INVESTIGATING VIOLATIONS OF SOME FUNDAMENTAL SYMMETRIES OF NATURE VIA DALITZ PLOTS AND DALITZ PRISMS

By

Dibyakrupa Sahoo

(PHYS10200904007)

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

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Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Mr. Dibyakrupa Sahoo entitled "Investigating Violations of Some Fundamental Symmetries of Nature via Dalitz Plots and Dalitz Prisms" and recommend that it maybe accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

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Place: IMSc, Chennai Guide: Prof. Rahul Sinha

STATEMENT BY AUTHOR

Dibyakrupa Sahoo

DECLARATION

I, Dibyakrupa Sahoo, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Dibyakrupa Sahoo

List of Publications arising from the thesis

Journal

1. "Model independent method for a quantitative estimation of $SU(3)$ flavor symmetry breaking using Dalitz plot distributions", Dibyakrupa Sahoo, Rahul Sinha and N. G. Deshpande,
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2. "Testing violations of Bose and CPT symmetries via Dalitz plots and their prismatic generalizations", Dibyakrupa Sahoo, Rahul Sinha and N. G. Deshpande,
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3. "Technique to observe CP Violation in D mesons using Bose Symmetry and Dalitz Plots", Dibyakrupa Sahoo, Rahul Sinha, N. G. Deshpande and Sandip Pakvasa,
Physical Review D **89**, 071903(R) (2014).

Chapters in books and lecture notes

Conferences

Others

Dibyakrupa Sahoo

Dedicated to my parents

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Contents

Synopsis

Introduction: Nature has many fundamental symmetries and corresponding symmetry operations; such as translations in spacetime, rotation in space, uniform velocity along a straight line (or Lorentz transformation), space-inversion (or parity P), time reversal T, interchange of identical bosons (or Bose symmetry), interchange of identical fermions (or Fermi antisymmetry), quantum-mechanical phase, matter-antimatter exchange (or charge conjugation C), the combined operations of charge conjugation, parity and time-reversal (CP and CPT). Symmetries not only help in the formulation of underlying physical laws but also provide an understanding of their mechanism. Therefore, the concept of symmetry have occupied the central role in our search for and eventual formulation of various fundamental laws of physics. It is not only very important to look for new symmetries of Nature, but it is also important to find out violations of existing symmetries. A quantitative estimate of the violation of a symmetry measures the extent to which the concerned symmetry operation can be applied.

Motivation: In this thesis we are concerned with violations of some fundamental symmetries in the realm of elementary particles, namely, Bose symmetry, the combined operations of charge conjugation and parity (and time-reversal), and the $SU(3)$ flavor symmetry. These four symmetries influence various elementary particle processes. So by studying some specific particle processes one can look for signatures of violations of these symmetries. In particular, we carefully exploit those particle processes which have three or more particles (mostly mesons) in their final states. For three-body decays, it is well know that the phase-space plot or Dalitz plot has a lot of information about the underlying dynamics [1–4]. Multi-body decays (with more than three final particles) can be treated as some effective three-body decay by introducing 'effective' particles [5]. In this case the concept of Dalitz plot can be generalized to a three-dimensional plot which has a prism-like appearance suggesting that ‘Dalitz prism’ might be an apt name for this plot [6]. In some specific decay modes, violations of the symmetries under our consideration leave their mark on the Dalitz plot and Dalitz prism, as an observable asymmetry in the distribution of events. Measuring these asymmetries would, therefore, provide quantitative estimates of the extent of breaking of the symmetries under consideration. Additionally, the Dalitz prism would help in gathering a huge amount of data that would help in a better quantitative estimation of the violation of the symmetry under consideration.

CP violation: The predominance of matter over anti-matter in our observable universe necessitates that some fundamental law of nature must violate the CP symmetry [7]. It is observed that in the weak decays of K and B mesons [8–15], the CP symmetry is indeed violated. In the standard model of particle physics (SM), these observed CP violations can be very well accounted for by the Kobayashi-Maskawa mechanism [5]. It also well known that the CP violation predicted by SM fails to explain the observed baryon asymmetry in our universe. Thus we must search for new sources of CP violation. SM predicts very small CP violation in the D meson sector, and experimental data are yet inconclusive about their size. New methods for observing CP violation are, therefore, most welcome. It was shown in Ref. [16,17] that for untagged neutral B meson decays to certain three-body, self-conjugate, hadronic final states, direct CP violation would appear as an asymmetry in the Dalitz plot. In Ref. [18] we have shown it explicitly, that for processes of the type $B \to D^0\bar{D}^0X$, where X can be π or K and both D^0 and $\bar{D}^0$ are reconstructed from flavor insensitive but distinct final states of same CP, any direct CP violation in the neutral D meson decays would appear as an asymmetry in the Dalitz plot under D^0 and $\bar{D}^0$ exchange. When the two neutral D mesons are reconstructed from final states of identical CP, they are fully Bose symmetric to one another under full CP symmetry. This would require that the Dalitz plot be symmetric under D^0 and $\bar{D}^0$ exchange. Once CP violation is allowed, there is no longer any Bose symmetry, and hence the Dalitz plot would exhibit asymmetry under D^0 and $\bar{D}^0$ exchange. In Ref. [18] it is shown that the asymmetry in the Dalitz plot is proportional to the difference in the direct CP violation parameters for the two D decays in the final state. Since the parent particle has no role here, continuum production of $c\bar{c}$ can also be taken into consideration. Including such production processes would help in increasing the statistics of the events and hence the precision of the CP violation parameters being probed would get enhanced.

CPT violation: CPT symmetry is one of the most fundamental symmetries of nature. Even though CP is violated in some weak decays, CPT is assumed to be fully conserved in all interactions. It was shown in Ref. [19,20] that in any interacting field theory, CPT violation invariably leads to associated breakdown of Lorentz invariance. We shall, however, not dwell upon the Lorentz violation in our discussion. In Ref. [6], it is shown that for three-body processes happening via either strong or electromagnetic interactions (in which CP is conserved) with self-conjugate, hadronic final states and two of the final particles being CP conjugate of one another, the Dalitz plot must be symmetric under exchange of the two particles. Any asymmetry in the Dalitz plot under the said exchange can happen, if CPT is violated (assuming CP is still conserved) or if CP is violated (assuming CPT holds good) or if both CP and CPT are violated. In Ref. [6], we introduce some CPT violating parameters and show that the asymmetry in the Dalitz plot is directly proportional to these. Finding an asymmetry in such a Dalitz plot is the best signature of new physics, since in these decay modes we do not expect any CP violation, and CPT itself is a very robust symmetry. Since we expect this violation to be extremely small, the precision required in the measurement would demand a gargantuan amount of events to be studied. The decay modes which one can use to look for CPT violation are $J/\psi \to N\pi^+\pi^-$, where N can be $\pi^0, \omega, \eta, \phi, \omega\pi^0, p\bar{p}, n\bar{n}, \pi^0 K^+K^-, \eta K^+K^-$ etc. For multi-body (> 3) decays which can be treated as effective three-body decays, the Dalitz plot can be generalized to a three dimensional plot which is equivalent to stacking up one Dalitz plot after another with increasing center-of-momentum energy. Since this plot resembles a prism we call it as 'Dalitz prism'. This Dalitz prism can now acquire immense number of events. For the purpose of our analysis, we take the projection of the full Dalitz prism onto its base and then look for any asymmetry in it under $\pi^+ \leftrightarrow \pi^-$ exchange.

Bose symmetry violation: Bose symmetry is another cornerstone of modern physics. Bose symmetry tells that a state made up of two identical bosons (particles with integer spin quantum numbers) remains the same when the two bosons are exchanged [21]. Bose symmetry is, in general, true for bosons which are stable. Nevertheless, it is used in particle physics to include unstable mesons also. Therefore, it is only pertinent to look for violations of Bose symmetry in mesons which are also composite particles. In Ref. [6] we consider a few three-body decays (all final particles are mesons) in which a minimum of two final particles are the same but they decay via different decay channels. If the two mesons were Bose symmetric, the Dalitz plot would remain symmetric under their exchange. Conversely, any asymmetry in the Dalitz plot under the exchange of the two particles is a tell-tale signature of Bose symmetry violation. One example of such a decay mode is $B^0 \to 3K_S^0$, in which two of the the K_S^0's are reconstructed from $\pi^+\pi^-$ and the remaining K_S^0 is reconstructed from $\pi^0\pi^0$. For this particular case, only half of the Dalitz plot can be obtained and all the three sextants of the Dalitz plot contained in this half would have to be symmetric with respect to one another. Any deviation would constitute a signature of violation of Bose symmetry.

Breaking of $SU(3)$ flavor symmetry: $SU(3)$ flavor symmetry is not an exact symmetry of nature. From its first proposal to explain light hadronic states in *the eightfold* *way* [22–26], it has always been considered to be broken in order to account for the mass differences amongst the hadronic states it relates. Full $SU(3)$ flavor symmetry implies that the up (u), down (d) and strange (s) quarks are identical. Hence, under full $SU(3)$ flavor symmetry these quarks can be exchanged with each other without affecting any physical observable. However, the mass difference between s quark and either of u or d quarks is substantial, the quarks u and d do not have same electric charge, and thus these quarks are not fully exchangeable with one another resulting in many observables which are sensitive to the $SU(3)$ flavor symmetry breaking. Nevertheless, an accurate quantitative estimate of $SU(3)$ flavor breaking has not been accomplished. In Ref. [27] we give a model independent prescription for quantitative estimation of $SU(3)$ flavor symmetry breaking using the Dalitz plots for a few specific kind of three-body decays. We know that the $SU(3)$ flavor symmetry has three non-commuting $SU(2)$ subgroups, namely, isospin, U-spin and V-spin. We consider only those three-body decays in which the final mesons are kaons or pions and particles inside two pairs of the final three mesons are connected to one another by two distinct $SU(2)$ symmetries. Since full $SU(3)$ flavor symmetry implies that all the three $SU(2)$ symmetries are individually and simultaneously valid, the Dalitz plot for the modes can be shown to have either fully symmetric distribution or fully anti-symmetric distribution. This implies that the alternate sextants of the Dalitz plot are identical to one another. Any deviation from this observation would constitute a violation of full $SU(3)$ flavor symmetry. In Ref. [27] we provide asymmetries which can quantify this violation.

Conclusion: In Ref. [6,18,27] we have provided new methods to look for violations of CP, CPT, Bose and $SU(3)$ flavor symmetries by using Dalitz plots and the new method of Dalitz prisms. These symmetries play some of the very vital roles in particle physics and any unusual violation of these symmetries would point out various new physics possibilities. Therefore, it would not be overemphasizing to belabour the point that accurate quantitative estimates of these symmetry violations constitute a significant step forward in our search for new physics.

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"Testing violations of Bose and CPT symmetries via Dalitz plots and their prismatic generalizations",
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(3) Dibyakrupa Sahoo, Rahul Sinha, N. G. Deshpande, and S. Pakvasa.
"Technique to observe direct CP violation in D mesons using Bose symmetry and the Dalitz plot",
Physical Review D **89**, 071903(R) (2014). arXiv:1310.7724 [hep-ph].

(4) Arjun Menon, Tanmoy Modak, Dibyakrupa Sahoo, Rahul Sinha and Hai-Yang Cheng,
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b. Accepted:

(5) Tanmoy Modak, Dibyakrupa Sahoo, Rahul Sinha, Hai-Yang Cheng and T.C. Yuan, "Disentangling the Spin-Parity of a Resonance via the Gold-Plated Decay Mode", arXiv:1408.5665[hep-ph]. *Accepted for publication in Chinese Physics C.*

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An Invitation to the Thesis

The immensely familiar but subtle concept of symmetry pervades our deepest understanding of Nature and is central to contemporary mathematics and physics. This thesis looks at some of the fundamental symmetries of Nature which are revealed in the realm of elementary particles and investigates how violations of these symmetries can be observed in some specific elementary particle processes.

The thesis is divided into five parts. Part I introduces the relevant background materials in two chapters. Chapter 1 provides a very concise introduction to the concept of symmetry as applied to physical laws and also introduces very briefly the symmetries of interest to this thesis, namely the Bose symmetry, CP and CPT symmetries and the flavor $SU(3)$ symmetry. Chapter 2 gives a concise description of the concept of Dalitz plot and finally introduces the kind of Dalitz plot that would be useful in our study of the violations of the symmetries.

Part II gives in detail all the original findings of this thesis in two chapters. Chapter 3 generalizes the concept of Dalitz plot to a three-dimensional plot, christened as Dalitz prism, which not only handles resonant three-body decays but also continuum production of three final particles and multi-body processes which can be treated as "effective" three-body processes. Thus, it works as a precision tool for looking at minute violations of symmetry, as it is rich with data by construction. In Chapter 4 we analyze in detail the signatures of all the symmetry violations under consideration in Dalitz plots and Dalitz prisms. Here various relevant Dalitz plot asymmetries are provided which would quantify the extent of breaking of our chosen symmetries.

Part III contains Chapter 5 which succinctly summarizes the research findings of Part II. This is followed by Part IV which contains the appendices. Appendix A describes in some detail a few concepts in flavor $SU(3)$ symmetry including introduction of two new G-parities. Appendix B summarizes the concept of ternary plot which is used in Chapter 2 to develop the Dalitz plot that would be helpful in Chapter 4. Finally Part V lists all the references and provides an index.

Nature and Nature's symmetry
Are as charming as any poetry.
The Physicist loves to ponder
A little over symmetry's wonder.
"From the farthest stars to the tiniest quarks,
How does symmetry rule over Nature's works?"

On beings animate to inanimate
Symmetry's spell is immaculate.
The Physicist finds and thinks
About symmetry's cracks and kinks.
"From the farthest stars to the tiniest quarks,
How does symmetry break in Nature's works?"

The Physicist's tryst with symmetry,
Has facilitated our pleasant entry,
Into the great grand treasury,
Of Nature's beautiful wizardry.

– *An Ode to Symmetry*
Dibyakrupa Sahoo

PART I

Introduction

In this part we shall briefly discuss the background materials upon which the research findings of Part II are based. In Chapter 1 we shall briefly explore the concept of symmetry in physical laws, its mathematical implementation and some symmetries that are of immediate interest to this thesis. In Chapter 2 we shall make a tour of the Dalitz plot technique used in three-body decays with the motive to improve upon this method and to use them for our study of violations of fundamental symmetries.

1

Symmetry in a nutshell

Symmetry is ubiquitous in Nature. Its obvious simplicity and profound subtlety fascinates the human mind. A spherical ball, a flower, a butterfly, the snowflakes, the five Platonic solids: tetrahedron, cube, octahedron, dodecahedron, and icosahedron are some out of the many objects that spring up in our mind when we hear the word symmetry. Nature has such a vast repertoire of objects with all sorts of geometric symmetries, that it would be foolish and insane to list them. Moreover, the kind of symmetry we are going to dwell upon in the next few pages is not a symmetry of objects of Nature, but the *symmetry in the physical laws* that govern some phenomena in the physical world[1]. Unlike the geometrical symmetries of material objects, the symmetries in physical laws are not perceptible by our sensory organs. Hence, the precise notion of such a symmetry needs to be carefully articulated.

[1]No physical law is permanent, they are all provisional, i.e. they are approximate statements about the ways Nature works to a fairly high degree of accuracy and are subject to modification or replacement whenever some better explanation is discovered. In this sense all the symmetries of physical laws are also provisional.

1.1 What is symmetry in physical laws?

The German mathematician and physicist Hermann Weyl in the preface of his book *Symmetry* [1] gives the generalized idea of symmetry as

> " ... that of invariance of a configuration of elements under a group of automorphic transformations."

Richard P. Feynman interprets Weyl's definition of symmetry in the following manner in his *Lectures on Physics* [2]:

> "... a thing is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation."

The same tone resonates when Dave Goldberg writes in the introduction of his book *The Universe in the Rearview Mirror* [3][2]:

> "A thing is symmetrical if there is something you can do to it so that after you have finished doing it, it looks the same as before."

It must be noted that there are three vital aspects to symmetry as defined above:

1. An *object* ('... a configuration of elements ...' or 'A thing ...') whose symmetry is the subject of our curiosity. This can be a geometrical object, a living or non-living entity, or something as abstract as a physical law.

2. An *operation* ('... a group of automorphic transformations.' or '... certain operation ...' or '... something you can do ...') which we call as the *symmetry operation* or *symmetry transformation*. This operation can be something like rotation by any angle in the plane for a circle, or reflection about any median of an equilateral triangle, or something like changing the physical situation of some physics experiment such as doing the Michelson-Morley experiment at different locations and in

[2] R.P. Feynman utters almost the same words in his book *The Character of Physical Law* [4]: "... a thing is symmetrical if there is something that you can do to it so that after you have finished doing it it looks the same as it did before."

several seasons on the earth. Steven Weinberg in his book *The Quantum Theory of Fields, Vol. I Foundations* [5] defines symmetry transformation as follows:

> "A symmetry transformation is a change in our point of view that does not change the results of possible experiments."

3. An *observable* ('… invariance of …' or '… it appears exactly the same …' or '… it looks the same as before.') which we study for quantitative variation under the symmetry operation. For the geometrical objects such as a circle or a triangle, the shape or appearance of the object itself is the observable. However, for a physics experiment it is the observed phenomenon, such as for the Michelson-Morley experiment the observable is the shift in the observed interference pattern.

Weyl's statement about symmetry, however, has one hidden but important aspect which gets manifest in both Feynman's and Goldberg's versions: the *observer* ('… one can …' or '… you can …') which always denotes an unbiased, competent experimenter. Where possible the human aspect can always be dispensed with quantitative experimental observations, otherwise known as data. Therefore, drawing inspiration from Weyl, one can define symmetry in physical laws as follows:

> "A physical law is said to have a certain symmetry, if all unbiased, competent observations agree that the experimental observable remains invariant under the appropriate symmetry operation."

The definition of symmetry in physical laws as given above is from an experimenter's point of view. The theorist's point of view has its foundations in the deeper mathematical aspects of symmetry and it must be in concurrence with that of the experimenter if the physical law has to describe some natural phenomena. From the theorist's point of view:

> "A physical law is said to have certain symmetry, if the equation describing the law retains its form (i.e. it is covariant) under the appropriate symmetry transformation."

Or as Steven Weinberg observes in his article "Symmetry: A 'Key to Nature's Secrets' " [6]:

> "A law of nature can be said to respect a certain symmetry if that law remains the same when we change the point of view from which we observe natural phenomena in certain definite ways. The particular set of ways that we can change our point of view without changing the law defines the symmetry."

In order to get a better understanding of the symmetry in physical laws, it is only pertinent that we analyse the various symmetry operations or transformations (the various "points of view") that have been explored by the theoretical physicists over the years. The profit of having an understanding of the symmetries of physical laws would also be clear from such analyses.

1.2 Classification of symmetries and symmetry operations

There are many transformations which keep various physical phenomena invariant. These transformations affect the variables of the equation describing the law in some appropriate manner. If the equation retains its form, then the law is said to have the particular symmetry under consideration. The various symmetry operations and, therefore, the various associated symmetries can be classified in many different ways [7]:

- If the symmetry transformation makes continuous (i.e. infinitesimally small) changes in the variables, it is called a *continuous symmetry operation*. When the transformation allows only certain discrete values for the variables, the symmetry operation is called *discrete*.

- There are some symmetry transformations which affect the space and time coordinates. These are usually called *spacetime* or *spatio-temporal symmetry operations*.

The non-spacetime symmetry operations (i.e. the ones where the variables are not spacetime coordinates), are called as the *internal symmetry operations*.

- If in some symmetry transformation the variables are independent of space and time coordinates, such transformations are called *global symmetry operations*. However, if the variables are dependent on space and time coordinates, the transformation is called *local symmetry operation*.

There exists a certain degree of quantitative mismatch in the symmetries proposed by the theorists and those observed by experimenters in many natural phenomena. This allows a further classification of symmetries and symmetry operations.

- A certain symmetry and the corresponding symmetry operation can be *exact* (or *perfect*), *approximate* (or *imperfect*), or *broken*, depending on the extent to which the theoretical ideas are supported by the experimental evidences. An exact or perfect symmetry applies well in all natural phenomena without any fail. An approximate or imperfect symmetry has some domain of validity, i.e. it is a symmetry of Nature manifest only in some specific natural phenomena. The concept of broken symmetry is more complicated than the other two. A symmetry would be said to be broken, if it no longer is a symmetry of the specific system in the specific case under consideration, but which would have been a perfect symmetry of the system in some completely different situation.

There are a lot other aspects of symmetry operations, which we have not yet considered:

1. *Subject of symmetry operation:* In case of symmetries in physical laws, the subject of operation for symmetry transformations can be equations of motion (equivalently Lagrangian or Hamiltonian), boundary conditions of the problem, or the solutions (such as wave functions or states or fields) themselves. It might also include

everything stated here. It is possible that a symmetry of the equations of motion would be different from the symmetry of the solutions.

2. *Scale of symmetry operation:* A system or a natural phenomenon might exhibit a multitude of nuances in its behaviour when some variable (known as the scale variable) of the system or phenomenon is varied. The equations describing the system or phenomenon must, therefore, evolve appropriately along the scale variable under consideration. In such cases, symmetry and symmetry operations are clearly defined at some given scale. The symmetries might also evolve along the variation of the scale.

3. *Nature of symmetry operation:* The symmetry operations can be classical or quantum mechanical by nature. Of course, there are some symmetry operations which are valid in both the classical and the quantum regime.

These aspects of symmetry help in a finer understanding of the broken symmetries and their further classification. If the Lagrangian or the Hamiltonian of a system or a phenomenon remains invariant under a given symmetry transformation except in the case of a specific situation, then the symmetry is said to be *explicitly broken* for the system in that particular situation. If the Lagrangian or the Hamiltonian of a system is symmetric under a certain symmetry transformation, but the solutions of the system, i.e. the wave functions or states or fields, lack that symmetry, then the symmetry under consideration is said to be *spontaneously broken*. If a system in its classical description has a symmetry but not in the corresponding quantum mechanical description, then that symmetry is said to be *anomalously broken*.

There is one very important symmetry transformation we have missed in the above discussion. In quantum field theory a continuous, internal, local symmetry transformation which keeps the Lagrangian invariant is christened as a *gauge symmetry operation* and the corresponding symmetry is called *gauge symmetry*. In his article on “The role of symmetry in fundamental physics" [14] David J. Gross provides the following insight on gauge symmetries:

> "Gauge symmetries are formulated only in terms of the laws of nature; the application of the symmetry transformation merely changes our description of the same physical situation, does not lead to a different physical situation."

This can be easily contrasted with *global* symmetries which lead to different physical situations without affecting any observation made on the system. Gauge symmetries play a very central role in quantum field theory and in our understanding of interactions amongst the elementary particles.

There is another kind of symmetry one comes across in the study of quantum field theories, namely *conformal symmetry*. Conformal symmetry deals with such spacetime coordinate transformations (aptly called *conformal transformations*) which result in only a positive rescaling of the metric (the distance function). There is a whole branch of field theories named *conformal field theories* which deal with conformal symmetries.

Some of the widely encountered symmetries in physics are listed in Table 1.1. We shall now briefly discuss the various properties of symmetry operations and how they are treated mathematically. This has revolutionized the way we explore any physical phenomenon.

1.3 Mathematical treatment of symmetry operations

All valid symmetry operations are found to satisfy the following four fundamental properties [8]:

(S.1) application of two symmetry operations is also another symmetry operation (i.e. the symmetry operations are said to *close*),

<table>
<tr><th>Symmetry transformation</th><th>Category</th></tr>
<tr><td>Translation in space</td><td rowspan="4">Continuous symmetry</td></tr>
<tr><td>Translation in time</td></tr>
<tr><td>Rotation in space</td></tr>
<tr><td>Uniform velocity along a certain direction
(Lorentz transformation)</td></tr>
<tr><td>Matter-antimatter exchange (charge conjugation or C)</td><td rowspan="3">Discrete symmetry</td></tr>
<tr><td>Space-inversion (parity or P)</td></tr>
<tr><td>Time reversal (T)</td></tr>
<tr><td>Interchange of identical bosons
(Bose-Einstein statistics or Bose symmetry)</td><td rowspan="2">Permutation symmetry
(or commutative symmetry)</td></tr>
<tr><td>Interchange of identical fermions
(Fermi-Dirac statistics or Fermi antisymmetry)</td></tr>
<tr><td>Quantum-mechanical phase (phase symmetry or $U(1)$)</td><td rowspan="3">Gauge symmetry</td></tr>
<tr><td>Charged lepton-neutrino symmetry ($SU(2)_L$)</td></tr>
<tr><td>Quark color symmetry ($SU(3)_{\text{color}}$)</td></tr>
<tr><td>Quark flavor symmetry ($SU(3)_{\text{flavor}}$)</td><td>Global symmetry</td></tr>
</table>

Table 1.1: Some symmetry operations frequently used in particle physics.

(S.2) application of the same set of three symmetry operations gives the same result whether two symmetry operations follow one symmetry operation, or one symmetry operation follows two symmetry operations, as long as the same sequence is maintained (i.e. the symmetry operations are said to be *associative*),

(S.3) there exists a symmetry operation, called *identity transformation*, which does not affect anything,

(S.4) for any symmetry operation there exists another symmetry operation, its *inverse transformation*, which nullifies (or reverses) its effect.

It is important to note that the sequence of combination of two symmetry operations is very important, and, in general, the final symmetry operation is different when the sequences are different.

All the above seemingly innocuous conditions are the foundational basis of an extremely rich mathematical concept called *group theory*. A set $G = \{g_1, g_2, \ldots, g_i, \ldots\}$ is said to form a group under a *group operation* (also known as *group composition* or *group multiplication*), denoted simply by juxtaposition of the group elements, if the following properties are satisfied:

(G.1) *Closure:* For any $g_i, g_j \in G$, there exists a unique $g_k \in G$ such that $g_i g_j = g_k$.

(G.2) *Associativity:* For any $g_i, g_j, g_k \in G$ the group operation is associative, i.e. $g_i(g_j g_k) = (g_i g_j)g_k$.

(G.3) *Existence of identity element:* There exists a unique element $\mathbf{1} \in G$, the identity element, such that for any $g_i \in G$, $\mathbf{1}g_i = g_i\mathbf{1} = g_i$.

(G.4) *Existence of inverse elements:* For any $g_i \in G$, there exists a unique element $(g_i)^{-1} \in G$, the inverse of g_i, such that $g_i(g_i)^{-1} = \mathbf{1} = (g_i)^{-1}g_i$.

Furthermore, if the group G satisfies the *commutativity* property, which states that for any $g_i, g_j \in G$, $g_i g_j = g_j g_i$, then G is said to be a *commutative* or an *Abelian* group, after the great Norwegian mathematician Niels Henrik Abel. Any group which does not satisfy commutativity property is called a *non-Abelian* group. The *order* of a group is the total number of distinct elements (or 'cardinality') of the group. It can be either finite or infinite; accordingly the group is said to be either a *finite* group or an *infinite* group. An infinite group with denumerably infinite number of elements is called a *discrete* group, and one with non-denumerably infinite number of elements is called a *continuous* group. If H is a subset of G (denoted as $H \subseteq G$) and forms a group under the same group operation as that of G, then H is called a *subgroup* of G. The identity element alone and the whole group itself, are the two 'trivial' subgroups of any group. Subgroups play a very important role in particle physics. Different fundamental particles and the various interactions amongst them are often related by some fundamental symmetry. However, the symmetry that manifests in Nature is most often a subgroup of the original symmetry group. Since subgroups are central to our scheme of things, let us explore them in a little more detail.

- The subgroup H of group G is said to be a *normal* subgroup, if $Hg = gH$, for all $g \in G$, where $Hg = \{hg : h \in H\}$ and $gH = \{gh : h \in H\}$, are called *cosets*, respectively the right-coset and left-coset of subgroup H with respect to g. Now $Hg = gH$ implies that for every $g \in G$ and $h_1 \in H$ there exists an element $h_2 \in H$ such that $h_1 g = g h_2$, or $g h_2 g^{-1} = h_1$. In this context, it is useful to introduce the notion of *conjugacy*. Two elements $g_i, g_j \in G$ are said to be *conjugate* to one another if there exists a $g_k \in G$ such that $g_j = g_k g_i g_k^{-1}$. The set of all elements of the group G which are conjugate to a given element of G is called a *conjugacy class*[3]. Since for a normal subgroup H we have $gHg^{-1} = H$, for any $g \in G$, it

[3]This is also simply called a *class*. No two classes of a group have any element in common. Thus we can decompose a finite group into a finite number of classes. The number of classes possible for a finite

can be said that all the elements of group G commute with the normal subgroup H. H is also called an *invariant*, or *self-conjugate* subgroup. All subgroups of an Abelian group are normal subgroups. Once again, the identity element alone and the whole group G itself are trivial normal subgroups of G. Any subgroup which is not a normal subgroup is called a non-normal subgroup.

- If G is a finite group and H is a subgroup of G, then the order of H divides the order of G. (This is known as Lagrange's theorem.) The set of all distinct right-cosets Hg (or left-cosets gH) of the subgroup H with respect to an element $g \in G$ is called the *right coset space*, $G/H^r = \{Hg : g \in G\}$ (or *left coset space*, $G/H^l = \{gH : g \in G\}$). When H is an invariant or normal subgroup, the coset space $G/H = G/H^l = G/H^r$ becomes a group and is called as the *factor* or *quotient group* of G by H. The group multiplication of Hg_1 and Hg_2 for different elements $g_1, g_2 \in G$, is given by $(Hg_1)(Hg_2) = H(g_1H)g_2 = H(Hg_1)g_2 = (HH)(g_1g_2) = H(g_1g_2)$, where we have used the fact that H is a normal subgroup.

- The *center* of a group G, denoted by $Z(G)$, is defined as the set of those elements of G that commute with all the elements of G: $Z(G) = \{z \in G : \forall g \in G, zg = gz\}$. The center of group G is always a normal subgroup of G. A *centerless* group is one whose center has only the identity element in it.

- Groups that do not have any non-trivial normal subgroups[4] are called *simple groups*. If G is a simple group and H is a normal subgroup of G, then H is either the group G itself or it only contains the identity. Simple groups are very fascinating groups and are considered as the basic building blocks of group theory. One of the simplest ways to combine groups to produce groups is via the concept

group is a characteristic of the group and is always less than or equal to the order of the group. An Abelian group of order n has n number of classes.

[4]Any normal subgroup except the group itself and the identity element alone are called non-trivial normal subgroups.

of *direct product group*. If G and H are two arbitrary groups, then their direct product group, $G \times H$, has ordered pairs (g, h) as its elements where $g \in G$ and $h \in H$, i.e. $G \times H = \{(g, h) : g \in G, h \in H\}$, and the rule for group multiplication between two elements $(g_1, h_1), (g_2, h_2) \in G \times H$ being given by $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$. This can be extended to include many groups. If $G_1, G_2, \ldots, G_n$ are a few groups, the direct product group is defined as $G_1 \times G_2 \times \ldots G_n = \{(g_1, g_2, \ldots, g_n) : g_i \in G_i \forall i \in \{1, 2, \ldots, n\}\}$, and the rule for group multiplication is $(g_1, g_2, \ldots, g_n)(g'_1, g'_2, \ldots, g'_n) = (g_1 g'_1, g_2 g'_2, \ldots, g_n g'_n)$. In particle physics, the frequently encountered groups are mostly product groups of some simple groups.

It is very interesting to find that looking for relationships among various groups is a very fruitful enterprise. A *mapping* or *function* ϕ from a group G_1 to another group G_2, denoted as $\phi : G_1 \to G_2$, is a subset of the direct product group $G_1 \times G_2$ subject to the condition that every element $g_1 \in G_1$ is the first component of one and only one ordered pair in the subset, i.e. every $g_1 \in G_1$ is uniquely associated with an element $g_2 = \phi(g_1) \in G_2$. Here G_1 is the *domain*, G_2 is the *codomain* and $g_2 = \phi(g_1)$ is called the *image* of g_1. Symbolically, the mapping of the element g_1 to its image $\phi(g_1)$ is written by $g_1 \mapsto \phi(g_1)$. The set of all elements in the codomain G_2 touched by the function $\phi : G_1 \to G_2$ is called the *image of the function* ϕ, denoted as $Im(\phi) \equiv \phi(G_1)$. The set of elements of domain G_1 that get mapped to the identity element of the codomain G_2 is called the *kernel* of the function, denoted as $Ker(\phi)$. It is important to note that a mapping $\phi : G_1 \to G_2$ can be of the following types:

(i) *injective* function if $\forall g_1, g'_1 \in G_1, g_1 \neq g'_1 \implies \phi(g_1) \neq \phi(g'_1)$,

(ii) *surjective* function if $\forall g_2 \in G_2, \exists g_1 \in G_1, \phi(g_1) = g_2$,

(iii) *bijective* function if ϕ is both injective and surjective.

A mapping ϕ from a group G_1 to another group G_2 is called a *homomorphism* if for all $g_1, g_1' \in G_1$, $\phi(g_1 g_1') = \phi(g_1)\phi(g_1')$, i.e. the image of a group multiplication in G_1 is the same as group multiplication of images in G_2. The image $Im(\phi)$ and kernel $Ker(\phi)$ of a homomorphism ϕ are subgroups. Moreover, the kernel $Ker(\phi)$ is in fact a normal subgroup. A bijective homomorphism is called an *isomorphism*. Furthermore, the homomorphism of a group into itself is known as an *endomorphism*, and the isomorphism of a group into itself is called an *automorphism*. An *inner automorphism* ϕ of a group G is the mapping $\phi : G \to G$ defined as $\forall g \in G, g \mapsto \phi(g) = g'g(g')^{-1}$, where g' is a fixed element of G. An *outer automorphism* of a group is one which is not an inner automorphism. The concept of homomorphism plays a very great role in application of group theoretic ideas in quantum field theories and hence in elementary particle physics. This would be clear soon with the introduction of the concept of *group representation*.

The *representation* of dimension n of a group G is defined as a homomorphism $T : G \to GL(n)$, where $GL(n)$ is the multiplicative group of non-singular $n \times n$ matrices[5]. If the homomorphism is in fact an isomorphism, the representation is then called a *faithful representation*. Every finite group of order n has a faithful representation in $GL(n)$. However, every finite-dimensional continuous group may not have faithful representation in some $GL(n)$. Nevertheless, they do have representations, albeit non-faithful ones and also locally[6], when the global aspects of the group do not concern us. There can be more than one representation of a given group. Two n-dimensional representations of a group G, namely $T^{(1)} : G \to GL(n)$ and $T^{(2)} : G \to GL(n)$, are said to be *equivalent* if for all $g \in G$ all the matrices $T^{(1)}(g)$ and $T^{(2)}(g)$ are related by the same similarity transformation: $T^{(1)}(g) = ST^{(2)}(g)S^{-1}$, where the similar matrix S is independent of g. There is a way to distinguish between various representations but treat two equivalent

[5] $GL(n)$ is the general linear group that describes the symmetries of a n-dimensional vector space. If the vector space is real (or complex), the entries for the $GL(n)$ matrices are all real (or complex) and we denote the group by $GL(n, \mathbb{R})$ (or $GL(n, \mathbb{C})$).

[6] Here locally implies that the elements of the $GL(n)$ matrices are only defined at a given space-time coordinate and can vary from one space-time point to another.

representations as the same. This is by the concept of *character*. The character (χ) of a n-dimensional representation of a group G, say $T : G \to GL(n)$, is the set of traces of the representation matrices $T(g)$ for all $g \in G$, i.e. $\chi = \{\chi(g) = \mathrm{Tr}(T(g)) : g \in G\}$. It is important to note that two equivalent representations have the same character[7], and if two representations have the same character they are definitely equivalent. Now, a representation of dimension $m + n$ is said to be a *reducible* representation, if all the matrices of the representation can be put into the form $T(g) = \begin{pmatrix} \mathbf{A}(g) & \mathbf{C}(g) \\ \mathbf{O} & \mathbf{B}(g) \end{pmatrix}, \forall g \in G$, where $\mathbf{A}(g)$, $\mathbf{B}(g)$ and $\mathbf{C}(g)$ are matrices of dimensions $m \times m$, $n \times n$ and $m \times n$ respectively, and $\mathbf{O}$ is a null matrix of dimension $n \times m$, for fixed m and n. Here matrices $\mathbf{A}$ and $\mathbf{B}$ individually constitute m-dimensional and n-dimensional representations of G respectively. When the representation is *completely reducible* or *decomposable*, $\mathbf{C}$ becomes a null matrix giving $T(g)$ the famous 'block-diagonal' form $T(g) = \begin{pmatrix} \mathbf{A}(g) & \mathbf{O} \\ \mathbf{O} & \mathbf{B}(g) \end{pmatrix}, \forall g \in G$, and it can be written as the *direct sum* of the subrepresentations $\mathbf{A}(g)$ and $\mathbf{B}(g)$: $T(g) = \mathbf{A}(g) \oplus \mathbf{B}(g)$. A representation is said to be *irreducible* if it is not reducible. All irreducible representations of a group can be classified by their characters. A representation is said to be *unitary* if all the matrices of the representation are unitary. It is important to note that every representation of a finite group is not only equivalent to a unitary representation but also completely reducible. Moreover, the number of irreducible representations of a finite group is equal to its number of conjugacy classes.

Most of the interesting groups we come across in physics are the *Lie groups*, named after the famous Norwegian mathematician Sophus Lie. Lie theory, which deals with Lie groups and the associated Lie algebras, is the standard formalism for the study of the local theory of continuous groups. A group G is a *Lie group*, if every element of G is specified by a set of real parameters and the parameters of a product element are analytic functions[8] of the parameters of the factors. Mathematically, a group G is a *Lie group* of dimension

[7]This follows directly from the cyclic property of the trace.

[8]By analytic parameters we mean that these are differentiable to all orders.

n (where n is finite), if every element of G is specified by n number of real parameters (denoted here by αs, βs and γs) in such a way that provided $g_1 \equiv g_1(\alpha_1, \ldots, \alpha_n)$, and $g_2 \equiv g_2(\beta_1, \ldots, \beta_n)$ are two elements of G and the product $g' = g_1 g_2^{-1}$ being parametrized as $g' \equiv g'(\gamma_1, \ldots, \gamma_n)$, imply that $\gamma_i = \gamma_i(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ are analytic functions of the αs and the βs. The parameters are very important, because of the following reasons:

(i) Two elements of the Lie group are same, if and only if their corresponding parameters are also the same, i.e. $g_1(\alpha_1, \ldots, \alpha_n) = g_2(\beta_1, \ldots, \beta_n)$ if and only if $\alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n$,

(ii) The analytic nature of the parameters allows one to study *infinitesimal* group elements (i.e. elements in the neighbourhood of the identity element) providing the complete information about the local structure of the group.

If the parameters vary in closed finite intervals, the Lie group is said to be *compact*.[9] The parameters are said to be *normal* if by putting them to zero, the group element corresponds to the identity element, i.e. $g(0, \ldots, 0) = \mathbf{1}$ ($\mathbf{1}$ being the identity element of the Lie group). The parameters $\alpha_1, \ldots, \alpha_n$ can be thought of as components of a n-dimensional vector $\boldsymbol{\alpha}$. Let us consider a representation of the Lie group G, $g(\boldsymbol{\alpha}) \mapsto T(\boldsymbol{\alpha})$ and $T(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\mathbf{0}} = \mathbf{1}$ (where $\mathbf{1}$ is the identity matrix). The elements in the representation of the n-dimensional Lie group G in the neighbourhood of the identity element can now be obtained by exponentiation, $T(\boldsymbol{\alpha}) = e^{i\alpha_a X_a}$, where $X_a \equiv -i\frac{\partial}{\partial \alpha_a} T(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\mathbf{0}}$ is called a *generator* of the group and $a = 1, \ldots, n$. The representation thus defined by exponentiation (with the imaginary i included in the exponent) is unitary and the generators of the group are hermitian and traceless. All the generators of a Lie group form the basis of a linear vector space. Most importantly, the generators of a Lie group G of dimension n

[9]The special orthogonal group $SO(n)$ of orthogonal $n \times n$ matrices of unit determinant and the special unitary group $SU(n)$ of unitary $n \times n$ matrices of unit determinant are examples of compact Lie groups. However, the special real linear group $SL(n, \mathbb{R})$, and the special complex linear group $SL(n, \mathbb{C})$ (which are groups of $n \times n$ unimodular (= unit determinant) matrices with real and complex entries respectively) are non-compact Lie groups.

are closed under commutation (action of the commutators also known as the *Lie brackets*), $[X_a, X_b] \equiv X_a X_b - X_b X_a = i f^c_{ab} X_c$, where we have used the summation convention (here c is summed from 1 to n) and f^c_{ab} are real constants called as *structure constants* for G. This commutator relation is called the *Lie algebra* of the group, and is completely determined by the structure constants. The Lie brackets satisfy the following properties:

(i) $[X_a, X_b] = -[X_b, X_a]$,

(ii) $[X_a + X_b, X_c] = [X_a, X_c] + [X_b, X_c]$, $[X_a, X_b + X_c] = [X_a, X_b] + [X_a, X_c]$,

(iii) $[\alpha X_a, X_b] = \alpha [X_a, X_b] = [X_a, \alpha X_b]$,

(iv) $[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$ (*Jacobi identity*).

The structure constants therefore satisfy the following relationships:

(i) $f^c_{ab} = -f^c_{ba}$,

(ii) $f^c_{ab} f^e_{cd} + f^c_{bd} f^e_{ca} + f^c_{da} f^e_{cb} = 0$.

It is important to note that for a simple Lie group G, the structure constants calculated for the group G are identical to those calculated using any nontrivial representation of G (i.e. representations in which the generators do not vanish). If all the structure constants are zero, then the group is Abelian, otherwise it is a non-Abelian group. Let us define some matrices whose elements are the structure constants, $[L_a]_{bc} \equiv -i f^c_{ab}$. These matrices satisfy the same commutator relation as the one by the generators of the Lie group: $[L_a, L_b] = i f^c_{ab} L_c$. Thus, the structure constants themselves furnish a representation of the Lie algebra, which is called the *adjoint representation.* Since the structure constants are real, all the generators of the adjoint representation are pure imaginary. The structure constants also lead to a useful concept, called the *Cartan tensor* or the *Killing form* (or *Cartan-Killing* form) defined as trace of the product of two generators $g_{ab} = \mathrm{Tr}\,(X_a X_b) = f^d_{ac} f^c_{bd} = g_{ba}$. In fact, the Cartan-Killing form is proportional to the identity matrix: $g_{ab} = C\delta_{ab}$, where C is a constant which depends on the choice of irreducible representation considered for the Lie algebra. We can now define the structure constant with all lower indices $f_{abc} = f_{ab}^{d}\delta_{cd} = \frac{1}{C} f_{ab}^{d} f_{cm}^{l} f_{dl}^{m} = -\frac{i}{C}\mathrm{Tr}\left([X_a, X_b]\, X_c\right)$, which explicitly shows that the structure constant is antisymmetric in all three indices. The constant C here is called the *quadratic Casimir operator* or *Dynkin index*. In general, a *Casimir operator* is a function $F(X)$ of the group generators X_i that commutes with all the generators, $[F(X), X_i] = 0$. A Casimir operator is always proportional to the identity and the constant of proportionality may depend on the choice of representation.

We have thus far covered most of the group theoretic concepts that are relevant to understand the symmetries we shall consider in this thesis. Details regarding what has been discussed above and more information can be found in References [8, 15–25].

Our current understanding of elementary particles is extremely well described by the so called standard model of particle physics (SM). The foundational basis of SM is a direct product group of three Lie groups, the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, where individually $SU(3)_c$ describes the strong interaction amongst the various quarks via exchange of gluons, $SU(2)_L \times U(1)_Y$ describes the electro-weak interaction amongst the quarks as well as the leptons via exchange of the massive weak gauge bosons $W^{\pm}$, Z^0, and massless photons.

1.4 Symmetry and conservation laws

It is true that symmetry operations act upon the variables of the theory and keep the Lagrangian or Hamiltonian or boundary conditions or the solutions (such as wave functions or states or fields) invariant. It was proven by the great German mathematician Emmy Noether in 1918, that if the Lagrangian or Hamiltonian of a system possesses some continuous symmetry, then there is an associated *conserved quantity*[10]. This is the famous Noether's theorem and it essentially provides a *conservation law* corresponding to the underlying symmetry. Thus, conservation laws can be directly obtained from symmetry or invariance principles alone without worrying about the laws of motion. Hence, the region of applicability of conservation laws is much wider than that of the laws of motion or the framework of any specific field or theory. Nevertheless, the region of applicability of conservation laws is as wide as that of the underlying symmetries under consideration. We shall now note down a few well known conservation laws as arising from some underlying symmetry principles.

(i) That the laws of physics are invariant under translation in space (homogeneity of space), implies that linear momentum is conserved.

(ii) That the laws of physics are invariant under rotation through a fixed angle in space (isotropy of space), implies that angular momentum is conserved.

(iii) That the laws of physics are invariant under translation in time (homogeneity of time), implies that energy is conserved.

(iv) That the laws of physics are invariant under any constant shift of the quantum-mechanical phase of any wave function or any field implies that some conserved "charge" (or quantum number) exists.

There is an important aspect in the way conservation laws are to be interpreted. All conservation laws are formulated as some kind of *prohibitory* rule and not as a guiding rule. What this means is that the conservation law prohibits any process or phenomenon which would change the conserved quantity. In some cases, when one observes that the experimental data does not allow certain types of processes involving some elementary

[10] By conserved quantity we mean a quantity which remains unchanged when the system moves or evolves over time.

particles, one can, in principle, formulate a conservation law as a law of prohibition even without knowing anything about the underlying symmetry. Prohibitory nature of conservation laws is, in deed, reminiscent of the fact that the concept of symmetry always reduces the number of different possible ways of formulating a theoretical model of the Nature.

1.5 Symmetries that are of interest to this thesis

In this thesis, we shall look at four specific symmetries: two discrete symmetries, namely, the combined operations of charge conjugation, parity and time reversal: CP and CPT; one permutation or commutative symmetry, namely, the Bose symmetry; and one continuous gauge symmetry, the quark flavor $SU(3)$ symmetry.

1.5.1 Bose symmetry

Bose symmetry [26] states that all measurable quantities of a physical system remain unchanged when two identical bosons (particles with integer spins) are swapped. Under Bose symmetry a state made of two identical bosons does not change under exchange of the two bosons. The bosons follow the Bose-Einstein statistics, which allows more than one bosons to share the same quantum state. The particles with half-integer spins, the fermions, follow the Fermi-Dirac statistics [27], which does not allow more than one fermion to occupy the same quantum state, and moreover the wavefunction for a state of two fermions picks up a negative sign when the two fermions are swapped. This goes by the name of Fermi antisymmetry. Both Bose symmetry and Fermi antisymmetry form one of the cornerstones of modern physics and they are combined into the famous spin-statistics theorem. Within the conventional Lorentz invariant and local quantum field theory, even a small violation of this symmetry is quite impossible. There have, therefore, been a lot of experimental interest in Bose symmetry violation as a means of testing the present theoretical framework. Theoretical ideas and experimental investigations for Bose symmetry violations have looked at the spin-0 nucleus of oxygen ^{16}O [28, 29], molecules such as $^{16}O_2$ and CO_2 [30–33], photons [34–39], pions [40], Bose symmetry violating transitions [41–47]. On the theoretical side, a scenario where Bose symmetry is not exact swings open doors to a plethora of avenues for new physics [48–52]. In this thesis we shall describe in detail in Chapter 4 how using Dalitz plots (discussed in Chapter 2) and the new concept of Dalitz prisms (developed in Chapter 3) we can look for signatures of Bose symmetry violation in some specific three-body processes.

1.5.2 Charge conjugation, parity and time reversal

Charge conjugation (C), parity (P), time reversal (T) and their combinations CP and CPT are the most important discrete symmetries in the whole of particle physics. Under charge conjugation a particle gets exchanged with its corresponding anti-particle (which has the same mass, spin and lifetime as those of the particle, but opposite sign of electric charge and magnetic moment). Under parity, or space inversion, all the spatial coordinates get reversed in sign, and under time reversal the time coordinate gets reversed. The parity operator P when operated twice brings the system back to itself, i.e. $P^2 = \mathbf{1}$, where $\mathbf{1}$ is the identity operator. Therefore, the eigenvalues of P are ± 1. Every elementary particle has an *intrinsic* parity, and if it is a composite particle it has *extrinsic* parity also. Fermionic particles and their corresponding antiparticles have opposite intrinsic parity. However bosonic particles and their corresponding antiparticles have the same parity. Parity is a multiplicative quantum number. So the extrinsic parity of a composite system in its ground state is the product of the intrinsic parities of its constituents. But when the composite system is in an excited state with orbital angular momentum l, there is an extra multiplicative factor of $(-1)^l$. Parity is a good symmetry for electromagnetic and strong interactions. But it was pointed out by Tsung Dao Lee and Chen Ning Yang [53] in 1956 and then proven experimentally by Chien-Shiung Wu and her collaborators [54] in 1957 that parity is violated in “the weak” interaction. Similar to parity, the charge conjugation operator C when operated twice brings the system to itself, i.e. $C^2 = \mathbf{1}$. Hence, the eigenvalues of C are ± 1. However, most particles are not eigenstates of charge conjugation operator. Only those particles which are antiparticles of themselves, are eigenstates of charge conjugation. Charge conjugation is also a multiplicative quantum number. A composite system consisting of a spin-$\frac{1}{2}$ particle and its antiparticle in a state with total spin s and orbital angular momentum l is an eigenstate of C with eigenvalue $(-1)^{l+s}$. Experimentally charge conjugation is found to be an excellent symmetry for electromagnetic and strong interactions, but it is not a symmetry for the weak interaction. Neutrinos which interact only via the weak interaction are observed to be left-handed and all the anti-neutrinos are found to be right-handed. Since charge-conjugation does not flip the handedness (but parity does), it is easy to see that the weak interaction does not oblige charge conjugation symmetry. Moreover, the combined operation of charge conjugation and parity, i.e. CP is also found to be a good symmetry in electromagnetic and strong interactions, but it is broken in weak interactions. It is well known that one of the Sakharov conditions [55], so fundamental to the explanation of the observed predominance of matter over antimatter in our universe, requires that the laws of nature are not invariant under CP. Indeed CP violation has been well established in the weak decays of K [56] and B mesons [57–63]. In the standard model of particle physics (SM) CP violation arises via the Kobayashi-Maskawa mechanism at a level consistent with that observed in the K mesons and B mesons [13]. However, it is also well known that CP violation in SM is not enough to account for the observed baryon asymmetry in the universe making it imperative to search for new sources of CP violation beyond SM. In the SM, CP violation in the D meson system is expected to be rather small in both mass mixing and in direct decays. An observation of sizeable CP violation in D mesons would hence open a window of opportunity to probe for new sources of CP violation. This in turn may lead to a more complete theory of CP violation that furthers our understanding of the observed baryon asymmetry. It is, however, challenging to observe an unambiguous signal of CP violation in D mesons. In this thesis we shall develop (in Chapter 4) a new technique to observe direct CP violation using Bose symmetry and Dalitz plots (or prisms). The method is completely general and can be applied to study CP violation in D mesons. Unlike parity and charge conjugation, it is not easy to check time reversal. One of the methods to check time reversal symmetry is by applying the principle of detailed balance to forward and backward reactions. But the two reactions are not equally probable and doing inverse reaction experiments in weak decays is very tough because it is practically impossible to prepare the system in the precise quantum state needed for the inverse process. Recently the BaBar collaboration has found an experimental evidence for time reversal violation [64]. There is a reason to believe that there must be some time reversal violation because of the CPT theorem. According to the CPT theorem, CPT is an exact symmetry for any fundamental interaction. The CPT symmetry is very closely related to the spin-statistics theorem and Lorentz invariance [65–78]. The CPT theorem implies that a particle and its antiparticle must have the same mass, decay width, lifetime and other such intrinsic properties. It is important to note that if CP is violated, then partial rate asymmetries for particle and antiparticle can be different, but their total decay rates would always be the same if CPT invariance holds good [79]. Similarly, the CPT invariance also implies that the total scattering cross-section of two particles would be equal to that of their antiparticles, but the partial scattering cross-sections need not be equivalent if CP is violated [80]. The best test of CPT invariance, to date, has come from the limit on the mass difference between the neutral kaons (K^0 and $\overline{K}^0$) [81–83] and there is no indication of any breakdown of CPT invariance. However, if there is even an extremely small violation of CPT, it has significant theoretical ramifications in various models of new physics. In this thesis we propose in Chapter 4 a new way to look for violations of CPT symmetry in some three-body decays via strong interaction using Dalitz plots and Dalitz prisms.

1.5.3 Flavor $SU(3)$ symmetry

The light hadronic states can be satisfactorily understood by using the quark flavor $SU(3)$ symmetry [84–88]. In its true essence the $SU(3)$ flavor symmetry denotes the full exchange symmetry amongst the up (u), down (d) and strange (s) quarks (or equivalently the exchange of the the anti-quarks $\bar{u}$, $\bar{d}$ and $\bar{s}$) which are referred to as the three *flavors* of light quarks (or anti-quarks). If $SU(3)$ flavor symmetry were an exact symmetry, then the mesons formed by combining the quarks u, d, s and the antiquarks $\bar{u}$, $\bar{d}$, $\bar{s}$ belonging to the same representation of $SU(3)$ would also be degenerate. Since the three quark masses differ from one another, the only way to treat the three quarks on the same footing is by allowing for a breaking of the symmetry. The Gell-Mann-Okubo mass formula relates the hadron masses by taking the small $SU(3)$ breaking into account but does not depend on the details of $SU(3)$ breaking effects. Such $SU(3)$ breaking effects cannot be calculated theoretically and must be estimated using experimental inputs. Usually, the mass differences between these mesons have been used as a measure of the extent of breaking of $SU(3)$ flavor symmetry. The masses of these mesons, which are bound states of a pair of quark and anti-quark, can be computed precisely by lattice QCD. It is not possible to estimate the binding energy[11] of a quark anti-quark pair in a meson from analytical QCD calculations since these resonances lie in the non-relativistic low energy regime in which QCD is essentially non-perturbative in nature. Moreover, the electro-magnetic interactions between the quark and the antiquark in the meson also contribute towards

[11] Here binding energy is not a very well defined quantity as strong interaction is a confining interaction implying that there are no free quark states at asymptotically large spatial separations.

its binding energy. Thus, by measuring the mass differences amongst the mesons one does not fully solicit the breaking of $SU(3)$ flavor symmetry.[12] Another usual method to explore the breaking of $SU(3)$ flavor symmetry is to look at specific loop diagrams where the down and strange quarks contribute. The loop effects affect the amplitude of the process under consideration and its physical manifestations are then studied for a quantitative estimation of the breaking of $SU(3)$ flavor symmetry. Since the up quark has different electric charge than the down and strange, it can not be treated in the same way in these studies of loop contributions. Therefore, such a method also fails to probe the full exchange symmetry of these three light quarks. Hence, all estimates of $SU(3)$ breaking in decay amplitudes are currently empirical. The $SU(3)$ flavor symmetry group has three non-commuting normal $SU(2)$ subgroups, namely isospin, U-spin and V-spin. Some details of the $SU(3)$ flavor symmetry are provided in Appendix A. There exist several studies in the literature which have used broken $SU(3)$ flavor symmetry (i) in various decay modes using the methods of amplitudes (usually isospin and U-spin amplitudes) and various quark diagrams [90–136], and (ii) in determinations of weak phases and CP violating phases [137–149]. These methods involve comparison of observables in distinct decay modes that are related by some underlying $SU(2)$ sub-symmetry, such as isospin, U-spin or V-spin. However, the full exchange symmetry amongst the three light quarks has not yet been fully exploited, in a single decay mode. Weak decays of hadrons involve several unknown parameters which can be reduced by using the $SU(3)$ flavor symmetry. Since $SU(3)$ flavor symmetry is still extensively used to relate the few decay modes of heavy mesons[13], it is important to realize other methods to experimentally measure the breaking of $SU(3)$ flavor symmetry and understand the complete nature of $SU(3)$ breaking in a better manner. In this thesis we propose a method in Chapter 4 to achieve

[12] It is, however, noteworthy that certain meson mass differences do reveal the size of the electromagnetic contributions via the Dashen's theorem [89].

[13] This is because the heavy mesons lie above the QCD confinement scale, thus having smaller $SU(3)$ flavor symmetry breaking than in the light meson cases.

precisely this by looking at asymmetries in the Dalitz plot (discussed in Chapter 2) under exchange of the mesons in the final state. These asymmetries can be measured in both resonant as well as in the non-resonant regions. A quantitative estimate of the variation of these asymmetries obtained experimentally would provide valuable understanding of $SU(3)$ breaking effects.

Here, we have only sketched the outlines of the symmetries under our consideration in this thesis. Much more details regarding these symmetries can be found in many References [150–158].

1.6 Summary

In this chapter, we have reviewed the concept of symmetry as applied to physical laws, classification of various symmetries, and discussed how symmetry can be applied mathematically by using the group theory. We have also looked at some salient features of conservation laws and how they are intimately connected to symmetries in Nature. Finally, we have briefly discussed all the symmetries under our consideration in this thesis and given references to the existing methodologies to study these symmetries, wherever applicable. Since we shall employ three-body processes in this thesis to study symmetry violations, the next chapter is devoted to a mini discussion on three-body decays and consequently a mini-review of the relevant concept of Dalitz plot.

2

Three-body decay and Dalitz plot

Three-body decays have long been used in various studies in particle physics. After the two-body decays, the three-body decay is arguably the simplest 'complicated' decay scenario. The higher multi-body decays are usually treated as some "effective" two- or three-body decays after combining their final particles into two or three 'effective particle' states. The Dalitz plot is a two-dimensional plot which encapsulates the phase-space for a three-body decay in its entirety and captures the signature of the underlying dynamics. Traditionally the Dalitz plot has played a very significant role in identification and characterization of many resonances observed as densely populated bands in the Dalitz plot. We shall very briefly discuss the general three-body decay and the concept of the Dalitz plot highlighting some of its salient features.

2.1 A general three-body decay

Let us consider a general three-body decay $X \to 1 + 2 + 3$ (see Fig. 2.1), where the 4-momentum of each particle is given by $p_i = (E_i, \vec{p}_i)$, where E_i is the energy and $\vec{p}_i$ is

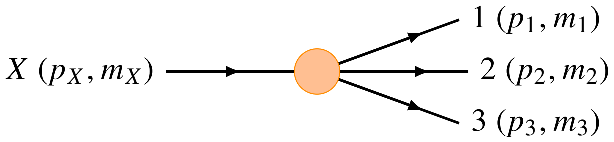


Figure 2.1: Four-momenta and masses of all the particles in the three-body decay $X \to 1 + 2 + 3$.

the 3-momentum of particle i (with $i \in \{X, 1, 2, 3\}$) and $p_i^2 = E_i^2 - |\vec{p}_i|^2 = m_i^2$, m_i being the mass of particle i. Conservation of energy and 3-momentum implies that

$$E_X = E_1 + E_2 + E_3, \quad \text{and} \quad \vec{p}_X = \vec{p}_1 + \vec{p}_2 + \vec{p}_3. \tag{2.1}$$

In case of three-body decays, the energy and momentum of the daughter particles are not completely fixed (unlike two-body decays) by applying the conservation of energy and momentum. This is because the three-body decay has additional degrees of freedom. In any three-body decay the orientation of the final particles depends on the relative orbital angular momentum between them. A complete description of the final state is possible by knowing the final state 4-momenta: p_1^μ, p_2^μ and p_3^μ. So we need to specify 12 parameters (4 parameters for each of the three 4-momenta). However, conservation of energy and momenta provide 4 constraint equations: $p_X^\mu = p_1^\mu + p_2^\mu + p_3^\mu$. Each of the three 4-momenta also satisfy the on-shell condition $m_i^2 = E_i^2 - |\vec{p}_i|^2$, giving 3 more constraint equations. Now if we consider the decay in the rest frame of the decaying particle (particle X), then all the final particles must lie in one plane and the final state is symmetric about any rotations about an axis which passes through particle X and is perpendicular to the decay plane. The orientation of the axis has two degrees of freedom and the rotational symmetry about the axis constitutes one more degree of freedom (see Fig. 2.2). So together these constitute 3 extra constraints. Thus in total we have $4 + 3 + 3 = 10$ constraints. This leaves only 2 free parameters out of the initial 12 parameters. So we can describe a three-body final state with only 2 parameters, and there are many choices for the type of variables one can choose. The Dalitz plots, the original one by Dalitz [9, 10] (which was further explored by E. Fabri [11] and Charles Zemach [12] and explained well in many books [151–154]) as well as the one used in current studies [13], exploit this feature and are different from each other in their choice of the two variables.

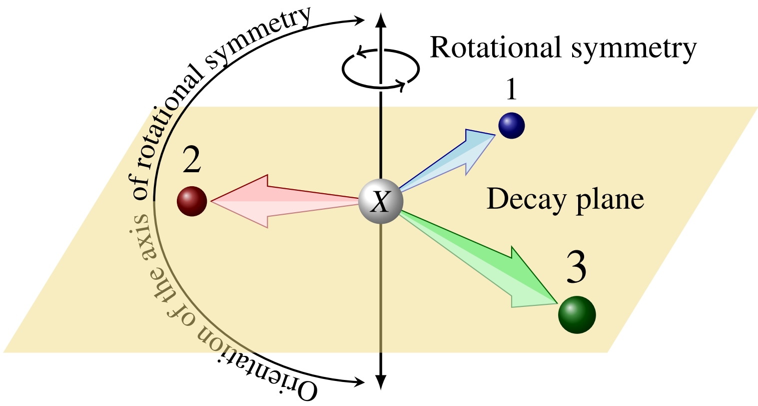


Figure 2.2: A representative plane of decay for $X \to 1 + 2 + 3$ showing the axis of rotational symmetry and its two possible orientations in the rest frame of X.

2.2 The Original ternary plot of Dalitz

Let us consider the three-body decay $X \to 1 + 2 + 3$. The 4-momenta of the particles are given by $p_i = (E_i, \vec{p}_i)$, where E_i and $\vec{p}_i$ are the energy and 3-momentum of the particle i respectively, with $i \in \{X, 1, 2, 3\}$. Conservation of energy and momentum are enshrined in Eq. (2.1). Let us denote the mass of the particle i by m_i. The Q-value of this decay is defined as $Q = m_X - (m_1 + m_2 + m_3)$. The '*kinetic energy*' of the particle i is defined as $T_i = E_i - m_i$. If we consider the parent particle to be at rest, then $T_X = 0$, as $E_X = m_X$. It is easy to see that

$$\begin{aligned}\sum_{i=1}^{3} T_i &= (E_1 + E_2 + E_3) - (m_1 + m_2 + m_3) = E_X - (m_1 + m_2 + m_3) \\ &= m_X - (m_1 + m_2 + m_3) = Q. \end{aligned} \tag{2.2}$$

Thus we find a scalar linear relationship amongst the kinetic energies of the final particles and the Q-value of the decay: $T_1 + T_2 + T_3 = Q$. R. H. Dalitz exploited this to present a ternary plot[1] with Cartesian coordinates (T_1, T_2, T_3) which can also be described by the following barycentric coordinates:

$$\left(X = \frac{\sqrt{3}\,(T_1 - T_2)}{Q}, \quad Y = \frac{2T_3 - T_2 - T_1}{Q} \right). \tag{2.3}$$

The equilateral triangle for the present case is shown in Fig. 2.3. Any point inside the

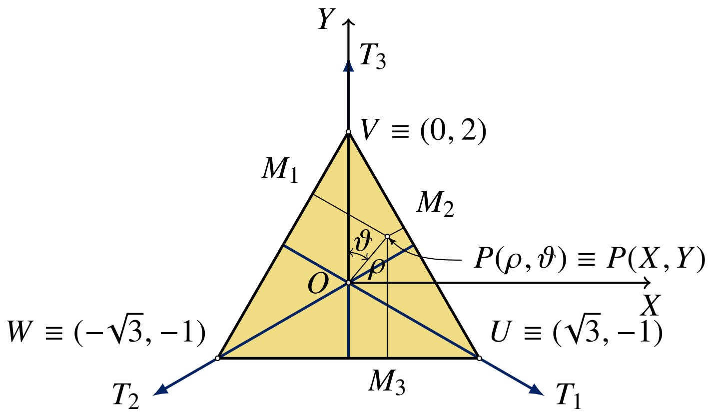


Figure 2.3: The ternary plot using the original parametrization of Dalitz for the three-body decay $X \to 1+2+3$. Each side of this triangle is of length $2\sqrt{3}$ units and the height of the triangle is 3 units.

equilateral triangle $\triangle UVW$ is allowed by conservation of energy. Moreover, the distance of any point, say $P(X, Y)$, from the three sides of the triangle are given by

$$d_1 = \left|\overline{PM_1}\right| = \frac{T_1}{Q/3}, \quad d_2 = \left|\overline{PM_2}\right| = \frac{T_2}{Q/3}, \quad d_3 = \left|\overline{PM_3}\right| = \frac{T_3}{Q/3}, \tag{2.4}$$

such that $d_1 + d_2 + d_3 = 3$. The equilateral triangle of Fig. 2.3 can also be described in terms of polar coordinates (ρ, ϑ) with the pole at the center of the triangle and the polar axis passing through one of the vertices, here V. In terms of the polar coordinates we

[1] For details of ternary plot see Appendix B.

have $X = \rho \sin\vartheta$ and $Y = \rho\cos\vartheta$, which lead to

$$T_1 = \frac{Q}{3}\left(1 + \rho\cos\left(\frac{2\pi}{3} - \vartheta\right)\right) = \frac{Q}{6}\left(2 - \sqrt{3}X - Y\right), \tag{2.5}$$

$$T_2 = \frac{Q}{3}\left(1 + \rho\cos\left(\frac{2\pi}{3} + \vartheta\right)\right) = \frac{Q}{6}\left(2 + \sqrt{3}X - Y\right), \tag{2.6}$$

$$T_3 = \frac{Q}{3}\left(1 + \rho\cos\vartheta\right) = \frac{Q}{3}\left(1 + Y\right). \tag{2.7}$$

It must be noted that even though the full region inside the equilateral triangle is energetically allowed, the full area is not physical. We have got to find out the *boundary* of this physically allowed region inside the triangle on and inside which both energy and 3-momentum are conserved. Outside this boundary, energy (not 3-momentum), is conserved, and outside the equilateral triangle neither energy nor 3-momentum is conserved. The conservation of 3-momentum in the rest frame of X implies that $\vec{p}_1 = -\left(\vec{p}_2 + \vec{p}_3\right)$. Let the angle between $\vec{p}_2$ and $\vec{p}_3$ be Θ. So we have $\left|\vec{p}_1\right|^2 = \left|\vec{p}_2\right|^2 + \left|\vec{p}_3\right|^2 + 2\left|\vec{p}_2\right|\left|\vec{p}_3\right|\cos\Theta$. This implies that

$$\cos\Theta = \frac{\left|\vec{p}_1\right|^2 - \left|\vec{p}_2\right|^2 - \left|\vec{p}_3\right|^2}{2\left|\vec{p}_2\right|\left|\vec{p}_3\right|} \tag{2.8}$$

Since $|\cos\Theta| \leqslant 1$, the boundary of the physical region is determined by $\cos\Theta = \pm 1$, i.e. when the 3-momenta are collinear. So the equation of the boundary corresponds to $\cos^2\Theta = 1$. Expressed in terms of the 3-momentum of the final particle, this directly leads to the following expression:

$$\left(\left|\vec{p}_1\right|^2 - \left|\vec{p}_2\right|^2 - \left|\vec{p}_3\right|^2\right)^2 = 4\left|\vec{p}_2\right|^2\left|\vec{p}_3\right|^2 \quad \Longrightarrow \lambda\left(\left|\vec{p}_1\right|^2, \left|\vec{p}_2\right|^2, \left|\vec{p}_3\right|^2\right) = 0, \tag{2.9}$$

where $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ca)$, is called the Källén function or triangle function which has the following properties:

1. It is symmetric under $a \leftrightarrow b \leftrightarrow c$.

2. $\lambda(a, b, c) = \lambda(-a, -b, -c)$.

3. $\lambda(\alpha\, a, \alpha\, b, \alpha\, c) = \alpha^2\, \lambda(a, b, c)$.

4. If $a \gg b, c$, then $\lambda(a, b, c) \to \lambda(a, 0, 0) = a^2$.

5. If $b = c$, then $\lambda(a, b, b) = a^2 - 4ab$.

6. If $c = 0$, then

   (a) $\lambda(a, b, 0) = \lambda(a - b, 0, 0) = (a - b)^2$,

   (b) $\lambda(a + b, b, 0) = \lambda(a - b, -b, 0) = \lambda(a, 0, 0) = a^2$.

7. $\lambda(a + b, b + c, c + a) = -4\,(ab + bc + ca)$.

8. $\lambda(a - b, b - c, c - a) = 4\left(a^2 + b^2 + c^2 - ab - bc - ca\right)$.

9. It can be re-expressed as: $\lambda(a, b, c) = \frac{1}{4}\,(\lambda(a + b, b + c, c + a) + \lambda(a - b, b - c, c - a))$.

10. Another useful property:

$$\begin{aligned}\lambda(a_1 + a_2, b_1 + b_2, c_1 + c_2) = {} & \lambda(a_1, b_1, c_1) + \lambda(a_2, b_2, c_2) \\ & - 2\,(a_1 + b_1 + c_1)\,(a_2 + b_2 + c_2) + 4\,(a_1 a_2 + b_1 b_2 + c_1 c_2)\,.\end{aligned}$$

Thus the physically allowed region is described by the region in which $\lambda\left(\left|\vec{p}_1\right|^2, \left|\vec{p}_2\right|^2, \left|\vec{p}_3\right|^2\right) \leqslant 0$. We know that $\left|\vec{p}_i\right|^2 = E_i^2 - m_i^2$ and $E_i = T_i + m_i$, and thus $\left|\vec{p}_i\right|^2 = T_i^2 + 2T_i m_i$. Thus the allowed region is described by

$$\lambda\left(T_1^2 + 2T_1 m_1, T_2^2 + 2T_2 m_2, T_3^2 + 2T_3 m_3\right) \leqslant 0. \tag{2.10}$$

Using the properties of the Källén function we have

$$\begin{aligned}\lambda\left(T_1^2+2T_1m_1,T_2^2+2T_2m_2,T_3^2+2T_3m_3\right) = {} & \lambda\left(T_1^2,T_2^2,T_3^2\right)+4\lambda\left(T_1m_1,T_2m_2,T_3m_3\right)\\ & -4\left(T_1^2+T_2^2+T_3^2\right)\left(T_1m_1+T_2m_2+T_3m_3\right)\\ & +8\left(T_1^3m_1+T_2^3m_2+T_3^3m_3\right). \qquad (2.11)\end{aligned}$$

If we substitute the expressions for the kinetic energies in terms of the barycentric coordinates, then the expression for the boundary of the physically allowed region can be written as $\Phi(X,Y)=0$, where

$$\Phi(X,Y)=C_{00}+C_{10}X+C_{01}Y+C_{11}XY+C_{20}X^2+C_{02}Y^2+C_{21}X^2Y+C_{12}XY^2+C_{30}X^3+C_{03}Y^3, \qquad (2.12)$$

with C_{ij} being the coefficient of X^iY^j, and they are given by

$$C_{00}=\frac{1}{27}Q^2\left(12\,\lambda\left(m_1,m_2,m_3\right)-4Qm_X+3Q^2\right), \qquad (2.13a)$$

$$C_{10}=-\frac{2}{3\sqrt{3}}Q^2\left(m_1-m_2\right)\left(m_X+m_1+m_2-3m_3\right), \qquad (2.13b)$$

$$C_{01}=-\frac{1}{9}Q^2\left(4\lambda\left(m_1,m_2,m_3\right)+2Q\left(m_1+m_2-2m_3\right)\right), \qquad (2.13c)$$

$$C_{11}=-\frac{2}{3\sqrt{3}}Q^2\left(m_1-m_2\right)\left(m_1+m_2-m_X\right), \qquad (2.13d)$$

$$C_{20}=\frac{1}{9}Q^2\left(3\left(m_1+m_2\right)^2+2Q\left(2m_1+2m_2-m_3\right)+Q^2\right), \qquad (2.13e)$$

$$C_{02}=\frac{1}{9}Q^2\left(\lambda\left(m_1,m_2,m_3\right)+3m_3\left(2m_X-m_3\right)+Q^2\right), \qquad (2.13f)$$

$$C_{21}=-\frac{2}{9}m_XQ^3=-3C_{03}, \qquad (2.13g)$$

$$C_{12}=0=C_{30}. \qquad (2.13h)$$

It is important to note that both C_{12} and C_{30} are identically zero. Therefore, the equation for the boundary is

$$\Phi(X,Y) = C_{00} + C_{10}X + C_{01}Y + C_{11}XY + C_{20}X^2 + C_{02}Y^2 + C_{21}X^2Y + C_{03}Y^3 = 0. \tag{2.14}$$

We can also express the boundary in terms of the polar coordinates. This is given by $\Phi(\rho,\vartheta) = 0$, where

$$\begin{aligned}\Phi(\rho,\vartheta) = {}& P_{00}(\rho) + P_{10}(\rho)\sin\vartheta + P_{01}(\rho)\cos\vartheta + P_{20}(\rho)\sin 2\vartheta + P_{02}(\rho)\cos 2\vartheta \\ &+ P_{30}(\rho)\sin 3\vartheta + P_{03}(\rho)\cos 3\vartheta,\end{aligned} \tag{2.15}$$

with P_{i0} (or P_{0j}) being the coefficient of $\sin i\vartheta$ (or $\cos j\vartheta$), and P_{00} being the term independent of ϑ:

$$P_{00}(\rho) = \frac{1}{27}Q^2\Big(12\lambda(m_1,m_2,m_3) - 4m_XQ + 3Q^2 + 3\left(m_1^2 + m_2^2 + m_3^2 + m_X^2\right)\rho^2\Big), \tag{2.16a}$$

$$P_{10}(\rho) = \frac{2}{3\sqrt{3}}Q^2(m_1 - m_2)(m_X + m_1 + m_2 - 3m_3)\rho, \tag{2.16b}$$

$$P_{01}(\rho) = -\frac{2}{9}Q^2\Big(2\lambda(m_1,m_2,m_3) + 6m_3(m_1 + m_2 - m_3) + Q(m_1 + m_2 - 2m_3)\Big)\rho, \tag{2.16c}$$

$$P_{20}(\rho) = \frac{1}{3\sqrt{3}}Q^2(m_1 - m_2)(m_1 + m_2 - 2m_X)\rho^2, \tag{2.16d}$$

$$P_{02}(\rho) = -\frac{1}{9}Q^2\Big(\lambda(m_1,m_2,m_3) + 6m_1m_2 - 3m_3^2 + 2Q(m_1 + m_2 - 2m_3)\Big)\rho^2, \tag{2.16e}$$

$$P_{30}(\rho) = 0, \tag{2.16f}$$

$$P_{03}(\rho) = \frac{2}{27}m_XQ^3\rho^3. \tag{2.16g}$$

It is important to notice that the coefficient P_{30} is identically zero. Thus in polar coordinates the boundary is given by

$$\begin{aligned}\Phi(\rho,\vartheta) = P_{00}(\rho) + P_{10}(\rho)\sin\vartheta + P_{01}(\rho)\cos\vartheta + P_{20}(\rho)\sin 2\vartheta + P_{02}(\rho)\cos 2\vartheta \\ + P_{03}(\rho)\cos 3\vartheta = 0.\end{aligned} \tag{2.17}$$

2.2.1 Special case of $m_1 = m_2$

When $m_1 = m_2$, the coefficients C_{10}, C_{11}, P_{10} and P_{20} are identically equal to zero. Thus in this special case, the boundary is given by $\Phi(X,Y) = C_{00} + X^2(C_{20} + C_{21}Y) + Y\left(C_{01} + C_{02}Y + C_{03}Y^2\right) = 0$, and $\Psi(\rho,\vartheta) = \sum_{i=0}^{3} P_{0i}(\rho)\,\cos(i\vartheta) = 0$. It is very easy to see that in this special case, the boundary is symmetric under $X \leftrightarrow -X$ (or $\vartheta \leftrightarrow -\vartheta$), which is nothing but reflection about the Y-axis (or equivalently the polar axis).

2.2.2 Some more special cases

Whatever we have discussed this far makes no assumptions about the momenta of the final particles. However, there are two interesting special cases, in which the momenta involved can be either non-relativistic or relativistic. Below we discuss both these cases.

(a) Non-relativistic case: For the non-relativistic case the kinetic energies of the final particles are given by $T_i = \dfrac{|\vec{p}_i|^2}{2m_i}$, where $i \in \{1,2,3\}$. So the expression for the boundary is given by

$$\lambda(T_1 m_1, T_2 m_2, T_3 m_3) = 0. \tag{2.18}$$

The equation for the boundary is still a function of the kinetic energies. But what we really need is the equation for the boundary in terms of X and Y. Substituting the values for T_1, T_2 and T_3 in the equation for the boundary we get the following expression:

$$C'_{20}\, X^2 + C'_{11}\, XY + C'_{02}\, Y^2 + C'_{10}\, X + C'_{01}\, Y + C'_{00} = 0, \tag{2.19}$$

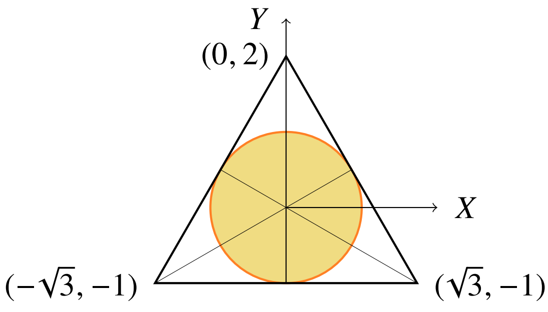
which is the equation for a *conic section* and here

$$C'_{20} = \frac{1}{12} Q^2 \left(m_1 + m_2\right)^2, \tag{2.20a}$$
$$C'_{11} = \frac{1}{6\sqrt{3}} Q^2 (m_1 - m_2)(m_1 + m_2 + 2m_3), \tag{2.20b}$$
$$C'_{02} = \frac{1}{36} Q^2 \Big(\lambda\left(m_1, m_2, m_3\right) + 3m_3\left(2m_1 + 2m_2 + m_3\right)\Big), \tag{2.20c}$$
$$C'_{10} = -\frac{1}{3\sqrt{3}} Q^2 (m_1 - m_2)(m_1 + m_2 - m_3), \tag{2.20d}$$
$$C'_{01} = -\frac{1}{9} Q^2 \Big(\lambda\left(m_1, m_2, m_3\right) + 3m_3\left(m_1 + m_2 - m_3\right)\Big), \tag{2.20e}$$
$$C'_{00} = \frac{1}{9} Q^2 \lambda\left(m_1, m_2, m_3\right). \tag{2.20f}$$

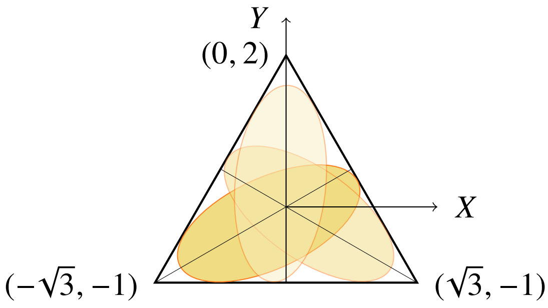
When $m_1 = m_2$, then $C'_{11} = 0 = C'_{10}$ and hence the boundary is symmetric under $X \leftrightarrow -X$ (i.e. reflection about Y-axis). However, in order to ascertain what type of conic section the boundary is, in general, let us evaluate the discriminant $C'^2_{11} - 4C'_{20}C'_{02}$:

$$C'^2_{11} - 4C'_{20}C'_{02} = -\frac{4}{27} m_1\, m_2\, m_3\, (m_1 + m_2 + m_3) Q^4. \tag{2.21}$$

So for all values of m_2, m_3 and m_4 the discriminant is negative: $C'^2_{11} - 4C'_{20}C'_{02} < 0$, this implies that the conic section is an *ellipse*. One can also arrive at this same conclusion by arguing that the only conic section which is bounded is, in general, an ellipse. Since our boundary is inscribed inside a triangle, it has to be bounded; since it is also a conic section, therefore it must be an ellipse. For the boundary to become a circle the necessary conditions are $C'_{20} = C'_{02}$ and $C'_{11} = 0$. Now $C'_{11} = 0$ implies that $m_1 = m_2$ and $C'_{20} = C'_{02}$ implies that $m_1 = m_2 = m_3$. So if we put $m_1 = m_2 = m_3 = m$ (say), then the equation for the boundary becomes $X^2 + Y^2 = 1$, which is a circle of unit radius centered at the origin. The boundary for the special case when all the final state particles are of equal mass, is shown in Fig. 2.4a. A few representative ellipses are drawn in Fig. 2.4b for the general case.



(a) When the daughter particles have exactly the same masses: $m_1 = m_2 = m_3 = m$.



(b) When the masses of the daughter particles are not the same: $m_1 \neq m_2 \neq m_3$.

Figure 2.4: The shaded area corresponds to the physical region allowed by conservation of both energy and 3-momentum. The boundary is in general an ellipse. Under the special case of identical particles in the final state, it becomes a circle. The three ellipses in (b) are shown for illustration only and they roughly correspond to the cases ($m_2/m_1 = 2, m_3/m_1 = 3$), ($m_1/m_3 = 2, m_2/m_3 = 3$) and ($m_1/m_2 = 3, m_3/m_2 = 2$).

The equation of the boundary, in terms of the polar coordinates, is given by

$$P'_{20}(\rho)\sin 2\vartheta + P'_{02}(\rho)\cos 2\vartheta + P'_{10}(\rho)\sin\vartheta + P'_{01}(\rho)\cos\vartheta + P'_{00}(\rho) = 0, \qquad (2.22)$$

where

$$P'_{20} = -\frac{1}{12\sqrt{3}}\,(m_1 - m_2)\,(m_1 + m_2 + 2m_3)\,Q^2\rho^2, \tag{2.23a}$$

$$P'_{02} = -\frac{1}{36}\left(\lambda\,(m_1, m_2, m_3) + 6m_1m_2 - 3m_3^2\right)Q^2\rho^2, \tag{2.23b}$$

$$P'_{10} = \frac{1}{3\sqrt{3}}(m_1 - m_2)(m_1 + m_2 - m_3)Q^2\rho, \tag{2.23c}$$

$$P'_{01} = -\frac{1}{9}\Big(\lambda\,(m_1, m_2, m_3) + 3m_3(m_1 + m_2 - m_3)\Big)Q^2\rho, \tag{2.23d}$$

$$P'_{00} = \frac{1}{18}Q^2\Big(2\lambda(m_1, m_2, m_3) + \left(\lambda(m_1, m_2, m_3) + 3(m_1m_2 + m_2m_3 + m_3m_1)\right)\rho^2\Big). \tag{2.23e}$$

Here again $m_1 = m_2$ implies that $P'_{10} = 0 = P'_{20}$ and hence the boundary would be symmetric under $\vartheta \leftrightarrow -\vartheta$ (i.e. reflection about the polar axis) in this case. For the special case of $m_1 = m_2 = m_3 = m$ (say), the equation of the boundary becomes $\rho^2 = 1$, which describes a circle of unit radius centered at the center of the triangle.

(b) Ultra-relativistic case: In the ultra-relativistic case, we can neglect the mass of the daughter particles in comparison to their energy. So for ultra-relativistic cases $T_i = E_i = \left|\vec{p}_i\right|$. The boundary conditions can therefore be restated as follows:

$$\lambda\left(T_1^2, T_2^2, T_3^2\right) = 0 \implies (2Y - 1)\left((Y + 1)^2 - 3X^2\right) = 0. \tag{2.24}$$

This expression gives equations for three straight lines $Y = 1/2$, $Y = \pm\sqrt{3}X - 1$. The region bounded by these straight lines is another equilateral triangle inscribed inside the bigger equilateral triangle. Fig. 2.3 shows the region and the bounding straight lines. It is important to note that here we have *not* considered the final particles to have equal masses. However, even if the final particles are equally massive, Fig. 2.5 remains unchanged.

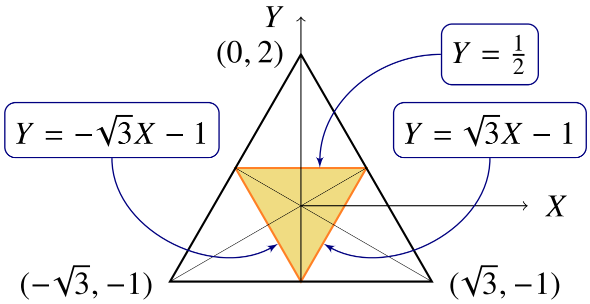


Figure 2.5: The shaded region is an equilateral triangle and its interior is the allowed region for extremely relativistic cases. These ultra-relativistic cases cannot have any points outside this smaller equilateral triangle.

2.2.3 The general case with particles of equal masses

In general, the physical region lies somewhere in between the two extremes we have just considered. This is clearly evident from Eq. (2.11), which we rewrite below for clarity:

$$\begin{aligned}\lambda\left(T_1^2+2T_1m_1, T_2^2+2T_2m_2, T_3^2+2T_3m_3\right) = {}& \underbrace{\lambda\left(T_1^2, T_2^2, T_3^2\right)}_{\text{ultra-relativistic case}} + 4\underbrace{\lambda\left(T_1m_1, T_2m_2, T_3m_3\right)}_{\text{non-relativistic case}} \\ &- 4\left(T_1^2+T_2^2+T_3^2\right)\left(T_1m_1+T_2m_2+T_3m_3\right) \\ &+ 8\left(T_1^3m_1+T_2^3m_2+T_3^3m_3\right). \end{aligned} \tag{2.25}$$

When all the final state particles are equally massive (i.e. $m_1 = m_2 = m_3 = m$) the equation for the boundary in terms of the barycentric coordinates X, Y is given by

$$C''_{03}Y^3 + C''_{21}X^2Y + C''_{20}X^2 + C''_{02}Y^2 + C''_{00} = 0, \tag{2.26}$$

where the coefficients C''_{ij} are

$$C''_{21} = -3C''_{03} = -\frac{2}{9}Q^3 m_X, \tag{2.27a}$$

$$C''_{20} = C''_{02} = \frac{1}{9}Q^2\left(12m^2 + 6mQ + Q^2\right), \tag{2.27b}$$

$$C''_{00} = -\frac{1}{27} Q^2 \, (6m + Q)^2 \,, \tag{2.27c}$$

with $Q = m_X - 3m$. In terms of the polar coordinates, the boundary for equally massive final states is

$$P''_{03}(\rho) \cos 3\vartheta + P''_{00}(\rho) = 0, \tag{2.28}$$

where $P''_{03}(\rho) = \frac{2}{27} m_X \, Q^3 \, \rho^3$, and $P''_{00}(\rho) = \frac{1}{27} Q^2 \left(3Q^2 - 4m_X Q - 36m^2 + 3\left(3m^2 + m_X^2\right)\rho^2\right)$. It is easy to invert this equation and solve for ρ^2 which gives

$$\rho^2 = (1 + \epsilon)^{-1} \left(1 - \epsilon \, \rho^3 \cos 3\vartheta\right) , \tag{2.29}$$

where

$$\epsilon = \frac{2m_X Q}{(2m_X - Q)^2} = \frac{2m_X (m_X - 3m)}{(m_X + 3m)^2}. \tag{2.30}$$

The physically allowed region is thus described by $\rho \leqslant R(\vartheta)$, where

$$R^2 = (1 + \epsilon)^{-1} \left(1 - \epsilon \, R^3 \cos 3\vartheta\right)$$

and graphically it looks like a shield as shown in Fig.2.6.

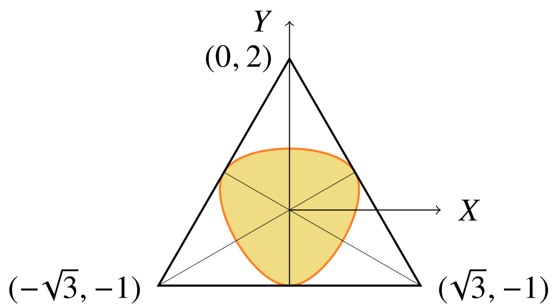


Figure 2.6: The physically allowed region for a three-body decay where all the three particles are identical.

Thus far we have discussed the kind of ternary plots that Dalitz proposed for study of three-body decays. In order to understand how important and relevant they are we would have to make a field theoretic study of the three-body decays.

2.3 Decay rate and definition of Dalitz plot

The differential decay rate for a general three-body decay $X(p_X) \to 1(p_1)+2(p_2)+3(p_3)$ is given by

$$d\Gamma = \mathsf{S}\frac{\langle|\mathscr{M}|^2\rangle}{2E_X}d\mathrm{PS}_f, \tag{2.31}$$

where $\langle|\mathscr{M}|^2\rangle$ is the Lorentz invariant square of the decay amplitude $\mathscr{M}$ averaged over initial spins and summed over final spins, $d\mathrm{PS}_f$ is the phase-space volume element in the final state:

$$d\mathrm{PS}_f = \prod_{i=1}^{3}\left(\frac{d^3\left|\vec{p}_i\right|}{(2\pi)^3\, 2E_i}\right)(2\pi)^4\,\delta^{(4)}\left(p_X - p_1 - p_2 - p_3\right), \tag{2.32}$$

S is the symmetry factor and is given by $\mathsf{S} = \prod_a \frac{1}{n_a!}$ where n_a is the number of particles of type a in the final state. Therefore, in the rest frame of the parent particle X we have

$$d\Gamma = \frac{\mathsf{S}}{16\,(2\pi)^5}\frac{\langle|\mathscr{M}|^2\rangle}{m_X}\prod_{i=1}^{3}\left(\frac{d^3\left|\vec{p}_i\right|}{E_i}\right)\delta^{(4)}\left(p_X - p_1 - p_2 - p_3\right). \tag{2.33}$$

Taking apart the delta function,

$$\delta^{(4)}\left(p_X - p_1 - p_2 - p_3\right) = \delta\left(m_X - E_1 - E_2 - E_3\right)\,\delta^{(3)}\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3\right),$$

and then doing the $\left|\vec{p}_3\right|$ integral we get

$$d\Gamma = \frac{\mathsf{S}}{16\,(2\pi)^5}\frac{\langle|\mathscr{M}|^2\rangle}{m_X}\left(\frac{d^3\left|\vec{p}_1\right|\, d^3\left|\vec{p}_2\right|}{E_1E_2\sqrt{\left|\vec{p}_1+\vec{p}_2\right|^2+m_3^2}}\right)\delta\left(m_X - E_1 - E_2 - \sqrt{\left|\vec{p}_1+\vec{p}_2\right|^2+m_3^2}\right). \tag{2.34}$$

Let the angle between $\vec{p}_1$ and $\vec{p}_2$ be α. (See Fig. 2.7.) Thus

$$\left|\vec{p}_1 + \vec{p}_2\right|^2 = \left(E_1^2 + E_2^2\right) - \left(m_1^2 + m_2^2\right) + 2\sqrt{\left(E_1^2 - m_1^2\right)\left(E_2^2 - m_2^2\right)}\cos\alpha. \tag{2.35}$$

Let us now do the $\left|\vec{p}_2\right|$ integral. Fixing the polar axis along $\vec{p}_1$, we get $d^3\left|\vec{p}_2\right| =$

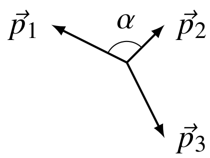


Figure 2.7: Decay plane of $X \to 1 + 2 + 3$ showing the angle α between the 3-momenta $\vec{p}_1$ and $\vec{p}_2$. All the 3-momenta are in the rest frame of the parent particle X.

$\left|\vec{p}_2\right|^2 \; d\left|\vec{p}_2\right| \; d\cos\alpha \; d\phi$. The integration over ϕ gives 2π. For the integration over $\cos\alpha$ let us do a change of variable, by defining $u^2 = \left|\vec{p}_1 + \vec{p}_2\right|^2 + m_3^2$, such that $d\cos\alpha = \dfrac{u\,du}{\sqrt{\left(E_1^2 - m_1^2\right)\left(E_2^2 - m_2^2\right)}}$. Now the limits on u are given by

$$u_\pm = \sqrt{\left|\vec{p}_1\right|^2 + \left|\vec{p}_2\right|^2 \pm 2\left|\vec{p}_1\right|\left|\vec{p}_2\right| + m_3^2} = \sqrt{\left(\left|\vec{p}_1\right| \pm \left|\vec{p}_2\right|\right)^2 + m_3^2}. \tag{2.36}$$

Therefore

$$d\Gamma = \frac{\mathsf{S}}{16\,(2\pi)^4}\frac{\langle\left|\mathscr{M}\right|^2\rangle}{m_X}\left(\frac{1}{E_1E_2}\sqrt{\frac{E_2^2 - m_2^2}{E_1^2 - m_1^2}}\right) d^3\left|\vec{p}_1\right| \; d\left|\vec{p}_2\right| \int_{u_-}^{u_+} \delta\left(m_X - E_1 - E_2 - u\right)\, du. \tag{2.37}$$

Finally we have

$$d^3\left|\vec{p}_1\right| = (4\pi)\left|\vec{p}_1\right|^2 \; d\left|\vec{p}_1\right| = (4\pi)\left(E_1^2 - m_1^2\right)\; d\left|\vec{p}_1\right|. \tag{2.38}$$

Thus the differential decay width is given by

$$d\Gamma = \frac{\mathsf{S}}{8\,(2\pi)^3}\frac{\langle|\mathscr{M}|^2\rangle}{m_X}\left(\frac{\sqrt{\left(E_1^2-m_1^2\right)\left(E_2^2-m_2^2\right)}}{E_1E_2}\right)d\left|\vec{p}_1\right|\,d\left|\vec{p}_2\right| \times \int_{u_-}^{u_+}\delta\left(m_X-E_1-E_2-u\right)\,du. \tag{2.39}$$

Using $\left|\vec{p}_i\right|^2 = E_i^2 - m_i^2$ it is easy to show that

$$\frac{\sqrt{\left(E_1^2-m_1^2\right)\left(E_2^2-m_2^2\right)}}{E_1E_2}d\left|\vec{p}_1\right|d\left|\vec{p}_2\right| = dE_1\,dE_2. \tag{2.40}$$

Therefore, the differential decay width is now given by

$$d\Gamma = \frac{\mathsf{S}}{8\,(2\pi)^3}\int_{u_-}^{u_+}\frac{\langle|\mathscr{M}|^2\rangle}{m_X}\times\delta\left(m_X-E_1-E_2-u\right)\,du\,dE_1\,dE_2. \tag{2.41}$$

It is also possible to cast the differential decay rate in terms of integrals over the kinetic energy T_i instead of the energy E_i: $T_i = E_i - m_i$, such that $dE_i = dT_i$. Hence

$$d\Gamma = \frac{\mathsf{S}}{8\,(2\pi)^3}\int_{u_-}^{u_+}\frac{\langle|\mathscr{M}|^2\rangle}{m_X}\delta\left(m_X+m_1+m_2-T_1-T_2-u\right)\,du\,dT_1\,dT_2, \tag{2.42}$$

where the limits on u are now given by,

$$u_\pm^2 = \left(\sqrt{T_1^2+2T_1m_1}\pm\sqrt{T_2^2+2T_2m_2}\right)^2 + m_3^2. \tag{2.43}$$

The delta function here determines a boundary in the T_1T_2 plane outside which the decay rate vanishes because the phase-space volume is zero there. It is the same boundary as we have found before by applying conservation of both energy and 3-momentum. Thus every decay observed in an experiment corresponds to a point in the physically allowed region of the equilateral triangle. The accumulation of such points inside the boundary is called as the *Dalitz plot*. It is important to note that the phase-space volume corresponding to a region in the Dalitz plot is proportional to the area of that region. Thus, phase-space alone would give a completely uniform distribution of points inside the boundary. If there is any departure from this uniformity, it must, therefore, arise from the matrix element. Thus observing the various patterns in the Dalitz plot gives us information about the properties of the matrix element.

2.4 New Dalitz Plot

So far we have dealt with a Dalitz plot in which we have used the kinetic energies of the final particles in the analysis [9–12, 151–154]. It is sometimes useful to make a change of variable to invariant masses of pairs of particles in order to get the Dalitz plot. This approach is thus different from Dalitz's original prescription [9, 10].

We define the following invariants $m_{ij}^2 = \left(p_i + p_j\right)^2$, where $i, j \in \{1, 2, 3\}$. These invariant masses are not independent quantities. If we sum these invariant masses we get

$$m_{12}^2 + m_{23}^2 + m_{31}^2 = (p_X - p_3)^2 + (p_X - p_1)^2 + (p_X - p_2)^2 = m_X^2 + m_1^2 + m_2^2 + m_3^2, \quad (2.44)$$

where we have used the conservation of 4-momentum: $p_X = p_1 + p_2 + p_3$. One can rewrite the expression relating all the seven masses in the problem as follows:

$$m_X^2 = m_{12}^2 + m_{23}^2 + m_{31}^2 - m_1^2 - m_2^2 - m_3^2. \quad (2.45)$$

We can express the invariant masses as $m_{ij}^2 = m_X^2 + m_k^2 - 2m_X E_k$, where $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$. Therefore, we have $dm_{ij}^2 = -2m_X\, dE_k$. It is therefore quite possible to replace $dE_1\, dE_2$ in terms of $dm_{23}^2\, dm_{31}^2$:

$$dm_{23}^2\, dm_{31}^2 = 4m_X^2\, dE_1\, dE_2. \quad (2.46)$$

Thus the area on the $m_{23}^2\, m_{31}^2$ invariant mass plot is proportional to the corresponding area on the $E_1 E_2$ (equivalently $T_1 T_2$) plot. Doing this change of variable we get

$$d\Gamma = \frac{\mathsf{S}}{32\,(2\pi)^3\, m_X^3} \int_{u_-}^{u_+} \left\langle \left|\mathscr{M}\right|^2 \right\rangle \times \delta\left(m_X - E_1 - E_2 - u\right) \times du\; dm_{23}^2\; dm_{31}^2, \qquad (2.47)$$

where

$$u_\pm^2 = \left(\sqrt{E_1^2 - m_1^2} \pm \sqrt{E_2^2 - m_2^2}\right)^2 + m_3^2, \qquad (2.48)$$

with the energies given by

$$E_k = \frac{m_X^2 + m_k^2 - m_{ij}^2}{2m_X}. \qquad (2.49)$$

Thus phase-space alone would again give a uniform distribution of points in the $m_{23}^2\, m_{31}^2$ plot. Any deviation from this uniformity would be completely due to the matrix element. If we suppose the particles 2 and 3 can come from the decay of a resonance with mass M, then we expect a concentration of events (points) along the $m_{23}^2 = M^2$ line in the Dalitz plot. However, due to unstable nature of the resonance the concentration of points about the $m_{23}^2 = M^2$ line form a band instead of a sharp line. The width of this band would be proportional to the decay-width of the resonance. In general, this Dalitz plot is not a ternary plot. One keeps the two variables m_{23}^2 and m_{31}^2 along the X and Y axes respectively, or vice versa. Now let us find out the allowed range of m_{ij}^2. Since $m_{ij}^2 = m_X^2 + m_k^2 - 2m_X E_k$ and $E_k > m_k$, the maximum of m_{ij}^2 is when $E_k = m_k$:

$$m_{ij,\max}^2 = (m_X - m_k)^2. \qquad (2.50)$$

In order to get the minimum of of m_{ij}^2 we need to go to a frame of reference which is the *center-of-momentum* frame for the particles i and j. This special frame is also called as the '*Gottfried-Jackson frame*'. In this frame

$$m_{ij}^2 = (p_i + p_j)^2 = (E_i + E_j)^2 > (m_i + m_j)^2 \implies m_{ij,\mathrm{min}}^2 = (m_i + m_j)^2. \tag{2.51}$$

Therefore, the allowed range for the invariant mass is

$$m_{ij}^2 \in \left[(m_i + m_j)^2, (m_X - m_k)^2\right], \tag{2.52}$$

where $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$. Thus the region of $m_{23}^2\, m_{31}^2$ plot available for our consideration is shown in Fig. 2.8. Please note that this is not the physically allowed region.

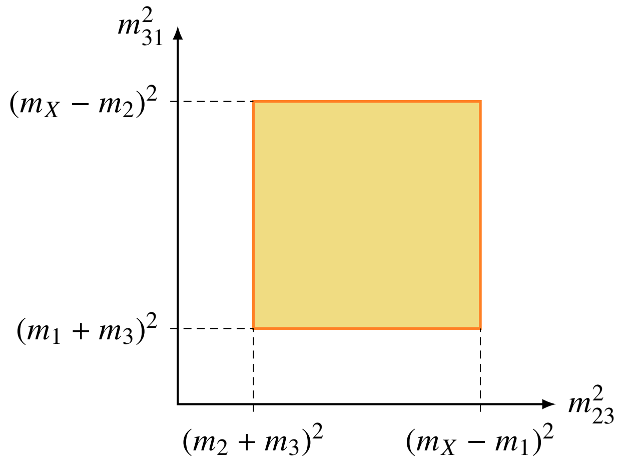


Figure 2.8: Plot showing the allowed region of m_{31}^2 and m_{23}^2. Please note that this is *not* the physically allowed region.

For a given value of, say m_{23}^2, the maximum and minimum allowed values for m_{31}^2 (or m_{12}^2) can be different from the ones shown above. The way to find out the maxima and minima for any of the three invariant masses, when one of the invariant masses is fixed, is given below. For this let us go to the Gottfried-Jackson frame of particles 2 and 3. (See Fig. 2.9.) This frame is defined by $\vec{p}_2{}' = -\vec{p}_3{}'$. Primes are used to distinguish quantities in the Gottfried-Jackson frame from those in the rest frame of particle X. Momentum

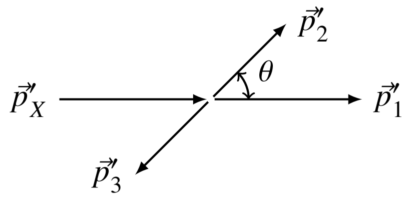


Figure 2.9: A general three-body decay in the Gottfried-Jackson frame. It is clear in the figure that $\vec{p}'_X = \vec{p}'_1$ and $\vec{p}'_2 = -\vec{p}'_3$. There is an angle θ between $\vec{p}'_1$ and $\vec{p}'_2$.

conservation, implies that in the Gottfried-Jackson frame $\vec{p}'_X = \vec{p}'_1$. Then it follows that the invariant mass of particles 2 and 3 is given by

$$\begin{aligned}
m_{23}^2 &= (p'_X - p'_1)^2 = (E'_X - E'_1)^2 \qquad \left(\because\ \vec{p}'_X = \vec{p}'_1\right)\\
&= \left(\sqrt{m_X^2 + \left|\vec{p}'_X\right|^2} - \sqrt{m_1^2 + \left|\vec{p}'_1\right|^2}\right)^2\\
&= \left(\sqrt{m_X^2 + \left|\vec{p}'_X\right|^2} - \sqrt{m_1^2 + \left|\vec{p}'_X\right|^2}\right)^2 \qquad \left(\because\ \left|\vec{p}'_1\right| = \left|\vec{p}'_X\right|\right).
\end{aligned}$$

Solving for $\left|\vec{p}'_X\right|^2$ we get

$$\left|\vec{p}'_X\right|^2 = \frac{m_X^4 + m_1^4 + m_{23}^4 - 2m_X^2 m_1^2 - 2m_X^2 m_{23}^2 - 2m_1^2 m_{23}^2}{4m_{23}^2} = \frac{\lambda(m_{23}^2, m_X^2, m_1^2)}{4m_{23}^2} = \left|\vec{p}'_1\right|^2. \tag{2.53}$$

Similarly using the other expression for m_{23}^2, we get

$$\begin{aligned}
m_{23}^2 &= (p'_2 + p'_3)^2 = (E'_2 + E'_3)^2 \qquad \left(\because\ \vec{p}'_2 = -\vec{p}'_3\right)\\
&= \left(\sqrt{m_2^2 + \left|\vec{p}'_2\right|^2} + \sqrt{m_3^2 + \left|\vec{p}'_3\right|^2}\right)^2\\
&= \left(\sqrt{m_2^2 + \left|\vec{p}'_2\right|^2} + \sqrt{m_3^2 + \left|\vec{p}'_2\right|^2}\right)^2 \qquad \left(\because\ \left|\vec{p}'_2\right| = \left|\vec{p}'_3\right|\right).
\end{aligned}$$

Solving for $\left|\vec{p}_2'\right|^2$ we get

$$\left|\vec{p}_2'\right|^2 = \frac{m_2^4 + m_3^4 + m_{23}^4 - 2m_2^2 m_3^2 - 2m_2^2 m_{23}^2 - 2m_3^2 m_{23}^2}{4m_{23}^2} = \frac{\lambda(m_{23}^2, m_2^2, m_3^2)}{4m_{23}^2} = \left|\vec{p}_3'\right|^2 . \tag{2.54}$$

Now let us consider the invariant m_{31}^2:

$$m_{31}^2 = (p_1 + p_3)^2 = m_1^2 + m_3^2 + 2\left(E_1' E_3' - \left|\vec{p}_1'\right|\left|\vec{p}_3'\right|\cos\theta\right), \tag{2.55}$$

where θ is the angle between $\vec{p}_1'$ and $\vec{p}_3'$, see Fig. 2.9. Since we have $\left|\vec{p}_1'\right|^2 = \frac{\lambda(m_{23}^2, m_X^2, m_1^2)}{4m_{23}^2}$ and $E_1' = \sqrt{m_1^2 + \left|\vec{p}_1'\right|^2}$, $\left|\vec{p}_3'\right|^2 = \frac{\lambda(m_{23}^2, m_2^2, m_3^2)}{4m_{23}^2}$ and $E_3' = \sqrt{m_3^2 + \left|\vec{p}_3'\right|^2}$ for a given value of m_{23}^2, we can easily see that m_{31}^2 is only a function of θ. It is also clear from the expression for m_{31}^2 that it has a maximum value when $\theta = \pi$ and a minimum value when $\theta = 0$:

$$m_{31,\max}^2 = m_1^2 + m_3^2 + 2\left(E_1' E_3' + \left|\vec{p}_1'\right|\left|\vec{p}_3'\right|\right), \tag{2.56}$$

$$m_{31,\min}^2 = m_1^2 + m_3^2 + 2\left(E_1' E_3' - \left|\vec{p}_1'\right|\left|\vec{p}_3'\right|\right). \tag{2.57}$$

Substituting the values for $\left|\vec{p}_1'\right|, \left|\vec{p}_3'\right|$ we get

$$E_1' = \frac{\left|m_{23}^2 - m_X^2 + m_1^2\right|}{2m_{23}}, \text{ and } E_3' = \frac{\left|m_{23}^2 - m_2^2 + m_3^2\right|}{2m_{23}}$$

. The maximum and minimum values for m_{31}^2 are therefore given by

$$m_{31,\max}^2 = m_1^2 + m_3^2 + \frac{1}{2m_{23}^2}\Big(\left|m_{23}^2 - m_X^2 + m_1^2\right|\left|m_{23}^2 - m_2^2 + m_3^2\right| + \sqrt{\lambda(m_{23}^2, m_X^2, m_1^2)}\sqrt{\lambda(m_{23}^2, m_2^2, m_3^2)}\Big), \tag{2.58}$$

$$m^2_{31,\text{min}} = m_1^2 + m_3^2 + \frac{1}{2m^2_{23}}\left(\left|m^2_{23} - m_X^2 + m_1^2\right|\left|m^2_{23} - m_2^2 + m_3^2\right| - \sqrt{\lambda(m^2_{23}, m_X^2, m_1^2)}\sqrt{\lambda(m^2_{23}, m_2^2, m_3^2)}\right). \tag{2.59}$$

The curves defined by Eqs. (2.58) and (2.59) give the boundary of the physically allowed region in the $m^2_{23}\, m^2_{31}$ plane. The accumulation of points inside this area constitutes the *Dalitz plot*. A hypothetical Dalitz plot is shown in Fig. 2.10. Most of the Dalitz plots that are currently in use are of this type.

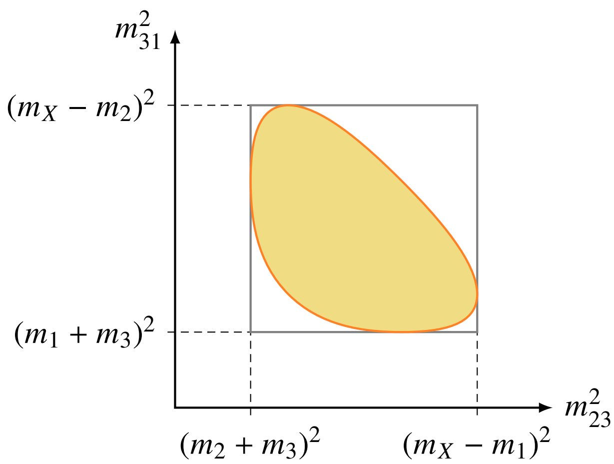


Figure 2.10: A hypothetical Dalitz plot showing the allowed region (shaded) for a three-body decay.

An aside: It is possible to find out what would be the maximum values of the momenta of the daughter particles, in the rest frame of the parent particle. We know that maximum momentum corrsponds to maximum energy. We also have $m^2_{ij} = m^2_X + m^2_k - 2m_X E_k$, where $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$. So when energy E_k is maximum, the invariant mass m^2_{ij} is at its minimum. We have observed that $m^2_{ij,\text{min}} = (m_i + m_j)^2$. This corresponds to $E_{k,\text{max}} = \dfrac{1}{2m_X}\left(m_X^2 - (m_i + m_j)^2 + m_k^2\right)$, and $\left|\vec{p}_{k,\text{max}}\right| = \dfrac{1}{2m_X}\sqrt{\lambda\left(m_X^2, (m_i + m_j)^2, m_k^2\right)}$.

2.5 Another Dalitz Plot

It is also possible to construct a ternary plot out of the three invariant masses of the pairs of final particles that we have considered, because they satisfy the relation

$$\sum_{\substack{i,j=1\\i<j}}^{3} m_{ij}^2 = m_X^2 + \sum_{k=1}^{3} m_k^2 = M^2 \text{ (say).} \tag{2.60}$$

We shall also work in the Gottfried-Jackson frame as shown in Fig. 2.9. We shall also use the following Mandelstam-like variables for notational simplicity: $s \equiv m_{23}^2$, $t \equiv m_{13}^2$, $u \equiv m_{12}^2$. Thus $s + t + u = M^2$. It is easy to show that

$$\left|\vec{p}_X'\right|^2 = \left|\vec{p}_1'\right|^2 = \frac{\lambda\left(s, m_X^2, m_1^2\right)}{4s}, \tag{2.61a}$$

$$\left|\vec{p}_2'\right|^2 = \left|\vec{p}_3'\right|^2 = \frac{\lambda\left(s, m_2^2, m_3^2\right)}{4s}, \tag{2.61b}$$

$$E_X' = \frac{\left|\left(m_X^2 - m_1^2\right) + s\right|}{2\sqrt{s}}, \tag{2.61c}$$

$$E_1' = \frac{\left|\left(m_X^2 - m_1^2\right) - s\right|}{2\sqrt{s}}, \tag{2.61d}$$

$$E_2' = \frac{\left|s + \left(m_2^2 - m_3^2\right)\right|}{2\sqrt{s}}, \tag{2.61e}$$

$$E_3' = \frac{\left|s - \left(m_2^2 - m_3^2\right)\right|}{2\sqrt{s}}. \tag{2.61f}$$

Using these one can show that

$$\begin{aligned} t = m_1^2 + m_3^2 + \frac{1}{2s}\Bigg(&\left|\left(m_X^2 - m_1^2\right) - s\right|\left|s - \left(m_2^2 - m_3^2\right)\right| \\ &+ \sqrt{\lambda\left(s, m_X^2, m_1^2\right)}\sqrt{\lambda\left(s, m_2^2, m_3^2\right)}\cos\theta\Bigg), \end{aligned} \tag{2.62}$$

$$u = m_1^2 + m_2^2 + \frac{1}{2s}\Bigg(\left|\left(m_X^2 - m_1^2\right) - s\right| \left|s + \left(m_2^2 - m_3^2\right)\right| \\ - \sqrt{\lambda\left(s, m_X^2, m_1^2\right)}\sqrt{\lambda\left(s, m_2^2, m_3^2\right)} \cos\theta \Bigg). \tag{2.63}$$

We can now rewrite these expressions as follows

$$t \equiv a_t + b\ \cos\theta, \text{ and } u \equiv a_u - b\ \cos\theta, \tag{2.64}$$

where

$$a_t = m_1^2 + m_3^2 + \frac{1}{2s}\left|\left(m_X^2 - m_1^2\right) - s\right| \left|s - \left(m_2^2 - m_3^2\right)\right|, \tag{2.65}$$

$$a_u = m_1^2 + m_2^2 + \frac{1}{2s}\left|\left(m_X^2 - m_1^2\right) - s\right| \left|s + \left(m_2^2 - m_3^2\right)\right|, \tag{2.66}$$

$$b = \frac{1}{2s}\sqrt{\lambda\left(s, m_X^2, m_1^2\right)}\sqrt{\lambda\left(s, m_2^2, m_3^2\right)}. \tag{2.67}$$

When $m_2 = m_3$, then $a_t = a_u$. From the fact that $s + t + u = M^2$, it follows that $a_t + a_u = M^2 - s$. Following the original spirit of the Dalitz plot, we can draw a ternary plot with Cartesian coordinates (t, u, s) which can also be described by the following barycentric coordinates:

$$\left(X = \frac{\sqrt{3}(t-u)}{M^2}, \quad Y = \frac{2s - t - u}{M^2} \right). \tag{2.68}$$

The equilateral triangle for the present case is shown in Fig. 2.11. Any point inside the equilateral triangle $\triangle UVW$ is allowed by conservation of energy. Moreover, the distance of any point, say $P(X, Y)$, from the three sides of the triangle are given by

$$d_t = \left|\overline{PM_1}\right| = \frac{t}{M^2/3}, \quad d_u = \left|\overline{PM_2}\right| = \frac{u}{M^2/3}, \quad d_s = \left|\overline{PM_3}\right| = \frac{s}{M^2/3}, \tag{2.69}$$

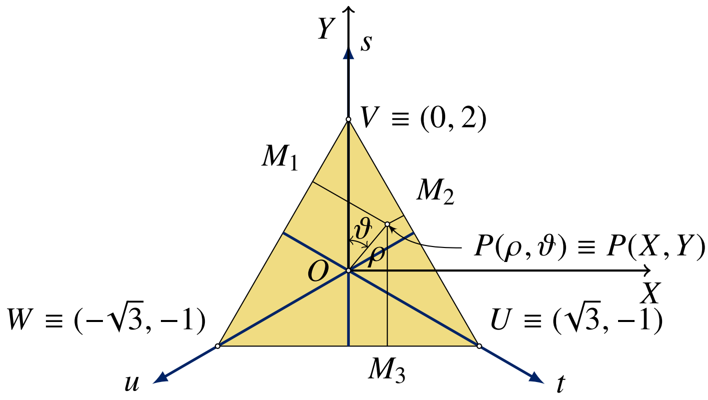


Figure 2.11: The ternary plot obtained by using a parametrization that is inspired from the original parametrization of Dalitz, but in terms of invariant masses of pair of final state particles. Each side of this triangle is of length $2\sqrt{3}$ units. The height of the triangle is 3 units.

such that $d_t + d_u + d_s = 3$. The equilateral triangle of Fig. 2.11 can also be described in terms of polar coordinates (ρ, ϑ) with the pole at the center of the triangle and the polar axis passing through one of the vertices, here V. In terms of the polar coordinates we have $X = \rho \sin\vartheta$ and $Y = \rho\cos\vartheta$, which lead to

$$t = \frac{M^2}{3}\left(1 + \rho\cos\left(\frac{2\pi}{3} - \vartheta\right)\right) = \frac{M^2}{6}\left(2 - \sqrt{3}X - Y\right), \tag{2.70}$$

$$u = \frac{M^2}{3}\left(1 + \rho\cos\left(\frac{2\pi}{3} + \vartheta\right)\right) = \frac{M^2}{6}\left(2 + \sqrt{3}X - Y\right), \tag{2.71}$$

$$s = \frac{M^2}{3}(1 + \rho\cos\vartheta) = \frac{M^2}{3}(1 + Y). \tag{2.72}$$

As before, the full region of the equilateral triangle is *not* the physically allowed region. In order to find out the boundary of the physically allowed region, we would proceed as follows. From Eqs. (2.64), we get

$$\cos\theta = \frac{(t - a_t) - (u - a_u)}{2b}. \tag{2.73}$$

The boundary of the physically allowed region is now given by $\cos^2\theta = 1$, or in other words by

$$((t-u)-(a_t-a_u))^2 - 4b^2 = 0. \tag{2.74}$$

Using Eqs. (2.65) and (2.66) we get $a_t - a_u = \left(m_3^2 - m_2^2\right)\left(1 + \frac{1}{s}\left|(m_X^2 - m_1^2) - s\right|\right)$. Since $s < (m_X - m_1)^2$, it also true that $s < (m_X^2 - m_1^2)$, such that

$$a_t - a_u = \frac{1}{s}\left(m_3^2 - m_2^2\right)\left(m_X^2 - m_1^2\right). \tag{2.75}$$

Thus the equation for the boundary becomes

$$\left(s(t-u) + \left(m_2^2 - m_3^2\right)\left(m_X^2 - m_1^2\right)\right)^2 = \lambda\left(s, m_X^2, m_1^2\right)\ \lambda\left(s, m_2^2, m_3^2\right). \tag{2.76}$$

Substituting the expressions for s, t and u in terms of the barycentric coordinates we get

$$\begin{aligned}\widetilde{\Phi}(X,Y) \equiv{}& \tilde{C}_{00} + \tilde{C}_{10}X + \tilde{C}_{01}Y + \tilde{C}_{11}XY + \tilde{C}_{20}X^2 + \tilde{C}_{02}Y^2 + \tilde{C}_{21}X^2Y + \tilde{C}_{12}XY^2 \\ &+ \tilde{C}_{30}X^3 + \tilde{C}_{03}Y^3 + \tilde{C}_{31}X^3Y + \tilde{C}_{22}X^2Y^2 + \tilde{C}_{13}XY^3 + \tilde{C}_{40}X^4 + \tilde{C}_{04}Y^4 = 0,\end{aligned} \tag{2.77}$$

where $\tilde{C}_{ij}$ is the coefficient of X^iY^j in $\widetilde{\Phi}(X,Y)$ and they are given by

$$\begin{aligned}\tilde{C}_{00} = -\frac{4}{81}\Big(&-8M^8 + 9M^6m_X^2 + 9M^4\lambda\left(m_1^2, m_2^2, m_3^2\right) + 81M^4\left(m_1^2m_2^2 + m_2^2m_3^2 + m_3^2m_1^2\right) \\ &- 54M^2\left(m_1^2 + m_2^2\right)\left(m_2^2 + m_3^2\right)\left(m_3^2 + m_1^2\right)\Big),\end{aligned} \tag{2.78a}$$

$$\tilde{C}_{10} = \tilde{C}_{11} = \frac{2}{3\sqrt{3}}M^4\left(m_2^2 - m_3^2\right)\left(m_1^2 - m_X^2\right), \tag{2.78b}$$

$$\tilde{C}_{01} = \frac{2}{81}\Big(7M^8 + 9M^6\left(3m_1^2 - m_X^2\right) - 9M^4\lambda\left(m_1^2, m_2^2, m_3^2\right) \\ - 27M^4\left(4m_2^2m_3^2 + m_1^2\left(m_1^2 + 6m_2^2 + 6m_3^2\right)\right) \\ + 108M^2\left(m_1^2 + m_2^2\right)\left(m_2^2 + m_3^2\right)\left(m_3^2 + m_1^2\right)\Big), \tag{2.78c}$$

$$\tilde{C}_{02} = \frac{1}{27}M^4\Big(4M^4 - 3\Big(\lambda\left(m_1^2, m_2^2, m_3^2\right) + 6m_1^2\left(m_2^2 + m_3^2\right) \\ + m_X^2\left(4M^2 - 3\left(2m_1^2 + m_X^2\right)\right)\Big)\Big), \tag{2.78d}$$

$$\tilde{C}_{21} = 2\tilde{C}_{20} = 3\tilde{C}_{03} = 2\tilde{C}_{22} = -6\tilde{C}_{04} = (2/27)M^8, \tag{2.78e}$$

$$\tilde{C}_{12} = \tilde{C}_{30} = \tilde{C}_{31} = \tilde{C}_{13} = \tilde{C}_{40} = 0. \tag{2.78f}$$

$$\tag{2.78g}$$

Since the coefficients $\tilde{C}_{12}$, $\tilde{C}_{30}$, $\tilde{C}_{31}$, $\tilde{C}_{13}$, $\tilde{C}_{40}$ vanish, the expression for the boundary is given by

$$\widetilde{\Phi}(X, Y) \equiv \tilde{C}_{00} + \tilde{C}_{10}X + \tilde{C}_{01}Y + \tilde{C}_{11}XY + \tilde{C}_{20}X^2 + \tilde{C}_{02}Y^2 \\ + \tilde{C}_{21}X^2Y + \tilde{C}_{03}Y^3 + \tilde{C}_{22}X^2Y^2 + \tilde{C}_{04}Y^4 = 0. \tag{2.79}$$

We can also express the boundary in terms of the polar coordinates. This is given by

$$\widetilde{\Phi}(\rho, \vartheta) = \tilde{P}_{00}(\rho) + \tilde{P}_{10}(\rho)\sin\vartheta + \tilde{P}_{01}(\rho)\cos\theta + \tilde{P}_{20}(\rho)\sin 2\vartheta + \tilde{P}_{02}(\rho)\cos 2\vartheta \\ + \tilde{P}_{30}(\rho)\sin 3\vartheta + \tilde{P}_{03}(\rho)\cos 3\vartheta + \tilde{P}_{40}(\rho)\sin 4\vartheta + \tilde{P}_{04}(\rho)\cos 4\vartheta = 0, \tag{2.80}$$

where $\tilde{P}_{i0}$ (or $\tilde{P}_{0j}$) is the coefficient of $\sin i\vartheta$ (or $\cos j\vartheta$) in $\widetilde{\Psi}(\rho,\vartheta)$ and they are given by

$$\begin{aligned}
\tilde{P}_{00}(\rho) = \frac{1}{162}M^2\Big(&432\left(m_1^2+m_2^2\right)\left(m_2^2+m_3^2\right)\left(m_3^2+m_1^2\right) \\
&- 9M^2\Big(12\left(m_2^2+m_3^2\right)^2 + 48m_2^2m_3^2 - 6m_X^4 \\
&+ 12m_1^2\left(5m_2^2+5m_3^2-m_X^2\right) + 3\left(2m_1^2\left(m_2^2+m_3^2-m_X^2\right)-m_X^4\right)\rho^2 \\
&+ \left(2+\rho^2\right)\lambda\left(m_1^2,m_2^2,m_3^2\right)\Big) \\
&+ 36M^4\left(3\left(m_2^2+m_3^2\right)-m_X^2\left(2+\rho^2\right)\right) + 5M^6\left(2+3\rho^2\right)\Big), \qquad (2.81)
\end{aligned}$$

$$\tilde{P}_{10}(\rho) = -\frac{2}{3\sqrt{3}}M^4\left(m_2^2-m_3^2\right)\left(m_1^2-m_X^2\right)\rho, \qquad (2.82)$$

$$\begin{aligned}
\tilde{P}_{01}(\rho) = \frac{1}{162}M^2\rho\Big(&432\left(m_1^2+m_2^2\right)\left(m_2^2+m_3^2\right)\left(m_3^2+m_1^2\right) \\
&- 36M^2\Big(\lambda\left(m_1^2,m_2^2,m_3^2\right) + 3m_1^4 + 6\left(3m_1^2m_2^2+3m_2^2m_3^2+2m_3^2m_1^2\right)\Big) \\
&+ 36M^4\left(3m_1^2-m_X^2\right) + 2M^6\left(14+3\rho^2\right)\Big), \qquad (2.83)
\end{aligned}$$

$$\tilde{P}_{20}(\rho) = -\frac{1}{3\sqrt{3}}M^4\left(m_2^2-m_3^2\right)\left(m_1^2-m_X^2\right)\rho^2, \qquad (2.84)$$

$$\begin{aligned}
\tilde{P}_{02}(\rho) = -\frac{1}{162}M^4\rho^2\Big(&27\left(2m_1^2\left(m_2^2+m_3^2-m_X^2\right)-m_X^4\right) + 9\lambda\left(m_1^2,m_2^2,m_3^2\right) \\
&+ 36M^2m_X^2 - M^4\left(9-\rho^2\right)\Big), \qquad (2.85)
\end{aligned}$$

$$\tilde{P}_{30}(\rho) = \tilde{P}_{40} = 0, \qquad (2.86)$$

$$\tilde{P}_{03}(\rho) = -\frac{1}{81}M^8\rho^3, \qquad (2.87)$$

$$\tilde{P}_{04}(\rho) = -\frac{1}{162}M^8\rho^4. \qquad (2.88)$$

Since $\tilde{P}_{30}(\rho)$ and $\tilde{P}_{40}(\rho)$ are zero, the boundary is given by

$$\begin{aligned}
\widetilde{\Phi}(\rho,\vartheta) = {}& \tilde{P}_{00}(\rho) + \tilde{P}_{10}(\rho)\sin\vartheta + \tilde{P}_{01}(\rho)\cos\theta + \tilde{P}_{20}(\rho)\sin 2\vartheta + \tilde{P}_{02}(\rho)\cos 2\vartheta \\
&+ \tilde{P}_{03}(\rho)\cos 3\vartheta + \tilde{P}_{04}(\rho)\cos 4\vartheta = 0. \qquad (2.89)
\end{aligned}$$

2.5.1 Special case of $m_2 = m_3$

In the special case when particles 2 and 3 have the same mass, we get $\tilde{C}_{10} = \tilde{C}_{11} = 0 = \tilde{P}_{10} = \tilde{P}_{20}$. In this case, the boundary of the physically allowed region is symmetric under $X \leftrightarrow -X$ or $\vartheta \leftrightarrow -\vartheta$, and the boundary is given by $\widetilde{\Phi}(X, Y) = \tilde{C}_{00} + \tilde{C}_{01}Y + \tilde{C}_{20}X^2 + \tilde{C}_{02}Y^2 + \tilde{C}_{21}X^2Y + \tilde{C}_{03}Y^3 + \tilde{C}_{22}X^2Y^2 + \tilde{C}_{04}Y^4 = 0$, and $\widetilde{\Psi}(\rho, \vartheta) = \sum_{i=0}^{4} P_{0i}(\rho)\ \cos(i\vartheta) = 0$. Now $X \leftrightarrow -X$ or $\vartheta \leftrightarrow -\vartheta$ exchange implies $t \leftrightarrow u$ exchange, which is equivalent to $\theta \leftrightarrow \pi - \theta$.

2.5.2 Relationship between θ and ϑ

We have two theta's in our calculation, one polar angle in the Dalitz plot (ϑ) and the other angle is between $\vec{p}_1'$ and $\vec{p}_2'$ (θ). These two angles are related to each other. Below we derive the relationship between them. Using the expressions for t and u in terms of polar coordinates we get

$$t - u = \frac{2}{3} M^2 \rho \sin\left(\frac{2\pi}{3}\right) \sin\vartheta = \sqrt{3} M^2 \rho \sin\vartheta. \tag{2.90}$$

However, we know that

$$t - u = \frac{1}{s}\left(\left(m_3^2 - m_2^2\right)\left(m_X^2 - m_1^2\right) + \sqrt{\lambda\left(s, m_X^2, m_1^2\right)}\sqrt{\lambda\left(s, m_2^2, m_3^2\right)} \cos\theta \right). \tag{2.91}$$

Therefore

$$\sin\vartheta = \frac{1}{\sqrt{3} s M^2 \rho}\left(\left(m_3^2 - m_2^2\right)\left(m_X^2 - m_1^2\right) + \sqrt{\lambda\left(s, m_X^2, m_1^2\right)}\sqrt{\lambda\left(s, m_2^2, m_3^2\right)} \cos\theta \right). \tag{2.92}$$

When $m_2 = m_3 = m$ (say), and $\vartheta \leftrightarrow -\vartheta$, then let us suppose that $\theta \leftrightarrow \theta'$:

$$-\sin\vartheta = \frac{1}{\sqrt{3} s M^2 \rho} \sqrt{\lambda\left(s, m_X^2, m_1^2\right)}\sqrt{\lambda\left(s, m^2, m^2\right)} \cos\theta'. \tag{2.93}$$

This would be consistent with our definition of θ, if $\theta' = \pi - \theta$. This can also be clearly seen from Fig. 2.9, where exchanging particles 2 and 3 amounts to changing the angle θ to $\pi - \theta$.

2.5.3 Usage of this new Dalitz plot

The new kind of Dalitz plot discussed in this section, will be applied to study violations of some of the fundamental symmetries of nature. Such a Dalitz plot is also easy to construct as invariant masses of pairs of final particles are routinely measured in various particle physics experiments.

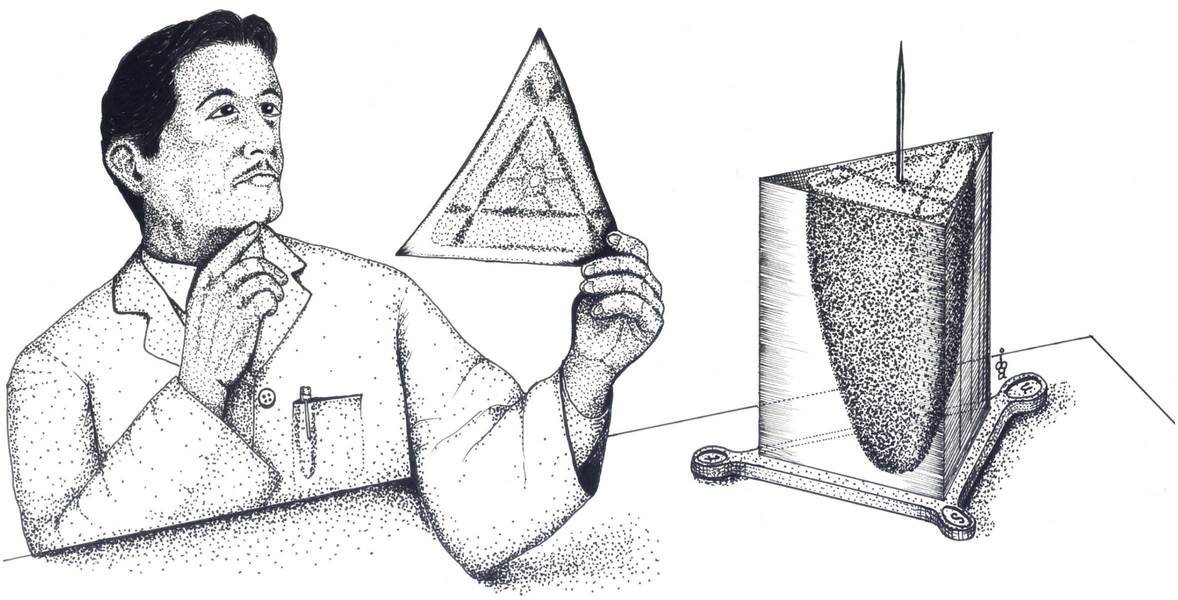
2.6 Summary

In this chapter, we have presented an overview of three-body decays as well as the important concept of the Dalitz plot. The Dalitz plot, contrary to the popular belief that it is only a three-body phase-space plot, carries much information about the underlying dynamics. If there is any symmetry dictating the dynamics of the process, it would leave its signature in the distribution of events inside the Dalitz plot. We will exploit such features for observing some symmetry violations in Chapter 4. But the Dalitz plot is, indeed, a very versatile tool, and as we shall observe in Chapter 3 it can be generalized to handle three-body and multi-body processes.

Part II

The Research Findings

In this part we shall elaborate all the main research findings of this thesis. In Chapter 3 we shall explore how the scope of the Dalitz plot can be enhanced by generalizing it to a three-dimensional plot, the Dalitz 'prism'. In Chapter 4 we shall show mathematically that the Dalitz plots and Dalitz prisms can be used to study violations of CP, CPT, Bose and $SU(3)$ flavor symmetries.



3

The concept of Dalitz 'prism'

As noted in Chapter 2, the phase-space plot for three-body decays, popularly known as the Dalitz plot, is very useful for deciphering various aspects of the underlying dynamics of the process. The scope of application of Dalitz plot can be broadened if it could accommodate non-resonant as well as resonant processes with all three final particles. This demands that the Dalitz plot be suitably modified to include the center-of-momentum energy or mod-square of the total initial four-momentum, in the new plot. This is easy to implement by taking an axis perpendicular to the plane of the triangular Dalitz plot, passing through the center of the equilateral triangle, and then we literally stack up Dalitz plots with increasing center-of-momentum energy along the new axis. The resulting plot would then have a prism-like appearance. Hence, we call this new three-dimensional plot as the Dalitz 'prism', as a humble tribute to Richard Henry Dalitz. The Dalitz prism can also handle multi-body processes as some "effective" three-body processes. In this chapter we shall lay down some details of the Dalitz prism, and in Chapter 4 we shall see how the Dalitz prism can be applied to observe violations of CP, CPT and Bose symmetries in some elementary particle processes.

3.1 Construction of the Dalitz prism

Let us consider a process $a + b \rightarrow 1 + 2 + 3$, where a and b are two initial particles and $1, 2, 3$ are the three final particles. The process will (or will not) proceed via some intermediate resonance X depending on whether the center-of-momentum energy is close to (or farther from) m_X, the mass of the resonance. Pairs of the final particles may themselves have some resonant origin also. Let us denote the center-of-momentum energy of the process by E_{CM}. When $a+b \rightarrow X \rightarrow 1+2+3$, we have, in Natural units, $E_{\mathrm{CM}} = m_X$. Ideally, for every value of E_{CM} we would get a triangular (or ternary) Dalitz plot. Let us now stack up all these Dalitz plots with increasing E_{CM}. This gives us the Dalitz prism for the process $a + b \rightarrow 1 + 2 + 3$.

The Dalitz prism is constructed in the usual three-dimensional Cartesian coordinate system (x, y, z) or in the cylindrical coordinate system (r, θ, z). The z axis denotes variation of E_{CM} (see Fig. 3.1). The x and y coordinates (or equivalently the r and θ coordinates) are obtained by solving the following equations:

$$s = \frac{M^2}{3}\Big(1 + r\,\cos\theta\Big) = \frac{M^2}{3}\Big(1 + y\Big), \tag{3.1}$$

$$t = \frac{M^2}{3}\left(1 + r\,\cos\left(\frac{2\pi}{3} + \theta\right)\right) = \frac{M^2}{6}\Big(2 + \sqrt{3}x - y\Big), \tag{3.2}$$

$$u = \frac{M^2}{3}\left(1 + r\,\cos\left(\frac{2\pi}{3} - \theta\right)\right) = \frac{M^2}{6}\Big(2 - \sqrt{3}x - y\Big), \tag{3.3}$$

where s, t, u denote the invariant masses of the pairs of particles $(2\,3)$, $(3\,1)$ and $(1\,2)$ respectively, $M^2 = m_X^2 + m_1^2 + m_2^2 + m_3^2$ with m_i being the mass of particle i, and θ is measured in the anti-clockwise direction from the y-axis which is also the s-axis of the ternary plot. The allowed values of (x, y) always lie inside the equilateral triangle with vertices at $(0, 2)$, $(\sqrt{3}, -1)$ and $(-\sqrt{3}, -1)$. Thus the complete range of s, t, u are covered

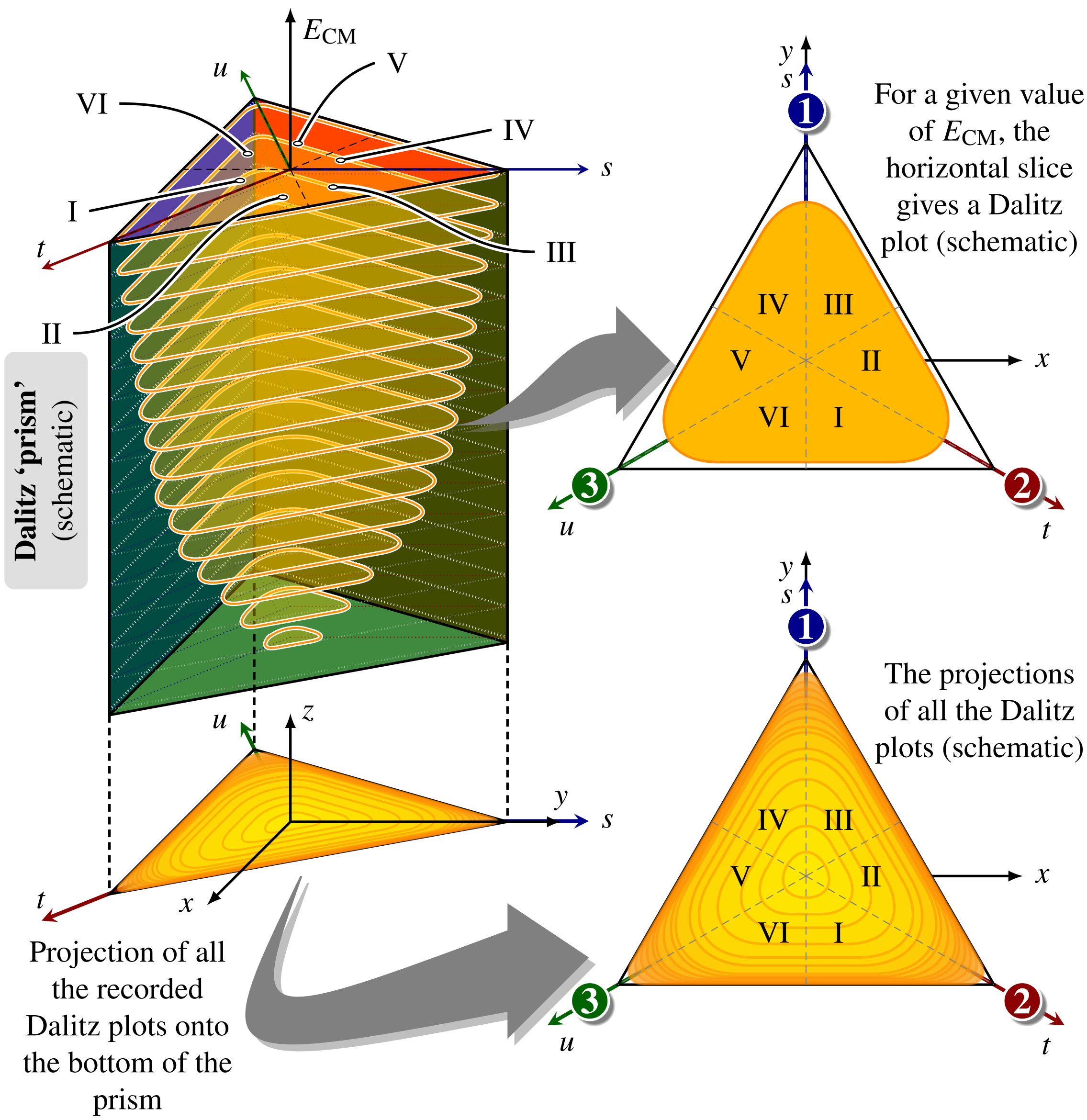


Figure 3.1: Schematic drawing of a Dalitz prism explaining its essential features and showing its intended usage (which will be discussed in Chapter 4). The six identical wedges of the prism and the six sextants of the equilateral triangle are numbered analogously. Here s, t, u denote the invariant masses of the pairs of particles (2 3), (3 1) and (1 2) respectively. Thus, the exchange of variables $s \leftrightarrow t \leftrightarrow u$ is carried out by an exchange of the 4-momenta of the final three particles $p_1 \leftrightarrow p_2 \leftrightarrow p_3$, where p_i is the 4-momentum of particle i. The blobs with **1**, **2** and **3** are mnemonic for showing that the exchanges $s \leftrightarrow t \leftrightarrow u$ and $1 \leftrightarrow 2 \leftrightarrow 3$ are the same.

in this ternary plot:

$$\left.\begin{array}{c} -\sqrt{3} \leqslant x \leqslant \sqrt{3} \\ \\ -1 \leqslant y \leqslant 2 \end{array}\right\} \implies 0 \leqslant s, t, u \leqslant M^2, \tag{3.4}$$

and only the physically allowed regions of s, t, u make up the Dalitz plot. Our Dalitz plot is in the (x, y) coordinate system. When we want to record an event in the Dalitz plot, we evaluate the x and y values corresponding to that event and then register the point (x, y) inside the equilateral triangle. We show in Fig. 3.1 the s, t and u axes to specify their directions in this (x, y) coordinate system.

3.2 Salient features of the Dalitz prism

The Dalitz prism has the following salient features.

- The sides of the ternary plot and the three faces of the Dalitz prism that run parallel to the z-axis, correspond to $s = 0$, $t = 0$ and $u = 0$. Similarly, at the vertices of the ternary plots and, hence, at the three edges of the Dalitz prism that run parallel to the z-axis, we have $s = M^2$, $t = M^2$ and $u = M^2$.

- Since the Dalitz prism records events that include both resonant and non-resonant or continuum production of the three final particles, it is a gargantuan storehouse of data. In order to look at the Dalitz plot at a given center-of-momentum energy, we just need to pull out a slice of the Dalitz prism at that energy, as shown schematically in Fig. 3.1.

- By construction, there is no top ceiling of Dalitz prism, but the experimental reach does put a limit on the height of the Dalitz prisms, which, in general, varies from mode to mode.

- The Dalitz prism can be divided into six identical wedges analogous to the six sextants of a ternary plot (see Fig. 3.1). If there is any underlying symmetry that correlates the distribution of events in the six sextants of the Dalitz plot, it would also get manifest in the Dalitz prism as an analogous correlation amongst the six wedges.

3.3 Dalitz prism and multi-body processes

When multi-body processes, depending on the context, can be treated as "effective" three-body processes by considering all but two of the final particles as arising from an "effective" particle, then one can construct an "effective" Dalitz prism for the process. Since we are fixing two final particles and keeping one 'fictitious' effective third particle in the final state, many multi-body processes can, in principle, contribute to the effective Dalitz prism. The Dalitz prism, say for $a + b \rightarrow 1 + 2 + 3$ where 3 is an effective particle (i.e. the fictitious particle 3 may represent many particles in combination) can be constructed if we know the 4-momenta of particles $1, 2$ and that of the effective particle 3. The 4-momentum of the effective particle can be known either by precisely measuring the 4-momenta of its constituent particles or by using the conservation law for 4-momentum with the 4-momenta of the particles $a, b, 1, 2$ as inputs. The second method is best, because by measuring 4-momenta of particles a, b, 1 and 2 precisely, we can use conservation of 4-momentum to assign $p_a + p_b - p_1 - p_2$ as the 4-momentum of the "effective" particle 3; here p_i is the 4-momentum of particle i. Thus all initial and final state radiations can be considered as part of the "effective" particle. Therefore, the Dalitz prism is a very robust method in handling initial state radiation and final state radiation. Since we are now dealing with multi-body decays, the slices of the Dalitz prism are no longer any Dalitz plots.

3.4 Summary

The concept of Dalitz plot can be generalized to a new three-dimensional plot called Dalitz prism when we consider both resonant and non-resonant production of the final three particles. The concept of Dalitz prism can also be adopted to study multi-body processes when they can be treated as "effective" three-body processes. By including the initial and final state radiations in the definition of the "effective" third final particle the Dalitz prism becomes capable of handling an amazingly large number of events. This enables Dalitz prism to be a natural tool of choice in investigating violations of some fundamental symmetries of nature, as we will discuss in Chapter 4.

4

Study of some symmetry violations

Study of symmetry violations or breakdowns is an essential part of elementary particle physics as it helps us in a better understanding of the workings of Nature at its minutest level. In this chapter we shall look at violations of Bose symmetry, CP, CPT and $SU(3)$ flavor symmetries. We shall employ the Dalitz plots and the Dalitz prisms (concepts discussed and developed in Chapters 2 and 3), and analyse the distribution of events in them to study the above said symmetry violations in some three-body processes.

4.1 Bose symmetry violation

The basic idea here is that if we consider a general three-body decay, say $X \to 1 + 2 + 3$, where all of the final particles are bosons and two of them, say 2 and 3, are identical bosons (but reconstructed from distinct and unique final states), then the Dalitz plot distribution must exhibit symmetry under $2 \leftrightarrow 3$ exchange. Mathematically, the exchange symmetry between 2 and 3 would manifest as a symmetry across the $m_{12}^2 = m_{13}^2$ line in the Dalitz plot distribution m_{12}^2 vs. m_{13}^2, where $s_{ij}^2 = (p_i + p_j)^2$, with p_i being the 4-

momentum of the particle i in the final state. For notational simplicity and mathematical clarity, we shall introduce the following Mandelstam-like variables: $s \equiv m_{23}^2 = (p_2 + p_3)^2 = (p - p_1)^2$, $t \equiv m_{13}^2 = (p_1 + p_3)^2 = (p - p_2)^2$, $u \equiv m_{12}^2 = (p_1 + p_2)^2 = (p - p_3)^2$, where p is the 4-momentum of the parent particle X. So, $2 \leftrightarrow 3$ exchange symmetry leads to symmetry in Dalitz plot distribution under $t \leftrightarrow u$ exchange. Any asymmetry observed in these Dalitz plots under $t \leftrightarrow u$ exchange would be a measure of the extent to which the Bose symmetry is violated.

Let us denote the amplitude for the decay $X \to 1 + 2 + 3$ by $A(t,u)$ and the amplitude with particles 2 and 3 exchanged by $A(u,t)$. If the underlying symmetry allows us to exchange particles 2 and 3, then $A(t,u) = A(u,t)$. However, if the underlying symmetry is not perfectly valid in the present context, we can break up the amplitude $A(t,u)$ into a part which is symmetric under $t \leftrightarrow u$ exchange and another part which is nonsymmetric under the same exchange:

$$A(t,u) = A^S + A^N, \tag{4.1}$$

where

$$A^S \equiv \frac{1}{2}\Big(A(t,u) + A(u,t)\Big), \quad \text{and} \quad A^N \equiv \frac{1}{2}\Big(A(t,u) - A(u,t)\Big). \tag{4.2}$$

It is important to note that if the underlying symmetry were exact, then $A^N = 0$ identically.[1] The differential decay rate for the three-body process $X \to 1 + 2 + 3$ (if X were a spin-0 particle) is therefore given by

$$\frac{d^2\Gamma}{dt\,du} = \frac{\left(\left|A^S\right|^2 + \left|A^N\right|^2 + 2\,\mathrm{Re}\left(A^S A^{N*}\right)\right)}{(2\pi)^3\, 32 M_X^3}, \tag{4.3}$$

where M_X is the mass of the parent particle X. Since A^N is odd under the exchange $t \leftrightarrow u$, the interference term will produce an observable asymmetry in the Dalitz plot.

[1] A misidentification of either particle 2 or 3 can result in a non-zero A^N. Here in our discussion we do assume that the particle 2 and 3 are identified correctly.

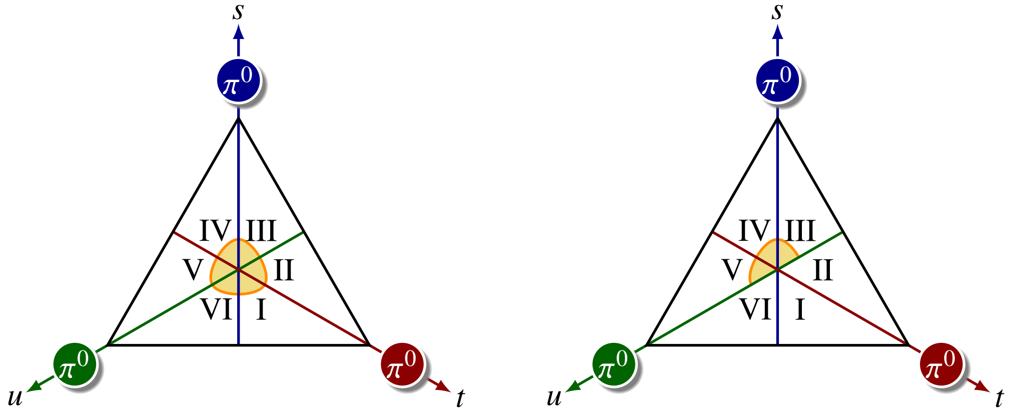
Even if X were not a spin-0 particle, the Dalitz plot distribution is always proportional to the square of the modulus of amplitude and the phase-space. The phase-space gives a uniform Dalitz distribution throughout and the interference term $A^S A^{N*}$ in the square of the modulus of amplitude would give an observable asymmetry in the Dalitz plot under $t \leftrightarrow u$ exchange. Observation of this asymmetry is crucial to test Bose symmetry. This observation also points out that this test of symmetry in the Dalitz plot is independent of what X actually is. Thus we can, indeed, replace the decay by a process, such as the scattering of two particles a and b giving rise to the same final states $1, 2, 3$, i.e. $a + b \to 1 + 2 + 3$, and the said symmetries would now be applicable to the Dalitz prism also.

Let us consider the decay $\eta \to 3\pi^0$, where two out of the three final pions are reconstructed from $\gamma\gamma$ with the remaining one from $e^+e^-\gamma$ final states: $\eta \to \underbrace{\pi^0(p_1)}_{\gamma\gamma}\ \underbrace{\pi^0(p_2)}_{\gamma\gamma}\ \underbrace{\pi^0(p_3)}_{e^+e^-\gamma}$. The Dalitz plot m_{12}^2 vs. m_{13}^2 should be completely symmetric about the $m_{12}^2 = m_{13}^2$ line if the two pions that are reconstructed differently are identical. In this particular case $\pi^0(p_1)$ and $\pi^0(p_2)$ are completely indistinguishable from each other. Therefore, only half of the Dalitz plot of Fig. 4.1 can be reconstructed. However, all the three sextant regions occupying that half of the Dalitz plot must be fully symmetric with respect to each other. Any observed asymmetry in the Dalitz plot can only be attributed to Bose symmetry violation.

Similarly one can look at the following decays:

$$(K^+, D^+, D_s^+) \to \underbrace{\pi^+(p_1)}_{\mu^+\nu_\mu}\ \underbrace{\pi^0(p_2)}_{e^+e^-\gamma}\ \underbrace{\pi^0(p_3)}_{\gamma\gamma}\,, \qquad (K^+, D^+, D_s^+) \to \underbrace{\pi^-(p_1)}_{\mu^-\bar{\nu}_\mu}\ \underbrace{\pi^+(p_2)}_{e^+\nu_e}\ \underbrace{\pi^+(p_3)}_{\mu^+\nu_\mu}\,,$$

and the Dalitz plots should again be symmetric about the $m_{12}^2 = m_{13}^2$ line. Any asymmetry in these Dalitz plots can not appear unless the Bose symmetry is violated. Thus the Dalitz plot asymmetry can be used in these cases to probe the validity of Bose symmetry in the case of pions. Some more decay modes where the Bose symmetry violations can be



(a) If each of the three final π^0s could be reconstructed from distinct and unique final states, the complete Dalitz plot would be available for study.

(b) When two out of the three final π^0s (say the ones carrying momenta p_1 and p_2), are reconstructed from identical final states, only half of the Dalitz plot is available for study.

Figure 4.1: The sextant regions of the Dalitz plot (schematic) for the decay $\eta \to 3\pi^0$.

searched for are

$$\eta' \to \begin{cases} \eta(p_1)\, \pi^0(p_2)\, \pi^0(p_3) \\ \\ \pi^0(p_1)\, \pi^0(p_2)\, \pi^0(p_3) \end{cases},$$

$$B^0 \to K_S^0(p_1)\; K_S^0(p_2)\; K_S^0(p_3)\;,$$

$$B^+ \to \begin{cases} D_s^-(p_1)\; K^+(p_2)\; K^+(p_3) \\ \\ D^-(p_1)\, \pi^+(p_2)\, \pi^+(p_3) \end{cases},$$

$$D_s^+ \to K^-(p_1)\; K^+(p_2)\; K^+(p_3).$$

It is important to note that the Dalitz prism can be used to study the violation of Bose symmetries in the above said final states. To apply the Dalitz prism we need to project the Dalitz prism onto its base, i.e. we integrate over the full E_{CM} under study and analyse the symmetry properties of the resulting two-dimensional plot, as one would do if it were a Dalitz plot (see Fig. 3.1).

4.2 Direct CP violation and CPT violation in mixing

The symmetries CP and CPT are two extremely important symmetries in physics, as discussed in Chapter 1. Mesons, specifically the K, B and D mesons, are most frequently probed in CP violation studies as they primarily decay via the weak interaction. In the analysis below we shall give a general formalism without considering any specific meson.

Let us consider the following decay process

$$X(p_X) \to Y(p_1)\, P^0(p_2)\, \bar{P}^0(p_3) \to Y(p_1)\, f_1(p_2)\, f_2(p_3), \tag{4.4}$$

where all the particles are spin-0 particles, P^0 and $\bar{P}^0$ are both neutral and are antiparticles of each other. The neutral particles P^0 and $\bar{P}^0$ are reconstructed from final states f_1 and f_2 as denoted above. Neither P^0 nor $\bar{P}^0$ are mass eigenstates. They are flavour eigenstates and the mass eigenstates are defined as linear combinations of the flavor eigenstates. Let us denote the mass eigenstates by P_1 and P_2. Allowing for both *CP* and *CPT* violations in the mixing, we can write down the following expressions for the mass eigenstates

$$\left|P_{1,2}\right\rangle = N_{1,2}\Big(p\sqrt{1 \mp z}\left|P^0\right\rangle \pm q\sqrt{1 \pm z}\left|\bar{P}^0\right\rangle\Big), \tag{4.5}$$

where p, q are in general complex and are responsible for *CP* violation in mixing, z is also complex and is responsible for *CPT* violation in mixing, and

$$N_{1,2} = \frac{1}{\sqrt{|p|^2(1 \mp z) + |q|^2(1 \pm z)}} = \frac{1}{\sqrt{\left(|p|^2 + |q|^2\right) \mp z\left(|p|^2 - |q|^2\right)}}, \tag{4.6}$$

with $|p|^2 + |q|^2 = 1$ and $N_1 = N_2 = 1$ for $z = 0$ (no *CPT* violation); and the *CP* eigenstates are

$$|P_\pm\rangle = \frac{1}{\sqrt{2}}\Big(\left|P^0\right\rangle \pm \left|\bar{P}^0\right\rangle\Big). \tag{4.7}$$

Now we can rewrite the flavour eigenstates in terms of the mass and *CP* eigenstates as follows:

$$\left|P^0\right\rangle = \frac{N_2\sqrt{1-z}\,|P_1\rangle + N_1\sqrt{1+z}\,|P_2\rangle}{2N_1N_2p} = \frac{1}{\sqrt{2}}\Big(\,|P_+\rangle + |P_-\rangle\,\Big), \tag{4.8}$$

and

$$\left|\bar{P}^0\right\rangle = \frac{N_2\sqrt{1+z}\,|P_1\rangle - N_1\sqrt{1-z}\,|P_2\rangle}{2N_1N_2q} = \frac{1}{\sqrt{2}}\Big(\,|P_+\rangle - |P_-\rangle\,\Big), \tag{4.9}$$

where

$$N_1N_2 = \frac{1}{\sqrt{(1-z^2)\left(|p|^4+|q|^4\right)+(1+z^2)\left(2\,|p|^2\,|q|^2\right)}}. \tag{4.10}$$

Finally the mass eigenstates can also be written in terms of the *CP* eigenstates as follows:

$$\begin{aligned}\left|P_{1,2}\right\rangle &= \frac{N_{1,2}}{\sqrt{2}}\bigg(\Big(p\sqrt{1\mp z}\pm q\sqrt{1\pm z}\Big)\,|P_+\rangle + \Big(p\sqrt{1\mp z}\mp q\sqrt{1\pm z}\Big)\,|P_-\rangle\bigg)\\ &= \frac{N_{1,2}}{\sqrt{2}}\Big(U_{1,2}\,|P_+\rangle + V_{1,2}\,|P_-\rangle\Big)\,,\end{aligned} \tag{4.11}$$

where

$$U_{1,2} = p\sqrt{1\mp z}\pm q\sqrt{1\pm z}\,, \quad \text{and} \quad V_{1,2} = p\sqrt{1\mp z}\mp q\sqrt{1\pm z}\,. \tag{4.12}$$

It is easy to see that

$$U_{1,2}+V_{1,2} = p\sqrt{1\mp z}\,, \tag{4.13a}$$

$$U_{1,2}-V_{1,2} = \pm q\sqrt{1\pm z}\,, \tag{4.13b}$$

$$U_{1,2}V_{1,2} = p^2(1\mp z)-q^2(1\pm z) = \left(p^2-q^2\right)\mp z\left(p^2+q^2\right)\,, \tag{4.13c}$$

$$U_{1,2}^2 = \left(p^2+q^2\right)\mp z\left(p^2-q^2\right)\pm 2pq\sqrt{1-z^2}\,, \tag{4.13d}$$

$$V_{1,2}^2 = \left(p^2+q^2\right)\mp z\left(p^2-q^2\right)\mp 2pq\sqrt{1-z^2}\,. \tag{4.13e}$$

The states $\left|P_{1,2}\right\rangle$ have only an exponential time dependence corresponding to their mass and decay width, and do not depend on time in any other way. This time dependence is given by

$$|P_i(t)\rangle = g_i(t)\,|P_i\rangle = e^{-im_it-\frac{1}{2}\Gamma_it}\,|P_i\rangle = e^{-i\mu_it}\,|P_i\rangle\,, \tag{4.14}$$

where $\mu_i \equiv m_i - \Gamma_i/2$ with m_i and Γ_i being the mass and the decay width of the mass eigenstate P_i. Now this implies that the combination of $P_\pm$ on the right hand side of Eq. (4.11) must also exhibit the same exponential time dependence. We choose to work in the mass eigenstates as the time evolution is a simple exponential. We also choose to work in the center-of-momentum frame of $P^0\bar{P}^0$ which is also the Gottfried-Jackson frame (see Fig. 4.2). The $\hat{z}$-axis is the direction of flight of the particle X. We define the

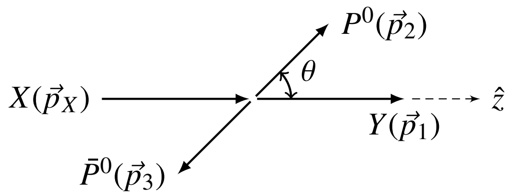


Figure 4.2: The Gottfried-Jackson frame used for the decay $X(p_X) \to Y(p_1)P^0(p_2)\bar{P}^0(p_3)$. It is clear in the figure that $\vec{p}_X = \vec{p}_1$ and $\vec{p}_2 = -\vec{p}_3$. There is an angle θ between $\vec{p}_1$ and $\vec{p}_2$.

invariant mass squares the Mandelstam way:

$$s = (p_2 + p_3)^2 = (p_X - p_1)^2, \tag{4.15a}$$

$$t = (p_1 + p_3)^2 = (p_X - p_2)^2, \tag{4.15b}$$

$$u = (p_1 + p_2)^2 = (p_X - p_3)^2. \tag{4.15c}$$

The two variables t and u can be written as

$$t = a + b\,\cos\theta, \quad \text{and} \quad u = a - b\,\cos\theta, \tag{4.16}$$

where

$$a = \frac{m_X^2 + m_Y^2 + 2m_P^2 - s}{2}, \quad \text{and} \quad b = \frac{\sqrt{\left(s - 4m_P^2\right)\ \lambda\left(m_X^2, m_Y^2, s\right)}}{2\sqrt{s}}, \tag{4.17}$$

where m_X, m_Y are the masses of particles X and Y respectively, m_P is the average mass[2] of the mass eigenstates P_1 and P_2. The particles P^0 and $\bar{P}^0$ produced in the decay of the particle X at time $t = 0$ oscillate amongst each other before finally decaying to final states f_1 and f_2 at times t_1 and t_2 respectively. For our calculation we would consider those final states that have definite CP. We denote the CP-even final state by f_i^+ and the CP-odd final state by f_i^-. We can express the final state $\left|Y(p_1)P^0(p_2)\bar{P}^0(p_3)\right\rangle$ and the corresponding momentum exchanged state $\left|Y(p_1)P^0(p_3)\bar{P}^0(p_2)\right\rangle$ in terms of the mass eigenstates $P_{1,2}$ as follows:

$$\left|Y(p_1)P^0(p_2)\bar{P}^0(p_3)\right\rangle = \frac{\left|YP^0\bar{P}^0\right\rangle_{\text{even}} - \left|YP^0\bar{P}^0\right\rangle_{\text{odd}}}{4N_1^2N_2^2pq}, \tag{4.18}$$

$$\left|Y(p_1)P^0(p_3)\bar{P}^0(p_2)\right\rangle = \frac{\left|YP^0\bar{P}^0\right\rangle_{\text{even}} + \left|YP^0\bar{P}^0\right\rangle_{\text{odd}}}{4N_1^2N_2^2pq}, \tag{4.19}$$

where the subscripts 'even' and 'odd' denote the behaviour of the concerned state under the momentum exchange $p_2 \leftrightarrow p_3$ and these states are given by

$$\begin{aligned}\left|YP^0\bar{P}^0\right\rangle_{\text{even}} = {} & \sqrt{1-z^2}\Big(N_2^2\, |Y\ P_1(p_2)\ P_1(p_3)\rangle - N_1^2\, |Y\ P_2(p_2)\ P_2(p_3)\rangle\Big) \\ & + zN_1N_2\Big(|Y\ P_1(p_2)\ P_2(p_3)\rangle + |Y\ P_2(p_2)\ P_1(p_3)\rangle\Big),\end{aligned} \tag{4.20}$$

$$\left|YP^0\bar{P}^0\right\rangle_{\text{odd}} = N_1N_2\Big(|Y\ P_1(p_2)\ P_2(p_3)\rangle - |Y\ P_2(p_2)\ P_1(p_3)\rangle\Big). \tag{4.21}$$

It is important to notice that the 'even' state $\left|YP^0\bar{P}^0\right\rangle_{\text{even}}$ has two bracketed terms, the

[2] The effect of the mass difference between P_1 and P_2 is taken into account in the analysis. However, in t and u the mass that enters is the average mass m_P because we assign this mass to the flavor eigenstates P^0 and $\bar{P}^0$ which participate in the decay. The flavor eigenstates are different from mass eigenstates P_1 and P_2 which, of course, have different masses and that mass difference is taken care of in our calculation.

first one of which is completely Bose symmetric under the exchange of particles with momenta p_2 and p_3, and the second bracketed term is clearly not Bose symmetric under the same exchange. Since the exchange $p_2 \leftrightarrow p_3$ is equivalent to the exchange $t \leftrightarrow u$, the Bose symmetry is realised as a symmetry under $t \leftrightarrow u$ exchange on the Dalitz plot, if and only if there is no *CPT* violation (i.e. $z = 0$) in the decay[3].

Let us now define the amplitudes for the decay of $P_\pm$ to a final state $f_i^\pm$ of definite *CP* as follows,

$$\text{Amp}(P_+ \to f_i^+) = \left\langle f_i^+ | P_+ \right\rangle = A_i^+, \tag{4.22a}$$

$$\text{Amp}(P_- \to f_i^+) = \left\langle f_i^+ | P_- \right\rangle = \epsilon_i^+ A_i^+, \tag{4.22b}$$

$$\text{Amp}(P_- \to f_i^-) = \left\langle f_i^- | P_- \right\rangle = A_i^-, \tag{4.22c}$$

$$\text{Amp}(P_+ \to f_i^-) = \left\langle f_i^- | P_+ \right\rangle = \epsilon_i^- A_i^-, \tag{4.22d}$$

where $\epsilon_i^\pm$ quantifies the amount of direct *CP* violation in the decays of the neutral particles P. The amplitudes for the decay of the mass eigenstates $P_{1,2}$ to $f_i^\pm$ can now be written as

$$\text{Amp}(P_{1,2} \to f_i^+) = \frac{N_{1,2}}{\sqrt{2}}\left(U_{1,2}\, A_i^+ + V_{1,2}\, \epsilon_i^+ A_i^+\right) = \frac{N_{1,2}}{\sqrt{2}}\left(U_{1,2} + V_{1,2}\, \epsilon_i^+\right) A_i^+, \tag{4.23}$$

$$\text{Amp}(P_{1,2} \to f_i^-) = \frac{N_{1,2}}{\sqrt{2}}\left(U_{1,2}\, \epsilon_i^- A_i^- + V_{1,2}\, A_i^-\right) = \frac{N_{1,2}}{\sqrt{2}}\left(U_{1,2}\, \epsilon_i^- + V_{1,2}\right) A_i^-. \tag{4.24}$$

Thus, the decay amplitude for $X \to Y(p_1)\underbrace{f_1^{s_1}(p_2)}_{P_i(p_2)}\underbrace{f_2^{s_2}(p_3)}_{P_j(p_3)}$ where $i, j \in \{1, 2\}$ and $s_{1,2} = \pm$ is given by

$$\begin{aligned}
&\text{Amp}\left(X \to Y(p_1)\left(f_1^{s_1}(p_2)\right)_{P_i(p_2)}\left(f_2^{s_2}(p_3)\right)_{P_j(p_3)}\right) \\
&= \frac{1}{4N_1^2 N_2^2 pq}\left(2A^p \text{Amp}\left((P^0\bar{P}^0)_{\text{even}} \to f_1^{s_1} f_2^{s_2}\right) + 2A^m \text{Amp}\left((P^0\bar{P}^0)_{\text{odd}} \to f_1^{s_1} f_2^{s_2}\right)\cos\theta\right),
\end{aligned} \tag{4.25}$$

[3]If *CPT* were violated, we could not assign the average mass m_P to both P^0 and $\bar{P}^0$.

where the $(P^0\bar{P}^0)_{\text{even}}$ and $(P^0\bar{P}^0)_{\text{odd}}$ states can be easily read out from Eqs. (4.20) and (4.21), and the amplitudes A^p ('a-plus') and A^m ('a-minus') are given by

$$A^p = \frac{1}{2}\Big(A(t,u) + A(u,t)\Big), \quad A^m = \frac{1}{2\cos\theta}\Big(A(t,u) - A(u,t)\Big), \tag{4.26}$$

with the amplitudes $A(t,u)$ and $A(u,t)$ being defined as

$$A(t,u) = \text{Amp}\left(X \to Y(p_1)P^0(p_2)\bar{P}^0(p_3)\right) = \left\langle Y(p_1)P^0(p_2)\bar{P}^0(p_3)|X\right\rangle, \tag{4.27}$$

$$A(u,t) = \text{Amp}\left(X \to Y(p_1)P^0(p_3)\bar{P}^0(p_2)\right) = \left\langle Y(p_1)P^0(p_3)\bar{P}^0(p_2)|X\right\rangle. \tag{4.28}$$

Now the amplitudes for $P_i P_j \to f_1^{s_1} f_2^{s_2}$ (with $i,j \in \{1,2\}$ and $s_{1,2} = \pm$), after taking into account the time-evolution of the mass eigenstates, are given by

$$\text{Amp}(P_1P_1 \to f_1^+ f_2^+) = e^{-i\mu_1(t_1+t_2)}\frac{1}{2}N_1^2\Big(U_1^2 + U_1V_1\left(\epsilon_1^+ + \epsilon_2^+\right) + V_1^2\epsilon_1^+\epsilon_2^+\Big)\, A_1^+\, A_2^+, \tag{4.29a}$$

$$\text{Amp}(P_1P_1 \to f_1^+ f_2^-) = e^{-i\mu_1(t_1+t_2)}\frac{1}{2}N_1^2\Big(U_1^2\epsilon_2^- + V_1^2\epsilon_1^+ + U_1V_1\left(1 + \epsilon_1^+\epsilon_2^-\right)\Big)\, A_1^+\, A_2^-, \tag{4.29b}$$

$$\text{Amp}(P_1P_1 \to f_1^- f_2^+) = e^{-i\mu_1(t_1+t_2)}\frac{1}{2}N_1^2\Big(U_1^2\epsilon_1^- + V_1^2\epsilon_2^+ + U_1V_1\left(1 + \epsilon_1^-\epsilon_2^+\right)\Big)\, A_1^-\, A_2^+, \tag{4.29c}$$

$$\text{Amp}(P_1P_1 \to f_1^- f_2^-) = e^{-i\mu_1(t_1+t_2)}\frac{1}{2}N_1^2\Big(U_1^2\epsilon_1^-\epsilon_2^- + U_1V_1\left(\epsilon_1^- + \epsilon_2^-\right) + V_1^2\Big)\, A_1^-\, A_2^-, \tag{4.29d}$$

$$\begin{aligned}\text{Amp}(P_1P_2 \to f_1^+ f_2^+) = e^{-i\mu_1 t_1}e^{-i\mu_2 t_2}\frac{1}{2}N_1N_2\Big(&U_1U_2 + U_2V_1\epsilon_1^+ + U_1V_2\epsilon_2^+ \\ &+ V_1V_2\epsilon_1^+\epsilon_2^+\Big)\, A_1^+\, A_2^+,\end{aligned} \tag{4.29e}$$

$$\begin{aligned}\text{Amp}(P_1P_2 \to f_1^+ f_2^-) = e^{-i\mu_1 t_1}e^{-i\mu_2 t_2}\frac{1}{2}N_1N_2\Big(&U_1V_2 + U_1U_2\epsilon_2^- + V_1V_2\epsilon_1^+ \\ &+ U_2V_1\epsilon_1^+\epsilon_2^-\Big)\, A_1^+\, A_2^-,\end{aligned} \tag{4.29f}$$

$$\mathrm{Amp}(P_1P_2 \to f_1^- f_2^+) = e^{-i\mu_1 t_1} e^{-i\mu_2 t_2} \frac{1}{2} N_1 N_2 \Big(U_2 V_1 + U_1 U_2 \epsilon_1^- + V_1 V_2 \epsilon_2^+ \\ + U_1 V_2 \epsilon_1^- \epsilon_2^+ \Big)\, A_1^- \, A_2^+, \tag{4.29g}$$

$$\mathrm{Amp}(P_1P_2 \to f_1^- f_2^-) = e^{-i\mu_1 t_1} e^{-i\mu_2 t_2} \frac{1}{2} N_1 N_2 \Big(V_1 V_2 + U_1 V_2 \epsilon_1^- + U_2 V_1 \epsilon_2^- \\ + U_1 U_2 \epsilon_1^- \epsilon_2^- \Big)\, A_1^- \, A_2^-, \tag{4.29h}$$

$$\mathrm{Amp}(P_2P_1 \to f_1^+ f_2^+) = e^{-i\mu_2 t_1} e^{-i\mu_1 t_2} \frac{1}{2} N_1 N_2 \Big(U_1 U_2 + U_1 V_2 \epsilon_1^+ + U_2 V_1 \epsilon_2^+ \\ + V_1 V_2 \epsilon_1^+ \epsilon_2^+ \Big)\, A_1^+ \, A_2^+, \tag{4.29i}$$

$$\mathrm{Amp}(P_2P_1 \to f_1^+ f_2^-) = e^{-i\mu_2 t_1} e^{-i\mu_1 t_2} \frac{1}{2} N_1 N_2 \Big(U_2 V_1 + U_1 U_2 \epsilon_2^- + V_1 V_2 \epsilon_1^+ \\ + U_1 V_2 \epsilon_1^+ \epsilon_2^- \Big)\, A_1^+ \, A_2^-, \tag{4.29j}$$

$$\mathrm{Amp}(P_2P_1 \to f_1^- f_2^+) = e^{-i\mu_2 t_1} e^{-i\mu_1 t_2} \frac{1}{2} N_1 N_2 \Big(U_1 V_2 + U_1 U_2 \epsilon_1^- + V_1 V_2 \epsilon_2^+ \\ + U_2 V_1 \epsilon_1^- \epsilon_2^+ \Big)\, A_1^- \, A_2^+, \tag{4.29k}$$

$$\mathrm{Amp}(P_2P_1 \to f_1^- f_2^-) = e^{-i\mu_2 t_1} e^{-i\mu_1 t_2} \frac{1}{2} N_1 N_2 \Big(V_1 V_2 + U_2 V_1 \epsilon_1^- + U_1 V_2 \epsilon_2^- \\ + U_1 U_2 \epsilon_1^- \epsilon_2^- \Big)\, A_1^- \, A_2^-, \tag{4.29l}$$

$$\mathrm{Amp}(P_2P_2 \to f_1^+ f_2^+) = e^{-i\mu_2 (t_1+t_2)} \frac{1}{2} N_2^2 \Big(U_2^2 + V_2^2 \epsilon_1^+ \epsilon_2^+ + U_2 V_2 \left(\epsilon_1^+ + \epsilon_2^+ \right) \Big)\, A_1^+ \, A_2^+, \tag{4.29m}$$

$$\mathrm{Amp}(P_2P_2 \to f_1^+ f_2^-) = e^{-i\mu_2 (t_1+t_2)} \frac{1}{2} N_2^2 \Big(U_2^2 \epsilon_2^- + V_2^2 \epsilon_1^+ + U_2 V_2 \left(1 + \epsilon_1^+ \epsilon_2^- \right) \Big)\, A_1^+ \, A_2^-, \tag{4.29n}$$

$$\mathrm{Amp}(P_2P_2 \to f_1^- f_2^+) = e^{-i\mu_2 (t_1+t_2)} \frac{1}{2} N_2^2 \Big(U_2^2 \epsilon_1^- + V_2^2 \epsilon_2^+ + U_2 V_2 \left(1 + \epsilon_1^- \epsilon_2^+ \right) \Big)\, A_1^- \, A_2^+, \tag{4.29o}$$

$$\mathrm{Amp}(P_2P_2 \to f_1^- f_2^-) = e^{-i\mu_2 (t_1+t_2)} \frac{1}{2} N_2^2 \Big(V_2^2 + U_2^2 \epsilon_1^- \epsilon_2^- + U_2 V_2 \left(\epsilon_1^- + \epsilon_2^- \right) \Big)\, A_1^- \, A_2^-. \tag{4.29p}$$

We shall now make the substitutions $\mu_1 = \mu + \Delta\mu$, $\mu_2 = \mu - \Delta\mu$, where

$$\mu = \frac{m_1 + m_2}{2} - \frac{i}{2}\frac{\Gamma_1 + \Gamma_2}{2} = m_P - \frac{i}{2}\Gamma_P, \tag{4.30}$$

$$\Delta\mu = \frac{m_1 - m_2}{2} - \frac{i}{2}\frac{\Gamma_1 - \Gamma_2}{2} = (x - i\,y)\frac{\Gamma_P}{2}, \tag{4.31}$$

with m_P and Γ_P being the average mass and decay width of P_1 and P_2, and $x\Gamma_P$, $2y\Gamma_P$ being the differences in masses and decay widths of P_1 and P_2. Let us also define the following functions:

$$H(T) = e^{-i\mu(t_1+t_2)}\cos(\Delta\mu\, T), \quad \text{and} \quad G(T) = e^{-i\mu(t_1+t_2)}\, i\,\sin(\Delta\mu\, T), \tag{4.32}$$

such that

$$e^{-i\mu_1(t_1+t_2)} = H(t_1 + t_2) + G(t_1 + t_2), \tag{4.33a}$$

$$e^{-i\mu_2(t_1+t_2)} = H(t_1 + t_2) - G(t_1 + t_2), \tag{4.33b}$$

$$e^{-i(\mu_1 t_1+\mu_2 t_2)} = H(t_1 - t_2) + G(t_1 - t_2), \tag{4.33c}$$

$$e^{-i(\mu_2 t_1+\mu_1 t_2)} = H(t_1 - t_2) - G(t_1 - t_2). \tag{4.33d}$$

Physically it is very plausible that the masses of two particles that mix with each other would be almost equal. Therefore, we can assume safely that $\Delta\mu \to 0$, in which case $\cos(\Delta\mu\, T) \approx 1$ and $\sin(\Delta\mu\, T) \approx \Delta\mu\, T$. Thus, we can always approximate $H(T)$ and $G(T)$ as $H(T) \approx e^{-i\mu(t_1+t_2)}$ and $e^{-i\mu(t_1+t_2)} i\,\Delta\mu\, T$ respectively. Thus to first order approximation in the mass and width difference between P_1 and P_2, we have

$$e^{-i\mu_1(t_1+t_2)} \approx e^{-i\mu(t_1+t_2)}\left(1 + i\,\Delta\mu\,(t_1 + t_2)\right), \tag{4.34a}$$

$$e^{-i\mu_2(t_1+t_2)} \approx e^{-i\mu(t_1+t_2)}\left(1 - i\,\Delta\mu\,(t_1 + t_2)\right), \tag{4.34b}$$

$$e^{-i(\mu_1 t_1+\mu_2 t_2)} \approx e^{-i\mu(t_1+t_2)}\left(1 + i\,\Delta\mu\,(t_1 - t_2)\right), \tag{4.34c}$$

$$e^{-i(\mu_2 t_1+\mu_1 t_2)} \approx e^{-i\mu(t_1+t_2)} \left(1 - i\,\Delta\mu\,(t_1 - t_2)\right). \tag{4.34d}$$

Using these expressions we can rewrite the amplitudes for $P_i\,P_j \to f_1^{s_1} f_2^{s_2}$ (with $i, j = 1, 2$ and $s_{1,2} = \pm$), as follows (again up to first order approximation in the mass and width difference between P_1 and P_2):

$$\begin{aligned}
\mathrm{Amp}(P_1P_1 \to f_1^+ f_2^+) = {} & \frac{1}{2} N_1^2 \Big(U_1^2 + U_1 V_1 \left(\epsilon_1^+ + \epsilon_2^+\right) + V_1^2 \epsilon_1^+ \epsilon_2^+ \Big) \\
& \Big(e^{-i\mu(t_1+t_2)} \left(1 + i\,\Delta\mu\,(t_1 + t_2)\right) \Big)\, A_1^+\, A_2^+, && (4.35a) \\
\mathrm{Amp}(P_2P_2 \to f_1^+ f_2^+) = {} & \frac{1}{2} N_2^2 \Big(U_2^2 + V_2^2 \epsilon_1^+ \epsilon_2^+ + U_2 V_2 \left(\epsilon_1^+ + \epsilon_2^+\right) \Big) \\
& \Big(e^{-i\mu(t_1+t_2)} \left(1 - i\,\Delta\mu\,(t_1 + t_2)\right) \Big)\, A_1^+\, A_2^+, && (4.35b) \\
\mathrm{Amp}(P_1P_2 \to f_1^+ f_2^+) = {} & \frac{1}{2} N_1 N_2 \Big(U_1 U_2 + U_2 V_1 \epsilon_1^+ + U_1 V_2 \epsilon_2^+ + V_1 V_2 \epsilon_1^+ \epsilon_2^+ \Big) \\
& \Big(e^{-i\mu(t_1+t_2)} \left(1 + i\,\Delta\mu\,(t_1 - t_2)\right) \Big)\, A_1^+\, A_2^+, && (4.35c) \\
\mathrm{Amp}(P_2P_1 \to f_1^+ f_2^+) = {} & \frac{1}{2} N_1 N_2 \Big(U_1 U_2 + U_1 V_2 \epsilon_1^+ + U_2 V_1 \epsilon_2^+ + V_1 V_2 \epsilon_1^+ \epsilon_2^+ \Big) \\
& \Big(e^{-i\mu(t_1+t_2)} \left(1 - i\,\Delta\mu\,(t_1 - t_2)\right) \Big)\, A_1^+\, A_2^+, && (4.35d) \\
\mathrm{Amp}(P_1P_1 \to f_1^+ f_2^-) = {} & \frac{1}{2} N_1^2 \Big(U_1^2 \epsilon_2^- + V_1^2 \epsilon_1^+ + U_1 V_1 \left(1 + \epsilon_1^+ \epsilon_2^-\right) \Big) \\
& \Big(e^{-i\mu(t_1+t_2)} \left(1 + i\,\Delta\mu\,(t_1 + t_2)\right) \Big)\, A_1^+\, A_2^-, && (4.35e) \\
\mathrm{Amp}(P_2P_2 \to f_1^+ f_2^-) = {} & \frac{1}{2} N_2^2 \Big(U_2^2 \epsilon_2^- + V_2^2 \epsilon_1^+ + U_2 V_2 \left(1 + \epsilon_1^+ \epsilon_2^-\right) \Big) \\
& \Big(e^{-i\mu(t_1+t_2)} \left(1 - i\,\Delta\mu\,(t_1 + t_2)\right) \Big)\, A_1^+\, A_2^-, && (4.35f) \\
\mathrm{Amp}(P_1P_2 \to f_1^+ f_2^-) = {} & \frac{1}{2} N_1 N_2 \Big(U_1 V_2 + U_1 U_2 \epsilon_2^- + V_1 V_2 \epsilon_1^+ + U_2 V_1 \epsilon_1^+ \epsilon_2^- \Big) \\
& \Big(e^{-i\mu(t_1+t_2)} \left(1 + i\,\Delta\mu\,(t_1 - t_2)\right) \Big)\, A_1^+\, A_2^-, && (4.35g) \\
\mathrm{Amp}(P_2P_1 \to f_1^+ f_2^-) = {} & \frac{1}{2} N_1 N_2 \Big(U_2 V_1 + U_1 U_2 \epsilon_2^- + V_1 V_2 \epsilon_1^+ + U_1 V_2 \epsilon_1^+ \epsilon_2^- \Big) \\
& \Big(e^{-i\mu(t_1+t_2)} \left(1 - i\,\Delta\mu\,(t_1 - t_2)\right) \Big)\, A_1^+\, A_2^-, && (4.35h) \\
\mathrm{Amp}(P_1P_1 \to f_1^- f_2^+) = {} & \frac{1}{2} N_1^2 \Big(U_1^2 \epsilon_1^- + V_1^2 \epsilon_2^+ + U_1 V_1 \left(1 + \epsilon_1^- \epsilon_2^+\right) \Big) \\
& \Big(e^{-i\mu(t_1+t_2)} \left(1 + i\,\Delta\mu\,(t_1 + t_2)\right) \Big)\, A_1^-\, A_2^+, && (4.35i)
\end{aligned}$$

$$\mathrm{Amp}(P_2P_2 \to f_1^- f_2^+) = \frac{1}{2}N_2^2\Big(U_2^2\epsilon_1^- + V_2^2\epsilon_2^+ + U_2V_2\left(1+\epsilon_1^-\epsilon_2^+\right)\Big)$$
$$\Big(e^{-i\mu(t_1+t_2)}\left(1 - i\,\Delta\mu\,(t_1+t_2)\right)\Big)\,A_1^-\,A_2^+, \tag{4.35j}$$
$$\mathrm{Amp}(P_1P_2 \to f_1^- f_2^+) = \frac{1}{2}N_1N_2\Big(U_2V_1 + U_1U_2\epsilon_1^- + V_1V_2\epsilon_2^+ + U_1V_2\epsilon_1^-\epsilon_2^+\Big)$$
$$\Big(e^{-i\mu(t_1+t_2)}\left(1 + i\,\Delta\mu\,(t_1-t_2)\right)\Big)\,A_1^-\,A_2^+, \tag{4.35k}$$
$$\mathrm{Amp}(P_2P_1 \to f_1^- f_2^+) = \frac{1}{2}N_1N_2\Big(U_1V_2 + U_1U_2\epsilon_1^- + V_1V_2\epsilon_2^+ + U_2V_1\epsilon_1^-\epsilon_2^+\Big)$$
$$\Big(e^{-i\mu(t_1+t_2)}\left(1 - i\,\Delta\mu\,(t_1-t_2)\right)\Big)\,A_1^-\,A_2^+, \tag{4.35l}$$
$$\mathrm{Amp}(P_1P_1 \to f_1^- f_2^-) = \frac{1}{2}N_1^2\Big(U_1^2\epsilon_1^-\epsilon_2^- + U_1V_1\left(\epsilon_1^- + \epsilon_2^-\right) + V_1^2\Big)$$
$$\Big(e^{-i\mu(t_1+t_2)}\left(1 + i\,\Delta\mu\,(t_1+t_2)\right)\Big)\,A_1^-\,A_2^-, \tag{4.35m}$$
$$\mathrm{Amp}(P_2P_2 \to f_1^- f_2^-) = \frac{1}{2}N_2^2\Big(V_2^2 + U_2^2\epsilon_1^-\epsilon_2^- + U_2V_2\left(\epsilon_1^- + \epsilon_2^-\right)\Big)$$
$$\Big(e^{-i\mu(t_1+t_2)}\left(1 - i\,\Delta\mu\,(t_1+t_2)\right)\Big)\,A_1^-\,A_2^-, \tag{4.35n}$$
$$\mathrm{Amp}(P_1P_2 \to f_1^- f_2^-) = \frac{1}{2}N_1N_2\Big(V_1V_2 + U_1V_2\epsilon_1^- + U_2V_1\epsilon_2^- + U_1U_2\epsilon_1^-\epsilon_2^-\Big)$$
$$\Big(e^{-i\mu(t_1+t_2)}\left(1 + i\,\Delta\mu\,(t_1-t_2)\right)\Big)\,A_1^-\,A_2^-, \tag{4.35o}$$
$$\mathrm{Amp}(P_2P_1 \to f_1^- f_2^-) = \frac{1}{2}N_1N_2\Big(V_1V_2 + U_2V_1\epsilon_1^- + U_1V_2\epsilon_2^- + U_1U_2\epsilon_1^-\epsilon_2^-\Big)$$
$$\Big(e^{-i\mu(t_1+t_2)}\left(1 - i\,\Delta\mu\,(t_1-t_2)\right)\Big)\,A_1^-\,A_2^-. \tag{4.35p}$$

Let us now introduce for brevity of expression the notation for the following two sets of amplitudes:

$$\mathcal{E}^{s_1s_2} \equiv \mathrm{Amp}\left((P^0\bar{P}^0)_{\rm even} \to f_1^{s_1}f_2^{s_2}\right), \quad \text{and} \quad \mathcal{O}^{s_1s_2} \equiv \mathrm{Amp}\left((P^0\bar{P}^0)_{\rm odd} \to f_1^{s_1}f_2^{s_2}\right), \tag{4.36}$$

such that the master equation Eq. (4.25) can now be written as

$$\mathcal{A}^{s_1s_2} \equiv \mathrm{Amp}\left(X \to Y(p_1)\left(f_1^{s_1}(p_2)\right)_{P_i(p_2)}\left(f_2^{s_2}(p_3)\right)_{P_j(p_3)}\right)$$
$$= \frac{1}{4N_1^2N_2^2pq}\left(2A^p\mathcal{E}^{s_1s_2} + 2A^m\mathcal{O}^{s_1s_2}\cos\theta\right). \tag{4.37}$$

Therefore

$$\begin{aligned}
\mathcal{E}^{++} = e^{-i\mu(t_1+t_2)}\frac{1}{2}N_1^2N_2^2\Bigg(\sqrt{1-z^2}\Big(& \left(U_1^2 - U_2^2\right) + (U_1V_1 - U_2V_2)\left(\epsilon_1^+ + \epsilon_2^+\right) \\
& + \left(V_1^2 - V_2^2\right)\epsilon_1^+\epsilon_2^+ \\
& + i\Delta\mu\,(t_1+t_2)\Big(\left(U_1^2 + U_2^2\right) \\
& \qquad + (U_1V_1 + U_2V_2)\left(\epsilon_1^+ + \epsilon_2^+\right) \\
& \qquad + \left(V_1^2 + V_2^2\right)\epsilon_1^+\epsilon_2^+\Big)\Big) \\
& + z\Big(2U_1U_2 + (U_2V_1 + U_1V_2)\left(\epsilon_1^+ + \epsilon_2^+\right) + 2V_1V_2\epsilon_1^+\epsilon_2^+ \\
& + i\Delta\mu\,(t_1-t_2)\,(U_2V_1 - U_1V_2)\left(\epsilon_1^+ - \epsilon_2^+\right)\Big)\Bigg)A_1^+A_2^+, \qquad (4.38a)
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}^{++} = e^{-i\mu(t_1+t_2)}\frac{1}{2}N_1^2N_2^2\Bigg(& (U_2V_1 - U_1V_2)\left(\epsilon_1^+ - \epsilon_2^+\right) \\
& + i\Delta\mu\,(t_1-t_2)\Big(2U_1U_2 + (U_2V_1 + U_1V_2)\left(\epsilon_1^+ + \epsilon_2^+\right) \\
& + 2V_1V_2\epsilon_1^+\epsilon_2^+\Big)\Bigg)A_1^+A_2^+, \qquad (4.38b)
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}^{+-} = e^{-i\mu(t_1+t_2)}\frac{1}{2}N_1^2N_2^2\Bigg(\sqrt{1-z^2}\Big(& \left(U_1^2 - U_2^2\right)\epsilon_2^- + \left(V_1^2 - V_2^2\right)\epsilon_1^+ \\
& + (U_1V_1 - U_2V_2)\left(1 + \epsilon_1^+\epsilon_2^-\right) \\
& + i\Delta\mu\,(t_1+t_2)\big(\left(U_1^2 + U_2^2\right)\epsilon_2^- + \left(V_1^2 + V_2^2\right)\epsilon_1^+ \\
& \qquad + (U_1V_1 + U_2V_2)\left(1 + \epsilon_1^+\epsilon_2^-\right)\big)\Big) \\
& + z\Big((U_1V_2 + U_2V_1) + 2U_1U_2\epsilon_2^- + 2V_1V_2\epsilon_1^+ \\
& \qquad + (U_2V_1 + U_1V_2)\,\epsilon_1^+\epsilon_2^- \\
& + i\Delta\mu\,(t_1-t_2)\Big((U_1V_2 - U_2V_1) \\
& \qquad + (U_2V_1 - U_1V_2)\,\epsilon_1^+\epsilon_2^-\Big)\Big)\Bigg)A_1^+A_2^-,
\end{aligned}$$

(4.38c)

$$\begin{aligned}
\mathcal{O}^{+-} = e^{-i\mu(t_1+t_2)}\frac{1}{2}N_1^2N_2^2\Big(&(U_1V_2 - U_2V_1) + (U_2V_1 - U_1V_2)\,\epsilon_1^+\epsilon_2^- \\
&+ i\Delta\mu(t_1-t_2)\Big((U_1V_2 + U_2V_1) + 2U_1U_2\epsilon_2^- + 2V_1V_2\epsilon_1^+ \\
&+ (U_2V_1 + U_1V_2)\,\epsilon_1^+\epsilon_2^-\Big)\Big)\, A_1^+\, A_2^-,
\end{aligned} \tag{4.38d}$$

$$\begin{aligned}
\mathcal{E}^{-+} = e^{-i\mu(t_1+t_2)}\frac{1}{2}N_1^2N_2^2\Big(&\sqrt{1-z^2}\Big(\left(U_1^2 - U_2^2\right)\epsilon_1^- + \left(V_1^2 - V_2^2\right)\epsilon_2^+ \\
&+ (U_1V_1 - U_2V_2)\left(1 + \epsilon_1^-\epsilon_2^+\right) \\
&+ i\Delta\mu\,(t_1+t_2)\Big(\left(U_1^2 + U_2^2\right)\epsilon_1^- + \left(V_1^2 + V_2^2\right)\epsilon_2^+ \\
&+ (U_1V_1 + U_2V_2)\left(1 + \epsilon_1^-\epsilon_2^+\right)\Big)\Big) \\
&+ z\,\Big((U_1V_2 + U_2V_1) + 2U_1U_2\epsilon_1^- + 2V_1V_2\epsilon_2^+ \\
&+ (U_2V_1 + U_1V_2)\,\epsilon_1^-\epsilon_2^+ \\
&+ i\Delta\mu\,(t_1-t_2)\Big((U_2V_1 - U_1V_2) \\
&+ (U_1V_2 - U_2V_1)\,\epsilon_1^-\epsilon_2^+\Big)\Big)\Big)\, A_1^-\, A_2^+,
\end{aligned} \tag{4.38e}$$

$$\begin{aligned}
\mathcal{O}^{-+} = e^{-i\mu(t_1+t_2)}\frac{1}{2}N_1^2N_2^2\Big(&(U_2V_1 - U_1V_2) + (U_1V_2 - U_2V_1)\,\epsilon_1^-\epsilon_2^+ \\
&+ i\Delta\mu(t_1-t_2)\Big((U_1V_2 + U_2V_1) + 2U_1U_2\epsilon_1^- + 2V_1V_2\epsilon_2^+ \\
&+ (U_1V_2 + U_2V_1)\,\epsilon_1^-\epsilon_2^+\Big)\Big)\, A_1^-\, A_2^+,
\end{aligned} \tag{4.38f}$$

$$\begin{aligned}
\mathcal{E}^{--} = e^{-i\mu(t_1+t_2)}\frac{1}{2}N_1^2N_2^2\Bigg(\sqrt{1-z^2}\Bigg(&\left(V_1^2 - V_2^2\right) + (U_1V_1 - U_2V_2)\left(\epsilon_1^- + \epsilon_2^-\right)\\
&+ \left(U_1^2 - U_2^2\right)\epsilon_1^-\epsilon_2^-\\
&+ i\Delta\mu\,(t_1 + t_2)\Bigg(\left(V_1^2 + V_2^2\right)\\
&\qquad + (U_1V_1 + U_2V_2)\left(\epsilon_1^- + \epsilon_2^-\right)\\
&\qquad + \left(U_1^2 + U_2^2\right)\epsilon_1^-\epsilon_2^-\Bigg)\Bigg)\\
+ z\Bigg(&2V_1V_2 + (U_1V_2 + U_2V_1)\left(\epsilon_1^- + \epsilon_2^-\right) + 2U_1U_2\epsilon_1^-\epsilon_2^-\\
&+ i\Delta\mu\,(t_1 - t_2)(U_1V_2 - U_2V_1)\left(\epsilon_1^- - \epsilon_2^-\right)\Bigg)\Bigg)A_1^-A_2^-,
\end{aligned} \tag{4.38g}$$

$$\begin{aligned}
\mathcal{O}^{--} = e^{-i\mu(t_1+t_2)}\frac{1}{2}N_1^2N_2^2\Big(&(U_1V_2 - U_2V_1)\left(\epsilon_1^- - \epsilon_2^-\right)\\
&+ i\Delta\mu\,(t_1 - t_2)\Big(2V_1V_2 + (U_1V_2 + U_2V_1)\left(\epsilon_1^- + \epsilon_2^-\right)\\
&\qquad + 2U_1U_2\epsilon_1^-\epsilon_2^-\Big)\Big)A_1^-A_2^-.
\end{aligned} \tag{4.38h}$$

Using these expressions in Eq. (4.37) we get

$$
\begin{aligned}
\mathcal{A}^{++} = \frac{e^{-i\mu(t_1+t_2)}}{4pq}\Bigg(\Bigg(&\sqrt{1-z^2}\left(U_1^2 - U_2^2\right) + z\,2U_1U_2 \\
&\qquad + \left(\epsilon_1^+ + \epsilon_2^+\right)\left(\sqrt{1-z^2}\,(U_1V_1 - U_2V_2) + z\,\,(U_2V_1 + U_1V_2)\right) \\
&\qquad\qquad + \epsilon_1^+\epsilon_2^+\left(\sqrt{1-z^2}\left(V_1^2 - V_2^2\right) + z\,2V_1V_2\right)\Bigg)A^p \\
&+ \left(\epsilon_1^+ - \epsilon_2^+\right)(U_2V_1 - U_1V_2)\,A^m\cos\theta \\
&+ i\Delta\mu\,(t_1+t_2)\sqrt{1-z^2}\Big(\left(U_1^2 + U_2^2\right) + (U_1V_1 + U_2V_2)\left(\epsilon_1^+ + \epsilon_2^+\right) \\
&\qquad\qquad + \left(V_1^2 + V_2^2\right)\epsilon_1^+\epsilon_2^+\Big)A^p \\
&+ i\Delta\mu\,(t_1-t_2)\Big(z\,\,(U_2V_1 - U_1V_2)\left(\epsilon_1^+ - \epsilon_2^+\right)A^p \\
&\qquad + (2U_1U_2 + (U_2V_1 + U_1V_2)\left(\epsilon_1^+ + \epsilon_2^+\right) \\
&\qquad\qquad + 2V_1V_2\epsilon_1^+\epsilon_2^+)\,A^m\cos\theta\Big)\Bigg)\,A_1^+\,A_2^+,
\end{aligned}
\tag{4.39a}
$$

$$
\begin{aligned}
\mathcal{A}^{+-} = \frac{e^{-i\mu(t_1+t_2)}}{4pq}\Bigg(\Bigg(\Bigg(&\sqrt{1-z^2}\,(U_1V_1 - U_2V_2) + z\,\,(U_1V_2 + U_2V_1)\Bigg)\left(1 + \epsilon_1^+\epsilon_2^-\right) \\
&+ \left(\sqrt{1-z^2}\left(U_1^2 - U_2^2\right) + z\,2U_1U_2\right)\epsilon_2^- \\
&\qquad + \left(\sqrt{1-z^2}\left(V_1^2 - V_2^2\right) + z\,2V_1V_2\right)\epsilon_1^+\Bigg)A^p \\
&+ (U_1V_2 - U_2V_1)\left(1 - \epsilon_1^+\epsilon_2^-\right)A^m\cos\theta \\
&+ i\Delta\mu\,(t_1+t_2)\sqrt{1-z^2}\Big((U_1V_1 + U_2V_2)\left(1 + \epsilon_1^+\epsilon_2^-\right) \\
&\qquad + \left(U_1^2 + U_2^2\right)\epsilon_2^- + \left(V_1^2 + V_2^2\right)\epsilon_1^+\Big)A^p \\
&+ i\Delta\mu\,(t_1-t_2)\Big(z\,\,(U_1V_2 - U_2V_1)\left(1 - \epsilon_1^+\epsilon_2^-\right)A^p \\
&\qquad + \Big((U_1V_2 + U_2V_1)\left(1 + \epsilon_1^+\epsilon_2^-\right) \\
&\qquad\qquad + 2U_1U_2\epsilon_2^- + 2V_1V_2\epsilon_1^+\Big)A^m\cos\theta\Big)\Bigg)\,A_1^+\,A_2^-,
\end{aligned}
\tag{4.39b}
$$

$$
\begin{aligned}
\mathcal{A}^{-+} = \frac{e^{-i\mu(t_1+t_2)}}{4pq}\Bigg(\Bigg(\Big(&\Big(\sqrt{1-z^2}\,(U_1V_1 - U_2V_2) + z\ (U_1V_2 + U_2V_1)\Big)\left(1+\epsilon_1^-\epsilon_2^+\right) \\
&+\Big(\sqrt{1-z^2}\left(U_1^2 - U_2^2\right) + z\,2U_1U_2\Big)\epsilon_1^- \\
&\qquad + \Big(\sqrt{1-z^2}\left(V_1^2 - V_2^2\right) + z\,2V_1V_2\Big)\epsilon_2^+\Big)A^p \\
&+ (U_2V_1 - U_1V_2)\left(1-\epsilon_1^-\epsilon_2^+\right)A^m\cos\theta \\
&+ i\Delta\mu\,(t_1+t_2)\sqrt{1-z^2}\Big((U_1V_1 + U_2V_2)\left(1+\epsilon_1^-\epsilon_2^+\right) \\
&\qquad + \left(U_1^2 + U_2^2\right)\epsilon_1^- + \left(V_1^2+V_2^2\right)\epsilon_2^+\Big)A^p \\
&+ i\Delta\mu\,(t_1-t_2)\Big(z\ (U_2V_1 - U_1V_2)\left(1-\epsilon_1^-\epsilon_2^+\right)A^p \\
&\qquad + (\,(U_1V_2 + U_2V_1)\left(1+\epsilon_1^-\epsilon_2^+\right) \\
&\qquad\qquad + 2U_1U_2\epsilon_1^- + 2V_1V_2\epsilon_2^+)\,A^m\cos\theta\Big)\Bigg)A_1^-\,A_2^+,
\end{aligned}
\tag{4.39c}
$$

$$
\begin{aligned}
\mathcal{A}^{--} = \frac{e^{-i\mu(t_1+t_2)}}{4pq}\Bigg(\Big(&\sqrt{1-z^2}\left(V_1^2 - V_2^2\right) + z\,2V_1V_2 \\
&+\left(\epsilon_1^- + \epsilon_2^-\right)\Big(\sqrt{1-z^2}\,(U_1V_1 - U_2V_2) + z\ (U_2V_1 + U_1V_2)\Big) \\
&+\epsilon_1^-\epsilon_2^-\Big(\sqrt{1-z^2}\left(U_1^2 - U_2^2\right) + z\,2U_1U_2\Big)\Big)A^p \\
&+\left(\epsilon_1^- - \epsilon_2^-\right)(U_1V_2 - U_2V_1)\,A^m\cos\theta \\
&+ i\Delta\mu\,(t_1+t_2)\sqrt{1-z^2}\big(\left(V_1^2+V_2^2\right) + (U_1V_1 + U_2V_2)\left(\epsilon_1^- + \epsilon_2^-\right) \\
&\qquad + \left(U_1^2 + U_2^2\right)\epsilon_1^-\epsilon_2^-\big)A^p \\
&+ i\Delta\mu\,(t_1-t_2)\Big(z\ (U_1V_2 - U_2V_1)\left(\epsilon_1^- - \epsilon_2^-\right)A^p \\
&\qquad + (2V_1V_2 + (U_2V_1 + U_1V_2)\left(\epsilon_1^- + \epsilon_2^-\right) \\
&\qquad\qquad + 2U_1U_2\epsilon_1^-\epsilon_2^-)\,A^m\cos\theta\Big)\Bigg)A_1^-\,A_2^-.
\end{aligned}
\tag{4.39d}
$$

From the expressions for $U_{1,2}$ and $V_{1,2}$ it is easy to get the following expressions:

$$U_1^2 - U_2^2 = -2z\left(p^2 - q^2\right) + 4pq\sqrt{1-z^2}, \tag{4.40a}$$

$$V_1^2 - V_2^2 = -2z\left(p^2 - q^2\right) - 4pq\sqrt{1-z^2}, \tag{4.40b}$$

$$U_1^2 + U_2^2 = 2\left(p^2 + q^2\right) = V_1^2 + V_2^2, \tag{4.40c}$$

$$U_1U_2 = \left(p^2 - q^2\right)\sqrt{1-z^2} + 2pqz, \tag{4.40d}$$

$$V_1V_2 = \left(p^2 - q^2\right)\sqrt{1-z^2} - 2pqz, \tag{4.40e}$$

$$U_1U_2 - V_1V_2 = 4pqz, \tag{4.40f}$$

$$U_1U_2 + V_1V_2 = 2\left(p^2 - q^2\right)\sqrt{1-z^2}, \tag{4.40g}$$

$$U_1V_1 - U_2V_2 = -2\left(p^2 + q^2\right)z, \tag{4.40h}$$

$$U_1V_1 + U_2V_2 = 2\left(p^2 - q^2\right), \tag{4.40i}$$

$$U_1V_2 - U_2V_1 = 4pq, \tag{4.40j}$$

$$U_1V_2 + U_2V_1 = 2\left(p^2 + q^2\right)\sqrt{1-z^2}. \tag{4.40k}$$

Substituting these expressions in the amplitudes and simplifying we get

$$
\begin{aligned}
\mathcal{A}^{++} = e^{-i\mu(t_1+t_2)}\Bigg(& A^p\left(1-\epsilon_1^+\epsilon_2^+\right) - A^m\cos\theta\left(\epsilon_1^+-\epsilon_2^+\right) \\
& + i\Delta\mu\,(t_1+t_2)\sqrt{1-z^2}\,A^p\Bigg(\left(\frac{p^2+q^2}{2pq}\right)\left(1+\epsilon_1^+\epsilon_2^+\right) \\
& \qquad + \left(\frac{p^2-q^2}{2pq}\right)\left(\epsilon_1^+ + \epsilon_2^+\right)\Bigg) \\
& + i\Delta\mu\,(t_1-t_2)\Bigg(- A^p\,z\left(\epsilon_1^+-\epsilon_2^+\right) \\
& \qquad + A^m\cos\theta\Bigg(z\left(1-\epsilon_1^+\epsilon_2^+\right) \\
& \qquad\qquad + \sqrt{1-z^2}\Bigg(\left(\frac{p^2-q^2}{2pq}\right)\left(1+\epsilon_1^+\epsilon_2^+\right) \\
& \qquad\qquad\qquad + \left(\frac{p^2+q^2}{2pq}\right)\left(\epsilon_1^+ + \epsilon_2^+\right)\Bigg)\Bigg)\Bigg)\Bigg) A_1^+ A_2^+,
\end{aligned}
\tag{4.41a}
$$

$$
\begin{aligned}
\mathcal{A}^{+-} = e^{-i\mu(t_1+t_2)}\Bigg(& - A^p\left(\epsilon_1^+-\epsilon_2^-\right) + A^m\cos\theta\left(1-\epsilon_1^+\epsilon_2^-\right) \\
& + i\Delta\mu\,(t_1+t_2)\sqrt{1-z^2}\,A^p\Bigg(\left(\frac{p^2-q^2}{2pq}\right)\left(1+\epsilon_1^+\epsilon_2^-\right) \\
& \qquad + \left(\frac{p^2+q^2}{2pq}\right)\left(\epsilon_1^+ + \epsilon_2^-\right)\Bigg) \\
& + i\Delta\mu\,(t_1-t_2)\Bigg(A^p\,z\left(1-\epsilon_1^+\epsilon_2^-\right) \\
& \qquad + A^m\cos\theta\Bigg(- z\left(\epsilon_1^+-\epsilon_2^-\right) \\
& \qquad\qquad + \sqrt{1-z^2}\Bigg(\left(\frac{p^2+q^2}{2pq}\right)\left(1+\epsilon_1^+\epsilon_2^-\right) \\
& \qquad\qquad\qquad + \left(\frac{p^2-q^2}{2pq}\right)\left(\epsilon_1^+ + \epsilon_2^-\right)\Bigg)\Bigg)\Bigg)\Bigg) A_1^+ A_2^-,
\end{aligned}
\tag{4.41b}
$$

$$
\begin{aligned}
\mathcal{A}^{-+} = e^{-i\mu(t_1+t_2)}\Bigg(& A^p \left(\epsilon_1^- - \epsilon_2^+\right) - A^m \cos\theta \left(1 - \epsilon_1^-\epsilon_2^+\right) \\
& + i\Delta\mu\,(t_1+t_2)\sqrt{1-z^2}\, A^p \Bigg(\left(\frac{p^2-q^2}{2pq}\right)\left(1+\epsilon_1^-\epsilon_2^+\right) \\
& \qquad + \left(\frac{p^2+q^2}{2pq}\right)\left(\epsilon_1^- + \epsilon_2^+\right)\Bigg) \\
& + i\Delta\mu\,(t_1-t_2)\Bigg(- A^p\, z \left(1 - \epsilon_1^-\epsilon_2^+\right) \\
& \qquad + A^m \cos\theta \Bigg(z \left(\epsilon_1^- - \epsilon_2^+\right) \\
& \qquad\qquad + \sqrt{1-z^2}\Bigg(\left(\frac{p^2+q^2}{2pq}\right)\left(1+\epsilon_1^-\epsilon_2^+\right) \\
& \qquad\qquad\qquad + \left(\frac{p^2-q^2}{2pq}\right)\left(\epsilon_1^- + \epsilon_2^+\right)\Bigg)\Bigg)\Bigg)\Bigg)\, A_1^-\, A_2^+,
\end{aligned}
\tag{4.41c}
$$

$$
\begin{aligned}
\mathcal{A}^{--} = e^{-i\mu(t_1+t_2)}\Bigg(& - A^p \left(1 - \epsilon_1^-\epsilon_2^-\right) + A^m \cos\theta \left(\epsilon_1^- - \epsilon_2^-\right) \\
& + i\Delta\mu\,(t_1+t_2)\sqrt{1-z^2}\, A^p \Bigg(\left(\frac{p^2+q^2}{2pq}\right)\left(1+\epsilon_1^-\epsilon_2^-\right) \\
& \qquad + \left(\frac{p^2-q^2}{2pq}\right)\left(\epsilon_1^- + \epsilon_2^-\right)\Bigg) \\
& + i\Delta\mu\,(t_1-t_2)\Bigg(A^p\, z \left(\epsilon_1^- - \epsilon_2^-\right) \\
& \qquad + A^m \cos\theta \Bigg(- z \left(1 - \epsilon_1^-\epsilon_2^-\right) \\
& \qquad\qquad + \sqrt{1-z^2}\Bigg(\left(\frac{p^2-q^2}{2pq}\right)\left(1+\epsilon_1^-\epsilon_2^-\right) \\
& \qquad\qquad\qquad + \left(\frac{p^2+q^2}{2pq}\right)\left(\epsilon_1^- + \epsilon_2^-\right)\Bigg)\Bigg)\Bigg)\Bigg)\, A_1^-\, A_2^-.
\end{aligned}
\tag{4.41d}
$$

Let us now introduce two functions $f_1(\epsilon_1, \epsilon_2)$ and $f_2(\epsilon_1, \epsilon_2)$ defined as follows:

$$
f_1(\epsilon_1, \epsilon_2) = \sqrt{1-z^2}\left(\left(\frac{p^2+q^2}{2pq}\right)(1+\epsilon_1\epsilon_2) + \left(\frac{p^2-q^2}{2pq}\right)(\epsilon_1+\epsilon_2)\right), \tag{4.42}
$$

$$f_2(\epsilon_1, \epsilon_2) = \sqrt{1 - z^2}\left(\left(\frac{p^2 - q^2}{2pq}\right)(1 + \epsilon_1\epsilon_2) + \left(\frac{p^2 + q^2}{2pq}\right)(\epsilon_1 + \epsilon_2)\right). \tag{4.43}$$

In terms of these two functions, the amplitudes can now be written as

$$\begin{aligned}\mathcal{A}^{++} = e^{-i\mu(t_1+t_2)}\Big(&A^p\left(1 - \epsilon_1^+\epsilon_2^+\right) - A^m\cos\theta\left(\epsilon_1^+ - \epsilon_2^+\right) + i\Delta\mu\,(t_1 + t_2)\,A^p f_1(\epsilon_1^+, \epsilon_2^+)\\ &+ i\Delta\mu\,(t_1 - t_2)\Big(- A^p\,z\left(\epsilon_1^+ - \epsilon_2^+\right)\\ &\qquad + A^m\cos\theta\left(z\left(1 - \epsilon_1^+\epsilon_2^+\right) + f_2(\epsilon_1^+, \epsilon_2^+)\right)\Big)\Big)\,A_1^+\,A_2^+,\end{aligned} \tag{4.44a}$$

$$\begin{aligned}\mathcal{A}^{+-} = e^{-i\mu(t_1+t_2)}\Big(& - A^p\left(\epsilon_1^+ - \epsilon_2^-\right) + A^m\cos\theta\left(1 - \epsilon_1^+\epsilon_2^-\right) + i\Delta\mu\,(t_1 + t_2)\,A^p f_2(\epsilon_1^+, \epsilon_2^-)\\ &+ i\Delta\mu\,(t_1 - t_2)\Big(A^p\,z\left(1 - \epsilon_1^+\epsilon_2^-\right)\\ &\qquad + A^m\cos\theta\left(- z\left(\epsilon_1^+ - \epsilon_2^-\right) + f_1(\epsilon_1^+, \epsilon_2^-)\right)\Big)\Big)\,A_1^+\,A_2^-,\end{aligned} \tag{4.44b}$$

$$\begin{aligned}\mathcal{A}^{-+} = e^{-i\mu(t_1+t_2)}\Big(&A^p\left(\epsilon_1^- - \epsilon_2^+\right) - A^m\cos\theta\left(1 - \epsilon_1^-\epsilon_2^+\right) + i\Delta\mu\,(t_1 + t_2)\,A^p f_2(\epsilon_1^-, \epsilon_2^+)\\ &+ i\Delta\mu\,(t_1 - t_2)\Big(- A^p\,z\left(1 - \epsilon_1^-\epsilon_2^+\right)\\ &\qquad + A^m\cos\theta\left(z\left(\epsilon_1^- - \epsilon_2^+\right) + f_1(\epsilon_1^-, \epsilon_2^+)\right)\Big)\Big)\,A_1^-\,A_2^+,\end{aligned} \tag{4.44c}$$

$$\begin{aligned}\mathcal{A}^{--} = e^{-i\mu(t_1+t_2)}\Big(& - A^p\left(1 - \epsilon_1^-\epsilon_2^-\right) + A^m\cos\theta\left(\epsilon_1^- - \epsilon_2^-\right) + i\Delta\mu\,(t_1 + t_2)\,A^p f_1(\epsilon_1^-, \epsilon_2^-)\\ &+ i\Delta\mu\,(t_1 - t_2)\Big(A^p\,z\left(\epsilon_1^- - \epsilon_2^-\right)\\ &\qquad + A^m\cos\theta\left(- z\left(1 - \epsilon_1^-\epsilon_2^-\right) + f_2(\epsilon_1^-, \epsilon_2^-)\right)\Big)\Big)\,A_1^-\,A_2^-.\end{aligned} \tag{4.44d}$$

The time-integrated, squares of the moduli of amplitudes can now be easily obtained. Here we shall neglect the second and higher order powers of $\Delta\mu$, and consider only the leading order (i.e. first order) contribution of $\Delta\mu$. The following integration results would be used in the evaluation of time-integrated squares of the moduli of amplitudes:

$$\int_0^\infty \int_0^\infty \left|e^{-i\mu(t_1+t_2)}\right|^2 dt_1\, dt_2 = \frac{1}{\Gamma^2}, \tag{4.45a}$$

$$\int_0^\infty \int_0^\infty \left|e^{-i\mu(t_1+t_2)}\right|^2 (t_1+t_2)\, dt_1\, dt_2 = \frac{2}{\Gamma^3}, \tag{4.45b}$$

$$\int_0^\infty \int_0^\infty \left|e^{-i\mu(t_1+t_2)}\right|^2 (t_1-t_2)\, dt_1\, dt_2 = 0. \tag{4.45c}$$

Thus, the time-integrated squares of the moduli of amplitudes are given by:

$$\begin{aligned}
\left|\mathcal{A}^{++}\right|^2 = \frac{\left|A_1^+\right|^2\left|A_2^+\right|^2}{\Gamma^2}\Bigg(&\left|A^p\right|^2\left|1-\epsilon_1^+\epsilon_2^+\right|^2 + \left|A^m\right|^2\left|\epsilon_1^+-\epsilon_2^+\right|^2\cos^2\theta \\
&- 2\mathrm{Re}\left(\left(1-\epsilon_1^+\epsilon_2^+\right)\left(\epsilon_1^{+*}-\epsilon_2^{+*}\right)A^pA^{m*}\right)\cos\theta \\
&+ \frac{4}{\Gamma}\mathrm{Re}\bigg(i\Delta\mu\, f_1(\epsilon_1^+,\epsilon_2^+)\Big(\left|A^p\right|^2(1-\epsilon_1^{+*}\epsilon_2^{+*}) \\
&\qquad\qquad - A^pA^{m*}(\epsilon_1^{+*}-\epsilon_2^{+*})\cos\theta\Big)\bigg)\Bigg) \\
= \frac{\left|A_1^+\right|^2\left|A_2^+\right|^2}{\Gamma^2}\Bigg(&\left|A^p\right|^2\left(\left|1-\epsilon_1^+\epsilon_2^+\right|^2 + \frac{4}{\Gamma}\mathrm{Re}(i\Delta\mu\, f_1(\epsilon_1^+,\epsilon_2^+)\left(1-\epsilon_1^{+*}\epsilon_2^{+*}\right))\right) \\
&+ \left|A^m\right|^2\left|\epsilon_1^+-\epsilon_2^+\right|^2\cos^2\theta \\
&- 2\mathrm{Re}\left(\left(1-\epsilon_1^+\epsilon_2^+ + \frac{2}{\Gamma}i\Delta\mu\, f_1(\epsilon_1^+,\epsilon_2^+)\right)\left(\epsilon_1^{+*}-\epsilon_2^{+*}\right)A^pA^{m*}\right)\cos\theta\Bigg),
\end{aligned} \tag{4.46a}$$

$$\begin{aligned}
\left|\mathcal{A}^{+-}\right|^2 = \frac{\left|A_1^+\right|^2\left|A_2^-\right|^2}{\Gamma^2}\Bigg(&\left|A^p\right|^2\left|\epsilon_1^+-\epsilon_2^-\right|^2 + \left|A^m\right|^2\left|1-\epsilon_1^+\epsilon_2^-\right|^2\cos^2\theta \\
&- 2\mathrm{Re}\left((1-\epsilon_1^{+*}\epsilon_2^{-*})(\epsilon_1^+-\epsilon_2^-)A^pA^{m*}\right)\cos\theta \\
&+ \frac{4}{\Gamma}\mathrm{Re}\bigg(i\Delta\mu\, f_2(\epsilon_1^+,\epsilon_2^-)\Big(-\left|A^p\right|^2(\epsilon_1^{+*}-\epsilon_2^{-*}) \\
&\qquad\qquad + A^pA^{m*}(1-\epsilon_1^{+*}\epsilon_2^{-*})\cos\theta\Big)\bigg)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left|A_1^+\right|^2\left|A_2^-\right|^2}{\Gamma^2}\Bigg(\left|A^p\right|^2\left(\left|\epsilon_1^+-\epsilon_2^-\right|^2-\frac{4}{\Gamma}\mathrm{Re}\left(i\Delta\mu\, f_2(\epsilon_1^+,\epsilon_2^-)\,(\epsilon_1^{+*}-\epsilon_2^{-*})\right)\right)\\
&\qquad+\left|A^m\right|^2\left|1-\epsilon_1^+\epsilon_2^-\right|^2\cos^2\theta\\
&\qquad-2\mathrm{Re}\Big((\epsilon_1^+-\epsilon_2^--\frac{2}{\Gamma}i\Delta\mu\, f_2(\epsilon_1^+,\epsilon_2^-))\left(1-\epsilon_1^{+*}\epsilon_2^{-*}\right)A^pA^{m*}\Big)\cos\theta\Bigg),
\end{aligned}
\tag{4.46b}
$$

$$
\begin{aligned}
\left|\mathcal{A}^{-+}\right|^2 &= \frac{\left|A_1^-\right|^2\left|A_2^+\right|^2}{\Gamma^2}\Bigg(\left|A^p\right|^2\left|\epsilon_1^--\epsilon_2^+\right|^2+\left|A^m\right|^2\left|1-\epsilon_1^-\epsilon_2^+\right|^2\cos^2\theta\\
&\qquad-2\mathrm{Re}\left((1-\epsilon_1^{-*}\epsilon_2^{+*})(\epsilon_1^--\epsilon_2^+)A^pA^{m*}\right)\cos\theta\\
&\qquad+\frac{4}{\Gamma}\mathrm{Re}\Big(i\Delta\mu\, f_2(\epsilon_1^-,\epsilon_2^+)\Big(-\left|A^p\right|^2(\epsilon_1^{-*}-\epsilon_2^{+*})\\
&\qquad\qquad+A^pA^{m*}(1-\epsilon_1^{-*}\epsilon_2^{+*})\cos\theta\Big)\Big)\Bigg)\\
&= \frac{\left|A_1^-\right|^2\left|A_2^+\right|^2}{\Gamma^2}\Bigg(\left|A^p\right|^2\left(\left|\epsilon_1^--\epsilon_2^+\right|^2-\frac{4}{\Gamma}\mathrm{Re}\left(i\Delta\mu\, f_2(\epsilon_1^-,\epsilon_2^+)\,(\epsilon_1^{-*}-\epsilon_2^{+*})\right)\right)\\
&\qquad+\left|A^m\right|^2\left|1-\epsilon_1^-\epsilon_2^+\right|^2\cos^2\theta\\
&\qquad-2\mathrm{Re}\Big((\epsilon_1^--\epsilon_2^+-\frac{2}{\Gamma}i\Delta\mu\, f_2(\epsilon_1^-,\epsilon_2^+))\left(1-\epsilon_1^{-*}\epsilon_2^{+*}\right)A^pA^{m*}\Big)\cos\theta\Bigg),
\end{aligned}
\tag{4.46c}
$$

$$
\begin{aligned}
\left|\mathcal{A}^{--}\right|^2 &= \frac{\left|A_1^-\right|^2\left|A_2^-\right|^2}{\Gamma^2}\Bigg(\left|A^p\right|^2\left|1-\epsilon_1^-\epsilon_2^-\right|^2+\left|A^m\right|^2\left|\epsilon_1^--\epsilon_2^-\right|^2\cos^2\theta\\
&\qquad-2\mathrm{Re}\left(\left(1-\epsilon_1^-\epsilon_2^-\right)\left(\epsilon_1^{-*}-\epsilon_2^{-*}\right)A^pA^{m*}\right)\cos\theta\\
&\qquad+\frac{4}{\Gamma}\mathrm{Re}\Big(i\Delta\mu\, f_1(\epsilon_1^-,\epsilon_2^-)\Big(\left|A^p\right|^2(1-\epsilon_1^{-*}\epsilon_2^{-*})\\
&\qquad\qquad-A^pA^{m*}(\epsilon_1^{-*}-\epsilon_2^{-*})\cos\theta\Big)\Big)\Bigg)\\
&= \frac{\left|A_1^-\right|^2\left|A_2^-\right|^2}{\Gamma^2}\Bigg(\left|A^p\right|^2\left(\left|1-\epsilon_1^-\epsilon_2^-\right|^2+\frac{4}{\Gamma}\mathrm{Re}(i\Delta\mu\, f_1(\epsilon_1^-,\epsilon_2^-)\left(1-\epsilon_1^{-*}\epsilon_2^{-*}\right))\right)\\
&\qquad+\left|A^m\right|^2\left|\epsilon_1^--\epsilon_2^-\right|^2\cos^2\theta\\
&\qquad-2\mathrm{Re}\Big(\left(1-\epsilon_1^-\epsilon_2^-+\frac{2}{\Gamma}i\Delta\mu\, f_1(\epsilon_1^-,\epsilon_2^-)\right)\left(\epsilon_1^{-*}-\epsilon_2^{-*}\right)A^pA^{m*}\Big)\cos\theta\Bigg).
\end{aligned}
\tag{4.46d}
$$

We can now introduce a few more definitions (just in order to make the expressions look simpler) as follows:

$$\phi_1(\epsilon_1, \epsilon_2) = \frac{4}{\Gamma}\mathrm{Re}\left(i\Delta\mu\, f_1(\epsilon_1, \epsilon_2)\,(1 - \epsilon_1^*\epsilon_2^*)\right), \tag{4.47}$$

$$\phi_2(\epsilon_1, \epsilon_2) = \frac{4}{\Gamma}\mathrm{Re}\left(i\Delta\mu\, f_2(\epsilon_1, \epsilon_2)\,(\epsilon_1^* - \epsilon_2^*)\right), \tag{4.48}$$

$$\psi_1(\epsilon_1, \epsilon_2) = \frac{2}{\Gamma} i\Delta\mu\, f_1(\epsilon_1, \epsilon_2), \tag{4.49}$$

$$\psi_2(\epsilon_1, \epsilon_2) = \frac{2}{\Gamma} i\Delta\mu\, f_2(\epsilon_1, \epsilon_2), \tag{4.50}$$

such that we have

$$\begin{aligned}\left|\mathcal{A}^{++}\right|^2 = \frac{\left|A_1^+\right|^2\left|A_2^+\right|^2}{\Gamma^2}\Bigg(&\left|A^p\right|^2\left(\left|1 - \epsilon_1^+\epsilon_2^+\right|^2 + \phi_1(\epsilon_1^+, \epsilon_2^+)\right) + \left|A^m\right|^2\left|\epsilon_1^+ - \epsilon_2^+\right|^2\cos^2\theta \\ &- 2\mathrm{Re}\Big(\left(1 - \epsilon_1^+\epsilon_2^+ + \psi_1(\epsilon_1^+, \epsilon_2^+)\right)\left(\epsilon_1^{+*} - \epsilon_2^{+*}\right) A^p A^{m*}\Big)\cos\theta\Bigg),\end{aligned} \tag{4.51}$$

$$\begin{aligned}\left|\mathcal{A}^{+-}\right|^2 = \frac{\left|A_1^+\right|^2\left|A_2^-\right|^2}{\Gamma^2}\Bigg(&\left|A^p\right|^2\left(\left|\epsilon_1^+ - \epsilon_2^-\right|^2 - \phi_2(\epsilon_1^+, \epsilon_2^-)\right) + \left|A^m\right|^2\left|1 - \epsilon_1^+\epsilon_2^-\right|^2\cos^2\theta \\ &- 2\mathrm{Re}\Big((\epsilon_1^+ - \epsilon_2^- - \psi_2(\epsilon_1^+, \epsilon_2^-))\left(1 - \epsilon_1^{+*}\epsilon_2^{-*}\right) A^p A^{m*}\Big)\cos\theta\Bigg),\end{aligned} \tag{4.52}$$

$$\begin{aligned}\left|\mathcal{A}^{-+}\right|^2 = \frac{\left|A_1^-\right|^2\left|A_2^+\right|^2}{\Gamma^2}\Bigg(&\left|A^p\right|^2\left(\left|\epsilon_1^- - \epsilon_2^+\right|^2 - \phi_2(\epsilon_1^-, \epsilon_2^+)\right) + \left|A^m\right|^2\left|1 - \epsilon_1^-\epsilon_2^+\right|^2\cos^2\theta \\ &- 2\mathrm{Re}\Big((\epsilon_1^- - \epsilon_2^+ - \psi_2(\epsilon_1^-, \epsilon_2^+))\left(1 - \epsilon_1^{-*}\epsilon_2^{+*}\right) A^p A^{m*}\Big)\cos\theta\Bigg),\end{aligned} \tag{4.53}$$

$$\begin{aligned}\left|\mathcal{A}^{--}\right|^2 = \frac{\left|A_1^-\right|^2\left|A_2^-\right|^2}{\Gamma^2}\Bigg(&\left|A^p\right|^2\left(\left|1 - \epsilon_1^-\epsilon_2^-\right|^2 + \phi_1(\epsilon_1^-, \epsilon_2^-)\right) + \left|A^m\right|^2\left|\epsilon_1^- - \epsilon_2^-\right|^2\cos^2\theta \\ &- 2\mathrm{Re}\Big(\left(1 - \epsilon_1^-\epsilon_2^- + \psi_1(\epsilon_1^-, \epsilon_2^-)\right)\left(\epsilon_1^{-*} - \epsilon_2^{-*}\right) A^p A^{m*}\Big)\cos\theta\Bigg).\end{aligned} \tag{4.54}$$

We wish to write down everything in terms of branching ratio of $P^0 \to f_i^{s_i}$:

$$\mathrm{Br}_i^{s_i} \equiv \mathrm{Br}(P^0 \to f_i^{s_i}) \propto \frac{\left|A_i^{s_i}\right|^2}{\Gamma} \left|1 + \epsilon_i^{s_i}\right|^2 , \tag{4.55}$$

which stem from the fact that

$$\mathrm{Amp}(P^0 \to f_i^{s_i}) = (1 + \epsilon_i^{s_i}) A_i^{s_i} . \tag{4.56}$$

Since our expressions already include all the phase-space integrations, we can simply write down the differential decay rates as follows

$$\begin{aligned} D^{s_1 s_2} \equiv \frac{d\Gamma\left(X \to Y \left(f_1^{s_1}\right)_P \left(f_2^{s_2}\right)_P\right)}{dt\, du} &= \frac{1}{(2\pi)^3} \frac{1}{32 m_X^3} \left|\mathrm{Amp}\left(X \to Y \left(f_1^{s_1}\right)_P \left(f_2^{s_2}\right)_P\right)\right|^2 \\ &= \frac{1}{(2\pi)^3} \frac{1}{32 m_X^3} \left|\mathcal{A}^{s_1 s_2}\right|^2 , \end{aligned} \tag{4.57}$$

where in the expression for $|\mathcal{A}^{s_1 s_2}|^2$ the term $\dfrac{\left|A_1^{s_1}\right|^2 \left|A_2^{s_2}\right|^2}{\Gamma^2}$ is replaced by $\dfrac{\mathrm{Br}_1^{s_1}}{\left|1 + \epsilon_1^{s_1}\right|^2} \dfrac{\mathrm{Br}_2^{s_2}}{\left|1 + \epsilon_2^{s_2}\right|^2}$. Thus

$$\begin{aligned} D^{++} = & \frac{1}{256 \pi^3 m_X^3} \frac{\mathrm{Br}_1^+}{\left|1 + \epsilon_1^+\right|^2} \frac{\mathrm{Br}_2^+}{\left|1 + \epsilon_2^+\right|^2} \\ & \times \Bigg(\left|A^p\right|^2 \left(\left|1 - \epsilon_1^+ \epsilon_2^+\right|^2 + \phi_1(\epsilon_1^+, \epsilon_2^+) \right) + \left|A^m\right|^2 \left|\epsilon_1^+ - \epsilon_2^+\right|^2 \cos^2\theta \\ & \qquad - 2\mathrm{Re}\Big(\left(1 - \epsilon_1^+ \epsilon_2^+ + \psi_1(\epsilon_1^+, \epsilon_2^+)\right) \left(\epsilon_1^{+*} - \epsilon_2^{+*}\right) A^p A^{m*} \Big) \cos\theta \Bigg), \end{aligned} \tag{4.58}$$

$$\begin{aligned} D^{+-} = & \frac{1}{256 \pi^3 m_X^3} \frac{\mathrm{Br}_1^+}{\left|1 + \epsilon_1^+\right|^2} \frac{\mathrm{Br}_2^-}{\left|1 + \epsilon_2^-\right|^2} \\ & \times \Bigg(\left|A^p\right|^2 \left(\left|\epsilon_1^+ - \epsilon_2^-\right|^2 - \phi_2(\epsilon_1^+, \epsilon_2^-) \right) + \left|A^m\right|^2 \left|1 - \epsilon_1^+ \epsilon_2^-\right|^2 \cos^2\theta \\ & \qquad - 2\mathrm{Re}\Big((\epsilon_1^+ - \epsilon_2^- - \psi_2(\epsilon_1^+, \epsilon_2^-)) \left(1 - \epsilon_1^{+*} \epsilon_2^{-*}\right) A^p A^{m*} \Big) \cos\theta \Bigg), \end{aligned} \tag{4.59}$$

$$
\begin{aligned}
D^{-+} = {} & \frac{1}{256\pi^3 m_X^3} \frac{\mathrm{Br}_1^-}{\left|1+\epsilon_1^-\right|^2} \frac{\mathrm{Br}_2^+}{\left|1+\epsilon_2^+\right|^2} \\
& \times \Bigg(\left|A^p\right|^2 \left(\left|\epsilon_1^- - \epsilon_2^+\right|^2 - \phi_2(\epsilon_1^-, \epsilon_2^+) \right) + \left|A^m\right|^2 \left|1 - \epsilon_1^- \epsilon_2^+\right|^2 \cos^2\theta \\
& \qquad - 2\mathrm{Re}\Big((\epsilon_1^- - \epsilon_2^+ - \psi_2(\epsilon_1^-, \epsilon_2^+)) \left(1 - \epsilon_1^{-*} \epsilon_2^{+*}\right) A^p A^{m*} \Big) \cos\theta \Bigg), \qquad (4.60)
\end{aligned}
$$

$$
\begin{aligned}
D^{--} = {} & \frac{1}{256\pi^3 m_X^3} \frac{\mathrm{Br}_1^-}{\left|1+\epsilon_1^-\right|^2} \frac{\mathrm{Br}_2^-}{\left|1+\epsilon_2^-\right|^2} \\
& \times \Bigg(\left|A^p\right|^2 \left(\left|1 - \epsilon_1^- \epsilon_2^-\right|^2 + \phi_1(\epsilon_1^-, \epsilon_2^-) \right) + \left|A^m\right|^2 \left|\epsilon_1^- - \epsilon_2^-\right|^2 \cos^2\theta \\
& \qquad - 2\mathrm{Re}\Big(\left(1 - \epsilon_1^- \epsilon_2^- + \psi_1(\epsilon_1^-, \epsilon_2^-)\right) \left(\epsilon_1^{-*} - \epsilon_2^{-*}\right) A^p A^{m*} \Big) \cos\theta \Bigg). \qquad (4.61)
\end{aligned}
$$

The Dalitz plot density corresponding to each of the decays would be given by the appropriate $D^{s_1 s_2}$ which are defined above. It is important to notice that, if there is any direct *CP* violation, i.e. $\epsilon_i^{s_i} \neq 0$, then there is an asymmetry in the t vs. u Dalitz plot under the exchange $t \leftrightarrow u \equiv \cos\theta \leftrightarrow -\cos\theta$.

Special Cases

If we reconstruct the two neutral P particles from identical final states, then $f_1^{s_1} = f_2^{s_2}$, which implies that $\epsilon_1^{s_1} = \epsilon_2^{s_2} = \epsilon^s$ (say). Therefore in such a case the Dalitz plot asymmetry for D^{++} and D^{--} become identically zero, and we have

$$
D^{++} = \frac{1}{256\pi^3 m_X^3} \left(\frac{\mathrm{Br}^+}{|1+\epsilon^+|^2} \right)^2 \left(\left|1 - (\epsilon^+)^2\right|^2 + \phi_1(\epsilon^+, \epsilon^+) \right) \left|A^p\right|^2, \qquad (4.62)
$$

$$
D^{--} = \frac{1}{256\pi^3 m_X^3} \left(\frac{\mathrm{Br}^-}{|1+\epsilon^-|^2} \right)^2 \left(\left|1 - (\epsilon^-)^2\right|^2 + \phi_1(\epsilon^-, \epsilon^-) \right) \left|A^p\right|^2, \qquad (4.63)
$$

where

$$
\begin{aligned}
\phi_1(\epsilon^s, \epsilon^s) &= \frac{4}{\Gamma}\mathrm{Re}\left(i\Delta\mu\, f_1(\epsilon^s, \epsilon^s)(1-(\epsilon^s)^2)\right) \\
&= \frac{4}{\Gamma}\mathrm{Re}\Bigg(i\Delta\mu\,\sqrt{1-z^2}\left(1-(\epsilon^s)^2\right) \\
&\quad \times \left(\left(\frac{p^2+q^2}{2pq}\right)\left(1+(\epsilon^s)^2\right)+\left(\frac{p^2-q^2}{2pq}\right)(2\epsilon^s)\right)\Bigg). \qquad (4.64)
\end{aligned}
$$

If *CPT* is conserved, then $z = 0$, but this hardly affects anything here. In fact, if we consider both *CP* and *CPT* violation, then it is impossible to distinguish them from the asymmetry of the Dalitz plot. However, if we assume that *CPT* is conserved in the process, then the asymmetry in the Dalitz plot can be attributed to be arising from pure *CP* violation only.

It is interesting to note that, even if we were to assume that the two neutral *P* particles have identically the same mass and decay width, i.e. $\Delta\mu = 0$, then also the asymmetry in the Dalitz plot survives. Assuming that there is neither any *CPT* violation, nor any mass or width difference between the two neutral *P* particles, we get the following expressions for the Dalitz distributions:

$$
\begin{aligned}
D^{++} = \frac{1}{256\pi^3\, m_X^3}\frac{\mathrm{Br}_1^+}{\left|1+\epsilon_1^+\right|^2}\frac{\mathrm{Br}_2^+}{\left|1+\epsilon_2^+\right|^2}\Bigg(&\left|A^p\right|^2\left|1-\epsilon_1^+\epsilon_2^+\right|^2+\left|A^m\right|^2\left|\epsilon_1^+-\epsilon_2^+\right|^2\cos^2\theta \\
&-2\mathrm{Re}\Big(\left(1-\epsilon_1^+\epsilon_2^+\right)\left(\epsilon_1^{+*}-\epsilon_2^{+*}\right)A^pA^{m*}\Big)\cos\theta\Bigg),
\end{aligned} \qquad (4.65a)
$$

$$
\begin{aligned}
D^{+-} = \frac{1}{256\pi^3\, m_X^3}\frac{\mathrm{Br}_1^+}{\left|1+\epsilon_1^+\right|^2}\frac{\mathrm{Br}_2^-}{\left|1+\epsilon_2^-\right|^2}\Bigg(&\left|A^p\right|^2\left|\epsilon_1^+-\epsilon_2^-\right|^2+\left|A^m\right|^2\left|1-\epsilon_1^+\epsilon_2^-\right|^2\cos^2\theta \\
&-2\mathrm{Re}\Big((\epsilon_1^+-\epsilon_2^-)\left(1-\epsilon_1^{+*}\epsilon_2^{-*}\right)A^pA^{m*}\Big)\cos\theta\Bigg),
\end{aligned} \qquad (4.65b)
$$

$$D^{-+} = \frac{1}{256\,\pi^3\,m_X^3}\frac{\mathrm{Br}_1^-}{\left|1+\epsilon_1^-\right|^2}\frac{\mathrm{Br}_2^+}{\left|1+\epsilon_2^+\right|^2}\Bigg(\left|A^p\right|^2\left|\epsilon_1^- - \epsilon_2^+\right|^2 + \left|A^m\right|^2\left|1-\epsilon_1^-\epsilon_2^+\right|^2\cos^2\theta$$
$$- 2\mathrm{Re}\Big((\epsilon_1^- - \epsilon_2^+)\left(1-\epsilon_1^{-*}\epsilon_2^{+*}\right)A^p A^{m*}\Big)\cos\theta\Bigg), \tag{4.65c}$$

$$D^{--} = \frac{1}{256\,\pi^3\,m_X^3}\frac{\mathrm{Br}_1^-}{\left|1+\epsilon_1^-\right|^2}\frac{\mathrm{Br}_2^-}{\left|1+\epsilon_2^-\right|^2}\Bigg(\left|A^p\right|^2\left|1-\epsilon_1^-\epsilon_2^-\right|^2 + \left|A^m\right|^2\left|\epsilon_1^- - \epsilon_2^-\right|^2\cos^2\theta$$
$$- 2\mathrm{Re}\Big(\left(1-\epsilon_1^-\epsilon_2^-\right)\left(\epsilon_1^{-*} - \epsilon_2^{-*}\right)A^p A^{m*}\Big)\cos\theta\Bigg). \tag{4.65d}$$

Measuring the *CP* asymmetry

We can neglect the $\epsilon_i^{s_i}$'s with respect to unity and doing so the Dalitz distributions can be written as

$$D^{++} = \frac{\mathrm{Br}_1^+\mathrm{Br}_2^+}{256\,\pi^3\,m_X^3}\Bigg(\left|A^p\right|^2 + \left|A^m\right|^2\left|\epsilon_1^+ - \epsilon_2^+\right|^2\cos^2\theta - 2\mathrm{Re}\Big(\left(\epsilon_1^{+*} - \epsilon_2^{+*}\right)A^p A^{m*}\Big)\cos\theta\Bigg), \tag{4.66a}$$

$$D^{+-} = \frac{\mathrm{Br}_1^+\mathrm{Br}_2^-}{256\,\pi^3\,m_X^3}\Bigg(\left|A^p\right|^2\left|\epsilon_1^+ - \epsilon_2^-\right|^2 + \left|A^m\right|^2\cos^2\theta - 2\mathrm{Re}\Big(\left(\epsilon_1^+ - \epsilon_2^-\right)A^p A^{m*}\Big)\cos\theta\Bigg), \tag{4.66b}$$

$$D^{-+} = \frac{\mathrm{Br}_1^-\mathrm{Br}_2^+}{256\,\pi^3\,m_X^3}\Bigg(\left|A^p\right|^2\left|\epsilon_1^- - \epsilon_2^+\right|^2 + \left|A^m\right|^2\cos^2\theta - 2\mathrm{Re}\Big(\left(\epsilon_1^- - \epsilon_2^+\right)A^p A^{m*}\Big)\cos\theta\Bigg), \tag{4.66c}$$

$$D^{--} = \frac{\mathrm{Br}_1^-\mathrm{Br}_2^-}{256\,\pi^3\,m_X^3}\Bigg(\left|A^p\right|^2 + \left|A^m\right|^2\left|\epsilon_1^- - \epsilon_2^-\right|^2\cos^2\theta - 2\mathrm{Re}\Big(\left(\epsilon_1^{-*} - \epsilon_2^{-*}\right)A^p A^{m*}\Big)\cos\theta\Bigg). \tag{4.66d}$$

We can split each distribution into two parts: 'even' E and 'odd' O as follows:

$$D^{s_1 s_2} \equiv E^{s_1 s_2} - O^{s_1 s_2}. \tag{4.67}$$

The special case, when $f_1^{s_1} = f_2^{s_2}$, we denote the branching ratio by $\mathrm{Br}_{\mathrm{sym}}$ and the distributions is fully 'even' and is denoted by E_{sym}:

$$E_{\mathrm{sym}} = \frac{\mathrm{Br}_{\mathrm{sym}}^2}{256\,\pi^3\, m_X^3} \left|A^p\right|^2 . \tag{4.68}$$

Therefore

$$\left|A^p\right| = \sqrt{256\,\pi^3\, m_X^3 \left(\frac{E_{\mathrm{sym}}}{\mathrm{Br}_{\mathrm{sym}}^2}\right)}. \tag{4.69}$$

The 'even' terms E^{+-} or E^{-+} are predominantly proportional to $|A^m|^2$:

$$E^{+-} = \frac{\mathrm{Br}_1^+ \mathrm{Br}_2^-}{256\,\pi^3\, m_X^3} \left(\left|A^p\right|^2 \left|\epsilon_1^+ - \epsilon_2^-\right|^2 + \left|A^m\right|^2 \cos^2\theta\right) \approx \frac{\mathrm{Br}_1^+ \mathrm{Br}_2^-}{256\,\pi^3\, m_X^3} \left|A^m\right|^2 \cos^2\theta. \tag{4.70}$$

This implies that

$$\left|A^m\right| \cos\theta = \sqrt{256\,\pi^3\, m_X^3 \left(\frac{E^{+-}}{\mathrm{Br}_1^+ \mathrm{Br}_2^-}\right)} = \sqrt{256\,\pi^3\, m_X^3 \left(\frac{E^{-+}}{\mathrm{Br}_1^- \mathrm{Br}_2^+}\right)}. \tag{4.71}$$

Now let us assume that

$$\epsilon_1^{s_1} - \epsilon_2^{s_2} = \left|\epsilon_1^{s_1} - \epsilon_2^{s_2}\right| e^{i\delta^{s_1 s_2}} \tag{4.72}$$

and

$$A^p A^{m*} = \left|A^p\right| \left|A^m\right| e^{i\varphi}, \tag{4.73}$$

where $\delta^{s_1 s_2}$ and φ are some phases. Therefore we can write the 'odd' parts of the differential decay rates as follows:

$$\begin{aligned} O^{++} &= \frac{\mathrm{Br}_1^+ \mathrm{Br}_2^+}{128\,\pi^3\, m_X^3} \,\mathrm{Re}\left(\left(\epsilon_1^+ - \epsilon_2^+\right)^* A^p A^{m*}\right) \cos\theta \\ &= \frac{\mathrm{Br}_1^+ \mathrm{Br}_2^+}{128\,\pi^3\, m_X^3} \left|\epsilon_1^+ - \epsilon_2^+\right| \left|A^p\right| \left|A^m\right| \cos(\varphi - \delta^{++}) \cos\theta, \end{aligned} \tag{4.74}$$

$$\begin{aligned}
O^{+-} &= \frac{\mathrm{Br}_1^+\mathrm{Br}_2^-}{128\,\pi^3\,m_X^3}\,\mathrm{Re}\left(\left(\epsilon_1^+ - \epsilon_2^-\right) A^p A^{m*}\right)\cos\theta \\
&= \frac{\mathrm{Br}_1^+\mathrm{Br}_2^-}{128\,\pi^3\,m_X^3}\,\left|\epsilon_1^+ - \epsilon_2^-\right|\left|A^p\right|\left|A^m\right|\cos(\varphi + \delta^{+-})\cos\theta, \qquad (4.75) \\
O^{-+} &= \frac{\mathrm{Br}_1^-\mathrm{Br}_2^+}{128\,\pi^3\,m_X^3}\,\mathrm{Re}\left(\left(\epsilon_1^- - \epsilon_2^+\right) A^p A^{m*}\right)\cos\theta \\
&= \frac{\mathrm{Br}_1^-\mathrm{Br}_2^+}{128\,\pi^3\,m_X^3}\,\left|\epsilon_1^- - \epsilon_2^+\right|\left|A^p\right|\left|A^m\right|\cos(\varphi + \delta^{-+})\cos\theta, \qquad (4.76) \\
O^{--} &= \frac{\mathrm{Br}_1^-\mathrm{Br}_2^-}{128\,\pi^3\,m_X^3}\,\mathrm{Re}\left(\left(\epsilon_1^- - \epsilon_2^-\right)^* A^p A^{m*}\right)\cos\theta \\
&= \frac{\mathrm{Br}_1^-\mathrm{Br}_2^-}{128\,\pi^3\,m_X^3}\,\left|\epsilon_1^- - \epsilon_2^-\right|\left|A^p\right|\left|A^m\right|\cos(\varphi - \delta^{--})\cos\theta. \qquad (4.77)
\end{aligned}$$

Now substituting expressions for $|A^p|$ and $|A^m|\,\cos\theta$ from Eqs. (4.69) and (4.71) respectively, we get

$$O^{++} = 2\mathrm{Br}_1^+\mathrm{Br}_2^+\,\left|\epsilon_1^+ - \epsilon_2^+\right|\sqrt{\frac{E_{\mathrm{sym}}\,E^{\pm\mp}}{\mathrm{Br}_{\mathrm{sym}}^2\,\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}\,\cos(\varphi - \delta^{++}), \qquad (4.78)$$

$$O^{+-} = 2\mathrm{Br}_1^+\mathrm{Br}_2^-\,\left|\epsilon_1^+ - \epsilon_2^-\right|\sqrt{\frac{E_{\mathrm{sym}}\,E^{\pm\mp}}{\mathrm{Br}_{\mathrm{sym}}^2\,\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}\,\cos(\varphi + \delta^{+-}), \qquad (4.79)$$

$$O^{-+} = 2\mathrm{Br}_1^+\mathrm{Br}_2^+\,\left|\epsilon_1^- - \epsilon_2^+\right|\sqrt{\frac{E_{\mathrm{sym}}\,E^{\pm\mp}}{\mathrm{Br}_{\mathrm{sym}}^2\,\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}\,\cos(\varphi + \delta^{-+}), \qquad (4.80)$$

$$O^{--} = 2\mathrm{Br}_1^+\mathrm{Br}_2^+\,\left|\epsilon_1^- - \epsilon_2^-\right|\sqrt{\frac{E_{\mathrm{sym}}\,E^{\pm\mp}}{\mathrm{Br}_{\mathrm{sym}}^2\,\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}\,\cos(\varphi - \delta^{--}). \qquad (4.81)$$

These equations can be inverted to measure the difference between the *CP* asymmetries as follows

$$\left|\epsilon_1^+ - \epsilon_2^+\right| = \frac{O^{++}}{2\mathrm{Br}_1^+\mathrm{Br}_2^+}\frac{\mathrm{Br}_{\mathrm{sym}}\sqrt{\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}{\sqrt{E_{\mathrm{sym}}\,E^{\pm\mp}}}\frac{1}{\cos(\varphi - \delta^{++})}, \qquad (4.82)$$

$$\left|\epsilon_1^+ - \epsilon_2^-\right| = \frac{O^{+-}}{2\mathrm{Br}_1^+\mathrm{Br}_2^-}\frac{\mathrm{Br}_{\mathrm{sym}}\sqrt{\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}{\sqrt{E_{\mathrm{sym}}\,E^{\pm\mp}}}\frac{1}{\cos(\varphi + \delta^{+-})}, \qquad (4.83)$$

$$\left|\epsilon_1^- - \epsilon_2^+\right| = \frac{O^{-+}}{2\mathrm{Br}_1^-\mathrm{Br}_2^+}\frac{\mathrm{Br}_{\mathrm{sym}}\sqrt{\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}{\sqrt{E_{\mathrm{sym}}\,E^{\pm\mp}}}\frac{1}{\cos(\varphi+\delta^{-+})}, \tag{4.84}$$

$$\left|\epsilon_1^- - \epsilon_2^-\right| = \frac{O^{--}}{2\mathrm{Br}_1^-\mathrm{Br}_2^-}\frac{\mathrm{Br}_{\mathrm{sym}}\sqrt{\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}{\sqrt{E_{\mathrm{sym}}\,E^{\pm\mp}}}\frac{1}{\cos(\varphi-\delta^{--})}. \tag{4.85}$$

These expressions enable us to put the following lower limits which can be experimentally observed

$$\left|\epsilon_1^+ - \epsilon_2^+\right| \geqslant \frac{O^{++}}{2\mathrm{Br}_1^+\mathrm{Br}_2^+}\frac{\mathrm{Br}_{\mathrm{sym}}\sqrt{\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}{\sqrt{E_{\mathrm{sym}}\,E^{\pm\mp}}}, \tag{4.86}$$

$$\left|\epsilon_1^+ - \epsilon_2^-\right| \geqslant \frac{O^{+-}}{2\mathrm{Br}_1^+\mathrm{Br}_2^-}\frac{\mathrm{Br}_{\mathrm{sym}}\sqrt{\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}{\sqrt{E_{\mathrm{sym}}\,E^{\pm\mp}}}, \tag{4.87}$$

$$\left|\epsilon_1^- - \epsilon_2^+\right| \geqslant \frac{O^{-+}}{2\mathrm{Br}_1^-\mathrm{Br}_2^+}\frac{\mathrm{Br}_{\mathrm{sym}}\sqrt{\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}{\sqrt{E_{\mathrm{sym}}\,E^{\pm\mp}}}, \tag{4.88}$$

$$\left|\epsilon_1^- - \epsilon_2^-\right| \geqslant \frac{O^{--}}{2\mathrm{Br}_1^-\mathrm{Br}_2^-}\frac{\mathrm{Br}_{\mathrm{sym}}\sqrt{\mathrm{Br}_1^\pm\mathrm{Br}_2^\mp}}{\sqrt{E_{\mathrm{sym}}\,E^{\pm\mp}}}. \tag{4.89}$$

Physical meaning of the *CP* asymmetries

It is easy to relate the *CP* asymmetries we have just obtained to the known *CP* asymmetries, called as A_{CP}. To derive these relations we first note that

$$\mathrm{Amp}(P^0 \to f_i^+) = \frac{1+\epsilon_i^+}{\sqrt{2}}A_i^+, \tag{4.90}$$

$$\mathrm{Amp}(P^0 \to f_i^-) = \frac{1+\epsilon_i^-}{\sqrt{2}}A_i^-, \tag{4.91}$$

$$\mathrm{Amp}(\bar{P}^0 \to f_i^+) = \frac{1-\epsilon_i^+}{\sqrt{2}}A_i^+, \tag{4.92}$$

$$\mathrm{Amp}(\bar{P}^0 \to f_i^-) = -\frac{1-\epsilon_i^-}{\sqrt{2}}A_i^-. \tag{4.93}$$

The usual *CP* asymmetry, the A_{CP} is defined as

$$A_{CP} = \frac{\left|A_f\right|^2 - \left|\bar{A}_f\right|^2}{\left|A_f\right|^2 + \left|\bar{A}_f\right|^2}, \tag{4.94}$$

where $A_f = \mathrm{Amp}(P^0 \to f)$ and $\bar{A}_f = \mathrm{Amp}(\bar{P}^0 \to f)$. In our case we shall denote the A_{CP} with a superscript $\pm$ and subscript i for obvious reasons. Thus we have

$$A^{+}_{CP,i} = \frac{2\,\mathrm{Re}(\epsilon_i^+)}{1 + \left|\epsilon_i^+\right|^2} \approx 2\,\mathrm{Re}(\epsilon_i^+), \tag{4.95}$$

$$A^{-}_{CP,i} = \frac{2\,\mathrm{Re}(\epsilon_i^-)}{1 + \left|\epsilon_i^-\right|^2} \approx 2\,\mathrm{Re}(\epsilon_i^-). \tag{4.96}$$

We can now define the difference between the two asymmetries as

$$\Delta A^{s_1 s_2}_{CP,12} = A^{s_1}_{CP,1} - A^{s_2}_{CP,2} \approx 2\,\mathrm{Re}\left(\epsilon_1^{s_1} - \epsilon_2^{s_2}\right) \leqslant 2\left|\epsilon_1^{s_1} - \epsilon_2^{s_2}\right|. \tag{4.97}$$

Therefore

$$\left|\epsilon_1^{s_1} - \epsilon_2^{s_2}\right| \geqslant \frac{1}{2}\,\Delta A^{s_1 s_2}_{CP,12}. \tag{4.98}$$

Thus we find that it is possible to look for direct CP violation, by analysing the Dalitz plot. However, the CPT violation in mixing can not be separated from CP violation from the Dalitz plot asymmetry. Thus assuming that CPT is conserved, we can associate the asymmetry in the Dalitz plot with CP violation. The methodology developed here can be applied to look for CP violation in the D meson sector, by looking at the three-body processes with $D^0\overline{D}^0P$ in the final state, where P can be a pion or a kaon or any other particle other than D^0 and $\overline{D}^0$. The D^0 and $\overline{D}^0$ are to be reconstructed from final states of definite CP, such as from CP-even final states like K^+K^- or $\pi^+\pi^-$ etc., or from CP-odd final states like $K_S^0\pi^0$, $K_S^0\omega$, $K_S^0\phi$ etc. If we want to measure the $\Delta A_{CP} = A_{CP}(K^+K^-) - A_{CP}(\pi^+\pi^-)$, then one neutral D meson should be reconstructed from K^+K^-

and the other from $\pi^+\pi^-$. Then the Dalitz plot and the Dalitz prism would explicitly exhibit an asymmetry under exchange of D^0 and $\overline{D}^0$ and this would lead to a quantitative estimation of ΔA_{CP}. It is interesting to note that signatures for CP violation in B decays are known to exist in the Dalitz plots for untagged B decays [159, 160]. However, these signatures do not rely on Bose symmetry and are unrelated to the approach discussed here. Yet another unrelated approach [161] has previously used the Dalitz plot of D decay to observe evidence of CP violation in a B decay (e.g., $B^+ \to DK^+$, $D \to K\pi\pi$). The approach as given in this thesis is fundamentally different from these other works.

4.3 Direct CPT violation

In this section we shall discuss the signatures of direct CPT violation in a Dalitz plot and a Dalitz prism. For this we consider only those three-body processes in which the process and its CP conjugate process are the same: $X \to N\,M\,\bar{M}$, with $X \equiv \bar{X}$ and $N \equiv \bar{N}$. If $\mathscr{H}$ is the Hamiltonian governing the transition, then the amplitude is given by

$$A = \langle N(p_1)\,M(p_2)\,\bar{M}(p_3)\big|\mathscr{H}\big|X(p)\rangle. \tag{4.99}$$

The amplitude for the CP conjugate process is given by

$$\bar{A} = \langle N(p_1)\,\bar{M}(p_2)\,M(p_3)\big|\mathscr{H}\big|X(p)\rangle. \tag{4.100}$$

In the Dalitz plot the amplitude that contributes is the sum of the above two amplitudes:

$$\mathcal{A}(r,\theta) = A(r,\theta) + \bar{A}(r,-\theta), \tag{4.101}$$

where r and θ are the polar coordinates of the event point in the Dalitz plot distribution (as shown in Fig. 4.3). Here $M \leftrightarrow \bar{M}$ implies $\theta \leftrightarrow -\theta$.

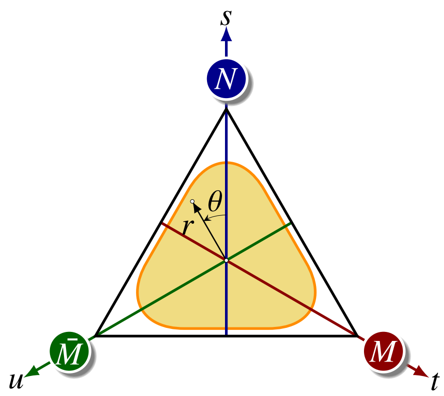


Figure 4.3: (Schematic) Dalitz plot for $X \to NM\bar{M}$. The angle θ is measured from the s-axis in anticlockwise direction. The Mandelstam-like variables s, t, u are defined in the same manner as in Eq. (4.15).

Let us now observe how CP and CPT relate the amplitudes A and $\bar{A}$. For this, let us define the CP and CPT operators as $\boldsymbol{\Theta}$ and Θ respectively. Now if both CP and CPT are conserved in the process, the Hamiltonian $\mathscr{H}$ commutes with both $\boldsymbol{\Theta}$ and Θ. Considering CP invariance we have

$$
\begin{aligned}
A(r,\theta) &= \left\langle N(p_1)M(p_2)\bar{M}(p_3)\middle|\mathscr{H}\middle|X(p)\right\rangle = \left\langle N(p_1)M(p_2)\bar{M}(p_3)\middle|\boldsymbol{\Theta}^{-1}\,\boldsymbol{\Theta}\mathscr{H}\boldsymbol{\Theta}^{-1}\,\boldsymbol{\Theta}\middle|X(p)\right\rangle \\
&= \eta_{CP}\left\langle \bar{N}(E_1,-\vec{p}_1)\bar{M}(E_2,-\vec{p}_2)M(E_3,-\vec{p}_3)\middle|\mathscr{H}\middle|\bar{X}(E,-\vec{p})\right\rangle \\
&= \eta_{CP}\left\langle N(E_1,-\vec{p}_1)\bar{M}(E_2,-\vec{p}_2)M(E_3,-\vec{p}_3)\middle|\mathscr{H}\middle|X(E,-\vec{p})\right\rangle \qquad \left(\because\ \bar{X}\equiv X, \bar{N}\equiv N\right) \\
&= \eta_{CP}\,\bar{A}(r,-\theta),
\end{aligned}
$$

where $\eta_{CP} = \pm 1$ is the product of CP of the initial and final state. When CP is conserved,

$\eta_{CP} = 1$. Considering CPT invariance we get

$$\begin{aligned}
A(r,\theta) &= \Big\langle N(p_1)M(p_2)\bar{M}(p_3)\Big|\mathscr{H}\Big|X(p)\Big\rangle = \Big\langle N(p_1)M(p_2)\bar{M}(p_3)\Big|\Theta^{-1}\,\Theta\mathscr{H}\Theta^{-1}\,\Theta\Big|X(p)\Big\rangle \\
&= \eta_{CP}\Big\langle \bar{N}(p_1)\bar{M}(p_2)M(p_3)\Big|\mathscr{H}\Big|\bar{X}(p)\Big\rangle^* \\
&= \eta_{CP}\Big\langle N(p_1)\bar{M}(p_2)M(p_3)\Big|\mathscr{H}\Big|X(p)\Big\rangle^* \;\; \Big(\because\ \bar{X} \equiv X, \bar{N} \equiv N\Big) \\
&= \eta_{CP}\,\bar{A}^*(r,-\theta).
\end{aligned} \tag{4.102}$$

If, however, CPT is not conserved in the decay, then the Hamiltonian $\mathscr{H}$ does not commute with Θ. In such a case, the two amplitudes A and $\bar{A}^*$ are not related, but $\bar{A}^*$ is given by

$$\bar{A}^*(r,-\theta) = \Big\langle N(p_1)M(p_2)\bar{M}(p_3)\Big|\overline{\mathscr{H}}\Big|X(p)\Big\rangle^*, \tag{4.103}$$

where $\overline{\mathscr{H}}$ is the Hamiltonian for the CP conjugate process. Only when CPT is conserved, do we have $\overline{\mathscr{H}} = \mathscr{H}$.

4.3.1 Expression for Dalitz distribution (no CPT violation).

The amplitude for the process $X \to N\,M\,\bar{M}$ can be Fourier decomposed as

$$A(r,\theta) = \sum_{n=0}^{\infty}\Big(s_n\,\sin(n\theta) + c_n\,\cos(n\theta)\Big), \tag{4.104}$$

where s_n and c_n are complex Fourier coefficients. These Fourier coefficients can be given in terms of the weak phase (ϕ) and the strong phases (δ_n^s, δ_n^c) as follows

$$s_n = |s_n|\,e^{i\phi}e^{i\delta_n^s}, \qquad c_n = |c_n|\,e^{i\phi}e^{i\delta_n^c}. \tag{4.105}$$

Therefore, the Fourier decomposition of the CP conjugate amplitude (assuming no CPT violation) is given by

$$\bar{A}(r,\theta) = \sum_{n=0}^{\infty} \Big(\bar{s}_n\ \sin(n\theta) + \bar{c}_n\ \cos(n\theta)\Big), \tag{4.106}$$

where

$$\bar{s}_n = |s_n|\, e^{-i\phi} e^{i\delta^s_n}, \qquad \bar{c}_n = |c_n|\, e^{-i\phi} e^{i\delta^c_n}. \tag{4.107}$$

Note that the weak phase leads to CP violation in the decay mode under consideration. This will be demonstrated below mathematically.

Since we cannot distinguish the process and its CP conjugate process, the amplitude that contributes to the Dalitz plot density is given by

$$\begin{aligned}
\mathcal{A}(r,\theta) = A(r,\theta) + \bar{A}(r,-\theta) &= \sum_{n=0}^{\infty} \Big((s_n - \bar{s}_n)\sin(n\theta) + (c_n + \bar{c}_n)\cos(n\theta)\Big) \\
&= 2\sum_{n=0}^{\infty} \Big(i\,|s_n| \sin\phi\ \sin(n\theta)\, e^{i\delta^s_n} + |c_n| \cos\phi\ \cos(n\theta) e^{i\delta^c_n}\Big) \\
&= \mathcal{A}^S(r,\theta) + \mathcal{A}^N(r,\theta),
\end{aligned} \tag{4.108}$$

where

$$\mathcal{A}^S(r,\theta) = 2\sum_{n=0}^{\infty} |c_n| \cos\phi\ \cos(n\theta)\, e^{i\delta^c_n}, \tag{4.109}$$

$$\mathcal{A}^N(r,\theta) = 2\sum_{n=0}^{\infty} i\,|s_n| \sin\phi\ \sin(n\theta)\, e^{i\delta^s_n}. \tag{4.110}$$

It is easy to observe that $\mathcal{A}^S(r,-\theta) = \mathcal{A}^S(r,\theta)$ and $\mathcal{A}^N(r,-\theta) = -\mathcal{A}^N(r,\theta)$. The Dalitz plot distribution is proportional to $|\mathcal{A}(r,\theta)|^2$:

$$|\mathcal{A}(r,\theta)|^2 = \left|\mathcal{A}^S(r,\theta)\right|^2 + \left|\mathcal{A}^N(r,\theta)\right|^2 + 2\,\mathrm{Re}\left(\mathcal{A}^S(r,\theta)\cdot \mathcal{A}^{N*}(r,\theta)\right). \tag{4.111}$$

It is only the interference term which can give rise to any asymmetry in the Dalitz plot under $\theta \leftrightarrow -\theta$:

$$\begin{aligned}
&2\,\mathrm{Re}\left(\mathcal{A}^S(r,\theta)\cdot\mathcal{A}^{N*}(r,\theta)\right)\\
&= 4\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|c_n|\,|s_m|\sin 2\phi\ \sin\left(\delta_n^c-\delta_m^s\right)\ \cos(n\theta)\ \sin(m\theta)\\
&\neq 0. \quad \text{(unless } \phi = 0\text{)}
\end{aligned} \tag{4.112}$$

It is easy to see that the interference term would vanish if and only if there is no weak phase or equivalently if there is no CP violation in the process. Therefore, assuming CPT invariance, any asymmetry in the Dalitz plot under $\theta \leftrightarrow -\theta \equiv t \leftrightarrow u$ would be due to CP violation only.

4.3.2 Expression for Dalitz distribution (with CPT violation).

The CPT violating amplitude for the process $X \to N\,M\,\bar{M}$ can be Fourier decomposed as

$$A(r,\theta) = \sum_{n=0}^{\infty}\left(s_n\ \sin(n\theta) + c_n\ \cos(n\theta)\right), \tag{4.113}$$

where s_n and c_n are complex Fourier coefficients given in terms of the weak phase ϕ and the strong phase δ as follows

$$s_n = \left(|s_n| + \epsilon_n^s\right) e^{i\phi} e^{i\delta_n^s}, \qquad c_n = \left(|c_n| + \epsilon_n^c\right) e^{i\phi} e^{i\delta_n^c}, \tag{4.114}$$

where ϵ_n^s and ϵ_n^c are CPT violating terms, i.e. for the case of CPT invariance they vanish identically. The Fourier decomposition of the CP conjugate amplitude is given by

$$\bar{A}(r,\theta) = \sum_{n=0}^{\infty}\left(\bar{s}_n\ \sin(n\theta) + \bar{c}_n\ \cos(n\theta)\right), \tag{4.115}$$

where

$$\bar{s}_n = (|s_n| - \epsilon_n^s)\, e^{-i\phi} e^{i\delta_n^s}, \qquad \bar{c}_n = (|c_n| - \epsilon_n^c)\, e^{-i\phi} e^{i\delta_n^c}. \tag{4.116}$$

Thus we now have both CP and CPT violating terms in our amplitudes.

Again since we cannot distinguish the process and its CP conjugate process, the amplitude that contributes to the Dalitz plot density is given by

$$\begin{aligned}
\mathcal{A}(r,\theta) = A(r,\theta) + \bar{A}(r,-\theta) &= \sum_{n=0}^{\infty} \Big((s_n - \bar{s}_n)\sin(n\theta) + (c_n + \bar{c}_n)\cos(n\theta) \Big) \\
&= 2\sum_{n=0}^{\infty} \bigg(\Big(i\,|s_n| \sin\phi + \epsilon_n^s \cos\phi \Big) \sin(n\theta) e^{i\delta_n^s} \\
&\qquad + \Big(|c_n| \cos\phi + i\epsilon_n^c \sin\phi \Big) \cos(n\theta) e^{i\delta_n^c} \bigg) \\
&= \mathcal{A}^S(r,\theta) + \mathcal{A}^N(r,\theta),
\end{aligned} \tag{4.117}$$

where

$$\mathcal{A}^S(r,\theta) = 2\sum_{n=0}^{\infty} \Big(|c_n| \cos\phi + i\epsilon_n^c \sin\phi \Big) \cos(n\theta)\, e^{i\delta_n^c}, \tag{4.118}$$

$$\mathcal{A}^N(r,\theta) = 2\sum_{n=0}^{\infty} \Big(i\,|s_n| \sin\phi + \epsilon_n^s \cos\phi \Big) \sin(n\theta)\, e^{i\delta_n^s}. \tag{4.119}$$

Now let us look at the interference term which can give rise to any asymmetry in the Dalitz plot under $\theta \leftrightarrow -\theta$:

$$\begin{aligned}
&2\,\mathrm{Re}\Big(\mathcal{A}^S(r,\theta) \cdot \mathcal{A}^{N*}(r,\theta) \Big) \\
&= 8\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \bigg(\Big(|c_n|\, \epsilon_m^s \cos^2\phi + |s_m|\, \epsilon_n^c \sin^2\phi \Big) \cos\left(\delta_n^c - \delta_m^s\right) \\
&\qquad + \frac{1}{2}\Big(|c_n|\, |s_m| - \epsilon_n^c \epsilon_m^s \Big) \sin 2\phi\, \sin\left(\delta_n^c - \delta_m^s\right) \bigg) \cos(n\theta)\, \sin(m\theta) \\
&\neq 0. \qquad \text{(when } \epsilon_n^c \neq 0 \text{ and } \epsilon_n^s \neq 0, \text{ even if } \phi = 0\text{)}
\end{aligned} \tag{4.120}$$

Therefore, if there is no CP violation in the decay mode under consideration, then presence of CPT violation can lead to an asymmetry in the Dalitz plot under $\theta \leftrightarrow -\theta$ (which is equivalent to $t \leftrightarrow u$).[4]

4.3.3 Processes suitable for observing direct CPT violation

The processes that can be used to study direct CPT violation by the methodology suggested here are necessarily those that do not have any CP violation in them. Thus we are forced to consider processes that occur via electromagnetic or strong interaction, which are known so far to be CP conserving interactions. One can, for example, look at the following three-body decays of J/ψ: $J/\psi \to N\pi^+\pi^-$, where N can be π^0, ω, η or ϕ. The violation of CPT symmetry, if it would ever be observed, is expected to be extremely minuscule[5]. So observation of CPT violation necessarily demands that the sample of events to be analysed and studied be extremely large. Here we make one important observation that the Dalitz plot asymmetry we have discussed so far in this section, does not depend on whether N is a single particle or a multi-particle state as long as $\bar{N} = N$. Thus, in principle, we could consider decays such as $J/\psi \to N\pi^+\pi^-$, with N being $\omega\pi^0, p\bar{p}, n\bar{n}, K^+K^-, \pi^0K^+K^-, \eta K^+K^-$ etc., as well as continuum production of all the final states mentioned here. Thus we can analyse multi-body decays and treat them as effective three-body decays and record all such events in a Dalitz prism, which should also have the corresponding asymmetry if CPT were violated in these decays. It is also important to notice that the asymmetry under the assumption of CPT invariance implies that CP is then definitely violated. This itself would also be a clear signature of new physics if found in processes involving electromagnetic and strong interactions.

[4]The signature here is similar to the ones noted in Refs. [159, 160, 163] for CP violation studies.

[5]Recently, values for $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ (in which z is the CPT violating parameter, see Eq. (4.5)) have been measured by the BaBar collaboration [162] in the neutral B meson sector to be $\mathrm{Re}(z)(\mathrm{Re}\lambda_{CP}/|\lambda_{CP}|) = 0.014 \pm 0.035(\mathrm{stat.}) \pm 0.034(\mathrm{syst.})$ and $\mathrm{Im}(z) = 0.038 \pm 0.029(\mathrm{stat.}) \pm 0.025(\mathrm{syst.})$, with λ_{CP} being the traditional variable that characterizes the decays of neutral B mesons into $J/\psi K_S^0$ or $J/\psi K_L^0$.

4.4 $SU(3)$ flavor symmetry violation

Finally, we shall illustrate how Dalitz plots can be used to study violation of the $SU(3)$ flavor symmetry in some three-body meson decays of the type $P \to M_1M_2M_3$, where P can be a B or a D meson and the final particles M_1, M_2 and M_3 are different members of the lightest pseudo-scalar $SU(3)$ multiplet (shown in Fig. 4.4). Our method described here relies on the simultaneous application of two of the $SU(2)$ symmetries subsumed in the $SU(3)$ flavor symmetry (which are isospin or T-spin, U-spin and V-spin), to the three body decay $P \to M_1M_2M_3$, where M_1, M_2 and M_3 are chosen such that M_1 and M_2 belong to the triplet of one of the $SU(2)$ subgroups and M_2 and M_3 belongs to another. Thus M_2 needs to be a part of all the different $SU(2)$ triplets under consideration. To be definite M_2 is always chosen to be the π^0 and the modes we consider are listed in Table 4.1. Under the limit of exact $SU(2)$ all the mesons belonging to the triplet are identical bosons and must exhibit an overall Bose symmetry under exchange. Thus in the final states under consideration exchange of M_1 and M_2, as well as the exchange of M_2 and M_3 should be fully symmetric. This implies that if the wave-function were symmetric under $SU(2)$ exchange, it must also be even under exchange in space; whereas if it were anti-symmetric in $SU(2)$, it must be odd under exchange in space too. We shall exploit this exchange symmetry to deduce some simple relations that predict a pattern in the distribution of events in the concerned Dalitz plot. Any deviation from this predicted Dalitz plot distribution would, therefore, constitute a test of breaking of $SU(3)$ flavor symmetry.

Let the 4-momenta of particles P and M_i (where $i \in \{1, 2, 3\}$) be p and p_i and their masses be m and m_i, respectively. A triangular Dalitz plot can now be constructed out of

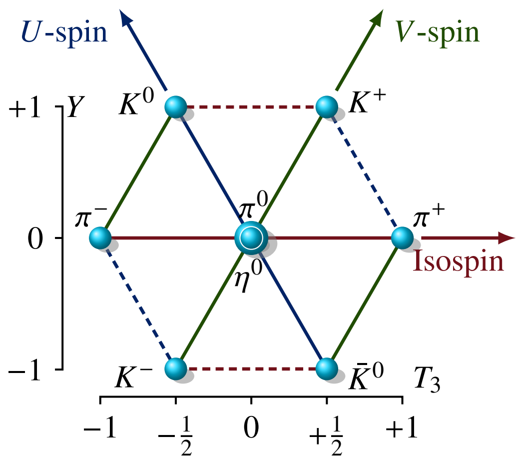


Figure 4.4: The $SU(3)$ meson octet of light pseudo-scalar mesons. Here the horizontal axis shows the eigenvalues of isospin (T_3) and the vertical axis shows the eigenvalues of hypercharge ($Y = B + S$, with B being baryon number and S being the strangeness number). The dotted lines parallel to U-spin (or isospin) axis signify that in no two-body decays of B or D meson can the two connected mesons appear together in the final state as that would violate conservation of electric charge (or strangeness by two units).

Final state			Kind of $SU(2)$ exchange	
M_1	M_2	M_3	$M_1 \leftrightarrow M_2$	$M_2 \leftrightarrow M_3$
K^0	π^0	π^+	U-spin	Isospin
K^+	π^0	π^-	V-spin	Isospin
K^+	π^0	$\overline{K}^0$	V-spin	U-spin
π^+	π^0	$\overline{K}^0$	Isospin	U-spin

Table 4.1: The final states $M_1M_2M_3$ in the decays under our consideration are given here. The particle M_2, which is always π^0, being at the center of the pseudoscalar meson octet, belongs to all the three triplets of the constituent $SU(2)$ symmetries under consideration. The last mode in the table with final state $\pi^+\pi^0\overline{K}^0$ has another exchange symmetry, namely exchange of π^+ and $\overline{K}^0$ under V-spin.

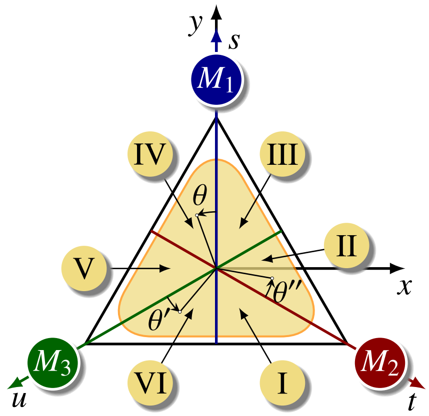


Figure 4.5: A hypothetical Dalitz plot for the decay $P \to M_1 M_2 M_3$. The three medians divide the interior of the equilateral triangle into six regions or six sextants of equal area which are denoted by I, II, III, IV, V and VI. The barycentric rectangular coordinate system has its origin at the center of the equilateral triangle and the y-axis is along the s-axis as shown here. The angles θ, θ' and θ'' are measured in anticlockwise direction from s-axis, u-axis and t-axis respectively. The physically allowed region is always inside the equilateral triangle as shown, schematically, by the shaded region.

the Mandelstam-like variables s, t and u:

$$\begin{aligned}
s &= (p - p_1)^2 = (p_2 + p_3)^2 \,, \\
t &= (p - p_2)^2 = (p_1 + p_3)^2 \,, \\
u &= (p - p_3)^2 = (p_1 + p_2)^2 \,.
\end{aligned} \tag{4.121}$$

It is clear that $(m_2+m_3)^2 \leqslant s \leqslant (m-m_1)^2$, $(m_1+m_3)^2 \leqslant t \leqslant (m-m_2)^2$, $(m_1+m_2)^2 \leqslant u \leqslant (m-m_3)^2$, and $s+t+u = m^2+m_1^2+m_2^2+m_3^2 = M^2$ (say). Giving equal weight to s, t and u we shall work with a ternary plot of which s, t, u form the three axes, as shown in Fig. 4.5. For ultra-relativistic final particles, the entire interior of the equilateral triangle tends to get occupied. In any case the Dalitz plot under our consideration (schematically shown by shading in Fig. 4.5) would always lie inside the equilateral triangle. The boundary of the Dalitz plot for a three-body decay process under consideration would not look symmetric under the exchanges $s \leftrightarrow t \leftrightarrow u$ due to the breaking of flavor $SU(3)$ symmetry on account of masses m_1, m_2 and m_3 being different. Any event inside the Dalitz plot, as illustrated in Fig. 4.5, can be specified by its radial distance (r) from the center of the equilateral triangle and the angle subtended by its position vector with any of the three axes s, t, or u. The angle subtended by the position vector with s-axis is denoted by θ, the one with u-axis is denoted by θ' and the one with t-axis is denoted by θ''. It is easy to see that $\theta = \theta' + 2\pi/3$ and $\theta = \theta'' + 4\pi/3$ (see Fig. 4.5). An event described by some values of s, t and u corresponds to some values of r and θ as obtainable from the relations given below:

$$s = \frac{M^2}{3}\Big(1 + r\ \cos\theta\Big), \tag{4.122}$$

$$t = \frac{M^2}{3}\left(1 + r\ \cos\left(\frac{2\pi}{3} + \theta\right)\right), \tag{4.123}$$

$$u = \frac{M^2}{3}\left(1 + r\ \cos\left(\frac{2\pi}{3} - \theta\right)\right). \tag{4.124}$$

It is straightforward to change the basis from (r, θ) to either (r, θ') or (r, θ'').

Before we analyze the specific decay modes, a few points regarding the notation to be used and the neutral pion are essential to put down.

- The particle states under a particular $SU(2)$ symmetry are labelled with subscripts for clarity, e.g. the state $|I = 1, I_3 = 0\rangle$ is denoted as $|1, 0\rangle_I$, the state $|U = 1, U_3 = +1\rangle$ is denoted as $|1, +1\rangle_U$, the state $|V = 1, V_3 = -1\rangle$ is denoted as $|1, -1\rangle_V$, etc. This helps to distinguish the states in different $SU(2)$ bases.

- The neutral pion is a pure isotriplet state $|1, 0\rangle_I \equiv \frac{1}{\sqrt{2}}\left(d\bar{d} - u\bar{u}\right)$:[6]

$$\left|\pi^0\right\rangle = -\,|1, 0\rangle_I\,. \tag{4.125}$$

Under U-spin it is a linear combination of the U-spin triplet state $|1, 0\rangle_U \equiv \frac{1}{\sqrt{2}}\left(s\bar{s} - d\bar{d}\right)$

[6]Here we assume perfect isospin symmetry, as otherwise the mixing of π^0-η-η' arising out of isospin violation must also be taken into account. This effect is however is not going to affect the kind of Dalitz plot asymmetries we will analyse ahead.

and the U-spin singlet but $SU(3)$ octet state $|0,0\rangle_{U,8} \equiv \frac{1}{\sqrt{6}}\left(d\bar{d} + s\bar{s} - 2u\bar{u}\right)$:

$$\left|\pi^0\right\rangle = \frac{1}{2}\,|1,0\rangle_U - \frac{\sqrt{3}}{2}\,|0,0\rangle_{U,8}\,. \tag{4.126}$$

Similarly under V-spin, π^0 is given by a linear combination of the V-spin triplet state $|1,0\rangle_V \equiv \frac{1}{\sqrt{2}}\,(s\bar{s} - u\bar{u})$ and the V-spin singlet but $SU(3)$ octet state $|0,0\rangle_{V,8} \equiv \frac{1}{\sqrt{6}}\left(u\bar{u} + s\bar{s} - 2d\bar{d}\right)$:

$$\left|\pi^0\right\rangle = -\frac{1}{2}\,|1,0\rangle_V + \frac{\sqrt{3}}{2}\,|0,0\rangle_{V,8}\,. \tag{4.127}$$

We have tabulated the expressions for combined states of pairs of final particles in their relevant $SU(2)$ symmetry bases in Table 4.2.

Final state	Expression for the state	
$M_1M_2M_3$	$\|M_1M_2\rangle$	$\|M_2M_3\rangle$
$K^0\pi^0\pi^+$	$\frac{1}{2\sqrt{2}}\left(\|2,+1\rangle_U + \|1,+1\rangle_U\right) - \frac{\sqrt{3}}{2}\,\|1',+1\rangle_U$	$-\frac{1}{\sqrt{2}}\left(\|2,+1\rangle_I - \|1,+1\rangle_I\right)$
$K^+\pi^0\pi^-$	$-\frac{1}{2\sqrt{2}}\left(\|2,+1\rangle_V + \|1,+1\rangle_V\right) + \frac{\sqrt{3}}{2}\,\|1',+1\rangle_V$	$\frac{1}{\sqrt{2}}\left(\|2,-1\rangle_I + \|1,-1\rangle_I\right)$
$K^+\pi^0\overline{K}^0$	$-\frac{1}{2\sqrt{2}}\left(\|2,+1\rangle_V + \|1,+1\rangle_V\right) + \frac{\sqrt{3}}{2}\,\|1',+1\rangle_V$	$\frac{1}{2\sqrt{2}}\left(\|2,-1\rangle_U + \|1,-1\rangle_U\right) - \frac{\sqrt{3}}{2}\,\|1',-1\rangle_U$
$\pi^+\pi^0\overline{K}^0$	$-\frac{1}{\sqrt{2}}\left(\|2,+1\rangle_I + \|1,+1\rangle_I\right)$	$-\frac{1}{2\sqrt{2}}\left(\|2,-1\rangle_U + \|1,-1\rangle_U\right) + \frac{\sqrt{3}}{2}\,\|1',-1\rangle_U$

Table 4.2: Expressions for states of pairs of final particles in their respective $SU(2)$ basis. As discussed in the text, the primed states such as $|1',\pm1\rangle$ arise from the $|0,0\rangle$ component of π^0 under U-spin and V-spin considerations. For the last mode we have $\left|\pi^+\overline{K}^0\right\rangle = \frac{1}{\sqrt{2}}\,(|1,0\rangle_V + |0,0\rangle_V)$ under V-spin.

4.4.1 Decay Mode with final state $K^0\pi^0\pi^+$

Let us consider the decay mode $B^+ \to K^0\pi^0\pi^+$. In the final state the π^0 and π^+ are identical under isospin and hence the final state must be totally symmetric under their exchange. Under U-spin (see Fig. 4.4) the K^0 and π^0 behave as identical bosons and hence the final state must similarly be totally symmetric under their exchange. It is easy to notice that in the Dalitz plot:

$$\begin{aligned} U\text{-spin exchange} &\equiv K^0 \leftrightarrow \pi^0 \implies s \leftrightarrow t, \\ \text{isospin exchange} &\equiv \pi^0 \leftrightarrow \pi^+ \implies t \leftrightarrow u. \end{aligned}$$

Therefore, considering U-spin and isospin to be exact symmetries, the final state $K^0\pi^0\pi^+$ has the following two possibilities:

1. $K^0\pi^0$ would exist in either symmetrical or anti-symmetrical state under $s \leftrightarrow t$ exchange, and
2. $\pi^0\pi^+$ would exist in either symmetrical or anti-symmetrical state under $t \leftrightarrow u$ exchange.

The amplitude for this decay, can thus be described by four independent functions:

(i) $\mathcal{A}_{SS}(s,t,u)$ which is symmetric under both $s \leftrightarrow t$ and $t \leftrightarrow u$,

(ii) $\mathcal{A}_{AA}(s,t,u)$ which is anti-symmetric under both $s \leftrightarrow t$ and $t \leftrightarrow u$,

(iii) $\mathcal{A}_{SA}(s,t,u)$ which is symmetric under $s \leftrightarrow t$ and anti-symmetric under $t \leftrightarrow u$,

(iv) $\mathcal{A}_{AS}(s,t,u)$ which is anti-symmetric under $s \leftrightarrow t$ and symmetric under $t \leftrightarrow u$.

Let us now analyze the above four amplitude functions in the most general manner. It is easy to show that $\mathcal{A}_{SS}(s,t,u)$, which is a function symmetric under both $s \leftrightarrow t$ and $t \leftrightarrow u$ must also be symmetric under $s \leftrightarrow u$:

$$\mathcal{A}_{SS}(s,t,u) \overset{s\leftrightarrow t}{=\!=} \mathcal{A}_{SS}(t,s,u) \overset{t\leftrightarrow u}{=\!=} \mathcal{A}_{SS}(u,s,t) \overset{s\leftrightarrow t}{=\!=} \mathcal{A}_{SS}(u,t,s). \tag{4.128}$$

Hence, $\mathcal{A}_{SS}(s,t,u)$ is a fully symmetric amplitude function. Let us now consider $\mathcal{A}_{AA}(s,t,u)$ which is anti-symmetric under both $s \leftrightarrow t$ and $t \leftrightarrow u$. It is easy to show that it is also anti-symmetric under $s \leftrightarrow u$:

$$\mathcal{A}_{AA}(s,t,u) \overset{s\leftrightarrow t}{=\!=} -\mathcal{A}_{AA}(t,s,u) \overset{t\leftrightarrow u}{=\!=} +\mathcal{A}_{AA}(u,s,t) \overset{s\leftrightarrow t}{=\!=} -\mathcal{A}_{AA}(u,t,s). \tag{4.129}$$

Hence, $\mathcal{A}_{AA}(s,t,u)$ is a fully anti-symmetric amplitude function. Using the same arguments as above it is easy to show that both $\mathcal{A}_{SA}(s,t,u)$ and $\mathcal{A}_{AS}(s,t,u)$ identically vanish.

$$\begin{aligned}\mathcal{A}_{SA}(s,t,u) &\overset{s\leftrightarrow t}{=\!=} \mathcal{A}_{SA}(t,s,u) \overset{t\leftrightarrow u}{=\!=} -\mathcal{A}_{SA}(u,s,t) \overset{s\leftrightarrow t}{=\!=} -\mathcal{A}_{SA}(u,t,s)\\ &\overset{t\leftrightarrow u}{=\!=} +\mathcal{A}_{SA}(t,u,s) \overset{s\leftrightarrow t}{=\!=} +\mathcal{A}_{SA}(s,u,t) \overset{t\leftrightarrow u}{=\!=} -\mathcal{A}_{SA}(s,t,u) = 0, \end{aligned}\tag{4.130}$$

$$\begin{aligned}\mathcal{A}_{AS}(s,t,u) &\overset{s\leftrightarrow t}{=\!=} -\mathcal{A}_{AS}(t,s,u) \overset{t\leftrightarrow u}{=\!=} -\mathcal{A}_{AS}(u,s,t) \overset{s\leftrightarrow t}{=\!=} +\mathcal{A}_{AS}(u,t,s)\\ &\overset{t\leftrightarrow u}{=\!=} +\mathcal{A}_{AS}(t,u,s) \overset{s\leftrightarrow t}{=\!=} -\mathcal{A}_{AS}(s,u,t) \overset{t\leftrightarrow u}{=\!=} -\mathcal{A}_{AS}(s,t,u) = 0. \end{aligned}\tag{4.131}$$

This implies that the amplitudes $\mathcal{A}_{SA}(s,t,u)$ and $\mathcal{A}_{AS}(s,t,u)$ never contribute to the distribution of events on the Dalitz plot. Thus the amplitude and hence the function describing the distribution of events in the Dalitz plot which is proportional to the square of the modulus of the amplitude, have only two parts, one which is fully symmetric under $s \leftrightarrow t \leftrightarrow u$, and another which is fully anti-symmetric under $s \leftrightarrow t \leftrightarrow u$.

Let us now examine the decay mode $B^+ \to K^0\pi^0\pi^+$ by writing down the decay amplitude in terms of isospin and U-spin amplitudes. The $\pi^0\pi^+$ combination can exist in isospin states $|2,+1\rangle_I$ and $|1,+1\rangle_I$ (see Table 4.2). If isospin were an exact symmetry, the state $\left|\pi^0\pi^+\right\rangle$ would stay unchanged under $\pi^0 \leftrightarrow \pi^+$ exchange. Thus the $|2,+1\rangle_I$ state would be in a space symmetric (even partial wave) state, and the $|1,+1\rangle_I$ state would be in a space anti-symmetric (odd partial wave) state. The final state $\left|K^0\pi^0\pi^+\right\rangle$ in isospin decomposition is given by

$$\left|K^0\pi^0\pi^+\right\rangle = -\frac{1}{\sqrt{5}}\left|\frac{5}{2},+\frac{1}{2}\right\rangle_I^{\mathrm{e}} + \frac{\sqrt{3}}{\sqrt{10}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_I^{\mathrm{e}} + \frac{1}{\sqrt{6}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_I^{\mathrm{o}} - \frac{1}{\sqrt{3}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle_I^{\mathrm{o}}, \tag{4.132}$$

where the superscripts e, o denote the even, odd nature of the state under the exchange $\pi^0 \leftrightarrow \pi^+$. The odd states above change sign under $\pi^0 \leftrightarrow \pi^+$ exchange because the $|1,+1\rangle_I$ isospin component of the $\left|\pi^0\pi^+\right\rangle$ state is odd under this exchange, whereas the $|2,+1\rangle_I$ state is even under the same exchange. Since B^+ has isospin state $\left|\frac{1}{2},+\frac{1}{2}\right\rangle_I$, and only $\Delta I = 0, 1$ currents are allowed by the Hamiltonian in the standard model of particle physics, we would have no contributions from $\left|\frac{5}{2},+\frac{1}{2}\right\rangle_I$ state. The $\left|\frac{3}{2},+\frac{1}{2}\right\rangle_I$ state can arise from both $\left|\frac{1}{2},-\frac{1}{2}\right\rangle_I \otimes |2,+1\rangle_I$ and $\left|\frac{1}{2},-\frac{1}{2}\right\rangle_I \otimes |1,+1\rangle_I$, with the first contribution being symmetric and the later being anti-symmetric. The state $\left|\frac{1}{2},+\frac{1}{2}\right\rangle_I$ on the other-hand is purely anti-symmetric. Even though we shall work with the standard model Hamiltonian, our conclusions are general and are valid even when more general Hamiltonians exist. The isospin $I = 1/2$ initial state decays to a final state that can be decomposed into either $I = 1/2$ or $I = 3/2$ states via a Hamiltonian that allows $\Delta I = 0$ or $\Delta I = 1$ transitions. The transition with $\Delta I = 1$ results in two amplitudes with $I = 1/2$ or $I = 3/2$ represented as $T_{1,\frac{1}{2}}$ and $T_{1,\frac{3}{2}}$ respectively, whereas $\Delta I = 0$ transition results only in a single amplitude with final state $I = 1/2$ labelled as $T_{0,\frac{1}{2}}$. The isospin amplitudes $T_{1,\frac{1}{2}}$, $T_{1,\frac{3}{2}}$ and $T_{0,\frac{1}{2}}$ are themselves defined [100] in terms of the Hamiltonian as,

$$T_{1,\frac{3}{2}} = \sqrt{\frac{1}{3}}\left\langle\frac{3}{2},\pm\frac{1}{2}\right|\mathscr{H}_{\Delta I=1}\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle, \tag{4.133a}$$

$$T_{1,\frac{1}{2}} = \pm\sqrt{\frac{2}{3}}\left\langle\frac{1}{2},\pm\frac{1}{2}\right|\mathscr{H}_{\Delta I=1}\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle, \tag{4.133b}$$

$$T_{0,\frac{1}{2}} = \sqrt{\frac{2}{3}}\left\langle\frac{1}{2},\pm\frac{1}{2}\right|\mathscr{H}_{\Delta I=0}\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle. \tag{4.133c}$$

The amplitude for the decay $B^+ \to K^0\pi^0\pi^+$ can then be written in terms of the isospin amplitudes as

$$A(B^+ \to K^0\pi^0\pi^+) = \frac{3}{\sqrt{10}}T^{\mathrm{e}}_{1,\frac{3}{2}}\,X - \frac{1}{\sqrt{2}}\left(T^{\mathrm{o}}_{1,\frac{3}{2}} + T^{\mathrm{o}}_{1,\frac{1}{2}} + T^{\mathrm{o}}_{0,\frac{1}{2}}\right)Y\sin\theta, \tag{4.134}$$

where X and $Y\sin\theta$ take care of the spatial and kinematic contributions as is seen from the discussion above (see Eqns. (4.123) and (4.124)). The functions X and Y are, in general, mode dependent arbitrary functions of r and $\cos\theta$, and remain the same for modes related by isospin symmetry. The subscripts ‘e’ and ‘o’ are retained to merely to track the even or odd isospin state of the two pion in the three-body final state.

Let us now apply U-spin to the same final state. If U-spin were an exact symmetry, the state $K^0\pi^0$ would remain unchanged under $K^0 \leftrightarrow \pi^0$ exchange. Under U-spin the $K^0\pi^0$ state is a linear combination of $|2,+1\rangle_U$ and $|1,+1\rangle_U$ (see Table 4.2). The state $|1,+1\rangle_U$ has a contribution from the $|0,0\rangle_{U,8}$ admixture in π^0 which is denoted by $|1',+1\rangle_U$. Both the states $|2,+1\rangle_U$ and $|1,+1\rangle_U$ coming from the $|0,0\rangle_{U,8}$ contribution of π^0, exist in space symmetric (even partial wave) states, and that part of $|1,+1\rangle_U$ arising out of $|1,0\rangle_U$ part of π^0 exists in space anti-symmetric (odd partial wave) state. Thus the final state $\left|K^0\pi^0\pi^+\right\rangle$ under U-spin is given by

$$\begin{aligned}\left|K^0\pi^0\pi^+\right\rangle = &-\frac{1}{2\sqrt{5}}\left|\frac{5}{2},+\frac{1}{2}\right\rangle_U^{\mathrm{e}} - \frac{\sqrt{3}}{2\sqrt{10}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_U^{\mathrm{e}} - \frac{1}{2\sqrt{6}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_U^{\mathrm{o}} - \frac{1}{2\sqrt{3}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle_U^{\mathrm{o}}\\ &+\frac{1}{2}\left|\frac{3'}{2},+\frac{1}{2}\right\rangle_U^{\mathrm{e}} + \frac{1}{\sqrt{2}}\left|\frac{1'}{2},+\frac{1}{2}\right\rangle_U^{\mathrm{e}},\end{aligned} \tag{4.135}$$

where the superscripts e, o denote that the state is even, odd under the exchange $K^0 \leftrightarrow \pi^0$. The reason for sign change in the odd terms above can be easily understood from the U-spin decomposition of the $\left|K^0\pi^0\right\rangle$ state:

$$\left|K^0\pi^0\right\rangle = \frac{1}{2\sqrt{2}}\Big(|2,+1\rangle_U + |1,+1\rangle_U\Big) - \frac{\sqrt{3}}{2}\left|1',+1\right\rangle_U, \tag{4.136}$$

which under the $K^0 \leftrightarrow \pi^0$ exchange becomes

$$\left|\pi^0K^0\right\rangle = \frac{1}{2\sqrt{2}}\Big(|2,+1\rangle_U - |1,+1\rangle_U\Big) - \frac{\sqrt{3}}{2}\left|1',+1\right\rangle_U. \tag{4.137}$$

We know that the state $|1,+1\rangle_U$ is odd under $K^0 \leftrightarrow \pi^0$ exchange, whereas the states $|2,+1\rangle_U$ and $|1',+1\rangle_U$ are even under the same exchange. Again the states $\left|\frac{5}{2},+\frac{1}{2}\right\rangle_U$ and $\left|\frac{3}{2},+\frac{1}{2}\right\rangle_U$ do not contribute, since the parent particle B^+ is a U-spin singlet, and only the $\Delta U = 1/2$ current contributes to the decay. This unique feature follows from the fact that the electroweak penguin does not violate U-spin as d and s quarks carry the same electric charge (see [118, 139]). Hence, only the $\left|\frac{1}{2},\frac{1}{2}\right\rangle_U$ and $\left|\frac{1'}{2},\frac{1}{2}\right\rangle_U$ can contribute to the decay amplitude and they correspond to anti-symmetric and symmetric contributions under $K^0 \leftrightarrow \pi^0$ respectively. The U-spin amplitudes

$$\begin{aligned} U_{\frac{1}{2},\frac{1}{2}} &= \pm\sqrt{\frac{2}{3}}\left\langle\frac{1}{2},\pm\frac{1}{2}\middle|\mathscr{H}_{\Delta U=\frac{1}{2}}\middle|0,0\right\rangle, \\ U'_{\frac{1}{2},\frac{1}{2}} &= \sqrt{\frac{1}{3}}\left\langle\frac{1'}{2},\pm\frac{1}{2}\middle|\mathscr{H}_{\Delta U=\frac{1}{2}}\middle|0,0\right\rangle. \end{aligned} \tag{4.138a}$$

Using these U-spin amplitudes the decay amplitude for $B^+ \to K^0\pi^0\pi^+$ can now be written as

$$A(B^+ \to K^0\pi^0\pi^+) = \frac{3}{\sqrt{10}} U'^{\mathrm{e}}_{\frac{1}{2},\frac{1}{2}} X' + U^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}} Y' \sin\theta', \tag{4.139}$$

where X' and Y' are, in general, arbitrary functions of r and $\cos\theta'$, and they take care of spatial and kinematic contributions to the decay amplitude. The subscripts 'e' and 'o' are again retained to keep track of the contributions of even or odd U-spin state of K^0 and π^0 in the three-body final state. As argued before the amplitude for the decay has two parts, one fully symmetric under the exchanges $s \leftrightarrow t \leftrightarrow u$ (i.e. $\mathcal{A}_{SS}(s,t,u)$) and another fully anti-symmetric under the same exchanges (i.e. $\mathcal{A}_{AA}(s,t,u)$). Comparing Eqs. (4.134) and (4.139) we obtain:

$$\mathcal{A}_{SS} = \frac{3}{\sqrt{10}} T^{\mathrm{e}}_{1,\frac{3}{2}} X = \frac{3}{\sqrt{10}} U'^{\mathrm{e}}_{\frac{1}{2},\frac{1}{2}} X' \tag{4.140}$$

$$\mathcal{A}_{AA} = -\frac{1}{\sqrt{2}}\left(T^{\mathrm{o}}_{1,\frac{3}{2}} + T^{\mathrm{o}}_{1,\frac{1}{2}} + T^{\mathrm{o}}_{0,\frac{1}{2}}\right) Y \sin\theta = U^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}} Y' \sin\theta'. \tag{4.141}$$

The exchange $s \leftrightarrow t \leftrightarrow u$ is equivalent to $\theta \leftrightarrow \theta' \leftrightarrow \theta''$. This implies that the fully anti-symmetric amplitude $\mathcal{A}_{AA}(s,t,u)$ must be proportional to $\sin 3\theta$ because $\sin 3\theta = \sin 3\theta' = \sin 3\theta''$ as $\theta = \theta' + 2\pi/3 = \theta'' + 4\pi/3$. It is easy to show that $\sin 3\theta = \sin\theta \left(4\cos^2\theta - 1\right)$, where the factor $\left(4\cos^2\theta - 1\right)$ is an even function of $\cos\theta$ and it must be a part of both Y and Y' in Eq. (4.141), i.e. $Y = y\left(4\cos^2\theta - 1\right)$ and $Y' = y'\left(4\cos^2\theta' - 1\right)$ for some y and y' such that

$$\mathcal{A}_{AA} = -\frac{1}{\sqrt{2}}\left(T^{\mathrm{o}}_{1,\frac{3}{2}} + T^{\mathrm{o}}_{1,\frac{1}{2}} + T^{\mathrm{o}}_{0,\frac{1}{2}}\right) y \sin 3\theta = U^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}}\, y' \sin 3\theta'. \tag{4.142}$$

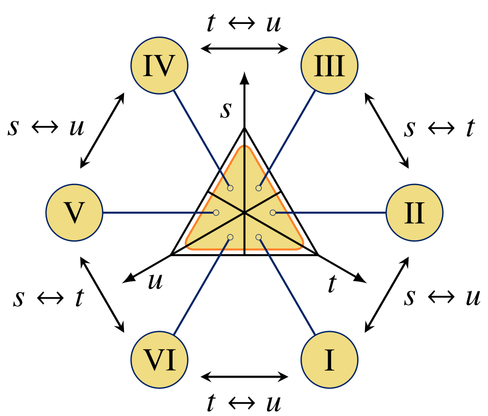


Figure 4.6: Exchanges that take us from one sextant to another in the Dalitz plot. It must be noted that the following exchanges are also equivalent: $s \leftrightarrow t \leftrightarrow u \equiv \theta \leftrightarrow \theta' \leftrightarrow \theta''$ as well as $t \leftrightarrow u \equiv \theta \leftrightarrow -\theta$, $s \leftrightarrow t \equiv \theta' \leftrightarrow -\theta'$ and $u \leftrightarrow s \equiv \theta'' \leftrightarrow -\theta''$.

Since the Dalitz plot distribution function is proportional to the square of the modulus of the amplitude, it would also have a part which is fully symmetric under $s \leftrightarrow t \leftrightarrow u$ (denoted by $f_{SS}(s,t,u)$) and another part which is fully anti-symmetric under the same exchanges (denoted by $f_{AA}(s,t,u)$):

$$f_{SS}(s,t,u) \propto |\mathcal{A}_{SS}(s,t,u)|^2 + |\mathcal{A}_{AA}(s,t,u)|^2 \,, \tag{4.143}$$

$$f_{AA}(s,t,u) \propto 2\,\mathrm{Re}\Big(\mathcal{A}_{SS}(s,t,u) \cdot \mathcal{A}^*_{AA}(s,t,u)\Big). \tag{4.144}$$

The Dalitz plot gets divided into six sextants by means of the s, t and u axes which lie along the medians of an equilateral triangle as shown in Figs. 4.5 and 4.6. Let us denote the function describing distribution of events in any sextant, say in the ith one, of the Dalitz plot by $f_i(r,\theta)$, where the coordinates (r,θ) lie in the sextant i. We could have as well used the other equivalent choices θ' or θ'' instead of θ. However, we shall judiciously choose the angles that would be best to describe the underlying symmetry being considered (see Fig. 4.6). Henceforth, we shall drop (r,θ) from the distribution functions, except when necessary, as we implicitly assume the r and θ dependence in them. The distribution function must have only two parts as observed above, the fully symmetric and the fully anti-symmetric parts. Let us assume that in sextant I the Dalitz plot distribution is given by the function

$$f_I = f_{SS}(s,t,u) + f_{AA}(s,t,u). \tag{4.145}$$

It is then trivial to find that the Dalitz plot distributions in the even numbered sextants are identical to one another, and the odd numbered sextants are also identically populated, because

$$f_I = f_{III} = f_V = f_{SS}(s,t,u) + f_{AA}(s,t,u), \tag{4.146}$$

$$f_{II} = f_{IV} = f_{VI} = f_{SS}(s,t,u) - f_{AA}(s,t,u). \tag{4.147}$$

This is the signature of exact $SU(3)$ flavor symmetry in the Dalitz plots under our consideration. Any observed deviation from this would constitute an evidence for violation of the $SU(3)$ flavor symmetry.

Thus far we have used the exchange properties of $K^0 \leftrightarrow \pi^0$ under U-spin and $\pi^0 \leftrightarrow \pi^+$ under isospin to obtain the even and odd amplitudes contributing to $B^+ \to K^0\pi^0\pi^+$. It is, therefore, only natural to ask what can we say about $K^0 \leftrightarrow \pi^+$ exchange. Since

K^0 and π^+ belong to different multiplets of V-spin (see Fig. 4.4), in order to consider the symmetry properties under $K^0 \leftrightarrow \pi^+$ exchange one needs to extend the concept of G-parity to accommodate V-spin, denoted by G_V and defined in the Appendix A. Since charge conjugation is a good symmetry in strong interaction, G_V is as good as the V-spin itself. The state $\left|K^0\pi^+\right\rangle$ is composed of states which are even and odd under G_V-parity:

$$\left|K^0\pi^+\right\rangle = \frac{1}{2}\left(\left|K^0\pi^+\right\rangle_{\mathrm{e}} + \left|K^0\pi^+\right\rangle_{\mathrm{o}}\right), \tag{4.148}$$

where

$$\left|K^0\pi^+\right\rangle_{\mathrm{e}} = \left|K^0\pi^+\right\rangle - \left|\pi^+K^0\right\rangle, \quad \left|K^0\pi^+\right\rangle_{\mathrm{o}} = \left|K^0\pi^+\right\rangle + \left|\pi^+K^0\right\rangle, \tag{4.149}$$

and

$$G_V\left|K^0\pi^+\right\rangle_{\mathrm{e}} = +\left|K^0\pi^+\right\rangle_{\mathrm{e}}, \quad G_V\left|K^0\pi^+\right\rangle_{\mathrm{o}} = -\left|K^0\pi^+\right\rangle_{\mathrm{o}}. \tag{4.150}$$

Note that the state even (or odd) under G_V is in odd (or even) partial wave state. We have already proven that the amplitudes for the decay $B^+ \to K^0\pi^0\pi^+$ has two parts one even and the other odd under the exchange of any two particles in the final state. Hence, $\mathcal{A}_{SS}$ is odd under G_V and $\mathcal{A}_{AA}$ is even under G_V. Since the two G_V-parity amplitudes do not interfere, the interference between $\mathcal{A}_{SS}$ and $\mathcal{A}_{AA}$, i.e. the f_{AA} term becomes zero (Eq. (4.144)). Therefore if G_V is a good symmetry of nature, it is interesting to conclude that the Dalitz plot is completely symmetric under $s \leftrightarrow t \leftrightarrow u$. This implies that

$$f_I = f_{II} = f_{III} = f_{IV} = f_V = f_{VI} \equiv f_{SS}(s,t,u). \tag{4.151}$$

This expression holds only if isospin, U-spin and V-spin are all exact symmetries. However, when G_V is broken but isospin and U-spin are still exact symmetries, the Dalitz plot distribution would still follow Eqs. (4.146) and (4.147). In the case when G_V is exact, the exchange properties of the distribution functions f_I to f_{VI} imply that if, (a) U-spin is an exact symmetry, then $f_{II} = f_{III}$, $f_I = f_{IV}$ and $f_V = f_{VI}$ irrespective of the exactness of isospin symmetry, (b) isospin is an exact symmetry, then $f_{II} = f_V$, $f_I = f_{VI}$ and $f_{III} = f_{IV}$ irrespective of the exactness of U-spin symmetry. However, when both G_V and either isospin or U-spin is broken, then the Eqs. (4.146) and (4.147) are no longer valid. In such a case, we have the following possibilities:

1. *Test for isospin symmetry:* By isospin symmetry, the Dalitz plot distribution of sextants I, II, III get mapped to those in sextants VI, V, IV respectively. We note that when isospin is *not* broken, then

$$f_I + f_{VI} = f_{III} + f_{IV} = f_V + f_{II} = 2f_{SS}(s,t,u), \tag{4.152}$$

$$f_I - f_{VI} = f_{III} - f_{IV} = f_V - f_{II} = 2f_{AA}(s,t,u). \tag{4.153}$$

However, once isospin is broken, the values of f_{SS} and f_{AA} extracted from sextants I and VI need not be same with those extracted from either II and V or III and IV. To quantify the variation of values of f_{SS} and f_{AA}, we introduce two quantities Σ^i_j and Δ^i_j defined as

$$\Sigma^i_j(r,\theta) = f_i + f_j, \tag{4.154a}$$

$$\Delta^i_j(r,\theta) = f_i - f_j, \tag{4.154b}$$

where i and j are two sextants and $i \neq j$. For brevity we shall not explicitly write the (r,θ) dependence of Σ^i_j and Δ^i_j, it is implied. In terms of these quantities, the signature of isospin breaking can be succinctly summarized by the following inequalities

$$\Sigma^I_{VI} \neq \Sigma^{III}_{IV} \neq \Sigma^V_{II}, \text{ and } \Delta^I_{VI} \neq \Delta^{III}_{IV} \neq \Delta^V_{II}. \tag{4.155}$$

An asymmetry can now be constructed to measure the isospin breaking as follows:

$$\begin{aligned}\mathbb{A}_{\text{Isospin}} = & \left|\frac{\Sigma^{I}_{VI} - \Sigma^{III}_{IV}}{\Sigma^{I}_{VI} + \Sigma^{III}_{IV}}\right| + \left|\frac{\Sigma^{III}_{IV} - \Sigma^{V}_{II}}{\Sigma^{III}_{IV} + \Sigma^{V}_{II}}\right| + \left|\frac{\Sigma^{V}_{II} - \Sigma^{I}_{VI}}{\Sigma^{V}_{II} + \Sigma^{I}_{VI}}\right| \\ & + \left|\frac{\Delta^{I}_{VI} - \Delta^{III}_{IV}}{\Delta^{I}_{VI} + \Delta^{III}_{IV}}\right| + \left|\frac{\Delta^{III}_{IV} - \Delta^{V}_{II}}{\Delta^{III}_{IV} + \Delta^{V}_{II}}\right| + \left|\frac{\Delta^{V}_{II} - \Delta^{I}_{VI}}{\Delta^{V}_{II} + \Delta^{I}_{VI}}\right|. \end{aligned} \tag{4.156}$$

2. *Test for U-spin symmetry:* By U-spin symmetry, the sextants VI, I, II get mapped to the sextants V, IV, III respectively. We note that when U-spin is *not* broken, then

$$\Sigma^{I}_{IV} = \Sigma^{III}_{II} = \Sigma^{V}_{VI} = 2f_{SS}(s,t,u), \tag{4.157}$$

$$\Delta^{I}_{IV} = \Delta^{III}_{II} = \Delta^{V}_{VI} = 2f_{AA}(s,t,u). \tag{4.158}$$

Here it would be profitable to consider the Σ's and Δ's being functions of (r, θ') as we are considering $s \leftrightarrow t$ exchange which is equivalent to $\theta' \leftrightarrow -\theta'$. When U-spin is broken

$$\Sigma^{I}_{IV} \neq \Sigma^{III}_{II} \neq \Sigma^{V}_{VI}, \text{ and } \Delta^{I}_{IV} \neq \Delta^{III}_{II} \neq \Delta^{V}_{VI}. \tag{4.159}$$

The asymmetry for U-spin breaking is, therefore, given by

$$\begin{aligned}\mathbb{A}_{U\text{-spin}} = & \left|\frac{\Sigma^{I}_{IV} - \Sigma^{III}_{II}}{\Sigma^{I}_{IV} + \Sigma^{III}_{II}}\right| + \left|\frac{\Sigma^{III}_{II} - \Sigma^{V}_{VI}}{\Sigma^{III}_{II} + \Sigma^{V}_{VI}}\right| + \left|\frac{\Sigma^{V}_{VI} - \Sigma^{I}_{IV}}{\Sigma^{V}_{VI} + \Sigma^{I}_{IV}}\right| \\ & + \left|\frac{\Delta^{I}_{IV} - \Delta^{III}_{II}}{\Delta^{I}_{IV} + \Delta^{III}_{II}}\right| + \left|\frac{\Delta^{III}_{II} - \Delta^{V}_{VI}}{\Delta^{III}_{II} + \Delta^{V}_{VI}}\right| + \left|\frac{\Delta^{V}_{VI} - \Delta^{I}_{IV}}{\Delta^{V}_{VI} + \Delta^{I}_{IV}}\right|. \end{aligned} \tag{4.160}$$

3. *Test for V-spin symmetry:* As said before, G_V-parity is as badly broken as the V-spin because charge conjugation is a good symmetry of strong interaction. When V-spin is broken, then G_V is also broken, and the distribution of events in the Dalitz plot sextants would follow Eqs. (4.146) and (4.147). In addition to that, when

V-spin is broken, K^0 and π^+ need not be even under exchange. This leads to

$$\Sigma^{V}_{IV} \neq \Sigma^{III}_{VI} \neq \Sigma^{I}_{II}, \tag{4.161}$$

$$\Delta^{V}_{IV} \neq \Delta^{III}_{VI} \neq \Delta^{I}_{II}. \tag{4.162}$$

The asymmetry for V-spin breaking is, therefore, given by

$$\begin{aligned}\mathbb{A}_{V\text{-spin}} = &\left|\frac{\Sigma^{V}_{IV} - \Sigma^{I}_{II}}{\Sigma^{V}_{IV} + \Sigma^{I}_{II}}\right| + \left|\frac{\Sigma^{I}_{II} - \Sigma^{III}_{VI}}{\Sigma^{I}_{II} + \Sigma^{III}_{VI}}\right| + \left|\frac{\Sigma^{III}_{VI} - \Sigma^{V}_{IV}}{\Sigma^{III}_{VI} + \Sigma^{V}_{IV}}\right| \\ &+ \left|\frac{\Delta^{V}_{IV} - \Delta^{I}_{II}}{\Delta^{V}_{IV} + \Delta^{I}_{II}}\right| + \left|\frac{\Delta^{I}_{II} - \Delta^{III}_{VI}}{\Delta^{I}_{II} + \Delta^{III}_{VI}}\right| + \left|\frac{\Delta^{III}_{VI} - \Delta^{V}_{IV}}{\Delta^{III}_{VI} + \Delta^{V}_{IV}}\right|.\end{aligned} \tag{4.163}$$

Hence, the extent of the breaking of isospin, U-spin and V-spin can easily be measured from the Dalitz plot distribution. The asymmetries measuring isospin, U-spin and V-spin are functions of r and $3\theta \equiv 3\theta' \equiv 3\theta''$ (see the discussion leading to Eq. (4.142)). These asymmetries are, thus, valid in the full Dalitz plot, i.e. these asymmetries can be measured both along resonances and in the non-resonant regions. A quantitative estimate of the variation of these asymmetries obtained experimentally would be valuable for understanding the effects of the breaking of $SU(3)$ flavor symmetry. It would also be interesting to find regions of the Dalitz plots where $SU(3)$ is a good symmetry.[7] The procedure discussed above can also be applied to other decay modes with the same final state. In particular one can study the Dalitz plot distribution for the decay $D_s^+ \to K^0\pi^0\pi^+$ in a similar manner. The relevant $SU(2)$ amplitudes for this mode are tabulated in Table. 4.3.

4.4.2 Decay Mode with final state $K^+\pi^0\pi^-$

Let us now consider the decay B_d^0 or $\overline{B}{}_s^0 \to K^+\pi^0\pi^-$ in which isospin symmetry allows the exchange of π^0 and π^-, and V-spin symmetry allows exchange of K^+ and π^0. This

[7] It is presumed that the signature of breaking of the $SU(3)$ flavor symmetry would be minimum in regions in which the final particles have maximal momenta.

$B^+ \to K^0\pi^0\pi^+$							
Isospin $\left(\text{initial state } \left\|\frac{1}{2}, +\frac{1}{2}\right\rangle\right)$				U-spin $\left(\text{initial state } \|0, 0\rangle\right)$			
transition	final state	symmetry	Amplitude	transition	final state	symmetry	Amplitude
$\Delta I = 1$	$\left\|\frac{3}{2}, +\frac{1}{2}\right\rangle$	mixed	$\frac{3}{\sqrt{10}} T^{\mathrm{e}}_{1,\frac{3}{2}} X + \frac{1}{\sqrt{2}} T^{\mathrm{o}}_{1,\frac{3}{2}} Y \sin\theta$	$\Delta U = \frac{1}{2}$	$\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$	odd	$-\frac{1}{2\sqrt{2}} U^{\mathrm{o}}_{\frac{1}{2},\frac{3}{2}} Y' \sin\theta'$
$\Delta I = 1$	$\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$	odd	$-\frac{1}{\sqrt{2}} T^{\mathrm{o}}_{1,\frac{1}{2}} Y \sin\theta$	$\Delta U = \frac{1}{2}'$	$\left\|\frac{1}{2}, +\frac{1}{2}'\right\rangle$	even	$\frac{\sqrt{3}}{\sqrt{2}} U'^{e}_{\frac{1}{2},\frac{1}{2}'} X'$
$\Delta I = 0$	$\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$	odd	$-\frac{1}{\sqrt{2}} T^{\mathrm{o}}_{0,\frac{1}{2}} Y \sin\theta$				
$D_s^+ \to K^0\pi^0\pi^+$							
Isospin $\left(\text{initial state } \|0, 0\rangle\right)$				U-spin $\left(\text{initial state } \left\|\frac{1}{2}, +\frac{1}{2}\right\rangle\right)$			
transition	final state	symmetry	Amplitude	transition	final state	symmetry	Amplitude
$\Delta I = \frac{3}{2}$	$\left\|\frac{3}{2}, +\frac{1}{2}\right\rangle$	mixed	$\frac{\sqrt{3}}{\sqrt{10}} T^{\mathrm{e}}_{\frac{3}{2},\frac{3}{2}} X + \frac{1}{\sqrt{6}} T^{\mathrm{o}}_{\frac{3}{2},\frac{3}{2}} Y \sin\theta$	$\Delta U = 1$	$\left\|\frac{3}{2}, +\frac{1}{2}\right\rangle$	mixed	$-\frac{3}{2\sqrt{10}} U^{\mathrm{e}}_{1,\frac{3}{2}} X' - \frac{1}{2\sqrt{2}} U^{\mathrm{o}}_{1,\frac{3}{2}} Y' \sin\theta'$
$\Delta I = \frac{1}{2}$	$\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$	odd	$-\frac{1}{\sqrt{2}} T^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}} Y \sin\theta$	$\Delta U = 1$	$\left\|\frac{3}{2}', +\frac{1}{2}\right\rangle$	even	$\frac{\sqrt{3}}{2} U'^{e}_{1,\frac{3}{2}'} X'$
				$\Delta U = 1$	$\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$	odd	$-\frac{1}{2\sqrt{2}} U^{\mathrm{o}}_{1,\frac{1}{2}} Y' \sin\theta'$
				$\Delta U = 1$	$\left\|\frac{1}{2}', +\frac{1}{2}\right\rangle$	even	$\frac{\sqrt{3}}{2} U'^{e}_{1,\frac{1}{2}'} X'$
				$\Delta U = 0$	$\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$	odd	$-\frac{1}{2\sqrt{2}} U^{\mathrm{o}}_{0,\frac{1}{2}} Y' \sin\theta'$
				$\Delta U = 0$	$\left\|\frac{1}{2}', +\frac{1}{2}\right\rangle$	even	$\frac{\sqrt{3}}{2} U'^{e}_{0,\frac{1}{2}'} X'$

Table 4.3: Comparison of decay amplitudes for the decays of B^+ and D_s^+ to the final state $K^0\pi^0\pi^+$.

corresponds to the following exchanges in the Dalitz plot:

$$V\text{-spin} \equiv K^+ \leftrightarrow \pi^0 \implies s \leftrightarrow t,$$
$$\text{Isospin} \equiv \pi^0 \leftrightarrow \pi^- \implies t \leftrightarrow u.$$

When isospin and V-spin are exact, the final state $K^+\pi^0\pi^-$ has the following two possibilities:

1. $K^+\pi^0$ would exist in either symmetrical or anti-symmetrical state under $s \leftrightarrow t$ exchange, and
2. $\pi^0\pi^-$ would exist in either symmetrical or anti-symmetrical state under $t \leftrightarrow u$ exchange.

Following the same steps as enunciated in subsection 4.4.1, the amplitude for the decay can be shown to have two components, one fully symmetric and the other fully anti-symmetric under exchange of any pair of final particles. The final state can be expanded in terms of isospin and V-spin states as follows:

- Isospin

$$\left|K^+\pi^0\pi^-\right\rangle = \frac{1}{\sqrt{5}}\left|\frac{5}{2},-\frac{1}{2}\right\rangle_I^{\mathrm{e}} + \sqrt{\frac{3}{10}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle_I^{\mathrm{e}} + \frac{1}{\sqrt{6}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle_I^{\mathrm{o}} + \frac{1}{\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle_I^{\mathrm{o}}, \tag{4.164}$$

  where the superscripts e, o denote even, odd behaviour of the state under the exchange $\pi^0 \leftrightarrow \pi^-$.

- V-spin

$$\begin{aligned}\left|K^+\pi^0\pi^-\right\rangle = {} & \frac{1}{2\sqrt{5}}\left|\frac{5}{2},+\frac{1}{2}\right\rangle_V^{\mathrm{e}} + \frac{\sqrt{3}}{2\sqrt{10}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_V^{\mathrm{e}} + \frac{1}{2\sqrt{6}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_V^{\mathrm{o}} + \frac{1}{2\sqrt{3}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle_V^{\mathrm{o}} \\ & - \frac{1}{2}\left|\frac{3'}{2},+\frac{1}{2}\right\rangle_V^{\mathrm{e}} - \frac{1}{\sqrt{2}}\left|\frac{1'}{2},+\frac{1}{2}\right\rangle_V^{\mathrm{e}},\end{aligned} \tag{4.165}$$

where the superscripts e, o denote even, odd behaviour of the state under the exchange $K^+ \leftrightarrow \pi^0$.

The odd terms in the above expressions for the state $\left|K^+\pi^0\pi^-\right\rangle$ change sign under exchange of the appropriate pair of particles, because of the sign changes in the two particle states as shown below (also noted in Table 4.1):

- Isospin:

$$\left|\pi^0\pi^-\right\rangle = \frac{1}{\sqrt{2}}\Big(|2,-1\rangle_I + |1,-1\rangle_I \Big), \tag{4.166}$$

$$\left|\pi^-\pi^0\right\rangle = \frac{1}{\sqrt{2}}\Big(|2,-1\rangle_I - |1,-1\rangle_I \Big). \tag{4.167}$$

- V-spin:

$$\left|K^+\pi^0\right\rangle = -\frac{1}{2\sqrt{2}}\Big(|2,+1\rangle_V + |1,+1\rangle_V \Big) + \frac{\sqrt{3}}{2}\left|1',+1\right\rangle_V, \tag{4.168}$$

$$\left|\pi^0K^+\right\rangle = -\frac{1}{2\sqrt{2}}\Big(|2,+1\rangle_V - |1,+1\rangle_V \Big) + \frac{\sqrt{3}}{2}\left|1',+1\right\rangle_V. \tag{4.169}$$

If isospin were an exact symmetry, the $|2,-1\rangle_I$ and $|1,-1\rangle_I$ states of $\left|\pi^-\pi^0\right\rangle$ would exist in even and odd partial wave states respectively, as was the case in subsection 4.4.1 also. On the other hand, if V-spin were an exact symmetry the state $\left|K^+\pi^0\right\rangle$ must remain unchanged under $K^+ \leftrightarrow \pi^0$ exchange. Under V-spin the $\left|K^+\pi^0\right\rangle$ state can exist in $|2,+1\rangle_V$ and $|1,+1\rangle_V$, out of which $|1,+1\rangle_V$ has a contribution from the $|0,0\rangle_{V,8}$ admixture in π^0, denoted above by $|1',+1\rangle_V$. Both state $|2,+1\rangle_V$ and the state $|1',+1\rangle_V$ exist in space symmetric (even partial wave) states, and that part of $|1,+1\rangle_V$ arising from $|1,0\rangle_V$ part of π^0 exists in space anti-symmetric (odd partial wave) state.

If we consider the initial state to be B_d^0 which is an isospin $\left|\frac{1}{2},+\frac{1}{2}\right\rangle_I$ state but a V-spin singlet $|0,0\rangle_V$ state, the standard model Hamiltonian allows only $\Delta I = 0, 1$ and $\Delta V = \frac{1}{2}, \frac{3}{2}$

transitions. Therefore, in addition to the isospin amplitudes of Eq. 4.133, we can define the following V-spin amplitudes:

$$V_{\frac{3}{2},\frac{3}{2}} = \left\langle \frac{3}{2}, \pm\frac{1}{2} \middle| \mathscr{H}_{\Delta V=\frac{3}{2}} \middle| 0, 0 \right\rangle , \tag{4.170a}$$

$$V'_{\frac{3}{2},\frac{3}{2}} = \left\langle \frac{3}{2}', \pm\frac{1}{2} \middle| \mathscr{H}_{\Delta V=\frac{3}{2}} \middle| 0, 0 \right\rangle , \tag{4.170b}$$

$$V_{\frac{1}{2},\frac{1}{2}} = \pm\sqrt{\frac{2}{3}} \left\langle \frac{1}{2}, \pm\frac{1}{2} \middle| \mathscr{H}_{\Delta V=\frac{1}{2}} \middle| 0, 0 \right\rangle , \tag{4.170c}$$

$$V'_{\frac{1}{2},\frac{1}{2}} = \sqrt{\frac{1}{3}} \left\langle \frac{1'}{2}, \pm\frac{1}{2} \middle| \mathscr{H}_{\Delta V=\frac{1}{2}} \middle| 0, 0 \right\rangle . \tag{4.170d}$$

The amplitude for the decay $B_d^0 \to K^+\pi^0\pi^-$ can, therefore, be written as

$$A(B_d^0 \to K^+\pi^0\pi^-) = -\frac{3}{\sqrt{10}} T^{\mathrm{e}}_{1,\frac{3}{2}} X + \frac{1}{\sqrt{2}} \left(-T^{\mathrm{o}}_{1,\frac{1}{2}} + T^{\mathrm{o}}_{0,\frac{1}{2}} \right) Y \sin\theta, \tag{4.171}$$

$$\begin{aligned} A(B_d^0 \to K^+\pi^0\pi^-) = {} & \sqrt{\frac{3}{2}} \left(\frac{1}{\sqrt{20}} V^{\mathrm{e}}_{\frac{3}{2},\frac{3}{2}} - \frac{1}{\sqrt{6}} V'^{e}_{\frac{3}{2},\frac{3}{2}'} - V'^{e}_{\frac{1}{2},\frac{1}{2}'} \right) X'' \\ & + \frac{1}{2\sqrt{2}} \left(\frac{1}{\sqrt{3}} V^{\mathrm{o}}_{\frac{3}{2},\frac{3}{2}} + V^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}} \right) Y'' \sin\theta'', \end{aligned} \tag{4.172}$$

where X'' and Y'' are, in general, arbitrary functions of r and $\cos\theta''$, and they take care of spatial and kinematic contributions to the decay amplitude. As argued before, the part of the amplitude even (or odd) under isospin must also be even (or odd) under V-spin:

$$\mathcal{A}_{SS} = \frac{3}{\sqrt{10}} T^{\mathrm{e}}_{1,\frac{3}{2}} X = \sqrt{\frac{3}{2}} \left(\frac{1}{\sqrt{20}} V^{\mathrm{e}}_{\frac{3}{2},\frac{3}{2}} - \frac{1}{\sqrt{6}} V'^{e}_{\frac{3}{2},\frac{3}{2}'} - V'^{e}_{\frac{1}{2},\frac{1}{2}'} \right) X'', \tag{4.173}$$

$$\mathcal{A}_{AA} = \frac{1}{\sqrt{2}} (-T^{\mathrm{o}}_{1,\frac{1}{2}} + T^{\mathrm{o}}_{0,\frac{1}{2}}) Y \sin\theta = \frac{1}{2\sqrt{2}} \left(\frac{1}{\sqrt{3}} V^{\mathrm{o}}_{\frac{3}{2},\frac{3}{2}} + V^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}} \right) Y'' \sin\theta''. \tag{4.174}$$

It is now straightforward to conclude that the Dalitz plot distributions in the even numbered sextants would be identical to one another, and those of odd numbered sextants would also be similar. Any deviation from this would constitute a signature of simultaneous violations of isospin and V-spin.

Since K^+ and π^- belong to different multiplets of U-spin, in order to consider the symmetry properties under $K^+ \leftrightarrow \pi^-$ one needs to define the G-parity analogue of U-spin, denoted by G_U and defined in the Appendix A. Since charge conjugation is a good symmetry in strong interaction, G_U is as good as U-spin itself. The state $\left|K^+\pi^-\right\rangle$ is composed of states which are even and odd under G_U-parity:

$$\left|K^+\pi^-\right\rangle = \frac{1}{2}\Big(\left|K^+\pi^-\right\rangle_{\rm e} + \left|K^+\pi^-\right\rangle_{\rm o}\Big), \tag{4.175}$$

where

$$\left|K^+\pi^-\right\rangle_{\rm e} = \left|K^+\pi^-\right\rangle - \left|\pi^-K^+\right\rangle, \quad \left|K^+\pi^-\right\rangle_{\rm o} = \left|K^+\pi^-\right\rangle + \left|\pi^-K^+\right\rangle, \tag{4.176}$$

and

$$G_U\left|K^+\pi^-\right\rangle_{\rm e} = \left|K^+\pi^-\right\rangle_{\rm e}, \quad G_U\left|K^+\pi^-\right\rangle_{\rm o} = -\left|K^+\pi^-\right\rangle_{\rm o}. \tag{4.177}$$

Note that the state even (or odd) under G_U is in odd (or even) partial wave state. We have already proven that the amplitudes for the decay $B_d^0 \to K^+\pi^0\pi^-$ has two parts one even and the other odd under the exchange of any two particles in the final state. Hence, $\mathcal{A}_{SS}$ is odd under G_U and $\mathcal{A}_{AA}$ is even under G_U. Since two G_U-parity amplitudes do not interfere, the interference between $\mathcal{A}_{SS}$ and $\mathcal{A}_{AA}$, which is the f_{AA} term (see Eq. (4.144)), vanishes. Therefore, if G_U is a good symmetry of nature it is interesting to conclude that the Dalitz plot is completely symmetric under $s \leftrightarrow t \leftrightarrow u$. The Dalitz plot asymmetries which would be a measure of the extent of breaking of the $SU(3)$ flavor symmetry are, therefore, given by

$$\begin{aligned}\mathbb{A}_{\rm Isospin} = &\left|\frac{\Sigma_{VI}^{I} - \Sigma_{IV}^{III}}{\Sigma_{VI}^{I} + \Sigma_{IV}^{III}}\right| + \left|\frac{\Sigma_{IV}^{III} - \Sigma_{II}^{V}}{\Sigma_{IV}^{III} + \Sigma_{II}^{V}}\right| + \left|\frac{\Sigma_{II}^{V} - \Sigma_{VI}^{I}}{\Sigma_{II}^{V} + \Sigma_{VI}^{I}}\right| \\ &+ \left|\frac{\Delta_{VI}^{I} - \Delta_{IV}^{III}}{\Delta_{VI}^{I} + \Delta_{IV}^{III}}\right| + \left|\frac{\Delta_{IV}^{III} - \Delta_{II}^{V}}{\Delta_{IV}^{III} + \Delta_{II}^{V}}\right| + \left|\frac{\Delta_{II}^{V} - \Delta_{VI}^{I}}{\Delta_{II}^{V} + \Delta_{VI}^{I}}\right|,\end{aligned} \tag{4.178}$$

$$\begin{aligned}\mathbb{A}_{U\text{-spin}} = &\left|\frac{\Sigma^V_{IV}-\Sigma^I_{II}}{\Sigma^V_{IV}+\Sigma^I_{II}}\right| + \left|\frac{\Sigma^I_{II}-\Sigma^{III}_{VI}}{\Sigma^I_{II}+\Sigma^{III}_{VI}}\right| + \left|\frac{\Sigma^{III}_{VI}-\Sigma^V_{IV}}{\Sigma^{III}_{VI}+\Sigma^V_{IV}}\right| \\ &+ \left|\frac{\Delta^V_{IV}-\Delta^I_{II}}{\Delta^V_{IV}+\Delta^I_{II}}\right| + \left|\frac{\Delta^I_{II}-\Delta^{III}_{VI}}{\Delta^I_{II}+\Delta^{III}_{VI}}\right| + \left|\frac{\Delta^{III}_{VI}-\Delta^V_{IV}}{\Delta^{III}_{VI}+\Delta^V_{IV}}\right|, \end{aligned} \tag{4.179}$$

$$\begin{aligned}\mathbb{A}_{V\text{-spin}} = &\left|\frac{\Sigma^I_{IV}-\Sigma^{III}_{II}}{\Sigma^I_{IV}+\Sigma^{III}_{II}}\right| + \left|\frac{\Sigma^{III}_{II}-\Sigma^V_{VI}}{\Sigma^{III}_{II}+\Sigma^V_{VI}}\right| + \left|\frac{\Sigma^V_{VI}-\Sigma^I_{IV}}{\Sigma^V_{VI}+\Sigma^I_{IV}}\right| \\ &+ \left|\frac{\Delta^I_{IV}-\Delta^{III}_{II}}{\Delta^I_{IV}+\Delta^{III}_{II}}\right| + \left|\frac{\Delta^{III}_{II}-\Delta^V_{VI}}{\Delta^{III}_{II}+\Delta^V_{VI}}\right| + \left|\frac{\Delta^V_{VI}-\Delta^I_{IV}}{\Delta^V_{VI}+\Delta^I_{IV}}\right|, \end{aligned} \tag{4.180}$$

where the Σ's and Δ's are defined in Eqs. (4.154a) and (4.154b) respectively. It must again be noted that these asymmetries are in general functions of r and θ (or θ' or θ''), and are defined throughout the Dalitz plot region, including resonant regions. It would again be interesting to look for patterns in the variations of these asymmetries inside the Dalitz plot. Observation of these asymmetries would quantify the extent of breaking of $SU(3)$ flavor symmetry in the concerned decay mode. One can also look for such asymmetries in the Dalitz plot for $\overline{B}^0_s \to K^+\pi^0\pi^-$. The amplitudes for this process are given in Table 4.4.

4.4.3 Decay Mode with final state $K^+\pi^0\overline{K}^0$

For study of simultaneous violations of both U-spin and V-spin, we consider decays such as B^+ or $D^+ \to K^+\pi^0\overline{K}^0$ and their conjugate modes. In the final state $K^+\pi^0\overline{K}^0$, the particles K^+, π^0 are exchangeable under V-spin and $\pi^0, \overline{K}^0$ can be exchanged under U-spin. These correspond to the following exchanges in the Dalitz plot:

$$V\text{-spin} \equiv K^+ \leftrightarrow \pi^0 \implies s \leftrightarrow t,$$
$$U\text{-spin} \equiv \pi^0 \leftrightarrow \overline{K}^0 \implies t \leftrightarrow u.$$

Under V-spin, the $K^+\pi^0$ state can exist in $|2,+1\rangle_V$ and $|1,+1\rangle_V$, out of which the state $|1,+1\rangle_V$ has a contribution from the $|0,0\rangle_{V,8}$ admixture in π^0. Thus if V-spin were an

$B_d^0 \to K^+\pi^0\pi^-$

Isospin (initial state $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$)				V-spin (initial state $\|0,0\rangle$)			
transition	final state	symmetry	Amplitude	transition	final state	symmetry	Amplitude
$\Delta I = 1$	$\left\|\frac{3}{2}, -\frac{1}{2}\right\rangle$	mixed	$\frac{3}{\sqrt{10}} T^{\mathrm{e}}_{1,\frac{3}{2}} X + \frac{1}{\sqrt{2}} T^{\mathrm{o}}_{1,\frac{3}{2}} Y \sin\theta$	$\Delta V = \frac{3}{2}$	$\left\|\frac{3}{2}, +\frac{1}{2}\right\rangle$	mixed	$\frac{\sqrt{3}}{2\sqrt{10}} V^{\mathrm{e}}_{\frac{3}{2},\frac{3}{2}} X'' + \frac{1}{2\sqrt{6}} V^{\mathrm{o}}_{\frac{3}{2},\frac{3}{2}} Y'' \sin\theta''$
$\Delta I = 1$	$\left\|\frac{1}{2}, -\frac{1}{2}\right\rangle$	odd	$-\frac{1}{\sqrt{2}} T^{\mathrm{o}}_{1,\frac{1}{2}} Y \sin\theta$	$\Delta V = \frac{3}{2}$	$\left\|\frac{3}{2}', +\frac{1}{2}\right\rangle$	even	$-\frac{1}{2} V'^{e}_{\frac{3}{2},\frac{3}{2}} X''$
$\Delta I = 0$	$\left\|\frac{1}{2}, -\frac{1}{2}\right\rangle$	odd	$\frac{1}{\sqrt{2}} T^{\mathrm{o}}_{0,\frac{1}{2}} Y \sin\theta$	$\Delta V = \frac{1}{2}$	$\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$	odd	$\frac{1}{2\sqrt{2}} V^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}} Y'' \sin\theta''$
				$\Delta V = \frac{1}{2}$	$\left\|\frac{1}{2}', +\frac{1}{2}\right\rangle$	even	$-\frac{\sqrt{3}}{\sqrt{2}} V'^{e}_{\frac{1}{2},\frac{1}{2}} X''$

$\overline{B}_s^0 \to K^+\pi^0\pi^-$

Isospin (initial state $\|0,0\rangle$)				V-spin (initial state $\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$)			
transition	final state	symmetry	Amplitude	transition	final state	symmetry	Amplitude
$\Delta I = \frac{3}{2}$	$\left\|\frac{3}{2}, -\frac{1}{2}\right\rangle$	mixed	$\frac{\sqrt{3}}{\sqrt{10}} T^{\mathrm{e}}_{\frac{3}{2},\frac{3}{2}} X + \frac{1}{\sqrt{6}} T^{\mathrm{o}}_{\frac{3}{2},\frac{3}{2}} Y \sin\theta$	$\Delta V = 1$	$\left\|\frac{3}{2}, +\frac{1}{2}\right\rangle$	mixed	$\frac{3}{2\sqrt{10}} V^{\mathrm{e}}_{1,\frac{3}{2}} X'' + \frac{1}{2\sqrt{2}} V^{\mathrm{o}}_{1,\frac{3}{2}} Y'' \sin\theta''$
$\Delta I = \frac{1}{2}$	$\left\|\frac{1}{2}, -\frac{1}{2}\right\rangle$	odd	$-\frac{1}{\sqrt{2}} T^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}} Y \sin\theta$	$\Delta V = 1$	$\left\|\frac{3}{2}', +\frac{1}{2}\right\rangle$	even	$-\frac{\sqrt{3}}{2} V'^{e}_{1,\frac{3}{2}} X''$
				$\Delta V = 1$	$\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$	odd	$\frac{1}{2\sqrt{2}} V^{\mathrm{o}}_{1,\frac{1}{2}} Y'' \sin\theta''$
				$\Delta V = 1$	$\left\|\frac{1}{2}', +\frac{1}{2}\right\rangle$	even	$-\frac{\sqrt{3}}{2} V'^{\mathrm{e}}_{1,\frac{1}{2}} X''$
				$\Delta V = 0$	$\left\|\frac{1}{2}, +\frac{1}{2}\right\rangle$	odd	$\frac{1}{2\sqrt{2}} V^{\mathrm{o}}_{0,\frac{1}{2}} Y'' \sin\theta''$
				$\Delta V = 0$	$\left\|\frac{1}{2}', +\frac{1}{2}\right\rangle$	even	$-\frac{\sqrt{3}}{2} V'^{e}_{0,\frac{1}{2}} X''$

Table 4.4: Comparison of decay amplitudes for the decays of B_d^0 and $\overline{B}_s^0$ to the final state $K^+\pi^0\pi^-$.

exact symmetry the state $|2,+1\rangle_V$ and that part of the $|1,+1\rangle_V$ state coming from $|0,0\rangle_{V,8}$ contribution of π^0 would be in space symmetric (even partial wave) states. The remaining part of $|1,+1\rangle_V$ state would be in space anti-symmetric (odd partial wave) state. Similarly, the $\pi^0\overline{K}^0$ state would exist in $|2,-1\rangle_U$ and $|1,-1\rangle_U$, out of which the state $|1,-1\rangle_U$ has a contribution from the $|0,0\rangle_{U,8}$ admixture in π^0. Thus, if U-spin were assumed to be an exact symmetry, the states $|2,-1\rangle_U$ and the $|1,-1\rangle_U$ state coming from $|0,0\rangle_{U,8}$ part of π^0 would exist in space symmetric (even partial wave) states, and the other part of $|1,-1\rangle_U$ would exist in space anti-symmetric (odd partial wave) state. Therefore, under exact U-spin and V-spin, the final state $K^+\pi^0\overline{K}^0$ has, the following two possibilities:

1. $K^+\pi^0$ would exist in either symmetrical or anti-symmetrical state under $s \leftrightarrow t$ exchange, and

2. $\pi^0\overline{K}^0$ would exist in either symmetrical or anti-symmetrical state under $t \leftrightarrow u$ exchange.

Again, following the steps as enunciated in subsection 4.4.1 we can conclude that the Dalitz plot distribution in the even numbered sextants would be identical to one another, and those of odd numbered sextants would also be similar, as given in Eqs. (4.146) and (4.147). Any deviation from this would constitute a signature of simultaneous violations of U-spin and V-spin. We can once again reaffirm the same logic as given in subsections 4.4.1 and 4.4.2, by invoking the G_I-parity operator (see Appendix A) to connect K^+ and $\overline{K}^0$ belonging to two different isospin doublets. This would lead to a fully symmetric Dalitz plot which would be broken when G_I is broken. The amplitudes for the two decay modes under consideration are given in Table 4.5. The Dalitz plot asymmetries that can be useful in this case are given by

$$\begin{aligned}
\mathbb{A}_{\text{Isospin}} &= \left|\frac{\Sigma^{V}_{IV}-\Sigma^{I}_{II}}{\Sigma^{V}_{IV}+\Sigma^{I}_{II}}\right| + \left|\frac{\Sigma^{I}_{II}-\Sigma^{III}_{VI}}{\Sigma^{I}_{II}+\Sigma^{III}_{VI}}\right| + \left|\frac{\Sigma^{III}_{VI}-\Sigma^{V}_{IV}}{\Sigma^{III}_{VI}+\Sigma^{V}_{IV}}\right| \\
&\quad + \left|\frac{\Delta^{V}_{IV}-\Delta^{I}_{II}}{\Delta^{V}_{IV}+\Delta^{I}_{II}}\right| + \left|\frac{\Delta^{I}_{II}-\Delta^{III}_{VI}}{\Delta^{I}_{II}+\Delta^{III}_{VI}}\right| + \left|\frac{\Delta^{III}_{VI}-\Delta^{V}_{IV}}{\Delta^{III}_{VI}+\Delta^{V}_{IV}}\right|, \qquad (4.181)\\
\mathbb{A}_{U\text{-spin}} &= \left|\frac{\Sigma^{I}_{VI}-\Sigma^{III}_{IV}}{\Sigma^{I}_{VI}+\Sigma^{III}_{IV}}\right| + \left|\frac{\Sigma^{III}_{IV}-\Sigma^{V}_{II}}{\Sigma^{III}_{IV}+\Sigma^{V}_{II}}\right| + \left|\frac{\Sigma^{V}_{II}-\Sigma^{I}_{VI}}{\Sigma^{V}_{II}+\Sigma^{I}_{VI}}\right| \\
&\quad + \left|\frac{\Delta^{I}_{VI}-\Delta^{III}_{IV}}{\Delta^{I}_{VI}+\Delta^{III}_{IV}}\right| + \left|\frac{\Delta^{III}_{IV}-\Delta^{V}_{II}}{\Delta^{III}_{IV}+\Delta^{V}_{II}}\right| + \left|\frac{\Delta^{V}_{II}-\Delta^{I}_{VI}}{\Delta^{V}_{II}+\Delta^{I}_{VI}}\right|, \qquad (4.182)\\
\mathbb{A}_{V\text{-spin}} &= \left|\frac{\Sigma^{I}_{IV}-\Sigma^{III}_{II}}{\Sigma^{I}_{IV}+\Sigma^{III}_{II}}\right| + \left|\frac{\Sigma^{III}_{II}-\Sigma^{V}_{VI}}{\Sigma^{III}_{II}+\Sigma^{V}_{VI}}\right| + \left|\frac{\Sigma^{V}_{VI}-\Sigma^{I}_{IV}}{\Sigma^{V}_{VI}+\Sigma^{I}_{IV}}\right| \\
&\quad + \left|\frac{\Delta^{I}_{IV}-\Delta^{III}_{II}}{\Delta^{I}_{IV}+\Delta^{III}_{II}}\right| + \left|\frac{\Delta^{III}_{II}-\Delta^{V}_{VI}}{\Delta^{III}_{II}+\Delta^{V}_{VI}}\right| + \left|\frac{\Delta^{V}_{VI}-\Delta^{I}_{IV}}{\Delta^{V}_{VI}+\Delta^{I}_{IV}}\right|. \qquad (4.183)
\end{aligned}$$

Once again the asymmetries being, in general, functions of r and θ (or θ' or θ''), it would be very interesting to look for their variation across the Dalitz plot. These would be the visible signatures of the breaking of $SU(3)$ flavor symmetry.

4.4.4 Decay Mode with final state $\pi^+\pi^0\overline{K}^0$

Finally, we consider a mode where each pair of the particles in the final state can be directly related by one of the three $SU(2)$ symmetries, namely isospin, U-spin and V-spin. Here we do not need G_I, G_V or G_U to relate the final states. We consider, as an example, decays with final state $\pi^+\pi^0\overline{K}^0$ such as $D^+ \to \pi^+\pi^0\overline{K}^0$ and the conjugate mode. In the final state considered here, isospin exchange implies $\pi^0 \leftrightarrow \pi^+$, U-spin exchange implies $\pi^0 \leftrightarrow \overline{K}^0$ and V-spin exchange implies $\pi^+ \leftrightarrow \overline{K}^0$. The $SU(2)$ decompositions of all the pairs of particles under their respective $SU(2)$ symmetries have already been considered in subsections 4.4.1, 4.4.2, 4.4.3. Once again, the steps elaborated in subsection 4.4.1 are applicable to this case also. The amplitudes for this decay mode can be easily read

$B^+ \to K^+\pi^0\overline{K}^0$							
U-spin (initial state $\lvert 0,0\rangle$)				V-spin (initial state $\lvert\frac{1}{2},+\frac{1}{2}\rangle$)			
transition	final state	symmetry	Amplitude	transition	final state	symmetry	Amplitude
$\Delta U=\frac{1}{2}$	$\lvert\frac{1}{2},-\frac{1}{2}\rangle$	odd	$\frac{1}{2\sqrt{2}}U^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}}Y'\sin\theta'$	$\Delta V=1$	$\lvert\frac{3}{2},+\frac{1}{2}\rangle$	mixed	$-\frac{3}{2\sqrt{10}}V^{\mathrm{e}}_{1,\frac{3}{2}}X''-\frac{1}{2\sqrt{2}}V^{\mathrm{o}}_{1,\frac{3}{2}}Y''\sin\theta''$
$\Delta U=\frac{1}{2}$	$\lvert\frac{1}{2}',-\frac{1}{2}\rangle$	even	$\frac{\sqrt{3}}{\sqrt{2}}U'^{e}_{\frac{1}{2},\frac{1}{2}},X'$	$\Delta V=1$	$\lvert\frac{3}{2}',+\frac{1}{2}\rangle$	even	$\frac{\sqrt{3}}{2}V'^{e}_{1,\frac{3}{2}},X''$
$\Delta U=\frac{3}{2}$	$\lvert\frac{3}{2},-\frac{1}{2}\rangle$	mixed	$\frac{\sqrt{3}}{2\sqrt{10}}U^{\mathrm{e}}_{\frac{3}{2},\frac{3}{2}}X'-\frac{1}{2\sqrt{6}}U^{\mathrm{o}}_{\frac{3}{2},\frac{3}{2}}Y'\sin\theta'$	$\Delta V=1$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	odd	$-\frac{1}{2\sqrt{2}}V^{\mathrm{o}}_{1,\frac{1}{2}}Y''\sin\theta''$
$\Delta U=\frac{3}{2}$	$\lvert\frac{3}{2}',-\frac{1}{2}\rangle$	even	$\frac{1}{2}U'^{e}_{\frac{1}{2},\frac{1}{2}},X'$	$\Delta V=1$	$\lvert\frac{1}{2}',+\frac{1}{2}\rangle$	even	$\frac{\sqrt{3}}{2}V'^{e}_{1,\frac{1}{2}},X''$
				$\Delta V=0$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	odd	$-\frac{1}{2\sqrt{2}}V^{\mathrm{o}}_{0,\frac{1}{2}}Y''\sin\theta''$
				$\Delta V=0$	$\lvert\frac{1}{2}',+\frac{1}{2}\rangle$	even	$\frac{\sqrt{3}}{2}V'^{e}_{0,\frac{1}{2}}X''$

$D^+ \to K^+\pi^0\overline{K}^0$							
U-spin (initial state $-\lvert\frac{1}{2},-\frac{1}{2}\rangle$)				V-spin (initial state $\lvert 0,0\rangle$)			
transition	final state	symmetry	Amplitude	transition	final state	symmetry	Amplitude
$\Delta U=1$	$\lvert\frac{3}{2},-\frac{1}{2}\rangle$	mixed	$-\frac{3}{2\sqrt{10}}U^{\mathrm{e}}_{1,\frac{3}{2}}X'-\frac{1}{2\sqrt{2}}U^{\mathrm{o}}_{1,\frac{3}{2}}Y'\sin\theta'$	$\Delta V=\frac{3}{2}$	$\lvert\frac{3}{2},+\frac{1}{2}\rangle$	mixed	$-\frac{\sqrt{3}}{2\sqrt{10}}V^{\mathrm{e}}_{\frac{3}{2},\frac{3}{2}}X''-\frac{1}{2\sqrt{6}}V^{\mathrm{o}}_{\frac{3}{2},\frac{3}{2}}Y''\sin\theta''$
$\Delta U=1$	$\lvert\frac{3}{2}',-\frac{1}{2}\rangle$	even	$\frac{\sqrt{3}}{2}U'^{e}_{1,\frac{3}{2}},X'$	$\Delta V=\frac{3}{2}$	$\lvert\frac{3}{2}',+\frac{1}{2}\rangle$	even	$\frac{1}{2}V'^{e}_{\frac{3}{2},\frac{3}{2}},X''$
$\Delta U=1$	$\lvert\frac{1}{2},-\frac{1}{2}\rangle$	odd	$\frac{1}{2\sqrt{2}}U^{\mathrm{o}}_{1,\frac{1}{2}}Y'\sin\theta'$	$\Delta V=\frac{1}{2}$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	odd	$-\frac{1}{2}V^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}}Y''\sin\theta''$
$\Delta U=1$	$\lvert\frac{1}{2}',-\frac{1}{2}\rangle$	even	$-\frac{\sqrt{3}}{2}U'^{e}_{1,\frac{1}{2}},X'$	$\Delta V=\frac{1}{2}$	$\lvert\frac{1}{2}',+\frac{1}{2}\rangle$	even	$\frac{\sqrt{3}}{\sqrt{2}}V'^{e}_{\frac{1}{2},\frac{1}{2}},X''$
$\Delta U=0$	$\lvert\frac{1}{2},-\frac{1}{2}\rangle$	odd	$-\frac{1}{2\sqrt{2}}U^{\mathrm{o}}_{0,\frac{1}{2}}Y'\sin\theta'$				
$\Delta U=0$	$\lvert\frac{1}{2}',-\frac{1}{2}\rangle$	even	$\frac{\sqrt{3}}{2}U'^{e}_{0,\frac{1}{2}},X'$				

Table 4.5: Comparison of decay amplitudes for the decays of B^+ and D^+ to the final state $K^+\pi^0\overline{K}^0$.

Isospin $\left(\text{initial state } \left|\frac{1}{2},+\frac{1}{2}\right\rangle\right)$			
transition	final state	symmetry	Amplitude
$\Delta I=1$	$\left\lvert\frac{3}{2},+\frac{3}{2}\right\rangle$	mixed	$\frac{\sqrt{3}}{\sqrt{10}}T^{\mathrm{e}}_{1,\frac{3}{2}}X+\frac{\sqrt{3}}{\sqrt{2}}T^{\mathrm{o}}_{1,\frac{3}{2}}Y\sin\theta$

U-spin $\left(\text{initial state } -\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)$			
transition	final state	symmetry	Amplitude
$\Delta U=1$	$\left\lvert\frac{3}{2},-\frac{3}{2}\right\rangle$	mixed	$\frac{\sqrt{3}}{2\sqrt{5}}U^{\mathrm{e}}_{1,\frac{3}{2}}X'-\frac{\sqrt{3}}{2}U^{\mathrm{o}}_{1,\frac{3}{2}}Y'\sin\theta'$
	$\left\lvert\frac{3}{2}',-\frac{3}{2}\right\rangle$	even	$\frac{3}{2}U'^{\mathrm{e}}_{1,\frac{3}{2}'}X'$

V-spin $\left(\text{initial state } \lvert 0,0\rangle\right)$			
transition	final state	symmetry	Amplitude
$\Delta V=1$	$\lvert 1,0\rangle$	mixed	$-\frac{1}{2\sqrt{6}}V^{\mathrm{e}}_{1,1}X''-\frac{1}{2\sqrt{3}}V^{\mathrm{o}}_{1,1}Y''\sin\theta''$
	$\lvert 1',0\rangle$	even	$\frac{\sqrt{3}}{2\sqrt{2}}V'^{\mathrm{e}}_{1,1}X''$
$\Delta V=0$	$\lvert 0,0\rangle$	odd	$-\frac{1}{2\sqrt{6}}V^{\mathrm{o}}_{0,0}Y''\sin\theta''$
	$\lvert 0',0\rangle$	odd	$\frac{\sqrt{3}}{2\sqrt{2}}V'^{\mathrm{o}}_{0,0}Y''\sin\theta''$

Table 4.6: Amplitudes for the decay $D^+\to\pi^+\pi^0\overline{K}^0$. The V-spin amplitudes are defined as follows: $V_{1,1}=\langle 1,0|\mathscr{H}_{\Delta V=1}|0,0\rangle$, $V'_{1,1}=\langle 1',0|\mathscr{H}_{\Delta V=1}|0,0\rangle$, $V_{0,0}=\langle 0,0|\mathscr{H}_{\Delta V=0}|0,0\rangle$ and $V'_{0,0}=\langle 0',0|\mathscr{H}_{\Delta V=0}|0,0\rangle$.

off from Table 4.6. However, in this mode the even and odd contributions to the decay amplitude can interfere as they are not eigenstates of G-parity, resulting in even and odd numbered sextants to have distinctly different density of events as depicted in Eqs. (4.146) and (4.147). Note that the Dalitz plot distributions for the even (odd) numbered sextants of the Dalitz plot would still be identical if isospin and U-spin are exact symmetries. The breakdown of isospin, U-spin and V-spin could be quantitatively measured using the following asymmetries:

$$\begin{aligned}
\mathbb{A}_{\text{Isospin}} = & \left|\frac{\Sigma_{IV}^{I} - \Sigma_{II}^{III}}{\Sigma_{IV}^{I} + \Sigma_{II}^{III}}\right| + \left|\frac{\Sigma_{II}^{III} - \Sigma_{VI}^{V}}{\Sigma_{II}^{III} + \Sigma_{VI}^{V}}\right| + \left|\frac{\Sigma_{VI}^{V} - \Sigma_{IV}^{I}}{\Sigma_{VI}^{V} + \Sigma_{IV}^{I}}\right| \\
& + \left|\frac{\Delta_{IV}^{I} - \Delta_{II}^{III}}{\Delta_{IV}^{I} + \Delta_{II}^{III}}\right| + \left|\frac{\Delta_{II}^{III} - \Delta_{VI}^{V}}{\Delta_{II}^{III} + \Delta_{VI}^{V}}\right| + \left|\frac{\Delta_{VI}^{V} - \Delta_{IV}^{I}}{\Delta_{VI}^{V} + \Delta_{IV}^{I}}\right|, \qquad (4.184) \\
\mathbb{A}_{U\text{-spin}} = & \left|\frac{\Sigma_{VI}^{I} - \Sigma_{IV}^{III}}{\Sigma_{VI}^{I} + \Sigma_{IV}^{III}}\right| + \left|\frac{\Sigma_{IV}^{III} - \Sigma_{II}^{V}}{\Sigma_{IV}^{III} + \Sigma_{II}^{V}}\right| + \left|\frac{\Sigma_{II}^{V} - \Sigma_{VI}^{I}}{\Sigma_{II}^{V} + \Sigma_{VI}^{I}}\right| \\
& + \left|\frac{\Delta_{VI}^{I} - \Delta_{IV}^{III}}{\Delta_{VI}^{I} + \Delta_{IV}^{III}}\right| + \left|\frac{\Delta_{IV}^{III} - \Delta_{II}^{V}}{\Delta_{IV}^{III} + \Delta_{II}^{V}}\right| + \left|\frac{\Delta_{II}^{V} - \Delta_{VI}^{I}}{\Delta_{II}^{V} + \Delta_{VI}^{I}}\right|, \qquad (4.185) \\
\mathbb{A}_{V\text{-spin}} = & \left|\frac{\Sigma_{IV}^{V} - \Sigma_{II}^{I}}{\Sigma_{IV}^{V} + \Sigma_{II}^{I}}\right| + \left|\frac{\Sigma_{II}^{I} - \Sigma_{VI}^{III}}{\Sigma_{II}^{I} + \Sigma_{VI}^{III}}\right| + \left|\frac{\Sigma_{VI}^{III} - \Sigma_{IV}^{V}}{\Sigma_{VI}^{III} + \Sigma_{IV}^{V}}\right| \\
& + \left|\frac{\Delta_{IV}^{V} - \Delta_{II}^{I}}{\Delta_{IV}^{V} + \Delta_{II}^{I}}\right| + \left|\frac{\Delta_{II}^{I} - \Delta_{VI}^{III}}{\Delta_{II}^{I} + \Delta_{VI}^{III}}\right| + \left|\frac{\Delta_{VI}^{III} - \Delta_{IV}^{V}}{\Delta_{VI}^{III} + \Delta_{IV}^{V}}\right|. \qquad (4.186)
\end{aligned}$$

Once again these asymmetries being, in general, functions of r and θ (or θ' or θ'') it would be very interesting to look for their variation across the Dalitz plot. These would constitute the visible signatures of the breaking of $SU(3)$ flavor symmetry.

4.5 Summary

In this chapter we have elucidated a few new model independent methods to look for breaking of Bose, CP, CPT and $SU(3)$ flavor symmetry. We use the distribution of events in the Dalitz plot and the Dalitz prism, which carry information about the dynamics of the process and, hence, that of the underlying symmetry, and look for relevant asymmetries in them which quantify the extent of symmetry violations in the processes under consideration. The Dalitz prism can substantially help in looking for some of these symmetry violations which are expected to be extremely small in the decay modes considered.

PART III

Conclusion

5

Summary and Conclusion

Nature has many fundamental symmetries, and the corresponding symmetry operations help us to formulate the underlying physical laws. Thus the concept of symmetry plays a pivotal role in discovering new fundamental laws of physics. It is not only necessary to look for new hitherto unknown symmetries of Nature, but it also is important to search for places where the already existing symmetries do break down or are violated. This exercise provides information about the region of applicability of the symmetry and suggests necessary modifications, if any, to the existing laws of physics. In this thesis we are concerned with investigating violations of a few fundamental symmetries of Nature, namely the Bose, CP, CPT and quark flavor $SU(3)$ symmetries, by use of Dalitz plots (which implies that we shall exploit three-body decays) and their unique generalization the Dalitz prisms (by which processes with more than three particles in the final states can be easily handled and they do get considered). The basic idea is that the distribution of events inside a Dalitz plot or a Dalitz prism is a signature of the underlying dynamics. Symmetries affect the amplitude and hence the distribution of events in Dalitz plots and Dalitz prisms shows some characteristic signature of the underlying symmetry. If we observe any distribution of event contrary to the expectation from symmetry arguments, it is a clear signature of violation of the concerned symmetry. By means of various asymmetries, these violations can be quantified [163–165].

The concept of Dalitz prism is a natural extension of the Dalitz plot if we extend our discussion of a resonant three-body decay to include continuum production of three final particles as well. Dalitz prism in its purest form can be visualized as a stack of Dalitz plots which are stacked up with increasing center-of-momentum energy. Once, this concept is in place, it is fairly straightforward to see that multi-body processes which can be treated as "effective" three-body processes are also easy to consider and analyse via the Dalitz prism. The Dalitz prism works well even when initial or final state radiations are involved. Moreover, being extremely rich with data, it serves as a precision tool with which we can search for extremely small violations of the fundamental symmetries under our consideration.

Bose symmetry is an important cornerstone of modern physics, and states that a system of two identical bosons does not alter in any way when the two bosons are swapped. Bose symmetry is true for stable particles, but in reality it is also applied to particles that have a fleeting existence such as various mesons. Therefore, it is pertinent that we look for violations of Bose symmetry in mesons which are also composite particles. For this we consider three-body processes where all final particles are mesons and at least two mesons are of the same type but they are reconstructed from different final states. Now we can construct the Dalitz plot for such cases. If the Dalitz plot shows any asymmetry under exchange of the two identical mesons, it would be a clear signature of violation of Bose symmetry.

The Dalitz plot distribution also gets affected by the presence of CP violation. Let us consider a three-body process in which two final particles are two neutral mesons that mix with one another and have both direct as well as mixing CP violation. When these two particles are reconstructed from flavor insensitive final states with definite CP, then in the CP conserving scenario Bose symmetry would force the Dalitz plot to be completely symmetric under exchange of the two particles. However, the direct CP asymmetry would manifest as an asymmetry in the Dalitz plot under the exchange of the two neutral mesons under consideration. When the CP violation is tiny as in case of the D mesons, one can consider the Dalitz prism to advantage and look for the small asymmetry in the Dalitz prism.

The Dalitz plot distribution also gets affected if there is CPT violation. For this we consider only those three-body processes which are self-conjugate and have no CP violation in them, in other words the processes must take place via either strong or electromagnetic interactions only. In the kind of final state which are possible here, there is always a pair of CP conjugate particle and a particle which is CP conjugate of itself. If there were no direct CPT violation in this process, then the Dalitz plot or Dalitz prism would be completely symmetric under exchange of the two CP conjugate particles. Any asymmetry in the Dalitz plot under the said exchange is possible only if there is any direct CPT violation in the mode. CPT violation, if at all observable, is always expected to be extremely tiny. Thus the Dalitz prism with appropriate multi-body processes included could be helpful in searching for CPT violation. Even a small but visible asymmetry, can also be interpreted as CP violation in the strong or electromagnetic interaction that facilitates the process. This in itself would constitute a clear evidence of new physics, even though we were to still assume CPT invariance.

The $SU(3)$ flavor symmetry is not an exact symmetry of nature. From its inception to explain light hadronic states via *the eightfold way* [84–86], it has always been considered to be broken in order to account for the mass differences amongst the hadronic states it relates. Full $SU(3)$ flavor symmetry implies that the three flavors of light quarks, namely the up (u), down (d) and strange (s) quarks, are identical. Hence, under full $SU(3)$ flavor symmetry these quarks can be exchanged with each other without affecting any physical observable. However, the mass difference between s quark and either of u or d quarks is substantial, the quarks u and d also do not have same electric charge, and thus these quarks are not fully exchangeable with one another resulting in many observables which are sensitive to the $SU(3)$ flavor symmetry breaking. Nevertheless, an accurate quantitative estimate of $SU(3)$ flavor breaking has not yet been accomplished in dynamical processes. We give a model independent prescription for quantitative estimation of $SU(3)$ flavor symmetry breaking by using the Dalitz plots for a few specific kind of three-body decays. It is well known that the $SU(3)$ flavor symmetry has three non-commuting $SU(2)$ subgroups, namely, isospin, U-spin and V-spin. We consider only those three-body decays in which the final mesons are kaons or pions and particles inside two pairs of the final three mesons are connected to one another by two distinct $SU(2)$ symmetries. Since full $SU(3)$ flavor symmetry implies that all the three $SU(2)$ symmetries are individually and simultaneously valid, the Dalitz plot for the modes can be shown to have either fully symmetric distribution or fully anti-symmetric distribution. This implies that the alternate sextants of the Dalitz plot are identical to one another. Any deviation from this observation would constitute a violation of full $SU(3)$ flavor symmetry. Appropriate Dalitz plot asymmetries can be defined that quantify this violation.

We have thus provided new methods to look for violations of Bose, CP, CPT and $SU(3)$ flavor symmetries by using Dalitz plots and Dalitz prisms. These symmetries play some of the very vital roles in particle physics and any unusual violation of these symmetries would point out various new physics possibilities. Therefore, it would not be over emphasizing to belabour the point that accurate quantitative estimates of these symmetry violations constitute a significant step forward in our search for new physics.

PART IV

Appendices

A

Some details of the $SU(3)$ flavor symmetry

A.1 The generators of $SU(3)$ flavor symmetry

The generators G_a of the group SU(3) are related to Gell-Mann matrices λ_a [84, 85]:

$$G_a = \frac{\lambda_a}{2}. \tag{A.1}$$

The eight traceless and Hermitian Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.2a}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{A.2b}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{A.2c}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A.2d}$$

The Gell-Mann matrices satisfy the following properties:

- Commutation relation:

$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c, \tag{A.3}$$

where f_{abc} are the *structure constants*, and are totally antisymmetric under exchange of any pair of indices with the following non-vanishing members

$$f_{123} = 1, \tag{A.4a}$$

$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}, \tag{A.4b}$$

$$f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \tag{A.4c}$$

- Trace orthogonal relation:

$$\mathrm{Tr}\,(\lambda_a\lambda_b) = 2\delta_{ab}, \tag{A.5}$$

where δ_{ab} denotes the Kronecker delta.

- Anti-commutation relation:

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} + 2d_{abc}\lambda_c, \tag{A.6}$$

where d_{abc} are totally symmetric under exchange of any pair of indices with non-vanishing numbers

$$d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}, \tag{A.7a}$$

$$d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}, \tag{A.7b}$$

$$d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}. \tag{A.7c}$$

Both f_{abc} and d_{abc} can be written in terms of trace of products of Gell-Mann matrices as follows:

$$f_{abc} = -\frac{1}{4} i \,\mathrm{Tr}\left(\lambda_a \left[\lambda_b, \lambda_c\right]\right), \tag{A.8}$$

$$d_{abc} = \frac{1}{4} \mathrm{Tr}\left(\lambda_a \left\{\lambda_b, \lambda_c\right\}\right). \tag{A.9}$$

The generators, therefore, satisfy the following commutation and anti-commutation relations:

$$[G_a, G_b] = i f_{abc} G_c, \tag{A.10a}$$

$$\{G_a, G_b\} = \frac{1}{3}\delta_{ab} + d_{abc} G_c. \tag{A.10b}$$

It is easy to verify that

$$\mathbf{G}^2 = \sum_{a=1}^{8} G_a G_a = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.11}$$

There are two Casimir operators which commute with all the generators G_a of $SU(3)$:

$$\mathbf{C}_1 = \sum_{a=1}^{8} G_a^2 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.12}$$

$$\mathbf{C}_2 = \sum_{a,b,c=1}^{8} d_{abc} G_a G_b G_c = \frac{10}{9}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.13}$$

Since both $\mathbf{C}_1$ and $\mathbf{C}_2$ are proportional to the 3×3 unit matrix, they commute with the generators of $SU(3)$:

$$[\mathbf{C}_i, G_a] = [G_a, \mathbf{C}_i] = 0. \tag{A.14}$$

A.2 The $SU(2)$ subgroups of $SU(3)$ flavor symmetry

In the set of eight $SU(3)$ generators, one can notice the following three subsets which generate $SU(2)$ subgroups, and the three $SU(2)$ subgroups do not commute among them.

- **Isospin subgroup $SU(2)_I$:** Generators of the isospin subgroup $SU(2)_I$ are

$$T_1 = F_1 = \frac{1}{2}\lambda_1, \tag{A.15}$$
$$T_2 = F_2 = \frac{1}{2}\lambda_2, \tag{A.16}$$
$$T_3 = F_3 = \frac{1}{2}\lambda_3, \tag{A.17}$$

and the hypercharge operator is defined as

$$Y = \frac{2}{\sqrt{3}}F_8 = \frac{1}{\sqrt{3}}\lambda_8 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A.18}$$

These satisfy the commutation relations

$$[T_a, T_b] = i\epsilon_{abc}T_c, \tag{A.19}$$
$$[T_3, Y] = 0, \tag{A.20}$$

where ϵ_{abc} is the usual Levi-Civita tensor. So the hypercharge operator is scalar under isospin, i.e., it has $T = 0$. We can also define the isospin raising and lowering operators as

$$T_\pm = T_1 \pm iT_2. \tag{A.21}$$

These operators satisfy the SU(2) algebra

$$[T_3, T_\pm] = \pm T_\pm, \tag{A.22}$$
$$[T_+, T_-] = 2T_3. \tag{A.23}$$

The Casimir operator for the isospin $SU(2)$ subgroup of $SU(3)$ flavor symmetry is

$$\mathbf{T}^2 = T_1^2 + T_2^2 + T_3^2. \tag{A.24}$$

A legitimate isospin eigenstate is one which is eigenstate of both $\mathbf{T}^2$ and T_3, such a state is denoted by $|t, t_3\rangle$:

$$\mathbf{T}^2 |t, t_3\rangle = t(t+1) |t, t_3\rangle , \tag{A.25}$$

$$T_3 |t, t_3\rangle = t_3 |t, t_3\rangle , \tag{A.26}$$

where t and t_3 denote the isospin and 3rd component of isospin of the system. The Casimir operator $\mathbf{T}^2$ commutes with $T_\pm$ and T_3:

$$\left[\mathbf{T}^2, T_\pm\right] = 0 = \left[\mathbf{T}^2, T_3\right] . \tag{A.27}$$

- **U-spin subgroup** $SU(2)_U$**:** Generators of the U-spin subgroup $SU(2)_U$ are

$$U_1 = F_6 = \frac{1}{2}\lambda_6, \tag{A.28}$$

$$U_2 = F_7 = \frac{1}{2}\lambda_7, \tag{A.29}$$

$$U_3 = \frac{1}{2}\left(\sqrt{3}F_8 - F_3\right) = \frac{1}{4}\left(\sqrt{3}\lambda_8 - \lambda_3\right) , \tag{A.30}$$

and the U-spin scalar operator is the electric charge operator:

$$Q = F_3 + \frac{1}{\sqrt{3}}F_8 = \frac{1}{2}\left(\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8\right) = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{A.31}$$

Since $Y = \frac{1}{\sqrt{3}}\lambda_8$, we have

$$\lambda_3 = 2Q - Y. \tag{A.32}$$

These satisfy the commutation relations

$$[U_a, U_b] = i\epsilon_{abc}U_c, \tag{A.33}$$

$$[U_3, Q] = 0. \tag{A.34}$$

We can also define the U-spin raising and lowering operators as

$$U_\pm = U_1 \pm iU_2. \tag{A.35}$$

These operators satisfy the SU(2) algebra

$$[U_3, U_\pm] = \pm U_\pm, \tag{A.36}$$

$$[U_+, U_-] = 2U_3. \tag{A.37}$$

It is possible to relate U_3 to T_3 and Y as follows:

$$U_3 = \frac{1}{2}\left(\frac{3}{2}Y - T_3\right). \tag{A.38}$$

The Casimir operator for the U-spin $SU(2)$ subgroup of $SU(3)$ flavor symmetry is

$$\mathbf{U}^2 = U_1^2 + U_2^2 + U_3^2. \tag{A.39}$$

A legitimate U-spin eigenstate is one which is eigenstate of both $\mathbf{U}^2$ and U_3, such a state is denoted by $|u, u_3\rangle$:

$$\mathbf{U}^2 |u, u_3\rangle = u(u+1) |u, u_3\rangle, \tag{A.40}$$

$$U_3 |u, u_3\rangle = u_3 |u, u_3\rangle, \tag{A.41}$$

where u and u_3 denote the U-spin and 3rd component of U-spin of the system. The Casimir operator $\mathbf{U}^2$ commutes with $U_\pm$ and U_3:

$$\left[\mathbf{U}^2, U_\pm\right] = 0 = \left[\mathbf{U}^2, U_3\right]. \tag{A.42}$$

- **V-spin subgroup $SU(2)_V$:** Generators of the V-spin subgroup $SU(2)_V$ are

$$V_1 = F_4 = \frac{1}{2}\lambda_4, \tag{A.43}$$

$$V_2 = F_5 = \frac{1}{2}\lambda_5, \tag{A.44}$$

$$V_3 = \frac{1}{2}\left(\sqrt{3}F_8 + F_3\right) = \frac{1}{4}\left(\sqrt{3}\lambda_8 + \lambda_3\right), \tag{A.45}$$

and the V-spin scalar operator is called the 'Z-charge' operator:

$$Z = F_3 - \frac{1}{\sqrt{3}}F_8 = \frac{1}{2}\left(\lambda_3 - \frac{1}{\sqrt{3}}\lambda_8\right) = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.46}$$

In terms of hypercharge Y and electric charge Q this is given by

$$Z = Q - Y. \tag{A.47}$$

These satisfy the commutation relations

$$[V_a, V_b] = i\epsilon_{abc}V_c, \tag{A.48}$$

$$[V_3, Z] = 0. \tag{A.49}$$

We can also define the V-spin raising and lowering operators as

$$V_\pm = V_1 \pm iV_2. \tag{A.50}$$

These operators satisfy the SU(2) algebra

$$[V_3, V_\pm] = \pm V_\pm, \tag{A.51}$$

$$[V_+, V_-] = 2V_3. \tag{A.52}$$

It is possible to relate V_3 to T_3 and Y as follows:

$$V_3 = \frac{1}{2}\left(\frac{3}{2}Y + T_3\right). \tag{A.53}$$

The Casimir operator for the V-spin $SU(2)$ subgroup of $SU(3)$ flavor symmetry is

$$\mathbf{V}^2 = V_1^2 + V_2^2 + V_3^2. \tag{A.54}$$

A legitimate V-spin eigenstate is one which is eigenstate of both $\mathbf{V}^2$ and V_3, such a state is denoted by $|v, v_3\rangle$:

$$\mathbf{V}^2 |v, v_3\rangle = v(v+1) |v, v_3\rangle, \tag{A.55}$$

$$V_3 |v, v_3\rangle = v_3 |v, v_3\rangle, \tag{A.56}$$

where v and v_3 denote the V-spin and 3rd component of V-spin of the system. The Casimir operator $\mathbf{V}^2$ commutes with $V_\pm$ and V_3:

$$\left[\mathbf{V}^2, V_\pm\right] = 0 = \left[\mathbf{V}^2, V_3\right]. \tag{A.57}$$

Note 1. *It is important to note that the 3rd components of isospin, U-spin and V-spin operators commute with each other:*

$$[T_3, U_3] = [U_3, V_3] = [T_3, V_3] = 0. \tag{A.58}$$

It must also be noted that the Casimir operators for isospin, U-spin and V-spin also commute with each other:

$$\left[\mathbf{T}^2, \mathbf{U}^2\right] = \left[\mathbf{U}^2, \mathbf{V}^2\right] = \left[\mathbf{V}^2, \mathbf{T}^2\right] = 0. \tag{A.59}$$

Note 2. *It is also important to note that the 3rd components of isospin, U-spin and V-spin operators commute with any of the scalar operators Y, Q and Z:*

$$[T_3, Y] = [U_3, Y] = [V_3, Y] = 0, \tag{A.60}$$

$$[T_3, Q] = [U_3, Q] = [V_3, Q] = 0, \tag{A.61}$$

$$[T_3, Z] = [U_3, Z] = [V_3, Z] = 0. \tag{A.62}$$

Thus, the 3rd components of isospin, U-spin and V-spin, and the quantum numbers hypercharge, electric charge and Z charge can all be simultaneously measured.

Note 3. *Since the 3rd component of isospin T_3 and hypercharge Y commute, an element of a $SU(3)$ multiplet can be denoted by $|t_3, y\rangle$, where t_3 and y are the eigenvalues of T_3 and Y operators respectively:*

$$T_3 \, |t_3, y\rangle = t_3 \, |t_3, y\rangle \,, \tag{A.63}$$

$$Y \, |t_3, y\rangle = y \, |t_3, y\rangle \,. \tag{A.64}$$

Since T_3, U_3, V_3, Y, Q and Z commute with one another, this state must also be an eigenstate of U_3, V_3, Q and Z:

$$U_3 \, |t_3, y\rangle = \left(\frac{3y}{4} - \frac{t_3}{2}\right) |t_3, y\rangle \,, \tag{A.65}$$

$$V_3 \, |t_3, y\rangle = \left(\frac{3y}{4} + \frac{t_3}{2}\right) |t_3, y\rangle \,, \tag{A.66}$$

$$Q\,|t_3, y\rangle = \left(t_3 + \frac{y}{2}\right)|t_3, y\rangle\,, \tag{A.67}$$

$$Z\,|t_3, y\rangle = \left(t_3 - \frac{y}{2}\right)|t_3, y\rangle\,. \tag{A.68}$$

Note 4. *Just like $|t_3, y\rangle$ one can also use the state $|u_3, q\rangle$, where u_3 and q are the eigenvalues of U_3 and Q operators respectively, to denote an element of $SU(3)$ multiplet:*

$$U_3\,|u_3, q\rangle = u_3\,|u_3, q\rangle\,, \tag{A.69}$$

$$Q\,|u_3, q\rangle = q\,|u_3, q\rangle\,. \tag{A.70}$$

Again similar to $|t_3, y\rangle$ this new state $|u_3, q\rangle$ would also be an eigenstate of T_3, V_3, Y and Z:

$$T_3\,|u_3, q\rangle = \left(\frac{3q}{4} - \frac{u_3}{2}\right)|u_3, q\rangle\,, \tag{A.71}$$

$$V_3\,|u_3, q\rangle = \left(\frac{3q}{4} + \frac{u_3}{2}\right)|u_3, q\rangle\,, \tag{A.72}$$

$$Y\,|u_3, q\rangle = \left(\frac{q}{2} + u_3\right)|u_3, q\rangle\,, \tag{A.73}$$

$$Z\,|u_3, q\rangle = \left(\frac{q}{2} - u_3\right)|u_3, q\rangle\,. \tag{A.74}$$

Note 5. *Again similar to both $|t_3, y\rangle$ and $|u_3, q\rangle$ we can also use a state $|v_3, z\rangle$, where v_3 and z are eigenvalues of V_3 and Z operators respectively, to denote an element of $SU(3)$ multiplet:*

$$V_3\,|v_3, z\rangle = v_3\,|v_3, z\rangle\,, \tag{A.75}$$

$$Z\,|v_3, z\rangle = z\,|v_3, z\rangle \tag{A.76}$$

To belabour the point once again, this state $|v_3, z\rangle$ is also an eigenstate of T_3, U_3, Y and

Q:

$$T_3 \left|v_3, z\right\rangle = \left(\frac{v_3}{2} + \frac{3z}{4}\right) \left|v_3, z\right\rangle, \tag{A.77}$$

$$U_3 \left|v_3, z\right\rangle = \left(\frac{v_3}{2} - \frac{3z}{4}\right) \left|v_3, z\right\rangle, \tag{A.78}$$

$$Y \left|v_3, z\right\rangle = \left(v_3 - \frac{z}{2}\right) \left|v_3, z\right\rangle, \tag{A.79}$$

$$Q \left|v_3, z\right\rangle = \left(v_3 + \frac{z}{2}\right) \left|v_3, z\right\rangle. \tag{A.80}$$

It is, therefore, sufficient to denote an element of $SU(3)$ multiplet by either $|t_3, y\rangle$, or $|u_3, q\rangle$, or $|v_3, z\rangle$. In fact they are identical to one another,

$$|t_3, y\rangle \equiv |u_3, q\rangle \equiv |v_3, z\rangle, \tag{A.81}$$

in the sense that

$$t_3 = \frac{3q}{4} - \frac{u_3}{2} = \frac{v_3}{2} + \frac{3z}{4}, \tag{A.82}$$

$$u_3 = \frac{3y}{4} - \frac{t_3}{2} = \frac{v_3}{2} - \frac{3z}{4}, \tag{A.83}$$

$$v_3 = \frac{3y}{4} + \frac{t_3}{2} = \frac{3q}{4} + \frac{u_3}{2}, \tag{A.84}$$

$$y = \frac{q}{2} + u_3 = v_3 - \frac{z}{2}, \tag{A.85}$$

$$q = t_3 + \frac{y}{2} = v_3 + \frac{z}{2}, \tag{A.86}$$

$$z = t_3 - \frac{y}{2} = \frac{q}{2} - u_3. \tag{A.87}$$

It is easy to see that these satisfy the identities

$$t_3 + u_3 = v_3, \tag{A.88}$$

$$q - y = z. \tag{A.89}$$

A.3 Action of ladder operators on elements of $SU(3)$ multiplet

The ladder operators, namely the raising operators T_+, U_+ and V_+, as well as the lowering operators T_-, U_- and V_-, take an element of $SU(3)$ multiplet to another element of the same multiplet as given below:

$$T_\pm \,|t_3, y\rangle = \sqrt{(t \mp t_3)\,(t \pm t_3 + 1)}\,|t_3 \pm 1, y\rangle\,, \tag{A.90}$$

$$U_\pm \,|t_3, y\rangle = \sqrt{(u \mp u_3)\,(u \pm u_3 + 1)}\left|t_3 \mp \frac{1}{2}, y \pm 1\right\rangle, \tag{A.91}$$

$$V_\pm \,|t_3, y\rangle = \sqrt{(v \mp v_3)\,(v \pm v_3 + 1)}\left|t_3 \pm \frac{1}{2}, y \pm 1\right\rangle, \tag{A.92}$$

where $u_3 = (3y/4) - (t_3/2)$, $v_3 = (3y/4) + (t_3/2)$, and t, u, v are, respectively, the isospin, U-spin and V-spin of the element represented by the state $|t_3, y\rangle$. This can be easily generalized to cases where one starts with the other ways to write the sate, namely $|u_3, q\rangle$ and $|v_3, z\rangle$:

$$T_\pm \,|u_3, q\rangle = \sqrt{(t \mp t_3)\,(t \pm t_3 + 1)}\left|u_3 \mp \frac{1}{2}, q \pm 1\right\rangle, \tag{A.93}$$

$$U_\pm \,|u_3, q\rangle = \sqrt{(u \mp u_3)\,(u \pm u_3 + 1)}\,|u_3 \pm 1, q\rangle\,, \tag{A.94}$$

$$V_\pm \,|u_3, q\rangle = \sqrt{(v \mp v_3)\,(v \pm v_3 + 1)}\left|u_3 \pm \frac{1}{2}, q \pm 1\right\rangle, \tag{A.95}$$

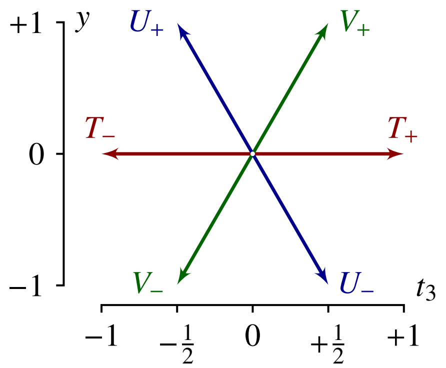
where $t_3 = (3q/4) - (u_3/2)$, $v_3 = (3q/4) + (u_3/2)$, and t, u, v are, respectively, the isospin, U-spin and V-spin of the element represented by $|u_3, q\rangle$, and similarly

$$T_\pm \,|v_3, z\rangle = \sqrt{(t \mp t_3)\,(t \pm t_3 + 1)}\left|v_3 \pm \frac{1}{2}, z \pm 1\right\rangle, \tag{A.96}$$

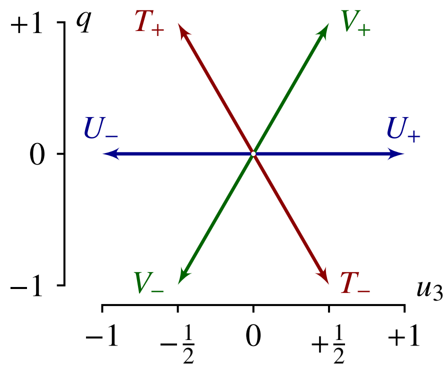
$$U_\pm \,|v_3, z\rangle = \sqrt{(u \mp u_3)\,(u \pm u_3 + 1)}\left|v_3 \pm \frac{1}{2}, z \mp 1\right\rangle, \tag{A.97}$$

$$V_{\pm}\,|v_3, z\rangle = \sqrt{(v \mp v_3)\,(v \pm v_3 + 1)}\,|v_3 \pm 1, z\rangle\,, \tag{A.98}$$

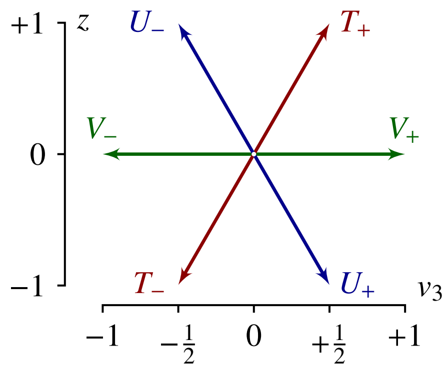
where $t_3 = (v_3/2) + (3z/4)$, $u_3 = (v_3/2) - (3z/4)$, and t, u, v are, respectively, the isospin, U-spin and V-spin of the element represented by $|v_3, z\rangle$. The actions of the ladder operators is schematically shown in Fig. A.1.



(a) 3rd component of isospin vs. hypercharge



(b) 3rd component of U-spin vs. electric charge



(c) 3rd component of V-spin vs. $z = q - y$

Figure A.1: Action of the ladder operators on states.

A.4 G-parity and $SU(3)$ triplets of quarks and antiquarks

The isospin, U-spin and V-spin concepts put two quarks (and corresponding anti-quarks also) in a doublet in a way which is reminiscent of the way the two spin states (spin up and spin down) of a spin-half fermion form a doublet under spin.

Note 6. *We shall implement the following notation for brevity of expression:*

$$|\uparrow\rangle_T = \left|t = \frac{1}{2}, t_3 = +\frac{1}{2}\right\rangle \equiv \left|\frac{1}{2}, +\frac{1}{2}\right\rangle_T , \tag{A.99}$$

$$|\downarrow\rangle_T = \left|t = \frac{1}{2}, t_3 = -\frac{1}{2}\right\rangle \equiv \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_T , \tag{A.100}$$

$$|\uparrow\rangle_U = \left|u = \frac{1}{2}, u_3 = +\frac{1}{2}\right\rangle \equiv \left|\frac{1}{2}, +\frac{1}{2}\right\rangle_U , \tag{A.101}$$

$$|\downarrow\rangle_U = \left|u = \frac{1}{2}, u_3 = -\frac{1}{2}\right\rangle \equiv \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_U , \tag{A.102}$$

$$|\uparrow\rangle_V = \left|v = \frac{1}{2}, v_3 = +\frac{1}{2}\right\rangle \equiv \left|\frac{1}{2}, +\frac{1}{2}\right\rangle_V , \tag{A.103}$$

$$|\downarrow\rangle_V = \left|v = \frac{1}{2}, v_3 = -\frac{1}{2}\right\rangle \equiv \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_V . \tag{A.104}$$

Under isospin, U-spin and V-spin we have the following doublets for quarks:

$$\text{Isospin:} \begin{pmatrix} |\uparrow\rangle_T \\ |\downarrow\rangle_T \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix}, \tag{A.105}$$

$$\text{U-spin:} \begin{pmatrix} |\uparrow\rangle_U \\ |\downarrow\rangle_U \end{pmatrix} = \begin{pmatrix} d \\ s \end{pmatrix}, \tag{A.106}$$

$$\text{V-spin:} \begin{pmatrix} |\uparrow\rangle_V \\ |\downarrow\rangle_V \end{pmatrix} = \begin{pmatrix} u \\ s \end{pmatrix}, \tag{A.107}$$

and for anti-quarks:

$$\text{Isospin:} \begin{pmatrix} |\uparrow\rangle_T \\ |\downarrow\rangle_T \end{pmatrix} = \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}, \tag{A.108}$$

$$\text{U-spin:} \begin{pmatrix} |\uparrow\rangle_U \\ |\downarrow\rangle_U \end{pmatrix} = \begin{pmatrix} \bar{s} \\ -\bar{d} \end{pmatrix}, \tag{A.109}$$

$$\text{V-spin:} \begin{pmatrix} |\uparrow\rangle_V \\ |\downarrow\rangle_V \end{pmatrix} = \begin{pmatrix} \bar{s} \\ -\bar{u} \end{pmatrix}. \tag{A.110}$$

The minus sign and arrangement of antiquarks in the doublets are in accordance with G-

parity. The G-parity operator (G) is defined as a rotation through π radian (180°) around the second axis of isospin/U-spin/V-spin space followed by a charge conjugation (C):

$$G = Ce^{i\pi\tau_2/2}, \tag{A.111}$$

where τ_2 is the second Pauli matrix: $\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. We shall put a subscript to $\mathcal{G}$ to show in which space it is working. Using the identity: $\exp\left(i\theta\frac{\tau_2}{2}\right) = \cos\frac{\theta}{2} + i\tau_2 \sin\frac{\theta}{2}$, we have $e^{i\pi\tau_2/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So the G-parity operator is given by:

$$G = C\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.112}$$

Under G-parity the particle doublet and the antiparticle doublet get related as follows:

$$G_I\begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}, \qquad G_I\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix} = -\begin{pmatrix} u \\ d \end{pmatrix}, \tag{A.113}$$

$$\Longrightarrow G_I^2\begin{pmatrix} u \\ d \end{pmatrix} = -\begin{pmatrix} u \\ d \end{pmatrix}, \qquad G_I^2\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix} = -\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}, \tag{A.114}$$

and

$$G_U\begin{pmatrix} d \\ s \end{pmatrix} = \begin{pmatrix} \bar{s} \\ -\bar{d} \end{pmatrix}, \qquad G_U\begin{pmatrix} \bar{s} \\ -\bar{d} \end{pmatrix} = -\begin{pmatrix} d \\ s \end{pmatrix}, \tag{A.115}$$

$$\Longrightarrow G_U^2\begin{pmatrix} d \\ s \end{pmatrix} = -\begin{pmatrix} d \\ s \end{pmatrix}, \qquad G_U^2\begin{pmatrix} \bar{s} \\ -\bar{d} \end{pmatrix} = -\begin{pmatrix} \bar{s} \\ -\bar{d} \end{pmatrix}, \tag{A.116}$$

and

$$G_V\begin{pmatrix} u \\ s \end{pmatrix} = \begin{pmatrix} \bar{s} \\ -\bar{u} \end{pmatrix}, \qquad G_V\begin{pmatrix} \bar{s} \\ -\bar{u} \end{pmatrix} = -\begin{pmatrix} u \\ s \end{pmatrix}, \tag{A.117}$$

$$\Longrightarrow G_V^2\begin{pmatrix} u \\ s \end{pmatrix} = -\begin{pmatrix} u \\ s \end{pmatrix}, \qquad G_V^2\begin{pmatrix} \bar{s} \\ -\bar{u} \end{pmatrix} = -\begin{pmatrix} \bar{s} \\ -\bar{u} \end{pmatrix}. \tag{A.118}$$

It is easy to notice that s and $\bar{s}$ are isospin singlets, u and $\bar{u}$ are U-spin singlets,

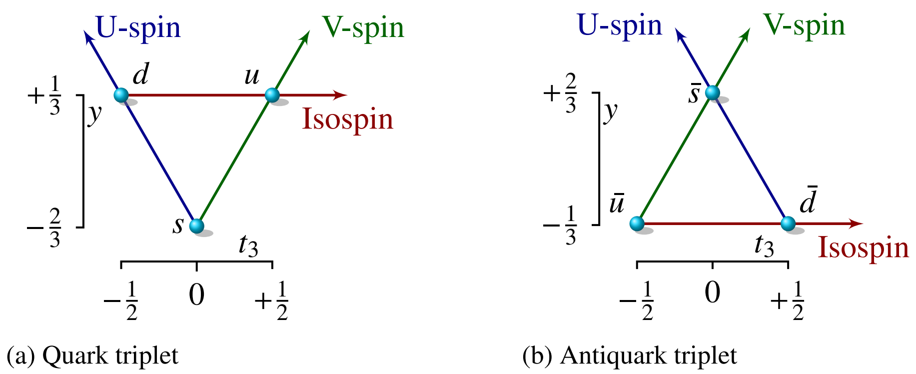


Figure A.2: $SU(3)$ flavor quark and antiquark triplets.

and d and $\bar{d}$ are V-spin singlets. The quarks u, d and s jointly form the fundamental representation of $SU(3)$ flavor, a triplet. The basic quark and antiquark triplets are shown in Fig. A.2. This figure also shows how isospin, U-spin and V-spin act on the quarks and antiquarks.

A.5 A study of the $SU(3)$ octet of lightest pseudoscalar mesons

Mesons, which are quark and antiquark bound states, can thus be assigned isospin, U-spin and V-spin quantum numbers. Given the isospin-doublet of up and down quarks, we can write down the following isotriplet and isosinglet of mesons:

$$\text{isotriplet:} \begin{cases} |1,+1\rangle_T & = |\uparrow\rangle_T \otimes |\uparrow\rangle_T = u\bar{d} \equiv \pi^+, \\ |1,0\rangle_T & = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_T \otimes |\downarrow\rangle_T + |\downarrow\rangle_T \otimes |\uparrow\rangle_T\right) = \frac{1}{\sqrt{2}}\left(d\bar{d} - u\bar{u}\right) \equiv -\pi^0, \\ |1,-1\rangle_T & = |\downarrow\rangle_T \otimes |\downarrow\rangle_T = -d\bar{u} \equiv -\pi^-, \end{cases} \tag{A.119}$$

$$\text{isosinglet:} \begin{cases} |0,0\rangle_T = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_T \otimes |\downarrow\rangle_T - |\downarrow\rangle_T \otimes |\uparrow\rangle_T\right) = -\frac{1}{\sqrt{2}}\left(d\bar{d} + u\bar{u}\right). \end{cases} \tag{A.120}$$

The components of the isotriplet can be easily identified with the three pions. Now combining u, d quarks with $\bar{s}$, and $\bar{u}, \bar{d}$ with s quark we get the doublets of kaons:

$$\text{Particle doublet: } \begin{pmatrix} |\uparrow\rangle_T \\ |\downarrow\rangle_T \end{pmatrix} = \begin{pmatrix} u\bar{s} \\ d\bar{s} \end{pmatrix} \equiv \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \tag{A.121}$$

$$\text{Antiparticle doublet : } \begin{pmatrix} |\uparrow\rangle_T \\ |\downarrow\rangle_T \end{pmatrix} = \begin{pmatrix} s\bar{d} \\ -s\bar{u} \end{pmatrix} \equiv \begin{pmatrix} \overline{K}^0 \\ -K^- \end{pmatrix}. \tag{A.122}$$

It is easy to see that the combination $s\bar{s}$ is also isosinglet. So the physical isosinglet states must be a combination of $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$. The isosinglet which is also $SU(3)$ singlet must contain each quark flavor on the same footing, i.e.

$$\begin{pmatrix} \text{Both } SU(3) \text{ singlet} \\ \text{and isosinglet:} \end{pmatrix} \quad |0,0\rangle_{T,1} = \frac{1}{\sqrt{3}} \left(u\bar{u} + d\bar{d} + s\bar{s} \right), \tag{A.123}$$

where the subscript 1 denotes that this isosinglet state is also $SU(3)$ singlet. This helps in distinguishing the state from $|0,0\rangle_8$ which is a part of $SU(3)$ octet of mesons but is an isosinglet. The state $|0,0\rangle_{T,8}$ must, therefore, be orthogonal to both $|0,0\rangle_{T,1}$ and $|1,0\rangle_T$. It is easy to see that such a state is given by

$$|0,0\rangle_{T,8} = \frac{1}{\sqrt{6}} \left(u\bar{u} + d\bar{d} - 2s\bar{s} \right).$$

It is customary to denote the isosinglet of $SU(3)$ octet as η_8 and that of the $SU(3)$ singlet as η_1. Therefore, the quark content of the various mesons occupying the ground level $SU(3)$ octet and singlet are: $\pi^+ = u\bar{d}$, $\pi^0 = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right)$, $\pi^- = d\bar{u}$, $K^+ = u\bar{s}$, $K^0 = d\bar{s}$, $\overline{K}^0 = s\bar{d}$, $K^- = s\bar{u}$, $\eta_8 = \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right)$, and $\eta_1 = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right)$. It is easy to put the quark-antiquark combinations into an $SU(3)$ octet part and an $SU(3)$ singlet part in matrix form as follows:

$$
\begin{aligned}
\begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes \begin{pmatrix} \bar{u} & \bar{d} & \bar{s} \end{pmatrix} &= \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3}\left(2u\bar{u} - d\bar{d} - s\bar{s}\right) & u\bar{d} & u\bar{s} \\ d\bar{u} & \frac{1}{3}\left(2d\bar{d} - u\bar{u} - s\bar{s}\right) & d\bar{s} \\ s\bar{u} & s\bar{d} & \frac{1}{3}\left(2s\bar{s} - u\bar{u} - d\bar{d}\right) \end{pmatrix} \\
&\quad + \frac{u\bar{u} + d\bar{d} + s\bar{s}}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}}_{\equiv M} + \frac{1}{\sqrt{3}}\eta_1 \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\equiv I} \\
&= \underbrace{M}_{\text{octet}} + \underbrace{\eta_1}_{\text{singlet}} \frac{I}{\sqrt{3}}.
\end{aligned}
\tag{A.124}
$$

The singlet and octet states, η_1 and η_8 respectively, mix in the following manner to give the physical states η and η':

$$
\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_1 \end{pmatrix},
$$

where θ is called the *nonet mixing angle*. Phenomenologically this angle is in the range $-10°$ to $-20°$ [166]. Lattice QCD simulations provide $\theta = -14.1(2.8)°$ [167].

Till now we have looked at isospin alone. Let us now consider the U-spin case also. The U-spin doublets are easily given by

$$
\text{Particle doublet: } \begin{pmatrix} |\uparrow\rangle_U \\ |\downarrow\rangle_U \end{pmatrix} = \begin{pmatrix} u\bar{s} \\ -u\bar{d} \end{pmatrix} \equiv \begin{pmatrix} K^+ \\ -\pi^+ \end{pmatrix}, \tag{A.125}
$$

$$
\text{Antiparticle doublet: } \begin{pmatrix} |\uparrow\rangle_U \\ |\downarrow\rangle_U \end{pmatrix} = \begin{pmatrix} d\bar{u} \\ s\bar{u} \end{pmatrix} \equiv \begin{pmatrix} \pi^- \\ K^- \end{pmatrix}. \tag{A.126}
$$

The U-spin triplet and U-spin singlet states are given by

$$\text{U-spin triplet}\begin{cases} |1,+1\rangle_U &= |\uparrow\rangle_U \otimes |\uparrow\rangle_U = d\bar{s} \equiv K^0, \\ |1,0\rangle_U &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_U \otimes |\downarrow\rangle_U + |\downarrow\rangle_U \otimes |\uparrow\rangle_U\right) = \frac{1}{\sqrt{2}}\left(s\bar{s} - d\bar{d}\right) \equiv \frac{1}{2}\pi^0 - \frac{\sqrt{3}}{2}\eta_8, \\ |1,-1\rangle_U &= |\downarrow\rangle_U \otimes |\downarrow\rangle_U = -s\bar{d} \equiv -\bar{K}^0, \end{cases} \tag{A.127}$$

$$\text{U-spin singlet}\begin{cases} |0,0\rangle_U = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_U \otimes |\downarrow\rangle_U - |\downarrow\rangle_U \otimes |\uparrow\rangle_U\right) = -\frac{1}{\sqrt{2}}\left(d\bar{d} + s\bar{s}\right). \end{cases} \tag{A.128}$$

We know that the state $u\bar{u}$ is also U-spin singlet. Therefore, the physical U-spin singlet $|0,0\rangle_{U,8}$ which belongs to the meson octet, which must be orthogonal to $|1,0\rangle_U$ and $|0,0\rangle_{U,1} = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right) \equiv \eta_1$, is given by

$$|0,0\rangle_{U,8} = \frac{1}{\sqrt{6}}\left(d\bar{d} + s\bar{s} - 2u\bar{u}\right) = -\frac{1}{2}\eta_8 - \frac{\sqrt{3}}{2}\pi^0. \tag{A.129}$$

Now it is easy to write down π^0 and η_8 in terms of U-spin multiplets as follows:

$$\begin{aligned} \pi^0 &= \frac{1}{2}\,|1,0\rangle_U - \frac{\sqrt{3}}{2}\,|0,0\rangle_{U,8}\,, \\ \eta_8 &= -\frac{\sqrt{3}}{2}\,|1,0\rangle_U - \frac{1}{2}\,|0,0\rangle_{U,8}\,. \end{aligned}$$

Let us now consider the V-spin case also. The V-spin doublets are easily given by

$$\begin{aligned} \text{First doublet: } \begin{pmatrix} |\uparrow\rangle_V \\ |\downarrow\rangle_V \end{pmatrix} &= \begin{pmatrix} d\bar{s} \\ -d\bar{u} \end{pmatrix} \equiv \begin{pmatrix} K^0 \\ -\pi^- \end{pmatrix}, \\ \text{Second doublet: } \begin{pmatrix} |\uparrow\rangle_V \\ |\downarrow\rangle_V \end{pmatrix} &= \begin{pmatrix} u\bar{d} \\ s\bar{d} \end{pmatrix} \equiv \begin{pmatrix} \pi^+ \\ \bar{K}^0 \end{pmatrix}. \end{aligned}$$

The V-spin triplet and V-spin singlet states are given by

$$\begin{array}{l}\text{V-spin}\\ \text{triplet}\end{array}\begin{cases} |1,+1\rangle_V &= |\uparrow\rangle_V \otimes |\uparrow\rangle_V = u\bar{s} \equiv K^+, \\ & \\ |1,0\rangle_V &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_V \otimes |\downarrow\rangle_V + |\downarrow\rangle_V \otimes |\uparrow\rangle_V\right) = \frac{1}{\sqrt{2}}\left(s\bar{s} - u\bar{u}\right) \equiv -\frac{1}{2}\pi^0 - \frac{\sqrt{3}}{2}\eta_8, \\ & \\ |1,-1\rangle_V &= |\downarrow\rangle_V \otimes |\downarrow\rangle_V = -s\bar{u} \equiv -K^-, \end{cases}$$

$$\begin{array}{l}\text{V-spin}\\ \text{singlet}\end{array}\Big\{ |0,0\rangle_V = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_V \otimes |\downarrow\rangle_V - |\downarrow\rangle_V \otimes |\uparrow\rangle_V\right) = -\frac{1}{\sqrt{2}}\left(u\bar{u} + s\bar{s}\right).$$

We know that the state $d\bar{d}$ is also V-spin singlet. Therefore, the physical V-spin singlet $|0,0\rangle_{V,8}$ which belongs to the meson octet, which must be orthogonal to $|1,0\rangle_V$ and $|0,0\rangle_{V,1} = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right) \equiv \eta_1$, is given by

$$|0,0\rangle_{V,8} = \frac{1}{\sqrt{6}}\left(u\bar{u} + s\bar{s} - 2d\bar{d}\right) = -\frac{1}{2}\eta_8 + \frac{\sqrt{3}}{2}\pi^0.$$

Now it is easy to write down π^0 and η_8 in terms of V-spin multiplets as follows:

$$\begin{aligned} \pi^0 &= -\frac{1}{2}\,|1,0\rangle_V + \frac{\sqrt{3}}{2}\,|0,0\rangle_{V,8}\,, \\ \eta_8 &= -\frac{\sqrt{3}}{2}\,|1,0\rangle_V - \frac{1}{2}\,|0,0\rangle_{V,8}\,. \end{aligned}$$

The expressions for the particles in terms of isospin, U-spin and V-spin multiplets is given in Table A.1.

The SU(3) octet of lightest pseudoscalar mesons is given in Fig. A.3.

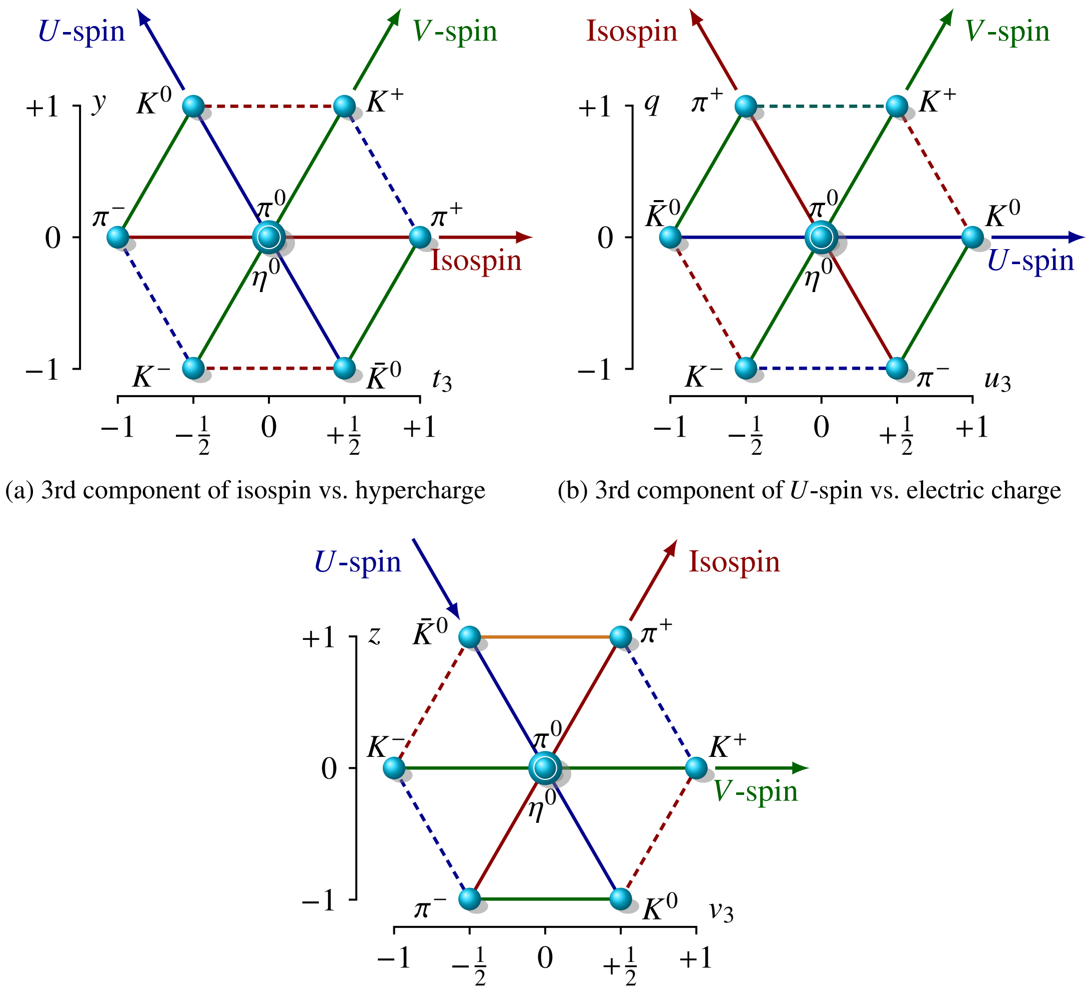
The various quantum numbers for the pions and kaons relevant from the $SU(3)$ flavor symmetry point of view are tabulated in Table A.2. Though in Table A.2 all the quantum numbers for π^0 and $\eta^0 \equiv \eta_8$ are the same, it is very well known that π^0 has isospin $t = 1$, but η^0 has isospin $t = 0$.

Particle	Isospin	U-spin	V-spin
π^+	$\|1,+1\rangle_T$	$-\left\|\frac{1}{2},-\frac{1}{2}\right\rangle_U$	$\left\|\frac{1}{2},+\frac{1}{2}\right\rangle_V$
π^0	$-\|1,0\rangle_T$	$\frac{1}{2}\|1,0\rangle_U-\frac{\sqrt{3}}{2}\|0,0\rangle_{U,8}$	$-\frac{1}{2}\|1,0\rangle_V+\frac{\sqrt{3}}{2}\|0,0\rangle_{V,8}$
π^-	$-\|1,-1\rangle_T$	$\left\|\frac{1}{2},+\frac{1}{2}\right\rangle_U$	$-\left\|\frac{1}{2},-\frac{1}{2}\right\rangle_V$
K^+	$\left\|\frac{1}{2},+\frac{1}{2}\right\rangle_T$	$\left\|\frac{1}{2},+\frac{1}{2}\right\rangle_U$	$\|1,+1\rangle_V$
K^0	$\left\|\frac{1}{2},-\frac{1}{2}\right\rangle_T$	$\|1,+1\rangle_U$	$\left\|\frac{1}{2},+\frac{1}{2}\right\rangle_V$
$\overline{K}^0$	$\left\|\frac{1}{2},+\frac{1}{2}\right\rangle_T$	$-\|1,-1\rangle_U$	$\left\|\frac{1}{2},-\frac{1}{2}\right\rangle_V$
K^-	$-\left\|\frac{1}{2},-\frac{1}{2}\right\rangle_T$	$\left\|\frac{1}{2},-\frac{1}{2}\right\rangle_U$	$-\|1,-1\rangle_V$
η_8	$\|0,0\rangle_{T,8}$	$-\frac{\sqrt{3}}{2}\|1,0\rangle_U-\frac{1}{2}\|0,0\rangle_{U,8}$	$-\frac{\sqrt{3}}{2}\|1,0\rangle_V-\frac{1}{2}\|0,0\rangle_{V,8}$
η_1	$\|0,0\rangle_{T,1}$	$\|0,0\rangle_{U,1}$	$\|0,0\rangle_{V,1}$

Table A.1: Isospin, U-spin and V-spin representation of pions, kaons and eta mesons.

Quantum Number	π^+	π^0	π^-	K^+	K^0	$\overline{K}^0$	K^-	η^0
t_3	$+1$	0	-1	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0
u_3	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$+1$	-1	$-\frac{1}{2}$	0
v_3	$+\frac{1}{2}$	0	$-\frac{1}{2}$	$+1$	$+\frac{1}{2}$	$-\frac{1}{2}$	-1	0
y	0	0	0	$+1$	$+1$	-1	-1	0
q	$+1$	0	-1	$+1$	0	0	-1	0
z	$+1$	0	-1	0	-1	$+1$	0	0

Table A.2: The various quantum number assignments significant from $SU(3)$ flavor symmetry point of view for the pseudoscalar mesons of the $SU(3)$ octet.



(a) 3rd component of isospin vs. hypercharge

(b) 3rd component of U-spin vs. electric charge

(c) 3rd component of V-spin vs. $z = q - y$

Figure A.3: The $SU(3)$ meson octet of pions and kaons.

Note 7. *For two particles to qualify as identical under a given symmetry transformation they must have some quantum numbers the same.*

- *When two particles are identical under isospin, they not only have identical isospin (t) but also identical hypercharge (y), but distinct 3rd components of isospin (t_3).*
- *When two particles are said to be identical under U-spin, they not only have identical U-spin (u) but also identical electric charge (q), but distinct 3rd components of U-spin (u_3).*
- *When two particles are said to be identical under V-spin, they not only have identical V-spin (v) but also identical 'Z charge' ($z = q - y$), but distinct 3rd components of V-spin (v_3).*

Note 8. *Since the operators T_3, U_3, V_3, Y, Q and Z commute with one another, states characterized by eigenstates of these operators can be used to study simultaneous application of isospin, U-spin and V-spin.*

Note 9. *G-parity as defined here transforms the various $SU(2)$ multiplets as follows:*

$$G_I \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} = - \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}, \qquad G_I \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} = \begin{pmatrix} \overline{K}^0 \\ -K^- \end{pmatrix}, \qquad G_I \begin{pmatrix} \overline{K}^0 \\ -K^- \end{pmatrix} = - \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \tag{A.130}$$

$$G_U \begin{pmatrix} K^0 \\ \pi^0 \\ \overline{K}^0 \end{pmatrix} = - \begin{pmatrix} K^0 \\ \pi^0 \\ \overline{K}^0 \end{pmatrix}, \qquad G_U \begin{pmatrix} K^+ \\ \pi^+ \end{pmatrix} = \begin{pmatrix} \pi^- \\ -K^- \end{pmatrix}, \qquad G_U \begin{pmatrix} \pi^- \\ -K^- \end{pmatrix} = - \begin{pmatrix} K^+ \\ \pi^+ \end{pmatrix}, \tag{A.131}$$

$$G_V \begin{pmatrix} K^+ \\ \pi^0 \\ K^- \end{pmatrix} = - \begin{pmatrix} K^+ \\ \pi^0 \\ K^- \end{pmatrix}, \qquad G_V \begin{pmatrix} \pi^+ \\ \overline{K}^0 \end{pmatrix} = \begin{pmatrix} K^0 \\ -\pi^- \end{pmatrix}, \qquad G_V \begin{pmatrix} K^0 \\ -\pi^- \end{pmatrix} = - \begin{pmatrix} \pi^+ \\ \overline{K}^0 \end{pmatrix}. \tag{A.132}$$

B

Ternary Plot

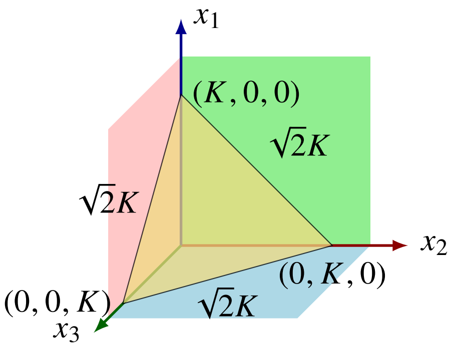
The ternary plot (also called as triangle plot) is a plot of three variables that sum up to a constant. The plot resembles an equilateral triangle, and hence the name ternary[1]. To get a concrete idea of the ternary plot let us consider three variables, say x_1, x_2 and x_3, which when added together give a constant, say K:

$$x_1 + x_2 + x_3 = K. \tag{B.1}$$

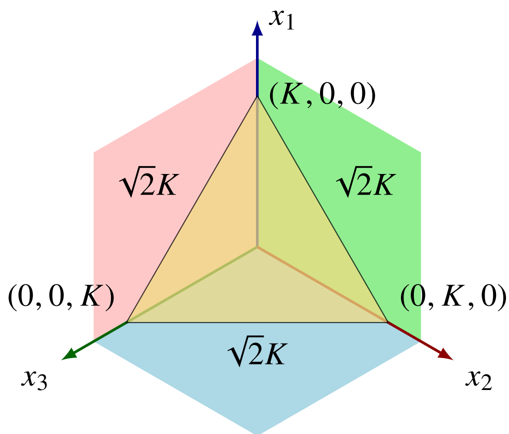
Now this can be geometrically viewed as an equilateral triangle in the plane that cuts the axes x_1, x_2 and x_3 at K intervals each and has each side of length $\sqrt{2}K$. This is shown in Fig. B.1a. The triangle, however, does not look as an equilateral triangle due to the angle of view. Changing the perspective to the one shown in Fig. B.1b we can clearly identify that the triangle is indeed an equilateral triangle visually. Since all the points that satisfy Eq. B.1 are now confined to this triangle only, we can as well define a coordinate system with two independent variables that also describes the triangle. One can choose such a coordinate system as one wishes. Here we choose the point of intersection of the medians (called *centroid*, where the center of mass of the triangle lies) as the origin of

[1]The word 'ternary' is derived from the Latin word '*ternarius*' which means 'having three parts'.

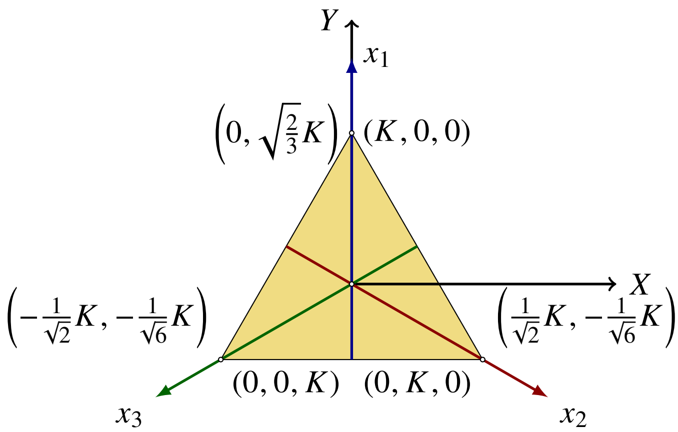
our new coordinate system. Hence this coordinate system goes by the name *barycentric coordinate system* (barycenter ≡ center of mass).



(a) Any point in the triangular region here satisfies $x_1 + x_2 + x_3 = K$.



(b) The same region as before, but viewed from an isometric perspective.



(c) The *barycentric coordinate system* defined for the equilateral triangle, and the original *Cartesian coordinate system*.

Figure B.1: Steps to get the ternary plot.

In order to relate the barycentric coordinates (X, Y) with the Cartesian coordinates (x_1, x_2, x_3) we note the corresponding points on the triangle in Table B.1. Let us assume that

$$X = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \tag{B.2}$$

$$Y = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3. \tag{B.3}$$

Barycentric Coordinates (X, Y)	Cartesian Coordinates (x_1, x_2, x_3)
$\left(0, \sqrt{\frac{2}{3}}K\right)$	$(K, 0, 0)$
$\left(\frac{1}{\sqrt{2}}K, -\frac{1}{\sqrt{6}}K\right)$	$(0, K, 0)$
$\left(-\frac{1}{\sqrt{2}}K, -\frac{1}{\sqrt{6}}K\right)$	$(0, 0, K)$
$(0, 0)$	$\left(\frac{1}{3}K, \frac{1}{3}K, \frac{1}{3}K\right)$

Table B.1: Some points on the triangle as shown in Fig. B.1c.

Solving for the coefficients α_i and β_i (with $i \in \{1, 2, 3\}$) we get

$$\alpha_1 = 0, \qquad \alpha_2 = -\alpha_3 = \frac{1}{\sqrt{2}}, \tag{B.4}$$

$$\beta_1 = \sqrt{\frac{2}{3}}, \; \beta_2 = \beta_3 = -\frac{1}{\sqrt{6}}, \tag{B.5}$$

which satisfy $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and $\beta_1 + \beta_2 + \beta_3 = 0$. Using these results we find that

$$X = \frac{1}{\sqrt{2}}\Big(x_2 - x_3\Big), \tag{B.6}$$

$$Y = \frac{1}{\sqrt{6}}\Big(2x_1 - x_2 - x_3\Big). \tag{B.7}$$

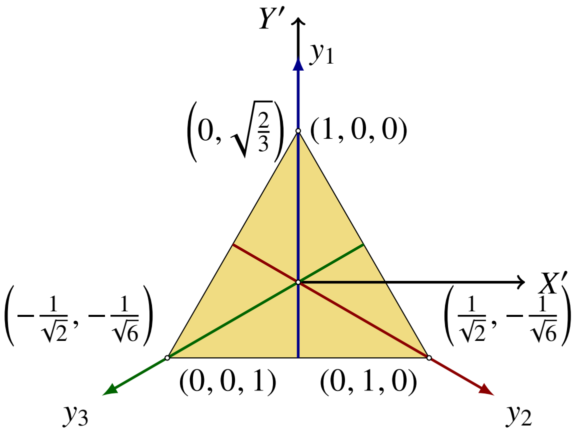
Instead of the variables x_1, x_2, x_3 one can always define a new set of variables y_1, y_2 and y_3 as $y_i = x_i/K$ such that

$$y_1 + y_2 + y_3 = 1. \tag{B.8}$$

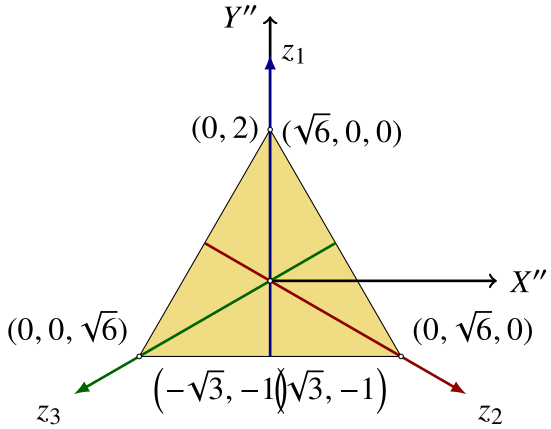
In such a case, the new barycentric coordinates (X', Y') for the equilateral triangle (as shown in Fig. B.2a) are given by

$$X' = \frac{1}{\sqrt{2}}\,(y_2 - y_3) = \frac{X}{K}, \tag{B.9}$$

$$Y' = \frac{1}{\sqrt{6}}\,(2y_1 - y_2 - y_3) = \frac{Y}{K}. \tag{B.10}$$



(a) The ternary plot corresponding to $y_1 + y_2 + y_3 = 1$.



(b) The ternary plot corresponding to $z_1 + z_2 + z_3 = \sqrt{6}$.

Figure B.2: Ternary plots after scaling of the variables.

It is clear from the relationships between X' (Y') and X (Y), that the equilateral triangle gets scaled according to the scaling of its barycentric coordinates. Therefore, one can further scale the coordinates (X', Y') to $(X'', Y'') = (\sqrt{6}X', \sqrt{6}Y')$, i.e.

$$X'' = \sqrt{6}X' = \sqrt{6}\frac{X}{K} = \sqrt{3}\,(y_2 - y_3) = \frac{\sqrt{3}\,(x_2 - x_3)}{K} = \frac{1}{\sqrt{2}}\,(z_2 - z_3)\,, \tag{B.11}$$

$$Y'' = \sqrt{6}Y' = \sqrt{6}\frac{Y}{K} = 2y_1 - y_2 - y_3 = \frac{2x_1 - x_2 - x_3}{K} = \frac{1}{\sqrt{6}}\,(2z_1 - z_2 - z_3)\,, \tag{B.12}$$

where $z_i = \sqrt{6}y_i = \sqrt{6}x_i/K$, and the corresponding equilateral triangle is shown in Fig. B.2b.

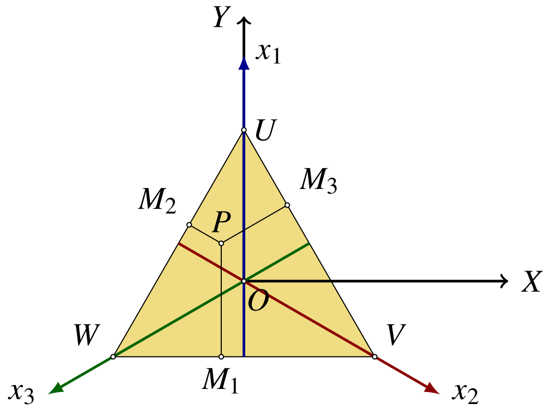


Figure B.3: The vertices of the equilateral triangle $\triangle UVW$ are $U \equiv \left(0, \frac{\sqrt{2}}{\sqrt{3}}K\right)$, $V \equiv \left(\frac{1}{\sqrt{2}}K, -\frac{1}{\sqrt{6}}K\right)$ and $W \equiv \left(-\frac{1}{\sqrt{2}}K, -\frac{1}{\sqrt{6}}K\right)$, and O is the origin of the barycentric coordinate system. Distances of any point $P(X, Y)$ from the three sides of the equilateral triangle are proportional to the Cartesian coordinates (x_1, x_2, x_3).

So far we have described the region allowed for the three variables. If a point is given inside the equilateral triangle, then how do we find out what are the corresponding Cartesian coordinates. For example let us consider a point P with barycentric coordinates (X, Y) in the triangle $\triangle UVW$ of Fig. B.3. One can drop perpendiculars from the point P onto the three sides of the triangle. The perpendiculars to sides $\overline{VW}$, $\overline{WU}$ and $\overline{UV}$ are denoted by $\overline{PM_1}$, $\overline{PM_2}$ and $\overline{PM_3}$ respectively, and the coordinates of M_1, M_2 and M_3 are given by

$$M_1 \equiv \left(X, -\frac{1}{\sqrt{6}}K\right), \tag{B.13}$$

$$M_2 \equiv \left(\frac{-\sqrt{2}K + X + \sqrt{3}Y}{4}, \frac{\sqrt{2}K + 3X + 3\sqrt{3}Y}{4\sqrt{3}}\right), \tag{B.14}$$

$$M_3 \equiv \left(\frac{\sqrt{2}K + X - \sqrt{3}Y}{4}, \frac{\sqrt{2}K - 3X + 3\sqrt{3}Y}{4\sqrt{3}}\right). \tag{B.15}$$

It is interesting to note that the lengths of the perpendiculars are proportional to the Cartesian coordinates of the point $P(x_1, x_2, x_3)$:

$$d_1 = \left|\overline{PM_1}\right| = \frac{\sqrt{3}}{\sqrt{2}} x_1, \tag{B.16}$$

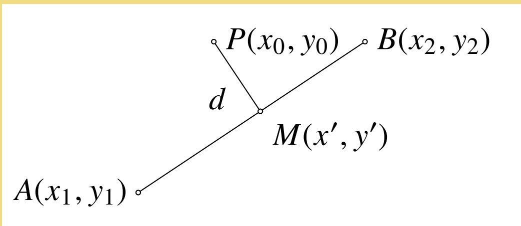
$$d_2 = \left|\overline{PM_2}\right| = \frac{\sqrt{3}}{\sqrt{2}} x_2, \tag{B.17}$$

$$d_3 = \left|\overline{PM_3}\right| = \frac{\sqrt{3}}{\sqrt{2}} x_3. \tag{B.18}$$

Therefore

$$d_1 + d_2 + d_3 = \sqrt{\frac{3}{2}}(x_1 + x_2 + x_3) = \sqrt{\frac{3}{2}} K. \tag{B.19}$$

Thus for any point on the ternary plot we can determine its Cartesian coordinates by knowing the normal distances of it from the sides of the triangle. The elementary geometrical relations used in deriving the above relations are summarized in Fig. B.4.



Equation for $\overline{AB}$: $y = mx + c$, where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $c = \dfrac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$.

The coordinates of M are given by $(x', y') = \left(\dfrac{1}{1+m^2}(x_0 + my_0 - mc), \dfrac{m}{1+m^2}(x_0 + my_0 - mc) + c\right)$.

The distance of the point P from the line $\overline{AB}$ is $d = \left|\overline{PM}\right| = \dfrac{|mx_0 - y_0 + c|}{\sqrt{1+m^2}}$.

Figure B.4: Some elementary results dealing with points and line from Euclidean geometry.

Similar relationships can be easily derived for the other two barycentric coordinate systems we have considered. The two scenario with (X', Y') and (X'', Y'') coordinates are depicted in Fig. B.5. The coordinates of the foots of the normals are

$$M_1' \equiv \left(X', -\frac{1}{\sqrt{6}}\right), \tag{B.20}$$

$$M_2' \equiv \left(\frac{-\sqrt{2} + X' + \sqrt{3}Y'}{4}, \frac{\sqrt{2} + 3X' + 3\sqrt{3}Y'}{4\sqrt{3}}\right), \tag{B.21}$$

$$M_3' \equiv \left(\frac{\sqrt{2} + X' - \sqrt{3}Y'}{4}, \frac{\sqrt{2} - 3X' + 3\sqrt{3}Y'}{4\sqrt{3}}\right), \tag{B.22}$$

$$M_1'' \equiv (X'', -1), \tag{B.23}$$

$$M_2'' \equiv \left(\frac{-2\sqrt{3} + X'' + \sqrt{3}Y''}{4}, \frac{2 + \sqrt{3}X'' + 3Y''}{4}\right), \tag{B.24}$$

$$M_3'' \equiv \left(\frac{2\sqrt{3} + X'' - \sqrt{3}Y''}{4}, \frac{2 - \sqrt{3}X'' + 3Y''}{4}\right). \tag{B.25}$$

The distances of the points P' and P'' from the three sides are

$$d_1' = \left|\overline{PM_1'}\right| = \frac{\sqrt{3}}{\sqrt{2}}y_1 = \frac{\sqrt{3}}{\sqrt{2}K}x_1, \tag{B.26}$$

$$d_2' = \left|\overline{PM_2'}\right| = \frac{\sqrt{3}}{\sqrt{2}}y_2 = \frac{\sqrt{3}}{\sqrt{2}K}x_2, \tag{B.27}$$

$$d_3' = \left|\overline{PM_3'}\right| = \frac{\sqrt{3}}{\sqrt{2}}y_3 = \frac{\sqrt{3}}{\sqrt{2}K}x_3, \tag{B.28}$$

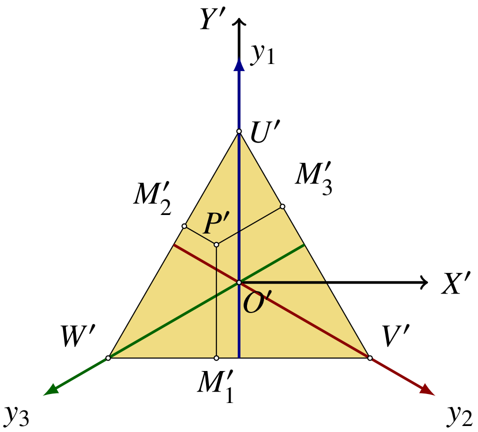
$$d_1'' = \left|\overline{PM_1''}\right| = \frac{\sqrt{3}}{\sqrt{2}}z_1 = \frac{3}{K}x_1, \tag{B.29}$$

$$d_2'' = \left|\overline{PM_2''}\right| = \frac{\sqrt{3}}{\sqrt{2}}z_2 = \frac{3}{K}x_2, \tag{B.30}$$

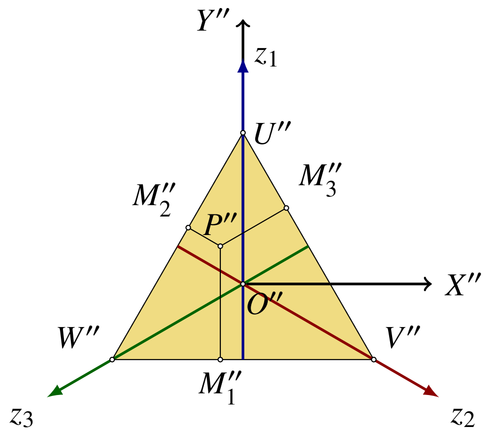
$$d_3'' = \left|\overline{PM_3''}\right| = \frac{\sqrt{3}}{\sqrt{2}}z_3 = \frac{3}{K}x_3. \tag{B.31}$$

Thus we find that

$$d_1' + d_2' + d_3' = \sqrt{\frac{3}{2}}, \text{ and } d_1'' + d_2'' + d_3'' = 3. \tag{B.32}$$



(a) The vertices of the equilateral triangle $\triangle U'V'W'$ are $U' \equiv \left(0, \frac{\sqrt{2}}{\sqrt{3}}\right)$, $V' \equiv \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\right)$ and $W' \equiv \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\right)$, and O' is the origin of the barycentric coordinate system of which $P'(X', Y')$ is a representative point.



(b) The vertices of the equilateral triangle $\triangle U''V''W''$ are $U'' \equiv (0, 2)$, $V'' \equiv \left(\sqrt{3}, -1\right)$ and $W'' \equiv \left(-\sqrt{3}, -1\right)$, and O'' is the origin of the barycentric coordinate system of which $P''(X'', Y'')$ is a representative point.

Figure B.5: Distances of any representative point from the three sides of the equilateral triangle are proportional to the Cartesian coordinates.

The barycentric coordinate system that we have used till now is a rectangular one. It is also possible to describe the ternary plot by polar coordinates also, say (r, θ). The polar axis can be either of the three medians of the triangle, and the polar angle is measured in the anti-clockwise direction from the chosen polar axis. Our choice of the polar axis is along x_1 direction, the polar angle is θ, and the origin of the polar coordinate system coincides with the origin O of the Cartesian coordinate system as shown in Fig. B.6a. We define

$$X = \frac{1}{\sqrt{2}}(x_2 - x_3) \equiv -r \sin\theta, \tag{B.33}$$

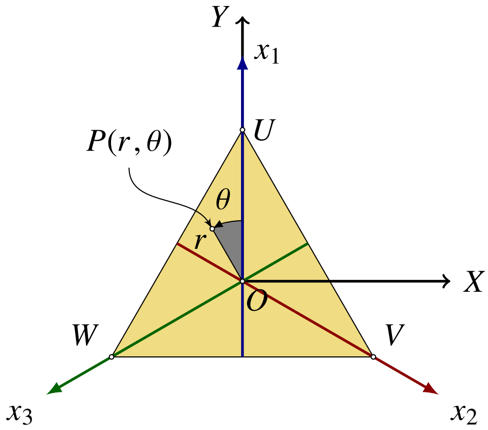
$$Y = \frac{1}{\sqrt{6}}(2x_1 - x_2 - x_3) \equiv r \cos\theta. \tag{B.34}$$

The Cartesian coordinates of any point inside the triangle are related to the polar coordinates by

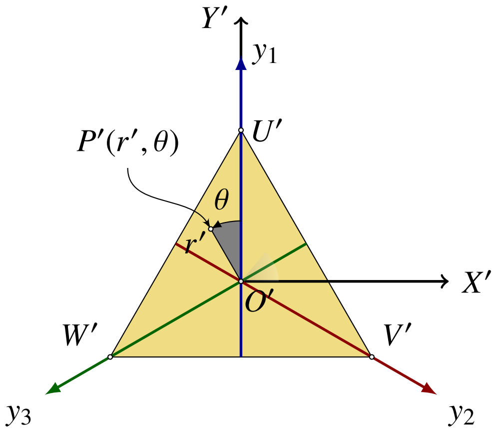
$$x_1 = \frac{K}{3} + \sqrt{\frac{2}{3}}\, r \cos\theta, \tag{B.35}$$

$$x_2 = \frac{K}{3} - \sqrt{\frac{2}{3}}\, r \sin\left(\frac{\pi}{6} + \theta\right) = \frac{K}{3} + \sqrt{\frac{2}{3}}\, r \cos\left(\frac{2\pi}{3} + \theta\right), \tag{B.36}$$

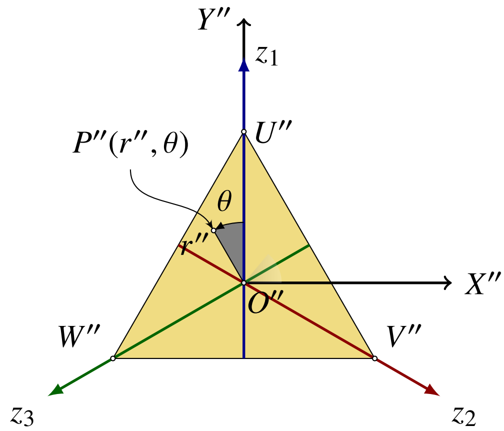
$$x_3 = \frac{K}{3} - \sqrt{\frac{2}{3}}\, r \sin\left(\frac{\pi}{6} - \theta\right) = \frac{K}{3} + \sqrt{\frac{2}{3}}\, r \cos\left(\frac{4\pi}{3} + \theta\right) = \frac{K}{3} + \sqrt{\frac{2}{3}}\, r \cos\left(\frac{2\pi}{3} - \theta\right). \tag{B.37}$$



(a) Polar coordinates r and θ.



(b) Polar coordinates $r' = r/K$ and θ.



(c) Polar coordinates $r'' = \sqrt{6}r' = \sqrt{6}r/K$ and θ.

Figure B.6: The ternary plots described in terms of suitably defined polar coordinates.

The Cartesian coordinates for the scaled ternary plots of Fig. B.6 are related to the corresponding polar coordinates as follows:

$$y_1 = \frac{1}{3} + \sqrt{\frac{2}{3}} r' \cos\theta, \tag{B.38}$$
$$y_2 = \frac{1}{3} + \sqrt{\frac{2}{3}} r' \cos\left(\frac{2\pi}{3} + \theta\right), \tag{B.39}$$
$$y_3 = \frac{1}{3} + \sqrt{\frac{2}{3}} r' \cos\left(\frac{4\pi}{3} + \theta\right), \tag{B.40}$$
$$z_1 = \sqrt{\frac{2}{3}} \left(1 + r'' \cos\theta\right), \tag{B.41}$$
$$z_2 = \sqrt{\frac{2}{3}} \left(1 + r'' \cos\left(\frac{2\pi}{3} + \theta\right)\right), \tag{B.42}$$
$$z_3 = \sqrt{\frac{2}{3}} \left(1 + r'' \cos\left(\frac{4\pi}{3} + \theta\right)\right), \tag{B.43}$$

where $r' = r/K$ and $r'' = \sqrt{6}r'$. It is interesting to note that instead of the scaled coordinates (z_1, z_2, z_3) one can use the unscaled coordinates (x_1, x_2, x_3), and use the barycentric coordinates defined as

$$X = \frac{\sqrt{3}(x_2 - x_3)}{K} = -\rho \sin\vartheta, \tag{B.44}$$
$$Y = \frac{2x_1 - x_2 - x_3}{K} = \rho \cos\vartheta, \tag{B.45}$$

to describe the equilateral triangle (see Fig. B.7).

In this case we have

$$x_1 = \frac{K}{3}\left(1 + \rho\cos\vartheta\right) = \frac{K}{3}\left(1 + Y\right), \tag{B.46}$$
$$x_2 = \frac{K}{3}\left(1 + \rho\cos\left(\frac{2\pi}{3} + \vartheta\right)\right) = \frac{K}{6}\left(2 + \sqrt{3}X - Y\right), \tag{B.47}$$
$$x_3 = \frac{K}{3}\left(1 + \rho\cos\left(\frac{4\pi}{3} + \vartheta\right)\right) = \frac{K}{3}\left(1 + \rho\cos\left(\frac{2\pi}{3} - \vartheta\right)\right) = \frac{K}{6}\left(2 - \sqrt{3}X - Y\right). \tag{B.48}$$

The ditances of any point $P(X, Y)$ inside this triangle from the sides of the triangle are

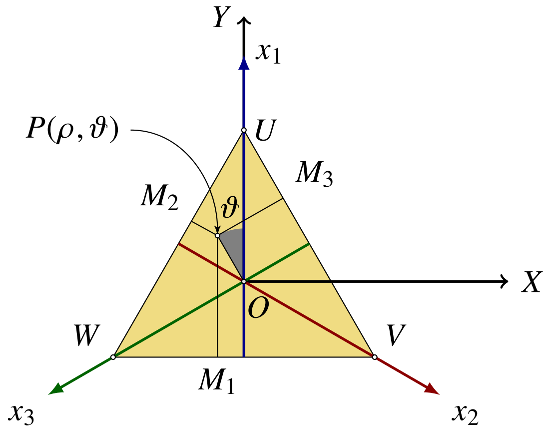


Figure B.7: A different parametrization for ternary plot. The vertices of the triangle $\triangle UVW$ are $U \equiv (0, 2)$, $V \equiv \left(\sqrt{3}, -1\right)$ and $W \equiv \left(-\sqrt{3}, -1\right)$.

given by

$$d_1 = \left|\overline{PM_1}\right| = \frac{3}{K} x_1, \tag{B.49}$$

$$d_2 = \left|\overline{PM_2}\right| = \frac{3}{K} x_2, \tag{B.50}$$

$$d_3 = \left|\overline{PM_3}\right| = \frac{3}{K} x_3. \tag{B.51}$$

Thus the distances sum up to

$$d_1 + d_2 + d_3 = 3. \tag{B.52}$$

This last kind of ternary plot is the one which was adopted by R. H. Dalitz [9, 10] for analysing various three-body decays.

PART V

References and Index

A

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