Aspects of joint measurement and interrelation of quantum correlations and channels

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PHYS10200804003

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A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

October, 2014

Homi Bhabha National Institute

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Rajarshi Pal

DEDICATIONS

To Ma

ACKNOWLEDGEMENTS

I thank kindred souls in Matscience for their friendship, particularly those who broke bread,coffee or else with me. I am very grateful to my adviser Sibasish for educating me and also for the remarkable freedom he gave. I am grateful to all the present and past members of the quantum science group and my teachers at Matscience. I thank Guruda and Somda for highly educative and enjoyable discussions about quantum theory. I am grateful to all the technical and administrative staff of the institute. I thank my father for kindly understanding that I am the best decision-maker concerning me. I dedicate this thesis to my mother whose love and support continues to be my greatest strength.

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Chapter 1

Synopsis

Quantum physics offers a myriad of phenomenon which challenges our intuition based on our experiences with the classical world. In this thesis we investigate some characteristic features of quantum theory.

This thesis represents work in two different directions. In one of them we explore aspects of joint measurability through an Arthur-Kelly like model for qubits. In the second we investigate the interrelation between quantum correlation and channels through two different problems.

Standard formulation of quantum mechanics does not allow the joint measurement of non-commuting observables. The uncertainty principle, usually described as lower bound on the product of standard deviations of the outcome statistics does not really capture this complementary feature as it (uncertainty principle) is a statement about the quantum state of the system and does not relate to an actual situation involving apparatus which can attempt a joint measurement. As intuition suggests one has to allow for some degree of imprecision in order to make room for a notion of joint measurement of non-commuting observables. The most general measurements possible in quantum mechanics can be described through POVMs. It turns out that the idea of imprecise or unsharp measurements can be appropriately investigated if one considers instead of projective measurements, POVMs. The observables corresponding to projective measurements form a subclass of those described by POVMs and are called *sharp*.

The first model for approximate joint measurement of position and momentum was given by Arthur and Kelly in (1965) by generalizing von Neumann's model for measurement.([1]) In the von Neumann's model for measurement, the position observable of the object is measured by coupling it to the momentum P_p of a probe system via the interaction evolution $U = e^{-\frac{i}{\hbar}\lambda Q\otimes P_p}$ and using the position Q_p of the probe as the readout observable. The idea of Arthur-Kelly was to couple two such probe systems respectively to the position and momentum of the system and then perform measurement on the commuting meter observables of the probe to gain information about the position and momenta of the system. This is in fact an unsharp joint measurement and inspired the later development of the formalism .

In the first part of the thesis we look at a model of approximate joint measurement of qubit observables through a Stern-Gerlach like setup. This allows us to investigate the effect of the initial detector states on different measures of unsharpness, entanglement between system and detectors and so on.

Entanglement breaking channels play a significant role in quantum information theory. In the second part of the thesis inspired by the notion of entanglement breaking([2],[3]) we look for a characterization of qubit channels based on their property of 'non-locality breaking'. By seeing how and to what extent 'non-locality breaking' is different from entanglement breaking we get additional insight into the relationship between entanglement and nonlocality. A state of a composite quantum system is known to be local iff the measurement statistics, for performing local measurements on the state, can be simulated by a local hidden variable model. A 'non-locality breaking channel' is thus one, which when acts on one subsystem of a composite system's state, brings every state to a local one.As *local* states can be entangled [4], a basic question of interest is: *Can we have a non-entanglement breaking channel which when acting on one side of any bipartite state produces a state which has a local model?*.This question is not already answered by the examples of mixed entangled states with local model that are known in the literature(([4])). One of the main results from [3] is that for a channel to be entanglement breaking it is necessary and sufficient for its dual state(in the Choi-Jamiolkowski sense) to be separable.It is apriori quite unclear if such a property holds good for a 'non-locality breaking channel'. Thus asking a channel to be 'non-locality breaking' is a stronger restriction than merely asking its dual-state to be local.

Further entanglement cannot be increased by LOCC(local operations with clasical communication) and composition of an SLOCC(stochastic LOCC) map with an entanglement breaking channel is again entanglement breaking. In the light of examples of genuine hidden quantum non-locality([5]) this is also unlikely to hold for 'non-locality breaking channels' and hence we also look at a stronger notion of non-locality breaking where not only are the output states of the channel required to be local, but they also do not show any hidden nonlocality under SLOCC. Such channels are said to be 'strongly non-locality breaking'. We study and characterise various properties of 'non-locality breaking' and 'strongly non-locality breaking' channels.

Shared entanglement between two spatially separated observers (Alice and Bob) is a critical resource for quantum information processing (QIP) tasks such as dense coding [6], cryptography [7], distributed quantum computation [8], and quantum teleportation [9]. Faithful implementation of QIP tasks require maximally entangled states, which can only be shared through noiseless quantum channels, where Alice prepares a maximally entangled state of two particles (say, qubits) and sends one of them to Bob through the channel. In practice, available channels are noisy resulting in mixed states. Entanglement distillation [10, 11, 12, 13, 14] provides a solution by converting these mixed states to fewer almost-perfect entangled states of purity close to unity while requiring many uses of the channel and joint measurements on many copies of the output. Clearly, the yield in an entanglement distillation protocol depends on the purity of the mixed states, which in turn is a function of the amount of noise present in the quantum channel. Thus, in the simplest case of entanglement sharing, a basic question is: Given a noisy quantum channel what is the maximum achievable purity for single use of the channel?

In this thesis , we answer the above question for qubit channels within the paradigm of trace-preserving local operations (TP-LOCC). By restricting to this class of operations, where no subsystem is thrown away, our results provide the conditions and an explicit protocol when every single use of the channel is maximally efficient. Our result also characterizes qubit channels by quantifying reliable transmission of quantum information via teleportation for single channel use and TP-LOCC.

The preliminary four chapters of the thesis describe basic notions and the various tools which allow us to study the above problems.

- In chapter 2 we introduce the original Arthur-Kelly model which was introduced in the context of joint measurement of position and momentum in [1]. We further introduce the formalism of approximate joint measurement and different notions of quality of joint measurement and measurement uncertainty relations.

- In chapter 3 we discuss non-locality in general and w.r.t the Bell-CHSH inequality. We discuss the connections between entanglement and non-locality and also known example of a family of mixed entangled states (Werner states) with local hidden variable theoretic description. Further we move on to how non-locality of certain mixed entangled states can be *revealed* by means of post-selection through local filtering operations, a phenomenon called as hidden no-locality in literature. We also discuss various results showing which are the two-qubit states showing maximal Bell violation among the states connected to an arbitrary two-qubit state by means of local filtering operations. These results are based on the the realisation of local filtering transformations in the Hilbert Schmidt basis as Lorentz transformations. We also discuss the normal forms of an arbitrary two-qubit state that can be obtained by local filtering transformations.

- In chapter 4 we provide an introduction to trace-preserving completely positive maps also known as quantum channels. We describe the characterisation of qubit channels following Ruskai et.al in reference and provide examples. We also introduce entanglement breaking channels.

- In chapter 5 we start with a discussion of the most general teleportation scheme and its connection with singlet fraction. We introduce the notion of twirling for states and channels and define channel fidelity and entanglement fidelity. We also show following ref. how the optimal teleportation fidelity for any given state is related to the optimal singlet fraction of the state that can be achieved by means of TP-LOCC. We define *negativity* of a bipartite state and show that it is an entanglement monotone. Next we discuss the characterisation of optimal singlet fraction for an arbitrary two-qubit state that was provided by

Verstaete et. al. in [15]. As we show, this characterisation is in the form of a semi-definite program.

In the next three chapters we describe the results that has been obtained in this thesis.

- In chapter 6, we consider characterization of unsharpness or quality of approximation by two ways existing in literature. Firstly by considering closeness of the marginal probability distribution of the probability distribution of joint measurement to that of the sharp observable being approximately measured .Secondly by considering suitably defined closeness of the observables themselves with the meter observables in the Heisenberg picture. We show that for a symmetric joint measurement where the two marginal probability distributions are equally close to the corresponding sharp probability distributions, the two measures of closeness are proportional($\eta_i = \frac{\pi}{4}a'$). The upper bound on the measure based on the Heisenberg picture ($\eta_i \leq \frac{1}{2}$, see [16]) provides a sharper bound on the measure based on closeness of marginal probability distribution than allowed in general by quantum theory indicating that the full freedom of joint measurement ($a' \leq \frac{1}{\sqrt{2}}$) cannot be achieved in this model. Error-disturbance relationship ([16]) does not seem to hold for the measure based on the Heisenberg picture for our choice of the pointer observable. Numerical analysis is performed for the two observable case to show the validity of measurement uncertainty relations, transition from POVM to projection-valued measurement and also the effect of the joint measurement on the system. The effect of the pre-measurement on the system turns out to be that of an asymmetric depolarising channel and this forms the basis for a physical understanding of the POVM to PVM transition. Entanglement between system and detectors is also investigated . We also prove a lemma showing the connection between joint measurement ,and the symmetries of the underlying Hamiltonian of the measurement interaction together with that of the initial detector states. This is then used to perform approximate joint measurement in arbitrary directions. Moving on to the case of three-observable joint unsharp measurement we prove a simple necessary condition that is sufficient for the case of three orthogonal observables.This necessary condition has also been proved to be sufficient for unbiased observables in [17] and the proof is included for the sake of completeness. This condition is derived from certain geometrical considerations based on the so called Fermat-Torricelli point. Finally, an extension of the Arthur Kelly like model to the three observable case is studied.

- In chapter 7 we begin by showing that for both the notions of 'non-locality breaking' it is enough to focus our attention on two-qubit pure state input. Also, we show that for non-locality breaking in the stronger sense it is enough to focus on local filtering through single qubit filters and maximally entangled input.

We provide examples to show that for non-unital channels to be non-locality breaking it is *not* enough for the output state of the channel for maximally entangled input to satisfy the Bell-CHSH inequality. In fact it may even not suffice to break the non-locality of all pure entangled states with a given Schmidt basis. There also seems to be exceptions like the amplitude damping channel which seems to be universally non-locality breaking for a certain parameter range. It is however true that channels breaking non-locality of a maximally entangled state also break that of all states whose reduction on the free side is maximally mixed.

Extending the work of Verstraete et al. in [18] and [19] we provide a closed form necessary-sufficient condition for any two-qubit state to show hidden non-locality [20] with respect to the Bell-CHSH inequality. Using this we provide an exact characterization of all strongly non-locality breaking qubit channels. It turns out that unital qubit channels breaking nonlocality of a maximally entangled state are strongly non-locality breaking while extremal qubit channels cannot be so unless they are entanglement breaking.We exploit a recent example of genuine hidden nonlocality [5] to show that a channel which *genuinely* breaks the non-locality of maximally entangled states (in the sense that its dual state has a local model) may not be strongly non-locality breaking. We also show numerically that the relative volume of entanglement breaking, non-locality breaking for maximally entangled states and strongly non-locality breaking channels in the six-dimensional real parameter space of all qubit channels is respectively about 0.24, 0.81 and 0.39 .

- In chapter 8, we find an exact formula of the maximum singlet fraction attainable for a qubit channel and give an explicit protocol to achieve the optimal value. The protocol distinguishes between unital and nonunital channels and requires no local post-processing. In particular, the optimal singlet fraction is achieved by transmitting part of an appropriate pure entangled state, which is maximally entangled if and only if the channel is unital. A linear function of the optimal singlet fraction is also shown to be an upper bound on the distillable entanglement of the mixed state dual to the channel.

- We end in chapter 9 with conclusions and future directions.

Chapter 2

Arthur-Kelly model and approximate joint measurement

In the quantum world an observer cannot jointly measure two non-commuting observables. What this means is that it is impossible to come up with a measurement setting (corresponding to a self-adjoint operator) which allows one to infer the statistics of measurement of both the observables for the system in an arbitrary quantum state. This property is called complemetarity and is unrelated to the usual uncertainty principle which is a statement about variances of statistics of counterfactual measurements performed on *two different ensembles* of a system corresponding to the same quantum state.

As intuition suggests one has to allow for some degree of imprecision in order to make room for a notion of joint measurement of non-commuting observables. In quantum mechanics of a single system observables correspond to self-adjoint operators with the outcome statistics given by the Born rule , i.e if P_i is the projector onto the eigenspace corresponding to the eigenvalue λ_i then in a measurement of the observable the probability of obtaining the ith outcome when the system is in state ρ is $Tr(\rho P_i)$. However this is not adequate. For example, suppose a system is allowed to interact with another system(ancilla) for a while and then measurement is done on the combined system. To describe the probabilities of obtaining various outcomes for such a measurement for different states of the original system before the interaction one has to replace the projectors above by positive operators E_i (acting on the states of the original system) with $\sum_i E_i = I$ for normalization. Such measurements are called POVM (positive operator valued measures). The most general measurements possible in quantum mechanics can be described through POVMs. It turns out that the idea of imprecise or unsharp measurements can be appropriately investigated if one considers instead of projective measurements, POVMs. The observables corresponding to projective measurements form a subclass of those described by POVMs and are called sharp.

The first model for approximate joint measurement of position and momentum was given by Arthur and Kelly in 1965 by generalizing von Neumann's model for measurement([1]). In the von Neumann's model for measurement, the position observable of the object is measured by coupling it to the momentum P_p of a probe system via the interaction evolution $U = e^{-\frac{i}{\hbar}\lambda Q \otimes P_p}$ and using the position Q_p of the probe as the readout observable. The idea of Arthur-Kelly was to couple two such probe systems respectively to the position and momentum of the system and then perform measurement on the commuting meter observables of the probe to gain information about the position and momenta of the system. This is in fact an unsharp joint measurement in the sense to be described later and inspired the later development of the formalism .

Let us begin with a discussion of the Arthur-Kelly model.

2.1 Arthur-Kelly Model

The Arthur-Kelly model was obtained by generalising von Neumann model of an approximate position measurement ([21]). Two independant probe systems with momenta p_1 and p_2 were coupled respectively to to the position $\hat{x}$ and momentum $\hat{p}$ of the system through the instantaneous coupling ,

$$\hat{H} = \hbar K \delta(t)(\hat{x}\hat{p_1} + \hat{p}\hat{p_2}) \tag{2.1}$$

. Assuming that the strong measurement interaction dominates all other parts of the Hamiltonian during its presence , we can integrate the time evolution operator to ,

$$U = exp[-i(K(\hat{x}\hat{p_1} + \hat{p}\hat{p_2}))). \tag{2.2}$$

After the evolution, ideal (projective) measurements are made on the commuting meter positions $\hat{x_1}$ and $\hat{x_2}$ to gain information about $\hat{x}$ and $\hat{p}$ of the system. The initial meter wave functions were taken to be,

$$\begin{aligned} M(x_1) &= \left(\frac{2}{\pi b}\right)^{\frac{1}{4}} e^{\frac{-x^2}{b}} \\ N(x_2) &= \left(\frac{2b}{\pi}\right)^{\frac{1}{4}} e^{-by^2} \end{aligned} \tag{2.3}$$

. Thus in the position representation the initial wave function of the combined system, with each part assumed to be independant is , $\psi_i(x, x_1, x_2) = F(x)M(x_1)N(x_2)$. The final joint wave func. after the evolution is ,

$$\psi_f(x, x_1, x_2) = \int_{-\infty}^{+\infty} F(x-w)M(x_1 - x + \frac{1}{2}w)\frac{e^{(-\frac{w^2}{4b})}}{(4\pi b)^{\frac{1}{2}}} e^{iwx_2} dw \tag{2.4}$$

The joint *observed* probability distribution for the commuting positions $\hat{x_1}$ and $\hat{x_2}$ is given by,

$$P(x_1, x_2) = \int_{-\infty}^{+\infty} |\psi_f|^2 dx, \tag{2.5}$$

which from eqn. (2.5) turns out to be ,

$$P(x_1,x_2)=\left(\frac{1}{4\pi^2 b}\right)^{\frac{1}{2}}|\left[\int_{-\infty}^{\infty}F(q)e^{-\frac{(x-q)^2}{2b}}e^{-iqy}dq\right]|^2. \tag{2.6}$$

The new wavefunction of the system after an observation x_1^m, x_2^m is given by,

$$\begin{aligned}F_1(x) &= \frac{\psi_f(x,x_1^m,x_2^m)}{\int|\psi_f(x,x_1^m,x_2^m)|^2dx} \\ &= \left(\frac{1}{\pi b}\right)^{\frac{1}{4}}e^{-\left(\frac{(x-x_1^m)^2}{2b}+ixx_2^m\right)}.\end{aligned} \tag{2.7}$$

Thus the post measurement state of the system is a minimum Gaussian wavepacket with mean position x_1^m and mean momentum x_2^m . Also, one can show from eqns. (2.5) and (2.6) that the expected value of x_1 is equal to that of x before the interaction and the expected value of x_2 is equal to the expected value of p before the interaction. The variances of x_1 and x_2 are related to the variances of x and p before the interaction by,

$$\begin{aligned}\sigma_{x_1}^2 &= \sigma_x^2+\frac{b}{2} \\ \sigma_{x_2}^2 &= \sigma_p^2+\frac{1}{2b}.\end{aligned} \tag{2.8}$$

The parameter b called "balance" represents a trade-off between how close the variances of the meters x_1 and x_2 are to the variances of x and p satisfying,

$$\sigma_{x_1}\sigma_{x_2}\geq 1. \tag{2.9}$$

Measurements made in Arthur-Kelly type models naturally fit into the formalism of approximate joint measuremnt which we now describe.

2.2 Approximate Joint Measurement

We begin with recalling the basic definitions and setting the notation. On a complex Hilbert space $\mathcal{H}$, a linear operator E with $0 \leq E \leq 1$ is called an *effect*. The set of effects is denoted by $\mathcal{EH}$. A general quantum *observable* (or *measurement*) is described by a POVM A, which is a countably additive mapping $A : \mathcal{A} \to \mathcal{EH}$ on a σ-algebra $\mathcal{A}$ of subsets of Ω_A satisfying $A(\Omega_A) = 1$. The set Ω_A represents the set of all possible outcomes of the measurement. For any input state ρ describing the initial preparation of the quantum system and for any $X \in \mathcal{A}$, the expression $\mathrm{tr}\big[\rho A(X)\big]$ gives then the probability of obtaining a measurement outcome $x \in X$. We denote by $\mathrm{ran}\,(A) := \{A(X)|X \in \mathcal{A}\}$ the set of effects corresponding to A.

Defn.: Observables A^1,A^2....A^n are said to be *jointly measurable* if there exists an observable $G : \mathcal{G} \to \mathcal{E}(\mathcal{H})$ on the σ-algebra generated by $\mathcal{A}^1 \times \mathcal{A}^2 \times ...\mathcal{A}^n$ such that,

$$\begin{aligned} G(X^1 \times \Omega_{A^2} \times ....\Omega_{A^n}) &= A^1(X^1) \\ G(\Omega_{A^1} \times X^2 \times ....\Omega_{A^n}) &= A^2(X^2) \\ &. \\ &. \\ &. \\ G(\Omega_{A^1} \times \Omega_{A^2} \times ...X^n) &= A^n(X^n) \end{aligned} \tag{2.10}$$

for all $X_1 \in \Omega_{A^1}$, $X_2 \in \Omega_{A^2}$,.. $X_n \in \Omega_{A^n}$.

In order to illustrate the definiton of joint measurability, let A and B be two observables with finite outcome spaces $\Omega_A = \{x_1, x_2, ..., x_n\}$ and $\Omega_B = \{y_1, y_2, ..., y_m\}$ respectively. The condition for joint measurability of A and B is the existence of such G defined on the outcome space $\Omega_A \times \Omega_B$, for which

$$\begin{aligned} A(x_i) &= \sum_{j=1}^{m} G(x_i, y_j), \quad \forall i = 1, 2, ..n \\ B(y_j) &= \sum_{i=1}^{n} G(x_i, y_j) \quad \forall j = 1, 2, ...m. \end{aligned} \tag{2.11}$$

Given the above definiton the natural question is given complementary (non-commuting) observables , A and B say considering the finite outcomes case, can we find a joint observable G satisfying eqn. (2.11) . As we will see this is impossible if A and B correspond to projective measurements on eigenbases of complementary observables. The question is then can we have some sort of approximations of A and B denoted by $\tilde{A}$ and $\tilde{B}$ for which we can find a joint observable ? This is indeed possible and the trade offs between how good the approximations $\tilde{A}$, $\tilde{B}$ are of respectively A and B gives rise to measurement uncertainty relations.

The connection to Arthur-Kelly model essentially comes through the fact that the probability of clicking of a projective measurement on the meters after an unitary evolution starting from a product state of the meter and system can be effectively described through a POVM (the joint POVM) on the system through the Naimark dilation theorem. A comparison of the moments of the marginal probability distribution of the meters to that of the complementary system observables for arbitrary initial system states is equivalent to comparison of the marginal POVM operators to the corresponding projectors on the eigenbases of the complementary observables .

We thus need a measure of closeness between two observables A and $\tilde{A}$. But before that let us consider the case of joint measurability of sharp observables . If observable A has projections as its values (i.e. $A(X)^2 = A(X)$ for all $X \in \mathcal{A}$), it is called sharp . Two sharp observables are jointly measurable if and only if they commute. Commutativity of A and B means that all pairs of their effects commute, i.e., $[A(X), B(Y)] = 0$for all X, Y.

Proposition 1 : Let A and B be jointly measurable observables. If (at least) one of them is sharp, then they commute and they have a unique joint observable G. This joint observable is determined by the condition

$$G(X \times Y) = A(X)B(Y) \quad \forall X \in \mathcal{A}, Y \in \mathcal{B} \tag{2.12}$$

.

Proof: Let, for instance **A** be a sharp observable and suppose that **G** is a joint observable for **A** and **B**. Since,

$$G(X \times Y) \leq G(X \times \Omega_B) = A(X) \tag{2.13}$$

it follows that the range of $G(X \times Y)$ is contained in the range of $A(X)$. Therefore,

$$A(X)G(X \times Y) = G(X \times Y), \tag{2.14}$$

and taking adjoint of both sides we also get ,

$$G(X \times Y)A(X) = G(X \times Y). \tag{2.15}$$

Applying this results to the complementary set X^c we get,

$$A(X)G(X^c \times Y) = (I - A(X^c))G(X^c \times Y) = 0$$

and similarly,

$$G(X^c \times Y)A(X) = 0.$$

It then follows that,

$$\begin{aligned} A(X)B(Y) &= A(X)G(\Omega_A \times Y) = A(X)(G(X \times Y) + G(X^c \times Y)) && (2.16) \\ &= G(X \times Y) && (2.17) \end{aligned}$$

and similarly,

$$B(Y)A(X) = G(X \times Y). \tag{2.18}$$

A comparison of these equations shows that A and B commute and eq. (2.12) holds.

□

2.3 Quality of joint measurement and measurement uncertainty relations

The notion of approximate joint measurement naturally demands a measure of proximity to the sharp observables being approximately measured. The restriction on the measures corresponding to non-commuting observables leads one to measurement uncertainty relations. For the rest of this section we consider only qubit observables though most of the definitons have natural generalisations to higher dimensions and continous variables.

2.3.1 Closeness of probabilities

Proximity between two observables can be characterised by the distance between the corresponding probability distributions for all states. Thus distance between two observables $\hat{A}$ and $\hat{B}$ can be defined as

$$D(\hat{A}, \hat{B}) := \max_j \sup_T |tr[TA_j] - tr[TB_j]| \tag{2.19}$$

where $A_j (or B_j)$ corresponds to the POVM element(or effect) of the observable A (or B) associated with the measurement outcome j . T is the density matrix of the system. [22]

For two single-qubit observables $\hat{\Upsilon}^{\alpha,\vec{a_1}}$,$\hat{\Upsilon}^{\beta,\vec{a_2}}$ with,

$$\hat{\Upsilon}^{\alpha,\vec{a_1}} = \frac{1}{2}(\alpha I + \vec{a_1}.\vec{\sigma}) \tag{2.20}$$

$$\hat{\Upsilon}^{\beta,\vec{a_2}} = \frac{1}{2}(\beta I + \vec{a_2}.\vec{\sigma}) \tag{2.21}$$

with $(\alpha, \vec{a_1})$ and $(\beta, \vec{a_2}) \in \mathcal{R}^4$. Due to positivity restrictions on $\hat{\Upsilon}^{\alpha,\vec{a_1}}$ and $\hat{\Upsilon}^{\beta,\vec{a_2}}$ we must have,

$$\begin{aligned} (2 - ||\vec{a_1}||) &\geq \alpha \geq ||\vec{a_1}|| \\ (2 - ||\vec{a_2}||) &\geq \beta \geq ||\vec{a_2}|| \end{aligned} \tag{2.22}$$

Now for observables $\hat{A}$, $\hat{B}$ with respective set of effects $\{A_+, A_-\}$, $\{B_+, B_-\}$ we have $|tr[TA_+] - tr[TB_+]| = |tr[TA_-] - tr[TB_-]|$. Taking T to be the $'+'$ state or $'-'$ state of $\vec{\sigma}.\frac{(\vec{a_1}-\vec{a_2})}{||(\vec{a-1}-\vec{a_2})||}$ according as whether $(\alpha - \beta) > 0$ or $(\alpha - \beta) < 0$ respectively one has,

$$D(\hat{\Upsilon}^{\alpha,\vec{a_1}}, \hat{\Upsilon}^{\beta,\vec{a_2}}) = \frac{1}{2}||\vec{a_1} - \vec{a_2}|| + \frac{1}{2}|\alpha - \beta| \tag{2.23}$$

This shows that the distance of a certain unsharp observable $\hat{\Upsilon}^{\alpha,\vec{a_1}}$ from any sharp observable $\hat{\Upsilon}^{1,\hat{n}}$ is minimum when $\vec{a_1}$ is along $\hat{n}$.

2.3.1.1 Unbiased Observables

Observables of the form $\hat{\Upsilon}^{1,\vec{a}}$ are called unbiased. As $\hat{\Upsilon}^{1,\vec{a}} = \frac{(1+|\vec{a}|)}{2}\frac{1}{2}(I + \frac{\vec{a}.\vec{\sigma}}{|\vec{a}|}) + \frac{(1-|\vec{a}|)}{2}\frac{1}{2}(I - \frac{\vec{a}.\vec{\sigma}}{|\vec{a}|})$, therefore the probability of occurence of outcomes +(or -) for the initial state $\frac{I}{2}$ (the maximally mixed state) is given by, $Tr[\frac{I}{2}.\frac{1}{2}(I + \frac{\vec{a}.\vec{\sigma})}{|\vec{a}|}] = \frac{1}{2}\left(\text{or } Tr[\frac{I}{2}.\frac{1}{2}(I - \frac{\vec{a}.\vec{\sigma})}{|\vec{a}|})] = \frac{1}{2}\right)$. So these two probabilities are same. Hence the name 'unbiased'.

For such an observable, both the outcomes are equally likely for a maximally mixed state . Also, the expectation value of the unsharp measurement when the system is in state T is given by $\langle\vec{\sigma}.\hat{a}\rangle_u := 1.tr(T\frac{1}{2}(I + \vec{a}.\vec{\sigma})) - 1.tr(T\frac{1}{2}(I - \vec{a}.\vec{\sigma})) = ||\vec{a}||\langle\vec{\sigma}.\hat{a}\rangle$. Again as, $D(\hat{\Upsilon}^{1,\hat{a}}, \hat{\Upsilon}^{1,\vec{a}}) = \frac{1}{2}(1 - ||\vec{a}||)$, $||\vec{a}||$ itself serves as a measure of proximity . We will often approximate sharp observables with unbiased unsharp ones.

2.3.1.2 Measurement uncertainties

We will choose jointly measurable observable pairs $(\hat{\Upsilon}^{\alpha,\vec{a}}, \hat{\Upsilon}^{\beta,\vec{b}})$ to approximate the sharp pair $(\hat{\Upsilon}^{1,\hat{n}}, \hat{\Upsilon}^{1,\hat{m}})$. The necessary and sufficient conditions on $\alpha, \beta, \vec{a}, \vec{b}$ so that the first pair is jointly measurable in the sense of eqn. 2.11 is formulated in [23],[24], [17]. When the observables are unbiased the conditions simplify to,

$$||\vec{a}+\vec{b}|| + ||\vec{a}-\vec{b}|| \leq 2. \tag{2.24}$$

It was further shown in [22] that,

$$D(\hat{\Upsilon}^{\alpha,\vec{a}}, \hat{\Upsilon}^{1,\hat{n}}) + D(\hat{\Upsilon}^{\beta,\vec{b}}, \hat{\Upsilon}^{1,\hat{m}}) \geq 2D_0 \tag{2.25}$$

with, $D_0 = \frac{1}{\sqrt{2}}(cos(\theta/2) + sin(\theta/2) - 1)$, θ being the angle between $\hat{a}$ and $\hat{b}$. The conditions for attainment of the lower bound $2D_0$ were also spelt out . The approximate observables should be unbiased, i.e $\alpha = \beta = 1$ in this later case. The other conditions([22]) imply that for optimality, $\vec{a}$ and $\vec{b}$ lie along $\hat{n}$ and $\hat{m}$ respectively, only when the latter are orthogonal.

The derivation of eqn .(2.24) follows as a special case of a necessary sufficient condition for approximate joint measurement of three unbiased qubit observables done later (ref of later chapter) .

2.3.2 Closeness of observables

A completely different approach was taken in references [25] and [26] in the context of the original Arthur-Kelly model to give a formulation of the complementary nature of the approximate joint measurement process. If $\hat{\mu}_x$ and $\hat{\mu}_p$ denote the pointer observables used to measure the system position and momentum respectively , then the retrodictive error operators are defined as ,

$$\epsilon\hat{_X}_i = \hat{\mu}_{xf} - \hat{x}_i, \ \epsilon\hat{_P}_i = \hat{\mu}_{pf} - \hat{p}_i \tag{2.26}$$

the predictive error operators as,

$$\epsilon\hat{_X}_f = \hat{\mu}_{xf} - \hat{x}_f, \ \epsilon\hat{_P}_f = \hat{\mu}_{pf} - \hat{p}_f \tag{2.27}$$

and the disturbance operators as,

$$\hat{\delta}_X = \hat{x}_f - \hat{x}_i, \ \hat{\epsilon}_P = \hat{p}_f - \hat{p}_i \tag{2.28}$$

where operators $\hat{O}_f$, appearing on right stands for the final Heisenberg picture operator $\hat{O}$ after the measurement interaction U, i.e, $\hat{O}_f = U^{\dagger}O_iU$ where $\hat{O}_i$ is the Heisenberg picture operator at the moment the interaction starts. Various errors were then defined by taking the square root of expectation of the square of the operators defined above and taking supremum over the system states. For example the maximal error of retrodiction was defined as ,

$$\Delta_{eix} = sup_{|\psi\rangle \in H_{sys}}(\langle\psi \otimes \Psi_1 \otimes \Psi_2|\epsilon\hat{_X}_i{}^2|\psi \otimes \Psi_1 \otimes \Psi_2\rangle)^{\frac{1}{2}}, \tag{2.29}$$

and similarly for Δ_{eip} , Δ_{ef_x}, Δ_{ef_p} , Δ_{dx} and Δ_{dp}(see [25] , [26] for details) .Measurement uncertainty or 'error principle' was shown to hold in the form of,

$$\Delta_{eix}\Delta_{eip} \geq \frac{\hbar}{2}, \tag{2.30}$$

$$\Delta_{ef_x}\Delta_{ef_p} \geq \frac{\hbar}{2}, \ \Delta_{eix}\Delta_{dp} \geq \frac{\hbar}{2}, \tag{2.31}$$

$$\Delta_{ef_x}\Delta_{dp} \geq \frac{\hbar}{2}, \tag{2.32}$$

and extensions of the above uncertainties in the obvious way. One of the important features of this approach is the difference between error of retrodiction and that of prediction .It was shown that these are not the same as long as there is a finite disturbance.

2.3.3 Qubit Observables

In a similar spirit, for the case of approximate joint measurement of qubit observables through an Arthur-Kelly like process fidelities were defined in the Heisenberg picture that would provide a notion of direction of spin [16] of the system. As in the previous case, distinction was made between errors of retrodiction and prediction. In this paper we consider only the type 1 measurements considered by the author of [16] . We next consider the fidelities as defined by the authors. The retrodictive fidelity is defined as,

$$\eta_i = inf_{|\chi\rangle \in H_{sys}}(\langle\psi| \otimes \langle\chi|\frac{1}{2}(\hat{n_f}.\hat{S_i} + \hat{S_i}.\hat{n_f})|\psi\rangle \otimes |\chi\rangle) \tag{2.33}$$

where, $\hat{S_i} = \hat{S}$ and $\hat{n_i} = \hat{n}$ are the initial values of the Heisenberg spin and pointer observables respectively and $\hat{n_f} = U^{\dagger}(\hat{n_i} \otimes 1_s)U$ and $\hat{S_f} = U^{\dagger}(I \otimes \hat{S_i})U$ respectively be the final Heisenberg pointer and spin direction observables after the measurement interaction.

The predictive fidelity is defined as,

$$\eta_f = inf_{|\chi\rangle \in H_{sys}}(\langle\psi| \otimes \langle\chi|\frac{1}{2}(\hat{n_f}.\hat{S_f} + \hat{S_f}.\hat{n_f})|\psi\rangle \otimes |\chi\rangle). \tag{2.34}$$

The measurement disturbance by,

$$\eta_d = inf_{|\chi\rangle \in H_{sys}}(\langle\psi| \otimes \langle\chi|\frac{1}{2}(\hat{S_f}.\hat{S_i} + \hat{S_i}.\hat{S_f})|\psi\rangle \otimes |\chi\rangle). \tag{2.35}$$

The intuition behind the above definitions is classical in the sense that it considers alignment of initial or final spin vector and initial or final pointer direction. But the above fidelities were used to define maximal rms error of retrodiction,

$$\begin{aligned}\Delta_{ei}S = sup_{|\chi\rangle \in H_{sys}}(\langle\psi| \otimes \langle\chi||\eta_i\hat{n_f} - \hat{S_i}|^2|\psi\rangle \otimes |\chi\rangle)^{\frac{1}{2}} \\ = (s + s^2 - \eta_i^2)^{\frac{1}{2}},\end{aligned} \tag{2.36}$$

maximal rms error of prediction,

$$\Delta_{ef}S = sup_{|\chi\rangle \in H_{sys}}(\langle\psi| \otimes \langle\chi||\eta_f \hat{n_f} - \hat{S}_f|^2|\psi\rangle \otimes |\chi\rangle)^{\frac{1}{2}}$$
$$= (s + s^2 - \eta_f^2)^{\frac{1}{2}}, \tag{2.37}$$

and maximal rms disturbance

$$\Delta_d S = sup_{|\chi\rangle \in H_{sys}}(\langle\psi| \otimes \langle\chi||\hat{S}_f - \hat{S}_i|^2|\psi\rangle \otimes |\chi\rangle)^{\frac{1}{2}}$$
$$= \sqrt{2}(s + s^2 - \eta_d^2)^{\frac{1}{2}}, \tag{2.38}$$

where the spin $s = \frac{1}{2}$ for our case. The quantities $\Delta_{ei}S$, $\Delta_{ef}S$, $\Delta_d S$ were expected to play the same role for these measurements as similar quantities defined for the original Arthur-Kelly model. It was also shown that,

$$\eta_i, \eta_f \leq s = \frac{1}{2}. \tag{2.39}$$

No measurement uncertainties were derived in [16] and the question was left open for further investigation.

Chapter 3

An introduction to quantum nonlocality

3.1 Nonlocality and the Bell-CHSH inequality

Suppose, for a composite system in the state ρ_{AB} (acting on Hilbert space $H_A \otimes H_B$) shared between Alice and Bob, each party can perform measurements characterised by POVMs $\sum_a M_{a|x} = I_A$ and $\sum_b M_{b|y} = I_B$ respectively with indices x and y characterising the possible choices of measurement settings for each party. Then ρ_{AB} is said to be *local* iff the probability of obtaining outcomes a and b for measurement choices x and y for Alice and Bob respectively can be written as,

$$p(ab|xy) = Tr(M_{a|x} \otimes M_{b|y}\rho) = \int_\Lambda p_A(a|x,\lambda)p_B(b|y,\lambda)p(\lambda)d\lambda. \tag{3.1}$$

Eqn. (3.1) reflects the fact that for 'local' states the correlation between Alice's and Bob's outcomes for a certain choice of measurement settings can be completely *explained away* by the 'hidden variable' λ (see [20]) so that we have, $p(ab|x,y,\lambda) = p(a|x,\lambda)p(b|y,\lambda)$. In a typical Bell-inequality scenario one considers measurement settings $M_{a|x}$ for Alice and $M_{b|y}$ for Bob with $x, y \in \{1, 2, ..m\}$ and outcomes $a, b = 1, 2, ...\Delta$. With the choice of outcome labels $a, b \in \{-1, 1\}$, and expectation $\langle a_x b_y \rangle = \sum_{a,b} abp(ab|xy))$ we have for a local state satisfying the condition in eqn.(3.1) the Bell-CHSH inequality([27]) given by,

$$\langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \leq 2. \tag{3.2}$$

Derivation:

In order to derive this inequality , we use the locality condition (3.1) in the definition of $\langle a_x b_y \rangle$ so that we have, $\langle a_x b_y \rangle = \int_\lambda \langle a_x \rangle_\lambda \langle b_y \rangle_\lambda d\lambda$, with the local expectations being $\langle a_x \rangle_\lambda = \sum_a ap(a|x,\lambda)$ and $\langle b_y \rangle_\lambda = \sum_b ap(b|y,\lambda)$ taking values in $[-1, 1]$. Thus the LHS of the Bell-CHSH inequality can be written as $S = \int_\lambda d\lambda S_\lambda p(\lambda)$, with $S_\lambda = \langle a_0 \rangle_\lambda \langle b_0 \rangle_\lambda + \langle a_0 \rangle_\lambda \langle b_1 \rangle_\lambda + \langle a_1 \rangle_\lambda \langle b_0 \rangle_\lambda - \langle a_1 \rangle_\lambda \langle b_1 \rangle_\lambda$. Since, $\langle a_0 \rangle_\lambda, \langle a_1 \rangle_\lambda \in$

$[-1,1]$ we have $S_\lambda \leq |\langle b_0\rangle_\lambda + \langle b_1\rangle_\lambda| + |\langle b_0\rangle_\lambda - \langle b_1\rangle_\lambda|$. Without loss of generality we can assume that $\langle b_0\rangle_\lambda \geq \langle b_1\rangle_\lambda \geq 0$ which yields $S_\lambda = 2\langle b_0\rangle_\lambda \leq 2$, and thus $\langle a_0b_0\rangle + \langle a_0b_1\rangle + \langle a_1b_0\rangle - \langle a_1b_1\rangle = \int_\lambda S_\lambda p(\lambda)d\lambda \leq 2$.

Motivations: The Bell-CHSH inequality can also be derived as a consequence of the following features of a local hidden variable theory, i) Realism: For a hidden variable state described by λ the values of all possible observables(and not just the observables actually measured) are specified ii) Locality: Values specified for Alice's particle are independent of the measurement choice for Bob's particle and vice versa.

Operationally speaking, the condition given by eqn. 3.1 reflects the following situation. Suppose, we have two separated parties Alice and Bob performing measurements x, y on their their respective systems in a quantum state ρ so that no information about the measurement performed by Bob is available for Alice before she obtains her result. Then, eqn. (3.1) reflects the situation where Alice and Bob *can* simulate the measurement statistics by sharing a common classical variable λ which varies across different simulation runs according to a probability distribution $p(\lambda)$([20]). The output of Alice will be characterized by a probability distribution $p(a|x,\lambda)$, which is fixed by their common strategy and the joint state ρ , but which may depend on the specific measurement x chosen and on the data λ shared with Bob. Similarly the output of Bob is given by a probabilistic function $p(b|y,\lambda)$.

3.1.1 Quantum Violation

Let now Alice and Bob share a two-qubit state ρ. Let the measurement choices x and y be associated with vectors $\vec{x}$ and $\vec{y}$ corresponding to measurements of $\vec{x}.\vec{\sigma}$ on the first qubit and $\vec{y}.\vec{\sigma}$ on the second qubit.

Consider the Hilbert space $H = \mathcal{C}^2 \otimes \mathcal{C}^2$ of a two-qubit system. Any state on H can be represented using the Hilbert-Schmidt basis as follows:

$$\rho = \frac{1}{4}(I \otimes I + \vec{r}.\vec{\sigma} \otimes I + I \otimes \vec{s}.\vec{\sigma} + \sum_{n,m=1}^{3} t_{nm}\sigma_n \otimes \sigma_m), \tag{3.3}$$

with the coefficients $\vec{s}, \vec{r} \in \mathcal{R}^3$ and

$$t_{nm} = tr(\rho\sigma_n \otimes \sigma_m) \tag{3.4}$$

form a real 3×3 matrix which we shall denote by T_ρ. Here I is the 2×2 identity matrix and σ_1, σ_2, σ_3 are Pauli spin matrices. $\vec{r}$,$\vec{s}$ and t_{nm} also satisfy the conditions for positivity of ρ [28]. Now, the Bell operator associated with the Bell-CHSH inequality (3.2) has the following general form [27] :

$$B_{CHSH} = \hat{a}.\vec{\sigma} \otimes (\hat{b} + \hat{b'}).\vec{\sigma} + \hat{a'}.\vec{\sigma} \otimes (\hat{b} - \hat{b'}).\vec{\sigma}, \tag{3.5}$$

where $\hat{a}$, $\hat{a'}$, $\hat{b}$, $\hat{b'}$ are unit vectors in $\mathcal{R}^3$. Then the Bell-CHSH inequality for ρ, following from the consideration of local hidden variable theory, is given by

$$Tr(\rho B_{CHSH}) \leq 2. \tag{3.6}$$

The matrix $U_\rho := T_\rho^T T_\rho$ is a real symmetric one, and so it can be diagonalized. We denote the two greater (obviously non-negative) eigenvalues of U_ρ by u and $\tilde{u}$. Then we define the quantity

$$M(\rho) = u + \tilde{u}. \tag{3.7}$$

Another matrix that will play an important role in the sequel is the real 4×4 matrix R defined as $R_{ij} = Tr(\rho\sigma_i \otimes \sigma_j), i, j = 0, 1, 2, 3$ (where $\sigma_0 = I_2$). The lower 3×3 block of R is of course the matrix T_ρ defined before.

Theorem 1. Any two-qubit density matrix ρ violates inequality (3.6) for some operator of the form (3.5) (i.e.,for some choice of $\hat{a}$, $\hat{a'}$, $\hat{b}$ and $\hat{b'}$) iff $M(\rho) > 1$.

The optimal Bell violation for ρ is given by $2\sqrt{M(\rho)}$.

Proof: This theorem was proved for the first time in [28]. Here we follow the treatment provided in ref. [29]. Now, let

$$B'_{CHSH} = \frac{1}{2}\sum_{i,j=1}^{3}[a_i(b_j + b'_j) + a'_i(b_j - b'_j)]\sigma_i \otimes \sigma_j, \tag{3.8}$$

where a_i,a'_i,b_j,b'_j respectively denote the components of $\hat{a}$, $\hat{a'}$, $\hat{b}$ and $\hat{b'}$. Thus defining the real 3×3 matrix , $X_{ij} = \frac{1}{2}\{(a_i+a'_i)b_j+(a_i-a'_i)b'_j\}$, i.e $X = \frac{1}{2}\begin{bmatrix}\hat{b} & \hat{b'}\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\hat{a}^T\\\hat{a'}^T\end{bmatrix}$, we have ,

$$Tr(\rho B'_{CHSH}) = \frac{1}{2}Tr(\rho B_{CHSH}) = Tr(T_\rho X). \tag{3.9}$$

For Bell violation we require, $Tr(T_\rho X) > 1$. Due to the structure of X it is constrained to be of rank 2 with $Tr(X^T X) = 1$. In order to maximize $Tr(T_\rho X)$ we must therefore choose X to be the best rank 2 least squares approximation of the matrix T_ρ. In the Hilbert space basis , where T_ρ is diagonal ($T_\rho = diag(s_1, s_2, s_3)$) we have for maximum violation $X = \frac{\text{diag}(s_1,s_2,0)}{\sqrt{s_1^2+s_2^2}}$ which leads to Theorem 1.

3.2 General Bell-inequality scenarios

In a general Bell-inequality scenario each observer has a choice of m measurements to perform on his subsystem. Each measurement additionally has Δ outcomes.

The inputs of Alice and Bob are labelled respectively as $x.y \in \{1, 2, ..m\}$ and the outputs are labelled as $a, b \in \{1, 2, ..\Delta\}$. Let $p(ab|xy)$ denote the joint probability of obtaining the output pair (a, b) given the input pair (x, y). A Bell scenario is then completely characterised by $\Delta^2 m^2$ no. of such joint probabilities, one for each possible pair of inputs and outputs. Any such probability can be viewed as a point $\mathbf{p} \in \mathcal{R}^{\Delta^2 m^2}$ belonging to the probability space $\mathcal{P} \subset \mathcal{R}^{\Delta^2 m^2}$ defined by the positivity constraints $p(ab|xy) \geq 0$ and the normalization constraints $\sum_{a,b=1}^{\Delta} p(ab|xy) = 1$ for all $x, y \in \{1, 2, ..m\}$. Due to the normalisation constraints $\mathcal{P}$ is a subset of $\mathcal{R}^{\Delta^2 m^2}$ of dimension $(\Delta^2 - 1)m^2$.

The existence of a physical model behind the correlations obtained in a Bell scenario puts additional constraints on the probabilities $\mathbf{p}$. Three main types of correaltions can be distinguished.

3.2.1 No-signalling correlations

The first natural constraints on the probabilities $\mathbf{p}$ are the non-signalling constraints formally expressed as

$$\begin{aligned} \sum_{b=1}^{\Delta} p(ab|xy) &= \sum_{b=1}^{\Delta} p(ab|xy'), \text{for all} a, x, y, y' \\ \sum_{a=1}^{\Delta} p(ab|xy) &= \sum_{a=1}^{\Delta} p(ab|x'y), \text{for all} b, x, y, x' \end{aligned} \tag{3.10}$$

Let $\mathcal{NS}$ denote the set of probabilities satisfying the no-signalling constraints given by eqn. (3.10). It can be seen that $\mathcal{NS}$ (a convex subset of $\mathcal{P}$) is an affine subspace of $\mathcal{R}^{\Delta^2 m^2}$ of dimension

$$dim\mathcal{NS} = 2(\Delta - 1)m + (\Delta - 1)^2 m^2 := t. \tag{3.11}$$

3.2.2 Local correlations

A more severe constraint is given by the locality condition described before by eqn. (3.1). Any local behavior satisfies the no-signaling constraint but the converse does not hold in general. There exists no-signaling correlations which do not satisfy the locality criterion (eqn. (3.1). The set of local correlations is thus strictly smaller than the set of non-signaling correlations, i.e, defining the set of probabilities satisfying eqn. 3.1 by $\mathcal{L}$ we have $\mathcal{L} \subset \mathcal{NS}$.

Correlations that cannot be written in the local form (3.1) are said to be nonlocal. As this can only happen for $\Delta \geq 2$ and $m \geq 2$ in what follows we will always assume $\Delta \geq 2$,$m \geq 2$.

3.2.3 Quantum correlations

Let us now consider the set of correlations achievable within quantum theory. Formally the quantum set $\mathcal{Q}$ corresponds to elements of $\mathcal{P}$ that can be written as,

$$p(ab|xy) = Tr(\rho_{AB} M_{a|x} \otimes M_{b|y}), \tag{3.12}$$

where ρ_{AB} is a quantum state in a joint Hilbert space $H_A \otimes H_B$ of arbitrary dimension, $M_{a|x}$ are measurement operators on H_A characterising Alice's measurements and similarly $M_{b|y}$ are measurement operators on H_B characterising Bob's measurements. Without loss of generality we can always assume the state to be pure and the measurement operators to be orthogonal projectors if necessary by increasing the dimension of the Hilbert space. That is we can equivalently write an element of the quantum set $\mathcal{Q}$ as a probability vector $\mathbf{p}$ with components

$$p(ab|xy) = \langle\psi|M_{a|x} \otimes M_{b|y}|\psi\rangle, \tag{3.13}$$

where $M_{a|x}M_{a'|x} = \delta_{aa'}M_{a|x}$, $\sum_a M_{a|x} = I_a$ and similarly for the operators $M_{b|y}$.

It can be easily shown that the local set $\mathcal{L}$ belongs to the quantum set $\mathcal{Q}$. Moreover any quantum correlation of course satisfies the no-signaling constraints. However as must be clear from the discussion on Bell-CHSH violation that there are quantum correlations that do not belong to the local set and also there are also no-signaling correlations that do not belong to the quantum set [30]. Furthermore it can be shown that, $dim\mathcal{NS} = dim\mathcal{Q} = dim\mathcal{L}$ [31] with their dimension t defined by eqn. (3.11).

3.2.4 Bell-inequalities

The sets $\mathcal{L}$, $\mathcal{Q}$ and $\mathcal{NS}$ are closed bounded and convex. By the hyperplane separation theorem it follows that for each $\mathbf{p} \in \mathcal{R}^t$ that does not belong to one of the sets $\mathcal{K} = \mathcal{L}, \mathcal{Q}$ or $\mathcal{NS}$ there exists a hyperplane that separates this $\mathbf{p}$ from the corresponding set. That is if $\mathbf{p} \notin \mathcal{K}$ then there exists an inequality of the form

$$\mathbf{s}.\mathbf{p} = \sum_{abxy} s^{ab}_{xy} p(ab|xy) \leq S_k \tag{3.14}$$

that is satisfied by all $\mathbf{p}$ belonging to $\mathcal{K}$ but which is violated by $\hat{\mathbf{p}} : \mathbf{s}.\hat{\mathbf{p}} > S_k$.

These inequalities for the local set $\mathcal{L}$ are the Bell inequalities an example of which is the Bell-CHSH inequality described before. The inequalities associated with the quantum set $\mathcal{Q}$ are called quantum Bell inequalities or Tsirelson's inequalities [20].

3.2.5 Violation of CHSH type inequalities by N-party states

A general quantum probability distribution with N parties instead of two can be described as

$$p(a_1..a_N|x_1...x_N) = tr(\rho \bigotimes_{n=1}^{N} A_n(a_n|x_n)), \tag{3.15}$$

where ρ acts on $\bigotimes_{n=1}^{N} H_n$ and A_n are POVMs for nth party with outcomes $a_n = 1, 2, ..\Delta$ for each choice of settings described by $x_n = 1, 2, ..m$. A generalized Bell inequality for N parties is again of the form of eqn.(3.14) with a,b and x,y replaced respectively by $a_n = 1, 2, ..\Delta$ and $x_n = 1, 2, ..m$. It was shown in [32] that the set of all quantum distributions (3.15) is convex.

In the following we consider the case of two two-outcome measurement settings per site, i.e, $\Delta = m = 2$. Following [33] we provide a characterization of all extreme points of the set of quantum probabilities for the case $\Delta = m = 2$ and arbitrary N. We begin with the following Lemma.

Lemma:- Let A_1, A_2, B_1, B_2 be four projectors acting on a Hilbert space $\mathcal{H}$ such that $A_1 + A_2 = I$ and $B_1 + B_2 = I$. Then there exists an orthonormal basis in $\mathcal{H}$ where the four projectors A_1, A_2, B_1, B_2 are simultaneously block diagonalized in blocks of size 1×1 or 2×2.

Proof: The three positive operators B_1, $B_1A_1B_1$, $B_1A_2B_1$ can be simultaneously diagonalized because their ranges are contained in the subspace where B_1 acts like the identity and we have, $B_1A_1B_1 + B_1A_2B_1 = B_1$. Let $|v\rangle$ be a simultaneous eigenvector for which $B_2|v\rangle = 0$ and hence $B_1|v\rangle = |v\rangle$ As, $A_1 + A_2 = I$ we cannot have both $A_1|v\rangle = 0$ and $A_2|v\rangle = 0$. If $A_1|v\rangle = 0$ then $A_2|v\rangle = B_1|v\rangle = |v\rangle$ and the span of the vector $|v\rangle$ (denoted by E_v) corresponds to a 1×1 diagonal block in which A_1,A_2,B_1,B_2 have eigenvalues 0,1,1,0 respectively. The case $A_2|v\rangle = 0$ is similar.Consider the case where, $A_1|v\rangle \neq 0$ and $A_2|v\rangle \neq 0$. Let us now consider the span(E_v) of the two orthogonal vectors $|a_1\rangle = A_1|v\rangle$ and $|a_2\rangle = A_2|v\rangle$. Using $A_1 + A_2 = I$ we have $|a_1\rangle + |a_2\rangle = |v\rangle$ and thus $|v\rangle \in E_v$. Also, $B_1|a_1\rangle = B_1A_1|v\rangle = B_1A_1B_1|v\rangle \propto |v\rangle$ and similarly $B_1|a_2\rangle \propto |v\rangle$. Hence it is possible to find a vector $|w\rangle \in E_v$ such that $B_1|w\rangle = 0$ and $B_2|w\rangle = |w\rangle$. Thus the subspace E_v corresponds to a 2×2 simultaneous diagonal block for A_1, A_2,B_1, B_2. The same can be done with the rest of simultaneous eigenvectors $|v\rangle$ as defined above. And analogous reasoning follows for the simultaneous eigenvectors of B_2, $B_2A_1B_2$, $B_2A_2B_2$ which are orthogonal to the vectors $|w\rangle$ that have appeared previously. Finally, the direct sum of the subspaces $E_1, E_2, ..$ is $\mathcal{H}$; each subspace E_i of dimension two contains two eigenvectors of each operator A_1, A_2, B_1, B_2.

This Lemma can be used to prove the following results:

Result 1: For the case $\Delta = m = 2$ all quantum extreme points (3.15) are achievable by measuring N-qubit pure states with projective observables.

Proof: Now it can be easily seen that all two-outcome POVMs are mixtures of two-outcome projective measurements. Hence, the distribution (eq. 3.15) can be written as a mixture of distributions where the operators $A_n(a|x)$ are projectors. According to the lemma, the four operators $A_n(a|x)$ for $a, x = 1, 2$ can be simultaneously block diagonalized in blocks of size 2×2 at most. Let us denote by $\{E_1^n, E_2^n, ...\}$ the projectors onto the subspaces corresponding to these diagonal blocks , for the nth party. If the nth party performs the measurements $\{E_i^n\}_i$ before measuring $\{A(1|x), A(2|x)\}$, the result does not change. But after performing $\{E_i^n\}_i$ the local system is contained in a two-dimensional subspace. Applying this to all parties , the distribution given by eqn .(3.15) becomes a mixture of distributions generated by measuring N-qubit systems with projective observables. The conclusion about pure states follows from the convex decomposition of mixed states to pure states.

Result 2: If an N-party state ρ violates a Bell inequality ,say S, for $\Delta = m = 2$, then ρ can be transformed by an SLO into an N-qubit state $\tilde{\rho}$ that violates S by an equal or larger amount.

Proof: If ρ violates S then it does it with projective measurements as they are more extremal. Proceeding exactly as the previous proof , the correlations obtained from ρ by eqn . (3.15) do not change if the nth party performs the measurement $\{E_i^n\}_i$ before measuring $A_n(a|x)$, for all n. The final distribution (3.15)becomes a mixture of distributions generated b family of two-qubit states $(E_{i1}^1 \otimes ... \otimes E_{iN}^N)\rho(E_{i1}^1 \otimes ... \otimes E_{iN}^N)$. By convexity at least one of these states violates S by at least the same amount.

3.3 Entanglement and nonlocality

Entanglement is a necessary pre-requirement of non-locality. For states of the separable form, $\rho_{AB} = \sum_\lambda p_\lambda \rho_\lambda^A \otimes \rho_\lambda^B$ we have for correlations obtained by making local measurements on the state

$$\begin{aligned} p(ab|xy) &= tr[\sum_\lambda p_\lambda(\rho_\lambda^A \otimes \rho_\lambda^B) M_{a|x} \otimes M_{b|y}] \\ &= \sum_\lambda p_\lambda tr(\rho_\lambda^A M_{a|x}) tr(\rho_\lambda^B M_{b|y}) \\ &= \sum_\lambda p_\lambda p(a|x,\lambda) p(b|y,\lambda), \end{aligned} \tag{3.16}$$

which is of the form of (3.1). Now the question is if this is also true the other way round. For pure entangled states it was shown that if the measurement statistics for performing local measurements on the state satisfies eqn. (3.1) then the state must be of the product form [34]. For mixed entangled states the situation is more complicated. In ref. [4] Werner provided an example of a class of mixed entangled states which admit a local model as in eqn.(3.1) for any possible local measurements and hence the resulting correlations cannot violate any local realistic inequality. Werner's results were extended to the case of POVMs by Barrett in [35]. Using theorem 1 it is easy to see that not all two-qubit mixed entangled states violate the Bell-CHSH inequality. However for mixed states it is not enough to focus only on the Bell-CHSH inequality, as there are two qubit states which do not violate the Bell-CHSH inequality but violate an inequality involving three measurement settings per party called I_{3322} [36]. Another example is the Werner state,

$$\rho_W = p|\Phi^+\rangle\langle\Phi^+| + (1-p)\frac{I}{4}. \tag{3.17}$$

This state is separable for $p < \frac{1}{3}$ and violates the Bell-CHSH inequality for $p > \frac{1}{\sqrt{2}}$. However it was shown in [37] that the state violates a Bell inequality involving 465 settings per party for $p > 0.7056$.

It is also possible to provide lower bounds on $p$for the state above to be non-local by coming up with explicit local models. As mentioned before, Werner in [4] produced a class of local models for $p \leq \frac{1}{2}$, even though the above state is entangled for $p > \frac{1}{3}$. Various extensions of this have been done and currently for the interval $0.66 \lesssim p \lesssim 0.7056$ (or $\frac{5}{12} < p \lesssim 0.7056$ for POVMs) [20] it is not known if the nonlocality of the Werner-state in eqn. (5.6) can be revealed by performing measurements on a single copy of a state at a time.

3.3.1 Hidden Nonlocality

Popescu in [38] proposed a more general way of obtaining nonlocal correlations from an entangled quantum state. The idea is to apply some local physical operation to the quatum state and only select those states for further Bell tests which are accompanied by certain outcomes of the physical operation. The natural example of such an operation is of course measurement (ideal or non-ideal). Such post-selection operations are often referrred to as local filtering. If the post-selected state shows Bell-violation then the (hidden) nonlocality of the original state is said to have been *revealed* by means of local filtering. In particular Popescu showed that these happens for generalized Werner states([20]) of dimension $d \geq 5$. Also, Gisin in [39] gave examples of two-qubit entangled states which violate the CHSH inequality only after suitable local filtering.

A question left open in the work of Popescu ([38]) is whether hidden nonlocality can also be demonstrated for an entangled state admitting a local model for POVM measurements. Note that in the examples before Pospecu considered Werner states which admit a local model for projec-tive measurements, but are not known to be local when POVMs are considered. This question was answered recently by Hirsch et al. (2013)([5]), where it is shown that there exist entangled states featuring genuine hidden nonlocality, that is, states which admit a local model for POVMs, but violate a Bell inequality after judicious filtering.

A general local filtering operation taking a two qubit state ρ to another one ρ' can be described as,

$$\rho' = \frac{(A \otimes B)\rho(A^\dagger \otimes B^\dagger)}{Tr(A^\dagger A \otimes B^\dagger B\rho)}. \tag{3.18}$$

As the next Theorem shows , in the R matrix picture these local filtering operations act as Lorentz transformations.

Theorem 2 : Under local filtering transformations give by eqn. (3.18) the 4×4 matrix R transforms as

$$R' = L_A R L_B^T, \tag{3.19}$$

with $R'_{ij} = Tr(\rho'\sigma_i \otimes \sigma_j)$ for $i, j = 0, ..3$, L_A, L_B being proper orthocronous Lorentz transformations given by,

$$L_A = \frac{T(A \otimes A^*)T^\dagger}{|det(A)|}, \tag{3.20}$$

$$L_B = \frac{T(B \otimes B^*)T^\dagger}{|det(B)|}, \tag{3.21}$$

with the 4×4 matrix ,

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}. \tag{3.22}$$

Proof: Let us introduce the matrix, $\tilde{\rho} \equiv \tilde{\rho}_{ij,kl} = \rho_{ik,jl}$. Now it is easy to check that for $\tilde{\rho'} \equiv \tilde{\rho'}_{ij,kl} = \rho'_{ik,jl}$ corresponding to ρ' in eqn.(3.18) we have ,

$$\tilde{\rho'} = (A \otimes A^*)\tilde{\rho}(B \otimes B^*)^T. \tag{3.23}$$

Further we have, $R = 4T\tilde{\rho}T^T$.

Replacing with R in eqn. 3.23 we have,

$$\begin{aligned} T^\dagger R' T^* &= (A \otimes A^*)T^\dagger R T^*(B \otimes B^*)^T \\ R' &= \frac{T(A \otimes A^*)T^\dagger}{|det(A)|} R \, \frac{(T(B \otimes B^*)T^\dagger)^T}{|det(B)|} |det(A)||det(B)| \end{aligned} \tag{3.24}$$

In order to show that $L_A = \frac{T(A \otimes A^*)T^\dagger}{|det(A)|}$, we use $A\sigma_y A^T = \sigma_y det(A)$ and $T^\dagger M T^* = -\sigma_y \otimes \sigma_y$, with $M = diag(1, -1, -1, -1)$ so that

$$
\begin{aligned}
L_A M L_A^T &= \frac{T(A\otimes A^*)T^\dagger}{|det(A)|} M \frac{\{T(A\otimes A^*)T^\dagger\}^T}{|det(A)|} \\
&= \frac{T(A\otimes A^*)T^\dagger M T^*(A\otimes A^*)^T T^T}{|det(A)|^2} \\
&= -\frac{T(A\otimes A^*)\sigma_y\otimes\sigma_y(A^T\otimes(A^*)^T)T^T}{|det(A)|^2} \\
&= -T(\sigma_y\otimes\sigma_y)T^T \\
&= TT^\dagger M T^* T^T = M(TT^\dagger)^* = M. \qquad (3.25)
\end{aligned}
$$

Further we have $det(L_A)$, $det(L_B) = 1$ and $(T(A\otimes A^*)T^\dagger)_{00} = \sum_{\alpha,\beta} T_{0\alpha}(A\otimes A^*)_{\alpha\beta}T_{0\beta} = 2\langle\Phi^+|A\otimes A^*|\Phi^+\rangle = 2\langle\Phi^+|AA^\dagger\otimes I|\Phi^+\rangle \geq 0$, where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ and we have used $(A\otimes I)|\Phi^+\rangle = (I\otimes A^T)|\Phi^+\rangle$.

This completes the proof.

3.4 Canonical Forms

In [18] and [19] canonical forms of any two-qubit state ρ under local filtering operations of the form of 3.18 were derived . Here we reproduce the results.

Theorem 3: The 4×4 matrix R can be decomposed as

$$R = L_1 \Lambda L_2^T, \qquad (3.26)$$

with L_1, L_2 proper orthochronous Lorentz transformations and Λ either of the diagonal form $\Lambda = diag(s_0, s_1, s_2, s_3)$ with $s_0 \geq s_1 \geq s_2 \geq |s_3|$ or of the form

$$\Lambda = \begin{bmatrix} a & 0 & 0 & b \\ 0 & d & 0 & 0 \\ 0 & 0 & -d & 0 \\ c & 0 & 0 & (a+c-b) \end{bmatrix} \qquad (3.27)$$

with s_i, a, b, c, d real.

Proof: Let us consider the matrix $C = MRMR^T$.This matrix is M-self adjoint i.e, $MC = C^TM$. From Theorem 5.3 of ref. [18] it follows that there exists matrices X and J with $C = X^{-1}JX$, J consisting of a direct sum of real Jordan blocks and $XMX^T = N_J$ with N_J a direct sum of symmetric $n\times n$ matrices of the form $[S_{ij}] = \pm[\delta_{i+j,n+1}]$, with n the size of the corresponding Jordan block. From Sylvester's law of inertia we have that there exists orthogonal O_J such that $N_J = O_J^T M O_J$. It then follows that $O_J X = L_1^T$ is a Lorentz transformation. Hence, we have $C = MRMR^T = ML_1MO_JJO_J^TL_1^T$. Left multiplying by M, Sylvester's law of inertia implies that there exists a matrix Λ with the same rank as J such that $MO_JJO_J^T = \Lambda M \Lambda^T$. Thus we have $RMR^T = L_1 \Lambda M \Lambda^T L_1^T$.

If R has the same rank as RMR^T , this relation implies that there exists a Lorentz transformation L_2 such that $R = L_1 \Lambda L_2^T$. Let us now look at the possible forms of Λ. Since $N_J = O_J^T M O_J$ has the signature (+,-,-,-), we only have the followng possibilities,

i) J is a direct sum of four 1×1 blocks, $\Lambda = diag(\sqrt{|\lambda_0|}, \sqrt{|\lambda_1|}, \sqrt{|\lambda_2|}, \sqrt{|\lambda_3|})P$ with P being a permutation matrix permuting the first column with one another column.

ii) J is a direct sum of one orthogonal 2×2 block and two 1×1 blocks, $J = diag\left[\sqrt{|\lambda_0|}\begin{bmatrix} cos(\phi) & sin(\phi) \\ sin(\phi) & -cos(\phi) \end{bmatrix}, \sqrt{|\lambda_2|}, \sqrt{|\lambda_3}\right]$

iii) J is a direct sum of one 2×2 Jordan block and two 1×1 blocks,

$$J = diag\left[\sqrt{|\lambda_0|}\begin{bmatrix} a & b \\ c & (a+c-b) \end{bmatrix}, \sqrt{|\lambda_2|}, \sqrt{|\lambda_3}\right]$$

iv) J is a direct sum of one 3×3 block and one 1×1 block,

$$J = diag\left[\begin{bmatrix} a & 0 & 0 \\ b & \sqrt{a^2+b^2} & 0 \\ 0 & \frac{-ab}{\sqrt{a^2+b^2}} & \frac{a^2}{\sqrt{a^2+b^2}} \end{bmatrix}, \sqrt{|\lambda_0|}\right], \text{ with } a = |\sqrt{\lambda_0}|, b = -\frac{1}{2\sqrt{|\lambda_0|}}.$$

Let us now return to the relation $R = L_1 \Lambda L_2^T$. L_1 and L_2 can be made proper and orthochronous by absorbing -1 into the rows and columns of Λ, yielding Λ'. Theroem 2 the implies that Λ' corresponds to an unnormalized physical state which means that ρ' corrsponding to Λ' has no negative eigenvalues. This requirement excludes cases $ii)$ and $iv)$ of the possible forms of Λ. The third case corresponds to 3.27. Also in the first case, the permutation matrix has to be identity and $|\lambda_0| \geq max(|\lambda_1|, |\lambda_2|, |\lambda_3|)$. Multiplying by proper orthochronous Lorentz transformations the elements Λ of this diagonal $\{s_i\}$ can always be ordered so that we have $s_0 \geq s_1 \geq s_2 \geq |s_3|$.

Let us now consider the situation where the rank of C is lower than the ramk of R. This is only possible if the row space of R has an isotropic subspace Q for which $QMQ^T = 0$. The only physical states for which this holds is of canonical form given by 3.27 with $a = b = c$ and $d = 0$ or $a = b$ and $c = d = 0$.

□

In a different basis the non-diagonal canonical forms(unnormalized) can be written as [19],

$$\rho' = \frac{1}{2}\begin{bmatrix} b+c & 0 & 0 & 0 \\ 0 & a-b & d & 0 \\ 0 & d & (a-c) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.28}$$

with the possible sets of real values of b, c and d given by,

$$\begin{aligned} &\text{(i)} && b = c = \frac{a}{2}, \\ &\text{(ii)} && (d = 0 = c) \text{ and } (b = a), \\ &\text{(iii)} && (d = 0 = b) \text{ and } (c = a), \\ &\text{(iv)} && (d = 0) \text{ and } (a = b = c). \end{aligned} \tag{3.29}$$

Case (i) corrsponds to rank three or two states while the other cases correspond to either the product states $|00\rangle\langle 00|$ or the state $|0\rangle\langle 0| \otimes \frac{I}{2}$.

3.5 Optimal Bell-CHSH violation under local filtering operations

In [19] it was shown that the optimal Bell-violation among the states connected to ρ by filtering operations of the form of 3.18 is obtained from the Bell-diagonal state or a 'quasi-distilled' Bell-diagonal state for states whose canonical form in the R picture is non-diagonal.

Theorem 4: Given a single copy of a state ρ the optimal Bell violation among states connected to ρ by local filtering operations is obtained from the unique Bell-diagonal state or 'quasi-distilled' Bell-diagonal state depending on the canonical form of ρ.

Proof: In the R picture filtering transformations of the form of eqn. 3.18 correspond to Lorentz transformations and the function which we have to maximize w.r.t $L_1,$, L_2 and $X = diag(q, r, 0)$ in order to obtain the maximal Bell violation therefore becomes (see eqn.(3.9))

$$Tr\left[\frac{L_1 R L_2^T}{(L_1 R L_{2\,00}^T)}\begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix}\right], \tag{3.30}$$

with the constraint $q^2 + r^2 = 1$. Variation of the Lorentz transformations yields the extremal conditions,

$$Tr\left[G_1 R'\begin{bmatrix} -\beta & 0 \\ 0 & X \end{bmatrix}\right] = Tr\left[R' G_2\begin{bmatrix} -\beta & 0 \\ 0 & X \end{bmatrix}\right] = 0 \tag{3.31}$$

with $R' = L_1 R L_2^T$ for all possible generators G_1, G_2 of the Lorentz group and β being the Bell-violation amount given by eqn. 3.30. The generators are all of the form,

$$G = \begin{bmatrix} 0 & \vec{v} \\ \vec{v}^T & A \end{bmatrix}, \tag{3.32}$$

with $\vec{v} \in \mathcal{R}^3$ and A a real antisymmetric 3×3 block. For the case of Bell violation we have $\beta > 1$ and hence β cannot be equal to $|q|$ or $|r|$ and the form of the generators in eqn. (3.32) implies that eqn.(3.31) can only be satisfied iff R' is diagonal corresponding to a Bell-diagonal state. This together with theorem 2 completes the proof.

The non-standard canonical forms of eqn. (7.25) can be brought arbitrarily close to a Bell-diagonal state at the cost of the success probability of getting the state out of the filter going to zero. This process is referred to as 'quasi-distillation'[18], [14].

Chapter 4

Quantum channels

In this chapter we provide a short introduction to quantum channels.

Any physical process \$ in quantum theory which takes an arbitrary input density matrix ρ to an output density matrix $\rho' = \$(\rho)$ should satisfy the following constraints:

1) *Linearity* : $\$(\alpha\rho_1 + \beta\rho_2) = \alpha\$(\rho_1) + \beta\$(\rho_2)$ for any two input matrices ρ_1 and ρ_2 and for any two scalars α and β.

2) *Hermiticity preservation* : If ρ is Hermitian then ρ' should also be.

3) *Trace preservation* : $Tr(\rho') = Tr(\$(\rho))$ for all ρ

4) *Positivity* : ρ' is nonnegative if ρ is (by non-negativity of an operator we will henceforth refer to non-negativity of its eigenvalues) .

However the above restrictions are not enough for the process to be physical. When we are evolving a system we can never be sure that it is not entangled with any other system. Thus it is reasonable to demand that when \$ acts on any subsystem of a bipartite system in an entangled state the result should still be a valid density matrix. This property is known as *complete-positivity.* Formally, if $\$: \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ then the extended map $I \otimes \$: \mathcal{B}(H_C \otimes H_A) \to \mathcal{B}(H_C \otimes H_B)$ should be positive for an arbitrary dimensional Hilbert space H_C. A quantum channel is a trace preserving completely positive map.

An example of a positive map that is not completely positive : Consider the maximally entangled state $|\Phi^+\rangle\langle\Phi^+| = \frac{1}{N}\sum_{i,j=0}^{N-1}(|i\rangle\langle j| \otimes |i\rangle\langle j|)$ of two N dimensional systems. Further consider the action of the transpose map which of course satisfies (1)-(4) above. We have,

$$(I \otimes T)|\Phi^+\rangle\langle\Phi^+| = \frac{1}{N}\sum_{i,j=0}^{N-1}(|i\rangle\langle j| \otimes |j\rangle\langle i|) = \frac{1}{N}V, \tag{4.1}$$

with V being the swap operator defined by, $V|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle$. The swap operator has at least one eigenvalue of -1 and hence T is not completely positive.

4.1 Choi-Jamiolkowski Isomorphism

The set of all trace-preserving quantum channels \$ on the set of d-dimensional states is isomorphic to the set of density matrices ρ acting on the Hilbert space $H = H_1 \otimes H_2 = \mathcal{C}^d \otimes \mathcal{C}^d$ satisfying $Tr_{\mathcal{H}_2}\rho = \frac{I_d}{d}$, with I_d being the $d \times d$ identity matrix. This isomorphism is known as the Choi-Jamiolkowski isomorphism [40], [41]. Given a channel \$ one can ascribe a state to it by sending half of the maximally entagled state $|\Phi^+\rangle$ down the channel,

$$\rho_{\$} = (I \otimes \$)|\Phi^+\rangle\langle\Phi^+|. \tag{4.2}$$

Suppose now we have a state ρ on $H_1 \otimes H_2 \equiv \mathcal{C}^\lceil \otimes \mathcal{C}^\lceil$satisfying $Tr_{\mathcal{H}_2}\rho = \frac{I}{d}$. Let us now consider the spectral decomposition of the state ,

$$\rho = \sum_{k=1}^{d^2} p_k|\psi_k\rangle\langle\psi_k|. \tag{4.3}$$

Let $|\psi_1\rangle = \sum_{i,j=1}^{d} c_{ij}|i\rangle \otimes |j\rangle$. Then it can be represented as $\psi_1 = (I \otimes V_1)|\Phi^+\rangle$, where $\langle i|V|j\rangle = \sqrt{d}c_{ij}$. Defining analogously V_k for $k = 1, 2, 3, ..d^2$ we obtain,

$$\rho = \sum_{k=1}^{d^2} p_k(I \otimes V_k)|\Phi^+\rangle\langle\Phi^+|(I \otimes V_k^\dagger) = (I \otimes \$)|\Phi^+\rangle\langle\Phi^+|, \tag{4.4}$$

with the map \$ defined by $\$(\sigma) = \sum_k p_k V_k \sigma V_k^\dagger$. A map defined in this way is of course completely positive(see next section) as it is of the operator sum form. In order to show that it is also trace-preserving we need to show that $A = \sum_k p_k V_k^\dagger V_k = I$. As $Tr_{\mathcal{H}_2}\rho = \frac{I}{d}$ we have for any operator B, acting on $H_1 \equiv \mathcal{C}^d$,

$$Tr(B) = dTr(\rho B \otimes I) = d\sum_k p_k Tr(|\Phi^+\rangle\langle\Phi^+|(B \otimes V_k^\dagger V_k)) = dTr(|\Phi^+\rangle\langle\Phi^+|(B \otimes A)). \tag{4.5}$$

Now using the property that $(C \otimes I)|\Phi^+\rangle\langle\Phi^+| = (I \otimes C^T)|\Phi^+\rangle\langle\Phi^+|$ for any operator C and that reduction of $|\Phi^+\rangle\langle\Phi^+|$ is equal to $\frac{I}{d}$, we obtain $Tr(B) = Tr(B^T A) = Tr(A^T B)$ for any B. This implies $A^T = I$ and hence $A = I$. It is also easy to show that given a channel \$, the state $\rho_{\$}$ is unique and vice versa([14]).

4.2 Necessary and sufficient conditions for complete positivity

Suppose we have a quantum channel $\$: \mathcal{B}(\mathcal{C}^d) \to \mathcal{B}(\mathcal{C}^d)$. We would like to derive here the necessary and sufficient conditions for its complete positivity. From the definiton of complete positivity it is clear that it is necessary that the Choi-state of the channel $\rho_\$$ defined by eqn. 4.2 should be positive. We would like to now show that this condition is sufficient too. For this let us consder a map $\$$ for which $\rho_\$ \geq 0$. Then following the last section we can provide an operator sum representation for the map using the spectral decomposition of $\rho_\$$, i.e we can define the map as $\$(\sigma) = \sum_i K_i \sigma K_i^\dagger$ with $K_i = \sqrt{p_i} V_i$. Now this map is completely positive as for any state $|\psi\rangle \in \mathcal{C}^d \otimes \mathcal{C}^d$ and $\rho \in \mathcal{B}(\mathcal{C}^d \otimes \mathcal{C}^d)$ we have $\langle\psi|(I \otimes K_i)\rho(I \otimes K_i^\dagger)|\psi\rangle \geq 0$, by virtue of positivity of ρ.

The arguments can be extended to $\$: \mathcal{B}(\mathcal{C}^d) \to \mathcal{B}(\mathcal{C}^{d'})$ in a straightforward manner.

4.3 Characterisation of qubit channels

A qubit channel Λ is a linear , trace-preserving , hermiticity-preserving completely positive map from $\mathcal{B}(\mathcal{C}^2)$ to itself, where $\mathcal{B}(\mathcal{C}^2)$ is the space of all bounded linear operators on $\mathcal{C}^2$. A characterisation of all such qubit channels was provided by Ruskai et al. in ref. ([42]). Using the Bloch sphere representations of $\rho = \frac{1}{2}(I + x\sigma_1 + y\sigma_2 + z\sigma_3)$ and $\rho' = \frac{1}{2}(I + x'\sigma_1 + y'\sigma_2 + z'\sigma_3)$ of the input and the output states of the channel $\$$, one can write :

$$\begin{bmatrix} 1 \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{t} & \mathbf{M} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}, \tag{4.6}$$

where $\mathbf{0} = (0, 0, 0)$ and $\mathbf{t} = (t_1, t_2, t_3)^T \in \mathcal{R}^3$ are row and column vectors respectively and $\mathbf{M}$ is a 3×3 real matrix.

A singular value decomposition of M gives,

$$\begin{bmatrix} 1 \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{t} & O_1 D O_2 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix},$$

with $O_1, O_2 \in O(3)$, $D = diag(\lambda_1, \lambda_2, \lambda_3)$ with λ_i being the singular values of M or,

$$\begin{bmatrix} 1 \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & O_1 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{t}' & D \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & O_2 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}, \tag{4.7}$$

with $\mathbf{t}' = O_1^T\mathbf{t}$. Now, if $det(O_1) = -1$ or $det(O_2) = -1$ in the above decomposition we may consider $O_1' = -O_1$ in the above decomposition and take $D' = -D$ instead of D to ensure that $O_1', O_2' \in SO(3)$ .Henceforth we refer to O_1', O_2' and $\mathbf{t}'$as O_1 , O_2 and $\mathbf{t}$ respectively for brevity. Now, using the SU(2)-SO(3) connection the map, $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & O_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$ at the Bloch sphere level, corresponds to a unitary map, $\rho_2 = U_1 \rho U_1^\dagger$ at the density matrix level, so that the Bloch sphere representation of ρ_2 becomes, $\begin{bmatrix} 1 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & O_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$. Thus, eqn. (4.7) can be written at the density matrix level as ,

$$\Lambda(\rho) = U_1 \circ \Lambda' \circ U_2(\rho) \tag{4.8}$$

, with $U_1(\rho) \equiv U_1(\rho)U_1^\dagger$, similarly for U_2 and the Bloch sphere representation of $\Lambda'(\rho)$ being $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{t} & D \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$.

The importance of eqn. (4.8) lies in the fact that the map Λ is completely positive iff Λ' is (see below). Also as noted in [42], the signs of any two of the λ_i s can be changed by conjugating with a Pauli matrix. Thus, upto pre and post processing by unitary maps, the most general qubit channel is given in the Bloch sphere representation by,

$$\begin{bmatrix} 1 \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}, \tag{4.9}$$

with t_i and λ_i being real $\forall i$.

4.3.1 Complete positivity conditions

The map in eqn. (4.8) is completely positive iff its action on one side of a two-qubit maximally entangled state yields a valid density matrix, i.e, iff $\rho_{\Phi^+,\Lambda} \geq 0$ (Section II) .Now,

$$\rho_{\Phi^+,\Lambda} = (I \otimes \Lambda)(|\Phi^+\rangle\langle\Phi^+|) = (I \otimes U_1 \circ \Lambda' \circ U_2)(|\Phi^+\rangle\langle\Phi^+|) = (U_2^T \otimes U_1) \circ (I \otimes \Lambda')(|\Phi^+\rangle\langle\Phi^+|), \tag{4.10}$$

where we have used the fact that $(I \otimes U_1)|\Phi^+\rangle\langle\Phi^+| = (U_1^T \otimes I))|\Phi^+\rangle\langle\Phi^+|$ with U_1^T representing conjugation by the unitary U_1^T. As local unitaries do not affect positivity we require for the complete positivity of Λ,

$$\rho_{\Lambda'} = (I \otimes \Lambda')(|\phi^+\rangle\langle\phi^+|) \geq 0. \tag{4.11}$$

From eqn. (4.11) we have,

$$\rho_{\Lambda'} = \frac{1}{4}\begin{pmatrix} (1+\lambda_3+t_3) & (t_1-it_2) & 0 & (\lambda_1+\lambda_2) \\ (t_1+it_2) & (1-\lambda_3-t_3) & (\lambda_1-\lambda_2) & 0 \\ 0 & (\lambda_1-\lambda_2) & (1-\lambda_3+t_3) & (t_1-it_2) \\ (\lambda_1+\lambda_2) & 0 & (t_1+it_2) & (1+\lambda_3-t_3) \end{pmatrix}. \tag{4.12}$$

The condition for positivity of $\rho_{\Lambda'}$ is given by the following theorem from reference [42] which says, that $\rho_{\Lambda'}$ is positive and hence the map given by 4.9 (for which $|t_3|+|\lambda_3| \leq 1$) completely positive if and only if the equation,

$$\begin{bmatrix} t_1+it_2 & \lambda_1+\lambda_2 \\ (\lambda_1-\lambda_2) & (t_1+it_2) \end{bmatrix} = \begin{bmatrix} \sqrt{1+t_3+\lambda_3} & 0 \\ 0 & \sqrt{1+t_3-\lambda_3} \end{bmatrix} R_\Phi \begin{bmatrix} \sqrt{1-t_3-\lambda_3} & 0 \\ 0 & \sqrt{1-t_3+\lambda_3} \end{bmatrix} \tag{4.13}$$

has a solution wih R_Φ that is a contraction. For $|t_k|+|\lambda_k| < 1$ the solution can be obtained uniquely by inverting the matrices multiplying R_Φ in eqn. (4.13) .

4.3.2 Unital Channels

For unital channels (channels mapping I to I ,for which $\vec{t}=0$) , the eigenvalues of $\rho_{\Lambda'}$ determine the feasible region governed by the equations,

$$\begin{aligned} \lambda_1-\lambda_2-\lambda_3 &\leq 1 \\ \lambda_1+\lambda_2-\lambda_3 &\leq 1 \\ -\lambda_1+\lambda_2+\lambda_3 &\leq 1 \\ \lambda_1-\lambda_2+\lambda_3 &\leq 1. \end{aligned} \tag{4.14}$$

The allowed region is a tetrahedron in the λ_1,λ_2, λ_3 space with extreme points $(1,1,1,)$, $(1,-1,-1)$, $(-1,1,-1)$ and $(-1,-1,1)$ corresponding to maps which are conjugation respectively by I, σ_1, σ_2 and σ_3.

Thus any map of the form of (4.9), with parameters $\vec{\lambda}$ can be written as the convex combination ,

$$\rho' = \Lambda(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i \tag{4.15}$$

, with $\sigma_0 = I$, $\sum_i p_i = 1$.

4.3.3 Examples

4.3.3.1 Depolarizing Channel

This is a process in which the initial state is replaced by completely mixed state $\frac{I}{2}$ with probability p and left unchanged with probability $(1-p)$. Hence,

$$\Lambda'(\rho) = \rho' = \frac{p}{2}I + (1-p)\rho. \tag{4.16}$$

The channel is unital , and we have $\lambda_1 = \lambda_2 = \lambda_3 = (1-p)$.

4.3.3.2 Amplitude Damping channel

The amplitude damping channel models spontaneous emission and is characterised by Krauss operators $K_0 = \sqrt{p}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $K_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{(1-p)} \end{bmatrix}$. The $\vec{t}, \vec{\lambda}$ vectors of the channel are given respectively by $(0,0,p)$ and $(\sqrt{1-p}, \sqrt{1-p}, (1-p))$.

4.4 Entanglement breaking Channel

A channel Λ is said to be entaglement breaking if on acting on one side of any bipartite state Γ it produces a separable state $(I \otimes \Lambda)(\Gamma)$. In [3], it was shown that the necessary and sufficient condition for a qudit channel Λ to be entanglement breaking is that $(I \otimes \Lambda)(|\Phi^+\rangle\langle\Phi^+|)$ should be separable, where $|\Phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{(d-1)} |ii\rangle$. Sufficiency can be proved as follows. Necessity follows trivially.

As shown before in section I, any bipartite state $|\psi\rangle \in \mathcal{C}^d \otimes \mathcal{C}^d$ can be written as $(V \otimes I)|\Phi^+\rangle$ with V being a $d \times d$ matrix. Thus, $(I \otimes \Lambda)|\psi\rangle\langle\psi| = (V \otimes I)(I \otimes \Lambda)(|\Phi^+\rangle\langle\Phi^+|)(V^\dagger \otimes I) = (V \otimes I)\rho_{\Phi^+,\Lambda}(V^\dagger \otimes I)$. Now, as for any separable state ρ, $(V \otimes I)\rho(V^\dagger \otimes I)$ is again separable , $(I \otimes \Lambda)|\psi\rangle\langle\psi|$ is separable. For mixed input states Γ the argument is generalised by taking spectral decomposition.

4.4.1 Holevo form

As $\rho_{\Phi^+,\Lambda}$ is separable, one can find normalized vectors $|v_n\rangle$ and $|w_n\rangle$ for which $\rho_{\Phi^+,\Lambda} = \sum_n p_n|v_n\rangle\langle v_n| \otimes |w_n\rangle\langle w_n|$ with $\{p_n\}$ being a probability distribtuion. Now let Ω be the map, for which

$$\Omega(\rho) = d\sum_n |w_n\rangle\langle w_n|Tr(\rho p_n|v_n^*\rangle\langle v_n^*|), \tag{4.17}$$

with $|v_n^*\rangle = \sum_{j=0}^{d-1}\langle v_n|j\rangle|j\rangle$. It follows that $(I \otimes \Omega)|\Phi^+\rangle\langle\Phi^+| = \sum_{ij} |i\rangle\langle j| \otimes \Omega(|i\rangle\langle j|) = \sum_{ijn} |i\rangle\langle j| \otimes |w_n\rangle\langle w_n|p_n\langle i|v_n\rangle\langle v_n|j\rangle = \sum_n p_n|v_n\rangle\langle v_n| \otimes |w_n\rangle\langle w_n|$. Due to the Choi-Jamiolkowski isomophism we must have $\Lambda = \Omega$. Thus any entanglement breaking channel can be written in the form of eqn. (4.17) known as the Holevo form as a measure and prepare map, with the measurement POVM being $\{dp_n|v_n^*\rangle\langle v_n^*|\}$.

Chapter 5

Optimal singlet fraction under TP-LOCC

5.1 Teleportation and singlet fraction

Suppose Alice and Bob share a pair of particles 1 and 2 prepared in a state described by the density matrix ρ acting on the Hilbert space $H_A \otimes H_B = C^d \otimes C^d$ and Alice has a third particle in an unknown state $\psi \in H_3 = C^d$ to be teleported. In the most general teleporation scheme, Alice and Bob apply some trace-preserving LOCC(local operations with classical comunication) operations on all the particles they share. After the operation the state of Bob's particle should be as close to $|\psi\rangle\langle\psi|$ as possible. For the final state of Bob's particle we have,

$$\rho_\psi^{\text{Bob}} = Tr_{3,A}(T(|\psi\rangle_3\langle\psi| \otimes \rho)). \tag{5.1}$$

This produces a quantum channel $\Lambda_{T,\rho}$ so that we have,

$$\Lambda_{T,\rho}(|\psi\rangle\langle\psi|) = \rho_\psi^{\text{Bob}}. \tag{5.2}$$

The above protocol offers a procedure for associating a qudit channel to a two-qudit state that is different from the Choi-Jamiolkowski isomorphism described before. In general for a given TP-LOCC T, the map $\rho \to \Lambda_{T,\rho}$ may not be an isomorphism. In order to judge the efficiency of any teleporation scheme one needs to define the notion of fidelity of teleportation. In order to define the fidelity of a teleportation protocol we first need to define fidelity of a generic channel.

Fidelity of a channel: The fidelity of a channel Λ is defined as,

$$f(\Lambda) = \int d\phi\langle\phi|\Lambda(|\phi\rangle\langle\phi|)|\phi\rangle, \tag{5.3}$$

where the integral is with respect to the uniform distribtuion $d\phi$ over the set of all input states $|\phi\rangle$. The fidelity is a measure of how close the output states of the channel are to the input states when averaged over the set of all input states.

The fidelity of a teleportation protocol T is the fidelity of the asoociated channel $\Lambda_{T,\rho}$. The fidelity $f(\rho)$ of a two-qudit state ρ is defined by the fidelity of the standard teleportation protocol T_0 using the state ρ, i.e. $f(\rho) = f(\Lambda_{T_0,\rho})$. The standard teleportation protocol is the protocol give for first time in [9], where Alice and Bob sharing the state $|\Phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{(d-1)}|ii\rangle$, performing generalized Bell state measurements on Alice's two subsystems and exchanging d^2 classical bits one-way(from Alice to Bob) can faithfully transmit an unknown qudit state.

Singlet Fraction: The singlet fraction of a two-qudit state ρ, is just the overlap of the state with $|\Phi^+\rangle$ and will henceforth be denoted by

$$F_1(\rho) = \langle\Phi^+|\rho|\Phi^+\rangle. \tag{5.4}$$

5.2 Depolarizing channel and Werner states

The qudit depolarizing channel acting on a qudit state σ, leaves it intact with probability p and replaces it with the maximally mixed state $\frac{I}{d}$ with probability $(1-p)$. We have,

$$\Lambda_p^{\text{dep}}(\sigma) = p\sigma + (1-p)\frac{I}{d}. \tag{5.5}$$

The Choi-dual state of the channel, obtained by the action of the channel on one side of $|\Phi^+\rangle$ is given by the Werner state [4],

$$\rho_p = \rho_{\Phi^+,\Lambda} = pP_+ + (1-p)\frac{I\otimes I}{d^2}, \tag{5.6}$$

with $P_+ = |\Phi^+\rangle\langle\Phi^+|$ and I is the $d\times d$ identity matrix.

Let us now consider the standard teleportation prototcol using the state ρ_o. The scheme produces perfect fidelity for the singlet state. For the completely random noise represented by the state $\frac{I\otimes I}{d^2}$, the average final state of Bob's particle after the completion of the protocol is equal to $\frac{I}{d}$ and does not depend on the unknown state to be teleported. Thus in this case we have a fidelity equal to $\frac{1}{d}$. Thus,

$$f(\rho_p) = p + (1-p)\frac{1}{d}, \text{and so } \frac{1}{d} \leq f((\rho_p)) \leq 1. \tag{5.7}$$

and for the parameter F_1 we have,

$$F_1(\rho_p) = p + \frac{(1-p)}{d^2}, \text{ and so} \frac{1}{d^2} \leq F_1(\rho_p) \leq 1, \tag{5.8}$$

so that,

$$f(\rho_p) = \frac{F_1(\rho_p)d+1}{d+1}. \tag{5.9}$$

Remark: The Werner states are the only states invariant under $U \otimes U^*$ transformations and any two-qudit state ρ when subjected to $U \otimes U^*$ twirling produces the Werner state ([43]),

$$\Gamma(\rho) = \int dUU \otimes U^* \rho U^\dagger \otimes U^{*\dagger}, \tag{5.10}$$

with the singlet fraction $F_1(\Gamma(\rho)) = Tr(\rho P_+)$. Thus the singlet fraction is invariant under the twirling procedure. Here dU represents the Haar measure on the unitary group U over C^d.

5.3 Twirling channels and entanglement fidelity

Entanglement fidelity of a channel is defined to be the singlet fraction of the Choi-state of the channel $\rho_{\Phi^+,\Lambda}$, i.e $F(\Lambda) = \langle \Phi^+ | \rho_{\Phi^+,\Lambda} | \Phi^+ \rangle$. A twirling of a channel is defined as:

$$\Gamma_\Lambda(\sigma) \equiv \int dUU^\dagger \Lambda U(\sigma), \tag{5.11}$$

with $U^\dagger \Lambda U(\sigma) \equiv U^\dagger(\Lambda(U\sigma U^\dagger))U$ from the unitary group $\mathcal{U}$ over C^d, w.r.t the Haar measure.

5.3.1 Invariants of twirling

In this subsection we will show that $f(\Lambda)$ and $F(\Lambda)$ are invariants of twirling and we have, $f(\Lambda) = \frac{F(\Lambda)d+1}{d+1}$. In order to show this let us begin by showing that $\rho_{\Phi^+,\Gamma_\Lambda} = \Gamma(\rho_{\Phi^+,\Lambda})$. We have,

$$\begin{aligned}
\rho_{\Phi^+,\Gamma_\Lambda} &= (I \otimes \Gamma(\Lambda))P_+ \\
&= \int dU\{(I \otimes U^\dagger)(I \otimes \Lambda)((I \otimes U)P_+(I \otimes U^\dagger))(I \otimes U)\} \\
&= \int dU\{(I \otimes U^\dagger)(I \otimes \Lambda)((U^T \otimes I)P_+(U^* \otimes I))(I \otimes U)\} \\
&= \int dU(U \otimes U^*)\rho_{\Phi^+,\Lambda}(U^\dagger \otimes U^{*\dagger}) \\
&= \Gamma(\rho_{\Phi^+,\Lambda}),
\end{aligned} \tag{5.12}$$

where we have used $(A \otimes I)P_+ = (I \otimes A^T)P_+$ and the invariance of the Haar measure under Hermitian conjugation.

Now as $\Gamma(\rho_{\Phi^+,\Lambda}$ is a Werner state and hence Γ_Λ must be a depolarising channel. We thus have,

$$F(\Gamma_\Lambda) \equiv F_1(\rho_{\Phi^+,\Gamma_\Lambda}) = F_1(\Gamma(\rho_{\Phi^+,\Lambda})) = F_1(\rho_{\Phi^+,\Lambda}) = F(\Lambda). \tag{5.13}$$

We will now show that $f(\Lambda) = f(\Gamma_\Lambda)$. Writing $f(\Lambda)$ as,

$$f(\Lambda) = \int dUTr[U|\phi\rangle\langle\phi|U^\dagger \Lambda(U|\phi\rangle\langle\phi|U^\dagger)], \tag{5.14}$$

where $|\phi\rangle$ is any given pure state of the system and the integral is performed over the uniform distribution dU over the unitary group $\mathcal{U}$ on C^d. Hence, we have

$$\begin{aligned} f(\Gamma_\Lambda) &= \int dU Tr[U|\phi\rangle\langle\phi|U^\dagger \int dV (V^\dagger \Lambda(VU|\phi\rangle\langle\phi|U^\dagger V^\dagger)V)] & (5.15)\\ &= \int dV \int dU Tr[U|\phi\rangle\langle\phi|U^\dagger \Lambda(U|\phi\rangle\langle\phi|U^\dagger)] & (5.16)\\ &= \int dV f(\Lambda) = f(\Lambda) & (5.17)\end{aligned}$$

with V unitary and dV representing the unitary over the Haar measure. Thus for any channel we have shown that $f(\Lambda)$ and $F(\Lambda)$ are invariants of twirling and from the results on the values of these quantities for depolarizing channel and Werner states we have,

$$f(\Lambda) = \frac{F(\Lambda)d + 1}{d + 1}. \tag{5.18}$$

5.4 Teleportation and singlet fraction

The previous discussions culminate into the following theorem proved in [14],

Theorem: Let $F^*(\rho)$ be the maximum possible fidelity obtainable from a given state ρ by means of trace-preserving LOCC(TP-LOCC). Then the maximum fidelity $f_{\max}$ of teleportation obtainable via the state ρ by TP-LOCC is equal to,

$$f_{\max} = \frac{F^*(\rho)d + 1}{d + 1}. \tag{5.19}$$

Proof: Let us first prove that $f_{\max} \leq \frac{F^*(\rho)d+1}{d+1}$. Suppose we have a teleportation channel of fidelity $f_{\max}$. From 5.18 it follows that the entanglement fidelity $F(\Lambda)$ of that channel satisfies $f_{\max} = \frac{Fd+1}{d+1}$. Then by applying the channel on one side of Φ^+ we have a state with F satisfying eqn. 5.19 and we have $F_{\max}$ at least equal to F. To see the other way round now suppose that by trace-preserving LOCC a state ρ' of maximal F has been obtained. By applying twirling to this state we turn it into a Werner state with the same fidelity. Then, the standard teleportation protocol with this state satisfies eqn. (5.19).

5.5 Negativity

In [29] an upper bound was given on $F^*(\rho)$ as a linear function of the negativity of the state ρ. In this section we define negativity and describe various properties of it. Let us denote by $||\rho||_1 = Tr(\sqrt{\rho\rho^\dagger})$ the trace norm of ρ. The negativity of a state ρ is defined as,

$$N(\rho) = \frac{||\rho^{T_A}||_1 - 1}{2}. \tag{5.20}$$

, with ρ^{T_A} denoting partial transpose of the state ρ with respect to side A. Thus negativity of a state is the sum of the absolute values of the negative eigenvalues of a state and vanishes for separable states. It measures the amount by which ρ^{T_A} fails to be positive. Let us now prove cetain properties of negativity.

1) Convexity: We wish to show that ,

$$N(\sum_i p_i\rho_i) \leq \sum_i p_i N(\rho_i). \tag{5.21}$$

This follows immediately from the facts that the trace norm is homogeneous of degree 1 for positive factors and being a norm satisfies the triangle inequality.

2) Monotonicity under LOCC: It turns out that negativity of a state cannot be increased by LOCC. In order to prove this we would require the following lemma.

Lemma 1: For any Hermitian matrix A there is a decomposition of the form

$$A = a_+\rho^+ - a_-\rho^- \tag{5.22}$$

with ρ^+ and ρ^- being density density matrices and $a_\pm$ being positive nos. such that $a_+ + a_-$ is minimal. For this decomposition $||A||_1 = a_+ + a_-$ and a_- is the absolute sum of the negative eigenvalues of A.

Proof: The fact that such a decomposition exists follows immediately from the spectral decomposition. Now consider an arbitrary decomposition like above. Let P^- be the projector onto the negative eigenvalued subspace of A, and $N = -tr(AP^-)$ ÍŤ the absolute sum of the negative eigenvalues. Now inverting the decomposition in eqn. 5.22 we have, $A + a_-\rho^-$ is positive semi-definite. This implies that

$$0 \leq Tr[(a + a^-\rho_-)P^-] = -N + a^-tr(\rho^-P^-). \tag{5.23}$$

But as $tr(\rho^-P^-) \leq 1$ we have $a^- \geq N$. Now we can saturate this bound by the choice $a^-\rho^- \equiv -P^-AP^-$ (corresponding to the spectral decomposition of A). This completes the proof.

□

This gives us the following formulae for negativity,

$$N(A) = \inf\{a_-|A^{T_A} = a_+\rho^+ - a_-\rho^-\} \tag{5.24}$$

where the infimum is over all density matrices $\rho^\pm$ and $a^\pm \geq 0$.

We are now ready to provide a proof of monotonicity of negativity under LOCC.

An LOCC consists of a series of steps at the end of each of which the parties perform local measurements and broadcast the result to all other parties. The choice of measurements of course may depend on prior information about outcomes of previous measurements. If at an end of an LOCC protocol we have state ρ_i

occurring with probability p_i then we require of an entanglement monotone E [44] that,

$$E(\rho) \geq \sum_i p_i E(\rho'_i). \tag{5.25}$$

It is clear that this may be proved by looking at just one round of an LOCC protocol and since negativity is symmetric on the two parties, just one local measurement by Bob.

Now the most general local measurement by Bob is a set of completely positive maps M_i so that $M_i(\rho) = p_i\rho'_i$. These maps further satisfy the normalization condition $\sum_i tr[M_i(\rho)] = tr(\rho)$. These can be further simplified when some maps M_i can be decomposed further into completely positive maps so that for example $M_i = M'_i + M''_i$. Then we may consider the finer decomposition as a finer measurement with the result i replaced by two others i' and i''. Due to convexity it suffices to prove eqn. 5.25 for these finer measurements. Hence we can assume that M_i can be written by a single Krauss operator and we have,

$$M_i(\rho) = (I_A \otimes M_i)\rho(I_A \otimes M_i^\dagger), \tag{5.26}$$

with M_is satisfying the normalization condition $\sum_i M_i^\dagger M_i \leq I_B$. For computing the right hand side of eqn. (5.25) we need that

$$[M_i(\rho)]^{T_A} = M_i(\rho^{T_A}). \tag{5.27}$$

This follows straightforwardly by expanding ρ as a not necessarily positive sum of tensor decompositions. Let us consider now the decomposition,

$$\rho^{T_A} = (1+N)\rho^+ - N\rho_- \tag{5.28}$$

with density operators $\rho^\pm$ and $N = N(\rho)$. Then we can also decompose the partially transposed output states as,

$$p_i\rho_i'^{T_A} = [M_i(\rho)]^{T_A} = M_i(\rho^{T_A}) = (1+N)M_i(\rho^+) - NM_i(\rho^-) \tag{5.29}$$

Dividing by p_i we get exactly a decomposition of the form of eqn. (8.41) defining $N(\rho'_i)$. The coefficient $a_- = \frac{N}{p_i}$ must be larger than the infimum .i.e, $N(\rho'_i) \leq \frac{N}{p_i}$. Multiplying by p_i and summing we finally have,

$$N(\rho) \geq \sum_i p_i N(\rho'_i), \tag{5.30}$$

i.e, negativity is an entanglement monotone. □

5.5.1 Upper bound on $F^*(\rho)$

Let $F^*(\rho) = \langle\Phi^+|L(\rho)|\Phi^+\rangle$ with L being the optimal trace-preserving LOCC.Now from the fact that the optimal trace-preserving protocol for a convex combination of states is (generally) sub-optimal for the individual states it is easily seen that,

$$F^*(\sum_i p_i\rho_i) \leq \sum_i p_i F^*(\rho_i). \tag{5.31}$$

Now as singlet fraction is invariant under twirling , we can assume that $L(\rho)$ is a Werner state with some parameter p given by eqn.5.6 with singlet fraction $F^*(\rho)$.Hence we have,

$$F^*(\rho) = p + \frac{(1-p)}{d^2}. \tag{5.32}$$

However for the Werner state , $||\rho^{T_A}||_1 = dp + \frac{(1-p)}{d}$ and hence $F^*(\rho) = \frac{1+2N(L(\rho))}{d}$. Now using the monotonicity property of negativity we immediately have that $N(L(\rho)) \leq N(\rho)$ and hence,

$$F^*(\rho) \leq \frac{1+2N(\rho)}{d}. \tag{5.33}$$

Let us define,

Defn:

$$F(\rho) = \max_{\phi \in M.E} \langle\phi|\rho|\phi\rangle$$

, with the maximisation over the set M.E of all two qudit maximally entangled states.

We have,

$$F(\rho) \leq F^*(\rho) \leq \frac{1+2N(\rho)}{d}. \tag{5.34}$$

5.5.2 Negativity bound on $F(\rho)$ for two-qubit states

For two qubit states from the above discussion we have(by taking $d = 2$ in eqn.(5.34)),

$$F(\rho) \leq \frac{1}{2}(1+2N(\rho)). \tag{5.35}$$

We would like to show here that $F(\rho) = \frac{1}{2}(1+2N(\rho))$ if and only if the eigenvector corresponding to the negative eigenvalue of ρ^Γ denoting partially transposed ρ is maximally entangled.

Proof:

For a two-qubit state ρ we have $F(\rho)$ given by,

$$\max_{U,V \in SU(2)} Tr[(U \otimes V)|\Phi^+\rangle\langle\Phi^+|(U \otimes V)^\dagger \rho] = \frac{1}{2} \max_{U,V} Tr\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (U \otimes V^*)^\dagger \rho^\Gamma (U \otimes V^*)\right]. \tag{5.36}$$

Now we have,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = V_1 = I_4 - 2|\Psi^-\rangle\langle\Psi^-|, \tag{5.37}$$

with $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

Hence,

$$F(\rho) = \frac{1}{2}\max_{U,V} Tr[(I_4 - 2|\Psi^-\rangle\langle\Psi^-)(U\otimes V^*)^\dagger \rho^\Gamma (U\otimes V^*)] \tag{5.38}$$

Now extending the maximisation over all unitaries instead of just local unitaries we have,

$$F(\rho) \le \frac{1}{2}\max_{U_2\in SU(4)}(1-2\langle\Psi^-|(U_2^\dagger\rho^\Gamma U_2)|\Psi^-\rangle) = \max_{|\alpha\rangle\in C^2\otimes C^2}\frac{1}{2}(1-2\langle\alpha|\rho^\Gamma|\alpha\rangle) \tag{5.39}$$

and hence,

$$F(\rho) \le \frac{1+2N(\rho)}{2}. \tag{5.40}$$

Equality is achieved if and only if the eigenvector of ρ^Γ corresponding to the negative eigenvalue is maximally entangled.

5.6 Optimal singlet fraction for two-qubit states

In [15] an exact characterisation of the optimal singlet fraction under TP-LOCC, i.e.,$F^*(\rho)$ was provided for two qubit states . The solution was obtained as a semi-definite program by using techniques developed by Rains in [45] in the context of entanglement distillation through PPT(positivie-partial-transpose) operations. Let us consider the following protocol now. In this protocol Alice and Bob apply filters A and B on their respective particles . In case of success, when both the particles come out of the filter successfully they share the state $\rho_1 = \frac{(A\otimes B)\rho(A^\dagger\otimes B^\dagger)}{Tr((A^\dagger A\otimes B^\dagger B)\rho)}$. Suppose, $|\phi\rangle$ is the maximally entangled state which gives the maximum value of the singlet fraction, i.e, $F(\rho_1) = \langle\phi|\rho_1|\phi\rangle$. In case of failure Alice and Bob prepare the product state $|\alpha\beta\rangle$ so that $|\langle\phi|\alpha\beta\rangle|^2 = \frac{1}{2}$.

Thus the resulting singlet fraction obtained from the protocol is,

$$S = pF(\rho_1) + \frac{1-p}{2} \tag{5.41}$$

with $p = Tr((A^\dagger A\otimes B^\dagger B)\rho)$ being the probability of success of the filtering process, absorbing the unitaries into A and B we can write $F(\rho_1) = \langle\Phi^+|\rho_1|\Phi^+\rangle$. Using now the expression for p we have,

$$S = \frac{1}{2} - \frac{1}{2}Tr\{((A\otimes B)\rho(A^\dagger\otimes B^\dagger))(|\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-| + |\Phi^-\rangle\langle\Phi^-| - |\Phi^+\rangle\langle\Phi^+|)\} \tag{5.42}$$

with $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. Again, $|\Phi^+\rangle\langle\Phi^+|^\Gamma = \frac{1}{2}V = \frac{1}{2}(|\Psi^+\rangle\langle\Psi^+| - |\Psi^-\rangle\langle\Psi^-| + |\Phi^-\rangle\langle\Phi^-| + |\Phi^+\rangle\langle\Phi^+|)$ where V denotes the SWAP operator defined by, $V|ij\rangle = |ji\rangle$.

Now using the fact that the trace of product of two matrices is the product of their partial transposes we finally have,

$$S = \frac{1}{2} - \langle \Phi^+ | (C \otimes I) \rho^\Gamma (C^\dagger \otimes I) | \Phi^+ \rangle \tag{5.43}$$

with $C = B^\dagger \sigma_y A$ and ρ^Γ denotes partial transpose w.r.t the party B.

This cost function when maximised over all $-I \leq A, B \leq I$ provides a lower bound on $F^*(\rho)$. Also, as is clear from the above equation one can choose $B = I$ without loss of generality and hence the protocol can be carried out by only one-way LOCC.

As mentioned before in the beginning of this section in order to obtain an upper bound we enlarge the class of operations from trace-preserving LOCC to trace-preserving PPT operations. A quantum operation Λ is PPT if and only if the dual state ρ_λ associated with this state is PPT. The dual state ρ_Λ corresponding to a map Λ acting on two qubit states is defined on a $2 \times 2 \times 2 \times 2$ Hilbert space as, $\rho_\Lambda = (I \otimes \Lambda)\mathbf{P}(\sum_{i,j=0}^{1} |ij\rangle \otimes |ij\rangle$, with $\mathbf{P}(|\alpha\rangle)$ denoting projector on the state $|\alpha\rangle$. We also have,

$$(\Lambda(\rho))_{A'B'} = 4Tr_{AB}(\rho_\Lambda^{AA'BB'}(\rho_{AB} \otimes I_{A'B'})). \tag{5.44}$$

It is now possible to provide an upper bound on $F^*(\rho)$ by considering the following maximization problem.

Maximize:

$$4Tr(\rho_\Lambda(\rho \otimes |\phi\rangle\langle\phi|)), \tag{5.45}$$

under the constraints $\rho_\Lambda \geq 0$, ${\rho_\Lambda}^{T_{BB'}} \geq 0$, $4Tr_{A'B'}(\rho_\Lambda) = I_{AB}$, with $|\phi\rangle$ being a maximally entangled state and $T_{BB'}$ denoting partial transpose w.r.t the systems B and B'. The above is a semi-definite program and can be solved numerically. We will now exploit symmetries to reduce the complexity of the problem.As $|\phi\rangle$ remains invariant under a twirling we can apply this to ρ_Λ and get a state of the form,

$$\rho_\Lambda = \frac{1}{16}(I_4 \otimes I_4 + (4X - I_4) \otimes (4|\phi\rangle\langle\phi| - I_4)), \tag{5.46}$$

where X is a 4×4 matrix subject to the convex constraints $\frac{I}{6} \leq X \leq \frac{I}{2}$ and $0 \leq X^T \leq \frac{I}{3}$. Now substituting, $X \to \frac{I - X^T}{3}$, the optimisation problem reduces to the following semi-definite program,maximize:

$$\frac{1}{2} - Tr(X\rho^\Gamma) \tag{5.47}$$

subject to, $0 \leq X \leq I_4$ and $-\frac{I_4}{2} \leq X^\Gamma \leq \frac{I_4}{2}$. The constraint $\frac{-I_4}{2} \leq X^\Gamma$ is redundant as X^Γ has only one negative eigenvalue $\lambda_-$and $\lambda_- \leq \max[\lambda(X^\Gamma)]$. One can also show that the optimal X must have rank one. The proof of this follows by contradiction. Suppose X satisfies the constraints and have rank larger than one. Then, X has a separable state in its support as each two-dimensional subspace contains at least one separable state. Consider now, y^2 the largest positive scalar such that $X - y^2 S \geq 0$. The matrix $Y = X - y^2 S$ also satisfies the above constraints as S^Γ is positive due to its separability. Also, $Tr(S\rho^\Gamma) = Tr(S^\Gamma \rho)$ with S separable and ρ entangled is always positive. Hence the matrix Y will provide a larger value of the cost function than X. Hence the optimal X must be of rank 1 and can be writen as,

$$X = (A \otimes I_2)|\Phi^+\rangle\langle\Phi^+|(A^\dagger \otimes I_2), \tag{5.48}$$

with the constraints equivalent to $-I_2 \leq A \leq I_2$. Thus we see that the upper and lower bound on $F^*\rho$ matches, indicating that the singlet fraction obtainable from the one-way LOCC protocol described before is indeed the optimal trace-preserving LOCC.

The optimal filter can be obtained by solving the following semi-definite program,

Maximize:

$$F^* = \frac{1}{2} - Tr(X\rho^\Gamma), \tag{5.49}$$

subject to $0 \leq X \leq I_4$ and $-\frac{I_4}{2} \leq X^\Gamma \leq \frac{I_4}{2}$, with the optimal filter given by, $X^* = (A \otimes I_2)|\Phi^+\rangle\langle\Phi^+|(A^\dagger \otimes I_2)$ with X^* denoting the solution to the above SDP.

Chapter 6

Approximate joint measurement through an Arthur-Kelly type model

In this chapter we consider the joint measurement of two and three unsharp qubit observables through an Arthur-Kelly type joint measurement model for qubits. We investigate the effect of initial state of the detectors on the unsharpness of the measurement as well as the post-measurement state of the system. We also try to explain the POVM to PVM transition in the model and look at what happens to entanglement between system and detectors during such a transition. Two existing approaches for characterizing the unsharpness of the measurement and the resulting measurement uncertainty relations are considered.The corresponding measures of unsharpness are shown to be proportional when both the measurements are equally unsharp. The connection between the POVM elements and symmetries of the underlying Hamiltonian of the measurement interaction is made explicit and used to perform joint measurement in arbitrary directions. Finally, we also provide for the first time a necessary and sufficient condition for approximate joint measurement of three unbiased qubit observables.

6.1 The Model

We consider an instantaneous coupling interaction with the help of the Hamiltonian of the form,

$$H = -(\hat{q}_1 \otimes \sigma_x + \hat{q}_2 \otimes \sigma_y)\delta t \tag{6.1}$$

(in $\hbar$=1 units). Possible coupling constants in the above equation have been absorbed by rescaling q_1 and q_2. Like the original Arthur-Kelly interaction (see Chapter 2) the idea is to entangle the detectors with the system through H and then perform a projective measurement of $\hat{p}_1$, $\hat{p}_2$ to obtain the spin information. Now, as a consequence of the Ehrenfest theorem the average momentum change of a particle carrying spin that experiences the above interaction is given by $< \dot{\hat{p}}_1 >=< \sigma_x >$ and $< \dot{\hat{p}}_2 >=< \sigma_y >$. Thus, for an ensemble of particles whose spin state is $|+x>$ and which has a symmetric distribution of p_1 before the interaction will have a greater probability of having a positive p_1 after it.The signs in equation (6.1) have been chosen so that this fact is true for both x and y directions. The signs thus allow us to map the four quadrants of the momentum plane (p_1, p_2) to the four outcomes of joint measurement and take the signs of the momenta to correspond to the outcomes (+,+) , (+,-),(-,+) ,(-,-) of the joint measurement. Note that for a Stern-Gerlach situation the two terms in eqn. (6.1) should have opposite signs to satisfy divergenceless of magnetic field.

Models similar to above have been considered before for example in [46],[47],[48]. As shown in [47] this model naturally arises in the context of a Stern-Gerlach experiment with a linear magnetic field.

We further assume that the measurement interaction (6.1) is strong enough to dominate the other parts of the Hamiltonian during its presence (e.g, the kinetic energy part). In the Stern-Gerlach context this would mean to assume the atoms carrying spin to be sufficiently massive.

The unitary evolution corresponding to the Hamiltonian of equation (6.1) is given after integrating the time evolution operator by,

$$U = exp(i(\hat{q}_1 \otimes \sigma_x + \hat{q}_2 \otimes \sigma_y)) \tag{6.2}$$

Now, we have

$$(\hat{q}_1 \otimes \sigma_x + \hat{q}_2 \otimes \sigma_y)^2 = \hat{q_1}^2 \otimes I + \hat{q_2}^2 \otimes I$$

,

$$(\hat{q}_1 \otimes \sigma_x + \hat{q}_2 \otimes \sigma_y)^3 = \hat{q}_1(\hat{q_1}^2 + \hat{q_2}^2) \otimes \sigma_x + \hat{q}_2(\hat{q_1}^2 + \hat{q_2}^2) \otimes \sigma_y$$

and

$$\begin{aligned}(\hat{q}_1 \otimes \sigma_x + \hat{q}_2 \otimes \sigma_y)^4 &= (\hat{q}_1 \otimes \sigma_x + \hat{q}_2 \otimes \sigma_y)(\hat{q}_1(\hat{q_1}^2 + \hat{q_2}^2) \otimes \sigma_x + \hat{q}_2(\hat{q_1}^2 + \hat{q_2}^2) \otimes \sigma_y) \\ &= (\hat{q_1}^2 + \hat{q_2}^2)^2 \otimes I. \end{aligned} \tag{6.3}$$

Thus on expanding the exponential in eqn. (6.2) and writing,

$$U = e(\hat{q}_1, \hat{q}_2) \otimes I + f(\hat{q}_1, \hat{q}_2) \otimes \sigma_x + g(\hat{q}_1, \hat{q}_2) \otimes \sigma_y + h(\hat{q}_1, \hat{q}_2) \otimes \sigma_z \tag{6.4}$$

we have with $x = \hat{q_1}^2 + \hat{q_2}^2$

$$e(\hat{q}_1, \hat{q}_2) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + ... = cos(\sqrt{x}), \tag{6.5}$$

$$f(\hat{q}_1, \hat{q}_2) = -iq_1(1 - \frac{x}{3!} + \frac{x^2}{5!} - ...) = -i\hat{q}_1 \frac{sin(\sqrt{x})}{\sqrt{x}}. \tag{6.6}$$

By symmetry,

$$g(\hat{q}_1, \hat{q}_2) = -iq_2(\frac{sin(\sqrt{\hat{q_1}^2 + \hat{q_2}^2})}{\sqrt{\hat{q_1}^2 + \hat{q_2}^2}}), \tag{6.7}$$

and $h(\hat{q}_1, \hat{q}_2) = 0$.

Hence we have,

$$\begin{aligned} U &= e(\hat{q}_1, \hat{q}_2) \otimes 1_s + f(\hat{q}_1, \hat{q}_2) \otimes \sigma_x + g(\hat{q}_1, \hat{q}_2) \otimes \sigma_y \qquad (6.8) \\ \text{with, } e(\hat{q}_1, \hat{q}_2) &= cos(\sqrt{(\hat{q}_1^2 + \hat{q}_2^2)}) \\ f(\hat{q}_1, \hat{q}_2) &= i\hat{q}_1 \frac{sin(\sqrt{(\hat{q}_1^2 + \hat{q}_2^2)})}{\sqrt{(\hat{q}_1^2 + \hat{q}_2^2)}} \\ g(q_1, q_2) &= i\hat{q}_2 \frac{sin(\sqrt{(\hat{q}_1^2 + \hat{q}_2^2)})}{\sqrt{(\hat{q}_1^2 + \hat{q}_2^2)}}. \qquad (6.9) \end{aligned}$$

Thus, the final state of the system is given by

$$\begin{aligned} |\psi_f\rangle = & \int_{q_1,q_2=-\infty}^{+\infty} e(q_1, q_2)|q_1, q_2\rangle \psi_1(q_1)\psi_2(q_2) dq_1 dq_2 \otimes |\chi\rangle \\ + & \int_{q_1,q_2=-\infty}^{+\infty} f(q_1, q_2)|q_1, q_2\rangle \psi_1(q_1)\psi_2(q_2) dq_1 dq_2 \otimes \sigma_x|\chi\rangle \\ + & \int_{q_1,q_2=-\infty}^{+\infty} g(q_1, q_2)|q_1, q_2\rangle \psi_1(q_1)\psi_2(q_2) dq_1 dq_2 \otimes \sigma_y|\chi\rangle, \qquad (6.10) \end{aligned}$$

with initial state being, $|\psi_i\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\chi\rangle$. Let, $\rho = |\psi_f\rangle\langle\psi_f|$.

6.2 Approximate Joint measurement in orthogonal directions

In this section we consider the joint measurement of σ_x and σ_y.We choose the observables $\hat{p}_1$ and $\hat{p}_2$ to serve as meters. As mentioned before, $(p_1 \geq 0, p_2 \geq 0)$ is taken to correspond to the outcome $(\sigma_x = 1, \sigma_y = 1) \equiv (+,+)$ of joint measurement, $(p_1 \geq 0, p_2 \leq 0)$ to $(\sigma_x = 1, \sigma_y = -1) \equiv (+,-)$, $(p_1 \leq 0, p_2 \geq 0)$ to $(\sigma_x = -1, \sigma_y = 1) \equiv (-,+)$ and $(p_1 \leq 0, p_2 \leq 0)$ to $(\sigma_x = -1, \sigma_y = -1) \equiv (-,-)$.

After the interaction U between the system and meters , projective measurement is performed separately on the observables $\hat{p}_1$ and $\hat{p}_2$. The probability of obtaining outcome (p_1, p_2) is given by,

$$\begin{aligned} p(p_1, p_2) &= Tr(|p_1, p_2\rangle\langle p_1, p_2|Tr_s(\rho)) \\ &= (|e^0|^2 + |f^0|^2 + |g^0|^2)\langle\chi|\chi\rangle + 2Re(f^0 e^{0*})\langle\chi|\sigma_x|\chi\rangle + \\ & 2Re(g^0 e^{0*})\langle\chi|\sigma_y|\chi\rangle - 2Im(g^0 f^{0*})\langle\chi|\sigma_z|\chi\rangle \qquad (6.11) \end{aligned}$$

with e^0, f^0 and g^0 representing respectively the fourier transforms of $e\psi_1\psi_2$, $f\psi_1\psi_2$ and $g\psi_1\psi_2$.

The initial pointer states are taken to be Gaussian,

$$\psi_1(q_1) = (\frac{1}{\sqrt{2\pi}}e^{-\frac{q_1^2}{2a^2}})^{\frac{1}{2}}, \tag{6.12}$$

$$\psi_2(q_2) = (\frac{1}{\sqrt{2\pi}}e^{-\frac{q_2^2}{2b^2}})^{\frac{1}{2}}. \tag{6.13}$$

satisfying $\int_{-\infty}^{+\infty}|\psi_j(q_j)^2|dq_j = 1$ for $j = 1, 2$.

We have chosen the initial states to be even in q_1 , q_2. Now as Fourier transform of a real even function is a real even function and that of an imaginary odd function is a real odd function we have f^0 odd in p_1 and even in p_2, $g^0$the other way around and e^0 even in both. Also, each of them is real. Thus, the σ_z term in (6.11) vanishes. Henceforth in this paper we refer to these properties of e^0, f^0 and g^0 as "parity properties".

We have for the probability of outcome (+,+) from equation (6.11),

$$\begin{aligned} p(p_1 \geq 0, p_2 \geq 0) = & \int_{p_1=0}^{\infty}\int_{p_2=0}^{\infty}\{(|e^0|^2 + |f^0|^2 + |g^0|^2)\langle\chi|\chi\rangle + 2Re(f^0 e^{0*})\langle\chi|\sigma_x|\chi\rangle \\ & +2Re(g^0 e^{0*})\langle\chi|\sigma_y|\chi\rangle\}dp_1 dp_2. \end{aligned} \tag{6.14}$$

One also of course has to satisfy,

$$p(p_1 \geq 0) + p(p_1 \leq 0) = 1,$$

which yields, due to the "parity properties",

$$\int_{p_1=0}^{\infty}\int_{p_2=0}^{\infty}(|e^0|^2 + |f^0|^2 + |g^0|^2)\, dp_1\, dp_2 = \frac{1}{4}. \tag{6.15}$$

From equation (6.15) and "parity properties" we have,

$$p(p_1 \geq 0, p_2 \geq 0) = \langle\chi|\frac{I}{4} + \frac{a'\sigma_x}{4} + \frac{b'\sigma_y}{4}|\chi\rangle = \langle\chi|G_{++}|\chi\rangle \tag{6.16}$$

with, $G_{++} = (\frac{I}{4} + \frac{a'\sigma_x}{4} + \frac{b'\sigma_y}{4})$ and

$$a' = \int_{p_1=0}^{+\infty}\int_{p_2=-\infty}^{+\infty}4(f^0 e^0)dp_1 dp_2, \tag{6.17}$$

$$b' = \int_{p_2=0}^{+\infty}\int_{p_1=-\infty}^{+\infty}4(g^0 e^0)dp_1 dp_2. \tag{6.18}$$

For the other outcomes, we have from consideration of the corresponding momentum probabilities ,

$$\begin{aligned} G_{+-} &= \frac{I}{4} + \frac{a'\sigma_x}{4} - \frac{b'\sigma_y}{4}, \\ G_{--} &= \frac{I}{4} - \frac{a'\sigma_x}{4} - \frac{b'\sigma_y}{4}, \\ G_{-+} &= \frac{I}{4} - \frac{a'\sigma_x}{4} + \frac{b'\sigma_y}{4}. \end{aligned} \tag{6.19}$$

From equations (6.16) and (6.19) we get the marginal unsharp observables as, (see chapter 2)

$$
\begin{aligned}
\Upsilon^1_\pm &= \frac{1}{2}(I \pm a'\sigma_x) \\
\Upsilon^2_\pm &= \frac{1}{2}(I \pm b'\sigma_y) \qquad (6.20)
\end{aligned}
$$

Thus, we see that the approximate observables Υ^1 , Υ^2 are unbiased and a' and b' themselves serve as measures of proximity to the sharp observables $\frac{1}{2}(I \pm \sigma_x)$ and $\frac{1}{2}(I \pm \sigma_y)$, respectively.

In section 6.5 we will see how the approximate joint measurement, characterised by the marginals in eqn. (6.20) arise as a consequence of symmetries of the Hamiltonian in eqn,(6.1) and that of the initial detector states ψ_1 and ψ_2 (given in eqns. (6.12) and (6.13)).

6.2.1 Reduced density matrix of the system after pre-measurement

The interaction U (6.2) acting on the system and the detectors induce a completely-positive map on the system . The rdm of the system after the interaction is given by,

$$
\begin{aligned}
\rho^s_f &= Tr_{1,2}(|\psi_f\rangle\langle\psi_f|) \qquad (6.21) \\
&= \int_{q_1,q_2=-\infty}^{+\infty} |e|^2{\psi_1}^2{\psi_2}^2 dq_1 dq_2 |\chi\rangle\langle\chi| + \int_{q_1,q_2=-\infty}^{+\infty} |f|^2{\psi_1}^2{\psi_2}^2 dq_1 dq_2 \sigma_x|\chi\rangle\langle\chi|\sigma_x \qquad (6.22) \\
&+ \int_{q_1,q_2=-\infty}^{+\infty} |g|^2{\psi_1}^2{\psi_2}^2 dq_1 dq_2 \sigma_y|\chi\rangle\langle\chi|\sigma_y \qquad (6.23) \\
&= \sum_{j=1}^{3} K_j|\chi\rangle\langle\chi|{K_j}^\dagger \qquad (6.24)
\end{aligned}
$$

The action of the measurement interaction on the system is thus that of an asymmetric depolarising channel , with the Krauss operators given by $K_1 = 2\sqrt{(c_f)}\sigma_1$, $K_2 = 2\sqrt{(c_g)}\sigma_2$ and $K_3 = \sqrt{(1-4c_f-4c_g)}I$, where

$$
\begin{aligned}
c_f &= \int_{q_1,q_2=-\infty}^{+\infty} |f|^2{\psi_1}^2{\psi_2}^2 dq_1 dq_2, \\
c_g &= \int_{q_1,q_2=-\infty}^{+\infty} |g|^2{\psi_1}^2{\psi_2}^2 dq_1 dq_2. \qquad (6.25)
\end{aligned}
$$

If (x, y, z) denotes the Bloch vector of $|\chi\rangle\langle\chi|$, then in the Bloch sphere representation we have ,

$$
\rho^s_f = \frac{1}{2}(I + x(1-8c_g)\sigma_x + y(1-8c_f)\sigma_y) \qquad (6.26)
$$

6.3 Effect of initial detector states

The form of the POVM elements in eqns.(6.16) , (6.19) which leads to unbiased marginal observables (see section 2.3.1.1) only depends on the "parity properties" and not on the Gaussian form of the initial detector states given by eqns. (6.12) , (6.13).

The integrals for computing the coefficients c_f and c_g have been done using the Monte Carlo integrator included in GNU Scientific Library. The MISER algorithm has been used which uses stratified random sampling for doing importance sampling [49].

Results

As we are considering unbiased approximate joint measurement in orthogonal directions , eqn. (2.24) applied on the parameters a' and b' of the marginals in eqn. (6.20) yield ,

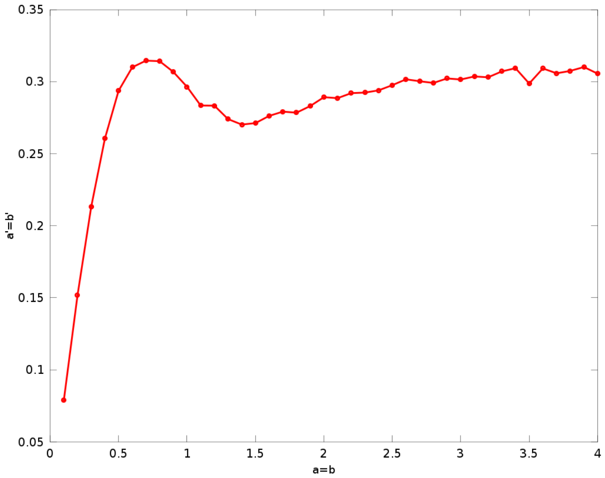
$$a'^2 + b'^2 \leq 1. \tag{6.27}$$

6.3.1 Symmetric case

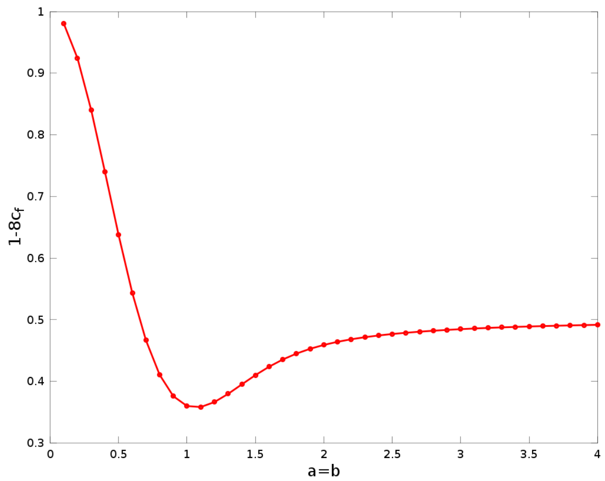
In this case the standard deviations a and b of the initial momentum wavefunctions in eqns. (6.12) and (6.13) are taken to be equal. As there is nothing else that differentiates between the detectors one must have here $a' = b'$. This is reflected in eqn. (6.17). Both the observables σ_x and σ_y are approximated equally well. From eqn. (6.27) we have, $a' \leq \frac{1}{\sqrt{2}} = 0.707$.

Quality:

In figure 6.1 we see that when a and b are close to zero, which corresponds to the fact that the initial momenta p_1 , p_2 have large spread we have almost no information about the spin state by looking at the momentum.In this case a' and b' are both close to zero and we have trivial marginal effects $(\frac{I}{2})$ which represent guessing the value of the corresponding sharp observable with equal probability for + and -. With increase in a, a' increases and touches the value 0.628 at $a = b = 0.7$.(see figure 6.1(a)). The graph of a' vrs. a is identical to the one obtained in [47] for the same case. The maxima in the curve is a feature of the interaction used and is explained in the next section.



(a) A plot of a' vrs. a



(b) A plot of $1 - 8c_f$ vrs. a

Figure 6.1: Effects of initial pointer states on the joint measurement and on the system state after the measurement interaction for $a = b$.

Disturbance due to the measurement interaction

Figure 6.1(b) shows the variation of $\frac{\langle\sigma_x\rangle_f}{\langle\sigma_x\rangle_i} = \frac{\langle\sigma_y\rangle_f}{\langle\sigma_y\rangle_i}$ with a, where $\langle\sigma_x\rangle_f = Tr(\rho_s^f \sigma_x)$ and $\langle\sigma_x\rangle_i = \langle\chi|\sigma_x|\chi\rangle$ and similarly for σ_y, (see eqn.(6.26)).

Corresponding to the maximum in a' we also have a minimum in the disturbance of the state characterised by $\frac{\langle\sigma_x\rangle_f}{\langle\sigma_x\rangle_i} = \frac{\langle\sigma_x\rangle_f}{\langle\sigma_x\rangle_i}$. Thus a sharper measurement seems to disturb the state lesser , though there is a slight difference in the value of a at which the maximum and the minimum occur. This disturbance is even more prominent in the asymmetric case to be discussed next.

$b \geq a$: POVM to PVM transition

a) Let us choose $a = 0.1$. The lhs of eqn. (6.27) starts off at a low value for $b = a$ and gradually increases and closes on the bound (1.0) as b becomes much greater than a.(see figure 6.2(a)) b much greater than a reflects the situation where the initial momentum wavefunction of apparatus 2 is much sharper than that of apparatus 1 . As shown clearly by figure 6.2(a) this marks a transition from a POVM measurement to a projection-valued measurement(PVM) in the sense that the unsharp measurement of σ_y becomes almost sharp. The requirement of complementarity is satisfied by the fact that the unsharp σ_x measurement becomes almost trivial. The fact that a' and b' depend both on a and b is a reflection of the correlations between p_1 and p_2 brought about by the unitary evolution given by eqn. (6.2).

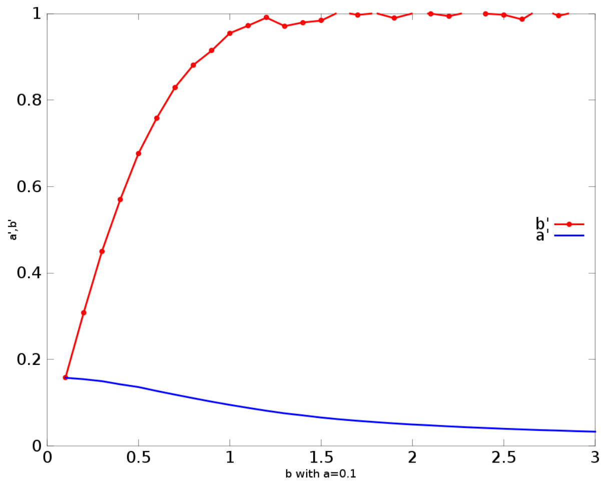
6.3.2 Disturbance due to the measurement

As seen in fig. 6.2(b) as the measurement becomes sharper (b increases with $a = 0.1$, b'tends to 1 while a' tends to 0) , the $< \sigma_y >$ information of the initial density matrix is almost kept intact,($\langle\sigma_y\rangle_f = (1 - 8c_g)\langle\chi|\sigma_y|\chi\rangle$) while the $< \sigma_x >$ information gets very disturbed ($\langle\sigma_x\rangle_f = (1 - 8c_f)\langle\chi|\sigma_x|\chi\rangle$) . This is reminiscent of a Heisenberg's Gammma-Ray microscope kind of a situation where using a short wavelength light to reduce the uncertainty in position measurement of an electron disturbs the momentum of the electron .

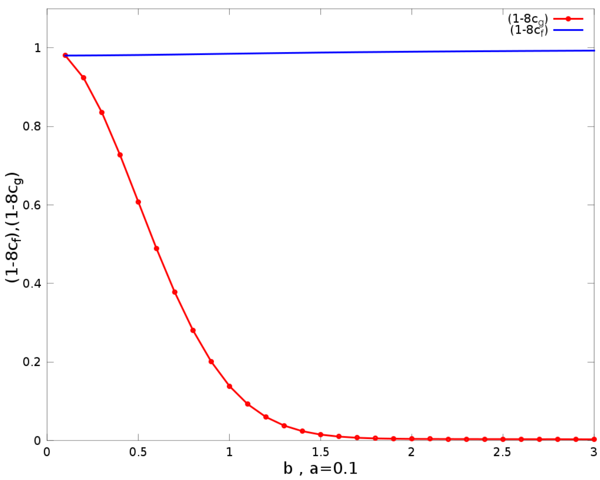
b) Next we choose $a = 1.0$. The conclusions are almost similar with the exception that the joint measurement uncertainty (i.e the l.h.s of eqn. (2.24) starts off at a much higher value compared to that of fig. 6.2(a) (see fig. 6.3).

6.4 Physics of the model and entanglement

In this section we first try to understand the results obtained in the previous section. It is instructive to first look at a single approximate measurement arising from a



(a) A plot of a' and b' vrs. b for a=0.1



(b) A plot of $(1-8c_f)$, $(1-8c_g)$(dotted line) vrs. b

Figure 6.2: Effects of initial pointer states on the joint measurement and post-measurement system state for $b \geq a = 0.1$.

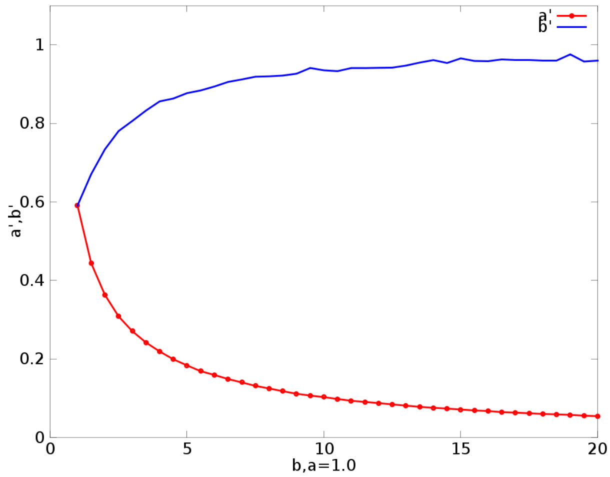


Figure 6.3: A plot of a' and b' vrs. b, a=1.0 . Bottom curve shows a' and top curve shows b'

von Neumann model. As mentioned before the situation is almost like a Stern-Gerlach Experiment. So we consider a neutral particle of mass m carrying spin-$\frac{1}{2}$ which propagating in the z-direction passes through a magnetic field $\vec{B} = -2B_0 x\hat{x}$ for a time interval τ. The position wavefunction of the particle before it enters the magnetic field is given by $\psi(x) = (\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2a^2}})^{\frac{1}{2}}$ .The interaction Hamiltonian is taken to be, $H = \frac{p^2}{2m} - \vec{S}.\vec{B}$. The unitary evolution corresponding to this interaction is given by,

$$U = exp(iB_0\tau x \otimes \sigma_x - i(\frac{p^2}{2m} \otimes 1_s)\tau) \text{ (in } \hbar = 1 \text{ units)} \tag{6.28}$$

The strong impulsive coupling approximation that we employed in section 6.2 to neglect the kinetic energy term amounts to $\frac{1}{2ma^2} \ll B_0 a$. We assume that the particle is sufficiently massive for this to hold for the values of a we have considered. We also note that in an usual Stern-Gerlach setting , initial position wavefunction is taken to be sharp and hence this assumption will breakdown for sufficiently small a . After neglecting the kinetic energy we have,

$$\begin{aligned} |\psi_f\rangle &= U(|\psi\rangle \otimes |\chi\rangle) \\ &= \int_{p_x=-\infty}^{\infty} dp_x \, {}_x\langle +|\chi\rangle \psi(p_x - \lambda)|p_x\rangle \otimes |+\rangle_x \\ &+ \int_{p_x=-\infty}^{\infty} dp_x \, {}_x\langle -|\chi\rangle \psi(p_x + \lambda)|p_x\rangle \otimes |-\rangle_x \end{aligned} \tag{6.29}$$

with, $\lambda = B_0\tau$ and $\sigma_x|\pm\rangle_x = \pm 1|\pm\rangle_x$.

If $\lambda \gg \frac{1}{a}$ then we can distinguish between $|+\rangle_x$ and $|-\rangle_x$ by looking at the momentum , as the Gaussian momentum distributions of width $\frac{1}{a}$ about $\pm\lambda$ do not essentially overlap. The mean of the distribution moves to $\pm\lambda$ depending on whether the state is $|\pm\rangle_x$. If we take ($p_x \geq 0$, $p_x \leq 0$) to correspond to an unsharp measurement of σ_x, then the POVM element G_+, characterising the unsharp measurement satisfies,

$$\begin{aligned} p(p_x \geq 0) &= \langle\chi|G_+|\chi\rangle \\ &= \langle\chi|\frac{1}{2}(I + a'\sigma_1)|\chi\rangle \end{aligned} \tag{6.30}$$

with, $a' = 2F(\lambda a) - 1$ and $F(x) = \int_{-\infty}^{x} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt$. Again as expected, as λa increases beyond 1 , $F(\lambda a)$ and a' moves closer to one and the measurement becomes sharper.

In the other limit , $\lambda \ll \frac{1}{a}$ distinguishability is lost and this is also reflected in a' going to zero.

Thus a necessary condition for an approximate measurement of $\vec{\sigma}.\hat{n}$ to be good can be taken to be its ability to distinguish between the eigenstates of $\vec{\sigma}.\hat{n}$. Distinguishability in turn depends on the two length scales λ , the distance by which the mean of the distribution moves and the width of the distribution being $\frac{1}{a}$. This is also reflected in the fact that in the limit $\lambda \gg \frac{1}{a}$, eqn. (6.29) becomes a Schmidt decomposition and $|\psi_f\rangle$ becomes maximally entangled.

Joint measurement

The Arthur-Kelly model that we have considered can be thought to come from a magnetic field $\vec{B} = -2B_0 x\hat{x} - 2B_0 y\hat{y}$ ($\lambda = B_0\tau$ has been taken to be 1). As mentioned before the non-zero divergence of this field is not a serious issue. We could also have taken $\vec{B} = -2B_0 x\hat{y} - 2B_0 y\hat{z}$ and measured p_x, p_y in order to have an approximate joint measurement of σ_y and σ_z .

After the measurement of momentum , the probability of obtaining p_x is given by (see eqn.(6.11))

$$p(p_x) = \int_{-\infty}^{\infty} (|e^0|^2 + |f^0|^2 + |g^0|^2) dp_y + 2\int_{-\infty}^{\infty} e^0 f^{0*} dp_y \langle\chi|\sigma_x|\chi\rangle \tag{6.31}$$

and

$$p(p_x \geq 0) = \frac{1}{2} + \frac{a'}{2}\langle\chi|\sigma_x|\chi\rangle \tag{6.32}$$

with

$$a' = \int_{p_x=0}^{+\infty} \int_{p_y=-\infty}^{+\infty} 4(f^0 e^0) dp_x dp_y. \tag{6.33}$$

The origin of complementarity between σ_x and σ_y in this model can be understood from the way the effective length scales governed by the movement of the mean momenta $< p_x >$ and $< p_y >$ change as we make one of the initial momentum wavefunctions sharper or broader. The Ehrenfest Theorem applied on the Hamiltonian (6.1) gives,

$$\begin{aligned} \langle \dot{p_x} \rangle &= \langle \sigma_x \rangle, \\ \langle \dot{p_y} \rangle &= \langle \sigma_y \rangle. \end{aligned} \tag{6.34}$$

Now as we saw in the section 6.2.1 and 6.3.2 ,the effect of the interaction (6.2) on the spin state of the system is that of an asymmetric depolarising channel that disturbs the $\langle \sigma_x \rangle$ component of the density matrix while keeps the $\langle \sigma_y \rangle$ almost intact as the initial y momentum wavefunction is made much sharper than the x momentum one. Eq. (6.34) shows that rate of change of average momentum in the x and y directions is similarly affected. Thus with the increasing sharpness of the initial y-momentum wavefunction, the movement of the p_y mean is not affected much , but the p_x mean moves very little. It becomes harder to distinguish between σ_x eigenstates by looking at p_x distribution after the interaction and unsharpness of σ_x measurement increases.

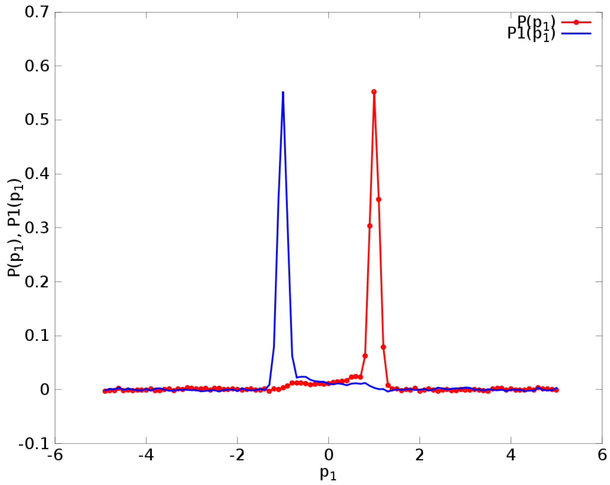
This is illustrated in fig. 6.4 .

Disturbance due to the measurement: In order to understand the asymmetric depolarising action of the interaction on the spin state we look at the rdm of the system after the interaction once more.

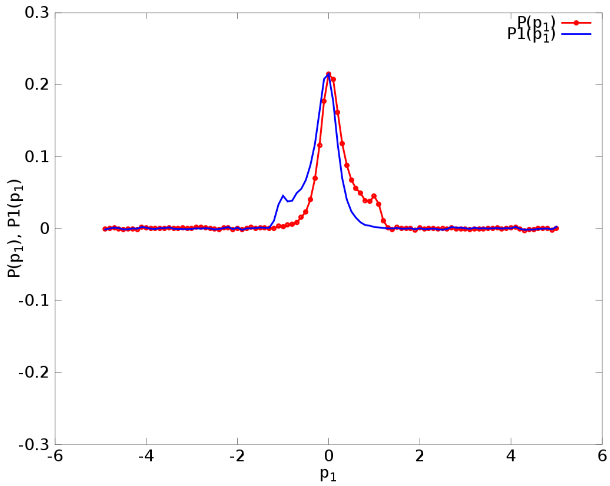
$$\begin{aligned} \rho_f^s &= Tr_{1,2}(|\psi_f\rangle\langle\psi_f|) \\ &= Tr_{1,2}(e^{i(x_{op}\otimes\sigma_x + y_{op}\otimes\sigma_y)}|\psi\rangle\langle\psi| \otimes |\chi\rangle\langle\chi| e^{-i(x_{op}\otimes\sigma_x + y_{op}\otimes\sigma_y)}) \\ &= \int_{x,y=-\infty}^{+\infty} e^{i(x\sigma_x + y\sigma_y)}|\chi\rangle\langle\chi|e^{-i(x\sigma_x+y\sigma_y)}|\psi_1(x)|^2|\psi_2(y)|^2 dxdy \\ &= \int_{x,y=-\infty}^{+\infty} e^{i\frac{\vec{\sigma}.\hat{r}}{2}(2r)}|\chi\rangle\langle\chi|e^{-i\frac{\vec{\sigma}.\hat{r}}{2}(2r)} \frac{e^{-(\frac{x^2}{a^2}+\frac{y^2}{b^2})}}{2\pi ab} dxdy \end{aligned} \tag{6.35}$$

with $r = \sqrt{x^2 + y^2}$, $\hat{r}$ denoting the radius and the unit radial vector respectively in polar coordinates . As eqn. (6.35) shows, the rdm of the spin part of the system after the interaction is a mixture of rotated states about $-\hat{r}$ by an angle ($2r$ mod 2π). The weight of a rotated state about $-\hat{r}$ in the mixture is the Gaussian probability density of the initial position of the particle. Thus as we increase b keeping a fixed , the weight of states which are rotated near about the y axis increases. Hence the $\langle \sigma_y \rangle$ component of the initial density matrix is disturbed more and more while the $\langle \sigma_x \rangle$ component is almost kept intact.

We earlier saw that the the quality of unsharpness a' increase and show a maximum in the symmetric case (fig. 6.1(a)). As we saw the disturbance due to the measurement governed by the plot of $(1-8c_f)$ vrs. a , also show a minimum. The maximum in the a' vrs. a curve was also there in [47].



(a) A plot of $P(p_1)$ and $P1(p_1)$ for a=5.0 , b=1.0



(b) A plot of $P(p_1)$ and $P1(p_1)$ for a=5.0 , b=25.0

Figure 6.4: A plot of probability of measuring p_1 after pre-measurement with p_1. $P(p_1)$ represents probability of p_1 for initial spin states $|+\rangle_x$ and $P1(p_1)$ represents that for initial spin state $|-\rangle_x$.

We argue that the maxima is due to the $2r$ factor in eq.(6.35) . For smaller values of a the rotations are constrained to smaller angles. To understand this we take a magnetic field of the form $\vec{B} = -2B_0\hat{r}$. This removes the r factor in 6.35 . a' is then seen to changes very little with a (from 0.27 to .285) . The disturbance characterised by c_f remains constant at .088 to about four orders of magnitude. The slight increase in a' is presumably due to the modification of the Ehrenfest eqns. (6.34) due to the non-linearity in the magnetic field.

Entanglement between detectors and system

We next consider the entanglement between the system and the two detectors. The detectors are infinite dimensional while the system of course is two-dimensional. Though strictly speaking only correlations between the detectors and system is required for the measurement on the detectors to reflect measurement statistics of the system , one expects entanglement to play a role in this kind of a scenario.

6.4.1 Entanglement between the joint detector system and the system

First, we consider the entanglement between the two detectors (considered as a single system) and the qubit system. As the final state $|\psi_f\rangle$ of the system and the detectors after pre-measurement is pure this entanglement is simply given by the von Neumann entropy of the reduced density of the system after the detectors are traced out. We have,

$$|\psi_f\rangle = \int dp_1 dp_2 e^0 |p_1, p_2\rangle \otimes |\chi\rangle + \int f^0 dp_1 dp_2 |p_1, p_2\rangle \otimes \sigma_x |\chi\rangle + \int dp_1 dp_2 g^0 |p_1, p_2\rangle \otimes \sigma_y |\chi\rangle$$

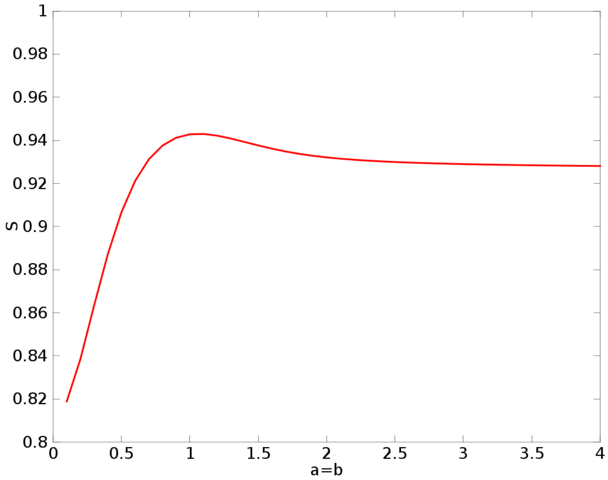
Let, $\rho_f^s = Tr_{12}|\psi_f\rangle\langle\psi_f|$. We have,

$$\begin{aligned} \rho_f^s = \quad & \int_{p1=-\infty}^{\infty} \int_{p2=-\infty}^{\infty} |e^0|^2 dp_1\, dp_2\, |\chi\rangle\langle\chi| + \\ & \int_{p1=-\infty}^{\infty} \int_{p2=-\infty}^{\infty} |f^0|^2 dp_1\, dp_2\, (\sigma_x|\chi\rangle\langle\chi|\sigma_x) + \\ & \int_{p1=-\infty}^{\infty} \int_{p2=-\infty}^{\infty} |g^0|^2 dp_1\, dp_2\, (\sigma_y|\chi\rangle\langle\chi|\sigma_y) \end{aligned} \tag{6.36}$$

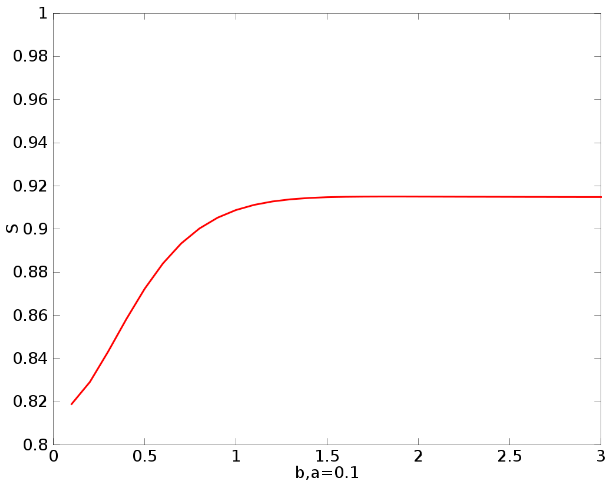
Now, for $|\chi\rangle\langle\chi| = \frac{1}{2}(I + x\sigma_1 + y\sigma_2 + z\sigma_3)$ we have ,

$$\rho_f^s = \frac{I}{2} + \frac{\sigma_x}{2}(x + 4c_{ff^*}(1-x) - 4xc_{gg^*}) + \frac{\sigma_y}{2}(y + 4c_{gg^*}(1-y) - 4yc_{ff^*}) + \frac{\sigma_z}{2}(1 - 4(c_{ff^*} + c_{gg^*})) \tag{6.37}$$

Taking $x, y = \frac{1}{2}$ and $z = 0$ the von Neumann entropy of the system is given by fig 6.5,



(a) A plot of $S(\rho_f^s)$ vrs. a=b



(b) A plot of $S(\rho_f^s)$ vrs. b for a=0.1

Figure 6.5: Entanglement between the joint detector system and the qubit system as reflected by the Von Neumann entropy of the r.d.m of the system after the configuration degrees are traced out.

The basic feature is that when the sharpnesses of the initial momentum wavefunctions are low the entanglement is relatively low , increasing as the sharpness increases. In both the symmetric and the asymmetric cases maximal entanglement is not reached.

Entanglement between detector and system

In this subsection we try to see the entanglement in the mixed state of one of the detectors and the system after the other detector is traced out. Now the situation is made difficult to handle by the fact that the detector is an infinite dimensional configuration degree of freedom . For this reason we project the state of the detector into two dimensional momentum subspaces. The average entanglement considering all such outcomes gives a lower bound on the entanglement of the $\infty \times 2$ state of the system and detector as projection being a local action cannot increase the the entanglement on the average.

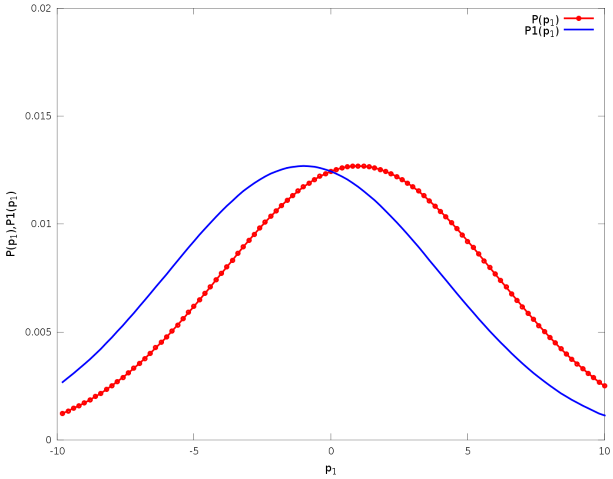
In fig. 6.6 we clearly see that the probability in eq. 6.32 becomes more and more greater than 0.5 as a increases with respect to b (for a $|+\rangle_x$ spin state), signifying an increase in a' as well. Depending on whether the state is $|+\rangle_x$ or $|-\rangle_x$ the probability peaks around $p_x = \pm 1$ (as the initial distribution is symmetric and we have taken the time of interaction to be unit , this follows from eq. (6.34) for almost sharp measurements). Thus correlation is likely to be highest between the system and the detector in the two dimensional momentum subspace around $p_x = \pm 1.0$.

As a measure of entanglement we consider concurrence which is defined below. We have considered projections in different two-dimensional momentum subspaces. The concurrence for states projected into different subspaces is qualitatively seen to follow the same behavior as state projected into $p_1 = \pm 1$ subspace. However for a much greater than b , numerics becomes difficult when we consider projections into subspaces far from ± 1. This is due to momentum distribution peaking around $p_1 = \pm 1$ (see fig. 6.6) and we expect the entanglement to fall for a much greater than b in these subspaces. Also, as the probability of obtaining such projections falls we do not expect the concurrence of states projected into momentum subspaces far from ± 1 to contribute much to the average concurrence.

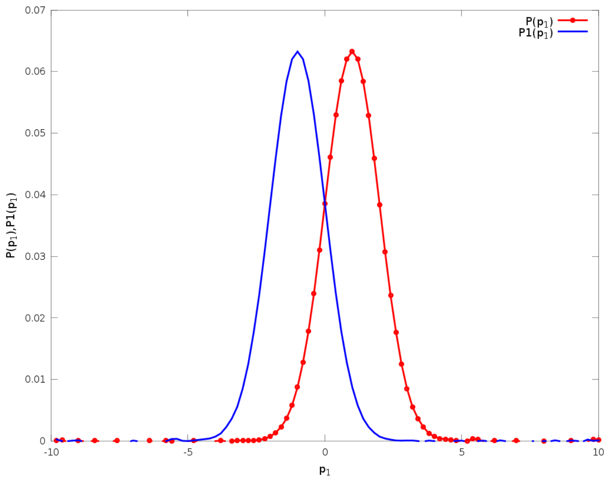
Thus we take the concurrence of the projected mixed state into $p_1 = \pm 1$ subspace to be an indicator of how the average concurrence for all possible projections on to two-dimensional subspaces behave (the first concurrence multiplied by probability of obtaining $p_1 = \pm 1$ is of course a lower bound for the average concurrence) as we increase sharpness of one initial detector momentum wavefunction (a) keeping the other fixed (b) (eqns. 6.12 and 6.13).

We consider the spin state to be the symmetric state $|\chi\rangle\langle\chi| = \frac{1}{2}(I + \frac{\sigma_x}{\sqrt{2}} + \frac{\sigma_y}{\sqrt{2}})$.

Let us first consider the entanglement between the first detector and the system . Let $\rho_f^1 = Tr_2(|\psi_f\rangle\langle\psi_f|)$. ($|\psi_f\rangle$ is the final state of the system and the detectors



(a) A plot of $P(p_1)$, $P1(p_1)$ vrs. p_1 for $a = b = 0.1$



(b) A plot of $P(p_1)$, $P1(p_1)$ vrs. p_1 for $a = 0.5, b = 0.1$

Figure 6.6: Probability distribution for obtaining p_1 . $P(p_1)$ and $P1(p_1)$ denotes respectively probability of obtaining p_1 for initial system state $|+\rangle_x$ and $|-\rangle_x$.

after the measurement interaction.). Because of “parity properties“ we have ,

$$\begin{aligned}
\int_{-\infty}^{\infty} e^0(1,p_2)e^0(-1,p_2)dp_2 &= \int_{-\infty}^{\infty} e^0(1,p_2)e^0(1,p_2)dp_2 := E^0(1), \\
\int_{-\infty}^{\infty} f^0(1,p_2)f^0(1,p_2)dp_2 &= -\int_{-\infty}^{\infty} f^0(1,p_2)f^0(-1,p_2)dp_2 := F^0(1), \\
\int_{-\infty}^{\infty} g^0(1,p_2)g^0(1,p_2)dp_2 &= \int_{-\infty}^{\infty} g^0(1,p_2)g^0(-1,p_2)dp_2 := G^0(1), \\
\int_{-\infty}^{\infty} e^0(1,p_2)f^0(1,p_2)dp_2 &= -\int_{-\infty}^{\infty} e^0(1,p_2)f^0(-1,p_2)dp_2 := E^0F^0(1) \quad (6.38)
\end{aligned}$$

Let, $\mathbf{P} = |p_1 = 1\rangle\langle p_1 = 1| + |p_1 = -1\rangle\langle p_1 = -1|$. We have,

$$\begin{aligned}
\rho^1 &= \mathbf{P}\rho_f^{p_1}\mathbf{P} \\
&= E^0(1)(|1\rangle\langle 1| + |1\rangle\langle -1| + |-1\rangle\langle 1| + |-1\rangle\langle -1|) \otimes |\chi\rangle\langle\chi| + \\
&+F^0(1)(|1\rangle\langle 1| - |-1\rangle\langle 1| - |1\rangle\langle -1| + |-1\rangle\langle -1|) \otimes \sigma_1|\chi\rangle\langle\chi|\sigma_1 \\
&+G^0(1)(|1\rangle\langle 1| + |1\rangle\langle -1| + |-1\rangle\langle 1| + |-1\rangle\langle -1|) \otimes \sigma_2|\chi\rangle\langle\chi|\sigma_2 \\
&+(E^0F^0(1)|1\rangle\langle 1| + E^0F^0(1)|-1\rangle\langle 1| - E^0F^0(1)|1\rangle\langle -1| \\
&-E^0F^0(1)|-1\rangle\langle -1|) \otimes |\chi\rangle\langle\chi|\sigma_1 \\
&+(E^0F^0(1)|1\rangle\langle 1| - E^0F^0(1)|-1\rangle\langle 1| + E^0F^0(1)|1\rangle\langle -1| \\
&-E^0F^0(1)|-1\rangle\langle -1|) \otimes \sigma_1|\chi\rangle\langle\chi|, \quad (6.39)
\end{aligned}$$

where $|1\rangle \equiv |p_1 = 1\rangle$ and $|-1\rangle \equiv |p_1 = -1\rangle$. Now for the spin state we have considered , we have the the unnormalized density matrix ρ^1 in the basis

$|1\rangle \equiv |p_1 = 1\rangle \otimes |0\rangle$, $|2\rangle \equiv |p_1 = 1\rangle \otimes |1\rangle$, $|3\rangle \equiv |p_1 = -1\rangle \otimes |0\rangle$ and $|4\rangle \equiv |p_1 = -1\rangle \otimes |1\rangle$ given by,

$$\begin{aligned}
\rho^1_{11} &= 0.5(E^0(1) + F^0(1) + G^0(1)) + 0.707E^0F^0(1) \\
\rho^1_{12} &= 0.5(E^0(1) - F^0(1) + G^0(1)) + 0.707iE^0F^0(1) \\
\rho^1_{13} &= 0.353(1-i)E^0(1) + 0.353(1+i)F^0(1) - 0.353(1+i)G^0(1) + E^0F^0(1) \\
\rho^1_{14} &= 0.353(1-i)E^0(1) - 0.353(1+i)F^0(1) - 0.353(1+i)G^0(1) \\
\rho^1_{22} &= 0.5(E^0(1) + F^0(1) + G^0(1)) - 0.707E^0F^0(1) \\
\rho^1_{23} &= 0.353(1-i)E^0(1) - 0.353(1+i)F^0(1) - 0.353(1+i)G^0(1) \\
\rho^1_{24} &= 0.353(1-i)E^0(1) - E^0F^0(1) + 0.353(1+i)F^0(1) - 0.353(1+i)G^0(1) \\
\rho^1_{33} &= 0.5(E^0(1) + F^0(1) + G^0(1)) + 0.707E^0F^0(1) \\
\rho^1_{34} &= 0.5(E^0(1) - F^0(1) + G^0(1)) - 0.707iE^0F^0(1) \\
\rho^1_{44} &= 0.5(E^0(1) + F^0(1) + G^0(1)) - 0.707E^0F^0(1) \quad (6.40)
\end{aligned}$$

with $Tr(\rho_1) = 2(E^0(1) + F^0(1) + G^0(1))$

After normalisation we apply the Peres-Horodecki PPT criterion to check for entanglement ([50]) . As a measurement of entanglement we use the concurrence which for a 2x2 density matrix is defined as $C(\rho) = \max\ (0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$, where λ_i

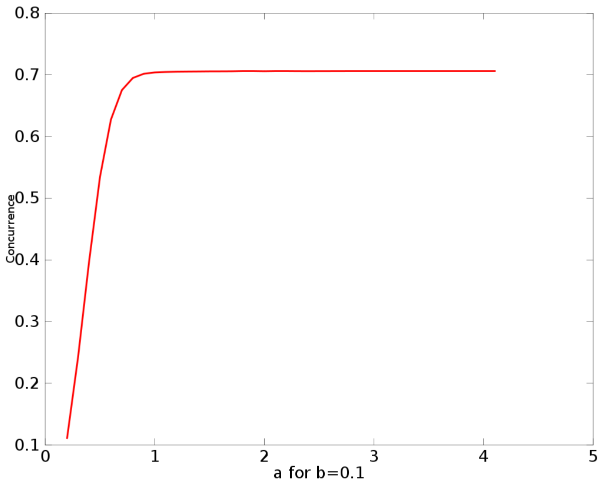


Figure 6.7: Concurrence of the rdm of system and first detector after projection into the subspace $p_1 = \pm 1$ vrs. a for b=0.1

denote in decreasing order the square root of the eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$,with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.([51])

In fig. 6.7 we see that the concurrence increases as a increases with fixed b, i,e as the σ_x measurement becomes sharper . We expect the entanglement between the system and the detector to behave similarly.

6.5 Effect of the symmetries of the underlying Hamiltonian on the POVM elements

It was shown in [46] in the context of a Stern-Gerlach type Hamiltonian how the symmetries of the underlying Hamiltonian like the one given in eqn. (6.1) can be used to do some particular unsharp measurement. The various cases considered there can be seen to follow from the following lemma.

Consider an Arthur-Kelly like measurement process and after the measurement on the detectors is made.

Lemma 1: Let H be an Arthur-Kelly like Hamiltonian of the form $H = f(q_1, q_2) \otimes \sigma_x + g(q_1, q_2) \otimes \sigma_y$, which has a symmetry given by $[A \otimes B, H] = 0$. Here A and B are unitaries acting respectively on the joint detector space and the spin space. Let $P_{|\chi\rangle}(p_1, p_2)$ represent the probability of obtaining the momenta values p_1 , p_2 if the system is initially in the state $|\chi\rangle$. Further let the initial joint detector state $|\psi\rangle$ have the symmetry A so that $A|\psi\rangle = e^{i\phi}|\psi\rangle$. Then for the initial system state $|\chi\rangle$ and a new basis in the joint detector space $|p_1', p_2'\rangle = A|p_1, p_2\rangle$, we have

$$P_{|\chi\rangle}(p_1, p_2) = P_{B|\chi\rangle}(p_1', p_2'). \tag{6.41}$$

Proof:

$$P_{|\chi\rangle}(p_1,p_2) = Tr_{1,2}[\mathbf{P}[|p_1,p_2\rangle]Tr_s(e^{-iH}\mathbf{P}[|\psi\rangle\otimes|\chi\rangle]e^{iH})] \tag{6.42}$$

with $\mathbf{P}[|\eta\rangle]$ denoting projector on $|\eta\rangle$.

Now replacing e^{-iH} by $(A^\dagger\otimes B^\dagger)e^{-iH}(A\otimes B)$ in (6.42) and using $A|\psi\rangle = e^{i\phi}|\psi\rangle$ we have ,

$$P_{|\chi\rangle}(p_1,p_2) = Tr_{1,2}[\mathbf{P}[|p_1,p_2\rangle]Tr_s((A^\dagger\otimes B^\dagger)e^{-iH}(|\psi\rangle\langle\psi|\otimes B|\chi\rangle\langle\chi|B^\dagger)e^{iH}(A\otimes B)] \tag{6.43}$$

Let ,

$$e^{-iH}(|\psi\rangle\langle\psi|\otimes B|\chi\rangle\langle\chi|B^\dagger)e^{iH} = e^{-iH}(\mathbf{P}[|\psi\rangle\otimes B|\chi\rangle])e^{iH} = \sum_j C_j\otimes D_j, \tag{6.44}$$

where the index j runs over a countable set.

C_j s and D_js are operators acting respectively on the joint Hilbert space of the two detectors and the system Hilbert space.

Using $B^\dagger B = I$ we have from eqns. (6.42) , (6.43) and (6.44):

$$Tr_s(e^{-iH}\mathbf{P}[|\psi\rangle\otimes|\chi\rangle]e^{iH}) = \sum_j Tr_s(D_j)(A^\dagger C_j A). \tag{6.45}$$

So,

$$\begin{aligned}
P_{|\chi\rangle}(p_1,p_2) &= \sum_j Tr_{1,2}[|p_1,p_2\rangle\langle p_1,p_2|Tr_s(D_j)(A^\dagger C_j A)] \\
&= \sum_j \langle p_1,p_2|A^\dagger C_j A|p_1,p_2\rangle Tr_s(D_j) \\
&= \sum_j \langle p_1',p_2'|C_j|p_1',p_2'\rangle Tr_s(D_j) \qquad (6.46)
\end{aligned}$$

Again,

$$\begin{aligned}
P_{B|\chi\rangle}(p_1',p_2') &= Tr_{1,2}[\mathbf{P}[|p_1',p_2'\rangle]Tr_s(e^{-iH}\mathbf{P}[|\psi\rangle\otimes B|\chi\rangle]e^{iH})] \\
&= Tr_{1,2}[\mathbf{P}[|p_1',p_2'\rangle]Tr_s(\sum_j C_j\otimes D_j)], \text{(from eqn. (6.44))} \\
&= \sum_j \langle p_1',p_2'|C_j|p_1',p_2'\rangle Tr_s(D_j) \qquad (6.47)
\end{aligned}$$

□

6.6 Approximate joint measurement in arbitrary directions

Let I_i (i=1,2) denote the operation of reflection in the detector space about q_i-axis (where $q_1 = x$ and $q_2 = y$). Consider a Hamiltonian of the entire system (i.e, two detectors plus the qubit) which satisfies $[I_1 \otimes \sigma_x, H] = 0$ and an initial joint state ψ of the two detectors jointly, that satisfies $I_1|\psi\rangle = |\psi\rangle$. Then eqn. (6.41) yields, (with E denoting the POVM element for a particular probability)

$$\begin{aligned}
P_{|\chi\rangle}(p_1, p_2) &= P_{\sigma_x|\chi\rangle}(p_1, -p_2). \\
\text{So, } \langle\chi|E(p_1, p_2)|\chi\rangle &= \langle\chi|\sigma_x E(p_1, -p_2)\sigma_x|\chi\rangle \\
\text{i.e } E(p_1, p_2) &= \sigma_x E(p_1, -p_2)\sigma_x \\
\text{and integrating over } p_2, \ [E(p_1), \sigma_x] &= 0. \\
\text{So we can write, } E(p_1) &= \frac{1}{2}(\alpha((p_1)I + \beta((p_1)\sigma_x) \\
\text{where } \alpha(p_1) \text{ and } \beta(p_1) \text{ are real nos.} & \\
\text{Hence we have, } E(p_1 \geq 0) &= \frac{1}{2}(\alpha' I + \beta' \sigma_x) \qquad (6.48)
\end{aligned}$$

with constant α' and β'.

As the Hamiltonian H of eqn. (6.1) satisfies $[I_1 \otimes \sigma_x, H] = 0$ and as the initial joint detector state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are given respectively by eqns. (6.12) and (6.13) satisfies both $I_1|\psi\rangle = |\psi\rangle$ and $I_2|\psi\rangle = |\psi\rangle$, it is clear from eqn. 6.48 that how the approximate joint measurement of σ_x and σ_y with the marginals given by eqn. (6.20) arises out of the measurement of p_1 and p_2.

Further consider an H which in addition to having the symmetry mentioned in the beginning of the present section, has a further rotational symmetry: $[R(\theta) \otimes S_z(\theta), H] = 0$, where $R(\theta)$ and $S_z(\theta)$ respectively denote rotation by an angle θ in the detector Hilbert space (i.e, the Hilbert space operation corresponding to rotation in the q_1, q_2 plane) and the spin space (about z-axis). From eqn. (6.41) we have,

$$P_{|\chi\rangle}(p_1, p_2) = P_{S_z(\theta)|\chi\rangle}(p'_1, p'_2) \qquad (6.49)$$

with $(p'_1, p'_2)^T = R(\theta)(p_1, p_2)^T$

Integrating both sides over the region $p_1 \geq 0$, $-\infty \leq p_2 \leq \infty$ we have of eqn. (6.49),

$$\begin{aligned}
P_{|\chi\rangle}(p_1 \geq 0) &= P_{S_z(\theta)|\chi\rangle}(p'_1 \geq 0), \\
\text{i.e , } P_{S_z(\theta)^\dagger|\chi\rangle}(p_1 \geq 0) &= P_{|\chi\rangle}(p'_1 \geq 0). \qquad (6.50)
\end{aligned}$$

Now, using eqn. 6.48 we have from eqn. 6.50 :

$$\begin{aligned}
P_{|\chi\rangle}(p'_1 \geq 0) &= \langle\chi|S_z(\theta)\frac{1}{2}(\alpha' I + \beta' \sigma_x)S_z(\theta)^\dagger|\chi\rangle, \qquad (6.51) \\
\text{i.e, } \langle\chi|E(p'_1 \geq 0)\chi\rangle &= \langle\chi|\frac{1}{2}(\alpha' I + \beta' \vec{\sigma}.\hat{n})|\chi\rangle \qquad (6.52)
\end{aligned}$$

where $\hat{n} = \hat{x}cos(\theta) + \hat{y}sin(\theta)$,with θ being the polar angle of a point in the x-y plane. In the last but one line we have also used $S_z(\theta) = exp[-i\frac{\sigma_z \theta}{2}]$.

The Hamiltonian in eqn. (6.1) satisfies both the above mentioned reflection as well as the rotational symmetry properties . Choosing the detector state to be a symmetric

Gaussian ,i.e, $b = a$ in eqn. (6.13) we have the required rotational invariance for all angles. Hence, we get an approximate joint measurement of spin in any direction by measuring the detector momentum in that direction.

6.6.1 POVM elements

Consider approximate joint measurement of σ_x and $\vec{\sigma}.\hat{n}$. We have already shown that the marginal probabilities of the joint measurement will be given by,

$$p(p_1' \geq 0) = \langle\chi|\frac{1}{2}(1 + a'\sigma_x)|\chi\rangle \quad (6.53)$$

$$p(p_2' \geq 0) = \langle\chi|\frac{1}{2}(1 + a'\vec{\sigma}.\hat{n})|\chi\rangle \quad (6.54)$$

with p_1' and p_2' denoting momenta in $\hat{x}$ and $\hat{n}$ direction respectively ($\hat{x}$ is along positive x-axis and $\hat{n}$ along $\hat{x}cos(\theta) + \hat{y}sin(\theta)$ in the momentum plane (p_1, p_2)).

6.6.1.1 Angular dependance of e^0 , f^0 and g^0

The rotational invariance of the initial state allows one to extract the polar angular dependance of e^0, f^0 and g^0 (introduced in section 6.2) in the detector space (p_1,p_2) ($p_1 = pcos(\theta_p)$, $p_2 = psin(\theta_p)$):

$$e^0(p, \theta_p) = \frac{1}{2\pi^3 a^2} \int e^{-iq_1 pcos\theta_p} e^{-iq_2 psin\theta_p} cos(\sqrt{q_1^2 + q_2^2}) e^{-\frac{q_1^2+q_2^2}{4a^2}} dq_1 dq_2 \quad (6.55)$$

Putting, $q_1 = rcos\theta$ and $q_2 = rsin\theta$ we have

$$\begin{aligned} e^0(p, \theta_p) &= C \int_0^{2\pi} \int_0^{+\infty} e^{-irpcos(\theta-\theta_p)} rcos(r) e^{\frac{-r^2}{4a^2}} dr d\theta \\ &= C \int_0^{+\infty} \int_0^{2\pi} e^{-irpcos(\theta-\theta_p)} d\theta g_1(r) dr \end{aligned} \quad (6.56)$$

with C being the constant part, $g_1(r) = rcos(r)e^{\frac{-r^2}{4a^2}}$. Now , taking $\theta - \theta_p = \theta'$ the theta integral in eqn. (6.56) becomes,

$$\begin{aligned} \int_0^{2\pi} e^{-irpcos(\theta-\theta_p)} d\theta &= \int_{-\theta_p}^{2\pi-\theta_p} e^{-irpcos(\theta')} d\theta' \\ &= \int_0^{2\pi} e^{-irpcos(\theta')} d\theta' \end{aligned} \quad (6.57)$$

The last equation follows from the fact that the integral of a periodic function over its period is independent of the limits of integration. Thus e^0 is only a function of p:

$$e^0 = e_1(p) \quad (6.58)$$

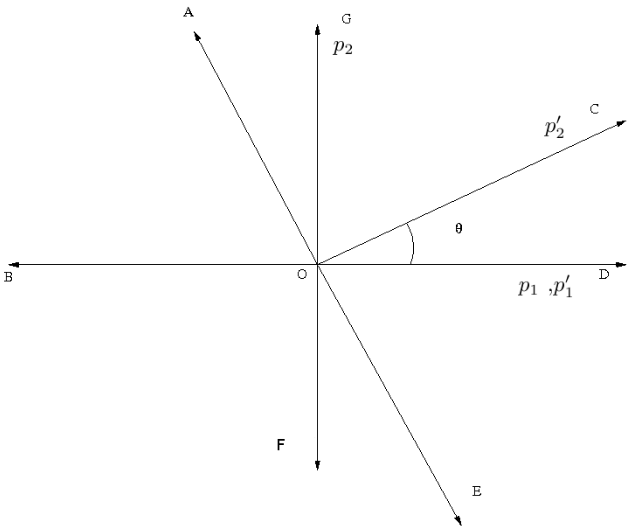


Figure 6.8: The above figure represents the detector momentum plane. The positive p_1 and p_2 axes are represented by OD and OG respectively. OD is also the p_1' axis. The p_2' axis OC makes an angle θ with OD. By DOE , we mean here the region of the plane bounded by OD and OE .

Again, similar transformation for f^0 yields,

$$f^0(p,\theta_p) = C' \int_0^{2\pi} \int_0^{+\infty} e^{-irpcos(\theta-\theta_p)} cos(\theta) sin(r) e^{\frac{-r^2}{4a^2}} r dr d\theta \tag{6.59}$$

with C' being the constant part. Using , $\theta - \theta_p = \theta'$ the theta integral in (6.59) becomes,

$$\int_{-\theta_p}^{2\pi-\theta_p} e^{-irpcos(\theta')} cos(\theta' + \theta_p) d\theta' = \int_{-\theta_p}^{2\pi-\theta_p} e^{-irpcos(\theta')} cos(\theta') cos(\theta_p) d\theta' - \int_{-\theta_p}^{2\pi-\theta_p} e^{-irpcos(\theta')} sin(\theta') sin(\theta_p) d\theta'$$

Now,

$$-\int_{-\theta_p}^{2\pi-\theta_p} e^{-irpcos(\theta')} sin(\theta') d\theta' = -\int_0^{2\pi} e^{-irpcos(\theta')} sin(\theta') d\theta'. \tag{6.60}$$

Taking, $cos(\theta') = x$, the above eqn. becomes $\int_1^{-1} e^{-irpx} dx + \int_{-1}^{1} e^{-irpx} dx = 0$. Therefore,

$$f^0(p,\theta_p) = f_1(p) cos(\theta_p). \tag{6.61}$$

Proceeding exactly similarly one can show that

$$g^0(p,\theta_p) = f_1(p) sin(\theta_p) \tag{6.62}$$

6.6.1.2 Joint measurement probabilities

Choosing the p_1' , p_2' axes according to fig. 6.8 , the joint probability is given by ,

$$\begin{aligned} p(p_1' \geq 0, p_2' \geq 0) &= \int_{GOD} p(p_1,p_2)dp_1dp_2 + \int_{DOE} p(p_1,p_2)dp_1dp_2 \\ &= \int_{p_1,p_2=0}^{\infty} p(p_1,p_2)dp_1dp_2 + \int_{DOE} p(p_1,p_2)dp_1dp_2 \end{aligned} \tag{6.63}$$

with, $p(p_1,p_2)$ given by (6.11),

where the last integral represents integral on the region DOE in the figure 6.8. In polar coordinates the region DOE is given by, the set of points $\{(p,\theta') : 0 \leq p \leq \infty, -(\frac{\pi}{2}-\theta) \leq \theta' \leq 0\}$. So, using eqn. (6.14) in (6.63) we have,

$$p(p_1' \geq 0, p_2' \geq 0) = \langle\chi| \left[\frac{1}{4}I + a'\frac{\sigma_x}{4} + a'\frac{\sigma_y}{4}\right] |\chi\rangle + \int_{DOE} p(p_1,p_2)dp_1dp_2 \tag{6.64}$$

Now using the θ dependance of e^0, f^0 and g^0 in 6.11 from (6.58) , (6.61) and (6.62) in (6.11) we have,

$$\begin{aligned} \int_{DOE} p(p_1,p_2)dp_1dp_2 = & \langle\chi|(\textstyle\int_0^\infty \int_0^{-(\frac{\pi}{2}-\theta)} [(|e_1(p)^2| + |f_1(p)^2|cos^2(\theta) + |f_1(p)^2|sin^2(\theta))1_s \\ & +2e_1(p)f_1(p)cos(\theta)\sigma_1 + 2e_1(p)f_1(p)sin(\theta)\sigma_2 pdpd\theta)|\chi\rangle \end{aligned} \tag{6.65}$$

Now, from eqn. (6.15) we have,

$$\int_0^\infty (e_1^2 + f_1^2)2\pi pdp = 1 \tag{6.66}$$

We also have from the definition of a' (eqn. (6.17)),

$$\int_0^\infty e_1 f_1 pdp = \frac{a'}{8}. \tag{6.67}$$

Hence ,

$$p(p_1' \geq 0, p_2' \geq 0) = \langle\chi| \left[(\frac{1}{2} - \frac{\theta}{2\pi})1_s + \frac{a'}{4}(1+cos(\theta))\sigma_x + \frac{a'}{4}sin(\theta)\sigma_y\right] |\chi\rangle \tag{6.68}$$

and similarly,

$$\begin{aligned} p(p_1' \leq 0, p_2' \geq 0) &= \langle\chi| \left[\frac{\theta}{2\pi}1_s + \frac{a'}{4}(cos(\theta)-1)\sigma_x + \frac{a'}{4}sin(\theta)\sigma_y\right] |\chi\rangle & (6.69) \\ p(p_1' \geq 0, p_2' \leq 0) &= \langle\chi| \left[|\frac{\theta}{2\pi}1_s + \frac{a'}{4}(-cos(\theta)+1)\sigma_x - \frac{a'}{4}sin(\theta)\sigma_y\right] |\chi\rangle & (6.70) \\ p(p_1' \leq 0, p_2' \leq 0) &= \langle\chi| \left[(\frac{1}{2} - \frac{\theta}{2\pi})1_s - \frac{a'}{4}(1+cos(\theta))\sigma_x - \frac{a'}{4}sin(\theta)\sigma_y\right] |\chi\rangle & (6.71) \end{aligned}$$

At $\theta = \frac{\pi}{2}$, we get back the POVM elements for the orthogonal case as derived in section 6.2 . For $\theta \to 0$ i.e, joint measurement along two almost same directions, the probabilities $p(p'_1 \leq 0, p'_2 \geq 0)$ and $p(p'_1 \geq 0, p'_2 \leq 0)$ vanish while $p(p'_1 \geq 0, p'_2 \geq 0)$ and $p(p'_1 \leq 0, p'_2 \leq 0)$ give the probabilities for single unsharp measurement along $\hat{x}$, as expected.

Remark: The joint measurement uncertainty relation is given by (2.24)

$$||\vec{a} + \vec{b}|| + ||\vec{a} - \vec{b}|| \leq 2. \tag{6.72}$$

If θ is the angle between $\vec{a}$ and $\vec{b}$, in the case $a = b$ (like above) one has,

$$a \leq \frac{1}{|sin(\frac{\theta}{2}) + cos(\frac{\theta}{2})|}. \tag{6.73}$$

The denominator in eqn. (6.73) is positive in $\theta \in [0, \pi]$ with maximum $\sqrt{2}$ at $\theta = \pi/2$. This is also our bound in the approximate joint measurement we have implemented, because for the symmetric initial detector state case (see section VI A) approximate joint measurement happens in orthogonal directions as well. In fact we just showed that measuring momenta in *any* two directions yields approximate joint measurement of spin in those two directions. We saw earlier that in our case (see section VI) a is able to reach about 0.628 (see fig.1a) . Thus, it is possible that one can have a different scheme with same but better quality of unsharp measurement in both the directions. For example, for $\theta = \pi/4$ the bound by the joint measurement inequality (6.73) is about 0.765.

6.7 Spin direction fidelities

In this section we consider the spin direction fidelities, defined in ([16]) . We consider type 1 measurement as defined by the author in ref.[16] in which the pointer observables are taken to be the commuting components of some unit vector $\hat{n}$. The measurement scheme considered by us in section 6.2 yields the momentum values p_1 ,p_2 (which can also be considered as components of $\vec{p}$). Alternatively, in polar coordinates we can look at the magnitude of the momentum $\vec{p}$ and its angle θ with the x-axis in the momentum plane. This angle uniquely fixes the direction of normalised momentum meter $\hat{p}$, and this is what we take for $\hat{n}$ in this section.

To begin with let us write eqn. (6.11) as,

$$p(p_1, p_2) = \langle \chi | T^\dagger(\vec{p}) T(\vec{p}) | \chi \rangle \tag{6.74}$$

with

$$\begin{aligned} T(\vec{p}) &= \langle \vec{p} | e | \psi \rangle 1_s + \langle \vec{p} | f | \psi \rangle \sigma_x + \langle \vec{p} | g | \psi \rangle \sigma_y \\ &= e^0 1_s + f^0 \sigma_x + g^0 \sigma_y \end{aligned} \tag{6.75}$$

The joint unsharp measurement on the system through measurement on the commuting meter observables after the meters have interacted with the system, yields the POVM described by eqns. (6.16) and (6.19) .The effect of the measurement on the system (a completely-positive map acting on the system density matrix) can be described by Krauss operators for the measurement $T(\vec{p})$. The average density matrix of the system (see eqn. (6.21- 6.24))ρ_f^s after the measurement on the meters, considering all possible outcomes can be written as (using Plancherel's Theorem of Fourier transforms):

$$\rho_f^s = \int pdpd\theta \; T(p,\theta)|\chi\rangle\langle\chi|T^\dagger(p,\theta) \tag{6.76}$$

The probability of registering the outcome at an angle θ in the (p_1 , p_2) plane is given by $p(\theta) = \langle\chi|E(\theta)|\chi\rangle$, with the angle POVM

$$E(\theta) = \int_0^\infty T^\dagger(p,\theta)T(p,\theta)pdp. \tag{6.77}$$

Let us now compute the fidelitites as defined in eqns. (2.33),(2.34) and (6.86) of Chapter 2. The retrodictive fidelity is defined as,

$$\eta_i = inf_{|\chi\rangle\in H_{sys}}(\langle\psi|\otimes\langle\chi|\frac{1}{2}(\hat{n_f}.\hat{S}_i + \hat{S}_i.\hat{n_f})|\psi\rangle\otimes|\chi\rangle) \tag{6.78}$$

where, $\hat{S}_i = \hat{S}$ and $\hat{n_i} = \hat{n}$ are the initial values of the Heisenberg spin and pointer observables respectively and $\hat{n_f} = U^\dagger(\hat{n_i}\otimes 1_s)U$ and $\hat{S}_f = U^\dagger(I\otimes\hat{S}_i)U$ respectively be the final Heisenberg pointer and spin direction observables after the measurement interaction. As mentioned before, we take $\hat{n_i} = \hat{p}$ with, $\hat{p}|\vec{p'}\rangle = \hat{p'}|\vec{p'}\rangle$ (where θ_p denotes the polar angle in the $p_1 - p_2$ plane and $\hat{x},\hat{y}$ denote respectively unit vectors along p_1 and p_2 axes). Also, in the above eqn. $\hat{n_f}.\hat{S}_i$ stands for the formal expansion $\sum_{i=1}^2 \hat{n_f}_i(I\otimes S_i)$.

Consider, the term $inf_{|\chi\rangle\in H_{sys}}\langle\psi|\otimes\langle\chi|(\hat{n_f}.\hat{S}_i)|\psi\rangle\otimes|\chi\rangle$.

We have,

$$inf_{|\chi\rangle\in H_{sys}}\langle\psi|\otimes\langle\chi|(\hat{n_f}.\hat{S}_i)|\psi\rangle\otimes|\chi\rangle = inf_{|\chi\rangle}\langle\psi|\otimes\langle\chi|\sum_{i=1}^{2}U^\dagger(\hat{p_i}\otimes 1_s)U(I\otimes S_i)|\psi\rangle\otimes|\chi\rangle \tag{6.79}$$

Now,

$$\begin{aligned} &inf_{|\chi\rangle}\left(\langle\psi|\otimes\langle\chi\right)|U^\dagger(\hat{p_1}\otimes 1_s)U(I\otimes S_1)\left(|\psi\rangle\otimes|\chi\rangle\right) \\ &= inf_{|\chi\rangle}\left(\langle\psi|\otimes\langle\chi|\right)U^\dagger(\hat{p_1}\otimes 1_s)\left(\int|\vec{p'}\rangle\langle\vec{p'}|dp_1dp_2\otimes 1_s\right)U(I\otimes S_1)\left(|\psi\rangle\otimes|\chi\rangle\right) \\ &= inf_{|\chi\rangle}\int\left(\langle\psi|\otimes\langle\chi|\right)U^\dagger\left((\hat{p'})_1|\vec{p'}\rangle\langle\vec{p'}|\otimes 1_s\right)U(I\otimes S_1)\left(|\psi\rangle\otimes|\chi\rangle\right)dp_1dp_2 \end{aligned} \tag{6.80}$$

Using the expansion of U as, $U = e\otimes 1_s + f\otimes\sigma_x + g\otimes\sigma_y$ (6.8) we have,

$$
\begin{aligned}
& inf_{|\chi\rangle}\left(\langle\psi|\otimes\langle\chi|\right)U^{\dagger}\left((\hat{p'})_1|\vec{p'}\rangle\langle\vec{p'}|\otimes 1_s\right)U(I\otimes S_1)\left(|\psi\rangle\otimes|\chi\rangle\right)\\
&\quad = inf_{|\chi\rangle}\{\langle\chi|\langle\vec{p'}|e|\psi\rangle^{*}\langle\vec{p'}|e|\psi\rangle(\hat{p'})_1 S_1|\chi\rangle +\\
&\quad \langle\chi|\langle\vec{p'}|e|\psi\rangle^{*}\langle\vec{p'}|f|\psi\rangle(\hat{p'})_1\sigma_x S_1\ |\chi\rangle +\\
&\quad \langle\chi|\langle\vec{p'}|e|\psi\rangle^{*}\langle\vec{p'}|g|\psi\rangle(\hat{p'})_1\sigma_y S_1\ |\chi\rangle + ...\}\\
&\quad = inf_{|\chi\rangle}\langle\chi|T^{\dagger}(\vec{p'})T(\vec{p'})(\hat{p'})_1 S_1\ |\chi\rangle
\end{aligned} \tag{6.81}
$$

Thus it follows that,

$$
\begin{aligned}
inf_{|\chi\rangle} &\quad \left(\langle\psi|\otimes\langle\chi\right)|U^{\dagger}(\hat{p_1}\otimes 1_s)U(I\otimes S_1)\left(|\psi\rangle\otimes|\chi\rangle\right)\\
&= inf_{|\chi\rangle}\int\langle\chi|T^{\dagger}(\vec{p'})T(\vec{p'})p'dp'd\theta_{p'}(\hat{p'})_1 S_1\ |\chi\rangle\\
&= inf_{|\chi\rangle}\langle\chi|\int E(\hat{p'})(\hat{p'})_1 S_1 d\theta_{p'}\ |\chi\rangle
\end{aligned} \tag{6.82}
$$

Hence we have,

$$
inf_{|\chi\rangle\in H_{sys}}\langle\psi|\otimes\langle\chi|(\hat{n_f}.\hat{S_i})|\psi\rangle\otimes|\chi\rangle = inf_{|\chi\rangle}\langle\chi|\int E(\hat{p'})\vec{S}.\hat{p'}d\theta_{p'}\ |\chi\rangle. \tag{6.83}
$$

In a similar way one can show that,

$$
inf_{|\chi\rangle\in H_{sys}}\langle\psi|\otimes\langle\chi|(\hat{S_i}.\hat{n_f})|\psi\rangle\otimes|\chi\rangle = inf_{|\chi\rangle}\langle\chi|\int \vec{S}.\hat{p'}E(\hat{p'})d\theta_{p'}\ |\chi\rangle. \tag{6.84}
$$

Hence ,

$$
\eta_i = inf_{|\chi\rangle}\langle\chi|\int\frac{1}{2}(\vec{S}.\hat{p'}E(\hat{p'}) + E(\hat{p'})\vec{S}.\hat{p'})d\theta_{p'}\ |\chi\rangle. \tag{6.85}
$$

Again, the measurement disturbance is defined as

$$
\eta_d = inf_{|\chi\rangle\in H_{sys}}(\langle\psi|\otimes\langle\chi|\frac{1}{2}(\hat{S_f}.\hat{S_i} + \hat{S_i}.\hat{S_f})|\psi\rangle\otimes|\chi\rangle). \tag{6.86}
$$

Let us consider the term , $\langle\psi|\otimes\langle\chi|(\hat{S_f}.\hat{S_i})|\psi\rangle\otimes|\chi\rangle$. $\hat{S_f}.\hat{S_i}$ stands for the expansion $\sum_j U^{\dagger}(I\otimes S_j)U(I\otimes S_j)$.

Thus,

$$
\begin{aligned}
\langle\psi|\otimes\langle\chi|(\hat{S_f}.\hat{S_i})|\psi\rangle\otimes|\chi\rangle &= \langle\psi|\otimes\langle\chi|\sum_j U^{\dagger}(I\otimes S_j)U(I\otimes S_j)|\psi\rangle\otimes|\chi\rangle\\
&= \langle\psi|\otimes\langle\chi|\sum_j(e^{*}\otimes 1_s + f^{*}\otimes\sigma_x + g^{*}\otimes\sigma_y)(I\otimes S_j)(e\otimes 1_s\\
&+ f\otimes\sigma_x + g\otimes\sigma_y)(I\otimes S_j)|\psi\rangle\otimes|\chi\rangle\\
&= \langle\psi|e^{*}e|\psi\rangle\langle\chi|S_jS_j|\chi\rangle + \langle\psi|e^{*}f|\psi\rangle\langle\chi|S_j\sigma_x S_j|\chi\rangle\\
&+ \langle\psi|e^{*}g|\psi\rangle\langle\chi|S_j\sigma_y S_j|\chi\rangle + ...
\end{aligned} \tag{6.87}
$$

Now $\langle\psi|e^* f|\psi\rangle = \int \langle\psi|e^*|\vec{p'}\rangle\langle\vec{p'}|f|\psi\rangle dp_1 dp_2 = \int e^{0*} f^0 dp_1 dp_2$ and so on. Hence,

$$\langle\psi| \otimes \langle\chi|(\hat{S_f}.\hat{S_i})|\psi\rangle \otimes |\chi\rangle = \langle\chi| \int \sum_{i=1}^{3} T^\dagger S_i T S_i dp_1 dp_2 \; |\chi\rangle. \tag{6.88}$$

Proceeding along exactly similar lines for the $(\hat{S_i}.\hat{S_f})$ term in eqn. (6.86) we finally have,

$$\eta_d = inf_{|\chi\rangle \in H_{sys}} \langle\chi| \left[\int \sum_{i=1}^{3} \frac{1}{2}(T^\dagger S_i T S_i + S_i T^\dagger S_i T) p dp d\theta \right] |\chi\rangle. \tag{6.89}$$

The predictive fidelity is defined as,

$$\eta_f = inf_{|\chi\rangle \in H_{sys}} (\langle\psi| \otimes \langle\chi| \frac{1}{2}(\hat{n_f}.\hat{S_f} + \hat{S_f}.\hat{n_f})|\psi\rangle \otimes |\chi\rangle). \tag{6.90}$$

Again, let us consider the term $inf_{|\chi\rangle \in H_{sys}} \langle\psi| \otimes \langle\chi|(\hat{n_f}.\hat{S_f})|\psi\rangle \otimes |\chi\rangle$. We have,

$$inf_{|\chi\rangle \in H_{sys}} \langle\psi| \otimes \langle\chi|(\hat{n_f}.\hat{S_f})|\psi\rangle \otimes |\chi\rangle = inf_{|\chi\rangle} \langle\psi| \otimes \langle\chi| \sum_{i=1}^{2} U^\dagger (\hat{p_i} \otimes 1_s)(I \otimes S_i)|U\psi\rangle \otimes |\chi\rangle \tag{6.91}$$

and hence the two terms in the R.H.S of eqn. 6.90 commute .

Now,

$$\begin{aligned} & inf_{|\chi\rangle} \left(\langle\psi| \otimes \langle\chi\rangle |U^\dagger (\hat{p_1} \otimes 1_s)(I \otimes S_1) U \left(|\psi\rangle \otimes |\chi\rangle\right) \\ = \; & inf_{|\chi\rangle} \left(\langle\psi| \otimes \langle\chi|\right) U^\dagger (\hat{p_1} \otimes 1_s) \left(\int |\vec{p'}\rangle\langle\vec{p'}| dp_1 dp_2 \otimes 1_s \right) (I \otimes S_1) U \left(|\psi\rangle \otimes |\chi\rangle\right) \\ = \; & inf_{|\chi\rangle} \int \left(\langle\psi| \otimes \langle\chi|\right) U^\dagger \left((\hat{p'})_1 |\vec{p'}\rangle\langle\vec{p'}| \otimes 1_s \right) (I \otimes S_1) U \left(|\psi\rangle \otimes |\chi\rangle\right) dp_1 dp_2 \end{aligned} \tag{6.92}$$

Using the expansion of U as, $U = e \otimes 1_s + f \otimes \sigma_x + g \otimes \sigma_y$ (6.8) we have,

$$\begin{aligned} & inf_{|\chi\rangle} \left(\langle\psi| \otimes \langle\chi|\right) U^\dagger \left((\hat{p'})_1 |\vec{p'}\rangle\langle\vec{p'}| \otimes 1_s \right) U (I \otimes S_1) \left(|\psi\rangle \otimes |\chi\rangle\right) \\ & = inf_{|\chi\rangle} \{ \langle\chi| \langle\vec{p'}|e|\psi\rangle^* \langle\vec{p'}|e|\psi\rangle (\hat{p'})_1 S_1 |\chi\rangle + \\ & \langle\chi| \langle\vec{p'}|e|\psi\rangle^* \langle\vec{p'}|f|\psi\rangle (\hat{p'})_1 S_1 \sigma_x \; |\chi\rangle + \\ & \langle\chi| \langle\vec{p'}|e|\psi\rangle^* \langle\vec{p'}|g|\psi\rangle (\hat{p'})_1 S_1 \sigma_y |\chi\rangle + ... \} \\ & = inf_{|\chi\rangle} \langle\chi| T^\dagger(\vec{p'}) S_1 (\hat{p'})_1 T(\vec{p'}) \; |\chi\rangle. \end{aligned} \tag{6.93}$$

Hence we have,

$$inf_{|\chi\rangle \in H_{sys}} \langle\psi| \otimes \langle\chi|(\hat{n_f}.\hat{S_f})|\psi\rangle \otimes |\chi\rangle = inf_{|\chi\rangle} \langle\chi| T^\dagger(\vec{p'}) \vec{S}.(\hat{p'}) T(\vec{p'}) |\chi\rangle \tag{6.94}$$

and

$$\eta_f = inf_{|\chi\rangle \in H_{sys}} \langle\chi| \left[\int T^\dagger \hat{p}\cdot\vec{S} T p dp d\theta \right] |\chi\rangle. \tag{6.95}$$

From (6.77) we have,

$$\begin{aligned} E(\theta) &= \int_0^{+\infty} (|e^0|^2 + |f^0|^2 + |g^0|^2) p dp 1_s + \int_0^{+\infty} 2Re(e^0 f^{0*}) p dp \sigma_x + \\ & \int_0^{+\infty} 2Re(e^0 g^{0*}) p dp \sigma_y = a(\theta) 1_s + b(\theta)\sigma_x + c(\theta)\sigma_y(\text{ say }). \end{aligned} \tag{6.96}$$

Hence,

$$\frac{1}{2}(E(\hat{n})\hat{n}.\vec{S} + \hat{n}.\vec{S}E(\hat{n})) = \frac{1}{2}[(b(\theta)cos(\theta) + c(\theta)sin(\theta))1_s + a(\theta)cos(\theta)\sigma_x + a(\theta)sin(\theta)\sigma_y]. \tag{6.97}$$

So from eqn. (6.85),

$$\begin{aligned} \eta_i = & \; inf_{|\chi\rangle\in H_{sys}} \int_0^{2\pi} d\theta[\tfrac{1}{2}(b(\theta)cos(\theta) + c(\theta)sin(\theta)) + \tfrac{1}{2}a(\theta)cos(\theta)\langle\chi|\sigma_x|\chi\rangle + \tfrac{1}{2}a(\theta)sin(\theta)\langle\chi|\sigma_y|\chi\rangle] \\ = & \int_0^\infty \int_0^{2\pi} e^0 f^{0*} cos(\theta) + e^0 g^{0*} sin(\theta) p dp d\theta + \int_0^\infty \int_0^{2\pi} (|e^0|^2 + |f^0|^2 + |g^0|^2) cos(\theta) p dp d\theta \langle\chi|\sigma_x|\chi\rangle \\ & + \int_0^\infty \int_0^{2\pi} (|e^0|^2 + |f^0|^2 + |g^0|^2) sin(\theta) p dp d\theta \langle\chi|\sigma_y|\chi\rangle \end{aligned} \tag{6.98}$$

Changing back to Cartesian coordinates we have,

$$\eta_i = \int_{-\infty}^{\infty} e^0 f^{0*} \frac{p_1}{(p_1^2+p_2^2)^{\frac{1}{2}}} dp_1 dp_2 + \int_{-\infty}^{\infty} e^0 g^{0*} \frac{p_2}{(p_1^2+p_2^2)^{\frac{1}{2}}} dp_1 dp_2. \tag{6.99}$$

The last two terms on the RHS of eqn. (6.98) vanish because the 3rd term is is odd in p_1 and the 4th one is odd in p_2 .

Now using eqn. (6.67) we have,

$$\begin{aligned} \eta_i &= \int e^0 f^0 cos(\theta) p dp d\theta + \int e^0 g^0 sin(\theta) p dp d\theta \\ &= \int e_1 f_1 p dp \int_{\theta=0}^{2\pi} cos^2(\theta) d\theta + \int e_1 f_1 p dp \int_{\theta=0}^{2\pi} sin^2(\theta) d\theta \\ &= 2\pi \int e_1 f_1 p dp. \end{aligned} \tag{6.100}$$

Thus,

$$\eta_i = \frac{\pi}{4} a'. \tag{6.101}$$

Bound on a': It was shown in [16] that η_i is bounded by the value $s = \frac{1}{2}$. From here we get the bound on a' to be about .64 which is almost exactly what is got in the simulation of a' (0.6292 in fig. 1a). The joint measurement uncertainty relation (6.27) allows a' to go till .707. This is because eqn. (6.27) refers to the most general approximate joint measurement in orthogonal directions without reference to any

Arthur-Kelly kind of an implementation. Also eqn. (6.101) shows that the error of retrodiction (given in eqn. (2.36)) falls as the measurement becomes sharp.

A similar calculation shows that,

$$\begin{aligned}\eta_f &= inf_{|\chi\rangle \in H_{sys}} \int pd\theta dp \langle\chi|[[cos(\theta)(|e^0|^2+|f^0|^2-|g^0|^2)+2Re(f^0 g^{0^*})sin(\theta)]\sigma_x \\ &+ [sin(\theta)(|e^0|^2-|f^0|^2|g^0|^2)+2Re(f^0 g^{0^*})cos(\theta)]\sigma_y \\ &+ [e^0 f^0 cos(\theta)+e^0 g^0 sin(\theta)]1_s]|\chi\rangle \qquad (6.102)\end{aligned}$$

and,

$$\eta_d = inf_{|\chi\rangle \in H_{sys}} \int pd\theta dp \langle\chi \left[|\frac{1}{2}(|e^0|^2+|f^0|^2+|g^0|^2)1_s + \frac{e^0 f^{0*}}{2}\sigma_x + \frac{e^0 g^{0*}}{2}\sigma_y \right] |\chi\rangle. \qquad (6.103)$$

Thus, as in the case of original Arthur-Kelly model, the fidelities are independent of the initial system state[25].

Again, because of the parity of various terms present there η_f turns out to be the same as η_i .

From the probability normalisation condition given in eqn. (6.15) we have $\eta_d = \frac{1}{2}$, *independent of initial apparatus state and* η_i ,η_f .

This shows that any error-disturbance relationship between error of retrodiction/ prediction and error of disturbance does not hold for all choices of the direction pointer observable.

6.8 Approximate joint measurement for three qubit observables

An approximate joint measurement observable for three unsharp qubit observables is obtained from (2.10) in the obvious way. Unlike the two observable case no necessary-sufficient condition for the approximate joint measurement on the marginal effect parameters in $\mathcal{R}^4$ (like the ones derived for example in [17]) is known. No measurement uncertainty relation like equation (2.25) , but which is stronger than the relations applied pairwise is known. Here, we derive a necessary condition on the approximate joint measurement of three qubit observables which yields the familiar necessary-sufficient condition known in literature for the case of mutually orthogonal unbiased observables (to be discussed below). We show that it holds even when the observables are biased. Also when one of the measurements is trivial i.e the corresponding marginal effect is $\frac{I}{2}$ representing equiprobable guesses of the values of the observable , our condition reproduces eqn. (2.24).

The eight joint effects G_{+++}, G_{++-},..........,G_{---} have to satisfy six marginality conditions like

$$\Upsilon^1_+ = G_{+++} + G_{++-} + G_{+-+} + G_{+--} \qquad (6.104)$$

A joint measurement of three observables gives rise to three two-observable joint measurements. Let G^{12}_{++} denote the joint measurement marginal effect corresponding to the outcome $(++)$ in directions 1 and 2. Then seven of the the effects (G_{+++},G_{++-} , etc.) can be written in terms of the three single-observable marginal effects , three pairwise joint measurement marginal effects and G_{+++}, as follows :

$$\begin{aligned} G_{++-} &= G^{12}_{++} - G_{+++}, && (6.105)\\ G_{-+-} &= G^{23}_{+-} - G^{12}_{++} + G_{+++}, && (6.106)\\ G_{--+} &= G^{13}_{-+} + G^{23}_{+-} + G_{+++} - \Upsilon^2_{+}, && (6.107)\\ G_{-++} &= \Upsilon^2_{+} - G^{23}_{+-} - G_{+++}, && (6.108)\\ G_{+-+} &= G^{13}_{++} - G_{+++}, && (6.109)\\ G_{---} &= G^{23}_{++} + G^{12}_{--} - G^{13}_{-+} - G_{+++}, && (6.110)\\ G_{+--} &= G^{12}_{+-} - \Upsilon^3_{+} + G^{13}_{-+} + G_{+++}. && (6.111) \end{aligned}$$

It was shown in [17] that any set of effects for joint measurement of two observables can be written in the form

$$G_{sgn(a)sgn(b)}(Z, \vec{z}) = \frac{(1 + ax + by + abZ)I + (ab\vec{z} + a\vec{m} + b\vec{n}).\vec{\sigma}}{4} \qquad (6.112)$$

with $a, b \in \{1, -1\}$, all other scalars in $\mathcal{R}$, all vectors in $\mathcal{R}^3$ and the two single-qubit marginal effects given by , $\hat{\Upsilon}_{\pm}(x, \vec{m}) = \frac{1}{2}((1 \pm x)I \pm \vec{m}.\vec{\sigma})$ and $\hat{\Upsilon}_{\pm}(y, \vec{n}) = \frac{1}{2}((1 \pm y)I \pm \vec{n}.\vec{\sigma})$ with positivity constraint being $|x| + m \leq 1$ and $|y| + n \leq 1$. Condition 6.112 is true because for the two observable case all the effects can be expressed in terms of one effect (say G_{++}) and the marginals ([22]) . The freedom in $(Z, \vec{z})$ suffices to specify an arbitrary G_{++}.

It then follows from equation 6.112 that the set of joint effects for the three observable case is of the form,

$$\begin{aligned} G_{sgn(a)sgn(b)sgn(c)}([Z], [\vec{z}]) \;=\; & ((1 + ax + by + cz + abZ_1 + bcZ_2 + caZ_3 + abcZ_4)I + \\ & (ab\vec{z_1} + bc\vec{z_2} + ca\vec{z_3} + abc\vec{z_4} + a\vec{l} + b\vec{m} + c\vec{n}).\vec{\sigma})/8 \end{aligned} \qquad (6.113)$$

with, $a, b, c \in \{1, -1\}$, the two-marginals $G_{12} = G(\vec{z_1}, Z_1)$, $G_{23} = G(\vec{z_2}, Z_2)$, $G_{13} = G(\vec{z_3}, Z_3)$ given by eqn. (6.112), and the one marginals being $\hat{\Upsilon}_{\pm}(x, \vec{l}) = \frac{1}{2}((1 \pm x)I \pm \vec{l}.\vec{\sigma})$, $\hat{\Upsilon}_{\pm}(y, \vec{m}) = \frac{1}{2}((1 \pm y)I \pm \vec{m}.\vec{\sigma})$, $\hat{\Upsilon}_{\pm}(z, \vec{n}) = \frac{1}{2}((1 \pm z)I \pm \vec{n}.\vec{\sigma})$. Given the positivity of the one marginals , the positivity of the eight joint effects places restrictions on the marginal effect parameters which are interpreted as measurement uncertainty relations in contrast to the usual state uncertainty relations.

6.8.1 Necessary condition

From eqn. (6.113) , $G_{+++} \geq 0$ gives,

$$||\vec{l} + \vec{m} + \vec{n} + \vec{z_1} + \vec{z_2} + \vec{z_3} + \vec{z_4}|| \leq 1 + x + y + z + Z_1 + Z_2 + Z_3 + Z_4 \qquad (6.114)$$

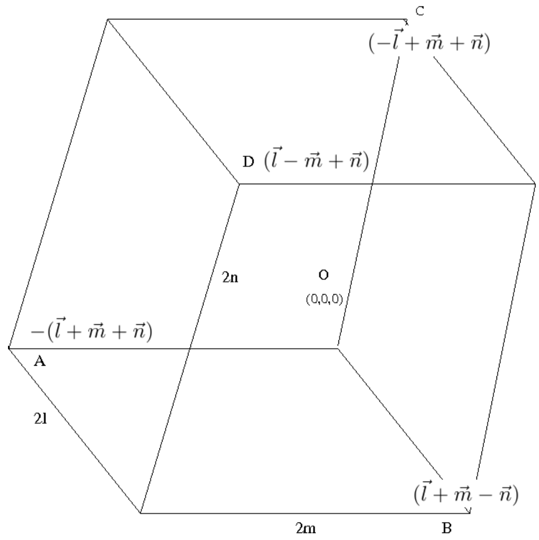


Figure 6.9: Geometric interpretation of inequalities (6.117) - (6.119) . Solid spheres about the centers A (with $\vec{OA} = -(\vec{l}+\vec{m}+\vec{n})$) , B (with $\vec{OB} = (\vec{l}+\vec{m}-\vec{n})$), C (with $\vec{OC} = (-\vec{l}+\vec{m}+\vec{n})$) and D (with $\vec{OD} = (\vec{l}-\vec{m}+\vec{n})$) and radii $(1+Z_1+Z_2+Z_3)$, $(1+Z_1-Z_2-Z_3)$, $(1-Z_1-Z_2+Z_3)$, $(1-Z_1+Z_2-Z_3)$ respectively have to intersect so that the sum of their radii is 4.

while $G_{---} \geq 0$ gives,

$$||-\vec{l}-\vec{m}-\vec{n}+\vec{z_1}+\vec{z_2}+\vec{z_3}-\vec{z_4}|| \leq 1-x-y-z+Z_1+Z_2+Z_3-Z_4 \quad (6.115)$$

Equations (6.114) and (6.115) together can be interpreted as the collection of all points $(\vec{z_1}+\vec{z_2}+\vec{z_3}) \in \mathcal{R}^3$, each of which lies within two spheres in $\mathcal{R}^3$ with their centers at $(\vec{l}+\vec{m}+\vec{n}+\vec{z_4})$ and $-(\vec{l}+\vec{m}+\vec{n}+\vec{z_4})$ and radii $(1-x-y-z+Z_1+Z_2+Z_3-Z_4)$, $(1+x+y+z+Z_1+Z_2+Z_3+Z_4)$ respectively. Thus, the distance between their centers should be less than or equal to the sum of their radii implying,

$$||-\vec{l}-\vec{m}-\vec{n}-\vec{z_4}|| \leq 1+Z_1+Z_2+Z_3 \quad (6.116)$$

Similarly, by consideration of the other complementary pairs G_{pqr} and $G_{(-p)(-q)(-r)}$ for $p,q,r=\pm$ we get three other equations,

$$\begin{aligned} ||\vec{l}+\vec{m}-\vec{n}-\vec{z_4}|| &\leq 1+Z_1-Z_2-Z_3, \quad (6.117)\\ ||\vec{l}-\vec{m}+\vec{n}-\vec{z_4}|| &\leq 1-Z_1-Z_2+Z_3, \quad (6.118)\\ ||-\vec{l}+\vec{m}+\vec{n}-\vec{z_4}|| &\leq 1-Z_1+Z_2-Z_3. \quad (6.119)\end{aligned}$$

6.8.2 Geometric interpretation

The above inequalities (6.116) - (6.119) imply that we have a cuboid with edges $2\vec{l}$, $2\vec{m}$ and $2\vec{n}$ with origin at the centre of this cuboid and solid spheres about points

A $(-\vec{l}-\vec{m}-\vec{n})$, B$(\vec{l}+\vec{m}-\vec{n})$, C $(-\vec{l}+\vec{m}+\vec{n})$ and D $(\vec{l}-\vec{m}+\vec{n})$have to intersect so that the sum of the radii of the spheres is 4 (see fig. 5). That is we necessarily have:

$$||-\vec{l}-\vec{m}-\vec{n}-\vec{z_4}||+||\vec{l}+\vec{m}-\vec{n}-\vec{z_4}||+||\vec{l}-\vec{m}+\vec{n}-\vec{z_4}||+||-\vec{l}+\vec{m}+\vec{n}-\vec{z_4}|| \leq 4 \quad (6.120)$$

Now, for any given set of points in $\mathcal{R}^3$, the Fermat-Torricelli (F-T) point of the set is defined to be the point which minimises the sum of distances from that point in $\mathcal{R}^3$ to the points of the set. The F-T point has been studied for quite long and its properties for any set of four non-coplanar points are well known. ([52]). For example, it is known that the F-T point either belongs to the set itself or else it is the point at which the gradient of the sum of distances vanish. Thus choosing $\vec{z_4}$ to be the F-T point of the set of points A,B,C,D one necessarily has (according to the definition of the F-T point),

$$||-\vec{l}-\vec{m}-\vec{n}-\vec{z}_{4F}||+||\vec{l}+\vec{m}-\vec{n}-\vec{z}_{4F}||+||\vec{l}-\vec{m}+\vec{n}-\vec{z}_{4F}||+||-\vec{l}+\vec{m}+\vec{n}-\vec{z}_{4F}|| \leq 4, \quad (6.121)$$

where $\vec{z}_{4F}$ denotes the F-T point.

6.8.3 Sufficiency

In order to evaluate the F-T point for a particular set of points A,B,C,D defined above we will use the following theorem from ref. [52] (Theorem 2.1), due to Lorentz Lindelf and Sturm.

Theorem 2: Suppose $\vec{z_4}$ is the F-T point for a set of points $S_n \equiv \{\vec{a_i} \in \mathcal{R}^3 : i = 1, 2, ...n\}$. Then either $\vec{z_4}$ belongs to the set S_n or $\vec{z_4} \notin S_n$.

i) If $\vec{z_4} \in S_n$ then for $\vec{z_4} = \vec{a_j}$, for some $j \in \{1, 2, ...n\}$, with $||\sum_{i=1(i\neq j)}^{n} \frac{(\vec{a_i}-\vec{a_j})}{||\vec{a_i}-\vec{a_j}||}|| \leq 1$.

ii) If $\vec{z_4} \notin S_n$ then $\vec{z_4}$ is the point at which $||\sum_{i}^{n} \frac{(\vec{a_i}-\vec{z_4})}{||\vec{a_i}-\vec{z_4}||}|| = 0$

Thus, for the first case the resultant of unit vectors to the FT point from other points of the set has magnitude less than 1. In the second case the unit vectors from the F-T point to the points of the set add up to the null vector. The condition for the second case is also the condition for the gradient of the function representing the sum of distances from $\vec{z_4}$ to $\vec{a_i}$, for $i = 1, 2, ...n$, to vanish at $\vec{z_4}$.

$\vec{l}$, $\vec{m}$, $\vec{n}$ mutually orthogonal

Suppose we are considering the case of approximate joint measurement in three orthogonal directions $\vec{l}$, $\vec{m}$ and $\vec{n}$. In this case we have, $||-\vec{l}-\vec{m}-\vec{n}|| = ||\vec{l}+\vec{m}-\vec{n}|| = ||\vec{l}-\vec{m}+\vec{n}|| = ||-\vec{l}+\vec{m}+\vec{n}||$. Hence, at the origin($\vec{z_4} = \vec{0}$) of the cuboid (see fig.

(6.9)) we have,

$$\sum_i \frac{(\vec{a_i}-\vec{z_4})}{||\vec{a_i}-\vec{z_4}||} = (-\vec{l}-\vec{m}-\vec{n}+\vec{l}-\vec{m}+\vec{n}+\vec{l}+\vec{m}-\vec{n}-\vec{l}+\vec{m}+\vec{n})/(l^2+m^2+n^2) = 0 \tag{6.122}$$

Hence equation (6.121) gives,

$$l^2+m^2+n^2 \leq 1 \tag{6.123}$$

This condition was shown to be sufficient in [53] . The necessity of this condition was shown in [54] assuming unbiasedness of the marginals (i.e $x = y = z = 0$ in eqn. (6.113)) by considering measurements by two parties on a singlet state. We have shown above that this is true for biased cases as well. Sufficiency of the above condition (6.123) shows that condition (6.121) is sufficient for the case of approximate joint measurement in three orthogonal directions. Also note that condition (6.123) is stronger than pairwise conditions for two-observable joint measurement in orthogonal directions like (6.27) which when added produces the bound of 1.5 on the lhs of equation (6.123).

Reduction to two-observable inequality

Suppose our joint measurement scheme is such that some approximate joint measurement on two observables is performed while the value of the third observable is guessed with probability of $+$ being $\frac{1+z}{2}$ and that of - being $\frac{1-z}{2}$. This will correspond to $\vec{n}=0$ i.e the corresponding marginal $\Upsilon_\pm(z,\vec{n}) = \frac{(1\pm z)I}{2}$. In this case the points A$(-\vec{l}-\vec{m})$, B$(\vec{l}+\vec{m}$), C$(\vec{l}-\vec{m})$ and D $(-\vec{l}+\vec{m})$ form a parallelogram of length $|2\vec{l}|$ and $|2\vec{m}|$ about the origin O . OA and OB lie opposite to each other and so does OC and OD. Thus, condition (ii) of the theorem 2 is satisfied and the origin is the F-T point. Hence, condition (6.121) reproduces equation (2.24) which shows that the bounds on the unsharpnesses is not more stringent than the two -observable case, as is to be expected.

Unbiased observables

The condition given by eqn. (6.121) also turns out to be sufficient for unbiased observables. The following choice of parameters is made to construct the particular joint observable in the form given by eqn. (6.113): $\vec{z_1} = \vec{z_2} = \vec{z_3} = 0$, $\vec{z_4} = \vec{z_{4F}}$, $Z_4 = 0$, $x, y, z = 0$(by definition of unbiased observables) and $Z_1 = 1 - \frac{||\vec{l}-\vec{m}+\vec{n}-\vec{z_4}_F||+||-\vec{l}+\vec{m}+\vec{n}-\vec{z_4}_F||}{2}$, $Z_2 = 1-\frac{||\vec{l}+\vec{m}-\vec{n}-\vec{z_4}_F||+||\vec{l}-\vec{m}+\vec{n}-\vec{z_4}_F||}{2}$, $Z_3 = 1-\frac{||-\vec{l}+\vec{m}+\vec{n}-\vec{z_4}_F||+||\vec{l}+\vec{m}-\vec{n}-\vec{z_4}_F||}{2}$. By this choice we obtain equality in conditions 6.117-6.119 , while inequality is obtained for condition given by eqn. 6.116 by using equalitites in eqn. 6.117-6.119 and 6.121 .

6.8.4 Arthur-Kelly model

The Arthur-Kelly model for the case of joint measurement of spin along three directions proceeds exactly as that of joint measurement of spin along two directions with the Hamiltonian of the measurement interaction of the form, $H = -(q_1 \otimes \sigma_x + q_2 \otimes \sigma_y + q_3 \otimes \sigma_z)$. The nature of the interaction is impulsive as in eqn. (6.1) and the corresponding unitary evolution is given by,

$$U = e(q_1, q_2, q_3) \otimes \sigma_x + f(q_1, q_2, q_3) \otimes \sigma_y + g(q_1, q_2, q_3) \otimes \sigma_z. \tag{6.124}$$

with,

$$\begin{aligned}
e(q_1, q_2, q_3) &= cos(\sqrt{q_1^2 + q_2^2 + q_3^2}) && (6.125)\\
f(q_1, q_2, q_3) &= iq_1 \frac{sin(\sqrt{q_1^2 + q_2^2 + q_3^2})}{\sqrt{(q_1^2 + q_2^2 + q_3^2)}}, && (6.126)\\
g(q_1, q_2, q_3) &= iq_2 \frac{sin(\sqrt{q_1^2 + q_2^2 + q_3^2})}{\sqrt{(q_1^2 + q_2^2 + q_3^2)}}, && (6.127)\\
h(q_1, q_2, q_3) &= iq_3 \frac{sin(\sqrt{q_1^2 + q_2^2 + q_3^2})}{\sqrt{(q_1^2 + q_2^2 + q_3^2)}}. && (6.128)
\end{aligned}$$

The initial joint state of the three detectors and system is $|\psi_{in}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \otimes |\chi\rangle$. As before, the initial detector states are Gaussians: $\psi_1(q_1) = (\frac{1}{\sqrt{2\pi}} e^{-\frac{q_1^2}{2a^2}})^{\frac{1}{2}}$, $\psi_2(q_2) = (\frac{1}{\sqrt{2\pi}} e^{-\frac{q_2^2}{2b^2}})^{\frac{1}{2}}$ and $\psi_3(q_3) = (\frac{1}{\sqrt{2\pi}} e^{-\frac{q_3^2}{2c^2}})^{\frac{1}{2}}$.

The detector outcome (($p_1 \geq 0, p_2 \geq 0, p_3 \geq 0$) is taken here to correspond to the outcome (+,+,+) for the joint unsharp measurement of the system observables σ_x, σ_y, σ_z ; ($p_1 \geq 0, p_2 \geq 0, p_3 \leq 0$) to (+,+,-) and so on. The POVM elements corresponding to the outcomes (+++), (++-) , etc. of the joint unsharp measurement of σ_x , σ_y and σ_z have a similar structure to that for the case of two observables : $(+++) \leftrightarrow \frac{1}{8}(I + \frac{a'}{8}\sigma_x + \frac{b'}{8}\sigma_y + \frac{c'}{8}\sigma_z)$, $(++-) \leftrightarrow \frac{1}{8}(I + \frac{a'}{8}\sigma_x + \frac{b'}{8}\sigma_y - \frac{c'}{8}\sigma_z)$,...... $(---) \leftrightarrow \frac{1}{8}(I - \frac{a'}{8}\sigma_x - \frac{b'}{8}\sigma_y - \frac{c'}{8}\sigma_z)$ with,

$$\begin{aligned}
a' &= \int_{p_1=0}^{+\infty} \int_{p_2=-\infty}^{+\infty} \int_{p_3=-\infty}^{+\infty} 4(f^0 e^0) dp_1 dp_2 dp_3, && (6.129)\\
b' &= \int_{p_2=0}^{+\infty} \int_{p_1=-\infty}^{+\infty} \int_{p_3=-\infty}^{+\infty} 4(g^0 e^0) dp_1 dp_2 dp_3, && (6.130)\\
c' &= \int_{p_3=0}^{+\infty} \int_{p_1=-\infty}^{+\infty} \int_{p_2=-\infty}^{+\infty} 4(h^0 e^0) dp_1 dp_2 dp_3. && (6.131)
\end{aligned}$$

with e^0, f^0, g^0, h^0 being the fourier transforms of e, f, g, h .

6.8.4.1 Symmetric case with a=b=c

With the initial joint pointer state a symmetric Gaussian, as in the two observable case, we have for the unsharp measurement along any three directions $\hat{n}, \hat{m}, \hat{l}$, the

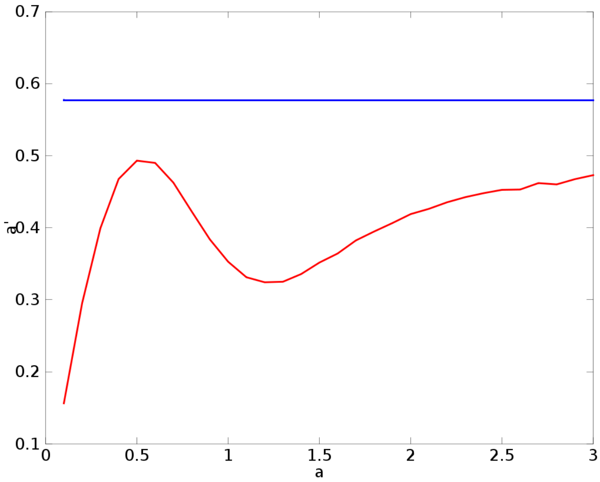


Figure 6.10: A plot of a' vrs a for symmetric Gaussian initial state

marginal effects given by $\Upsilon_+^{\hat{n}} = \frac{1}{2}(I + a'\hat{n}.\vec{\sigma})$, $\Upsilon_+^{\hat{m}} = \frac{1}{2}(I + a'\hat{m}.\vec{\sigma})$ and $\Upsilon_+^{\hat{l}} = \frac{1}{2}(I + a'\hat{l}.\vec{\sigma})$. Thus from eqn. (6.123) we have a' to be bounded by $\frac{1}{\sqrt{(3)}} = 0.577$. Numerically , for our scheme, a' is seen to be able to reach up to about 0.49 . It cannot reach the bound 0.577,as for two-observable joint measurement.(see fig. 6).

For approximate joint measurement in the directions $\hat{l} = (1, 0, 0)$, $\hat{m} = (cos(\phi), sin(\phi), 0)$, $\hat{n} = (sin(\theta)cos(\phi_1), sin(\theta)sin(\phi_1), cos(\theta))$ with $\theta = 0.414\pi$, $\phi_1 = 0.159\pi$ and $\phi = 0.477\pi$, the Fermat- Toricelli point of the set of points A,B,C,D, as defined before, is seen to be at $\vec{l} + \vec{m} - \vec{n}$. This yields, from inequality (6.121), $a' \leq 0.667$. Thus, as for the two observable case, considerably more freedom is available for unbiased measurement in non-orthogonal directions which our scheme cannot take advantage of.

6.9 Discussion

Non-commuting observables cannot be measured jointly. However it is possible in the POVM formalism to do joint measurements of (generally non-commuting) observables which are approximations of the actual non-commuting observables and called unsharp observables. We have considered in this chapter approximate joint measurement of two and three qubit observables separately, through an Arthur-Kelly like model for qubit observables.This model comes naturally when one considers a Stern-Gerlach setup with a linear magnetic field. In the Stern-Gerlach setup the momenta of the atoms acts as pointer observables for their spin degrees of freedom. Considering approximate joint measurement of σ_x and σ_y through this model, we have shown here numerically that the measurement uncertainty relations derived elsewhere(see ref.[22]) hold. It has also been shown how increasing the relative sharpness of the initial momentum wavefunctions of the detectors leads measurement of one observable to become almost sharp while that of the other to become almost trivial. Effect of initial detector states on the post-measurement state of the system has also been considered. The action of the measurement interaction on the system turns out to be that of an asymmetric depolarising channel. This forms the basis of a physical understanding of the origin of complementarity (between σ_x and σ_y measurement)in the model. We also see an indication of the entanglement between the system and detectors increasing as one of the measurements becomes sharper.

We have considered two different characterisations of unsharpness. Firstly, by comparing the probability distribution of the values of observable to be approximately measured with that of the approximate observable. Secondly, by considering the alignment of the momentum direction with the spin observable in the Heisenberg picture. We have shown that for the case in which both the observables are approximated equally well, the corresponding measures of unsharpness are proportional. For our choice of pointer observable the measures checking the alignment and disturbance due to measurement do not seem to satisfy an error-disturbance relationship.

We have expounded the connection between the symmetries of the underlying Hamiltonian for measurement interaction and initial detector states with the joint measurement in a lemma.This was first stated by Martens et al.([46]). It has then been used to perform approximate joint measurement in arbitrary directions. The POVM elements calculated for the same match with that which were found earlier in the orthogonal case. They also turn as expected to that of a single unsharp measurement when the directions are taken to be almost same.

For the case of joint unsharp measurement of three qubit observables we have given a necessary condition to be satisfied by the parameters of the marginal POVM elements . This condition has been derived from certain geometric considerations involving the so called Fermat-Toricelli point . The condition is sufficient for the case of three orthogonal observables and identical to the only necessary-sufficient condition known for three -observable joint measurements. Our proof shows that this also holds for biased unsharp measurements , namely those in which the probability of obtaining 'up' or 'down' is different for a maximally mixed state.

The measurement scheme employed by us for joint measurement in non-orthogonal directions cannot take advantage of the greater freedom available for better approximation compared to the orthogonal case. It will be interesting to see such a scheme in the Arthur-Kelly setup that is able to get close to the bound set by eqn.(2.24) for arbitrary directions. We have shown that the retrodictive fidelity η_i and the unsharpness a' are proportional for the symmetric joint unsharp measurement case that we have considered. It may be possible to connect the two pictures in a more general setting starting with certain symmetries of the Hamiltonian and initial detector states. The problem of determining necessary-sufficient conditions for the most general joint measurement of three observables by extension of our approach or otherwise is also left open.

Chapter 7

Non-locality breaking qubit channels

In this chapter we investigate qubit channels through their property of 'non-locality breaking', defined in a natural way but within the purview of CHSH nonlocality. This also provides a different perspective on the relationship between entanglement and nonlocality through the dual picture of quantum channels instead of through states. For a channel to be entanglement breaking it is sufficient to 'break' the entanglement of maximally entangled states. We provide examples to show that for CHSH nonlocality breaking such a property does not hold in general, though for certain channels and for a restricted class of states for all channels this holds.We also consider channels whose output remains local under SLOCC and call them 'strongly non-locality breaking'. We provide a closed form necessary-sufficient condition for any two-qubit state to show hidden CHSH nonlocality. This in turn allows us to characterize all strongly non-locality breaking qubit channels. It turns out that unital qubit channels breaking nonlocality of maximally entangled states are strongly non-locality breaking while extremal qubit channels cannot be so unless they are entanglement breaking.

7.1 Non-locality breaking and strongly non-locality breaking channels

As described in chapter 3 a Bell inequality in a scenario consisting of Δ measurements per site and m outcomes per measurement can be written as a hyperplane separation condition in a $t = 2(\Delta-1)m+(\Delta-1)^2m^2$ dimensional subspace of $\mathcal{R}^{\Delta^2m^2}$(the space of Δ^2m^2 dimensional probability vectors $\vec{p}$ with components $p(ab|xy)$)as,

$$\underline{\mathrm{s}}.\underline{\mathrm{p}} = \sum_{abxy} s^{ab}_{xy} p(ab|xy) \leq S_k \tag{7.1}$$

with the inequality being satisfied by all probability vectors $\vec{p}$ satisfying eqn. (3.1) and violated otherwise(see [20] for more details).

Defn.:

(i) A channel $\$_A : \mathcal{B}(\mathcal{C}^d) \rightarrow \mathcal{B}(\mathcal{C}^{d'})$, is said to be *non-locality breaking*, if acting on side A of *any* bipartite state ρ_{BA}, it produces a state $\rho'_{BA} = (I \otimes \$_A)(\rho_{BA})$ which satisfies an inequality/inequalities of the form of eqn. 7.1 .

(ii)Again, it is said to be *strongly non-locality breaking* if under any SLOCC operation Ω , $\Omega(\rho'_{BA})$ satisfies the same inequality/inequalities for an arbitrary choice of ρ_{BA}.

Remark: Both the definitions are w.r.t a set of inequalities like eqn. (7.1) and for *genuine*, non-locality breaking or non-locality breaking in the stronger sense one should consider the locality criterion(eqn. 3.1 in intro chapter).

7.1.1 Qubit channels and CHSH nonlocality

In this subsection we focus on qubit channels and non-locality breaking with respect to the Bell-CHSH inequality.

We begin by showing that for both the notions of non-locality breaking it is sufficient to consider only two-qubit pure states as input and single qubit filters. This follows from Lemma 1 and 2 proved below.

Suppose we have a qubit channel \$ that breaks non-locality of arbitrary two-qubit states $|\alpha\rangle \in \mathcal{C}^2 \otimes \mathcal{C}^2$ i.e.,

$$p(ab|xy) = Tr((I \otimes \$)(|\alpha\rangle\langle\alpha|)(M_{a|x} \otimes M_{b|y})), \tag{7.2}$$

satisfies eqn. (7.1) for all $|\alpha\rangle$ and any qubit POVMs $M_{a|x}$ and $M_{b|y}$ for all $x, y \in \{1, 2, ...\Delta\}$ and $a, b \in \{1, 2, .., m\}$.

The following Lemma tells us that the channel also breaks the non-locality of *any* qudit-qubit state.

Lemma 1: Let \$ be a qubit channel for which $p(ab|xy)$ given by eqn. (7.2) satisfies an inequality of the form (7.1) for an arbitrary two-qubit state $|\alpha\rangle$ and qubit POVMs $M_{a|x}$ and $M_{b|y}$ as described before. Then for any arbitrary state $\rho \in \mathcal{B}\left(\mathcal{C}^d \otimes \mathcal{C}^2\right)$ and arbitrary set of POVMs $\sum_a N_{a|x} = I_d$ and $\sum_b N_{b|y} = I_2$, $p'(a, b|x, y) = \mathrm{Tr}(((I \otimes \$)\rho)(N_{a|x} \otimes N_{b|y}))$ also satisfies eqn. (7.1).

Proof:

Consider any qudit-qubit state $|\beta\rangle \in \mathcal{C}^d \otimes \mathcal{C}^2$. Choosing the Schmidt basis for the state so that $|\beta\rangle = \sqrt{\lambda}|e_0 f_0\rangle + \sqrt{(1-\lambda)}|e_1 f_1\rangle$ where $\{|e_0\rangle, |e_1\rangle\}$ is a two-dimensional ONB in $\mathcal{C}^d$, $\{|f_0\rangle, |f_1\rangle\}$ an ONB of the qubit on which the channel acts and $\lambda \in [0, 1]$

we have $\rho_{\$,\beta} = (I \otimes \$)(|\beta\rangle\langle\beta|) \in \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{C}^2)$ where the first $\mathcal{C}^2$ is the space spanned by $\{|e_0\rangle, |e_1\rangle\}$. Again, suppose $\sum_a N_{a|x} = I_d$ and $\sum_b N_{b|y} = I_2$ are two sets of POVMs acting respectively on qudit and qubit systems. The probability of clicking of $N_{a|x}$ and $N_{b|y}$ for the system in the state $\rho_{\$,\beta}$ is given by,

$$\begin{aligned}
p'(a,b|x,y) &= \mathrm{Tr}(\rho_{\$,\beta}(N_{a|x} \otimes N_{b|y})) \\
&= \mathrm{Tr}((P_2 \otimes I_2)\rho_{\$,\beta}(P_2 \otimes I_2)(N_{a|x} \otimes N_{b|y})), (\text{where } P_2 = |e_0\rangle\langle e_0| + |e_1\rangle\langle e_1|) \\
&= \mathrm{Tr}((P_2 \otimes I_2)\rho_{\$,\beta}(P_2 \otimes I_2)((P_2 + (I_d - P_2)) \otimes I)(N_{a|x} \otimes N_{b|y})) \\
&= \mathrm{Tr}(\rho_{\$,\beta}((P_2 N_{a|x} P_2) \otimes N_{b|y})) \\
&= \mathrm{Tr}(\rho_{\$,\beta}(N'_{a|x} \otimes N_{b|y}))
\end{aligned} \tag{7.3}$$

with $N'_{a|x} = P_2 N_{a|x} P_2$ being a qubit POVM satisfying $\sum_a N'_{a|x} = I_2$. Thus $p'(ab|xy)$ with x and y indexing respectively a set of qudit and qubit POVMs, obtained from $|\beta\rangle$ through the action of $N_{a|x}$ and $N_{b|y}$ through eqn. (7.2) will satisfy the Bell inequality given by eqn. (7.1), due to \$ being non-locality breaking for all two-qubit pure states. The arguments extend to mixed qudit-qubit input states through convexity.

□

Let us move on to non-locality breaking in the stronger sense. Here we only consider inequalities with two two-outcome measurement settings per site (i.e, eqn. (7.1) with $\Delta = m = 2$). Let

$$\rho' = \frac{(A \otimes B)((I \otimes \$)(|\alpha\rangle\langle\alpha|))(A^\dagger \otimes B^\dagger)}{Tr((A^\dagger A \otimes B^\dagger B)(I \otimes \$)(|\alpha\rangle\langle\alpha|))} \tag{7.4}$$

with \$ and $|\alpha\rangle$ being a qubit channel and an arbitrary two-qubit pure state respectively and A, B being 2×2 complex matrices acting as qubit filters. Let us suppose now that $p(ab|xy) = Tr(\rho'(M_{a|x} \otimes M_{b|y}))$ obtained through qubit POVMs $\{M_{a|x} : a = 1, 2\}$, $\{M_{b|y} : b = 1, 2\}$ satisfies eqn. (7.1) for any $|\alpha\rangle$, A, B. The next Lemma shows that this condition ensures that \$ is strongly nonlocality breaking.

Lemma 2: Let \$ be an arbitrary qubit channel for which $p(ab|xy)$ obtained from any ρ' defined above satisfies eqn. (7.1) with $\Delta = m = 2$ i.e, two two-outcome measurement settings per site. Then \$ is strongly non-locality breaking w.r.t such an inequality.

Proof: To begin with, let us show that locality of ρ' in eqn. (7.4) is strong enough to ensure locality of $\rho_1 = (I \otimes \$)(|\alpha\rangle\langle\alpha|)$ for any SLOCC operation $\Omega : \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{C}^2) \to \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{C}^2)$. Let,

$$\Omega(\rho_1) = \frac{\sum_k (A_k \otimes B_k)\rho_1(A_k^\dagger \otimes B_k^\dagger)}{Tr(\sum_k (A_k^\dagger A_k \otimes B_k^\dagger B_k)\rho_1)} \tag{7.5}$$

with A_k, B_k being 2×2 complex matrices. Defining,

$p_k = Tr((A_k^\dagger A_k \otimes B_k^\dagger B_k)\rho_1))$ and $\rho'_k = (A_k \otimes B_k)\rho_1(A_k^\dagger \otimes B_k^\dagger)/p_k$ we have, $\Omega(\rho_1) = \frac{\sum_k p_k \rho'_k}{\sum_k p_k}$. As this is a convex combination for $\Omega(\rho_1)$ to violate inequality (7.1) we must have at least one ρ'_k which violates inequality (7.1) for some choice of measurement setting. The aforesaid condition on ρ' in eqn. (7.4) guarantees that this does not happen for any ρ'_k.

The general proof follows by contradiction . Let us assume that there is a SLOCC operation $\Omega' : \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{C}^2) \to \mathcal{B}(\mathcal{C}^d \otimes \mathcal{C}^d)$ under which $\rho_1 = (I \otimes \$)(|\alpha\rangle\langle\alpha|)$ violates inequality (7.1 with $\Delta = m = 2$) for some choice of measurement setting. Now by Result 2 of [33] (also shown in chapter 3)$\Omega'(\rho_1)$ can be transformed by a SLO (Ω_1, say)to a two-qubit state which violates inequality (7.1 with $\Delta = m = 2$) by an equal or larger amount. Thus $\Omega_1 \circ \Omega' = \Omega_2$ is an SLOCC for which $\Omega_2(\rho_1) \in \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{C}^2)$ violates inequality (7.1). As we saw in the previous paragraph, locality of ρ' in eqn. (7.4) ensures that this does not happen and hence we have a contradiction. The argument can be generalized to consider qudit-qubit mixed input states using Schmidt decomposition and convexity as before. If there exists a deterministic LOCC for which our proposition is not true then by using the representation of that LOCC with separable superoperators, like in eqn . (7.5), with the added restriction $\sum_k A_k^\dagger A_k \otimes B_k^\dagger B_k = I \otimes I$ we can construct a local filtering transformation with one of the pairs (A_k, B_k) for which our proposition will also be violated by convexity.

□

One of the most important properties of entanglement breaking channels proved in [3](also shown in chapter 4) is that for a channel acting on density operators of a d-dimensional system it is enough to look at separability of the dual state of the channel, $\rho_{\Phi^+,\$} = (I \otimes \$)(|\Phi^+\rangle\langle\Phi^+|)$,with $|\Phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$. For non-locality breaking we have no such thing. But the following Lemma shows that for "strongly non-locality breaking channels" within the purview of CHSH-nonlocality it is enough to check if the Choi-state of the channel (i.e.,$\rho_{\Phi^+,\$}$) show any hidden non-locality under a single local filtering operation.

Lemma 3: An arbitrary qubit channel is strongly non-locality breaking w.r.t an inequality of the form of eqn. (7.1) with $\Delta = m = 2$ iff ρ' defined in eqn. (7.4) satisfies (7.1) for arbitrary A, B and $|\alpha\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}\sum_{i=0}^{1}|ii\rangle$.

Proof: The most general two-qubit pure state in the Schmidt form is given by, $|\alpha\rangle = \sqrt{\lambda}|e_1 f_1\rangle + \sqrt{1-\lambda}|e_2 f_2\rangle = (U \otimes V)(\sqrt{\lambda}|00\rangle + \sqrt{(1-\lambda)}|11\rangle)$, with $\lambda \in [0,1]$ and the 2×2 unitary matrices U and V being given by: $U|0\rangle = |e_1\rangle, V|0\rangle = |f_1\rangle, U|1\rangle = |e_2\rangle$ and $V|1\rangle = |f_2\rangle$ with $\{|0\rangle, |1\rangle\}$ being the standard ONB for $\mathcal{C}^2$.

For $\lambda \in [0,1]$, let

$$W_\lambda = \sqrt{\lambda}|0\rangle\langle 0| + \sqrt{(1-\lambda)}|1\rangle\langle 1|. \tag{7.6}$$

Now, we have $(U \otimes V)(W_\lambda \otimes I)|\Phi^+\rangle = (\sqrt{\lambda}|e_1 f_1\rangle + \sqrt{1-\lambda}|e_2 f_2\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}}|\alpha\rangle$.

Now using the facts that a qubit channel Λ is a trace-preserving map and that for any operator A, $(I \otimes A)|\Phi^+\rangle = (A^T \otimes I)|\Phi^+\rangle$ it is easy to show that,

$$\begin{aligned} \rho_{\alpha,\Lambda} &= (I \otimes \Lambda)(|\alpha\rangle\langle\alpha|) \\ &= \frac{(A_1 \otimes I)\rho_{\Phi^+,\Lambda}(A_1^\dagger \otimes I)}{Tr((A_1^\dagger A_1 \otimes I)\rho_{\Phi^+,\Lambda})}, \end{aligned} \tag{7.7}$$

with the filter $A_1 = UW_\lambda V^T$, $\rho_{\Phi^+,\Lambda} = (I \otimes \Lambda)\left(|\Phi^+\rangle\langle\Phi^+|\right)$.

As local filtering operations can be composed it follows from Lemma 2 that in order to check if \$ is strongly non-locality breaking it is sufficient to check if $\rho_{\Phi^+,\Lambda}$ show any hidden non-locality.

□

Henceforth, a channel $\$_A$, acting on qubit A, will be said to be *non-locality breaking*, if acting on side A of *any* two-qubit state ρ_{BA}, it produces a state $\rho'_{BA} = (I \otimes \$_A)(\rho_{BA})$ which satisfies the Bell-CHSH inequality, *i.e.*, we have

$$M(\rho') \leq 1. \tag{7.8}$$

7.2 Breaking nonlocality of maximally entangled states

In this section we produce various examples to show that breaking non-locality of a maximally entangled state is typically not enough for a channel to be non-locality breaking for all states.

In the Hilbert-Schmidt basis , $\rho_{\Phi^+,\Lambda'}$ with Λ' being the canonical map in (eqn. (4.9 of introduction to quantum channel chapter) is given by,

$$\rho_{\Phi^+,\Lambda'} = \frac{1}{4}(I \otimes I + I \otimes \vec{t}.\vec{\sigma} + \lambda_1 \sigma_1 \otimes \sigma_1 - \lambda_2 \sigma_2 \otimes \sigma_2 + \lambda_3 \sigma_3 \otimes \sigma_3). \tag{7.9}$$

Now from eqn. (7.7) and (4.8) using $(I \otimes A)|\Phi^+\rangle = (A^T \otimes I)|\Phi^+\rangle$ we have

$$\rho_{\alpha,\Lambda} = (W_\lambda V^T \otimes I)\rho_{\Phi^+,\Lambda'}((W_\lambda V^T)^\dagger \otimes I)/tr((W_\lambda^\dagger W_\lambda \otimes I)|\Phi^+\rangle\langle\Phi^+|) \tag{7.10}$$

where we have neglected stray local unitaries which do not affect the Bell-violation of $\rho_{\alpha,\Lambda}$. We further have, on using $tr((W_\lambda^\dagger W_\lambda \otimes I)|\Phi^+\rangle\langle\Phi^+|) = \frac{1}{2}$,

$$\rho_{\alpha,\Lambda} = 2(W_\lambda \otimes I)\frac{1}{4}(I \otimes I + I \otimes \vec{t}.\vec{\sigma} + \lambda_1 V^T \sigma_1 (V^T)^\dagger \otimes \sigma_1 - \lambda_2 V^T \sigma_2 (V^T)^\dagger \otimes \sigma_2 + \lambda_3 V^T \sigma_3 (V^T)^\dagger \otimes \sigma_3)(W_\lambda^\dagger \otimes I). \tag{7.11}$$

Now let $V^T \sigma_i (V^T)^\dagger = \sum_{j=1}^3 R_{ij}\sigma_j$ where $R = (R_{ij})$ is the real rotation matrix in three dimensions corresponding to the matrix $V^T \in SU(2)$. Now using the action of W_λ on the basis elements $I, \sigma_i, i = 1, 2, 3$, we have from eqn. (7.11)

$$\rho_{\alpha,\Lambda} = \frac{1}{4}(I\otimes I + (2\lambda-1)\sigma_3\otimes I + I\otimes \vec{t}.\vec{\sigma} + \lambda_1 R_{13}(2\lambda-1)I\otimes\sigma_1 - \lambda_2 R_{23}(2\lambda-1)I\otimes\sigma_2 + \lambda_3 R_{33}(2\lambda-1)I\otimes\sigma_3 + \sum_{ij=1}^{3} t'_{ij}(\sigma_i\otimes\sigma_j)). \tag{7.12}$$

.

The entries for the correlation matrix t' for $\rho_{\alpha,\Lambda}$ are given by:

$$(T_{\rho_{\alpha,\Lambda}})_{ij} = \begin{pmatrix} 2\sqrt{\lambda(1-\lambda)}\lambda_1 R_{11} & -2\sqrt{\lambda(1-\lambda)}\lambda_2 R_{21} & 2\sqrt{\lambda(1-\lambda)}\lambda_3 R_{31} \\ 2\sqrt{\lambda(1-\lambda)}\lambda_1 R_{12} & -2\sqrt{\lambda(1-\lambda)}\lambda_2 R_{22} & 2\sqrt{\lambda(1-\lambda)}\lambda_3 R_{32} \\ \lambda_1 R_{13} + (2\lambda-1)t_1 & -\lambda_2 R_{23} + (2\lambda-1)t_2 & \lambda_3 R_{33} + (2\lambda-1)t_3 \end{pmatrix} \tag{7.13}$$

$$= \mathrm{Diag}(\alpha',\alpha',1)\begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix}\mathrm{Diag}(\lambda_1,-\lambda_2,\lambda_3) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (2\lambda-1)t_1 & (2\lambda-1)t_2 & (2\lambda-1)t_3 \end{pmatrix}$$

with $\alpha' = 2\sqrt{\lambda(1-\lambda)}$.

7.2.1 Non-locality breaking condition for maximally entangled states

The condition for a channel to break non-locality of a maximally entangled state follows from eqn. (7.8) by taking $R = I$ and $\lambda = \frac{1}{2}$ in eqn. (7.13) and is given by

$$\lambda_1^2 + \lambda_2^2 \leq 1, \tag{7.14}$$

assuming $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

7.2.2 Examples and counterexamples of universal non-locality breaking

In this subsection we provide three examples to show that breaking non-locality of maximally entangled state may or may not be sufficient to break non-locality of all states.

We choose the following channel parameters for non-unital qubit channels of the canonical form (4.9),

i)

$$\lambda_1 = \frac{1}{\sqrt{2}}, \lambda_2 = \frac{1}{\sqrt{2}}, \lambda_3 = \frac{1}{2}, t_1 = -0.12, t_2 = 0.047, t_3 = -0.210. \tag{7.15}$$

The channel with the above parameters saturates the non-locality breaking condition given by eqn.(7.14).

We choose $\lambda = 0.4$ (note $\lambda = 0.5$ corresponds to the maximally entangled state). Now, for any $R \in SO(3)$ we can write it (using Euler angles) as

$$R = R_z(\alpha_0)R_y(\beta_0)R_z(\gamma_0), \tag{7.16}$$

with $\alpha_0, \beta_0, \gamma_0 \in [0, 2\pi]$, $R_z(\alpha_0) = \begin{pmatrix} cos(\alpha_0) & -sin(\alpha_0) & 0 \\ sin(\alpha_0) & cos(\alpha_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

and $R_y(\beta_0) = \begin{pmatrix} cos(\beta_0) & 0 & sin(\beta_0) \\ 0 & 1 & 0 \\ -sin(\beta_0) & 0 & cos(\beta_0) \end{pmatrix}$.

We choose $\alpha_0 = 1.2, \beta_0 = 1.4, \gamma_0 = 3.5$,*i.e.*, resp. about 0.382π, 0.4456π and 1.11π .

For this choice of λ and R and the channel parameters given in eqn. (7.15) , we have for $\rho_{\alpha,\Lambda}$ in eqn. (7.12), $M(\rho_{\alpha,\Lambda}) = 1.01094$. Here R is the 3×3 real rotation matrix corresponding to the 2×2 $SU(2)$ matrix V^T appearing in the expression for $\rho_{\alpha,\Lambda}$ in eqn. (7.11).

Thus clearly the channel does not break the non-locality of the state arising from the action of $W_\lambda V^T \otimes I$ on the maximally entangled state $|\Phi^+\rangle$. However this channel does break the non-locality of all states of the form $\sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle$ as we have checked by taking $R = I$ and varying over λ. Hence, breaking non-locality of all entangled states with a given Schmidt basis is not enough.

ii)

$$\lambda_1 = 0.7, \lambda_2 = 0.71, \lambda_3 = 0.7, t_1 = 0.28, t_2 = 0.01, t_3 = -0.1 \tag{7.17}$$

.

We further choose $\lambda = 0.45$ and $R = I$ and we have for $\rho_{\alpha,\Lambda}$ in eqn. (7.12), $M(\rho_{\alpha,\Lambda}) = 1.0159$. Thus a channel breaking non-locality of a maximally entangled state may not even break it for all states with a given Schmidt basis.

iii) The *amplitude damping channel*: The vectors ($\vec{t}$ and $\vec{\lambda}$) of the amplitude-damping channel Φ (as defined in eqn.(4.9)) are given respectively by $(0, 0, p)$ and $(\sqrt{(1-p)}, \sqrt{(1-p)}, (1-p))$. Thus, to break non-locality of $|\Phi^+\rangle$ we must have from eqn. (7.14), $p \geq \frac{1}{2}$.

The correlation matrix for the state $\rho_{\alpha,\Lambda}$ for the amplitude-damping channel is given by,

$$\begin{aligned} T_{\rho_{\alpha,\Lambda}} &= \text{Diag}(\alpha,\alpha,1)\begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix}\text{Diag}(\sqrt{(1-p)},-\sqrt{(1-p)},(1-p)) \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (2\lambda-1)p \end{pmatrix}, \end{aligned} \tag{7.18}$$

with $\alpha = 2\sqrt{\lambda(1-\lambda)}$.

Now consider $|\tilde{\phi}(\lambda)\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle$ and let $\rho_1 = \rho_{\tilde{\phi}(\lambda),\Lambda}$(eqn. (7.11). So we have a diagonal correlation matrix $T_{\rho_1} = (t_{ij})$ with $t_{11} = 2\sqrt{\lambda(1-\lambda)(1-p)}$, $t_{22} = -2\sqrt{\lambda(1-\lambda)(1-p)}$ and $t_{33} = \lambda + (1-\lambda)(1-2p)$ .Thus the condition $M(\rho_1) = t_{11}^2 + t_{22}^2 \leq 1$ is satisfied. Also here,

$t_{11}^2 + t_{33}^2 = 4\lambda(1-\lambda)(1-p) + \lambda^2 + (1-\lambda)^2(1-2p)^2 + 2\lambda(1-\lambda)(1-2p)$, with the non-locality breaking condition $p \geq \frac{1}{2}$. We have therefore $t_{11}^2 + t_{33}^2 < 2\lambda(1-\lambda) + \lambda^2 + (1-\lambda)^2 = 1$. Thus we see that if the amplitude damping channel Φ breaks the non-locality of the maximally entangled state $|\Phi^+\rangle$, it then also breaks the non-locality in the states $|\tilde{\phi}(\lambda)\rangle = \sqrt{\lambda}|00\rangle + \sqrt{(1-\lambda)}|11\rangle$.

The maximal Bell violation for action of the amplitude damping channel on all pure entangled states is given by ,

$$M = \max_{\{\lambda\in[0,1],R\in SO(3)\}}\{\sigma_1(t')^2 + \sigma_2(t')^2\}, \tag{7.19}$$

where $\sigma_1(t')$ and $\sigma_2(t')$ denote the first two singular values in descending order of $T_{\rho_{\alpha,\Lambda}}$ in eqn.(7.18) .

Numerics: For the purpose of numerical investigation we take the same decomposition of R as in eqn. (7.16). The maximization in eqn.(7.19) has been done by choosing $\alpha_0, \beta_0, \gamma_0 \in [0, 2\pi]$ with an interval of 0.1 for each and $\lambda \in [0,1]$ with an interval of 0.05. Figure 7.1 shows the variation of M with respect to p varying between 0 and 1 . M is very close to $2(1-p)$ for $p \leq 0.5$ and exactly equals to 1 for all points $p > 0.5$. Thus we see from fig. 7.1 that the amplitude damping channel breaks non-locality of every state for $p \geq \frac{1}{2}$.

But as we will see in section V it fails to be strongly non-locality breaking for any value of p.

7.2.3 Channels breaking non-locality of maximally entangled states also break that of states whose free sided reduction is maximally mixed

It is however true that if for a qubit channel $\$_B$, $(I_A \otimes \$_B)(|\Phi^+\rangle\langle\Phi^+|)$ is a local state ($i.e$, eqn. (7.8) is satisfied) , then for any two-qubit state σ_{AB} , $(I_A \otimes \$_B)(\sigma_{AB})$ is

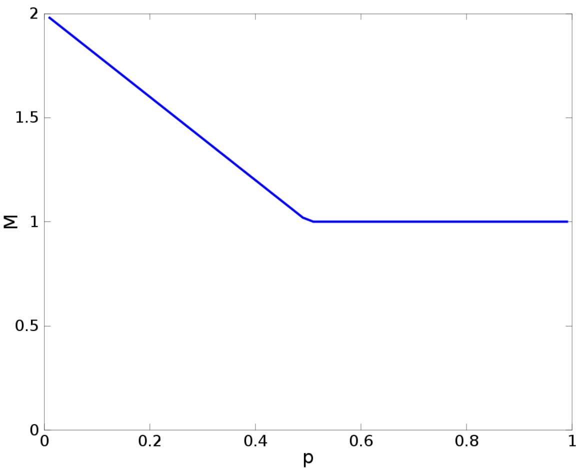


Figure 7.1: p vrs. M

also a local state provided $Tr_B(\sigma_{AB}) = \frac{I_2}{2}$, I_2 being the 2×2 identity matrix. This is proved in the Appendix.

In our aforesaid proof we see that $(I_A \otimes \$_B)((I_A \otimes \$_1)(|\Phi^+\rangle\langle\Phi^+|))$ is a local state if $(I_A \otimes \$_B)(|\Phi^+\rangle\langle\Phi^+|)$ is a local state. From the structure of the proof (see Appendix) it is also clear that $(I_A \otimes \$_1)((I_A \otimes \$_B)(|\Phi^+\rangle\langle\Phi^+|))$ is also a local state (we have to choose $A = R_W {D'_1}^2 R_W^T$, $B = D_1^2$ and use lemma 1). Thus we see that the composition of a qubit channel , that breaks non-locality of a maximally entangled state, with any other qubit channel also does the same job.

7.3 Stronger non-locality breaking

In this section we consider strongly non-locality breaking channels.As proved in Lemma 3 of Section III, for a channel Λ to be strongly non-locality breaking it is enough for the Choi-state of the channel $\rho_{\Phi^+,\Lambda}$ to not show any hidden non-locality.

Hence, building on the work done by Verstraete et. al in [29], [18] and [19] we first provide a necessary and sufficient condition for an arbitrary two-qubit state to show hidden CHSH non-locality (see chapter 3). This is proved in Theorem 1.

As shown by Theorem 2 of chapter 3 under a local filtering transformation taking any two-qubit state ρ to

$$\rho' = (A \otimes B)\rho(A^\dagger \otimes B^\dagger), \tag{7.20}$$

the real 4×4 matrix R with $R_{ij} = Tr(\rho\sigma_i \otimes \sigma_j), i, j = 0, 1, 2, 3$ (where $\sigma_0 = I_2$) transforms as

$$R' \equiv Tr(\rho'\sigma_i \otimes \sigma_j) = L_A R L_B^T |det(A)||det(B)| \tag{7.21}$$

with the Lorentz transformations L_A and L_B being given by,

$$\begin{aligned} L_A &= \frac{T(A \otimes A^*)T^\dagger}{|det(A)|}, \\ L_B &= \frac{T(B \otimes B^*)T^\dagger}{|det(B)|}, \end{aligned} \tag{7.22}$$

and $T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$, with the normalisation factor $R'_{00} = Tr(\rho') = Tr(A^\dagger A \otimes B^\dagger B \rho)$.

Remark: The filters should be of full rank, for the Lorentz transformations to be finite.

Further, let $C_\rho = MRMR^T$ with $M = \text{diag}(1, -1, -1, -1)$.

Theorem 1: Let $\lambda_i(C_\rho), (i = 0, 1, 2, 3)$ denote the eigenvalues of C_ρ in descending order for an arbitrary two-qubit state ρ. Then, ρ shows hidden CHSH nonlocality iff

$$\lambda_1(C_\rho) + \lambda_2(C_\rho) > \lambda_0(C_\rho). \tag{7.23}$$

The maximum Bell violation obtained from the optimal filtered (or quasi-distilled) Bell-diagonal state being $2\sqrt{\frac{(\lambda_1(C_\rho)+\lambda_2(C_\rho))}{\lambda_0(C_\rho)}}$.

Proof:

As shown by Theorem 3 of chapter 3([18] and [19]) that by suitably choosing A and B and hence proper orthochronous Lorentz transformations L_A , L_B for *any* ρ we can have R' to be either diagonal corresponding to a Bell-diagonal state ρ' or of the form,

$$R' = R_{\rho'} = \begin{bmatrix} a & 0 & 0 & b \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ c & 0 & 0 & (b+c-a) \end{bmatrix} \tag{7.24}$$

with the corresponding ρ'(unnormalized) being

$$\rho' = \frac{1}{2} \begin{bmatrix} b+c & 0 & 0 & 0 \\ 0 & a-b & d & 0 \\ 0 & d & (a-c) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.25}$$

The possible sets of real values of b, c and d are given by,

$$\begin{array}{ll}
\text{(i)} & b = c = \frac{a}{2}, \\
\text{(ii)} & (d = 0 = c) \text{ and } (b = a), \\
\text{(iii)} & (d = 0 = b) \text{ and } (c = a), \\
\text{(iv)} & (d = 0) \text{ and } (a = b = c).
\end{array} \tag{7.26}$$

Case (i) corrsponds to rank three or two states while the other cases corrrespond to either the product states $|00\rangle\langle 00|$ or the state $|0\rangle\langle 0| \otimes \frac{I}{2}$.

From eqn. (7.21) it follows that the spectrum of $MR'MR'^T$ is given by

$$\lambda(MR'MR'^T) = |det(A)|^2|det(B)|^2\lambda(ML_ARL_B^TML_BR^TL_A^T) = |det(A)|^2|det(B)|^2\lambda(MRMR^T), \tag{7.27}$$

where we have used $L_A^TML_A = M = L_B^TML_B$. Now as the filters A and B are of full rank i.e, $det(A), det(B) \neq 0$ we have for each for each $i \in \{0,1,2,3\}$

$$\frac{\lambda_i(C_{\rho'})}{\lambda_0(C_{\rho'})} = \frac{\lambda_i(C_\rho)}{\lambda_0(C_\rho)}. \tag{7.28}$$

Let us consider the following cases now.

(a) $R' = \text{diag}(s_0, s_1, s_2, s_3)$. ρ' corresponds to a Bell-diagonal state which in turn violates the Bell-CHSH inequality ([28]) after normalization provided,

$$1 < \frac{s_1^2}{s_0^2} + \frac{s_2^2}{s_0^2} = \frac{\lambda_1(C_{\rho'}) + \lambda_2(C_{\rho'})}{\lambda_0(C_{\rho'})} = \frac{\lambda_1(C_\rho) + \lambda_2(C_\rho)}{\lambda_0(C_\rho)}.$$

(by eqn. (7.28)). This proves Thoerem 1 for this case.

(b) ρ' is of the non Bell-diagonal form with $d \neq 0$ in eqn. (7.25) (case (i) of eqn. (7.26)) .

It is easy to see by partial transposition that ρ' must be entangled .

Further by using filters of the form of $A = \text{diag}(\sqrt{\frac{(a-c)}{(a-b)}}\frac{1}{n}, 1)$ and $B = \text{diag}(\frac{1}{n}, 1)$ we have,

$$\begin{aligned}
\rho_1 &= (A \otimes B)\rho(A^\dagger \otimes B^\dagger) \\
&= \frac{1}{2}\Bigg(\frac{(b+c)(a-c)}{(a-b)n^4}|00\rangle\langle 00| + \frac{(a-c)}{n^2}(|01\rangle\langle 01| + |10\rangle\langle 10|) \\
&+ \frac{d\sqrt{(a-c)}}{n^2\sqrt{(a-b)}}(|01\rangle\langle 10| + |10\rangle\langle 01|)\Bigg).
\end{aligned} \tag{7.29}$$

By taking a very large positive no. n , $\rho_2 = \frac{\rho_1}{Tr(\rho_1)}$ can be made to approach arbitrarily close to the Bell-diagonal state

$$\begin{aligned}\rho_3 &= \frac{1}{2}((|01\rangle\langle 01| + |10\rangle\langle 10|) + \frac{d}{\sqrt{(a-b)(a-c)}}(|01\rangle\langle 10| + |10\rangle\langle 01|)) \\ &= \frac{1}{4}(I \otimes I + \frac{d}{\sqrt{(a-c)(a-b)}}\sigma_1 \otimes \sigma_1 + \frac{d}{\sqrt{(a-c)(a-b)}}\sigma_2 \otimes \sigma_2 \\ &- \sigma_3 \otimes \sigma_3).\end{aligned} \tag{7.30}$$

Now, from eqn. (7.25) we have $\lambda(C_{\rho'}) = [(a-b)(a-c), (a-b)(a-c), d^2, d^2]$.

From theorem 4 of chapter 3 ([29]) we also know that the optimal Bell-violation among the states connected to ρ by local filtering transformations is obtained from the 'quasi-distilled' state ρ_3. Hence by using eqn. (7.28) we obtain an optimal Bell violation of amount

$$1 + \frac{d^2}{(a-b)(a-c)} = \frac{\lambda_1(C_{\rho'}) + \lambda_2(C_{\rho'})}{\lambda_0(C_{\rho'})} = \frac{\lambda_1(C_\rho) + \lambda_2(C_\rho)}{\lambda_0(C_\rho)} > 1 \tag{7.31}$$

(note that $(a-b)(a-c) \geq d^2$ by virtue of positivity of ρ')

Thus states for which ρ' is not Bell-diagonal ($d \neq 0$ case) will *always* violate the Bell-CHSH inequality after suitable local filtering transformation.

(c) ρ' is of the non Bell-diagonal form with $d = 0$ in eqn. (7.25) (cases (ii), (iii) and (iv) of eqn. (7.26)) . These states being of the product form must come from a separable ρ (local filtering with full rank filters being invertible) and from eqns. (7.25) and (7.27) we have $\lambda_i(C_\rho) = \lambda_i(C_{\rho'}) = 0$ for all i. Thus Theorem 1 holds.

Conversely, when eqn. (7.23) is satisfied we can either filter or quasi-distill ρ to a Bell-diagonal state with optimal Bell-violation $2\sqrt{\frac{(\lambda_1(C_\rho)+\lambda_2(C_\rho))}{\lambda_0(C_\rho)}}$.

□

7.3.1 Strongly non-locality breaking qubit channels

Theorem 1 allows us to characterise exactly all the strongly non-locality breaking qubit channels within the purview of CHSH nonlocality.

Let us first consider the amplitude damping channel which breaks the non-locality of all states for $p \geq \frac{1}{2}$(see fig. 7.1) . The Choi-state of the amplitude damping channel is given by,

$$\rho_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \sqrt{(1-p)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ \sqrt{(1-p)} & 0 & 0 & (1-p) \end{bmatrix}. \tag{7.32}$$

The eigenvalues of C_{ρ_1} are given by, $[(1-p),(1-p),(1-p),(1-p)]$ and eqn. (7.23) is satisfied provided $p<1$. As can be checked, the Choi-state can be quasi-distilled to a singlet state using the filters $A = \text{diag}(\frac{(1-p}{(2-p)}, \frac{1}{n})$ and $B = \text{diag}(\frac{1}{n}, 1)$. Hence, a channel of the amplitude damping form can never be strongly non-locality breaking for any non-zero value of p even though it breaks the nonlocality of all states for $p \geq \frac{1}{2}$. Note that amplitude damping channels are also not entanglement breaking for any non-zero p.

7.3.1.1 Unital channels

For unital qubit channels we have $\vec{t}=0$ in eqn. (4.9) and the Choi state $\rho_{\Phi^+,\Lambda}$ is local unitarily connected to the Bell-diagonal state,

$$\rho_{\Phi^+,\Lambda'} = \frac{1}{4}(I\otimes I + \lambda_1\sigma_1\otimes\sigma_1 - \lambda_2\sigma_2\otimes\sigma_2 + \lambda_3\sigma_3\otimes\sigma_3). \tag{7.33}$$

As this itself is of the normal form, from Theorem 1 and eqn .(7.14) it follows that unital channels breaking non-locality of maximally entangles states are also strongly non-locality breaking. Hence, as mentioned after eqn.(7.7), they also break the non-locality of any input state.

7.3.1.2 Extremal qubit channels

The set of all qubit channels is convex and as shown in the introduction to quantum channels chapter(Ruskai citation) that the closure of the set of extreme points of this set are given, upto pre and post-processing by unitaries, by a two parameter family with the canonical form (eqn. (4.9)),

$$\mathbf{T}_\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(u) & 0 & 0 \\ 0 & 0 & cos(v) & 0 \\ sin(u)sin(v) & 0 & 0 & cos(u)cos(v) \end{pmatrix}, \tag{7.34}$$

with $u\in[0,2\pi), v\in[0,\pi)$. The R matrix of Choi-state of this extremal channel is given by,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(u) & 0 & 0 \\ 0 & 0 & -cos(v) & 0 \\ sin(u)sin(v) & 0 & 0 & cos(u)cos(v) \end{pmatrix}, \tag{7.35}$$

, with $\lambda(C_{\rho_{\Phi^+,\Lambda}}) = [cos^2(u), cos^2(u), cos^2(v), cos^2(v)]$ assuming $cos(u) > cos(v)$. The $M-$value for the optimal Bell-violation (eqn. 7.8) of the filtered Bell-diagonal version of $\rho_{\Phi^+,\Lambda}$, from Theorem 1 is given by $\left(1 + \left(\frac{cos(v)}{cos(u)}\right)^2 > 1\right)$ and hence an extremal channel cannot be strongly non-locality breaking. As a check one can compute the eigenvectors of $C_{\rho_{\Phi^+,\Lambda}}$ and $C'_{\rho_{\Phi^+,\Lambda}}$ to obtain the filters(see [29] for details)

$$A = i\begin{pmatrix} 0 & \left(\frac{sin(\frac{(v+u)}{2})cos(\frac{(v-u)}{2})}{sin(\frac{(v-u)}{2})cos(\frac{(v+u)}{2})}\right)^{\frac{1}{4}} \\ \left(\frac{sin(\frac{(v-u)}{2})cos(\frac{(v+u)}{2})}{sin(\frac{(v+u)}{2})cos(\frac{(v-u)}{2})}\right)^{\frac{1}{4}} & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \left(\frac{sin(\frac{(v+u)}{2})cos(\frac{(v+u)}{2})}{sin(\frac{(v-u)}{2})cos(\frac{(v-u)}{2})}\right)^{\frac{1}{4}} & 0 \\ 0 & \left(\frac{sin(\frac{(v-u)}{2})cos(\frac{(v-u)}{2})}{sin(\frac{(v+u)}{2})cos(\frac{(v+u)}{2})}\right)^{\frac{1}{4}} \end{pmatrix},$$

through which the Choi-state of extremal channels can be brought to the Bell-diagonal state ,

$$\rho_3 = \frac{1}{4}(I \otimes I + \sigma_1 \otimes \sigma_1 + \frac{cos(v)}{cos(u)}\sigma_2 \otimes \sigma_2 - \frac{cos(v)}{cos(u)}\sigma_3 \otimes \sigma_3). \qquad (7.36)$$

If $cos(u)$ or $cos(v) = 0$ then the channel becomes entanglement breaking and hence also strongly non-locality breaking. The condition for an extremal channel to break nonlocality of a maximally entangled state is of course $cos^2(u) + cos^2(v) \leq 1$.

7.3.1.3 Channels breaking nonlocality of maximally entangled state genuinely, may not be strongly non-locality breaking

In ref. [5] an example was given of a one parameter family of two-qubit entangled states which have a local model for projective measurements but shows hidden CHSH nonlocality under suitable filtering. The one parameter family of states, defined by the parameter q, is given by $\rho = qP\{\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\} + (1-q)|0\rangle\langle 0| \otimes \frac{I}{2}$ where $P\{|\alpha\rangle\}$ denotes projector on $|\alpha\rangle$. This state is dual to the channel \$ with $\vec{t} = (0, 0, (1-q))$ and $\vec{\lambda} = (-q, q, -q)$ (see eqn. (4.9)), i.e., $\rho = (\$ \otimes I)(|\Phi^+\rangle\langle\Phi^+|)$. The state is entangled for all values of $q > 0$ and has a local model for projective measurements for all $q \leq \frac{1}{2}$. Hence \$ is not entangled breaking for any positive value of q and breaks the non-locality of maximally entangled states genuinely (in the sense that the output has a local model for projective measurement) for $q \leq \frac{1}{2}$. It turns out that $\lambda(C_\rho) = [q, q, q^2, q^2]$. Thus, by Theorem 1 and as mentioned in ref. [5], the optimal Bell-violation under local filtering is $2\sqrt{1+q}$ and the channel is not strongly nonlocality breaking for any positive value of q. The channel breaks CHSH nonlocality of maximally entangled states for $q \leq \frac{1}{\sqrt{2}}$. Interestingly, we find numerically that \$ *fails* to break the CHSH non-locality of an arbitrary input state for the range $0.62 < q \leq \frac{1}{\sqrt{2}}$ (the lower bound is correct upto two decimal places, for $q = 0.6236$

M-value of the state (eqn. (3.7)) $(I \otimes \$)(\sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle)$ for $\lambda = 0.95$ being $1.0 + 2.339 \times 10^{-5}$). Thus the question of whether the channel genuinely breaks the nonlocality of all states for $q \leq \frac{1}{2}$ remains open .

7.3.1.4 Example of non-unital strongly non-locality breaking channels

Using Theorem 1 one can easily generate examples of non-unital strongly non-locality breaking channels. For example the channel in canonical form (4.9) with parameters $t_1 = 0, t_2 = 0, t_3 = 0.29, \lambda_1 = \frac{1}{\sqrt{2}}, \lambda_2 = \frac{1}{\sqrt{10}}, \lambda_3 = \frac{1}{2}$ has for $\rho \equiv \rho_{\Phi^+,\Lambda}$, $\frac{\lambda_1(C_\rho)+\lambda_2(C_\rho)}{\lambda_0(C_\rho)} = 0.887$ and hence is strongly non-locality breaking by Theorem 1.

7.3.2 Relative vol. of strongly non-locality breaking channels and entanglement breaking channels

Entanglement breaking channels are isomorphic to the set of separable states whose one-sided reduction is maximally mixed. As Lemma 3 shows , through the Choi-Jamiolkowski isomorphism strongly non-locality breaking qubit channels are isomorphic to the set of states which do not show any hidden nonlocality, with one sided reduction maximally mixed. As a quantitative comparison of entanglement and non-locality it thus becomes interesting to compute the volume of this set and Theorem 1 allows us to achieve this. So, we compare the relative volume(w.r.t the volume of all qubit channels) of the set of all strongly non-locality breaking qubit channels(also a convex set) with that of the entanglement breaking channels. For this we sample uniformly within a six dimensional real hypercube of parameters $t_i, \lambda_i \in [-1, 1], i = 1, 2, 3$ and reject points [1] which do not satisfy complete-positivity criterion. Among the remaining points we count the fraction which correspond to strongly non-locality breaking channels using theorem 1 and the fraction that correspond to entanglement breaking channels. We also count the fraction which breaks just the non-locality of maximally entangled states. We sample 10^7 points for this purpose. The relative volume of the entanglement breaking channels turn out to be about 0.24, that of channels breaking non-locality of maximally entangled state turn out to be about 0.81 , while that of strongly non-locality breaking channels turn out to be about 0.39. If we restrict to the unital case then the vol. of entanglement breaking channels is 0.5 and that of non-locality breaking channels turns out to be 0.92. Thus though almost any unital channel is strongly non-locality breaking the chance that a generic qubit channel is strongly non-locality breaking is much closer to that of it being entanglement breaking.

[1] Complete positivity of qubit channels with the canonical form given by eqn.(4.9) demands that $|t_i|, |\lambda_i| \leq 1$,see chapter 4

7.4 Discussion

In this chapter inspired by the notion of entanglement breaking we investigate qubit channels through their property of 'non-locality breaking'. This additionally,provides a way to connect the notion of entanglement and non-locality through channel action—instead of the usual trend via states. We focus on CHSH nonlocality as this is the only inequality for which the necessary and sufficient conditions on the state for violation are known. One of the main properties of entanglement breaking channels is that it is sufficient to 'break' the entanglement of maximally entangled states. We provide examples to show that similar property does not hold for 'non-locality breaking'.Though there seems to be some channels and a certain restricted class of states for all channels for which this is true. We also consider a stronger notion of non-locality breaking, again taking cue from entanglement breaking where the output states of one-sided action of the channel are required to be local under SLOCC. We show that for a qubit channel to be strongly non-locality breaking it is enough for the dual-state of the channel to not show any hidden nonlocality under local filtering.

We provide a closed-form necessary sufficient condition for any two-qubit state to violate the Bell-CHSH inequality under local filtering,which is likely to be useful for other purposes as well. This is then used to study 'strongly non-locality breaking' qubit channels and compute their relative volume within the set of all channels. It turns out that unital qubit channels breaking non-locality of maximally entangled states are strongly non-locality breaking while extremal qubit channels cannot be so unless they are entanglement breaking. It may be mentioned here that each single mode entanglement breaking Gaussian channel is related to a single mode non-classicality breaking Gaussian channel via some squeezing transformation [55].

An interesting course of future study is to see how the gap between entanglement breaking and non-locality breaking qubit channels close as one considers more inequalities (e.g,I_{3322} [36]). We also, at present, do not have any example of a channel which breaks the non-locality of a maximally entangled state, *genuinely* but fails to break that of other states. This can be studied, for example, for the channel in Section V whose dual state has a local model.

Chapter 8

Entanglement sharing through noisy qubit channels

Maximally entangled states–a resource for quantum information processing–can only be shared through noiseless quantum channels, whereas in practice channels are noisy. In this chapter we ask : Given a noisy quantum channel, what is the maximum attainable purity (measured by singlet fraction) of shared entanglement for single channel use and local trace preserving operations? We find an exact formula of the maximum singlet fraction attainable for a qubit channel and give an explicit protocol to achieve the optimal value. The protocol distinguishes between unital and nonunital channels and requires no local post-processing. In particular, the optimal singlet fraction is achieved by transmitting part of an appropriate pure entangled state, which is maximally entangled if and only if the channel is unital. A linear function of the optimal singlet fraction is also shown to be an upper bound on the distillable entanglement of the mixed state dual to the channel.

In the simplest scenario, the general protocol of sharing entanglement works as follows: Alice prepares a bipartite pure entangled state $|\psi\rangle$ and sends one half of it to Bob through a quantum channel, say Λ. This results, in general, in a mixed entangled state $\rho_{\psi,\Lambda} = (I \otimes \Lambda)\rho_\psi$, where $\rho_\psi = |\psi\rangle\langle\psi|$. The *purity* of this state is characterized by its singlet fraction [14, 10, 12, 56] defined as:

$$F(\rho_{\psi,\Lambda}) = \max_{|\Phi\rangle} \langle\Phi|\rho_{\psi,\Lambda}|\Phi\rangle, \tag{8.1}$$

where $|\Phi\rangle$ is a maximally entangled state. The singlet fraction quantifies how close the state $\rho_{\psi,\Lambda}$ is to a maximally entangled state, and therefore how useful the state is for QIP tasks. For example, it is related to the teleportation fidelity f for teleportation of a qudit via the following relation:

$$f(\rho_{\psi,\Lambda}) = \frac{dF(\rho_{\psi,\Lambda}) + 1}{d+1} \tag{8.2}$$

as discussed in chapter 5.

In this part of the thesis we are interested in the *optimal singlet fraction* for the channel Λ defined as :

$$F(\Lambda) = \max_{|\psi\rangle} \max_{\mathrm{L}} F\left(L(\rho_{\psi,\Lambda})\right), \tag{8.3}$$

where the maximum is taken over all pure state transmissions and trace preserving LOCCs L. Note that, by virtue of Eq. (8.2) $F(\Lambda)$ also quantifies reliable transmission of quantum states via teleportation, albeit for single channel use, where the optimal teleportation fidelity for the channel is expressed as $f(\Lambda) = \frac{dF(\Lambda)+1}{d+1}$. This is in contrast with the known measures such as, channel fidelity [14], which quantifies, on an average, how close the output state is to the input state, and entanglement fidelity [57, 58], which captures how well the channel preserves entanglement [59] of the transmitted system with other systems.

For qubit channels such as depolarizing [14] and amplitude damping [60] the value of $F(\Lambda)$ is known, but no general expression has been found yet for a generic qubit channel. In this work, we obtain an exact formula of $F(\Lambda)$ for a qubit channel and give an explicit protocol to achieve this value. Surprisingly, we also find that to attain the optimal value no local post processing is required, even though it is known that local post-processing can increase the singlet fraction of a state. In particular, we show that the optimal value is attained by sending part of a maximally entangled state down the channel if and only if the channel is unital. This means that for nonunital channels one must necessarily transmit part of an appropriate nonmaximally entangled state. We also prove that the optimal singlet fraction is equal to a linear function of the negativity [56] of the mixed state $\rho_{\Phi^+,\Lambda}$, where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Thus a linear function of $F(\Lambda)$ is an upper bound on the distillable entanglement of the mixed state $\rho_{\Phi^+,\Lambda}$.

Let us note a couple of implications of our results. As noted earlier, an entanglement distillation [10, 11, 12, 13, 14] protocol uses many copies of the mixed state $\rho_{\psi,\Lambda}$(for some transmitted pure state $|\psi\rangle$) of purity $F(\rho_{\psi,\Lambda})$ and converts them to a fewer number of near-perfect entangled states of purity close to unity. Following the prescription in this paper, for a given noisy qubit channel Alice and Bob can now prepare states with maximum achievable purity for each channel use so as to maximize the yield in their distillation protocol. Second, by virtue of Eq. (8.2) we are able to provide the optimal teleportation fidelity for any qubit channel, albeit for single channel use.

8.1 Optimal singlet fraction for qubit channels.

8.1.1 Preliminaries

As we saw in chapter 4 a quantum channel Λ is a trace preserving completely positive map characterized by a set of Kraus operators $\{A_i\}$ satisfying $\sum A_i^\dagger A_i = I$. Its dual $\hat{\Lambda}$ is described in terms of the Kraus operators $\left\{A_i^\dagger\right\}$(the dual is the adjoint map with respect to the Hilbert-Schmidt inner product). A channel Λ is said to be *unital* if its action preserves Identity: $\Lambda(I) = I$, and *nonunital* if it does not, i.e., $\Lambda(I) \neq I$. A dual channel $\hat{\Lambda}$ is trace-preserving iff Λ is unital. Sending half of pure state $|\phi\rangle$ down the channel $\$ \in \left\{\Lambda, \hat{\Lambda}\right\}$ gives rise to a mixed state

$$\rho_{\phi,\$} = (I \otimes \$)\, \rho_{\phi}, \tag{8.4}$$

where $\rho_{\phi} = |\phi\rangle\langle\phi|$. For the channel \$ with a set of Kraus operators $\{K_i\}$, the above equation takes the form

$$\rho_{\phi,\$} = \sum_i (I \otimes K_i)\, \rho_{\phi} \left(I \otimes K_i^{\dagger}\right) \tag{8.5}$$

Recall that, by transmitting one half of a pure entangled state $|\psi\rangle$ through a noisy channel Λ results in a mixed state $\rho_{\psi,\Lambda}$ of singlet fraction $F(\rho_{\psi,\Lambda})$. Simply maximizing $F(\rho_{\psi,\Lambda})$ over all transmitted pure states $|\psi\rangle$ may not yield the optimal value we are looking for because it is known [61, 62, 15] that TP-LOCC can enhance singlet fraction of two qubit states. Thus for a given $\rho_{\psi,\Lambda}$, the maximum achievable singlet fraction is defined as [15]

$$F^{*}(\rho_{\psi,\Lambda}) = \max_{L} F(L(\rho_{\psi,\Lambda})), \tag{8.6}$$

where the maximization is over all TP-LOCC L carried out by Alice and Bob on their respective qubits. Note that,as discussed and proved in the introduction chapter unlike F, which can increase under TP-LOCC, F^{*} is an entanglement monotone [15] and can be exactly computed [15] by solving a convex semi-definite program for any given two-qubit density matrix. Maximizing F^{*} over all transmitted pure states $|\psi\rangle$ yields the *optimal singlet fraction* defined earlier in Eq. (8.3):

$$F(\Lambda) = \max_{|\psi\rangle} F^{*}(\rho_{\psi,\Lambda}). \tag{8.7}$$

It is clear from the above definitions that for any shared pure state $|\psi\rangle$, the following inequalities hold:

$$F(\Lambda) \geq F^{*}(\rho_{\psi,\Lambda}) \geq F(\rho_{\psi,\Lambda}). \tag{8.8}$$

Our first result gives an exact formula for the optimal singlet fraction defined in Eq. (8.7) and an explicit protocol by which the optimal value can be achieved. We show that for any qubit channel Λ there exists an "optimal" two-qubit pure state $|\psi_0\rangle$, not necessarily maximally entangled, such that all the inequalities in (8.8) become equalities.

Theorem 1. *The optimal singlet fraction of a qubit channel Λ is given by*

$$F(\Lambda) = \lambda_{\max}(\rho_{\Phi^{+},\Lambda}), \tag{8.9}$$

where $|\Phi^{+}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and $\lambda_{\max}(\rho_{\Phi^{+},\Lambda})$ is the maximum eigenvalue of the density matrix $\rho_{\Phi^{+},\Lambda}$. Moreover, the following equalities hold:

$$F(\Lambda) = F^{*}(\rho_{\psi_0,\Lambda}) = F(\rho_{\psi_0,\Lambda}), \tag{8.10}$$

where $|\psi_0\rangle$ is the eigenvector corresponding to the maximum eigenvalue of the density matrix $\rho_{\Phi^{+},\hat{\Lambda}}$

Proof. We begin by obtaining an exact expression of the maximum pre-processed singlet fraction. It is defined as

$$F_1(\Lambda) = \max_{|\psi\rangle} F(\rho_{\psi,\Lambda}), \tag{8.11}$$

$$= \max_{|\psi\rangle}\max_{|\Phi\rangle}\langle\Phi|\rho_{\psi,\Lambda}|\Phi\rangle, \tag{8.12}$$

where $|\Phi\rangle$ is maximally entangled. Noting that every maximally entangled state $|\Phi\rangle$ can be written as $U\otimes V|\Phi^+\rangle$, for some $U,V\in SU(2)$, we can rewrite Eq. (8.12) as

$$F_1(\Lambda) = \max_{|\psi\rangle,U,V}\langle\Phi^+|\left(U^\dagger\otimes V^\dagger\right)\rho_{\psi,\Lambda}(U\otimes V)|\Phi^+\rangle. \tag{8.13}$$

Let, $\rho_\psi = |\psi\rangle\langle\psi|$ and $\rho_{\Phi^+} = |\Phi^+\rangle\langle\Phi^+|$. Using the fact that $(I\otimes V)|\Phi^+\rangle = (V^T\otimes I)|\Phi^+\rangle$, we now simplify the above equation:

$$\begin{aligned}
F_1(\Lambda) &= \max_{|\psi\rangle,U,V}\langle\Phi^+|\left(U^\dagger\otimes V^\dagger\right)\rho_{\psi,\Lambda}(U\otimes V)|\Phi^+\rangle \\
&= \max_{|\psi\rangle,U,V}\langle\Phi^+|\left(U^\dagger\otimes V^\dagger\right)\sum_i(I\otimes A_i)\rho_\psi\left(I\otimes A_i^\dagger\right)(U\otimes V)|\Phi^+\rangle \\
&= \max_{|\psi\rangle,U,V}\langle\psi|\sum_i(I\otimes A_i^\dagger)(U\otimes V)\rho_{\Phi^+}(U^\dagger\otimes V^\dagger)(I\otimes A_i)|\psi\rangle \\
&= \max_{|\psi\rangle,U,V}\langle\psi|\sum_i(I\otimes A_i^\dagger)(UV^T\otimes I)\rho_{\Phi^+}(V^*U^\dagger\otimes I)(I\otimes A_i)|\psi\rangle \\
&= \max_{|\psi\rangle,U,V}\langle\psi|\left(UV^T\otimes I\right)\rho_{\Phi^+,\hat{\Lambda}}\left(V^*U^\dagger\otimes I\right)|\psi\rangle \\
&= \max_{|\psi\rangle}\langle\psi|\rho_{\Phi^+,\hat{\Lambda}}|\psi\rangle,
\end{aligned} \tag{8.14}$$

From the above equation it immediately follows that ,

$$F_1(\Lambda) = F(\rho_{\psi_0,\Lambda}) = \lambda_{\max}\left(\rho_{\Phi^+,\hat{\Lambda}}\right) \tag{8.15}$$

where $\lambda_{\max}$ denotes the maximum eigenvalue of $\rho_{\Phi^+,\hat{\Lambda}}$ and $|\psi_0\rangle$ the corresponding eigenvector. Using the result,

$$\lambda_{\max}\left(\rho_{\Phi^+,\hat{\Lambda}}\right) = \lambda_{\max}(\rho_{\Phi^+,\Lambda}) \tag{8.16}$$

proved in lemma 5(section A of Appendix) , we have therefore proven that

$$F(\Lambda) \geq F_1(\Lambda) = \lambda_{\max}(\rho_{\Phi^+,\Lambda}) \tag{8.17}$$

The following lemma now gives an upper bound on the optimal singlet fraction $F(\Lambda)$.

Lemma 1. *For a qubit channel* Λ

$$F(\Lambda) \leq \lambda_{\max}(\rho_{\Phi^+,\Lambda}), \tag{8.18}$$

where $\lambda_{\max}(\rho_{\Phi^+,\Lambda})$ *denotes the maximum eigenvalue of the density matrix* $\rho_{\Phi^+,\Lambda}$.

Proof. Recall that by definition, $F(\Lambda) = \max_\psi F^*(\rho_{\psi,\Lambda})$; in particular,

$$F^*(\rho_{\psi,\Lambda}) = \max_L F(L(\rho_{\psi,\Lambda})) = F\left(\rho^*_{\psi,\Lambda}\right), \tag{8.19}$$

where $\rho^*_{\psi,\Lambda}$ is the state obtained from $\rho_{\psi,\Lambda}$ by *optimal* TP-LOCC for a given $\rho_{\psi,\Lambda}$. As discussed in section VI of the introduction chapter (shown in ref.[15]) that the optimal TP-LOCC is an 1-way LOCC protocol, where any of the parties apply a state dependent filter. In case of success the other party does nothing, and in case of failure, Alice and Bob simply prepare a separable state. We have, therefore,

$$\rho^*_{\psi,\Lambda} = p\rho_1 + (1-p)\rho_s, \tag{8.20}$$

where $\rho_1 = \frac{1}{p}(A \otimes I)\rho_{\psi,\Lambda}\left(A^\dagger \otimes I\right)$ with A being the optimal filter, is the state arising with probability $p = \mathrm{Tr}\left[\left(A^\dagger A \otimes I\right)\rho_{\psi,\Lambda}\right]$ when filtering is successful and ρ_s is a separable state which Alice and Bob prepare when the filtering operation is not successful. F^* is given by ([15]),

$$\begin{aligned} F^*(\rho_{\psi,\Lambda}) &= F\left(\rho^*_{\psi,\Lambda}\right) \\ &= pF(\rho_1) + \frac{1-p}{2} \qquad (8.21) \\ &= p\langle\Phi^+|\rho_1|\Phi^+\rangle + \frac{1-p}{2}. \qquad (8.22) \end{aligned}$$

Observe that the filter is applied at Alice's end, that is, on the qubit she holds and not on the qubit that was sent through the channel to Bob. In eqns. (8.21) and (8.22) , the separable state ρ_s is chosen so that $\langle\Phi^+|\rho_s|\Phi^+\rangle = \frac{1}{2}$ and optimality of the filter A implies that $F(\rho_1) = \langle\Phi^+|\rho_1|\Phi^+\rangle$(if the latter is not the case we will get another filter unitarily connected with A which yields higher singlet fraction). We will now show that $F(\rho_1) \leq \lambda_{\max}(\rho_{\Phi^+,\Lambda})$. First we observe that

$$\begin{aligned} F(\rho_1) &= \frac{1}{p}\langle\Phi^+|(A\otimes I)(I\otimes\Lambda)(|\psi\rangle\langle\psi|)\left(A^\dagger\otimes I\right)|\Phi^+\rangle \\ &= \frac{1}{p}\langle\Phi^+|(I\otimes\Lambda)(A\otimes I)(|\psi\rangle\langle\psi|)\left(A^\dagger\otimes I\right)|\Phi^+\rangle. \end{aligned} \tag{8.23}$$

On the other hand, because Λ is a trace preserving map, we also observe that

$$\begin{aligned} p &= \mathrm{Tr}\left[(A^\dagger A\otimes I)\rho_{\psi,\Lambda}\right] \\ &= \mathrm{Tr}\left[(I\otimes\Lambda)(A^\dagger A\otimes I)|\psi\rangle\langle\psi|\right] \\ &= \mathrm{Tr}\left[(A^\dagger A\otimes I)|\psi\rangle\langle\psi|\right] \end{aligned} \tag{8.24}$$

.

We thus have $\rho_1 = (I\otimes\Lambda)(|\psi'\rangle\langle\psi'|)$ and from Eqns. (8.23) and (8.24) we get

$$\begin{aligned} F(\rho_1) &= \langle\Phi^+|(I\otimes\Lambda)(|\psi'\rangle\langle\psi'|)|\Phi^+\rangle \\ &= F(\rho_{\psi',\Lambda}), \end{aligned} \tag{8.25}$$

where $|\psi'\rangle = \frac{1}{\sqrt{q}}(A\otimes I)|\psi\rangle$ is a normalized vector with $q = p = \langle\psi|\left(A^{\dagger}A\otimes I\right)|\psi\rangle$. Hence from eqns. (8.11) and (8.17) we have,

$$F(\rho_1) \leq F_1(\Lambda) = \lambda_{\max}(\rho_{\Phi^+,\Lambda}). \tag{8.26}$$

Thus from Eq. (8.22) we have,

$$\begin{aligned} F^*(\rho_{\psi,\Lambda}) &\leq p\lambda_{\max}(\rho_{\Phi^+,\Lambda}) + \frac{1-p}{2} \\ &\leq \lambda_{\max}(\rho_{\Phi^+,\Lambda}) \end{aligned} \tag{8.27}$$

. The last inequality follows from the fact that $\lambda_{\max}(\rho_{\Phi^+,\Lambda}) > 1/2$ (as the channel is not entanglement breaking, this follows by applying Lemma 6 (section B of Appendix) on $\rho_{\Phi^+,\Lambda}$).

Since Inequality (8.27) holds for any transmitted pure state $|\psi\rangle$, we therefore conclude that

$$F(\Lambda) \leq \lambda_{\max}(\rho_{\Phi^+,\Lambda}) \tag{8.28}$$

This completes the proof of lemma 1. □

From Eqs. (8.17) and (8.18) we have, $F(\Lambda) = \lambda_{\max}(\rho_{\Phi^+,\Lambda})$.

Now, as $F(\Lambda) \geq F^*(\rho_{\psi_0,\Lambda}) \geq F(\rho_{\psi_0,\Lambda})$ from eqns. (8.15) and (8.17) we have,

$$F(\Lambda) = F^*(\rho_{\psi_0,\Lambda}) = F(\rho_{\psi_0,\Lambda}) \tag{8.29}$$

This completes the proof of theorem 1. □

What can we say about $|\psi_0\rangle$? Evidences so far are mixed: $|\psi_0\rangle$ can be either maximally entangled (e.g., for depolarizing channel [14]) or nonmaximally entangled (e.g., for amplitude damping channel[60]), but the answer for a generic qubit channel is not known. The following result completely characterizes the channels for which $|\psi_0\rangle$ is maximally entangled and for which it is not.

Theorem 2. *The state $|\psi_0\rangle$, as defined in Theorem 1, is maximally entangled if and only if the channel Λ is unital.*

Proof. Recall that $|\psi_0\rangle$ is the eigenvector corresponding to the maximum eigenvalue of $\rho_{\Phi^+,\hat{\Lambda}}$. Let $|\psi_0'\rangle$ be the eigenvector corresponding to the maximum eigenvalue of $\rho_{\Phi^+.\Lambda}$. The following lemma establishes the correspondence between the vectors $|\psi_0\rangle$ and $|\psi_0'\rangle$.

Lemma 2. *Let V be the swap operator defined by the action $V|\eta\rangle|\chi\rangle = |\chi\rangle|\eta\rangle$. Then $V|\psi_0\rangle^* = |\psi_0'\rangle$.*

Proof. Let us now consider the spectral decomposition of $\rho_{\Phi^+,\Lambda}$: Let

$$\rho_{\Phi^+,\Lambda} = \sum_{k=0}^{3} p_k |\psi'_k\rangle\langle\psi'_k|, \tag{8.30}$$

be the spectral decomposition.

From eqn. (8.67) in the appendix we have,

$$\rho_{\Phi^+,\hat{\Lambda}} = \sum_k \lambda_k (V^\dagger|\psi'_k\rangle\langle\psi'_k|V)^*. \tag{8.31}$$

For different values of k, $(V^\dagger|\psi'_k\rangle)^*$ are orthogonal as V is unitary .

Hence we see that eqn. (8.31) is in fact a spectral decomposition of $\rho_{\Phi^+,\hat{\Lambda}}$ with eigenvectors

$$|\psi_k\rangle = (V^\dagger|\psi'_k\rangle)^*. \tag{8.32}$$

The Schmidt coefficients of $|\psi'_k\rangle$ are same as that of $|\psi_k\rangle$. The entanglement of $|\psi'_k\rangle$ is thus also same as that of $|\psi_\alpha\rangle$.

Let ψ'_0 be the eigenvector corresponding to the maximum eigenvalue of $\rho_{\Phi^+,\Lambda}$. We have from eqn. (8.32) ,

$$|\psi_0\rangle = (V^\dagger|\psi'_0\rangle)^*. \tag{8.33}$$

This completes the proof of lemma 2. □

Therefore, if $|\psi'_0\rangle$ is maximally entangled, then so is $|\psi_0\rangle$ and vice versa. We will prove the theorem by showing that $|\psi'_0\rangle$ is maximally entangled if and only if Λ is unital.

We first show that if $|\psi'_0\rangle$ is maximally entangled then Λ must be unital. We first note that the Kraus operators of the channel Λ can be obtained from the action of the channel on the maximally entangled state $|\Phi^+\rangle$.

Now for every k, we can write $|\psi'_k\rangle$ as

$$|\psi'_k\rangle = (I \otimes G_k)\,|\Phi^+\rangle, \tag{8.34}$$

where G_k is a 2×2 complex matrix. As discussed in chapter 4, ([14]) the channel Λ can be described in terms of the Kraus operators $\left\{\sqrt{p_k}G_k\right\}$. Noting that (a) $\langle\psi'_i|\psi'_j\rangle = \delta_{ij}$, and (b) for any operator O, $\langle\Phi^+|I \otimes O|\Phi^+\rangle = \frac{1}{2}\mathrm{Tr}O$, it follows that the Kraus operators $\left\{\sqrt{p_k}G_k\right\}$ are trace orthogonal. That is,

$$\mathrm{Tr}A_k^\dagger A_l = 2\sqrt{p_k p_l}\delta_{kl}, \tag{8.35}$$

where $A_k = \sqrt{p_k}G_k$. The Kraus operators thus obtained through the spectral decomposition of $\rho_{\Phi^+,\Lambda}$ are trace orthogonal. They also satisfy $\sum A_k^\dagger A_k = I$, as Λ is a TPCP map.

Suppose now the channel Λ is non-unital, i.e., $\Lambda(I) \neq I$. This implies that

$$\sum A_k A_k^\dagger \neq I \tag{8.36}$$

None of our considerations change if we consider a channel $U \circ \Lambda$ with Kraus operators UA_k where $U \in SU(2)$. This is because the eigenvectors of $\rho_{\Phi^+,\Lambda}$ and $\rho_{\Phi^+,U\Lambda}$ are local unitarily connected and eigenvalues are same. Let us now assume that one of the eigenstates($|\psi_0'\rangle$ say) in the spectral decomposition of $\rho_{\Phi^+,\Lambda}$ in Eq. (8.30) is maximally entangled. This necessarily implies one of the Kraus operators say, A_0 is proportional to a unitary. Now because of the post-processing freedom, without any loss of generality we can take A_0 to be $\sqrt{p}I$, with $p \in [0,1]$. Due to trace-orthogonality [Eq. (8.35)] we will have

$$\mathrm{Tr}(A_k) = 0, k = 1,2,3. \tag{8.37}$$

We can thus take $A_k = \overrightarrow{\alpha_k}.\overrightarrow{\sigma}$, where $\overrightarrow{\alpha_k} \in \mathbb{C}^3$ and $\overrightarrow{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$, for $k = 1,2,3$. On using $(\vec{\sigma}\cdot\vec{a})\left(\vec{\sigma}\cdot\vec{b}\right) = (\vec{a}\cdot\vec{b})I + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$ the trace preservation condition $\sum A_k^\dagger A_k = I$ now becomes,

$$pI + \sum_{k=1}^{3}(\overrightarrow{\alpha_k}^* \cdot \overrightarrow{\alpha_k})I + i(\overrightarrow{\alpha_k}^* \times \overrightarrow{\alpha_k}) \cdot \overrightarrow{\sigma} = I, \tag{8.38}$$

from which we obtain,

$$\begin{aligned} p + \sum_{k=1}^{3}(\overrightarrow{\alpha_k}^* \cdot \overrightarrow{\alpha_k}) &= 1, \\ \sum_{k=1}^{3}\overrightarrow{\alpha_k}^* \times \overrightarrow{\alpha_k} &= 0. \end{aligned} \tag{8.39}$$

On the other hand the condition for non-unitality [Eq. (8.36)] of the channel gives us,

$$pI + \sum_{k=1}^{3}(\overrightarrow{\alpha_k}^* \cdot \overrightarrow{\alpha_k})I - i(\overrightarrow{\alpha_k}^* \times \overrightarrow{\alpha_k}) \cdot \overrightarrow{\sigma} \neq I. \tag{8.40}$$

which is clearly in contradiction with Eqn. (8.39) . Thus $\rho_{\Phi^+,\Lambda}$ cannot have a maximally entangled eigenvector if Λ is non-unital. Hence, $|\psi_0'\rangle$ is not maximally entangled. Therefore it follows that if $|\psi_0\rangle$ is maximally entangled, then the channel must be unital.

We will now show that if Λ is unital then $|\psi_0'\rangle$ is maximally entangled. As shown in chapter 4 ([42]) any unital qubit channel Λ, $\rho_{\Phi^+,\Lambda}$ is local unitarily connected to the Bell-diagonal state $\sum_{i=0}^{3} p_i (I\otimes\sigma_i)|\Phi^+\rangle\langle\Phi^+|(I\otimes\sigma_i)$ with $\sigma_0 = I$, $1 \geq p_i \geq 0$ and $\sum_i p_i = 1$. It immediately follows that $|\psi_0'\rangle$ is maximally entangled. This completes the proof of theorem 2. □

8.2 Optimal singlet fraction and the maximum output negativity

Here we show that $F(\Lambda)$ is related to the negativity of the density matrix $\rho_{\Phi^+,\Lambda}$. We first note that an upper bound on $F^*(\rho_{\psi,\Lambda})$ can be given in terms of its negativity, as discussed in chapter 5([56]), $N(\rho_{\psi,\Lambda})$:

$$F^*(\rho_{\psi,\Lambda}) \leq \frac{1}{2}\left[1 + N(\rho_{\psi,\Lambda})\right], \tag{8.41}$$

where $N(\rho_{\psi,\Lambda}) = \max\left\{0, -2\lambda_{\min}\left(\rho_{\psi,\Lambda}^{\Gamma}\right)\right\}$ and $\rho_{\psi,\Lambda}^{\Gamma}$ is the partially transposed matrix obtained from $\rho_{\psi,\Lambda}$.

Remark: Note that for convenience, we have considered an extra factor of 2 in the definition of negativity over the definition in Chapter 5.

Maximizing over all input states $|\psi\rangle$we get,

$$F(\Lambda) \leq \frac{1}{2}\left[1 + N(\Lambda)\right], \tag{8.42}$$

where $N(\Lambda) = \max_{\psi} N(\rho_{\psi,\Lambda})$. An interesting question here is, does the optimal singlet fraction always reach the above upper bound for all channels Λ? In order to answer this question, we first prove the following:

Lemma 3. *For a qubit channel Λ, the optimal singlet fraction $F(\Lambda)$ is related to the negativity $N(\rho_{\Phi^+,\Lambda})$ of the state $\rho_{\Phi^+,\Lambda}$ by the following relation:*

$$F(\Lambda) = \frac{1}{2}\left[1 + N(\rho_{\Phi^+,\Lambda})\right] \tag{8.43}$$

Proof. The proof follows by using the formula of negativity, simple application of Lemma 6 (see section B of Appendix) and Thm 1:

$$\begin{aligned} \frac{1}{2}\left[1 + N(\rho_{\Phi^+,\Lambda})\right] &= \frac{1}{2}\left[1 - 2\lambda_{\min}\left(\rho_{\Phi^+,\Lambda}^{\Gamma}\right)\right] \\ &= \lambda_{\max}(\rho_{\Phi^+,\Lambda}) = F(\Lambda) \end{aligned} \tag{8.44}$$

This completes the proof of lemma 3. □

Next we show that that, $F(\Lambda)$ does not reach the upper bound in Eq. (8.42) for all non-unital channels as there are examples for which $N(\Lambda) > N(\rho_{\Phi^+,\Lambda})$. Thus, even though the ordering of negativity may change under one-sided channel action, $I \otimes \Lambda$ the optimal singlet fraction obeys the bound in Eq. (8.41) for maximally entangled input. For unital channels however, as the next lemma shows, we have $N(\Lambda) = N(\rho_{\Phi^+,\Lambda})$.

Lemma 4. *For unital qubit channels we have $N(\Lambda) = N(\rho_{\Phi^+,\Lambda})$*

Proof. The most general two qubit pure state in the Schmidt form is given by, $|\alpha\rangle = \sqrt{\lambda}|e_1 f_1\rangle + \sqrt{1-\lambda}|e_2 f_2\rangle = (U \otimes V)(\sqrt{\lambda}|00\rangle + \sqrt{(1-\lambda)}|11\rangle)$, with $\lambda \in [0,1]$ and the 2×2 unitary matrices U and V being given by: $U|0\rangle = |e_1\rangle, V|0\rangle = |f_1\rangle, U|1\rangle = |e_2\rangle$ and $V|1\rangle = |f_2\rangle$.

For $\lambda \in [0,1]$, let

$$W_\lambda = \sqrt{\lambda}|0\rangle\langle 0| + \sqrt{(1-\lambda)}|1\rangle\langle 1|. \tag{8.45}$$

Now using the fact that Λ is a trace-preserving map it is easy to show that,

$$\begin{aligned} \rho_{\alpha,\Lambda} &= (I \otimes \Lambda)|\alpha\rangle\langle\alpha| \\ &= \frac{(A_1 \otimes I)\rho_{\Phi^+,\Lambda}(A_1^\dagger \otimes I)}{Tr((A_1^\dagger A_1 \otimes I)\rho_{\Phi^+,\Lambda})}, \end{aligned} \tag{8.46}$$

with the filter $A_1 = U W_\lambda V^T$.

For a unital channel Λ , $\rho_{\Phi^+,\Lambda}$ is Bell-diagonal (see proof of theorem 2). In ref. [18] it was shown that negativity of a Bell-diagonal state cannot be increased by local filtering. Hence, from eqn. (8.46) for a unital qubit channel Λ we have

$$N(\Lambda) = N(\rho_{\Phi^+,\Lambda}). \tag{8.47}$$

This completes the proof of lemma 4. □

8.2.0.1 Example of channel for which $N(\Lambda) > N(\rho_{\Phi^+,\Lambda})$

Let us consider the amplitude damping channel, with Kraus operators $K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$ and $K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$ with $1 \leq p \leq 0$. The channel is non-unital.

It was shown in [60] that the optimal input state for attaining optimal singlet fraction of the channel is given by, $|\chi\rangle = \frac{1}{\sqrt{(2-p)}}|00\rangle + \sqrt{\frac{1-p}{2-p}}|11\rangle$.

Using theorem 1 for the amplitude damping channel Λ we therefore get , $F(\Lambda) = \lambda_{\max}(\rho_{\Phi^+,\Lambda}) = F^*(\rho_{\chi,\Lambda}) = F(\rho_{\chi,\Lambda})$. Now from eqn. (8.41) we get $F^*(\rho_{\chi,\Lambda}) \leq \frac{1}{2}[1 + N(\rho_{\chi,\Lambda})]$, while from lemma 3 we get $F(\Lambda) = \frac{1}{2}[1 + N(\rho_{\Phi^+,\Lambda})]$. Hence we must have, $N(\rho_{\Phi^+,\Lambda}) \leq N(\rho_{\chi,\Lambda})$.

For the amplitude damping channels for input states $|\phi(\lambda)\rangle = \sqrt{\lambda}|00\rangle + \sqrt{(1-\lambda)}|11\rangle (\lambda \in [0,1])$ we have,

$$N\left(\rho_{\phi(\lambda),\Lambda}\right) = \sqrt{p^2(1-\lambda)^2 + 4\lambda(1-\lambda)(1-p)} - (1-\lambda)p. \tag{8.48}$$

Thus,

$$N(\rho_{\Phi^+,\Lambda}) = \sqrt{\left(\frac{p^2}{4} + 1 - p\right)} - \frac{p}{2}$$

and,

$$N\left(\rho_{\phi(\frac{1}{2-p}),\Lambda}\right) = \frac{1-p}{2-p}(\sqrt{p^2+4}-p).$$

It is easy to see that $N\left(\rho_{\Phi^+,\Lambda}\right) < N\left(\rho_{\phi(\frac{1}{2-p}),\Lambda}\right)$ for all $1 > p > 0$ and hence $N\left(\rho_{\Phi^+,\Lambda}\right) < N(\Lambda)$.

8.3 Nonunital channels and maximally entangled input

It is important to recognize that theorems 1 and 2 put together only prescribes a method to attain the optimal singlet fraction. It does not, however, rule out the possibility that the optimal singlet fraction for a nonunital channel may still be attained by sending part of a maximally entangled state followed by local post-processing. As it turns out this is not the case.

Theorem 3. *For a nonunital qubit channel* Λ*,*

$$F^*\left(\rho_{\Phi^+,\Lambda}\right) \quad < \quad F\left(\Lambda\right) \tag{8.49}$$

Proof. Using the bound in Eq. (8.41) for the density matrix $\rho_{\Phi^+,\Lambda}$ we have

$$F^*\left(\rho_{\Phi^+,\Lambda}\right) \quad \leq \quad \frac{1}{2}\left[1+N\left(\rho_{\Phi^+,\Lambda}\right)\right]. \tag{8.50}$$

It follows from lemma 3 that to prove theorem 3 it suffices to show that for a nonunital channel Λ,

$$F^*\left(\rho_{\Phi^+,\Lambda}\right) \quad < \quad \frac{1}{2}\left[1+N\left(\rho_{\Phi^+,\Lambda}\right)\right]. \tag{8.51}$$

As discussed in chapter 5([15]), for any two qubit density matrix ρ the optimal fidelity $F^*(\rho)$ can be found by solving the following convex semidefinite program:

$$\text{maximize} \quad F^* = \frac{1}{2} - \text{Tr}(X\rho^\Gamma), \tag{8.52}$$

under the constraints

$$0 \quad \leq X \leq \quad I_4, \tag{8.53}$$

$$-\frac{I_4}{2} \quad \leq X^\Gamma \leq \quad \frac{I_4}{2}, \tag{8.54}$$

with X^Γ being the partial transpose of X. In addition, the optimal X is known to be of rank one.

The proof is now by contradiction. Suppose that $F^*\left(\rho_{\Phi^+,\Lambda}\right) = \frac{1}{2}\left[1+N\left(\rho_{\Phi^+,\Lambda}\right)\right]$; thus to achieve this equality we must necessarily have,

$$\frac{1}{2} - \text{Tr}(X_{\text{opt}}\rho^\Gamma_{\Phi^+,\Lambda}) \quad = \quad \frac{1}{2}\left[1+N\left(\rho_{\Phi^+,\Lambda}\right)\right], \tag{8.55}$$

from which it follows that

$$\begin{aligned}\mathrm{Tr}(X_{\mathrm{opt}}\rho^{\Gamma}_{\Phi^+,\Lambda}) &= -\frac{N\left(\rho_{\Phi^+,\Lambda}\right)}{2} \\ &= \lambda_{\min}\left(\rho^{\Gamma}_{\Phi^+,\Lambda}\right). \end{aligned} \tag{8.56}$$

Using the facts that X_{opt} is a positive rank one operator (proved in [15]) and there is only one negative eigenvalue for $\rho^{\Gamma}_{\Phi^+,\Lambda}$ (which means $\lambda_{\min}$ is negative), we obtain

$$X_{\mathrm{opt}} = |\alpha\rangle\langle\alpha|, \tag{8.57}$$

where $\rho^{\Gamma}|\alpha\rangle = \lambda_{\min}(\rho^{\Gamma})|\alpha\rangle$. Clearly X_{opt} in the above eqn. is of rank one and satisfies $0 \leq X \leq I_4$. As eigenvalues of X and X^{Γ} are invariant under local unitaries it is sufficient to take ,

$$X = \mathbf{P}(\sqrt{\lambda}|00\rangle + \sqrt{(1-\lambda)}|11\rangle), \tag{8.58}$$

with $\mathbf{P}(|a\rangle)$ denoting projector on $|a\rangle$ and $\lambda \in (0,1)$. .

The spectrum of X^{Γ} for X in Eq. (8.58) is given by ,

$$\lambda(X^{\Gamma}) = \lambda, (1-\lambda), \pm\sqrt{\lambda(1-\lambda)}. \tag{8.59}$$

Thus the constraint (8.54) is only satisfied for $\lambda = \frac{1}{2}$, i.e, if $|\alpha\rangle$ is maximally entangled. Therefore, under the assumption $F^*\left(\rho_{\Phi^+,\Lambda}\right) = \frac{1}{2}\left[1 + N\left(\rho_{\Phi^+,\Lambda}\right)\right]$, the eigenvector $|\alpha\rangle$ corresponding to the negative eigenvalue $\lambda_{\min}\left(\rho^{\Gamma}_{\Phi^+,\Lambda}\right)$ is maximally entangled.

But then this implies that

$$F\left(\rho_{\Phi^+,\Lambda}\right) = \frac{1}{2}\left[1 + N\left(\rho_{\Phi^+,\Lambda}\right)\right] = \lambda_{\max}\left(\rho_{\Phi^+,\Lambda}\right) \tag{8.60}$$

because as we saw in section V B of the introduction chapter for any two qubit entangled density matrix σ, $F(\sigma) = \frac{1}{2}\left[1 + N(\sigma)\right]$ if and only if the eigenvector corresponding to the negative eigenvalue of σ^{Γ} is maximally entangled [56].The last equality in eqn. (8.60) follows from eqn. 8.44.

Now from theorem 1 we have,

$$F(\Lambda) = F\left(\rho_{\psi_0,\Lambda}\right) = \lambda_{\max}\left(\rho_{\Phi^+,\Lambda}\right) \tag{8.61}$$

where $|\psi_0\rangle$ is the eigenvector corresponding to the maximum eigenvalue of $\rho_{\Phi^+,\hat{\Lambda}}$. Now from Theorem 2 we know that $|\psi_0\rangle$ is necessarily non-maximally entangled when the channel Λ is nonunital. Thus for a nonunital channel Λ,

$$F(\rho_{\Phi^+,\Lambda}) < F(\Lambda) = \lambda_{\max}\left(\rho_{\Phi^+,\Lambda}\right) \tag{8.62}$$

which contradicts Eq. (8.60). □

This completes the proof of theorem 3.

8.4 Discussion

Shared entanglement is a critical resource for quantum information processing tasks such as quantum teleportation. Typically, quantum entanglement is shared by sending part of a pure entangled state through a quantum channel which, in practice is noisy. This results in mixed entangled states, purity of which is characterized by singlet fraction. Because faithful implementation of quantum information processing tasks require near-perfect entangled states (states with very high purity), a basic question is: What is the optimal singlet fraction attainable for a single use of a quantum channel Λ and trace-preserving local operations?

In this chapter, we obtained an exact expression of the optimal singlet fraction for a qubit channel and prescribed a protocol to attain the optimal value. The protocol consists of sending part of a pure entangled state $|\psi_0\rangle$ through the channel, where $|\psi_0\rangle$ is given by the eigenvector corresponding to the maximum eigenvalue of the density matrix $\rho_{\Phi^+,\hat{\Lambda}}$ ($\hat{\Lambda}$ is the channel dual to the qubit channel Λ). We have also shown that this "best" state $|\psi_0\rangle$ is maximally entangled for unital channels but must be nonmaximally entangled if the channel is nonunital. Interestingly, we find that in the optimal case no local post-processing is required even though it is known that TP LOCC can increase singlet fraction of a density matrix.

8.5 Appendix

A. Technical Lemma

Lemma 5. $\lambda_{\max}(\rho_{\Phi^+,\hat{\Lambda}}) = \lambda_{\max}(\rho_{\Phi^+,\Lambda})$

Proof. We first obtain a relationship between the states $\rho_{\Phi^+,\Lambda}$ and $\rho_{\Phi^+,\hat{\Lambda}}$. Recall that these states are given by

$$\rho_{\Phi^+,\Lambda} = \sum_i (I \otimes A_i)|\Phi^+\rangle\langle\Phi^+|(I \otimes A_i^\dagger). \tag{8.63}$$

$$\rho_{\Phi^+,\hat{\Lambda}} = \sum_i (I \otimes A_i^\dagger)|\Phi^+\rangle\langle\Phi^+|(I \otimes A_i). \tag{8.64}$$

Eqn. (8.64) can be written as,

$$\begin{aligned}\rho_{\Phi^+,\hat{\Lambda}} &= \sum_i ((A_i^\dagger)^T \otimes I)|\Phi^+\rangle\langle\Phi^+|(A_i^T \otimes I) \\ \implies \rho^*_{\Phi^+,\hat{\Lambda}} &= \sum_i (A_i \otimes I)|\Phi^+\rangle\langle\Phi^+|(A_i^\dagger \otimes I),\end{aligned} \tag{8.65}$$

where the complex conjugation is taken with respect to the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Now using the SWAP operator V defined by the action

$V|ij\rangle = |ji\rangle$, we have

$$\begin{aligned} (A_i \otimes I)|\Phi^+\rangle &= \frac{1}{\sqrt{2}}\sum_{k=0}^{1} A_i|k\rangle \otimes |k\rangle \quad \text{and so,} \\ V(A_i \otimes I)|\Phi^+\rangle &= \frac{1}{\sqrt{2}}\sum_{k=0}^{1} |k\rangle \otimes A_i|k\rangle \\ &= (I \otimes A_i)|\Phi^+\rangle. \end{aligned} \tag{8.66}$$

Hence,

$$\begin{aligned} \rho^*_{\Phi^+,\hat{\Lambda}} &= V^\dagger \rho_{\Phi^+,\Lambda} V, \\ \implies \rho_{\Phi^+,\hat{\Lambda}} &= \left(V^\dagger \rho_{\Phi^+,\Lambda} V\right)^*. \end{aligned} \tag{8.67}$$

From the above equation it therefore follows that

$$\lambda_{\max}(\rho_{\Phi^+,\hat{\Lambda}}) = \lambda_{\max}(\rho_{\Phi^+,\Lambda}). \tag{8.68}$$

Note that lemma 5 does not assume that Λ is a qubit channel. Also, from eqn. (8.67) it is clear that $\rho_{\Phi^+,\hat{\Lambda}}$ is a valid state even for a non-unital channel Λ for which the dual channel $\hat{\Lambda}$ is not trace preserving. But we will get unnormalized states if the dual channel acts on one side of some non-maximally entangled states.

B. Technical Lemma

Lemma 6. *Let $\sigma_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^2$ be a bipartite density matrix such that $Tr_B(\sigma_{AB}) = \frac{1}{2}I$. Then,*

$$\lambda_{\min}\left(\sigma_{AB}^\Gamma\right) + \lambda_{\max}\left(\sigma_{AB}\right) = \frac{1}{2} \tag{8.69}$$

where $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimum and maximum eigenvalue of $X \in \left\{\sigma_{AB}, \sigma_{AB}^\Gamma\right\}$ and Γ denotes partial transposition.

Proof. Let $\sigma_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^2$ be a bipartite density matrix such that $\mathrm{Tr}_B(\sigma_{AB}) = \frac{1}{2}I$. From the Choi-Jamiolkowski isomorphism ([41], [40]) we have that σ_{AB} can be written as ,

$$\sigma_{AB} = (I \otimes \Lambda)\left(|\Phi^+\rangle_{AB}\langle\Phi^+|\right),$$

where Λ is trace preserving completely positive map(TPCP), mapping $\mathcal{B}(\mathcal{C}^2)$ to itself.

In [42] it was shown that any such map Λ can be written as,

$$\Lambda(\rho) = U_1 \circ \Lambda' \circ U_2(\rho) \tag{8.70}$$

with Λ' being a canonical TPCP map and U_1 and U_2 being unitary maps. If $\rho = \frac{1}{2}(I + x\sigma_1 + y\sigma_2 + z\sigma_3)$ and $\rho' = \Lambda'(\rho) = \frac{1}{2}(I + x'\sigma_1 + y'\sigma_2 + z'\sigma_3)$ then in the Bloch sphere representation the map Λ' is given by,

$$\begin{bmatrix} 1 \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}, \tag{8.71}$$

with t_i and λ_i being real for all i.

Now as local unitaries do not affect the eigenvalues of σ_{AB} or σ_{AB}^{Γ} , for the rest of the proof we can focus on $(I \otimes \Lambda')(|\Phi^+\rangle\langle\Phi^+|) = \rho_{\Phi^+,\Lambda'}$ with the map Λ' given by eqn. (8.71) . We have,

$$\rho_{\Phi^+,\Lambda'} = \frac{1}{2} \begin{bmatrix} a & b & 0 & d \\ b^* & (1-a) & f & 0 \\ 0 & f & c & b \\ d & 0 & b^* & (1-c) \end{bmatrix} \tag{8.72}$$

, with $a = \frac{1+t_3+\lambda_3}{2}$, $b = \frac{t_1 - it_2}{2}$, $d = \frac{(\lambda_1+\lambda_2)}{2}$, $f = \frac{(\lambda_1-\lambda_2)}{2}$, $c = \frac{(1+t_3-\lambda_3)}{2}$. Now complete positivity of Λ' implies positivity of $\rho_{\Phi^+,\Lambda'}$ and hence the spectrum of $\rho_{\Phi^+,\Lambda'}$ is same as that of $\rho^*_{\Phi^+,\Lambda'}$. Now the eigenvalue equation of $\rho^*_{\Phi^+,\Lambda'}$ is

$$\begin{vmatrix} (\frac{a}{2} - \lambda) & \frac{b^*}{2} & 0 & \frac{d}{2} \\ \frac{b}{2} & (\frac{1-a}{2} - \lambda) & \frac{f}{2} & 0 \\ 0 & \frac{f}{2} & (\frac{c}{2} - \lambda) & \frac{b^*}{2} \\ \frac{d}{2} & 0 & \frac{b}{2} & (\frac{(1-c)}{2} - \lambda) \end{vmatrix} = 0. \tag{8.73}$$

Now, the partial transpose w.r.t first party of $\rho_{\Phi^+,\Lambda'}$ is given by,

$$\rho^{\Gamma}_{\Phi^+,\Lambda'} = \frac{1}{2} \begin{bmatrix} a & b & 0 & f \\ b^* & (1-a) & d & 0 \\ 0 & d & c & b \\ f & 0 & b^* & (1-c) \end{bmatrix}. \tag{8.74}$$

The eigenvalue equation of $\rho^{\Gamma}_{\Phi^+,\Lambda'}$ is given by,

$$\begin{vmatrix} (\frac{a}{2} - \lambda) & \frac{b}{2} & 0 & \frac{f}{2} \\ \frac{b^*}{2} & (\frac{(1-a)}{2} - \lambda) & \frac{d}{2} & 0 \\ 0 & \frac{d}{2} & (\frac{c}{2} - \lambda) & \frac{b}{2} \\ \frac{f}{2} & 0 & \frac{b^*}{2} & (\frac{(1-c)}{2} - \lambda) \end{vmatrix} = 0. \tag{8.75}$$

Replacing λ by $(\frac{1}{2} - \lambda')$, in eqn. (8.75) we have,

$$\begin{vmatrix} -(\frac{(1-a)}{2} - \lambda') & \frac{b}{2} & 0 & \frac{f}{2} \\ \frac{b^*}{2} & -(\frac{a}{2} - \lambda') & \frac{d}{2} & 0 \\ 0 & \frac{d}{2} & -(\frac{(1-c)}{2} - \lambda') & \frac{b}{2} \\ \frac{f}{2} & 0 & \frac{b^*}{2} & -(\frac{c}{2} - \lambda') \end{vmatrix} = 0. \tag{8.76}$$

In eqn. (8.76) performing the interchanges, column 1 $\Leftrightarrow$ column 2 and column 3 $\Leftrightarrow$ column 4 we have,

$$\begin{vmatrix} \frac{b}{2} & -(\frac{(1-a)}{2}-\lambda') & \frac{f}{2} & 0 \\ -(\frac{a}{2}-\lambda') & \frac{b^*}{2} & 0 & \frac{d}{2} \\ \frac{d}{2} & 0 & \frac{b}{2} & -(\frac{(1-c)}{2}-\lambda') \\ 0 & \frac{f}{2} & -(\frac{c}{2}-\lambda') & \frac{b^*}{2} \end{vmatrix} = 0. \tag{8.77}$$

In eqn. (8.77) performing the interchanges, row 1 $\Leftrightarrow$ row 2 and row 3 $\Leftrightarrow$ row 4 we have,

$$\begin{vmatrix} -(\frac{a}{2}-\lambda') & \frac{b^*}{2} & 0 & \frac{d}{2} \\ \frac{b}{2} & -(\frac{(1-a)}{2}-\lambda') & \frac{f}{2} & 0 \\ 0 & \frac{f}{2} & -(\frac{c}{2}-\lambda') & \frac{b^*}{2} \\ \frac{d}{2} & 0 & \frac{b}{2} & -(\frac{(1-c)}{2}-\lambda') \end{vmatrix} = 0. \tag{8.78}$$

Now multiplying the 1st row by -1, 2nd column by -1, 3rd row by -1 and 4th column by -1 successively in eqn. (8.78) we get back eqn. (8.73) . Thus if eigenvalues of $\rho_{\Phi^+,\Lambda'}$ are λ_i with $i = 1, 2, 3, 4$, that of $\rho^{\Gamma}_{\Phi^+,\Lambda'}$ are $(\frac{1}{2} - \lambda_i)$. Thus we have,

$$\begin{aligned} \lambda_{\min}(\rho^{\Gamma}_{\Phi^+,\Lambda'}) &= \frac{1}{2} - \lambda_{\max}(\rho_{\Phi^+,\Lambda'}) \\ \Rightarrow \lambda_{\min}(\rho^{\Gamma}_{\Phi^+,\Lambda'}) + \lambda_{\max}(\rho_{\Phi^+,\Lambda'}) &= \frac{1}{2} \\ \Rightarrow \lambda_{\min}(\sigma^{\Gamma}_{AB}) + \lambda_{\max}(\sigma_{AB}) &= \frac{1}{2}. \end{aligned} \tag{8.79}$$

Chapter 9

Conclusions

Here we provide a summary of the main results obtained in this thesis and discuss future directions of research that can be taken.

In chapter 6 , we have considered approximate joint measurement of two and three qubit observables separately, through an Arthur-Kelly like model for qubit observables.This model comes naturally when one considers a Stern-Gerlach setup with a linear magnetic field. In the Stern-Gerlach setup the momenta of the atoms acts as pointer observables for their spin degrees of freedom. Considering approximate joint measurement of σ_x and σ_y through this model, we have shown here numerically that the measurement uncertainty relations derived elsewhere(see ref.[22]) hold. It has also been shown how increasing the relative sharpness of the initial momentum wavefunctions of the detectors leads measurement of one observable to become almost sharp while that of the other to become almost trivial. Effect of initial detector states on the post-measurement state of the system has also been considered. The action of the measurement interaction on the system turns out to be that of an asymmetric depolarising channel. This forms the basis of a physical understanding of the origin of complementarity (between σ_x and σ_y measurement)in the model. We also see an indication of the entanglement between the system and detectors increasing as one of the measurements becomes sharper.

We have considered two different characterisations of unsharpness. Firstly, by comparing the probability distribution of the values of observable to be approximately measured with that of the approximate observable. Secondly, by considering the alignment of the momentum direction with the spin observable in the Heisenberg picture. We have shown that for the case in which both the observables are approximated equally well, the corresponding measures of unsharpness are proportional. For our choice of pointer observable the measures checking the alignment and disturbance due to measurement do not seem to satisfy an error-disturbance relationship.

We have expounded the connection between the symmetries of the underlying Hamiltonian for measurement interaction and initial detector states with the joint measurement in a lemma.This was first stated by Martens et al.([46]). It has then been used to perform approximate joint measurement in arbitrary directions. The POVM elements calculated for the same match with that which were found earlier in the orthogonal case. They also turn as expected to that of a single unsharp measurement when the directions are taken to be almost same.

For the case of joint unsharp measurement of three qubit observables we have given a necessary condition to be satisfied by the parameters of the marginal POVM elements . This condition has been derived from certain geometric considerations involving the so called Fermat-Toricelli point . The condition is sufficient for the case of three orthogonal observables and for *unbiased* observables in general.

The measurement scheme employed by us for joint measurement in non-orthogonal directions cannot take advantage of the greater freedom available for better approximation compared to the orthogonal case. It will be interesting to see such a scheme in the Arthur-Kelly setup that is able to get close to the bound set by eqn.(2.24) for arbitrary directions. We have shown that the retrodictive fidelity η_i and the unsharpness a' are proportional for the symmetric joint unsharp measurement case that we have considered. It may be possible to connect the two pictures in a more general setting starting with certain symmetries of the Hamiltonian and initial detector states. The problem of determining necessary-sufficient conditions for the most general joint measurement of three observables by extension of our approach or otherwise is also left open.

In chapter 7, inspired by the notion of entanglement breaking we have investigated qubit channels through their property of 'non-locality breaking'. We have focused on CHSH nonlocality as this is the only inequality for which the necessary and sufficient conditions on the state for violation are known. One of the main properties of entanglement breaking channels is that it is sufficient to 'break' the entanglement of maximally entangled states. We have provided examples to show that similar property does not hold for 'non-locality breaking'.Though there seems to be some channels and a certain restricted class of states for all channels for which this is true. We ahve also considered a stronger notion of non-locality breaking, again taking cue from entanglement breaking where the output states of one-sided action of the channel are required to be local under SLOCC. We show that for a qubit channel to be strongly non-locality breaking it is enough for the dual-state of the channel to not show any hidden nonlocality under local filtering.

We have provided a closed-form necessary sufficient condition for any two-qubit state to violate the Bell-CHSH inequality under local filtering,which is likely to be useful for other purposes as well. This is then used to study 'strongly non-locality breaking' qubit channels and compute their relative volume within the set of all channels. It turns out that unital qubit channels breaking non-locality of maximally entangled states are strongly non-locality breaking while extremal qubit channels cannot be so unless they are entanglement breaking.

An interesting course of future study is to see how the gap between entanglement breaking and non-locality breaking qubit channels close as one considers more inequalities (e.g,I_{3322} [36]). We also, at present, do not have any example of a channel which breaks the non-locality of a maximally entangled state, *genuinely* but fails to break that of other states. This can be studied, for example, for the channel in Section V whose dual state has a local model.

In chapter 8, we have obtained an exact expression of the optimal singlet fraction for a qubit channel and prescribed a protocol to attain the optimal value. The protocol consists of sending part of a pure entangled state $|\psi_0\rangle$ through the channel, where $|\psi_0\rangle$ is given by the eigenvector corresponding to the maximum eigenvalue of the density matrix $\rho_{\Phi^+,\hat{\Lambda}}$ ($\hat{\Lambda}$ is the channel dual to the qubit channel Λ). We have also shown that this "best" state $|\psi_0\rangle$ is maximally entangled for unital channels but must be nonmaximally entangled if the channel is nonunital. Interestingly, we find that in the optimal case no local post-processing is required even though it is known that TP LOCC can increase singlet fraction of a density matrix.

An obvious course of study is to see to what extent our results extend to optimal singlet fraction for channels acting on higher dimensional systems.In particular it will be interesting to check if we see the same kind of division, between channels which can be expressed as a convex combination of unitary channels and those which cannot be expressed in a similar manner, as we have seen for the case of qubit channels.

List of Publications

1. **Approximate joint measurement of qubit observables through an Arthur - Kelly model**,

   Rajarshi Pal and Sibasish Ghosh,

   2011 J. Phys. A: Math. Theor. 44 485303 doi:10.1088/1751-8113/44/48/485303

2. **Non-locality breaking qubit channels**,

   Rajarshi Pal and Sibasish Ghosh,

   under review in J.Phys.A

   A part of this work has got accepted as a poster in QIP 2015, arxiv-quantph:1410.7574

3. **Entanglement sharing through noisy qubit channels: One-shot optimal singlet fraction**,

   Rajarshi Pal, Somshubhro Bandyopadhyay, and Sibasish Ghosh

   Phys. Rev. A 90, 052304, 2014.

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