1)}+n_i^{(2)}+n_i^{(3)})l_i\Big]\right\rangle_X$$

$$+\sum_{i=1}^{N}(m_i^{(1)}+n_i^{(1)})\eta_i + \sum_{i=1}^{N}(m_i^{(2)}+n_i^{(2)})\xi_i$$

This is the same as

$$\Psi(\rho\hat{x}_{B^+}, \lambda) = \sum_{i=1}^{N}(m_i^{(1)} + n_i^{(1)})\eta_i + \sum_{i=1}^{N}(m_i^{(2)} + n_i^{(2)})\xi_i$$
$$- \rho \| \tilde{x}_{B^\perp} \|_X^2$$

which implies that

$$\inf_{x\in X} \Psi(x,\lambda) = -\infty$$

Then

$$\sup_{\lambda\in\Lambda} \inf_{x\in X} \Psi(x,\lambda) = \sup_{\lambda\in\Lambda_B} \Psi(\hat{x}(\lambda),\lambda)$$

and

$$\Psi(\hat{x}_B(\lambda),\lambda) \leq \Psi(s,\lambda) \quad \text{for all } \lambda\in\Lambda_B$$

Now for $\lambda\in\Lambda$,

$$\Psi(s,\lambda) \leq \Psi(s,\lambda^*)$$

Hence

$$\Psi(\hat{x}_B(\lambda),\lambda) \leq \Psi(s\ \lambda^*) \quad \text{for all } \lambda\in\Lambda_B$$

and

$$\sup_{\lambda\in\Lambda} \inf_{x\in X} \Psi(n,\lambda) \leq \Psi(s,\lambda^*)$$

Since $\lambda^* \in \Lambda_B$ and $s \in \hat{x}(\lambda^*)$

$$\Psi(\hat{x}(\lambda^*), \lambda^*) = \Psi(s, \lambda^*) \leq \sup_{\lambda \in \Lambda_B} \Psi(\hat{x}(\lambda), \lambda)$$

Thus

$$\Psi(s, \lambda^*) \leq \sup_{\lambda \in \Lambda} \inf_{x \in X} \Psi(x, \lambda) \leq \Psi(s, \lambda^*)$$

We can now conclude that

$$\sup_{\lambda \in \Lambda} \inf_{x \in X} \Psi(x, \lambda) = \Psi(s, \lambda^*)$$

.

3.8. More general constraint sets:

The existence of a set of splines whose image under τ minimises $\|\tau\phi\|_Z$ for $\phi \in \Phi(C; A^\perp)$ has been established in theorem 3.2.1, when C was compact and convex. Now we prove that if, for a subset $\mathcal{U}$ of A, the minimal element in $\tau\Phi(\mathcal{U}; A^\perp)$ exists, then it is the image of a set of interpolating splines.

LEMMA 3.8.1. Let $\mathcal{U}$ be any subset of A. Suppose there exists $z_1 \in Z$ such that

$$\|z_1\|_Z = \min\{\|\tau\phi\|_Z : \phi \in \Phi(\mathcal{U}; A^\perp)\}$$

and $\Delta_1 = \{x \in \Phi(\mathcal{U}; A^{\perp}) : \tau x = \gamma_1\}$ **then every element of** Δ_1 **is an interpolating spline.**

PROOF. Let $x_1 \in \Delta_1$. Then

$$\|\tau x_1\|_Z = \min\{\|\tau\phi\|_Z : \phi \in \Phi(\mathcal{U}; A^{\perp})\}$$

$x_1 \in X \Rightarrow x_1 = a_{x_1} + a x_1^{\perp}$ where $a_{x_1} \in A$ and $a_{x_1}^{\perp} \in A^{\perp}$. Let $S_{a_{x_1}}$ be the set of splines of $\Phi(a_{x_1}; A^{\perp})$ relative to τ . Now

$\Phi(a_{x_1}; A^{\perp}) \subset \Phi(\mathcal{U}; A^{\perp})$. Hence

$$\|\tau x_1\|_Z = \min\{\|\tau\phi\|_Z : \phi \in \Phi(a_{x_1}; A^{\perp})\} = \|\tau s_{a_{x_1}}\|_Z$$

for all $s_{a_{x_1}} \in S_{a_{x_1}}$

Thus $x_1 \in S_{a_{x_1}}$. Hence the lemma.

Let us now consider the case when it is closed and convex. Assume that

$$\mathcal{U} + A^{\perp} + B^{\perp} \text{ is closed.}$$

Then $\tau\Phi(\mathcal{U}; A^{\perp})$ is a closed and convex subset of of Z (from lemma 2.2.3) and the unique element of minimal norm in $\tau\Phi(\mathcal{U}; A^{\perp})$ is the image of a set of splines (from lemma 3.8.1)

Now

$$\Phi(U;A^{\perp}) = U + A^{\perp}$$
$$= \{\phi \in X \,/\, P_A\phi \in U\}$$
$$= \{\phi \in X \,/\, T\phi \in E\} \quad \text{where } E = T(U)$$

and since the set

$$\{\phi \in X \,/\, \langle e_i, \phi\rangle_X \leq \langle e_i, u_1\rangle_X , \; i \in I\} \quad u_1 \in U$$

and

$$\{\phi \in X \,/\, \langle e_i, \phi\rangle_X \geq \langle e_i, u_2\rangle_X \; i \in I\} \quad u_2 \in U$$

are of the form $\Phi(U;A^{\perp})$, all the results obtained for the interpolating splines of $\triangle$ can be extended with suitable modifications to the splines of $\hat{\triangle}$.

REMARK 3.8.1. We remark that analogous properties hold for interpolating splines relative to T.

CHAPTER IV

THE SMOOTHING SPLINE

4.1. Introduction:

Even as the definition of a generalised interpolating spline has been based on an extremal property of the polynomial spline and the fact that the constraint set under consideration is the translate of a closed subspace of a Hilbert space, the generalised smoothing spline owes its origin to another extremal property of the polynomial spline (see equation (1.2.2)) and the possibility of representing the same constraint set in terms of projection operators. Precisely, the set $\Phi(a;A^{\perp}) = a + A^{\perp}$ can also be represented as $\Phi(a;A^{\perp}) = \{\phi \in X \mid P_A\phi = a\}$ where P_A denotes the projection operator mapping X onto the closed subspace A of X. In this chapter, we introduce the concept of a smoothing spline and, as in chapter II, establish the existence of two classes of such splines. We further show that the two classes of smoothing splines coincide with the classes S and Σ of interpolating splines. Hence <u>a spline is 'interpolating' or 'smoothing' depending on the optimal property which it satisfies in the particular context</u>. A relation linking the two classes of splines and the operators associated with the smoothing and interpolating splines is derived. Properties of smoothing splines related to a sequence analogous to the sets constructed in chapter II are also obtained.

The notations introduced in the second paragraph of chapter II will be adhered to right through this chapter also. The closed subspaces A and B, the constraint sets $\Phi(a;A^{\perp})$ and $\Phi(b;B^{\perp})$ and the transformations T and τ are as in chapter II. In this chapter also we shall assume throughout that $A^{\perp} + B^{\perp}$ is closed.

4.2. Definition and existence of smoothing splines:

Let us consider the product spaces $G = Z \times A$ and $H = Y \times B$. If $g_1 = (z_1, a_1)$ and $g_2 = (z_2, a_2)$ are any two elements of G and α a real number, we define addition and scalar multiplication in G by

$$g_1 + g_2 = (z_1, a_1) + (z_2, a_2) = (z_1 + z_2, a_1 + a_2)$$

and

$$\alpha g = \alpha(z, a) = (\alpha z, \alpha a)$$

Further, the equation

$$\langle g_1, g_2 \rangle_G = \langle z_1, z_2 \rangle_Z + \rho \langle a_1, a_2 \rangle_X$$

where ρ is a fixed positive number, defines an inner product on G. G then becomes a Hilbert space and the norm in G is given by

$$\|g\|_G = \{\langle g, g\rangle_G\}^{1/2}$$

Similarly, H can be made a Hilbert space by defining an inner product in it suitably. Let L and Q be linear transformations of X into G and H respectively, defined by $Lx = (\tau x, P_A x)$ and $Qx = (Tx, P_B x)$. The linearity of L and Q follows from the linearity of the operators τ, P_A, T and P_B. Since

$$\|Lx\|_G \leq \{\|\tau\|^2 + \rho\}^{1/2} \|x\|_X \text{ and } \|Qx\|_H \leq \{\|T\|^2 + \rho\}^{1/2} \|x\|_X$$

L and Q are continuous transformations. We have

LEMMA 4.2.1. LX <u>is a closed subspace of</u> G.

PROOF. Following the proof given by Atteia ([9] , p.206), let us consider a sequence $\{Lx_n\} \subset LX$ such that $\lim_{n\to\infty} \|Lx_n - g\| = 0$ with $g = (z, a)$ $z \in Z$ $a \in A$. In order to prove that LX is a closed subspace, it is sufficient to show that $g \in LX$. We first notice that since P_A is a continuous linear mapping of X onto A, there exists, from a theorem of Banach (see [10]) a sequence $\{\mu_n\}$ of elements of X such that

$$\lim_{n \to \infty} \|\mu_n - \mu\|_X = 0 \text{ where } \mu \in X$$

with

$$P_A \mu = q \text{ and } P_A \mu_n = P_A x_n$$

Thus there exists a sequence of elements $\{a_n^\perp\} \subset A^\perp$ such that

$$x_n = \mu_n + a_n^\perp$$

Now

$$\|\tau a_n^\perp - (\gamma - \tau\mu)\|_Z \leq \|\tau a_n^\perp - (\gamma - \tau\mu_n)\|_Z + \|\tau\mu_n - \tau\mu\|_Z \qquad (4.2.1)$$

As $n \to \infty$, we have $\|\mu_n - \mu\|_X \to 0$ and $\|Lx_n - g\|_G \to 0$ so that

$$\lim_{n \to \infty} \|\tau\mu_n - \tau\mu\|_Z = 0 \text{ and } \lim_{n\to\infty} \|\tau a_n^\perp - (\gamma - \tau\mu_n)\|_Z = 0$$

Hence from (4.2.1) we obtain

$$\lim_{n\to\infty} \| \tau a_n^{\perp} - (z - \tau\mu) \|_2 = 0$$

Since $\tau A^{\perp}$ is a closed subspace, there exists an element $a^{\perp} \in A$ such that $z - \tau\mu = \tau a^{\perp}$. Set $x = \mu + a^{\perp}$. Then $z = \tau\mu + \tau a^{\perp} = \tau x$ and

$a = P_A \mu = P_A(x - a^{\perp}) = P_A x$. Thus

$$g = (z, a) = (\tau x, P_A x) \in LX$$

Hence LX is closed.

Analogously, we can prove

LEMMA 4.2.2. QX _is a closed subspace of_ H.

The operators L and Q have the same kernel, namely $A^{\perp} \cap B^{\perp}$ and the adjoint operators L^* and Q^* exist. L^* is a continuous linear transformation of LX onto $(A^{\perp} \cap B^{\perp})^{\perp}$ and Q^* is a continuous linear transformation of QX onto $(B^{\perp} \cap A^{\perp})^{\perp}$.

We now establish two extremal properties in the spaces G and H which are used for the definition of smoothing splines.

THEOREM 4.2.3. _If_ $g = (\theta_z, a) \in G$ _and_ $h = (\theta_y, b) \in H$ _then there exist two sets_ $\hat{S}_g$ _and_ $\hat{\Sigma}_h$ _of interpolating splines in_ X _such that_

$$\| L\hat{s}_g - g\|_G = \min\{\| Lx - g\|_G : x \in X\} \ \underline{\text{forall}}\ \hat{s}_g \in \hat{S}_g$$

and

$$\| Q\hat{\sigma}_h - h\|_H = \min\{\| Qx - h\|_H : x \in X\} \ \underline{\text{forall}}\ \hat{\sigma}_h \in \hat{\Sigma}_h$$

Further, the sets $\hat{S}_g$ **and** $\hat{\Sigma}_h$ **reduce to a single element each if and only if** $A^{+} \cap B^{\perp} = \{\theta_x\}$

PROOF. Using the fact that LX is a closed subspace of G, we find that if $g = (\theta_2, a) \in G$ is given, then there exists a unique element $u_g \in LX$ such that u_g is the best approximation in LX to g. Similarly, if $h = (\theta_Y, b) \in H$ is given, we can find a unique $v_g \in QX$ giving the best approximation to h. Hence, there exist two sets $\hat{S}_g$ and $\hat{\Sigma}_h$ of elements in X such that

$$\| L\hat{s}_g - g\|_G = \min\{\| Lx - g\|_G : x \in X\} \ \text{forall}\ \hat{s}_g \in \hat{S}_g$$

and

$$\| Q \hat{\sigma}_h - h \|_H = \min \{ \| Qx - h \|_H : x \in X \} \text{ for all } \hat{\sigma}_h \in \hat{\Sigma}_h$$

We assert that $\hat{S}_g$ and $\hat{\Sigma}_h$ are sets of interpolating splines. We show this only for $\hat{S}_g$, the case of $\hat{\Sigma}_h$ being similar. Now let $\hat{s}_g$ be any element of $\hat{S}_g$. Since $\hat{s}_g \in X$, we have $\hat{s}_g = a_g + a_g^{\perp}$ with $a_g \in A$ and $a_g^{\perp} \in A^{\perp}$. Denote by S_g the set of interpolating splines of $\Phi\,(a_g ; A^{\perp})$ relative to τ. Then

$$\| \tau s_g \|_Z = \min \{ \| \tau \phi \|_Z : \phi \in \Phi (a_g ; A^{\perp}) \} \text{ for all } s_g \in S_g$$

Consequently

$$\| \tau s_g \|_Z \leq \| \tau \hat{s}_g \|_Z$$

But

$$\| L \hat{s}_g - g \|_G \leq \| L s_g - g \|_G \quad \text{for all } s_g \in S_g$$

Hence

$$\| L \hat{s}_g - g \|_G = \| L s_g - g \|_G$$

or equivalently,

$$\| \tau \hat{s}_g \|_Z = \| \tau s_g \|_Z$$

Thus $\hat{s}_g = s_g + b^\perp$ where $b^\perp \in B^\perp$ and $\hat{s}_g \in S_g$

But $\hat{S}_g = \hat{s}_g + (A^\perp \cap B^\perp) = s_g + (A^\perp \cap B^\perp) = S_g$

This proves our assertion.

The last part of the theorem is a consequence of the fact that $A^\perp \cap B^\perp$ is the kernel of L.

DEFINITION 4.2.4. An element $\hat{S}_g$ is called <u>a smoothing spline relative to</u> L <u>and</u> $g = (\theta_z, a)$ and an element of $\hat{\Sigma}_h$ is <u>a smoothing spline relative to</u> Q <u>and</u> $h = (\theta_Y, b)$

<u>4.3. Properties of smoothing splines:</u>

We shall now derive various properties of smoothing splines.

THEOREM 4.3.1. <u>If</u> $\hat{S} = \bigcup_{g \in \theta_z \times A} \hat{S}_g$ <u>and</u>

$\hat{\Sigma} = \bigcup_{h \in \Theta_y \times B} \Sigma_h$

1) $\hat{S} \equiv S$ **and** $\hat{\Sigma} \equiv \Sigma$

2) $L^* L S = A$ **and** $Q^* Q \Sigma = B.$

PROOF. From theorem 4.2.3, it follows that if $\hat{S}_g \subset \hat{S}$, then $\hat{S}_g \subset S$ so that $S \subset S$.

To prove the converse, consider $s_{a_0} \in S$. It is sufficient to find some $a \in A$ such that $g \in (\Theta_z, a)$ and $\|L s_{a_0} - g\|_G = \min\{\|Lx - g\|_G : x \in X\}$ Since $\tau^* \tau S = A \cap B$ (from theorem 2.3.1), there exists $\hat{a} \in A \cap B$ such that $\tau^* \tau s_{a_0} = \hat{a}$. Let $\tilde{a} = a_0 + \frac{\hat{a}}{\rho}$. We claim that $\tilde{a}$ is the element a we are looking for. To this and, we first notice that

$$\langle \rho(\tilde{a} - a_0), a \rangle_X = \langle \hat{a}, a \rangle_X = \langle \hat{a}, a + a^\perp \rangle_X \text{ for all } a \in A$$
$$\text{and for all } a^\perp \in A^\perp$$

Substituting $\hat{a} = \tau^* \tau s_{a_0}$ in the above equation, we obtain

$$\langle \rho(\tilde{a} - a_0), a \rangle_X = \langle \tau^* \tau s_{a_0}, a + a^\perp \rangle_X \text{ for all } a \in A$$
$$\text{and for all } a^\perp \in A^\perp.$$

which implies that

$$\langle (-\tau s_{a_0},\ \tilde{a} - a_0),\ (\tau x, P_A x) \rangle_G = 0 \quad \text{for all } x \in X$$

Consequently

$$(-\tau s_{a_0},\ \tilde{a} - a_0) \in (LX)^{\perp}$$

From the relation

$$(\theta_Z, \tilde{a}) = (-\tau s_{a_0},\ \tilde{a} - a_0) + (\tau s_{a_0},\ P_A s_{a_0}),$$

it is clear that

$$\tilde{g} = (\theta_Z, \tilde{a}) = L s_{a_0} + (-\tau s_{a_0},\ \tilde{a} - a_0)$$

But $L s_{a_0} \in LX$ and $(-\tau s_{a_0},\ \tilde{a} - a_0) \in (LX)^{\perp}$

Hence

$$\| L s_{a_0} - \tilde{g} \|_G = \min \{ \| Lx - \tilde{g} \|_G : x \in X \}$$

This is what we wished to prove. Similarly, we can prove that $\hat{\Sigma} = \Sigma$

To prove the second part, we observe that for $s \in S$

$$\langle La^{\perp}, Ls \rangle_G = \langle \tau a^{\perp}, \tau s \rangle_Z = 0 \quad \text{for all } a^{\perp} \in A^{\perp}$$

and hence $L^* L S \subset A$.

Now L^* maps LX onto $(A^{\perp} \cap B^{\perp})^{\perp}$ and so given $a \in A$, there exists $x \in X$ such that $L^* L x = a$. Thus

$$\langle L^* L x, a^{\perp} \rangle_X = 0 \quad \text{for all } a^{\perp} \in A^{\perp}$$

or equivalently,

$$\langle Lx, La^{\perp} \rangle_G = 0 \quad \text{for all } a^{\perp} \in A^{\perp}$$

We have

$$\langle (\tau x, P_A x) , (\tau a^{\perp}, P_A a^{\perp}) \rangle_G = 0 \quad \text{for all } a^{\perp} \in A^{\perp}$$

By the definition of the inner product, we get

$$\langle \tau x, \tau a^{\perp} \rangle_Z + \rho \langle P_A x, P_A a^{\perp} \rangle_X = 0$$

Since for $a^{\perp} \in A^{\perp}$, $P_A a^{\perp} = \theta_X$, this implies

$$\langle \tau x, \tau a^{\perp} \rangle_Z = 0 \quad \text{for all } a^{\perp} \in A^{\perp}$$

Thus $\tau x \in (\tau A^{\perp})^{\perp}$ i.e., $\tau x \in \tau S$

(by theorem 2.3.1). Hence $A \subset L^* L S$.

Similarly, we can prove that $Q^* Q \Sigma = B$. This completes the proof of the theorem.

COROLLARY 4.3.2. 1) <u>If</u> $g = (\theta_2, a)$ <u>and</u> S_a <u>is the set of interpolating splines belonging to</u> $\Phi\,(a;A^{\perp})$ <u>then</u>

$$\| \tau s_a \|_Z = \min \{ \| \tau \phi \|_Z : \phi \in \Phi(a\,; A^{\perp}) \} \text{ if and only if}$$

$$\| L s_a - g \|_G = \min \{ \| L\phi - g \|_G : \phi \in \Phi(a : A^{\perp}) \}$$

<u>for all</u> $s_a \in S_a$

<u>and</u>

$$\| L\phi - L s_a \|_G^2 = \| L\phi \|_G^2 - \| L s_a \|_G^2 \quad \text{for all } \phi \in \Phi(a\,; A^{\perp})$$

$$\text{and for all } s_a \in S_a \qquad (4.3.1)$$

Equation (4.3.1) is an analogue of the first integral relation for smoothing splines.

2) <u>From the identity</u> (2.3.3) <u>and theorem</u> 4.3.1, <u>we have</u>

$$\boxed{\tau^*\tau S = (L^*LS)\cap(Q^*Q\Sigma) = T^*T\Sigma}$$

In the following, it will be assumed that $A^\perp \cap B^\perp = \{\theta_X\}$. In chapter II, we constructed a sequence of sets $\{\Phi(S_n;S_n^\perp)\}_{n=1}^{\infty}$

Corresponding to the operators $L: X \longrightarrow Z \times A$ and $Q: X \longrightarrow Y \times B$, transformations $L_n: X \to Z \times S_n$ and $Q_n: X \longrightarrow Y \times \Sigma_n$ can be constructed such that $L_n x = (\tau x, P_{S_n} x)$ and $Q_n(x) = (Tx, P_{\Sigma_n} x)$. $L_n X$ and $Q_n X$ are closed subspaces of $Z \times S_n$ and $Y \times \Sigma_n$ respectively since $S_n^\perp + B^\perp$ and $\Sigma_n^\perp + A^\perp$ ($n = 1,2,\ldots.$) are closed subsets of X. Moreover, for any positive integer n,

$$L_n^* L_n' S_{n+1} = S_n$$

Hence

$$\tau^*\tau S_{n+1} = (Q_n^* Q_n \Sigma_{n+1}) \cap A$$

and

$$T^* T \Sigma_{n+1} = (Q_n^* Q_n \Sigma_{n+1}) \cap A \,.$$

4.4. Remarks:

1) It has been proved in theorem 4.3.1 that the class $\hat{S}$ of smoothing splines coincides with the class S of interpolating splines. But the kernel $B^{\perp}$ of τ is a closed subspace of S (from corollary 2.3.2). Hence the image under L of any $b^{\perp} \in B^{\perp}$ must be the best approximation in LX to an element of $\theta_z \times A$. Since $L b^{\perp} = (\tau b^{\perp}, P_A b^{\perp}) = (\theta_z, P_A b^{\perp})$ for any $b^{\perp} \in B^{\perp}$, $L b^{\perp} \in \theta_z \times A$. In other words, the closed subspace $\theta_z \times P_A (B^{\perp})$ is precisely the set of all those elements of $\theta_z \times A$ which are contained in LX.

2) The transformation L mapping X onto G has been defined as $Lx = (\tau x, P_A x)$. The existence and uniqueness of the set of smoothing spline $\hat{S}_g$ depends only on $A^{\perp}$ and $B^{\perp}$, i.e., the kernels of τ and P_A. Let M and N be any two continuous linear transformation of X onto two real Hilbert spaces $\mathcal{U}$ and $\mathcal{V}$ isomorphic to B and A respectively such that the kernel of M is $B^{\perp}$ and the kernel of N is $A^{\perp}$. $\mathcal{U} \times \mathcal{V}$ is a Hilbert space with an inner product suitably

defined. Consider a transformation J taking X to $U \times V$ defined by $Jx = (Mx, Nx)$. Then the existence of smoothing splines relative to L guarantees the existence of smoothing splines relative to J.

3) The transformation associated with a smoothing spline involves two continuous linear transformations. For instance, the transformation L depends on τ and P_A. It has already been remarked in chapter II that $A^{\perp} + B^{\perp}$ is closed if either $A^{\perp}$ or $B^{\perp}$ is of finite dimension. Thus if either of the continuous linear transformations appearing in the definition of an operator of the type L has a finite - dimensional kernel, then the corresponding smoothing splines exist.

4.5. Special cases:

1) The $\mathcal{L}g$ -spline has been studied in [15] and in chapter II as an interpolating spline. It can also be obtained as a smoothing spline. The product space $L_2 \times M$ is a Hilbert space with the inner product defined as

$$\langle (f_1, m_1), (f_2, m_2) \rangle_{L_2 \times M} = \langle f_1, f_2 \rangle_{L_2} + \rho \langle m_1, m_2 \rangle_{\mathcal{H}^q} \quad \rho >$$

The set $\mathcal{L}\mathcal{H}^q \times M$ is a closed subspace of $L_2 \times M$ since $M^\perp + \mathcal{U}$ is closed. Thus, given $m \in M$, there exists a set S_m of $\mathcal{L}g$ - splines satisfying

$$\|(\mathcal{L} s_m, P_M s_m) - (\theta_{L_2}, m)\|_{L_2 \times M} = \min\{\|(\mathcal{L}x, P_M x) - (\theta_{L_2}, m)\|_{L_2 \times M} : x \in X\}$$
$$\text{for all } s_m \in S_m$$

Thus all the properties of the smoothing splines hold for $\mathcal{L}g$-splines also. In particular, the set S_m consists of a single $\mathcal{L}g$-spline if and only if $M^\perp \cap \mathcal{U} = \{\theta_{\mathcal{H}^q}\}$ The closed subspace $S_{\mathcal{L}}$ of $\mathcal{L}g$-splines is the union of S_m as m varies in M. Further, s is an $\mathcal{L}g$-spline if and only if

$$\langle \mathcal{L}^*\mathcal{L}s + \rho P_M s, m^\perp \rangle_{\mathcal{H}^q} = 0 \quad \text{for all } m^\perp \in M^\perp$$

2) Let K be the subspace spanned by a set $\{k_i\}_1^n$ of linearly independent elements in X and $H = Y \times \mathbb{R}^n$ the Hilbert space with the inner product

$$\langle h_1, h_2 \rangle_H = \langle y_1, y_2 \rangle_Z + \rho \langle r_1, r_2 \rangle_{\mathbb{R}^n} \quad \rho > 0$$

Consider the continuous linear operator F on X into Z defined by $Fx = [Tx, (\langle k_1, x\rangle, \cdots, \langle k_n, x\rangle)]$

If $r = (r_1, \cdots, r_n) \in \mathbb{R}^n$ and $h = [\theta_Y, (r_1, \cdots, r_n)]$,

then Atteia [7] and Anselone and Laurent [4] defined the smoothing spline relative to T and the point h as an element s of X satisfying

$$\|Fs - h\|_H = \min_{x \in X} \|Fx - h\|_H \tag{3.5.1}$$

Assuming that (1) $\dim N(T) = q$, (2) $n \geq q$ and (3) $K^{\perp} \cap N(T) = \{\theta_X\}$, they proved the existence of a unique smoothing spline satisfying (3.5.1). We observe that $N(T) + K^{\perp}$ is closed since $N(T)$ is of finite dimension. Thus the existence of a set of smoothing splines satisfying (3.5.1) is a consequence of theorem 4.2.3 and remark 4.4.2. Since it has also been assumed in [4] and [7] that $K^{\perp} \cap N(T) = \{\theta_X\}$, the smoothing spline is unique. Thus the results in sections 1 and 2 of chapter VIII in [7] and propositions 8.1 and 8.2 of [4] are a direct consequence of our theorems 4.2.3 and 4.3.1.

CHAPTER V

A FURTHER GENERALISATION OF THE INTERPOLATING SPLINE

5.1. Introduction:

In defining interpolating and smoothing splines, the best approximation property of closed subspaces of a Hilbert space has been exploited. However, this property is not restricted to subspaces alone. It is sufficient if the set under consideration is closed and convex. In this chapter, we shall define in a natural way what we shall call c-splines in a Hilbert space when the set F under consideration is closed and convex. If F happens to be a closed subspace, our c-splines give the interpolating splines defined in chapter II.

The notations used in this chapter are the same as those introduced in chapter II. As in the earlier chapters, A and B are two closed subspaces of a Hilbert space X, and Y and Z are two Hilbert spaces isomorphic to A and B with isomorphisms I_A and I_B respectively. The transformations T and τ are given by $T = I_A P_A$ and $\tau = I_B P_B$.

5.2. Definition and existence of c-splines:

If C is a closed and convex subset of a Hilbert space H and $h \in H$, we set

$$\Psi(h; C) = h - C$$

DEFINITION 5.2.1. Let T be a continuous linear transformation of a Hilbert space H onto a Hilbert space H'. Suppose that M is a closed subspace of H and C a closed and convex subset of $M^{\perp}$. If, for $m \in M$, there exists an element $\xi \in \Psi(m; C)$ satisfying

$$\|T\xi\|_{H'} = \min\{\|T\psi\|_{H'} : \psi \in \Psi(m; C)\}$$

then ξ is called a c - spline of $\Psi(m; C)$ relative to T.

We now have the following

THEOREM 5.2.2. Let F be a closed and convex subset of $A^{\perp}$. Assume that

$$F + B^{\perp} \text{ is closed} \tag{5.2.1}$$

and

$$A^{\perp} \cap B^{\perp} = \{\theta_x\} \tag{5.2.2}$$

Then, for $a \in A$, there exists a unique element ξ_a of X satisfying

$$\| \tau \xi_a \|_Z = \min \{ \| \tau \psi \|_Z : \psi \in \Psi(a; F) \}$$

PROOF. From lemma 2.2.3, $\tau(F)$ is closed. Since F is convex and τ is linear, $\tau(F)$ is also convex. $\tau(F)$ being a closed and convex subset of the Hilbert space Z, there exists a unique element $z_F \in \tau(F)$ such that

$$\| \tau a - z_F \|_Z = \min \{ \| \tau a - \tau f \|_Z : f \in F \}$$

Suppose that there exist $f_1, f_2 \in F$ such that $\tau f_1 = z_F = \tau f_2$. Then $f_1 - f_2 \in B^\perp \cap A^\perp = \{\theta_X\}$ (from the assumption (5.2.2)). Hence there exists a unique $f_a \in F$ such that $\tau f_a = z_F$. Now set

$$\xi_a = a - f_a$$

Then ξ_a is the unique element of $\Psi(a;F)$ satisfying

$$\| \tau \xi_a \|_Z = \min \{ \| \tau \psi \|_Z : \psi \in \Psi(a; F) \}$$

COROLLARY 5.2.3. ***If*** ξ_a ***is the c-spline of*** $\Psi(a;F)$ ***relative to*** T***, then***

$$\|\tau\psi - \tau\xi_a\|_Z^2 \leq \|\tau\psi\|_Z^2 - \|\tau\xi_a\|_Z^2 \quad \underline{\text{for all}} \ \psi \in \Psi(a;F)$$

PROOF. We have

$$\|\tau\psi - \tau\xi_a\|_Z^2 = \|\tau\psi\|_Z^2 - \|\tau\xi_a\|_Z^2 + 2\langle \tau\xi_a, \tau\xi_a - \tau\psi\rangle_Z$$

Since $\tau\xi_a$ is the unique element of minimal norm in $\tau\Psi(a;F)$, it is characterised by the following inequality

$$\langle \tau\psi - \tau\xi_a, \tau\xi_a\rangle_Z \geqslant 0 \quad \text{for all } \psi \in \Psi(a;F)$$

2) ***If*** L ***is the operator defined as in chapter III and*** $g = (\theta_2, a)$ ***, then***

$$\|\tau\xi_a\|_Z = \min\{\|\tau\psi\| : \psi \in \Psi(a;F)\} \ \underline{\text{if and only if}}$$

$$\|L\xi_a - g\|_G = \min\{\|L\psi - g\|_G : \psi \in \Psi(a;F)\}$$

5.3. A remark on theorem 5.2.1:

Using a result of Daniel and Schumaker (quoted below), we can prove the following

THEOREM 5.3.1. Let E and F be closed and convex subsets of $B^{\perp}$ and $A^{\perp}$ respectively. If

$$A^{\perp} + B^{\perp} \text{ is closed} \tag{5.3.1}$$

and

$$A^{\perp} \cap B^{\perp} = \{\theta_X\} \tag{5.3.2}$$

then for $a \in A$ and $b \in B$, there exist two c-splines ξ_a and ξ_b satisfying

$$\|T\xi_a\|_Z = \min\{\|T\psi\|_Z \,;\ \psi \in \Psi(a;F)\}$$

and

$$\|T\xi_b\|_Y = \min\{\|T\psi\|_Y \,:\ \psi \in \Psi(b;E)\}$$

The result of Daniel and Schumaker is the following

LEMMA 5.3.2. ([11] , p. 4) Let X be a Banach space having N and X_0 as closed subspaces such that $X_0 + N$ is closed in X. Let $\mathcal{U}$ be a subset

of X_0 **such that** $\mathcal{U} + (N \cap X_0)$ **is norm closed in** X_0.
Then $\mathcal{U} + N$ **is norm closed in** X.

5.4. Properties of c-splines:

THEOREM 5.4.1. **If** $S_F = \{ \xi_a : a \in A \}$ **, then**

(i) S_F **is closed**

and

(ii) $B_1^{\perp} \subset S_F$

where

$$B_1^{\perp} = \{ b^{\perp} \in B^{\perp} : b^{\perp} = a_b^{\perp} + a_{b^{\perp}}^{\perp} ,\quad a_{b^{\perp}} \in A ,\ (-a_{b^{\perp}}^{\perp}) \in F \}$$

PROOF. Let $\xi_a^{(n)}$ be a sequence in S_F converging to ξ in X. Now ξ can be represented as $\xi = a + a^{\perp}$ where $a \in A$ and $a^{\perp} \in A^{\perp}$. Since $\xi_a^{(n)} \in S_F$, for $n = 1,2,$ ---- , there exist sequences $\{a^{(n)}\}_{n=1}^{\infty} \subset A$ and $\{f_n\}_{n=1}^{\infty} \subset F$ such that $\xi_a^{(n)} = a^{(n)} - f_n$ where τf_n is the best approximation in τF to $\tau a^{(n)}$. Thus

$$\| a^{(n)} - f_n - \overline{a + a^{\perp}} \|_X \longrightarrow 0 \quad \text{as } n \to \infty$$

which implies that

$$\| a^{(n)} - a \|_X \to 0 \quad \text{and} \quad \| f_n + a^{\perp} \|_X \to 0 \quad \text{as } n \to \infty$$

Hence there exists $f_a \in F$ such that $\xi = a - f_a$

It remains to be shown that τf_a is the best approximation in τF to $\tau a^{(n)}$. Since τf_n is the best approximation in τF to $\tau a^{(n)}$, we have for every n

$$\langle \tau f - \tau f_n, \tau f_n - \tau a^{(n)} \rangle_Z \geq 0 \quad \text{for all } f \in F$$

Since the inner product is a continuous function of its arguments,

$$\langle \tau f - \tau f_a, \tau f_a - \tau a \rangle_Z \geq 0 \quad \text{for all } f \in F$$

In other words, τf_a is the best approximation in τF to τa. This proves (i).

To prove (ii), consider $b'^{\perp} \in B'^{\perp}$. Then there exists $f \in F$ such that $b'^{\perp} = a^{\perp}_{b'} - f$

If $\tau f_{b'^{\perp}}$ is the best approximation in τF to $\tau a_{b'^{\perp}}$, set $b''^{\perp} = a_{b'^{\perp}} - f_{b'^{\perp}}$. Suppose that $b''^{\perp} \neq b'^{\perp}$. By definition, $b''^{\perp}$ is the c-spline of $\bar{\Psi}(a_{b'^{\perp}}, F)$. But $b'^{\perp} \in \bar{\Psi}(a_{b'^{\perp}}; F)$

Hence

$$\| \tau b''^{\perp} \|_2 \leq \| \tau b'^{\perp} \|_2$$

Now $b'^{\perp} \in B^{\perp} \Longrightarrow \| \tau b'^{\perp} \|_2 = 0$ which in turn implies that

$$\| \tau b''^{\perp} \|_2 = \| \tau b'^{\perp} \|_2$$

From theorem 5.2.2, $b''^{\perp} = b'^{\perp}$ and consequently, $B_1^{\perp} \subset S_p$.

5.5. A more general constraint set:

If U and W are subsets of A and $A^{\perp}$ respectively, we set

$$\Psi(U\,;\,W) = U - W.$$

Analogous to theorem 3.2.1, we have the following

THEOREM 5.5.1. Let C be a compact, convex subset of A and F a closed convex subset of $A^{\perp}$. Assume that the conditions (5.2.1) and (5.2.2) hold. Then there exists $c_0 \in C$ such that if $\mathcal{B}_{c_0}$ denotes the c-spline belonging to $\Psi(\varsigma_0 ; F)$, then

$$\|\tau\xi\|_Z = \min\{\|\tau\psi\|_Z : \psi \in \Psi(C ; F)\} \tag{5.5.1}$$

for all ξ in the set $\Gamma = [(\xi_{c_0} + B^+) \cap \Psi(C; F)]$

Further $\tau(T)$ is the unique element of Z satisfying (5.5.1)

In order to prove theorem 5.5.1, we need the

LEMMA 5.5.2. If $\mathcal{U}$ is any subset of A, F a closed and convex subset of A^+ and if there exists a set $V \subset \Psi(\mathcal{U} ; F)$ such that

$$\|\tau v\|_Z = \min\{\|\tau\psi\|_Z ; \psi \in \Psi(\mathcal{U} ; F)\} \text{ for all } v \in V$$

then every element of V is a c-spline.

PROOF. We are given that, for $v \in V$

$$\|\tau v\|_Z = \min\{\|\tau\psi\|_Z ; \psi \in \Psi(\mathcal{U}; F)\}$$

Let $v_0 \in V$. Then there exists $u_0 \in \mathcal{U}$ and $f_0 \in F$ such that $v_0 = u_0 - f_0$. Let ξ_0 be the c-spline of $\Psi(u_0 ; F)$ relative to τ. Now $\Psi(u_0;F) \subset \Psi(\mathcal{U};F)$.

Since

$$\|\tau v_0\|_Z = \min\{\|\tau\psi\|_Z : \psi \in \Psi(u; F)$$

we have

$$\|\tau v_0\|_Z = \min\{\|\tau\psi\|_Z : \psi \in \Psi(u_0; F)\} = \|\tau\xi_0\|_Z$$

The uniqueness of the spline ξ_0 implies that $v_0 = \xi_0$

PROOF OF THEOREM 5.5.1. Since $F + B^{\perp}$ is closed, τF is closed. Hence, given $z \in Z$, a 'projection map' $\mathcal{P}_{\tau F}$ can be associated with τF defined by

$$\|z - \mathcal{P}_{\tau F} z\|_Z = \min\{\|z - \tau f\|_Z : f \in F\}$$

Corresponding to each $a \in C$, there exists a c-spline $\xi_a \in \Psi(a; F)$ satisfying

$$\|\tau\xi_a\|_Z = \min\{\|\tau\psi\|_Z : \psi \in \Psi(a; F)\}$$

Define $f(a) = \| \tau \xi_a \|_Z$ for $a \in C$ and set

$$\overline{\Gamma} = \{ \xi_a \in S_F : a \in C \}.$$

For $a_1, a_2 \in C$, we have

$$|f(a_1) - f(a_2)| \leq \| \tau \xi_{a_1} - \tau \xi_{a_2} \|_Z$$

$$\leq \| \tau a_1 - \tau a_2 \|_Z + \| \mathcal{P}_{\tau F}(\tau a_1) - \mathcal{P}_{\tau F}(\tau a_2) \|_Z$$

Since $\mathcal{P}_{\tau F}$ is Lipschitz continuous,

$$|f(a_1) - f(a_2)| \leq 2 \| \tau a_1 - \tau a_2 \|_Z$$

$$\leq 2 \| \tau \| \, \| a_1 - a_2 \|_X$$

Thus f is a continuous function on the compact set C. Consequently, there exists $c_0 \in C$ satisfying

$$\| \tau \xi_{c_0} \|_Z = \min \{ \| \tau \xi_a \|_Z : a \in C \} = \min \{ \| \tau \xi_a \|_Z ; \xi_a \in \overline{\Gamma} \}$$

But

$$\| \tau \xi_a \|_Z = \min \{ \| \tau \psi \|_Z : \psi \in \Psi(a; F) \}$$

Hence

$$\|\tau \xi_{c_0}\|_Z = \min \{ \|\tau \psi\|_Z : \psi \in \Psi (C : F) \} \quad (5.5.2)$$

Each element of the set $\Gamma = [(\xi_{c_0} + B^{\perp}) \cap \Psi(C ; F)]$ also satisfies (5.5.2) and from lemma 5.5.2, $\Gamma \in S_F$. Since Z is a Hilbert space and C is convex, if $\tau \xi_{c_0} = \gamma_{c_0}$ then γ_{c_0} is the unique element of minimal norm in $\tau \Psi(C ; F)$.

<u>5.6. An extension of theorem 5.5.1:</u>

If $\tau \mathcal{U}$ and τF are closed and convex, the problem of finding the minimal element in $\tau \Psi (\mathcal{U} ; F)$ is equivalent to the problem of finding the shortest distance between the two closed and convex sets $\tau \mathcal{U}$ and τF. This point of view enables us to obtain an extension of theorem 5.5.1. We need two results from A.A.Goldstein (12).

THEOREM 5.6.1 ([12] p.100) <u>Let</u> K_1 <u>and</u> K_2 <u>be two closed and convex sets in a Hilbert space</u> H. <u>Let</u> $\mathcal{P}_{K_i}$ <u>denote the projection operator for</u> K_i, $i = 1,2$ (i.e., <u>Given</u> $h \in H$,

$$\| h - \mathcal{P}_{K_i} h \|_H = \min \{ \| h - k_i \| : k_i \in K_i \ \ i = 1,2 \})$$

<u>Any fixed point of</u> $\mathcal{P} = \mathcal{P}_{K_1} \mathcal{P}_{K_2}$ <u>is a point of</u> K_1 <u>nearest</u> K_2 <u>and conversely.</u>

THEOREM 5.6.2. ([12] p.101) <u>Let</u> K_1, K_2 <u>and</u> $\mathcal{P}$ <u>be as in theorem</u> 5.6.1 <u>and</u> k_1 <u>an arbitrary element of</u> K_1. <u>Convergence of</u> $\{\mathcal{P}^n k_1\}$ <u>to a fixed point of</u> $\mathcal{P}$ <u>is assured when either</u> K_1 <u>or</u> K_2 <u>is compact.</u>

By virtue of lemma 5.5.2, the problem of finding a set of splines whose image under τ is the minimal element of $\tau\Psi(U ; F)$, (τU and τF being closed and convex) is equivalent to the problem of finding a fixed point of $\mathcal{P}_{\tau U} \mathcal{P}_{\tau F}$. Thus theorem 5.5.1. can be extended to the following

THEOREM 5.6.3. <u>If the conditions</u> (5.3.1) <u>and</u> (5.3.2) <u>hold, and either</u> U <u>or</u> F <u>is compact, and if both</u> U <u>and</u> F <u>are convex then there exists a set</u> $\tilde{\Gamma}$ <u>of c-splines satisfying</u>

$$\|\tau s\|_Z = \min\{\|\tau\psi\|_Z : \psi \in \Psi(U ; F)\}$$

<u>for all</u> $s \in \tilde{\Gamma}$.

<u>5.7. Connection with smoothing splines:</u>

Smoothing splines relative to an operator L and points of a Hilbert space G were introduced in chapter IV. If the conditions (5.3.1) and (5.3.2) hold, it is proved below that the smoothing splines relative to L and points belonging to a particular subset of G are also c-splines. We set

$$G_1 = \{ g \in G : g = (\theta_2, a),\ g = L\hat{s}_g + v_g^{\perp},\ L\hat{s}_g \in LX,\ v_g^{\perp} \in (LX)^{\perp},\ \hat{s}_g = a_g + a_g^{\perp},\ a_g \in A, (-a_g^{\perp}) \in F \}$$

We have the

THEOREM 5.7.1. __If the assumptions__ (5.3.1) __and__ (5.3.2) __are satisfied, then the smoothing spline relative to__ L __and any point of__ G_1 __is a c-spline.__

PROOF. Given $g = (\theta_2, a) \in G$, there exists a unique smoothing spline $\hat{s}_g$ satisfying

$$\| L\hat{s}_g - g \|_G = \min \{ \| Lx - g \|_G : x \in X \}$$

Since $s_g^{\perp}$ is also an interpolating spline, $\hat{s}_g = a_g - a_{a_g}^{\perp}$ where $\tau a_{a_g}^{\perp} = P_{\tau A^{\perp}}(\tau a_g)$. Denote by σ_g the c-spline of $\Psi(a_g ; F)$ relative to τ. If $g \in G_1$ then

$$\| \tau \sigma_g \|_Z \leq \| \tau \hat{s}_g \|_Z$$

Since $\hat{s}_g$ is a smoothing spline, we have

$$\| L \hat{s}_g - g \|_G \leq \| L \varsigma_g - g \|_G$$

We can now conclude that $\varsigma_g = \hat{s}_g$.

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