STUDIES IN

COLLECTIVE EXCITATIONS IN MANY-PARTICLE SYSTEMS

THESIS

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BY

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R Sridhar
(R.SRIDHAR)

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INTRODUCTION

The object of the present study is to investigate the collective phenomena in many-particle systems - both bosons and fermions. Such collective effects like the plasmon excitations in an electron gas and phonon and roton excitations in a many-boson system are important and wellknown[1]. An attempt is made in this thesis to study these phenomena from a somewhat different view point, which has led to some new and interesting results.

A theory in terms of currents and densities as co-ordinates has been proposed to describe an interacting Bose system[2]. For sufficiently low temperatures, the slope of the linear excitation spectrum is found to be independent of temperature. In this respect these excitations resemble the plasmon excitations in a Fermi liquid[3]. The new description in terms of the collective variables - the currents and the densities - employs a closure procedure which differs slightly from the ones used in the conventional theories. It is hoped that this will lead to more realistic and physically significant approximation procedures. Further the sum rules governing the many Boson system have been studied for the excited state proposed in this theory[4].

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1)D.Pines,The Many Body Problem - W.A.Benjamin,Inc.,N.Y.,(1961)
2)R.Vasudevan,R.Sridhar and N.R.Ranganathan,Phys.Letts.29A (1969) 138
3)D.Pines,Quantum Liquids - Ed. D.F.Brewer, North-Holland (1966)
4)R.Sridhar, MATSCIENCE Preprint (1971)

A system of interacting Fermions, in an externally produced, time independent and uniform magnetic field has been studied in the frame work of the formalism developed by Prigogine and coworkers[5]. The Liouville equation is solved formally and a subset of this formal solution is summed to get the dielectric function. An interesting feature of this formulation is a new initial Liouville density constructed by using the Landau wave functions and the Fermi distribution. Because of this choice, the formalism is quasi classical in nature. It is found that for purely magnetic interactions (through a transverse electromagnetic field) the thermodynamic quantities exhibit Bessel function oscillations as functions of the inverse of the magnitude of the externally applied magnetic field[6]. A study of the optics of the problem exhibits interesting features. Interactions through a scalar field[7] are important to discuss the plasma behaviour. A study in this regime yields stability criterion against instabilities that arise purely from the collective interactions. These are also characteristic of the initial preparation of the system. A study in longitudinal interactions[8], which is needed for a complete understanding, reveals similar

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5) (i) I.Prigogine, "Non-Equilibrium Statistical Mechanics" Interscience, New York (1963)
(ii) R.Balescu, "Statistical Mechanics of charged particles" Interscience, New York (1963)
6) R.Pratap, R.Vasudevan and R.Sridhar, IL Nuovo Cimento to be published (1971)
7) R.Pratap and R.Sridhar, IL Nuovo Cimento (1971) (to appear)
8) R.Sridhar, MATSCIENCE Preprint (1971)

features. The dispersion relation and the associated stability condition for these interactions have also been obtained.

The first chapter deals with collective excitations in a superfluid Bose system without putting in the Bose-Einstein condensation in a priori fashion[3]. An attractive feature of this treatment is the formulation in terms of the physically relevant quantities, the number density $\rho(\vec{x})$ and the current density in contrast to the conventional formalism in terms of the field operators $\psi(\vec{x})$ and $\psi^{\dagger}(\vec{x})$. Such a formulation is not completely unknown in superfluid physics. Landau[9], Khalatnikov[10] and later Gross[11] have studied the hydrodynamics of superfluid systems in terms of such variables. However the formalism proposed by Dashen and Sharp[12] based on currents and densities has aroused a new wave of interest in this formulation. In this formulation the many body hamiltonian is written in terms of currents and densities and the wave functions are the elements of a vector space formed by a complete set of eigenvectors of a commuting set of operators, namely, the number density operator $\rho(\vec{x})$. An unwelcome feature of this formulation, however, is the presence of the inverse of the density operator. A method has been devised to overcome this difficulty in the first approximation. This

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9) L.D.Landau, Jour.Phys. (U.S.S.R) 11 (1947) 91.
10) I.M.Khalatnikov,"Introduction to the theory of superfluidity" W.A.Benjamin Inc., (1965)
11) E.P.Gross, Quantum Fluids, Ed.D.F.Brewer, North-Holland Publishing Company, Amsterdam (1966)
12) R.F.Dashen and D.H.Sharp, Phys. Rev. 165 (1968) 1857

enquiry is motivated by the experimental results of Woods[13] which demonstrated the existence of the phonon spectrum even above the λ transition temperature contradicting the prediction of Bogoliubov like theories[14]. Later Pines[2] suggested that these excitations might be the analogue in a neutral system, of the plasmons in a Fermi liquid as proposed by Landau[15]. The present investigation is based on the equation of motion method in which the excitation operator is constructed on the basis of physical considerations in a manner analogous to that of Feynman[16]. In this excitation operator the backflow is taken into account to some extent. It is found that the slope of the phonon spectrum is independent of the condensate thus corraborating the experimental evidence of Woods. These results are compared with those of the extreme weak coupling limit as proposed by Bogoliubov.

Chapter II deals with the evaluvation of the first few moments of the dynamic form factor in the excited state constructed in Chapter I. An interesting feature of these moments is the presence of terms which can be interpreted as the quantum pressure terms. Further there are terms which are bilinear and higher order in these operators. Such terms describe the scatt-

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13) A.D.B.Woods, Phys.Rev.Lett. 14 (1965) 355.
14) P.C.Hohenberg and P.C.Martin, "Phys.Rev. Lett., 12 (1964) 69. K.Huang, "The Many Body Problem", John-Wiley & Sons Inc. (1958).
15) L.D.Landau, J.Phys. (U.S.S.R) 5 (1941) 71.
16) R.P.Feynman, Phys. Rev. 94 (1954) 262.

ering mechanism among higher order excitations as well as production of such excitations. These moments are evaluvated in weak coupling approximation also. By requiring that the f-sum rule be satisfied the co-efficients in the excitation operator are determined. Also the excitation spectrum is obtained in a form analogous to the one obtained by Feynman[16].

The statistical mechanics of an electron gas in a constant external magnetic field is discussed in Chapter III in the framework of the formalism developed by Brussels school[5]. The formal solution of the Liouville equation is obtained in the form of an infinite series. A finite subseries is selected on the basis of a diagram technique relevant for short time scales and this subset is summed exactly. This process gives the dielectric function which describes the collective effects in the system. An important aspect of this formulation is the new choice of the initial density distribution function. This takes into account the Landau wave functions of the electrons and the Fermi distribution. The Fermi distribution denotes the fact that at the initial time the system is in thermodynamic equilibrium. The distribution has an asymmetry in the y direction when the magnetic field is taken along the z -axis of the Cartesian co-ordinate system. As the quantum effects have been introduced by the inclusion of the Landau levels and the Fermi distribution, the motion variable (in the transverse plane) is restricted to take only discrete values consistent with the

quantisation of energy. The criteria for the choice of the subset of diagrams is discussed on the basis of time scales which are relevant for the problem. The nature of the vertex operators are discussed with reference to the initial condition. Another special feature of this formalism is the appearance of two non-commuting operators in the propagator for correlations. This special structure of the propagator makes it difficult for straightforward applications. Thus the two non commuting operators have to be decoupled. This decoupling is achieved through the use of Baker-Hausdorff relation. This procedure is discussed in detail in Appendix 1. Chapter III further, deals with a factorization theorem due to Resibois[17] which is useful in simplifying the higher order contributions. The time ordering of the interactions makes the calculations tedious. A slight simplification can be introduced by resorting to Laplace transform.

Chapters IV, V and VI deal with special aspects of the general formulation presented in chapter III. Collective effects due to purely magnetic interactions (transverse electromagnetic interactions) and the consequent oscillations in an electron plasma are discussed in chapter IV. The system may be compared with the conduction electrons in a metal where the ions play no dynamical role, The dielectric function exhibits an oscillation of the Bessel function of order zero in the inverse of the magnitude of the applied field. Further this function is related to

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17) Resibois, Phys.Fluids, 6 (1963) 817.

the refractive index and can be employed in the study of frequencies of absorption and emission.

Chapters V and VI deal essentially with the plasma aspect of the problem. The case of interactions through a scalar field are considered in chapter V, while chapter VI deals with the case of interactions through a longitudinal electromagnetic field. The dielectric function and hence the dispersion equations are obtained for these two cases. It is found that the instabilities that arise due to the dielectric function have their origin in the collective effects. For the case of interactions through a scalar potential it is found that there can be no instabilities when the electromagnetic waves propagate purely along the direction of the magnetic field or in any direction transverse to the direction of the magnetic field. The dispersion relation obtained are valid for weak magnetic fields only. These relations are discussed in the asymptotic time limit. Extension of this work to include a more detailed study of the instabilities is in progress. The modifications of these results at high magnetic fields and high temperatures are also to be studied in the future.

The only available reprint of the paper entitled "Study of interacting Bose gas with currents and densities as coordinates" (with R.Vasudevan and N.R.Ranganathan, Phys.Letts.29A (1969) 138) is enclosed at the end of the thesis (Please see back cover). Further the reprints of the following papers are

also enclosed in support of this thesis:

"Photoproduction of charged pions from ^{11}B" (with V.Devanathan and K.Srinivasa Rao, Phys.Letts.25B (1967) 456).

"Regge Pole model for A_2 meson production" (with A.Sundaram, IL Nuovo Cimento 50 (1967) 969).

CHAPTER. I

COLLECTIVE EXCITATIONS IN A SUPERFLUID BOSE SYSTEM+

1. Introduction:

Collective excitations in a superfluid system of bosons is studied without putting in the Bose-Einstein condensation (B.E. condensation) in an a priori fashion. The system is described in terms of the following physical quantities: the number density, $\rho(\vec{x})$ and the current $\vec{J}(\vec{x})$, which as formulated here in take into account the Bose commutation relations and hence reflect all the properties of a Bose system. Further it is examined whether the excitation spectrum for the system arrived at in this formulation depends upon the condensate at low momenta as has been found by Bogoliubov[1]. The enquiry is motivated by the experimental results of Woods[2] which do not corraborate the results obtained by the wellknown Bogoliubov theory in the phonon region.

Following the classic work of Bogoliubov it is usual to assume that N_0, the number of particles in the zero momentum (single particle) state is macroscopic and hence it plays a dominant role in determining the physical properties of the system at low temperatures. For a non-interacting system at low temperatures N_0 can be taken to be of the order of N, the total number of particles in the system. However Bogoliubov proposed that this

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+ R.Vasudevan, R.Sridhar and N.R.Ranganathan, Phys.Letts. 29A (1969), 138.
1) N.N.Bogoliubov, J.Phys. (U.S.S.R.) 9 (1947), 23.
2) A.D.B.Woods, Phys.Rev.Letts. 14 (1965), 355.

property continues to hold good even when the interactions are present provided they are weak. One-particle states with momenta greater than zero are not supposed to possess this property of macroscopic occupation though there is no convincing argument that this should be so. Recently there has been an attempt by Coniglio and Vasudevan[3] to study the consequences of Bose-Einstein condensation in a small strip above $k=0$ state in the momentum space. They find that for a particular choice of the potential, Bogoliubov type of transformations and procedures lead to a stable energy spectrum which exhibits a gap at finite volume. The gap, however, vanishes at infinite volume. Still, so far, no valid argument has been put forward for assuming $N_0 \sim N$ in the thermodynamic limit[4] for arbitrary interactions. Therefore it is worthwhile to develop a theory of superfluid Bose systems without "apriori" assumptions on the number distribution in the momentum space. In the well-known theories based on the formulation of Bogoliubov the annihilation and creation operators of the zero momentum state are replaced by their ground state expectation value $\sqrt{N_0}$. In the interaction part of the hamiltonian Bogoliubov retained only those terms which involve bilinear combinations of annihilation and creation operators for the zero momentum state and diagonalised the resulting hamiltonian by means of a unitary transformation

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3) A.Coniglio and R.Vasudevan, Il Nuovo Cimento 70 (1970), 39.
4) For a system of N particles occupying a volume Ω, the thermodynamic limit is specified as $N \to \infty$, $\Omega \to \infty$ with the condition that $(N/\Omega) \to$ a finite constant, d.

familiarly known as the Bogoliubov canonical transformation. Subsequently there have been attempts[5] to generalise this treatment by including some more terms which were neglected by Bogoliubov. The details of these approaches are not discussed here. Bogoliubov got the following excitation spectrum[6]

$$\omega(k) = \left[(k^4/4) + k^2 N_0 \bar{V}(k) \right]^{1/2} \tag{1.1}$$

where the Fourier transform $\bar{V}(k)$ of the interaction potential is assumed to be a positive constant and at low momenta the excitation energy is proportional to k, the proportionality constant being the velocity of propagation for the excitation in the medium.

$$\omega(k) \sim ck \quad \text{(for small } k\text{)} \tag{1.2}$$

the propagation velocity c being given by

$$c = \lim_{k \to 0} \left[N_0 \bar{V}(k) \right]^{1/2} \tag{1.3}$$

Because of the macroscopic occupation of the zero momentum state N_0 is often approximated by N itself. However, it should be emphasised here that the slope c of the phonon spectrum depends directly upon the condensate number. This is the characteristic feature of the Bogoliubov like theories. A detailed study by Hohenberg and Martin[7] predicted that the slope of the excitation

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5) M.Girardeau and R.Arnowitt, Phys.Rev. 113 (1959),755.
S.Butler and J.Valatin, Nuovo Cimento 10 (1958), 37.
These theories give rise to a gap in the excitation spectrum which is not physical.

6) Throughout this chapter the following units are employed: $\hbar = m = 1$, m being the mass of the particle.

7) P.C.Hohenberg and P.C.Martin, Phys.Rev.Letts. 12 (1964), 69.

spectrum in He II at low momenta is proportional to $\sqrt{\rho_s/\rho}$ where ρ_s is the superfluid component of the total density ρ. The quantity ρ_s is strongly dependent upon temperature. At T = 0, ρ_s is of the order of ρ and at $T = T_\lambda$ it is vanishingly small. Thus according to Hohenberg and Martin also the slope of the excitation spectrum should exhibit a strong temperature dependence. Huang[8], in an earlier calculation, also came to a similar conclusion.

However, this prediction, was not supported by an experiment of Woods[2,9] on the inelastic scattering of neutrons by liquid helium. The experiment was carried out using a rotating crystal spectrameter with an incident neutron wave length of 4.05 Å (energy = 5.0 mev). The specimen temperatures ranged from 1.5 to 4.2° K and the excitation spectrum was studied with help of neutron scattering at small momentum transfers. The experiment showed that the velocity of phonons with momentum 0.38 Å is essentially independent of temperature for all temperatures less than 2.57°K. The small increase of the velocity with the temperature that has been observed is not a significant one. The phonon branch of the spectrum was present even at 4.12°K, the boiling point of liquid helium. Woods further observed that neither the energy nor the life-time of these excitations change appreciably while going through the phase transition. The observations of Woods led to a review of the

8) K.Huang, The Many Body Problem, John Wiley & Son Inc. (1958)
9) For earlier experimental attempts at temperatures below T_λ see D.G.Henshaw, Phys.Rev.Letts. 1 (1958), 127.
D.G.Henshaw and A.D.B.Woods, Phys. Rev. 121 (1961), 1266.

earlier theoretical formulations with a special reference to the ideas on the B.E.condensation in the zero momentum state.

In the international symposium on quantum fluids held at Sussex in 1965, Pines[10] proposed a possible explanation for Woods' experimental results. He pointed out that the experiment was carried out in a "collisionless" regime for which

$$\omega_k \tau >> 1 \tag{1.4}$$

where ω_k is the phonon energy and τ is its life-time. Further the results obtained by Woods resemble very much the plasmon behaviour in an electron liquid. In the long wave length limit the plasmon energy ω_{pl} given by

$$\omega_{pl} = \sqrt{4\pi n e^2/m} \tag{1.5}$$

is not altered by a transition from the normal to the superconducting state. Further in super conductors the plasmon energy is independent of ρ_s . Thus Pines proposed that the observed excitations in liquid helium actually correspond to zero ~~soul~~ Sound oscillations. These oscillations in a neutral system are totally analogous to the plasmons in a Fermi liquid proposed by Landau[11]. There should be a restoring force which is responsible for these oscillations. In the present case, according to Pines, it is the averaged self consistent field of all the particles acting in unison and thus is very familiar in the high energy or "collisionless" regime. On the other hand in the low frequency or the

10) D.Pines, Quantum Fluids, edited by D.F.Brewer, North-Holland Publishing Company, Amsterdam (1966), Page.257
11) L.D.Landau, J.Phys. (U.S.S.R) 5 (1941), 71.

hydrodynamic regime the frequent collisions between the thermal excitations provide the restoring force. The first sound oscillations of the earlier theories are associated with the hydrodynamic regime for which $\omega_k \tau << 1$

Etters[12] has constructed a modified density fluctuation operator following the ideas of Pines. This operator is a linear combination of the planewave products $a^{\dagger}_{p+k} a_p$ viz.,

$$\mu^{\dagger}_k = \sum_p g(p,k)\, a^{\dagger}_{p+k} a_p \tag{1.6}$$

where the C -number coefficients $g(p,k)$ are to be determined from the requirement that $\mu^{\dagger}_k$ should satisfy the equation of motion to as good an approximation as possible. If in (1.6) $g(p,k)$ is put equal to unity the kth fourier component of the total density is obtained. Thus the choice of $\mu^{\dagger}_k$ may be considered as the kth fourier component of the density for a distribution of smears rather than point particles. The equation of motion leads to the following equation for $\omega(k)$

$$1 = \overline{V}(k) \sum_p \frac{N_p - N_{p+k}}{\omega(k) + T_p - T_{p+k}} \tag{1.7}$$

where $T_k = k^2/2$.

The right side of (1.7) is expanded in terms of the quantity

$$\lambda^2 = (T_{p+k} - T_p)^2 / \omega^2(k) \tag{1.8}$$

with the assumption that λ is small. For the second order in λ the following excitation spectrum is obtained under the

12) R.D.Etters, Nuot Cimento 44B (1966), 68. Nuovo

Random Phase Approximation (R.P.A)

$$\omega^2(k) = k^2 \left[\frac{N\tilde{V}(k)}{m} + 2\frac{\bar{T}_i}{m} \right] \tag{1.9}$$

$\bar{T}_i$ being the average kinetic energy of the individual particles. At low momenta the coefficients $g(p,k)$ are almost independent of p as a result of which

$$\mu_k^+ \sim \rho_k^+ \tag{1.10}$$

The aim of this chapter is to study the collective excitations by employing a theory which uses the currents and the densities as co ordinates in contradistinction to the conventional approaches which use the field operators ψ and ψ^+ as the basic quantities to build up a microscopic theory. This type of approach is not completely unknown in the theory of liquid helium and has been used to some extent by Landau[13], Khalatnikov[14] and later by Gross[15] in describing the superfluid flow properties. However interest in this method of approach has been revived by the recent work of Dashen and Sharp[16] who have suggested a formulation of non-relativistic quantum mechanics in terms of observables like currents and densities. They have printed out the

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13) L.D.Landau, J.Phys. (U.S.S.R) 11 (1947), 91.
14) I.M.Khalatnikov, "Introduction to the theory of superfluidity". W.A.Benjamin Inc. (1965).
15) (i) E.P.Gross, "Quantum Fluids" Edited by D.F.Brewer, North-Holland Publishing Company, Amsterdam (1966)
(ii) D.Amit and E.P.Gross, Phys. Rev. 145, (1966), 130.
16) R.F.Dashen and D.H.Sharp, Phys. Rev. 165, (1968), 1857.

inadequacy of the approaches using the fields as the "building blocks" to describe elementary particles when the number of these particles itself is ever increasing. They further suggest that a formulation in terms of currents and densities may lead to different and probably better approximation schemes.

After this suggestion Pardee et al[17] have made an attempt in the exact formulation and solution of the Schrodinger equation in terms of currents and densities. They have presented a single particle theory and have discussed the application to some many body problems.

In this chapter the many boson problem is formulated in terms of currents and densities. An excitation operator is constructed following the arguments of Feynman[18] and Pines[10] and the equation of motion is solved in a new approximation scheme. This approximation procedure is identical to the wellknown Random Phase Approximation[19] (R.P.A) when the equation of motion involving the interaction Hamiltonian is solved while the approximation procedure employed for solving the equation involving the free part of the Hamiltonian has no analogue in the conventional theories.

2. Formulation of the theory:

A system of N bosons interacting through a two particle potential in a cubic box of unit volume is considered. It is

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17) W.J.Pardee, L.Schlessinger and Jon Wright, Phys.Rev. 175, (1968), 2140.
18) R.P.Feynman, Phys.Rev. 94, (1954), 262.
19) See for instance H.Suhl and N.Werthamer, Phys.Rev. 122, (1961), 359.

assumed that the Fourier transform of the pairwise interaction exists. The usual formulation of this problem is in terms of the canonical field operators ψ and $\psi^{\dagger}$ which satisfy the following commutation relations.

$$[\psi(\vec{x}), \psi(\vec{y})] = [\psi^{\dagger}(\vec{x}), \psi^{\dagger}(\vec{y})] = 0$$

$$[\psi(\vec{x}), \psi^{\dagger}(\vec{y})] = \delta(\vec{x}-\vec{y}) \tag{2.1}$$

The Hamiltonian of the system is written as

$$H = \frac{1}{2}\int d\vec{x}\ \nabla\psi^{\dagger}(\vec{x}) . \nabla\psi(\vec{x})$$

$$+ \frac{1}{2}\int d\vec{x}\int d\vec{y}\ \psi^{\dagger}(\vec{x})\ \psi^{\dagger}(\vec{y})\ V(|\vec{x}-\vec{y}|)\ \psi(\vec{x})\ \psi(\vec{y}) \tag{2.2}$$

The object now is to go over to a representation in terms of currents and densities which are defined as

$$\rho(\vec{x}) = \psi^{\dagger}(\vec{x})\ \psi(\vec{x}) \tag{2.3}$$

$$\vec{J}(\vec{x}) = \frac{1}{2i}\left[\psi^{\dagger}(\vec{x})\ \nabla\psi(\vec{x}) - \nabla\psi^{\dagger}(\vec{x})\ \psi(\vec{x})\right] \tag{2.4}$$

$\rho(\vec{x})$ and $\vec{J}(\vec{x})$ satisfy the following commutation relations

$$[\rho(\vec{x}), \rho(\vec{y})] = 0 \tag{2.5}$$

$$[\rho(\vec{x}), J_{\mu}(\vec{y})] = -i\frac{\partial}{\partial x_{\mu}}\left[\delta(\vec{x}-\vec{y})\rho(\vec{x})\right] \tag{2.6}$$

$$[J_{\mu}(\vec{x}), J_{\nu}(\vec{y})] = -i\frac{\partial}{\partial x_{\nu}}\left[\delta(\vec{x}-\vec{y})J_{\mu}(\vec{x})\right] + i\frac{\partial}{\partial y_{\mu}}\left[\delta(\vec{x}-\vec{y})J_{\nu}(\vec{y})\right] \tag{2.7}$$

where μ, ν = 1,2,3 denote the spatial components of the

current operator. Dashen and Sharp[16] have shown that these operators form an irreducible set and that any observable can be expressed in terms of these operators. For example, the momentum $\vec{P}$ is given by

$$\vec{P} = \int d\vec{x} \; \vec{J}(\vec{x}) \tag{2.8}$$

and the total number N is given by

$$N = \int d\vec{x} \; \rho(\vec{x}) \tag{2.9}$$

The Hamiltonian of the system takes the following form

$$H = H_0 + H_I \tag{2.10}$$

where

$$H_0 = \frac{1}{8}\int d\vec{x} \left[\nabla\rho(\vec{x}) - 2i\vec{J}(\vec{x})\right]\left[\rho(\vec{x})\right]^{-1}\left[\nabla\rho(\vec{x}) + 2i\vec{J}(\vec{x})\right] \tag{2.11}$$

$$H_I = \frac{1}{2}\int d\vec{x}\int d\vec{y} \; \rho(\vec{x})\, V(|\vec{x}-\vec{y}|)\, \rho(\vec{y}) + \frac{1}{2}\int d\vec{x} \; \rho(\vec{x})\, V(\vec{x}=0) \tag{2.12}$$

In (2.10) H_0 is the kinetic energy part of the Hamiltonian and H_I gives the interaction energy. The second term on the right side of equation (2.12) may be neglected as it is a constant. It will not contribute to the excitation energy as it commutes with any excitation operator.

Sharp[20] has introduced a functional representation for the algebra given by equations (2.5),(2.6) and (2.7). An abstract

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20) D.H.Sharp, Phys.Rev. 165 (1968) 1867.

state in the Hilbert space is represented by its components along a basis formed by the eigenvectors of a complete set of commuting current operators. In this formalism, $\rho(\vec{x})$ is the only operator which satisfies this criterion and accordingly the states are labelled by the eigenvalues $\langle \rho(\vec{x}) \rangle$ of $\rho(\vec{x})$

$$\rho(\vec{x}) \, | \rho \rangle = \langle \rho(\vec{x}) \rangle \, | \rho \rangle \tag{2.13}$$

The components of an arbitrary state $| \Psi \rangle$ along such a basis is then a wave functional

$$\Psi(\rho) = \langle \rho | \Psi \rangle \tag{2.14}$$

In this basis the action of $\rho(\vec{x})$ on $| \Psi \rangle$ is just the multiplication of $\Psi(\rho)$ by the eigenvalue $\langle \rho(\vec{x}) \rangle$ and the action of $\vec{J}(\vec{x})$ on these states is represented by the following functional derivative.

$$J_\mu(\vec{x}) \longrightarrow i \, \rho(\vec{x}) \frac{\partial}{\partial x_\mu} \frac{\delta}{\delta \rho(\vec{x})} \qquad (\mu = 1,2,3) \tag{2.15}$$

The energy spectrum of the system is given by

$$H \Psi(\rho) = E \Psi(\rho) \tag{2.16}$$

Defining a scalar product in this functional basis requires a knowledge of a measure on the space of all functions satisfying the constraints

$$\rho(\vec{x}) > 0 \qquad \text{and} \qquad \int d\vec{x} \, \rho(\vec{x}) = N \tag{2.17}$$

This point need not be elaborated further as there is no need to evaluvate any scalar product in the calculations that follow.

3. <u>The Excitation Operator</u>:

The form of the excitation operator can be guessed from the arguments of Feynman[18] who has calculated in a general way the wave functions for the low-lying excited states. In the formulation of Feynman, the excited state $|k\rangle_F$ is obtained from the ground state by the application of the operator $\sum_j \exp(i\vec{k}.\vec{x}_j)$ which is just the Fourier transform of the particle density operator in the configuration space given by

$$\rho(\vec{x}) = \sum_j \delta(\vec{x} - \vec{x}_j) \tag{3.1}$$

where $\vec{x}_j$ denotes the position of the jth particle. It can be easily seen that in the second quantised version $\sum_j \exp(i\vec{k}.\vec{x}_j)$ is just the operator $\tilde{\rho}_{\vec{k}}$ defined by

$$\tilde{\rho}_{\vec{k}} = \sum_p a^{+}_{p+k} a_p \tag{3.2}$$

where a^{+}_p and a_p denote the creation and annihilation operators for the particle in state $|p\rangle$. Thus according to Feynman the excited state is given by

$$|k\rangle_F = \tilde{\rho}_{\vec{k}} |0\rangle \tag{3.3}$$

where $|0\rangle$ denotes the physical vacuum. With this excited state Feynman calculated the structure factor $S(\vec{k})$ and the

excitation spectrum was expressed in the following manner

$$\omega(\vec{k}) = k^2 / 2S(\vec{k}) \tag{3.4}$$

This spectrum agrees with the experiment in the phonon region but not in the roton region. Later the wave function given by (3.3) was improved by Feynman and Cohen[21] who took into account the backflow around a given particle as it moves through the medium. By this correction the fit with the experimental curve was improved. This clearly stresses the need to add more terms to the excitation operator $\tilde{\rho}^{\dagger}_{\vec{k}}$. The simplest choice that one can think of is the longitudinal projection of the kth Fourier component of the current density fluctuation. This will naturally include some of the consequences of the backflow. Thus a term $\vec{k}.\tilde{J}^{\dagger}(\vec{k})$ is included to the excitation given by (3.3). The density fluctuation $\tilde{\rho}^{\dagger}_{\vec{k}}$ is created by a moving particle in the superfluid and there will be a comoving cloud of virtual excitations which try to reduce the energy of this particle and conserve the current. It is easily seen that the addition of the term $\vec{k}.\tilde{J}^{\dagger}(\vec{k})$ will actually achieve the longitudinal current conservation.

These arguments lead to the following choice of the excitation operator

$$X^{\dagger}_{\vec{k}} = A\, X^{\dagger}_{1} + B\, X^{\dagger}_{2} \tag{3.5}$$

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21) R.P.Feynman and M.Cohen, Phys. Rev. 102 (1956) 1186.

with

$$X_1^{\dagger} = \int d\vec{x} \quad \exp(-i\vec{k}.\vec{x})\, \rho(\vec{x}) \tag{3.6}$$

$$X_2^{\dagger} = -i \int d\vec{x} \quad \exp(-i\vec{k}.\vec{x}) \frac{\partial}{\partial x_\mu} [J_\mu(\vec{x})] \tag{3.7}$$

A and B are C-number coefficients which are functions of k.

4. <u>Excitation Spectrum</u>:

The equation of motion approach is employed to calculate the excitation spectrum with the excitation operator given by (3.5). This method has been successfully employed in the theories of superconductivity as well as superfluidity[22]. However in the present context there is a difficulty because of the nature of the kinetic energy part of the Hamiltonian. The presence of the inverse of the density operator, $[\rho(\vec{x})]^{-1}$ is a somewhat unwelcome feature of this formulation as it does not allow a straight forward calculation in the momentum space. However under the assumption of B.E.condensation this operator can be replaced by its ground state expectation value. But in this section no " a priori" assumption is made regarding the number distribution in the momentum space. Hence the operator $[\rho(\vec{x})]^{-1}$ has to be retained in the Hamiltonian, and approximations have to be made only in simplifying the equation of motion.

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22) P.W.Anderson, Phys.Rev. 112 (1958) 1900.
G.Rickayzen, Phys.Rev. 115 (1959) 795.
K.Sawada, Phys.Rev. 106 (1957) 372.

It is assumed that the inverse of the density operator commutes with $\rho(\vec{x})$

$$[\rho^{-1}(\vec{y}), \rho(\vec{x})] = 0 \tag{4.1}$$

From this it is shown, on using (2.6)

$$[(\rho(\vec{x}))^{-1}, J_\mu(\vec{y})] = i \frac{\partial}{\partial x_\mu}[\delta(\vec{x}-\vec{y})\rho(\vec{x})]\rho^{-2}(\vec{x}) \tag{4.2}$$

$$(\mu = 1, 2, 3)$$

This commutation relation is very important in evaluvating the commutator of $X_k^\dagger$ with H.

$$\int d\vec{x} \int d\vec{y}\, e^{-i\vec{k}\cdot\vec{y}} \left[\frac{\partial\rho(\vec{x})}{\partial x_\mu}\frac{1}{\rho(\vec{x})}\frac{\partial\rho(\vec{x})}{\partial x_\mu}, \rho(\vec{y})\right] = 0 \tag{4.3}$$

$$\int d\vec{x} \int d\vec{y}\, e^{-i\vec{k}\cdot\vec{y}} \left[\left(\frac{\partial\rho(\vec{x})}{\partial x_\mu}\frac{1}{\rho(\vec{x})}J_\mu(\vec{x}) - J_\mu(\vec{x})\frac{1}{\rho(\vec{x})}\frac{\partial\rho(\vec{x})}{\partial x_\mu}\right), \rho(\vec{y})\right] = 0 \tag{4.4}$$

$$\frac{1}{2}\int d\vec{x} \int d\vec{y}\, e^{-i\vec{k}\cdot\vec{y}} \left[J_\mu(\vec{x})\frac{1}{\rho(\vec{x})}J_\mu(\vec{x}), \rho(\vec{y})\right]$$

$$= i\int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}} \frac{\partial J_\mu(\vec{x})}{\partial x_\mu} \tag{4.5}$$

$$-\frac{i}{8}\int d\vec{x} \int d\vec{y}\, e^{-i\vec{k}\cdot\vec{y}} \frac{\partial}{\partial y_\nu}\left[\left(\frac{\partial\rho(\vec{x})}{\partial x_\mu}\frac{1}{\rho(\vec{x})}\frac{\partial\rho(\vec{x})}{\partial x_\mu}\right), J_\nu(\vec{y})\right]$$

$$= -\frac{k^4}{4}\int d\vec{x}\, e^{-i\vec{k}\cdot\vec{y}} \rho(\vec{x})$$

$$+ \frac{1}{4}\int d\vec{x}\, e^{-i\vec{k}\cdot\vec{y}} \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\mu}\left[\frac{\partial\rho(\vec{x})}{\partial x_\mu}\frac{1}{\rho(\vec{x})}\frac{\partial\rho(\vec{x})}{\partial x_\nu}\right] \tag{4.6}$$

$$\frac{1}{4}\int d\vec{x}\int d\vec{y}\, e^{-i\vec{k}\cdot\vec{y}}\frac{\partial}{\partial y_\nu}\left[\left(\frac{\partial\rho(\vec{x})}{\partial x_\mu}\frac{1}{\rho(\vec{x})}J_\mu(\vec{x})-J_\mu(\vec{x})\frac{1}{\rho(\vec{x})}\frac{\partial\rho(\vec{x})}{\partial x_\mu}\right),J_\nu(\vec{y})\right]$$

$$= i\int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\mu}\left[\frac{\partial\rho(\vec{x})}{\partial x_\nu}\frac{1}{\rho(\vec{x})}J_\mu(\vec{x})-J_\mu(\vec{x})\frac{1}{\rho(\vec{x})}\frac{\partial\rho(\vec{x})}{\partial x_\mu}\right] \tag{4.7}$$

$$-\frac{i}{2}\int d\vec{x}\int d\vec{y}\, e^{-i\vec{k}\cdot\vec{y}}\frac{\partial}{\partial y_\nu}\left[J_\mu(\vec{x})\frac{1}{\rho(\vec{x})}J_\mu(\vec{x}),\ J_\nu(\vec{y})\right]$$

$$= \int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\mu}\left[J_\mu(\vec{x})\frac{1}{\rho(\vec{x})}J_\nu(\vec{x})\right] \tag{4.8}$$

Equations (4.3) to (4.8) determine the commutator of $X_k^\dagger$ with H_o. Evaluvation of the commutator of $X_k^\dagger$ with the interaction Hamiltonian H_I is easily done by going over to the momentum representation. In the momentum space commutation relations (2.5), (2.6) and (2.7) take the following forms.

$$[\tilde{\rho}(\vec{p}),\tilde{\rho}(\vec{q})]=0 \tag{4.9}$$

$$[\tilde{\rho}(\vec{p}),\tilde{J}_\mu(\vec{q})]=-p_\mu\tilde{\rho}(\vec{p}+\vec{q}) \tag{4.10}$$

$$[\tilde{J}_\mu(\vec{p}),\tilde{J}_\nu(\vec{q})]=-p_\nu\tilde{J}_\mu(\vec{p}+\vec{q})+q_\mu\tilde{J}_\nu(\vec{p}+\vec{q}) \tag{4.11}$$

and $$\tilde{\rho}^\dagger(\vec{k})=\tilde{\rho}(-\vec{k})\quad;\quad \tilde{J}_\mu^\dagger(\vec{k})=\tilde{J}_\mu(-\vec{k}) \tag{4.12}$$

$X_k^\dagger$ takes the form $$X_k^\dagger = A\,\tilde{\rho}^\dagger(\vec{k}) + B\,\vec{k}.\tilde{\vec{J}}^\dagger(\vec{k}) \tag{4.13}$$

and $$H_I = \frac{1}{2}\sum_{\vec{p}} \tilde{V}(\vec{p})\,\tilde{\rho}(\vec{p})\,\tilde{\rho}(-\vec{p}) \tag{4.14}$$

In the above equations $\tilde{V}$, $\tilde{\rho}$ and $\tilde{\vec{J}}$ denote the Fourier components of $V(|\vec{x}|)$, $\rho(\vec{x})$ and $\vec{J}(\vec{x})$ respectively.

Then

$$[H_I, \tilde{\rho}(\vec{k})] = 0 \tag{4.15}$$

$$[H_I, \vec{k}.\tilde{\vec{J}}^\dagger(\vec{k})] = -k_\nu \sum_{\vec{p}} \tilde{V}(\vec{p})\, p_\nu\, \tilde{\rho}^\dagger(\vec{p})\, \tilde{\rho}(\vec{p}-\vec{k}) \tag{4.16}$$

It is assumed that $$\tilde{V}(\vec{p}) = \tilde{V}(-\vec{p}) \tag{4.17}$$

The following approximation procedure is now adopted. In the commutators involving the free part H_0 of the Hamiltonian only those terms which are independent of the inverse of the density operator are retained. The commutator with H_I is given by (4.16). In the this it is assumed that the mode with momentum $\vec{k}$ gives the dominant contribution to the summation over $\vec{p}$. While the approximation scheme devised for the commutators involving H_0 has no analogue in the usual second quantised theories that for the interacting part of the Hamiltonian is identical with the usual R.P.A.

This approximation procedure leads to the equations

$$[H, X_1^\dagger] = -X_2^\dagger \tag{4.18}$$

$$[H, X_2^\dagger] = -F(\vec{k})\, X_1^\dagger ,$$

$$F(\vec{k}) = \left[\frac{k^4}{4} + \tilde{V}(\vec{k})\, k^2\, \tilde{\rho}(\vec{k}=0) \right] \tag{4.19}$$

Consequently,

$$[H, X_k^\dagger] = \omega(\vec{k})\, X_k^\dagger$$

$$= -F(\vec{k})\, B\, X_1^\dagger - A\, X_2^\dagger \tag{4.20}$$

If A and B should have non-trivial values the following secular equation has to be satisfied

$$\begin{vmatrix} \omega(\vec{k}) & F(\vec{k}) \\ 1 & \omega(\vec{k}) \end{vmatrix} = 0$$

or equivalently

$$\omega(\vec{k}) = \left[\frac{k^4}{4} + k^2\, \tilde{V}(\vec{k})\, N \right]^{1/2} \tag{4.21}$$

which for sufficiently small values of k (the long wavelength limit)

$$\omega(\vec{k}) \sim c\, k \tag{4.22}$$

with $\quad c = N \tilde{V}(\vec{k})$

It is found that the slope of the linear dispersion relation is proportional to the total number N rather than the number in the

condensate N_0. Thus (4.22) is consistent with the experimental observation of Woods[2] that the phonon branch of the excitation spectrum exists even after the λ- transition point and that the slope almost remains a constant with temperature. Further it remains supports the conjecture of Pines[10] that the low-lying excited states of the system may be the counterpart in a neutral system of the plasmon excitation in a Fermi liquid.

5. The Extreme Weak Coupling Limit:

In this section it is shown that the theory formulated earlier reduces, in the extreme weak coupling limit, to the theories of Bogoliubov[1] Brueckner and Sawada[22] which are based on the 23) "a priori" assumption that in the low lying excited states of the system all the particles are in the zero momentum (single particle) state. As has already been stated the basic assumption in these theories, in the thermodynamic limit, is:

$$\frac{N_0}{N} \to 1 \qquad \text{and} \qquad \frac{N_k}{N} \to 0 \quad (k \neq 0) \tag{5.1}$$

In the Bogoliubov approximation procedure the creation and annihilation operators for the zero momentum state are replaced by their ground state expectation value $\sqrt{N_0}$. This is justified because of the macroscopic occupation of $k=0$ state. In the present context a similar procedure may be adopted to overcome the difficulty because of the presence of the inverse of the density operator. The operator $[\rho(\vec{x})]^{-1}$ is replaced by its ground state

22) K.A.Brueckner and K.Sawada, Phys.Rev. 106 (1957) 1117. 23)

expectation value $\langle \rho(\vec{x}) \rangle_0^{-1}$. After this replacement it is easy to work in the momentum representation where the Hamiltonian takes the simple form

$$H = \sum_{\vec{p}} G(\vec{p})\, \tilde{\rho}^{\dagger}(\vec{p})\, \tilde{\rho}(\vec{p}) + \frac{1}{2\langle \rho \rangle_0} \sum_{\vec{p}} \tilde{J}_{\mu}^{\dagger}(\vec{p})\, \tilde{J}_{\mu}(\vec{p}) \qquad (\mu = 1,2,3) \tag{5.2}$$

with
$$G(\vec{p}) = \frac{p^2 + 4\langle \rho \rangle_0 \tilde{V}(\vec{p})}{8\langle \rho \rangle_0} \tag{5.3}$$

The commutation relations between the operators are given by (4.9), (4.10) and (4.11). Using these commutation relations $[H, X_k^{\dagger}]$ is evaluvated as follows.

$$\sum_{\vec{p}} G(\vec{p}) \left[\tilde{\rho}^{\dagger}(\vec{p})\, \tilde{\rho}(\vec{p}), \tilde{\rho}^{\dagger}(\vec{k}) \right] = 0 \tag{5.4}$$

$$\frac{1}{2\langle \rho \rangle_0} \sum_{\vec{p}} \left[\tilde{J}_{\mu}^{\dagger}(\vec{p})\, \tilde{J}_{\mu}(\vec{p}), \tilde{\rho}^{\dagger}(\vec{k}) \right] = \frac{1}{2\langle \rho \rangle_0} \sum_{\vec{p}} \left[k_{\nu} \tilde{\rho}^{\dagger}(\vec{p}+\vec{k}) J_{\nu}(\vec{p}) + k_{\nu} \tilde{J}_{\nu}(\vec{p}) \tilde{\rho}^{\dagger}(\vec{p}+\vec{k}) \right] \tag{5.5}$$

$$\sum_{\vec{p}} G(\vec{p}) \left[\tilde{\rho}^{\dagger}(\vec{p})\, \rho(\vec{p}), \tilde{J}_{\nu}^{\dagger}(\vec{k}) \right] = 2k_{\nu} \sum G(\vec{p})\, p_{\nu}\, \tilde{\rho}^{\dagger}(\vec{p})\, \tilde{\rho}(\vec{p}-\vec{k}) \tag{5.6}$$

$$\frac{1}{2\langle \rho \rangle_0} \sum_{\vec{p}} \left[\tilde{J}_{\mu}^{\dagger}(\vec{p})\, \tilde{J}_{\mu}(\vec{p}), \tilde{J}_{\nu}^{\dagger}(\vec{k}) \right] = \frac{k_{\nu}}{\langle \rho \rangle_0} \sum_{\vec{p}} \left\{ \tilde{J}_{\mu}^{\dagger}(\vec{p}) \left[p_{\nu} \tilde{J}_{\mu}(\vec{p}-\vec{k}) + k_{\mu} J_{\nu}(\vec{p}-\vec{k}) \right] \right. \tag{5.7}$$

$$+\left[-p_\nu \tilde{J}_\mu^{\dagger}(\vec{p}+\vec{k}) + k_\mu \tilde{J}_\nu^{\dagger}(\vec{p}+\vec{k})\right] \tilde{J}_\mu(\vec{p})\Big\} \tag{5.7}$$

In the spirit of R.P.A only those terms with $\vec{p}=\vec{k}$ (or $\vec{p}=-\vec{k}$ depending upon the term under consideration) are retained in the summation over $\vec{p}$ in the equations (5.5) to (5.7). Further the excitation operator is linear in the current and density operators. Hence bilinear terms in the current operator are omitted in the approximation procedure and the following equation of motion is obtained.

$$[H, X_k^{\dagger}] = \omega(\vec{k})\, X_k^{\dagger}$$

$$= 2Bk^2 G(\vec{k})\, \rho(0)\, \tilde{\rho}^{\dagger}(\vec{k}) + \frac{A}{\langle\rho\rangle_0}\, \rho(0)\, \vec{k}\cdot\tilde{\vec{J}}^{\dagger}(\vec{k}) \tag{5.8}$$

The following secular equation has to be satisfied if A and B have to be non trivial

$$\begin{vmatrix} \omega(\vec{k}) & -2k^2 G(\vec{k})\,\tilde{\rho}(0) \\ -\dfrac{\tilde{\rho}(0)}{\langle\rho\rangle_0} & \omega(\vec{k}) \end{vmatrix} = 0 \tag{5.9}$$

Equivalently

$$\omega^2(k) = k^2 \frac{[\rho(0)]^2}{(\langle\rho\rangle_0)^2}\left\{\frac{k^2}{4} + \langle\rho\rangle_0 \tilde{V}(\vec{k})\right\} \tag{5.10}$$

In the weak coupling limit it is justifiable to assume that $\langle \rho \rangle_0 \sim \rho(0)$ and hence for small k

$$\omega(k) \sim \left[\langle \rho \rangle_0 \tilde{V}(k)\right]^{1/2} k \tag{5.11}$$

This dispersion relation exhibits a strong dependence on the condensate in the longwave length limit. This is the characteristic feature of the Bogoliubov like theories.

6. Discussion:

It is interesting to note that the formulation presented here can be related to the earlier investigations[14,15]. It is well known that in the phenomenological model the hamiltonian for the Bose liquid may be represented as[24]

$$\begin{aligned} H - E_0 = & \frac{1}{2} \int (\nabla S)\, \rho\, (\nabla S)\, d\vec{x} \\ & + \frac{\hbar^2}{2} \int d\vec{x}\, (\nabla \sqrt{\rho})^2 \\ & + \frac{1}{2} \int d\vec{x} \int d\vec{y}\, [\rho(\vec{x}) - \rho_{eq}]\, V(|\vec{x}-\vec{y}|)\, [\rho(\vec{y}) - \rho_{eq}] \\ & + \text{non-linear terms} \end{aligned} \tag{6.1}$$

$\rho(\vec{x},t)$ is the quantised density field and $S(\vec{x},t)$ is the velocity potential and they are canonically conjugate

$$[\rho(\vec{x},t),\, S(\vec{y},t)] = i\hbar\, \delta(\vec{x}-\vec{y}) \tag{6.2}$$

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24) E.P.Gross points out that in the quantum hydrodynamic case, the two body interaction potential has to be interpreted as the average potential among the particles.

E_0 has to be adjusted to give the experimental cohesive energy for N particles. Further ρ_{eq} represents the equilibrium density of the Bose fluid. The first two terms are understood by studying the kinetic energy part of the usual second quantised Hamiltonian (2.2). Further

$$\left(\nabla\sqrt{\rho(\vec{x})}\right)^2 = \left[\frac{1}{2}\nabla\rho(\vec{x})\frac{1}{\sqrt{\rho(x)}}\right]^2 = \frac{1}{4}\left(\nabla\rho(\vec{x})\right)\frac{1}{\rho(\vec{x})}\left(\nabla\rho(\vec{x})\right)$$

and from the arguments that follow it is seen that the term $(\nabla S)\rho(\nabla S)$ corresponds to the operator $J\frac{1}{\rho}J$ of the Hamiltonian H_0 given by equation (2.11). In the classical theory both these Hamiltonians are related by the canonical transformation[25)]

$$\Psi = \sqrt{\rho(x)}\ \exp(iS) \tag{6.3}$$

This gives the hydrodynamical representation. In the present context the current density is employed in the place of the velocity field. Classically the density ρ and the velocity $\vec{v}$ are the

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25) It may be noted that this is analogous to the density phase formulation used, for example, by Bohm and Zilzel (See: P.R.Zilzel, "Symposium on the many body problem" edited by J.Percus, Interscience Publishers Inc., (1963)) Here the field operator is transformed as

$$\psi(\vec{r}) = \exp[i\phi(\vec{r})]\sqrt{\rho(\vec{r})}$$

where

$$\psi^{+}(\vec{r})\,\psi(\vec{r}) = \rho(\vec{r}) \quad \text{and} \quad [\rho(\vec{r}), \phi(\vec{r}')] = i\,\delta(\vec{r}-\vec{r}')$$

which leads to the following current density

$$\vec{J}(\vec{r}) = \frac{1}{2}\left[\rho(\vec{r})\nabla\phi + (\nabla\phi)\rho(\vec{r})\right]$$

See also R.Fanelli and R.E.Struzynski, Phys.Rev. 173 (1968)248.

fundamental dynamical variables and the current density is defined as

$$\vec{J}(\vec{x}) = \frac{1}{2}\left[\rho(\vec{x})\,\vec{v}(\vec{x}) + \vec{v}(\vec{x})\,\rho(\vec{x})\right] \tag{6.4}$$

$\vec{J}(\vec{x})$ is written in a symmetric form for use in a quantum theory. In a quantum treatment the velocity operator is not preferred. Because, it is not well defined as an observable as the current density operator is. Further the current density and the number density are related by the law of conservation of particles which actually leads to the famous f- sum rule. There are many complications in dealing with the velocity operator in a quantum hydrodynamical treatment. However by requiring that $\vec{J}(\vec{x})$ as defined by (6.4) should satisfy the commutation relations (2.6) viz.

$$[\rho(\vec{x}), J_\mu(\vec{y})] = -i\,\frac{\partial}{\partial x_\mu}\left[\delta(\vec{x}-\vec{y})\,\rho(\vec{x})\right] \qquad (\mu = 1,2,3)$$

the commutator of the velocity operator with $\rho(\vec{x})$ is determined[26].

$$[\rho(\vec{x}), v_\mu(\vec{y})] = i\,\frac{\partial}{\ }\left[\delta(\vec{x}-\vec{y})\right] \tag{6.5}$$

This is ofcourse consistent with (6.2) where the velocity potential is occurring instead of the velocity operator itself.

A more serious consequence of dealing with the density and velocity potential as the fundamental variables is that one loses the capacity to describe the B.E.condensation. However Landau[11]

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26) D.D.H.Yee, Phys.Rev. 184 (1969) 196.

and London[27] and later Kronig and Thellung[28] have adopted a two fluid model to describe the B.E.condensation in this representation. The same feature exists in the present context also.

As has already been discussed a rigorous treatment in terms of currents and densities has to deal with a Hilbert space in which the states are labelled by the eigenvalue $\overline{\rho(\vec{x})}$ of the number density operator and any abstract state in this space is represented by its components along this basis. Further the current density $\vec{j}(\vec{x})$ can be given a representation as a functional derivative with respect to $\rho(\vec{x})$. (See equation (2.15)). In the treatments of quantum hydrodynamics referred to above a similar situation exists. The velocity potential is given a functional representation $-i\,\delta/\delta\rho(\vec{x})$ and the Hamiltonian operates on the wave functionals $\Psi(\rho(\vec{x}))$ of the density.

On the basis of macroscopic occupation of the zero momentum state it was argued in section 5 that the operator $[\rho(\vec{x})]^{-1}$ can be replaced by its ground state expectation value $\langle\rho(\vec{x})\rangle_0^{-1}$. A more convincing argument can be given by the following expansion

$$\frac{1}{\rho(\vec{x})} = \frac{1}{\langle\rho\rangle_0}\,\frac{1}{1+\left[\dfrac{\rho(\vec{x})-\langle\rho\rangle_0}{\langle\rho\rangle_0}\right]}$$

$$= \frac{1}{\langle\rho\rangle_0}\left\{1-\left[\frac{\rho(\vec{x})-\langle\rho\rangle_0}{\langle\rho\rangle_0}\right]+\left[\frac{\rho(\vec{x})-\langle\rho\rangle_0}{\langle\rho\rangle_0}\right]^2-\cdots\right\} \qquad (6.6)$$

In the approximation procedure employed in section 5 only the

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27) F.London,"Superfluids" Vol.II, Dover (N.Y) (1964).
28) R.Kronig and A.Thellung, Physica 18 (1952) 749
Physica 19 (1953) 217.

first term in this expansion has been taken into account (6.4) makes it convenient to work in the momentum space. It can be shown that with the first term alone the Hamiltonian corresponds to the wellknown Hamiltonian of Bogoliubov[1] with B.E.condensation in the $k=0$ state if the current and density operators in the momentum representation are expressed in terms of the annihilation and creation operators.

$$\tilde{\rho}(\vec{k}) = \sum_{\vec{p}} a^{\dagger}_{\vec{p}-\vec{k}}\, a_{\vec{p}} \quad ; \quad J_{\mu}(\vec{k}) = \frac{1}{2} \sum (2\vec{p}-\vec{k})_{\mu}\, a^{\dagger}_{\vec{p}-\vec{k}}\, a_{\vec{p}} \tag{6.7}$$

Now the B.E.condensation is assumed so that, in the summation over $\vec{p}$ only those terms involving the operator for the zero momentum state is retained. Then

$$\tilde{\rho}(\vec{k}) \sim \sqrt{N_0}\,(a^{\dagger}_{-\vec{k}} + a_{\vec{k}})$$

$$\tilde{J}_{\mu}(\vec{k}) \sim \frac{\sqrt{N_0}}{2}\, k_{\mu}\,(a_{\vec{k}} - a^{\dagger}_{-\vec{k}}) \tag{6.8}$$

Substituting these expressions in (5.2) the anologue of the Bogoliubov Hamiltonian is obtained with the further assumption that $\langle \rho \rangle_0 = N_0$.

To conclude, the Hamiltonian for an interacting many boson system has been employed, in a representation in terms of currents and densities as coordinates, for studying the elementary excitation spectrum. An excitation operator has been constructed and the equation of motion has been solved under an approximation

scheme analogous to the RPA and the excitation spectrum has been obtained. This spectrum is consistent with the experimental results of Woods[2] in that the phonon velocity is independent of the condensation and hence is almost independent of temperature even above T_λ. The results have been discussed for the case of the weak coupling limit where the results of the earlier investigations have been compared with the present ones.

CHAPTER. II

SUM RULES FOR AN INTERACTING BOSE GAS+

1. Introduction:

In chapter I a system of interacting Bosons is studied with currents and densities as coordinates. An excitation operator $X^{\dagger}_{\vec{k}}$ of the form

$$X^{\dagger}_{\vec{k}} = A\int d\vec{x}\, \exp(-i\vec{k}.\vec{x})\, \rho(\vec{x}) - iB\int d\vec{x}\, \exp(-i\vec{k}.\vec{x})\, \vec{\nabla}.\vec{J}(\vec{x}) \tag{1.1}$$

is constructed and the excitation spectrum

$$\omega(k) = \left[(k^4/4) + k^2 \tilde{v}(k)\, \tilde{\rho}(0) \right]^{1/2} \tag{1.2}$$

is obtained under an approximation scheme similar to the R.P.A. For any approximate theory to be physically consistent it is necessary that the conservation laws are satisfied. It is possible in some cases, to express these conservation laws in the form of sum rules. Feynman[1)] has assumed that the excitated states are generated from the ground state by means of density fluctuation $\tilde{\rho}^{\dagger}(\vec{k})$. Later Miller, Nozieres and Pines[2)] have shown that these states satisfy the famous f -sum

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+ R.Sridhar, MATSCIENCE Preprint (1971) (to be published)
1)R.P.Feynman, Phys.Rev. 94 (1954) 262.
2)A.Miller, D.Pines and P.Nozieres, Phys.Rev.127 (1962)1452
For a detailed study of the sum rules see (for example) The theory of quantum liquids, D.Pines and P.Nozieres, W.A.Benjamin (New York) (1966).

rule. This sum rule is directly connected with the important requirement of particle number conservation and with the gauge invariance (of first kind) of the theory. It has been shown in chapter I that

$$[\tilde{\rho}(\vec{k}), H] = \vec{k}.\vec{\tilde{J}}(\vec{k}) \tag{1.3}$$

This relation has been obtained under the assumption that the interactions are velocity independent. It is known that

$$\frac{d\tilde{\rho}(k)}{dt} = i\,[H, \tilde{\rho}(\vec{k})] = -i\,\vec{k}.\tilde{J}(\vec{k}) \tag{1.4}$$

Taking the inverse Fourier transform, this equation can be written in the configuration space as

$$\frac{\partial \rho}{\partial t} + i\,\vec{\nabla}.\vec{J} = 0 \tag{1.5}$$

which is the continuity equation. Also

$$\langle [[\tilde{\rho}^{\dagger}(\vec{k}), H], \tilde{\rho}(\vec{k})] \rangle = k_{\mu} \langle [\tilde{J}_{\mu}(-\vec{k}), \tilde{\rho}(\vec{k})] \rangle$$

$$= k^2 \langle \tilde{\rho}(0) \rangle \tag{1.6}$$

and

$$\langle [[\tilde{\rho}^{\dagger}(\vec{k}), H], \tilde{\rho}(\vec{k})] \rangle = 2 \langle \tilde{\rho}^{\dagger}(\vec{k})\, H\, \tilde{\rho}(\vec{k}) \rangle \tag{1.7}$$

Equations (1.6) and (1.7) along with (1.3) lead to the famous

f - sum rule. This is useful in the following respect also. It may happen that a certain set of states exhaust the sum rule. Thus these set of states alone may be considered in any calculation to simplify the treatment.

Another important and interesting sum rule arises when the response of a system to a weakly interacting external probe is considered. In such a case in additions to the transitions that actually arise because of the interaction of the probe with the system (which correspond to irreversible dissipative processes) there are in addition virtual transitions which represent a reversible deformation of the system. For the case of liquid helium a density fluctuation can act as a probe and it is well known that in the long wavelength limit, the quantity $\langle \tilde{\varrho}^{\dagger}(\vec{k})\, H^{-1}\, \tilde{\varrho}(\vec{k}) \rangle$ is related to the compressibility.

$$\lim_{k \to 0} \frac{1}{N} \langle \tilde{\varrho}^{\dagger}(\vec{k})\, H^{-1}\, \tilde{\varrho}(\vec{k}) \rangle = 1/2c^2 \qquad (1.8)$$

c is the isothermal sound velocity and it is related to the compressibility.

The discussion so far has been based on the assumption that $\tilde{\varrho}^{\dagger}(\vec{k})$ generates the excited states. However in the present approach the operator $X^{\dagger}_{\vec{k}}$ given by (1.1) generates the excited states. In addition to $\tilde{\varrho}^{\dagger}(\vec{k})$, $X^{\dagger}_{\vec{k}}$ involves the longitudinal component of the density fluctuation and hence the backflow to some extent. Thus it is perhaps worthwhile to

seek for the modifications in these sum rules if this new excited state is used.

2. <u>The f -sum rule:</u>

The following identities have been derived in Chapter I.

$$[H, X_{\vec{k}}^{\dagger}] = -A\, X_2^{\dagger} - B\,\omega^2(k)\, X_1^{\dagger}$$

$$-B \sum_{\substack{p \neq k \\ p \neq 0}} \tilde{V}(|\vec{p}|)(\vec{k}.\vec{p})\, \tilde{\varrho}(\vec{p}-\vec{k})\, \tilde{\varrho}(-\vec{p}) + B\, T^{\dagger}(k) \tag{2.1}$$

where

$$X_1^{\dagger} = \int d\vec{x}\, \exp(-i\vec{k}.\vec{x})\, \varrho(\vec{x}) \tag{2.2}$$

$$X_2^{\dagger} = -i \int d\vec{x}\, \exp(-i\vec{k}.\vec{x})\, \vec{\nabla}.\vec{J}(\vec{x}) \tag{2.3}$$

$$T^{\dagger}(\vec{k}) = T(-k)$$

$$= \frac{1}{4}\int d\vec{x}\, \exp(-i\vec{k}.\vec{x}) \cdot$$

$$\cdot \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\left\{ \frac{\partial \varrho(\vec{x})}{\partial x_\mu}\frac{1}{\varrho(\vec{x})}\frac{\partial \varrho(\vec{x})}{\partial x_\nu} + 4 J_\mu(\vec{x})\frac{1}{\varrho(\vec{x})} J_\nu(\vec{x}) \right.$$

$$\left. + 2i\left[\frac{\partial \varrho(x)}{\partial x_\mu}\frac{1}{\varrho(\vec{x})} J_\nu(\vec{x}) - J_\nu(\vec{x})\frac{1}{\varrho(\vec{x})}\frac{\partial \varrho(\vec{x})}{\partial x_\mu}\right]\right\} \tag{2.4}$$

The quantity $\langle X_{\vec{k}}^{\dagger} H X_{\vec{k}} \rangle$ can be evaluvated by a procedure similar to the one used by Miller, Pines and Nozieres[2].

$$\langle X^{\dagger}_{\vec{k}} H X_{\vec{k}} \rangle = \frac{1}{2} \langle [[X^{\dagger}_{\vec{k}}, H], X_{\vec{k}}] \rangle \tag{2.5}$$

$$= \frac{Nk^2}{2} \Sigma' [\tilde{V}(p+k)(\vec{k}.(\vec{p}+\vec{k}))^2 - \tilde{V}(p)(k.p)^2] \; \tilde{\varrho}(\vec{p})\tilde{\varrho}(-\vec{p})$$

$$+ \frac{Nk^2}{2} [|A|^2 + |B|^2 \omega^2(k)]$$

$$+ 3|B|^2 k^2 k_\mu k_\nu \, t_{\mu\nu} \tag{2.6}$$

The prime over the summation denotes that the terms with $\vec{p} = \vec{k}$ and $\vec{p} = 0$ have been omitted[3]. The last term on the right side of (2.6) can be related to the average kinetic energy $\langle K.E \rangle$ in the following fashion

$$k_\mu k_\nu \, t_{\mu\nu} = k_\mu k_\nu \int d\vec{x} \left\langle \frac{1}{4} \frac{\partial \varrho(x)}{\partial x_\mu} \frac{1}{\varrho(\vec{x})} \frac{\partial \varrho(x)}{\partial x_\nu} + J_\mu(x) \frac{1}{\varrho(\vec{x})} J_\nu(x) \right\rangle$$

$$= k^2 \int d\vec{x} \left\langle \frac{1}{4} \frac{\partial \varrho(\vec{x})}{\partial x_\mu} \frac{1}{\varrho(x)} \frac{\partial \varrho(x)}{\partial x_\mu} + J_\mu(\vec{x}) \frac{1}{\varrho(\vec{x})} J_\mu(\vec{x}) \right\rangle$$

$$- \int d\vec{x} \left\{ \left\langle \frac{1}{4} (\vec{k} \times \nabla \varrho(\vec{x})) \frac{1}{\varrho(\vec{x})} (\vec{k} \times \nabla \varrho(\vec{x})) \right\rangle + \left\langle (\vec{k} \times \vec{J}(\vec{x})) \frac{1}{\varrho(\vec{x})} (\vec{k} \times \vec{J}(\vec{x})) \right\rangle \right\} \tag{2.7}$$

---

3) In the derivation of (2.6) the following terms have been omitted as the system is stationary

$$\langle \vec{P} \rangle = \int d\vec{x} \, \langle \vec{J}(\vec{x}) \rangle = \langle \vec{J}(0) \rangle = 0$$

$$\Sigma' \tilde{V}(p) (\vec{k}.\vec{p}) \langle \varrho(\vec{p}) \varrho(-\vec{p}) \rangle = 0$$

Further,

$$\int d\vec{x} \, \exp(-i\vec{k}.\vec{x}) \left\langle \frac{\partial \varrho(\vec{x})}{\partial x_\mu} \frac{1}{\varrho(\vec{x})} J_\nu(\vec{x}) - J_\nu(\vec{x}) \frac{1}{\varrho(\vec{x})} \frac{\partial \varrho(\vec{x})}{\partial x_\mu} \right\rangle = 0$$

Thus (2.6) may be written as

$$< X^{+}_{\vec{k}} H X_{\vec{k}} > = \frac{N k^2}{2} [|A|^2 + |B|^2 \omega^2(k)]$$
$$+ \frac{|B|^2}{2} \Sigma' [\tilde{V}(p+k) (k.(\vec{p}+\vec{k}))^2$$
$$- \tilde{V}(p) (\vec{k}.\vec{p})^2] \tilde{\varrho}(\vec{p}) \tilde{\varrho}(-\vec{p})$$
$$+ (3|B|^2) . 4 < K.E > \varepsilon(k)$$
$$- \frac{1}{2} 3|B|^2 . k^2 \int d\vec{x} \{ < \frac{1}{4} (\vec{k} \times \nabla \varrho(\vec{x})) .$$
$$. \frac{1}{\varrho(x)} (\vec{k} \times \nabla \varrho(\vec{x})) >$$
$$+ < (\vec{k} \times \vec{J}(\vec{x})) \frac{1}{\varrho(\vec{x})} (\vec{k} \times \vec{J}(\vec{x})) > \} \quad (2,8)$$

with $\varepsilon(k) = k^2/2$

Thus the f -sum rule takes the form given by (2.8) in this new excited state. The appearance of the term involving $\tilde{\varrho}(\vec{p}) \tilde{\varrho}(-\vec{p})$ is quite interesting. This occurs only in the higher order moments of the state $\tilde{\varrho}^{+}_{\vec{k}}$ is used instead of $X^{+}_{\vec{k}}$ .. The last term, however, is altogether a new one.

3. Higher order moments of the dynamic form factor:

With the use of the relation (2.1) and its hermitian adjoint the higher order moments $< X^{+}_{\vec{k}} H^n X_{\vec{k}} > (n=2,3,\ldots)$ can be evaluvated.

$$< X^{+}_{\vec{k}} H^2 X_{\vec{k}} > = < [X^{+}_{k}, H] [H, X_{\vec{k}}] >$$
$$= |A|^2 < X_2^{+} X_2 > + |B|^2 \omega^4(k) < X_1^{+} X_1 >$$
$$+ \omega^2(k) (AB^{*} + BA^{*}) < X_2^{+} X_1 >$$

(Contd ...)

$$+ |B|^2 \sum_{q}{}' \sum_{p}{}' \tilde{V}(|\vec{p}|)\tilde{V}(|\vec{q}|) < \tilde{\varrho}(\vec{p}-\vec{k})\varrho(-\vec{p})\varrho(\vec{k}-\vec{q})\varrho(\vec{q})>$$

$$+ |B|^2 < T^{\dagger}(k)\, T(k)>$$

$$+ (AB^* + BA^*) \Sigma' \tilde{V}(|\vec{p}|)(\vec{k}.\vec{p}) < x_2^{\dagger} \tilde{\varrho}(\vec{k}-\vec{p})\tilde{\varrho}(\vec{p})>$$

$$+ 2|B|^2 \omega^2(k) \Sigma' \tilde{V}(|\vec{p}|)(\vec{k}.\vec{p}) < x_1^{\dagger} \tilde{\varrho}(\vec{k}-\vec{p})\tilde{\varrho}(\vec{p})>$$

$$- (AB^* + A^*B) < x_2^{\dagger} T(k)> - 2|B|^2\omega^2(k) <x_1^{\dagger} T(k)>$$

$$- 2|B|^2 \Sigma' \tilde{V}(|\vec{p}|)(\vec{k}.\vec{p}) < \tilde{\varrho}(\vec{p}-\vec{k})\tilde{\varrho}(-\vec{p})\, T(k)> \qquad (3.1)$$

Equations (3.1) involves, in addition to bilinear terms in current and density operators, terms which have more than two operators. Such operators which are more than bilinear in these collective variables describe scattering as well as production of higher order excitations. As the present study restricts itself to low temperatures and to low momenta such higher order processes are not probably important. Similar argument has been employed by Gross[4] in his study of impurities in liquid helium. Thus the attention is confined only to the first three terms of (3.1). Further

$$< x_2^{\dagger} x_2> = < [x_1^{\dagger}, H] x_2> = <x_1^{\dagger} H x_1> \qquad (3.2)$$

$$< x_1^{\dagger} x_1> = N S(k) \qquad (3.3)$$

4) E.P.Gross, "Quantum Fluids", Edited by D.F.Brewer, North-Holland Publishing Company, Amsterdam (1966).

$$\langle X_2^\dagger X_1 \rangle = \langle [X_1^\dagger, H] X_1 \rangle = \langle X_1^\dagger H X_1 \rangle \tag{3.4}$$

From equations (2.5) and (2.6)

$$\langle X_1^\dagger H X_2 \rangle = \frac{1}{(AB^* + A^* B)} \Big[\langle X_{\vec{k}}^\dagger H X_{\vec{k}} \rangle - |A|^2 \langle X_1^\dagger H X_1 \rangle - |B|^2 \langle X_2^\dagger H X_2 \rangle \Big] \tag{3.5}$$

From the equations of Chapter I it is seen that

$$A = \omega(k)\, B \tag{3.6}$$

Thus

$$\langle X_{\vec{k}}^\dagger H^2 X_{\vec{k}} \rangle = |B|^2 N \omega^3(k) \left[2\,\mathcal{E}(k) + \omega(k)\, S(k) \right] \tag{3.7}$$

In a similar strain the third moment may also be evaluvated.

$$\langle X_{\vec{k}}^\dagger H^3 X_{\vec{k}} \rangle = \omega^2(k) \langle X_{\vec{k}}^\dagger H X_{\vec{k}} \rangle \tag{3.8}$$

These results, especially (3.8), may be compared with those obtained by Mihara and Puff[5] and Sposito[6]. There are some differences as $X_{\vec{k}}^\dagger$ includes the longitudinal component of the current density fluctuation. Especially the appearance of the term $t_{\mu\nu}$ is an interesting one as it can be

5) N.Mihara and R.D.Puff, Phys.Rev. 174 (1968) 221.
6) G.Sposito, Phys.Rev. 182 (1969) 284.

thought of as the quantum pressure tensor.

4. The "R.P.A" calculations:

The consequences of the approximation procedure used in the derivation of $\omega(k)$ are analysed in this section. The approximation scheme essentially amounts to the following:

$$H\,|X_{\vec{k}}\rangle = \omega(k)\,|X_{\vec{k}}\rangle \tag{4.1}$$

$$\langle X^{+}_{\vec{k}}\, H\, X_{\vec{k}}\rangle = \frac{1}{2}\langle [[X^{+}_{\vec{k}}, H], X_{\vec{k}}]\rangle$$

$$= N\,\omega^2(k)\,k^2\,|B|^2 \tag{4.2}$$

By demanding that the states $|X_{\vec{k}}\rangle$ should satisfy the f-sum rule $|B|^2$ can be determined and has the following value

$$|B|^2 = 1/2\,\omega^2(k) \tag{4.3}$$

(4.3) and (3.6) fix the value of $|A|$ also.

$$|A|^2 = 1/2 \tag{4.4}$$

Further

$$\langle X^{+}_{\vec{k}}\, H\, X_{\vec{k}}\rangle = \omega(k)\,\langle X^{+}_{\vec{k}}\, X_{k}\rangle \tag{4.5}$$

Using (4.2),(4.3) and (4.5),

$$\omega(k) = N k^2 / 2\langle X^{+}_{\vec{k}}\, X_{\vec{k}}\rangle \tag{4.6}$$

This is the analogue of the well known result obtained by Feynman[1]. The compressibility sum rule is obtained in a very natural and elegant manner with the assumption that $|X_{\vec{k}}>$ are the approximate eigenstates of the hamiltonian.

$$< X^{+}_{\vec{k}} \frac{1}{H} X_{\vec{k}} > = \frac{< X^{+}_{\vec{k}} X_{\vec{k}} >}{\omega(k)} = N k^{2} / 2\,\omega^{2}(k) \tag{4.7}$$

In the long wave length limit $k \to 0$ [7]

$$\underset{k \to 0}{lt} < X^{+}_{\vec{k}} \frac{1}{H} X_{\vec{k}} > = N / 2c^{2} \tag{4.8}$$

where c denotes the velocity of propagation of the excitation in the system.

$$c = \sqrt{N \tilde{V}(k)} \tag{4.9}$$

This approximation procedure however is not valid when higher order moments are considered.

5. Conclusion:

The various moments $< X^{+}_{\vec{k}} H^{n} X_{\vec{k}} >$ of the dynamic form factor have been evaluvated. Exact derivation of these quantities for any value of n is possible. However these quantities involve bilinear or higher order terms in currents and densities. Such terms describe the scattering and the production mechanisms of higher order excitations which are

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7) K.Huang and A.Klein, Annals of Phys. 30 (1964) 203.

probably not very significant at low momenta and low temperatures. Further these terms may be important when the superfluid is under the influence of a strong external field in which case the higher order excitations are essential to keep the superfluid in the fluid state.

CHAPTER . III

STATISTICAL MECHANICS OF ELECTRON PLASMA IN A MAGNETIC FILED - A GENERAL FORMULATION

1. Introduction:

In this chapter the Liouville equation, for the study of the statistical mechanics of an electron plasma in a constant external magnetic field, is set up and its formal solution obtained. The general formulation of the problem is given in this chapter and special cases have been discussed in the chapters to follow.

The Liouville equation has been the starting point of most of the investigations in the classical description of charged particles in external fields. It is wellknown that the reduced distributions of a finite number of particles, rather than the complete Liouville density, are the quantities which are really relevant for the study of the time evolution of physical systems. Thus in the famous BBGKY[1] (Bogoliubov - Born - Green - Kirkwood - Yvon) method of approach, equations for the reduced distributions of a finite number of particles are derived from the Liouville equation. In this way the reduced distribution f_n of n particles is related to $f_{(n+1)}$ of $(n+1)$ particles. Thus BBGKY approach leads to a hierarchy of equations in which the lower order densities are coupled to the higher order ones through the interactions. Some physical arguments or assumptions are needed to express the higher

1) N.N.Bogoliubov, "Problems of a dynamical theory in Statistical Physics", Moscow (1946); translated by E.K.Gerce in "Studies in Statistical Mechanics", Vol.1, edited by J.de Boer and G.E. Uhlenbeck, Interscience, New York (1962).

order densities in terms of the lower order ones and truncate the hierarchy at some convenient point. In a different method, developed by Prigogine and Coworkers,[2] one starts with the formal solution of the Liouville equation rather than the Liouville equation itself and the approximations are done on the formal solution. The formal solution, obtained by a direct iterative method, is in a form very much similar to the interaction representation in quantum mechanics.[3] In this and in the following chapters, the method due to Prigogine will be employed.

In the present context an aggregate of negetively charged particles is considered to be embedded in a uniform background of positive charges so that there is over-all charge neutrality. This aggregate is called the medium. The background is passive and is not supposed to take part in the dynamical evolution of the system. This system is under the influence of a constant external magnetic field. Another charged test particle is introduced in this system. The test particle interacts with the medium through an electromagnetic field and because of this, its propagation is modified. Henin[4] and Pratap[5] have earlier studied the Liouville equation and its formal solution for such a system and the present formalism is based on these observations.

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2) I.Prigogine,"Non-Equilibrium Statistical Mechanics",Interscience, New York (1963)
3) R.Brout and I.Prigogine, Physica 22 (1956) 621.
4) F.Henin, Physica 29 (1963) 1233; Phys.Letts. 7 (1963) 193.
5) R.Pratap, Nuovo Cimento 52B (1967) 63.

2. The Liouville Equation:

The Hamiltonian $\mathcal{H}$ of the system, is written as

$$\mathcal{H} = \mathcal{H}_P + \mathcal{H}_M + \mathcal{H}_F \tag{2.1}$$

where $\mathcal{H}_P$ denotes the test particle, $\mathcal{H}_M$ that of the medium and $\mathcal{H}_F$, that of the electromagnetic field which is responsible for the interactions. To be precise, it can be assumed that the particles interact through a transverse electromagnetic field. The case of longitudinal and scalar fields have been duscussed separately. For the case of transverse interactions

$$\mathcal{H}_P = \frac{1}{2m_P}\left[\vec{p}_P - \frac{e_P}{2c}\,\vec{q}_P \times \vec{H} - \frac{e_P}{c}\,\vec{A}(\vec{q}_P)\right]^2 \tag{2.2}$$

$$\mathcal{H}_M = \sum_{\ell} \frac{1}{2m_\ell}\left[\vec{p}_\ell - \frac{e_\ell}{2c}\,\vec{q}_\ell \times \vec{H} - \frac{e_\ell}{c}\,\vec{A}(\vec{q}_\ell)\right]^2 \tag{2.3}$$

$$\mathcal{H}_F = \sum_{\lambda} \nu_\lambda \left[J_\lambda + J_{-\lambda}\right] \tag{2.4}$$

where $e_P = \alpha_P e$; $e_\ell = \alpha_\ell e$

α_P and α_ℓ being the number of charges on the test particle and the medium particle respectively. In this ℓ is the particle index of the medium, m_P and m_ℓ denote the masses of the test particle and the medium particle respectively. $\vec{q}$ and $\vec{p}$ are

canonically conjugate position and momentum variables. $\vec{H}$ denotes the external magnetic field taken along the z-axis of the cartesian co-ordinate system. Further ν_λ denotes the frequency and J_λ, ω_λ denote respectively the action and angle variables of the transverse electromagnetic field. Following Heitler[6] and Henin[4] $\vec{A}$ can be given the following fourier decomposition

$$\vec{A} = \sum_\lambda \vec{A}_\lambda$$

$$= \left(\frac{8c^2}{\Omega}\right)^{\frac{1}{2}} \sum_\lambda \frac{\hat{e}_\lambda}{\sqrt{ck_\lambda}} \left\{ \sqrt{J_\lambda} \cos(\vec{k}_\lambda \cdot \vec{q}) \cos \omega_\lambda + \sqrt{J_{-\lambda}} \sin(\vec{k}_\lambda \cdot \vec{q}) \cos \omega_{-\lambda} \right\} \quad (2.5)$$

where $(\hat{e}_\lambda \cdot \vec{k}_\lambda) = 0$ (2.6)

Ω denotes the volume and $\vec{k}_\lambda$ and $\hat{e}_\lambda$ are respectively the propagation and the transverse polarisation vectors for the electromagnetic field. Equation (2.6) expresses the fact that the electromagnetic field is transverse.

It may be noted that the Hamiltonians $\mathcal{H}_P$, $\mathcal{H}_M$ and $\mathcal{H}_F$ are coupled through the electromagnetic field. If the interactions are switched off by putting $J_\lambda = 0$ the hamiltonians get decoupled and $\mathcal{H}_P$ and $\mathcal{H}_M$ reduce to the wellknown hamiltonian considered by Landau[7] for describing the diamagnetism of electrons viz,

$$\mathcal{H}_0 = \frac{P^2}{2m} = \frac{1}{2m}\left[\vec{p} - \frac{e}{2c}\vec{q} \times \vec{H}\right]^2 \quad (2.7)$$

6) W.Heitler, "The Quantum Theory of Radiation", Oxford University Press (1954).
7) L.D.Landau, Z.Phys. 64 (1930) 629.

The initial density distribution for the particles of the medium is constructed dusing the eigenfunctions of $\mathcal{H}_0$.

The hamiltonian $\mathcal{H}_F$, defined by (2.4), for the electromagnetic field is given written in terms of action and angle variables. It is assumed that the field can be thought of as a collection of oscillators with frequency ν_λ. The angle variable being a cyclic variable, does not appear explicitely.

In his classic paper Landau[7] has discussed in detail the dynamics of an electron gas in a constant magnetic field. It is wellknown that the motion of electrons along the direction of the magnetic field is unaffected. The motion in the plane perpendicular to the direction of the magnetic field ($x-y$ plane) is along discrete orbits with energies E_n given by

$$E_n = (n+\tfrac{1}{2})\hbar\varpi \tag{2.8}$$

where n is the index for the Landau orbit and ϖ is the cyclotron frequency defined by

$$\varpi = \frac{eH}{mc}$$

The motion in the $x-y$ plane is described by action and angle variables given by the following equations

$$P_x = \left(p_x - \frac{m\varpi}{2} q_x\right) = \sqrt{2m\varpi J}\ \cos\omega \tag{2.9}$$

$$P_y = \left(p_x + \frac{m\varpi}{2} q_y\right) = \sqrt{2m\varpi J}\ \sin\omega \tag{2.10}$$

(x,y,z) or equivalently (1,2,3) are used as indices to denote spatial components of vectors.

$$P_z = p_z \tag{2.11}$$

The hamiltonians $\mathcal{H}_P$ and $\mathcal{H}_M$ are expanded and the term $\vec{A}_\lambda^{\,2}$ is omitted in this expansion as it will not contribute in the approximation scheme that is to be employed. This point will be discussed during the choice of the diagrams. Thus

$$\mathcal{H}_P = \omega_P J_P + (p_{P3}^2/2m_P) - \frac{e}{c}\alpha_P(\vec{A}_\lambda \cdot \vec{P}_P) \tag{2.12}$$

$$\mathcal{H}_M = \sum_\ell \omega_\ell J_\ell + (p_{\ell 3}^2/2m_\ell) - \frac{e}{c}\alpha_\ell(\vec{A}_\lambda \cdot \vec{P}_\ell) \tag{2.13}$$

The Liouville equation now is written as

$$\frac{\partial \rho_N}{\partial t} + i\mathcal{L}\rho_N = \left(\frac{e}{c}\right)(i\delta L)\rho_N \tag{2.14}$$

where $(i\mathcal{L})$ and $(i\delta L)$ are operators corresponding to the unperturbed Landau hamiltonian and the perturbation parts respectively

$$i\mathcal{L} = L_P + L_M + L_F \tag{2.15}$$

with

$$L_P = \omega_P \frac{\partial}{\partial \omega_P} + \frac{\vec{P}_P}{m_P} \cdot \frac{\partial}{\partial \vec{q}_P} \tag{2.16a}$$

$$L_M = \sum_{\ell} \varpi_\ell \frac{\partial}{\partial \omega_\ell} + \frac{\vec{P}_\ell}{m_\ell} \cdot \frac{\partial}{\partial \vec{q}_\ell} \tag{2.16b}$$

$$L_F = \sum_{\lambda} \nu_\lambda \frac{\partial}{\partial \omega_\lambda} \tag{2.16c}$$

and $$(i\,\delta L) = \mathcal{A}_p + \mathcal{B}_p + \mathcal{A}_\ell + \mathcal{B}_\ell \tag{2.17}$$

$$\mathcal{A}_p^{\lambda} = -\alpha_p \sum_{\lambda} \left(\frac{8c^2}{\Omega \nu_\lambda} \right)^{\frac{1}{2}} (\sqrt{J_\lambda} \cos \omega_\lambda)\, \Theta_p^{\lambda} \tag{2.18a}$$

with $$\Theta_p^{\lambda} = \cos(\vec{k}_\lambda \cdot \vec{q}_p) \left[\frac{\partial}{\partial \omega_p} (\hat{e}_\lambda \cdot \vec{P}_p) \frac{\partial}{\partial J_p} - \frac{\partial}{\partial J_p} (\hat{e}_\lambda \cdot \vec{P}_p) \frac{\partial}{\partial \omega_p} - \hat{e}_\lambda \cdot \frac{\partial}{\partial \vec{q}_p} \right]$$

$$- k_{\lambda 3} (\hat{e}_\lambda \cdot \vec{P}_p) \sin(\vec{k}_\lambda \cdot \vec{q}_p) \frac{\partial}{\partial p_{p3}} \tag{2.18b}$$

$$\mathcal{B}_p^{\lambda} = -\alpha_p \sum \left(\frac{8c^2}{\Omega \nu_\lambda} \right)^{1/2} (\hat{e}_\lambda \cdot \vec{P}_p) \cos(\vec{k}_\lambda \cdot \vec{q}_p)$$

$$\left[\frac{\partial}{\partial \omega_\lambda} (\sqrt{J_\lambda} \cos \omega_\lambda) \frac{\partial}{\partial J_\lambda} - \frac{\partial}{\partial J_\lambda} (\sqrt{J_\lambda} \cos \omega_\lambda) \frac{\partial}{\partial \omega_\lambda} \right] \tag{2.19}$$

$\mathcal{A}_\ell$ and $\mathcal{B}_\ell$ are obtained just by replacing the index p by ℓ The operator $\mathcal{A}$ involves only derivatives with respect to the test particle or the medium particle, while $\mathcal{B}$ involves

derivatives with respect to the electromagnetic field. Further, the $\mathcal{B}$ operator is in the form of a Poisson bracket.

Multiplying both sides of (2.14) by $\exp[i\mathcal{L}t]$ on the left, the following equation is arrived at

$$\frac{\partial}{\partial t}\left[e^{i\mathcal{L}t}\,\rho_N(t)\right] = \left(\frac{e}{c}\right) e^{i\mathcal{L}t}\,(i\delta L)\,\rho_N$$

Integrating both sides with respect to t

$$\rho_N(t) = \left(\frac{e}{c}\right)\int_0^t dt_1\, e^{-i\mathcal{L}(t-t_1)}\,(i\delta L)\,\rho_N(t_1) + e^{-i\mathcal{L}t}\,\rho(0) \qquad (2.20)$$

Iterating this equation the formal solution of (2.14) is obtained as

$$\rho_N(t) = e^{-i\mathcal{L}t}\,\rho(0) + \sum_{j=1}^{\infty}\left(\frac{e}{c}\right)^j \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j \,[i\delta L(t_1)][i\delta L(t_2)]\ldots[i\delta L(t_j)]\,\rho(0) \qquad (2.21)$$

with

$$[i\delta L(t_j)] = \exp[i\mathcal{L}t_j]\,(i\delta L)\,[\exp(-i\mathcal{L}t_j) \qquad (2.21a)$$

$\rho(0)$ denotes the initial distribution. The following remarks about the series (2.21) are in order.

(i) It has already been pointed out that the operator exp $[i\mathcal{L}t]$ involves operators which arise due to the unperturbed (Landau) Hamiltonian and $(i\delta L)$ introduces the effect of the interactions. In this manner the above integral equation is the analogue of the integral equation obeyed by the quantum mechanical density matrix in the interaction representation.

(ii) The operator exp $[i\mathcal{L}t]$ in the above expansion effects a finite change in the respective variables as can be seen from (2.22)

$$\exp\left[a\frac{\partial}{\partial x}\right]f(x) = f(x) + a\left(\frac{\partial f}{\partial x}\right) + \frac{a^2}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right) + \cdots$$

$$= f(x+a) \qquad (2.22)$$

(iii) It may be noted that the displacement operators for the present problem as given by (2.16a) and (2.16b) contain two non commuting parts so much so the straightforward interpretation given in (ii) is applicable only if we decouple the operator $\exp\left[\alpha\frac{\partial}{\partial\omega} + \frac{\vec{P}}{m}\cdot\frac{\partial}{\partial\vec{q}}\right]$. The Baker-Hausdorff formula is used to achieve this decomposition. This is a special feature of the problem.

(iv) The structure of the series is simple to understand. The first term exp$[i\mathcal{L}t]\ \rho(0)$ expresses the free propagation of the test particle in vacuum while the other terms give the modification of the free propagation because of the presence of the medium. The factor $(e/c)^j$ corresponds to j successive interactions and the factor $(i\delta L)$ introduces the modification due to a single interaction. Between two successive interactions

the particle propagates freely. The appearance of the displacement operator expresses the fact that a finite momentum change occurs in a single step. This feature is wellknown in quantum theory.

(v) The perturbation expansion obtained in this way contains an infinity of terms each of which is a complicated expression. Therefore, in each physical problem one has to choose from among these terms, a sub series which gives the dominant contribution. The determination of the subset is done in section 4 where a diagram technique is developed for this purpose.

(vi) Further a knowledge of the initial Liouville density $\rho(0)$ is essential to obtain the time evolution of the system at later times. The choice of $\rho(0)$ is a special one for the present treatment and is discussed in the next section.

3. <u>The Initial Distribution</u>:

At $t=0$ the test particle, the medium and the electromagnetic field are assumed to be uncorrelated so that

$$\rho(0) = \rho_P(0)\,\rho_M(0)\,\rho_F(0) \tag{3.1}$$

where $\rho_P(0)$, $\rho_M(0)$ and $\rho_F(0)$ denote the initial distributions for the test particle, the medium and the electromagnetic field, respectively.

The distribution $\rho_M(0)$ is constructed from the wavefunctions of the Landau Hamiltonian (2.7) which satisfy the equation

$$\left[\left(p_x - \frac{m\omega}{2} q_y\right)^2 + \left(p_y + \frac{m\omega}{2} q_x\right)^2 + p_z^2\right]\psi$$

$$= 2mE\psi \qquad (3.2)$$

Following Landau[7] the momenta can be replaced by their corresponding coordinate space operators

$$\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial q_x} - \frac{m\omega}{2} q_y\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial q_y} + \frac{m\omega}{2} q_x\right)^2 + \frac{\hbar}{i}\frac{\partial}{\partial q_z}\right]\psi$$

$$= 2mE\psi \qquad (3.3)$$

As pointed before the external magnetic field has been taken along the z axis of the coordinate system and hence the motion of the electrons along this direction is not affected. Thus ψ can be split into two parts as:

$$\psi(q_x, q_y, q_z) = \bar{\psi}(q_x, q_y)\exp\left(\frac{i}{\hbar} p_z q_z\right) \qquad (3.4)$$

Further $\bar{\psi}(q_x, q_y)$ may be written as

$$\bar{\psi}(q_x, q_y) = \exp(\alpha q_x q_y)\, u(q_x, q_y) \qquad (3.5)$$

Then $u(q_x, q_y)$ satisfies the equation (3.6)

$$\left[\frac{\hbar}{i}\frac{\partial}{\partial q_y}+\frac{m\varpi}{2}q_x\right]^2 u(q_x,q_y)$$

$$=\left(\frac{\hbar}{i}\right)^2\frac{\partial^2 u}{\partial q_y^2}+\left(\frac{\hbar}{i}\right)\left(2\alpha\frac{\hbar}{i}+m\varpi\right)q_x\frac{\partial u}{\partial q_y}$$

$$+\left[\left(\frac{\hbar}{i}\right)^2\alpha^2+\frac{\hbar}{i}m\varpi\alpha+\left(\frac{m\varpi}{2}\right)^2\right]q_x^2 u \tag{3.6}$$

The explicit dependence of this equation on q_x is removed by choosing

$$\alpha=-\frac{i}{\hbar}\frac{m\varpi}{2} \tag{3.7}$$

and hence the dependence of $u(q_x,q_y)$ on q_x can be explicitly stated ·

$$u(q_x,q_y)=\exp\left(\frac{i}{\hbar}p_x q_x\right)v(q_y) \tag{3.8}$$

and $v(q_y)$ satisfies the equation

$$\frac{\partial^2 v}{\partial W^2}=\left[W^2+\left(\frac{p_z^2-2mE}{m\varpi\hbar}\right)\right]v \tag{3.9}$$

with

$$W=\sqrt{\frac{m\varpi}{4\hbar}}\left[q_y-\sqrt{\frac{8J}{m\varpi}}\cos\omega\right] \tag{3.10}$$

Now the following transformation is made

$$v = \exp\left(-\frac{W^2}{2}\right) f \tag{3.11}$$

as a result of which (3.9) reduces to the wellknown equation

$$\left[\frac{\partial^2 f}{\partial W^2} - 2W\frac{\partial f}{\partial W} + 2n f\right] = 0 \tag{3.12}$$

with
$$n = \left[\frac{2mE - p_z^2}{m\varpi\hbar} - \frac{1}{2}\right] \tag{3.13}$$

Equation (3.12) has Hermite polynomials $H_n(W)$ as its solutions. Thus the complete solution of (3.3) is given in the following form

$$\psi(q_x, q_y, q_z) = \exp\left[\frac{i}{\hbar}\left(p_x q_x + p_z q_z - \frac{m\varpi}{2}\right) - \frac{W^2}{2}\right] H_n(W) \tag{3.14}$$

Also (3.13) leads to the following energy levels

$$E_n = \left(n + \frac{1}{2}\right)\hbar\varpi + \left(p_z^2/2m\right) \tag{3.15}$$

The initial density function is constructed with the use of (3.14) with the assumption that the electrons are allowed to move in their Landau orbits only. This requires a restriction on the allowed values of the action variable J_ℓ which has

been introduced through the equations (2.9) and (2.10). Accordingly it is required that J_ℓ should take on values which are consistent with the allowed values for energy given by (3.15). This is achieved by introducing a delta function $\delta[J_\ell-(n+\frac{1}{2})\hbar]$ which shall be made use of while doing the integration over J_ℓ. Further the medium is assumed to be in thermodynamic equilibrium and has to be represented by a Fermi distribution. Thus $\rho_M(0)$ has the form

$$\rho_M(0) = \prod_\ell \sum_n \langle \psi_n | \frac{1}{1+\exp[\beta(\mathcal{H}_0-\mu)]} | \psi_n \rangle \cdot \delta[J_\ell-(n+\tfrac{1}{2})\hbar]$$

$$= \prod_\ell \sum_n e^{-W^2} H_n^2(W) \cdot [2^n n! \sqrt{\pi}]^{-1} H_n \left\{\exp[\beta(E_n-\mu)]+1\right\}^{-1} \delta[J_\ell-(n+\tfrac{1}{2})\hbar] \tag{3.16}$$

where $\beta = 1/KT$, K being the Boltzmann's constant and T being the absolute temperature. μ is the chemical potential.

The initial condition for the electromagnetic field is taken to be an electromagnetic vacuum. This is consistent with the assumption that at $t=0$ there exists a correlations

$$\rho_F(0) = \prod_\lambda \delta(J_\lambda) \tag{3.17}$$

As it is not required immediately, the initial distribution for

the test particle need not be specified at present. It may also be noted the quantum features enter in three stages, as follows:

(i) The introduction of the density function constructed with Landau wave functions. There is an asymmetry in the y direction which appears through the variable W. Because of this there is a quantisation of the energy in the plane perpendicular to the external magnetic field.

(ii) The initial condition is essentially of the Fermi type.

(iii) The restriction that the action variable can take only those values which are consistent with the quantisation condition (3.15)

Finally, it may be noted that the initial distribution is independent of the angle variable ω_λ of the electromagnetic field. Further the Fermi distribution introduced in (3.16) goes over to a Maxwellian in the high temperature limit as has been shown for example, by Huang[8].

4. <u>The Diagram Technique</u>:

As has been pointed out already it is not possible to take into account all the terms that appear in the formal solution (2.21). It is only possible to resort to a subset of terms which will give the dominant contribution for the particular physical situation under consideration. A diagram technique is developed to pick out the particular subset in a natural way.

The following remarks are essential to proceed further. The object is to study the statistical mechanics of an electron

8) K.Huang, "Statistical Mechanics".

gas in a constant magnetic field. This system may be the aggregate of conduction electrons in a metal where the dynamical role played by the ions is not considered and ions are supposed to maintain only overall charge neutrality. The system may thus be thought of as an electron plasma in a metal. The suppression of ion dynamics is valid at low temperatures.

The analysis of the time scales that are relevant for this problem has to be considered. These are very important for the choice of the subset of diagrams from the complete set of solutions given by (2.21). The arguments given by Balescu[9] are used extensively used in the choice of the subset.

An important time scale is given by the plasma frequency

$$\omega_{pl} = \left(4\pi e^2 d/m\right) \tag{4.1}$$

where d is the concentration. It is seen that this frequency is proportional to the charge which decisively shows the role of the interactions in producing collective effects. The corresponding time scale is

$$t_{pl} \approx \sqrt{m/e^2 d} \tag{4.2}$$

It may be noted that this time scale is independent of the velocity of the particles and may be considered as the characteristic time scalefor the interactions. Another time scale, which

9) R.Balescu,"Statistical Mechanics of charged particles", Interscience (N.Y) (1963).

can be thought of as the relaxation time, may also be constructed. This depends on the inverse of the concentration d and is given by

$$t_r = (e^4 d)^{-1} \sqrt{m/\beta^3} \tag{4.3}$$

For a plasma the actual life time is much shorter than the relaxation time. The accuracy of this statement increases actually for experimentally realisable plasmas. The system has no time to reach equilibrium and hence its behaviour can be determined from an equation valid for short times - that is for time scales of the order of t_{Pl}.

$$t \sim t_{Pl} << t_r \tag{4.4}$$

It may be noted that the inverse of the powers of the "duration of collision" depend necessaryly on the charge and concentration through the combination $(e^2 d)$. Thus for time scales of order t_{Pl} all contributions proportional to $(e^2 d)$ and its higher powers must be retained - ie., an infinite series of the form $\sum_n f_n (e^2 d)^n$ must be extracted and summed completely in order to get a physically consistent result. However, terms of the type $e^{2r}(e^2 d)^n \ (r > 1)$ may be neglected as they contain "uncompensated" powers of e^2.

Consider a term whose coefficient is $(e/c)^n$ in (2.21).

$$\exp[i\mathcal{L}t_1]\,[\mathcal{A}_p^{\lambda}+\mathcal{B}_p^{\lambda}+\mathcal{A}_\ell^{\lambda}+\mathcal{B}_\ell^{\lambda}]\,\exp[-i\mathcal{L}(t_1-t_2)]$$
$$\ldots\,[\mathcal{A}_p^{\lambda'}+\mathcal{B}_p^{\lambda'}+\mathcal{A}_\ell^{\lambda'}+\mathcal{B}_\ell^{\lambda'}]\,\exp[-i\mathcal{L}t_n]\,\rho(0)$$

It has already been pointed out that the operators $\mathcal{B}$ are in the form of Poisson brackets in the variables of the electromagnetic field: J_λ and ω_λ. If $[U,V]$ denotes the Poisson bracket for the functions U and V with respect to x and y then

$$\int dx \int dy\,[U,V]=0$$

where the integration is over the entire range of the variables x and y. (4.5) is true provided U and V vanish at the boundary. By using (4.5) it is readily seen that the operator $\mathcal{B}$ can not occur at the extreme left of the expressions. Thus

$$\int_0^\infty dJ_\lambda \int_0^{2\pi} d\omega_\lambda \ldots \mathcal{B}^{\lambda}\ldots\rho(0)=0 \qquad (4.5)$$

Further the initial distribution involves a delta function $\delta(J_\lambda)$ and the operator $\mathcal{A}^{\lambda}$ does not have any derivative with respect to the electromagnetic field variables. Thus

Generalising this under the assumption that only a pair, given by $(\mathcal{A}, \mathcal{B})$, of operators appear for any mode of the electromagnetic field the following equation is obtained

$$\int_0^\infty dJ_\lambda \int_0^{2\pi} d\omega_\lambda \; \ldots \; \mathcal{A}_p^\lambda \; \delta(J_\lambda) = 0 \tag{4.6}$$

However, $\mathcal{B}^\lambda$ occurring at the extreme right gives a non-vanishing contribution. In this case an integral of the following type occurs.

$$\int_0^\infty dJ_\lambda \int_0^{2\pi} d\omega_\lambda \; \ldots \; \sqrt{J_\lambda} \; \frac{\partial}{\partial J_\lambda} \delta(J_\lambda) \neq 0 \tag{4.7}$$

These observations give an important condition that the various terms in the expansion should have the operator $\mathcal{A}^\lambda$ on the left and $\mathcal{B}^\lambda$ should always succeed $\mathcal{A}^\lambda$ and never precede it. Further a random phase approximation is employed in which there are always pairs of vertices pertaining to the same mode of the electromagnetic field. With these remarks the nature of the various orders of contributions can be discussed.

Consider the first order term with the co-efficient (e/c).

$$\int_0^\infty dJ_\lambda \int_0^{2\pi} d\omega_\lambda \; \exp[i\mathcal{L}t] \, \mathcal{A}_p^\lambda \, \exp[-i\mathcal{L}t] \, \rho(0) \tag{4.8}$$

It has already been pointed out in (4.6) that a term of this type goes to zero. Further, any contribution of order $(e/c)^{2n+1}$ also goes to zero. This can be easily seen by integrating over the angle variables $\omega_\lambda, \omega_{\lambda'}, \ldots$ etc.

The second order term consists of the following possibilities:

(i) $$\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^\infty d\mathcal{J}_\lambda \int_0^{2\pi} d\omega_\lambda \; \exp[i\mathcal{L}t_1]\, \mathcal{A}_P^\lambda \exp[-i\mathcal{L}(t_1-t_2)]\, \mathcal{B}_P^\lambda \exp[-i\mathcal{L}t_2]\, \rho(0) \tag{4.9}$$

(ii) $$\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^\infty d\mathcal{J}_\lambda \int_0^{2\pi} d\omega_\lambda \; \exp[i\mathcal{L}t_1]\, \mathcal{A}_P^\lambda \exp[-i\mathcal{L}(t_1-t_2)]\, \mathcal{A}_P^\lambda \exp[-i\mathcal{L}t_2]\, \rho(0) \tag{4.10}$$

(iii) $$\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^\infty d\mathcal{J}_\lambda \int_0^{2\pi} d\omega_\lambda \; \exp[i\mathcal{L}t_1]\, \mathcal{B}_P^\lambda \exp[-i\mathcal{L}(t_1-t_2)]\, \mathcal{B}_P^\lambda \exp[-i\mathcal{L}t_2]\, \rho(0) \tag{4.11}$$

(iv)

$$\int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^\infty d\tau_\lambda \int_0^{2\pi} d\omega_\lambda \quad \exp[i\mathcal{L}t_1]\, \mathcal{B}_p^\lambda$$

$$\exp[-i\mathcal{L}(t_1-t_2)]\, e\mathcal{A}_p^\lambda \exp[-i\mathcal{L}t_2]\, \rho(0)$$

(4.12)

The possibility (ii) and (iv) give vanishing contributions by virtue of (4.6). (iii) also gives zero contribution because of the fact that $\mathcal{B}_p^\lambda$ is in the form of a Poisson bracket. (See (4.5)). The only non trivial contribution is given by (i). This is diagramatically represented in figure 1. In this diagram the time flows from the right to the left. The square at the right represents a $\mathcal{A}\mathcal{B}$ vertex and the circle at the left represents a $\mathcal{A}$ type vertex. A non-zero contribution is obtained only if the square vertex appears at the earliest time. For this reason it is called the creation vertex for correlations with the electromagnetic field. Similarly only if the circular vertex occurs at the later time, a non trivial result is obtained. Thus a circular vertex is called the annihilation vertex for correlations. The propagator for the particle (either the test particle or a particle of the medium) is denoted by a continuous line while that for the electromagnetic field is represented by a dashed line.

The physical picture corresponding to Fig.1 is the following: The test particle creates a correlation with the electromagnetic field at time t_2. This correlation is propagated in time till t_1. At time t_1 this correlation is destroyed. During the time interval $(t_1 - t_2)$ the test particle is said to have interacted with the electromagnetic field.

In the next higher order, i.e., the fourth order, there are a number of possibilities to choose from

$$\left(\frac{e}{c}\right)^4 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \int_0^\infty d\tau_\lambda \int_0^\infty d\tau_{\lambda'} \int_0^{2\pi} d\omega_\lambda \int_0^{2\pi} d\omega_{\lambda'}$$

$$\exp\left[-i\mathcal{L}(t-t_1)\right] \mathcal{A}_p \exp\left[-i\mathcal{L}(t_1-t_2)\right]$$

$$\left[\mathcal{A}_p + \mathcal{B}_p + \mathcal{A}_\ell + \mathcal{B}_\ell\right] \exp\left[-i\mathcal{L}(t_2-t_3)\right]$$

$$\left[\mathcal{A}_p + \mathcal{B}_p + \mathcal{A}_\ell + \mathcal{B}_\ell\right] \exp\left[-i\mathcal{L}(t_3-t_4)\right]$$

$$\left[\mathcal{A}_p + \mathcal{B}_p + \mathcal{A}_\ell + \mathcal{B}_\ell\right] \exp\left[-i\mathcal{L}t_4\right] \rho(0) \tag{4.13}$$

For the sake of simplicity in the following discussions the integrations over time and the electromagnetic field variables are not explicitely shown. Further the following abbreviation is employed.

$$X_1 X_2 X_3 X_4 \Longrightarrow \exp[-i\alpha(t-t_1)]\, X_1 \cdot \exp[-i\alpha(t_1-t_2)]\, X_2 \cdot \exp[-i\alpha(t_2-t_3)]\, X_3 \cdot \exp[-i\alpha(t_3-t_4)]\, X_4 \quad (4.14)$$

$$\exp[-i\alpha t_4]\, \rho(0) \, .$$

First consider the diagrams that do not involve the particles of the medium

(i) $\mathcal{A}_p^{\lambda} \quad \mathcal{B}_p^{\lambda} \quad \mathcal{A}_p^{\lambda'} \quad \mathcal{B}_p^{\lambda'}$ (4.15)

(ii) $\mathcal{A}_p^{\lambda} \quad \mathcal{A}_p^{\lambda'} \quad \mathcal{B}_p^{\lambda'} \quad \mathcal{B}_p^{\lambda}$ (4.16)

(iii) $\mathcal{A}_p^{\lambda} \quad \mathcal{A}_p^{\lambda'} \quad \mathcal{B}_p^{\lambda} \quad \mathcal{B}_p^{\lambda'}$ (4.17)

The other possibilities can be eliminated by arguments similar to the ones employed for the case of second order contribution. Contribution (i) is diagramatically represented in fig 2. It is easily seen that it has two disconnected cycles, each of which is identical to fig 1. Contribution (ii) is represented by fig 3. In this the test particle creates a correlation at

time t_4 through the mode λ . During the period of propagation of this correlation it creates yet another correlation through λ' at time t_3 . The latter one ceases to exist before the correlation through the mode λ is destroyed. The correlation through the mode λ is annihilated at time t_1 . Thus the duration of the second correlation is completely contained in the first one. There exists another possibility in which the correlationsçcreated at different times, are propagated together for some time, but one of the correlations is not contained wholly into the duration of the other. This possibility, given by (iii), is represented diagramatically in fig 4.

The diagrams involving the medium particle lines can be considered next. The various possibilities are

(i) $\mathcal{A}_p^{\lambda} \quad \mathcal{B}_\ell^{\lambda} \quad \mathcal{A}_\ell^{\lambda'} \quad \mathcal{B}_p^{\lambda'}$ (4.18)

(ii) $\mathcal{A}_p^{\lambda} \quad \mathcal{A}_p^{\lambda'} \quad \mathcal{B}_\ell^{\lambda} \quad \mathcal{B}_\ell^{\lambda'}$ (4.19)

(iii) $\mathcal{A}_p^{\lambda} \quad \mathcal{B}_\ell^{\lambda} \quad \mathcal{A}_p^{\lambda'} \quad \mathcal{B}_\ell^{\lambda'}$ (4.20)

(iv) $\mathcal{A}_p^{\lambda} \quad \mathcal{A}_p^{\lambda'} \quad \mathcal{B}_\ell^{\lambda'} \quad \mathcal{B}_\ell^{\lambda}$ (4.21)

Contribution (i) is associated with the diagram given in fig 5. In this the test particle creates a correlation with the electromagnetic field through the mode λ' at time t_4. This correlation is propagated till time t_3 at which instant it is

transferred to a particle in the medium. Then the correlation with the medium is propagated till time t_2 , at which instant it again goes over to a correlation with the test particle, through the mode λ . At time t_1 this correlation with the test particle is annihilated. This diagram corresponds to the first Born approximation (see Prigogine[2]). Contributions (II) to (iv) have been given diagramatically in figures 6 to 8 respectively.

These diagrams may now be considered in the light of the requirement that their dependence on e^2 and the concentration d should be in the form $(e^2 d)$. If in the formal expansion, there exists a term involving a pair of vertices $\mathcal{A}_\ell$ and $\mathcal{B}_\ell$ then on summing over the particles in the medium a factor N is obtained, N being the total number of particles in the medium. A factor $\Omega^{-1/2}$ has been included in the Fourier decomposition of the electromagnetic field. Because of this the volume dependence is Ω^{-1} . The presence of the two vertices thus yields a factor $(N/\Omega)e^2$, which in the thermodynamic limit becomes $(e^2 d)$ Thus a term involving α pairs of vertices of the medium has a factor $(e^2 d)^\alpha$. If there are α' pairs of vertices in this diagram $(\alpha' > \alpha)$ then this has a dependence on e^2 and d given by $(e^2)^{\alpha'-\alpha}(e^2 d)^\alpha$. In selecting the subset, only those diagrams for which $\alpha' - \alpha = 1$ (with no restriction on α') are considered. This is the so called ring approximation (See Balescu 9). It is for this reason that the term A_λ^2 was neglected in the expansion of the Hamiltonians $\mathcal{H}_P$ and $\mathcal{H}_M$ (See

equations (2.12) and (2.13)). Inclusion of this term yields terms of the order $e^2(e^4 d)$ which have to be omitted in the approximation procedure outlined above.

Figures 2,3 and 4 do not involve particles of the medium. Hence their dependence on volume is not supplemented by the total number of particles. Consequently these diagrams give negligibly small contributions in the thermodynamic limit. In the Born approximation only diagram 5 is considered.

The construction of higher order graphs is easily done. The diagram of order e^6 involves two particles of the medium. The correlation established by the test particle with the electromagnetic field is successively transferred to two particles of the medium before it is annihilated . This contribution is represented by fig 9. Proceeding in this manner it is seen that the modification in the propagation of the correlation (because of the medium) is felt in the propagator for the electromagnetic field. Accordingly the complete modification in the propagation is represented by a wavy line as shown in Fig 10. Fig 11 gives an equation for this wavy line and is analogous to the Dyson's equation in quantum treatments. The higher order terms have to be factored in terms of the lower order contributions, if the series has to be summed exactly. This is achieved by a factorisation theorem due to Resibois[10].

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10) P.Resibois, Phys.of Fluids 6, 817 (1963). Balescu in his book (Ref.11) gives a critical assessment of the theorem. A similar but much simpler factorisation theorem has been used by C.Block in nuclear physics. (Nucl.Phys. 1,451 (1958)). This theorem of course was given for a quantum case.

5. <u>A Factorisation Theorem</u>:

From the diagrams that have been chosen from the formal solution it is seen that the two sets of particles appearing in the upper and lower lines of the diagram do not interact between themselves. The factorisation theorem proposed by Resibois depends essentially upon this property.

Consider a contribution to the perturbation series which satisfies the following conditions:

(a) In a given time interval (t_i, t_j) $(t_i > t_j)$ the interactions can be split into two groups such that one group involves the particles of the set $\{\tau\}$ only and the other of the set $\{\beta\}$ only. It is important to realise that the two sets $\{\tau\}$ and $\{\beta\}$ have <u>no</u> particles in common.

(b) All the interactions involving the group $\{\tau\}$ occur after the $\{\beta\}$ interactions.

A diagram of this nature is called a diagram of the F type. The Fig 5 is the simplest diagram of the F type. All the higher order graphs built out of Fig 5 are also of this class. Actually the requirements (a) and (b) are very general and in the present context only a very special case of these conditions occurs. If conditions (a) and (b) are satisfied the relative time ordering between the sets $\{\tau\}$ and $\{\beta\}$ is irrelevant though the time ordering among the particles of the same set is still important. This is the main result of the factorisation theorem which can be explicitely stated in the following manner.

If

$$F = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{i-1}} dt_i \int_0^{t_i} dt_{i+1} \cdots \int_0^{t_{i+\ell+m-1}} dt_{i+\ell+m}$$

$$\int_0^{t_{i+\ell+m}} dt_j \cdots \int_0^{t_{n-1}} dt_n \; [i\delta L(t_1)] \cdots$$

$$\cdots [i\delta L(t_i)] \, [i\delta L^{(r_1)}(t_{i+1})] \cdots [i\delta L^{(r_\ell)}(t_{i+\ell})]$$

$$[i\delta L^{(s_1)}(t_{i+\ell+1})] \cdots [i\delta L^{(s_m)}(t_{i+\ell+m})]$$

$$[i\delta L(t_j)] \cdots [i\delta L(0)] \qquad (5.1)$$

where the notations $[i\delta L^{(r_i)}(t)]$ $(i = 1, 2, \ldots \ell)$ and $[i\delta L^{(s_i)}(t)]$ $(i = 1, 2, \ldots m)$ have been introduced to point out the time ordered interactions among the particles of the group $\{r\}$ and $\{s\}$ respectively while the relative time ordering between the two groups is not relevant. To this diagram one can associate a whole class Σ_F obtained by maintaining the same time ordering <u>inside</u> each group of interactions $\{r\}$ and $\{s\}$ respectively but taking into account all possible combinations of <u>mutual</u> ordering between these groups. Such a class is obtained as follows: If during a time interval $t_i < t < t_j$ two subgroups of particles do not interact with each other the mutual ordering of the interactions between these two subgroups in completely irrelevant. In other words the total contribution Σ_F factorises into two independent

quantities because these groups "ignore" each other during the entire interval under consideration. Thus

$$\Sigma_F = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{i-1}} dt_i \int_0^{t_i} dt_j \cdots \int_0^{t_{n-1}} dt_n$$

$$[i\delta L(t_1)][i\delta L(t_2)] \cdots [i\delta L(t_i)]$$

$$g^{(r)}(t_i - t_j)\; g^{(s)}(t_i - t_j)\,[i\delta L(t_j)] \cdots [i\delta L(0)] \tag{5.2}$$

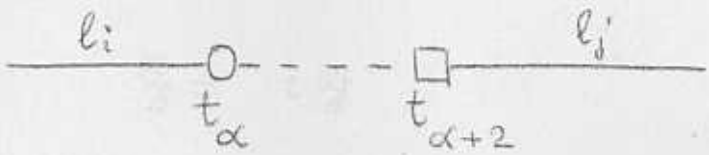
where

$$g^{(r)}(\tau) = \int_0^{\tau} dt_1 \cdots \int_0^{t_{\ell-1}} dt_\ell\, [i\delta L^{(r_1)}(t_1)] \cdots [i\delta L^{(r_\ell)}(t_\ell)] \tag{5.3}$$

A similar definition holds good for $g^{(s)}(\tau)$

6. The Reduced Distribution Function:

An examination of the diagramatic representation of the infinite series reveals that the basic unit in this series is given by the following diagram



The summation can be easily performed if this graph is factored out from all the higher order graphs. The various contributions, in their present forms, are complicated because of the time orderings. This difficulty can be removed by using the Laplace transform. In this way the use of the numerous time variables can be avoided. Following Prigogine[2)] the Laplace transform $\tilde{\rho}(\bar{z})$ of the distribution $\rho(t)$ is introduced by the equation

$$\tilde{\rho}(\bar{z}) = \int_0^\infty dt_1 \exp[-\bar{z}t]\,\rho(t) \tag{6.1}$$

and hence

$$\rho(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\bar{z}\; e^{t\bar{z}}\,\tilde{\rho}(\bar{z}) \tag{6.2}$$

Where the contour of integration is a straight line parallel to the imaginary axis with all singularities of $\tilde{\rho}(\bar{z})$ to its left. The motivation for this choice of the contour is the requirement that

$$\rho(t) = 0 \qquad \text{for } t < 0 \tag{6.3}$$

At this stage it is convenient to introduce a new variable z as

$$\bar{z} = iz \tag{6.4}$$

and consequently

$$\rho(t) = -\frac{1}{2\pi i}\int_{-\infty+i\gamma}^{\infty+i\gamma} dz\, e^{-izt}\,\tilde{\rho}(iz) \tag{6.5}$$

The integration is now along a line parallel to the real axis above all singularities of $\tilde{\rho}$. This contour is given in Fig 12. These remarks are sufficient to find out how the higher order graphs factor out. It has been shown in the following chapters

$$W_{P\ell\ell P} = W_{PP}\,\mathcal{E}(z) \tag{6.6}$$

Where $W_{P\ell\ell P}$ is the fourth order graph given by fig 5 and W_{PP} is the second order contribution given by fig 1. It may be noted that $\mathcal{E}(z)$ arises because of the presence of the medium and the resulting interaction between the test particle and the medium. Further the next higher order graph - the Sixth order one - can also be factored as

$$W_{P\ell\ell'\ell' P} = W_{PP}\,\mathcal{E}^2(z) \tag{6.7}$$

and so on. Thus

$$\tilde{\rho}(\vec{p}_p, \vec{q}_p; z) = W_{PP}\left[1 + \mathcal{E}(z) + \mathcal{E}^2(z) + \dots\right]$$

$$= W_{PP}\left[\frac{1}{1-\mathcal{E}(z)}\right] \tag{6.8}$$

$\varepsilon(z)$ is the dielectric function. It may be noted that in getting the dielectric function the integrations over the variables of the medium as well as the electromagnetic field have been performed. By virtue of its dependence on z, the dielectric function is time dependent. The poles z_j of the expression on the right side of (6.8) arising due to $\varepsilon(z)$ are of special interest. These yield the dispersion relation giving the frequencies z_j as functions of k. These are the natural or intrinsic frequencies which are independent of the initial preparation of the system, in the sense that interactions alone play a dominant role in determining them. This can be seen mathematically from the fact that an infinite series of diagrams has been summed to get the dielectric function and hence the collective effects play an important role in determining the nature of this function. When the interactions are purely magnetic - through a transverse electromagnetic field - $\varepsilon(z)$ can be made use of to study the phenomena such as the de Haas - van Alphen and related oscillations. The optics of the problem can also be studied as $\varepsilon(z)$ can be related to the refractive index. This has been discussed in the next chapter. When the interactions are through a scalar potential, this formalism can be employed for the study of microinstabilities in a plasma. Such instabilities do not depend upon the boundary conditions but only the nature and the strength of the interactions. The case of longitudinal interactions are important to study the Landau damping and the polarisation losses. These ideas have been discussed in chapters V and VI.

To conclude it may be noted that the treatment given here may be compared with the dielectric formulation of classical electrodynamics[8]. The presence of the test particle polarizes the medium by creating a non-uniform charge distribution around each particle of the medium. Because of this polarization the interactions between the particles are screened. This screening effect is described by the dielectric constant in phenomenological electrodynamics. The function $\varepsilon(z)$ obtained by summing an infinite series, does exactly represent the modification due to the presence of the medium. If the concentration d of the medium is put equal to zero, $\varepsilon(z)$ also goes to zero and the propagation of the test particle is given by W_{PP} which gives the free propagation of the particle in vacuum. $\varepsilon(z)$ is not a constant and for this reason it is called the dielectric function. However, in the asymptotic time limit which is equivalent to $z \to 0$, $\varepsilon(z)$ reduces to a constant.

FIGURE CAPTIONS

Fig.1 : The matrix element of order e^2 which is denoted as W_{PP} in the text. The solid line on the top indicates the test particle propagator and the dashed line below indicates the propagator for the electromagnetic field. The box indicates a creation vertex for correlations and the circle at left indicates an annihilation vertex.

Fig.2,3 and 4 : Fourth order contributions that do not involve particles of the medium.

Fig.5,6,7 and 8 : Fourth order contributions that include both the test particle and the medium particle. ℓ denotes a particle of the medium and P denotes the test particle. Fig.5 corresponds to first Born approximation and is represented as $W_{P\ell\ell P}$ in the text.

Fig.9 : The sixth order contribution that is not neglected in the approximation procedure. This diagram involves two particles of the medium.

Fig.10 : The effect of the summation of the infinite series is given by the wavy line.

Fig.11 : This diagram gives the structure of the wavy line appearing in Fig.10

Fig.12 : The contour for integration (6.5).

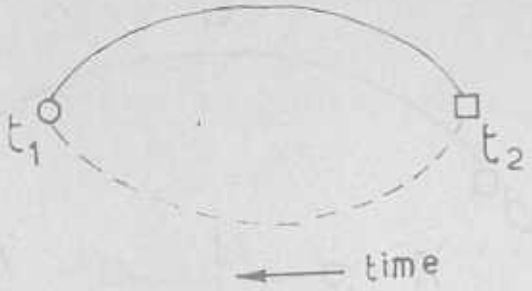


FIG.1

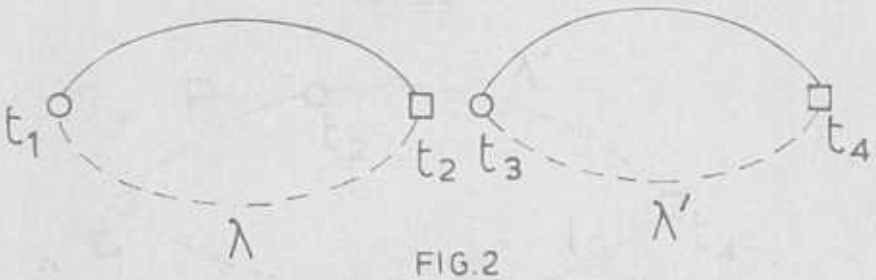


FIG.2

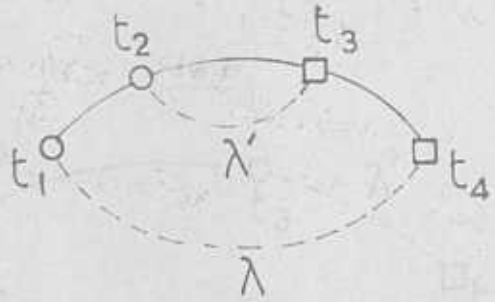


FIG.3

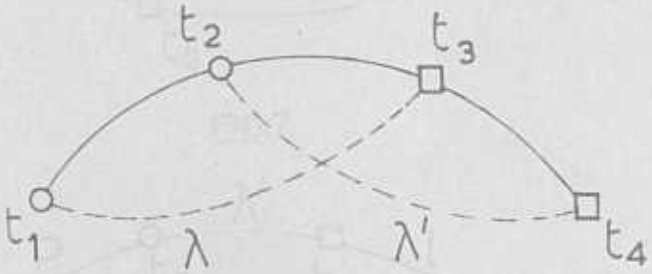


FIG. 4

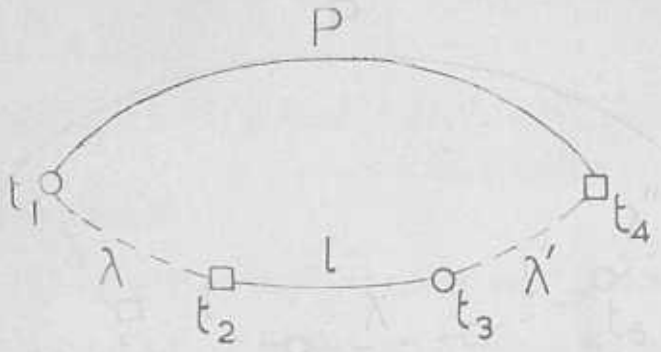


FIG.5

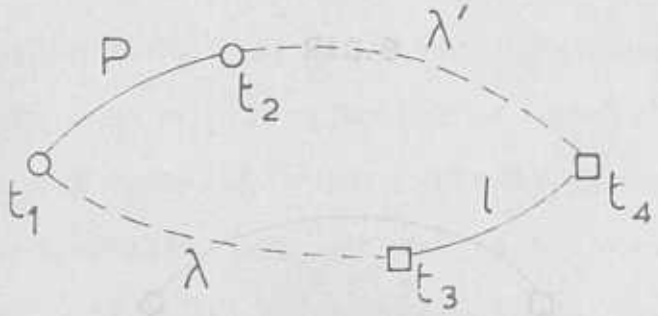


FIG.6

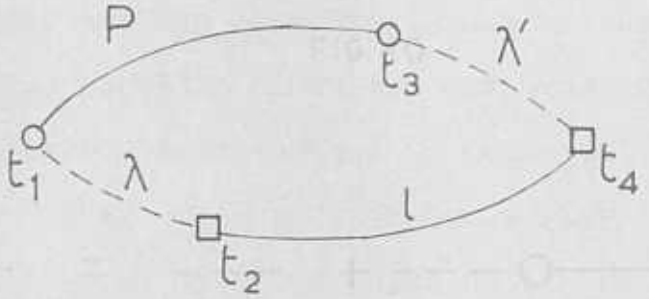


FIG.7

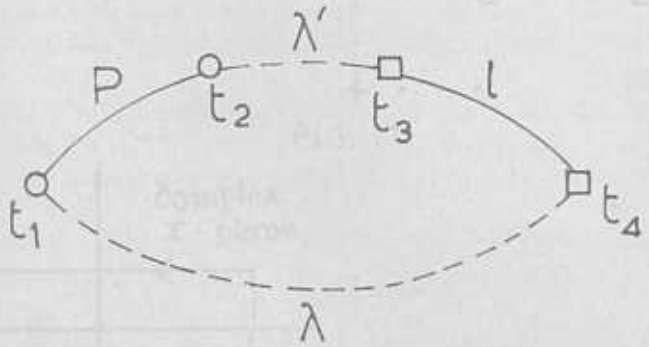


FIG.8

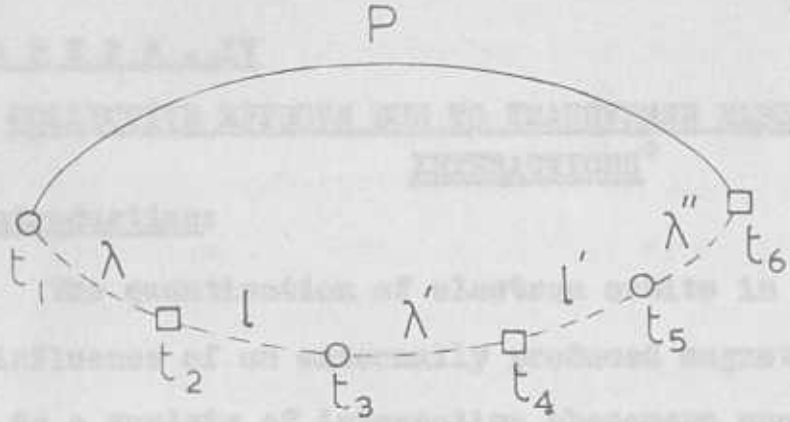


FIG. 9

FIG. 10

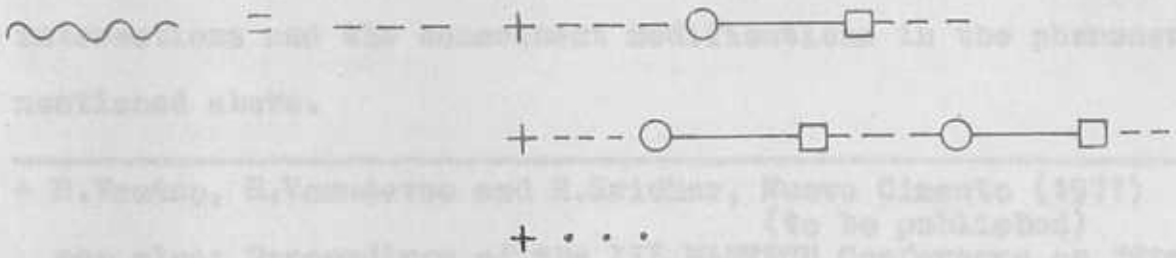


FIG. 11

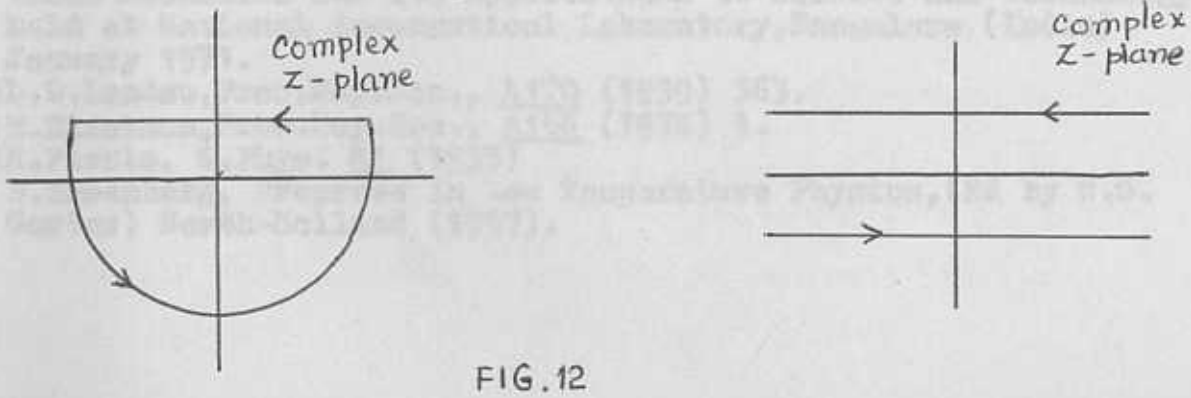


FIG. 12

CHAPTER. IV

COLLECTIVE EFFECTS DUE TO TRANSVERSE ELECTROMAGNETIC INTERACTIONS+

1. Introduction:

The quantisation of electron orbits in a metal due to the influence of an externally produced magnetic field gives rise to a variety of interesting phenomena such as the de Haas - van Alphen (DHVA) oscillations, the cyclotron resonance etc. Though the free electron theory of metals[1-3] is able to account for such phenomena with considerable success, the role of collective interactions can not be ignored. Thus it is only natural, to seek for the modifications, as well the new features that may arise because of the inclusion of collective effects. These phenomena are essentially due to the magnetic interactions among the conduction electrons and hence it is only appropriate to describe the interactions by means of a transverse electromagnetic field. This chapter deals with the inclusion of interactions and the consequent modifications in the phenomena mentioned above.

+ R.Pratap, R.Vasudevan and R.Sridhar, Nuovo Cimento (1971) (to be published)
see also: Proceedings of the III MASTECH Conference on "Statistical Mechanics and its Applications to Science and Technology" held at National Aeronautical Laboratory, Bangalore (India) January 1971.

1) L.D.Landau, Proc.Roy.Soc., A170 (1939) 363.
2) M.Blackman, Proc.Roy.Soc., A166 (1938) 1.
3) R.Pierls, Z.Phys. 81 (1933)
D.Shoenberg, Progress in Low Temperature Physics, (Ed by C.G. Gorter) North-Holland (1957).

Considerable progress has been made in this field after the pioneering work of Landau[1] and Blackman[2] who have considered a non-interacting system of electrons in an external magnetic field. Interest in this direction has, however, been revived by the work of Quinn[4] who, in a series of papers, has studied the transport properties of such a system. Dewell[5] has studied this problem with an interaction potential of the form $V(|\vec{r}_i - \vec{r}_j|)$. Arunasalam,[5a] in a different context, has studied this physical situation with the linearised Vlasov equation. The present attempt is different from these approaches in that an infinite series of diagrams is summed to get the collective effects and the initial density distribution takes into account the Landau levels in a proper fashion. Further the approach presented here is essentially quasi-classical.

The next section deals with the formulation of the problem in the frame work of the ideas discussed in Chapter III. Section 3 deals with the evaluvation of the various terms of the infinite series and it is shown how the higher order terms can be factored in terms of the lower order ones so that the summation of the series is possible. The dielectric function is derived in section 4. Section 5 deals with a discussion of the results. An interesting outcome is the nature of the refractive index as a function of frequency. The mathematical

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4) See for instance: J.J.Quinn and Sergio Rodriguez, Phys.Rev. 126 (1962) 2487.
5) G.Dewell, Ph.D. Thesis, University of Brussels (1970).
5a) V. Arunasalam, Jour. Math. Phys. 10 (1969) 1305.

formulation presented here leads to the conclusion that there are prefered frequencies for absorption and emission. This has been demonstrated in fig.1.

2. Formulation of the problem:

Magnetic interactions in an electron gas in an external magnetic field are considered. There is a positive uniform background which maintains over all charge neutrality. A test particle is introduced into this system. The hamiltonian and the Liouville equation for the composite system have already been discussed in Chapter II. For the sake of continuity a brief account of the same is given here.

The complete hamiltonian is written as

$$\mathcal{H} = \mathcal{H}_P + \mathcal{H}_M + \mathcal{H}_F \tag{2.1}$$

where the suffixes P, M and F denote respectively the test particle, the medium and the electromagnetic field.

$$\mathcal{H}_P = \frac{1}{2m_P}\left[\vec{p}_P - \frac{e_P}{2c}\,\vec{q}_P \times \vec{H} - \frac{e_P}{c}\,\vec{A}(\vec{q}_P)\right]^2 \tag{2.2}$$

$$\mathcal{H}_F = \sum_{\lambda} \nu_\lambda \left(J_\lambda + J_{-\lambda} \right) \tag{2.3}$$

The hamiltonian for the medium has a structure analogous to (2.2). Following Heitler[6] and Henin[7] $\vec{A}(\vec{q})$ is given the

6) W.Heitler, "The Quantum Theory of Radiation", Oxford University Press (1954).
7) F.Henin, Physica 29 (1963) 1233.

following decomposition

$$\vec{A} = \sum_{\lambda} \vec{A}_{\lambda}(\vec{q}_p)$$

$$= \left(\frac{8c^2}{\Omega}\right)^{\frac{1}{2}} \sum_{\lambda, \vec{k}_\lambda} \frac{\hat{e}_\lambda}{\sqrt{ck_\lambda}} \left\{ \sqrt{J_\lambda} \cos(\vec{k}_\lambda \cdot \vec{q}) \cos\omega_\lambda + \sqrt{J_{-\lambda}} \sin(\vec{k}_\lambda \cdot \vec{q}) \cos\omega_{-\lambda} \right\} \tag{2.4}$$

with the condition

$$(\hat{e}_\lambda \cdot \vec{k}_\lambda) = 0 \tag{2.5}$$

(2.5) indicates that the field is transverse.

The Liouville equation has the form

$$\frac{\partial \rho}{\partial t} + i\mathcal{L}\rho = \left(\frac{e}{c}\right)(i\delta L)\rho \tag{2.6}$$

$(i\mathcal{L})$ denotes the free propagation and $(i\delta L)$ includes interactions.

$$i\mathcal{L} = L_p + L_M + L_F \tag{2.7}$$

$$L_p = \omega_p \frac{\partial}{\partial \omega_p} + \frac{\vec{P}_p}{m_p} \cdot \frac{\partial}{\partial \vec{q}_p} \tag{2.8}$$

$$L_F = \sum_{\lambda} \nu_\lambda \frac{\partial}{\partial \omega_\lambda} \tag{2.9}$$

L_M has a structure similar to (2.8) with the replacement $p \to \ell$. Further there is a summation over ℓ . $(i\delta L)$ has the following structure.

$$(i\delta L) = \mathcal{A}_p + \mathcal{B}_p + \mathcal{A}_\ell + \mathcal{B}_\ell \tag{2.10}$$

with

$$\mathcal{A}_p = -\alpha_p \sum_{\vec{k}_\lambda} \left(\frac{8c^2}{\Omega \nu_\lambda}\right)^{\frac{1}{2}} \sqrt{J_\lambda} \cos\omega_\lambda \quad \theta_p^\lambda \tag{2.11}$$

$$\mathcal{B}_p = -\alpha_p \sum_{\vec{k}_\lambda} \left(\frac{8c^2}{\Omega \nu_\lambda}\right)^{-\frac{1}{2}} (e_\lambda \cdot P_p) \cos(\vec{k}_\lambda \cdot \vec{q}_p)$$

$$\left[\frac{\partial}{\partial \omega_\lambda}(\sqrt{J_\lambda} \cos\omega_\lambda) \frac{\partial}{\partial J_\lambda} - \frac{\partial}{\partial J_\lambda}(\sqrt{J_\lambda} \cos\omega_\lambda) \frac{\partial}{\partial \omega_\lambda} \right] \tag{2.12}$$

where

$$\theta_p^\lambda = \cos(\vec{k}_\lambda \cdot \vec{q}_p) \left[\frac{\partial}{\partial \omega_p}(\hat{e}_\lambda \cdot \vec{P}_p) \frac{\partial}{\partial J_p} - \frac{\partial}{\partial J_p}(\hat{e}_\lambda \cdot \vec{P}_p) \frac{\partial}{\partial \omega_p} - e_\lambda \cdot \frac{\partial}{\partial \vec{q}_p} \right]$$

$$- k_{\lambda_3} (\hat{e}_\lambda \cdot \vec{P}_p) \sin(\vec{k}_\lambda \cdot \vec{q}_p) \frac{\partial}{\partial p_{p_3}} \tag{2.13}$$

θ_ℓ^λ is obtained in a form identical to (2.13). The $(-\lambda)$ point of the interaction can be obtained by replacing cos $(\vec{k}_\lambda \cdot \vec{q}_p)$ by sin $(\vec{k}_\lambda \cdot \vec{q}_p)$ and sin $(\vec{k}_\lambda \cdot \vec{q}_p)$ by - cos $(\vec{k}_\lambda \cdot \vec{q}_p)$

3. <u>Evaluation of the series</u>:

The lowest order diagram has the following structure

$$W_{pp} = \left(\frac{e}{c}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \; \exp[-L_p(t-t_1)] \; \mathcal{A}_p^\lambda$$

$$\exp[-(L_p + L_\lambda)(t_1 - t_2)] \; \mathcal{B}_p^\lambda \; \exp[-L_p t_2] \; \rho_p(0) \tag{3.1}$$

The operators exp $[-L_\lambda(t-t_1)]$ and exp $[-L_\lambda t_2]$ are omitted because of the following reasons.

(i) the initial distribution is angle independent. Thus

$$\frac{\partial \rho(0)}{\partial \omega_\lambda} = 0 \tag{3.2}$$

(ii) Also

$$\int_0^{2\pi} d\omega_\lambda \frac{\partial F}{\partial \omega_\lambda} = 0 \tag{3.3}$$

Because of these reasons only first terms in the expansion of the operators exp $[-I_\lambda(t-t_1)]$ and exp $[-L_\lambda t_2]$ contribute. Now the integrations over the electromagnetic field variables can be performed.

$$\int_0^\infty dI_\lambda \; I_\lambda \frac{\partial}{\partial \omega_\lambda} \delta(I_\lambda) = -1 \tag{3.4}$$

$$\int_0^{2\pi} d\omega_\lambda (\cos\omega_\lambda) \exp[-\nu_\lambda \tau \frac{\partial}{\partial \omega_\lambda}] (\sin\omega_\lambda)$$

$$= \int_0^{2\pi} d\omega_\lambda (\cos\omega_\lambda) \sin(\omega_\lambda - \nu_\lambda \tau) = -\pi \sin\nu_\lambda \tau \tag{3.5}$$

It is convenient to represent sin $(\nu_\lambda \tau)$ in terms of its Laplace transform. Thus

$$W_{pp} = \left(\frac{e}{c}\right)^2 \left(\frac{8\pi\alpha_p^2 c^2}{\Omega}\right) \int_0^t dt_1 \int_0^{t_1} dt_2 \; \exp[-I_p(t-t_1)\theta_p^\lambda$$

$$\exp[-I_p(t_1-t_2)] (\hat{e}_\lambda \cdot \vec{P}_p) \cos(\vec{k}_\lambda \cdot \vec{q}_p) \exp[-I_p t_2]$$

$$\left\{ \oint dz \; \exp[-iz\tau] \frac{1}{z^2-\nu_\lambda^2} \right\} \rho_p(0)$$

$$(\tau = t_1 - t_2) \tag{3.6}$$

The fourth order contribution is given by

$$W_{P\ell\ell P} = \left(\frac{8\pi e^2 \alpha_p^2}{m_p^2}\right) \frac{1}{\Omega \nu_\lambda} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_{t_2}^{t_1} dT \int_{t_2}^{T} dT' \; \sin[\nu_\lambda (t_1 - T)]$$

$$\sin[\nu_{\lambda'} (T' - t_2)] \exp[-L_\ell (t - t_1)] \; \theta_P^{\lambda} \exp[-L_P (t_1 - t_2)]$$

$$\left\{ \frac{8\pi e_\ell^2}{m_\ell^2} \frac{1}{\Omega \nu_{\lambda'}} \int_0^\infty d\tau_\ell \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell 3} \int d\vec{q}_\ell \; (\hat{e}_\lambda \cdot \vec{p}_\ell) \cos(\vec{k}_\lambda \cdot \vec{q}_\ell) \right.$$

$$\left. \exp[-L_\ell (T - T')] \; \theta_\ell^{\lambda'} \exp[-L_\ell T'] \; \rho_\ell(0) \right\}$$

$$(\hat{e}_{\lambda'} \cdot \vec{p}_P) \cos(\vec{k}_{\lambda'} \cdot \vec{q}_P) \exp[-L_P t_2] \; \rho_P(0)$$

(3.7)

The curly brackets on the right side of (3.7) denotes the modification due to the presence of the medium. This will be denoted by $E(T, T')$. It may also be noted that the integrations over the electromagnetic field variables have already been performed. Further the ordering in the time integrations have been changed using the factorisation theorem of Resibois discussed in Chapter III. This theorem shall be extensively used in this and in the chapters to follow.

$$E(T, T') = \frac{8\pi e_\ell^2}{\Omega \nu_{\lambda'}} \frac{1}{m_\ell^2} \int_0^\infty d\tau_\ell \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell 3} \int d\vec{q}_\ell \; (\hat{e}_\lambda \cdot \vec{p}_\ell)$$

(Contd ...)

$$\cos(\vec{k}_{\lambda'}\cdot\vec{q}_\ell)\ \exp[-L_\ell\tau]$$

$$\Big\{\cos(\vec{k}_{\lambda'}\cdot\vec{q}_\ell)\Big[\frac{\partial}{\partial\omega_\ell}(\hat{e}_{\lambda'}\cdot\vec{P}_\ell)\frac{\partial}{\partial J_\ell}-\frac{\partial}{\partial J_\ell}(\hat{e}_{\lambda'}\cdot\vec{P}_\ell)\frac{\partial}{\partial\omega_\ell}-e_{\lambda'}\cdot\frac{\partial}{\partial\vec{q}_\ell}\Big]$$

$$-k_{\lambda'_3}(\hat{e}_{\lambda'}\cdot\vec{P}_\ell)\ \sin(\vec{k}_{\lambda'}\cdot\vec{q}_\ell)\frac{\partial}{\partial p_{\ell_3}}\Big\}$$

$$\exp[-L_\ell T']\ \frac{1}{2^n n!\sqrt{\pi}}\ \exp(-W^2)\ H_n^2(W)$$

$$F_n\,\delta[J_\ell-(n+\tfrac{1}{2})\hbar]$$

$$\text{with}\quad \tau = T-T' \tag{3.8}$$

where

$$F_n = \{1+\exp[\beta(E_n-\mu)]\}^{-1} \tag{3.9}$$

$$E_n = (n+\tfrac{1}{2})\hbar\varpi_\ell + p_{\ell_3}^2/2m_\ell \tag{3.10}$$

and

$$W = \sqrt{\frac{m_\ell\varpi_\ell}{4\hbar}}\left[q_{\ell_2}-\sqrt{\frac{8J_\ell}{m_\ell\varpi_\ell}}\ \cos\omega_\ell\right] \tag{3.11}$$

The initial distribution used in (3.8) has already been discussed in the previous chapter. The propagator exp $[-L_\ell t]$ has two non-commuting parts which have to be decoupled before it can be operated on the functions to its right. This decoupling has been discussed in Appendix 1.

$$\exp[-L_\ell t] = \exp[-\varpi_\ell t \frac{\partial}{\partial \omega_\ell}] \exp[-(p_{\ell_3} t/m_\ell)\, \partial/\partial q_{\ell_3}]$$

$$\exp\left\{\sqrt{\frac{2J_\ell}{m_\ell \varpi_\ell}}\left[\left(\sin\omega_\ell - \sin(\omega_\ell - \varpi_\ell t)\right)\frac{\partial}{\partial q_{\ell_1}} + \left(\cos(\omega_\ell - \varpi_\ell t) - \cos\omega_\ell\right)\frac{\partial}{\partial q_{\ell_2}}\right]\right\}$$

(3.12)

Let M_1 **denote the following integral**

$$M_1 = \frac{1}{2^n n!\sqrt{\pi}} \int_0^\infty dJ_\ell\, \delta(J_\ell - (n+\tfrac{1}{2})\hbar) \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell_3} \int d\vec{q}_\ell$$

$$(\hat{e}_\lambda \cdot \vec{P}_\ell)\, \cos(\vec{k}_\lambda \cdot \vec{q}_\ell)\, \exp[-L_\ell \tau]\, \cos(\vec{k}_{\lambda'} \cdot \vec{q}_\ell)$$

$$\left[\frac{\partial}{\partial \omega_\ell}(\hat{e}_{\lambda'} \cdot \vec{P}_\ell)\frac{\partial}{\partial J_\ell} - \frac{\partial}{\partial J_\ell}(\hat{e}_{\lambda'} \cdot \vec{P}_\ell)\frac{\partial}{\partial \omega_\ell} - e_{\lambda'} \cdot \frac{\partial}{\partial \vec{q}_\ell}\right] \exp[-L_\ell \tau']$$

$$\exp(-W^2)\, H_n^2(W)\, F_n$$

$$= \frac{1}{2^n n!\sqrt{\pi}} \int_0^\infty dJ_\ell\, \delta[J_\ell - (n+\tfrac{1}{2})\hbar] \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell_3} \int d\vec{q}_\ell$$

$$(\hat{e}_\lambda \cdot \vec{P}_\ell)\, \cos(\vec{k}_\lambda \cdot \vec{q}_\ell)\, \exp[-L_\ell \tau]\, \cos(\vec{k}_{\lambda'} \cdot \vec{q}_\ell)$$

$$\left[\frac{\partial}{\partial \omega_\ell}(\hat{e}_{\lambda'} \cdot \vec{P}_\ell)\frac{\partial \tilde{W}}{\partial J_\ell} - \frac{\partial}{\partial J_\ell}(\hat{e}_{\lambda'} \cdot \vec{P}_\ell)\frac{\partial \tilde{W}}{\partial \omega_\ell} - e_{\lambda'_2}\frac{\partial \tilde{W}}{\partial q_{\ell_2}}\right]$$

$$\frac{\partial}{\partial \tilde{W}} \exp(-\tilde{W}^2)\, H_n^2(\tilde{W})\, F_n$$

(3.13)

where

$$\tilde{W} = \sqrt{\frac{m_\ell \varpi_\ell}{4\hbar}}\left[q_{\ell_2} + \sqrt{\frac{2J_\ell}{m_\ell \varpi_\ell}}\left(\cos\omega_\ell - 3\cos(\omega_\ell - \varpi_\ell \tau'_\lambda)\right)\right]$$

(3.14)

Further the term within the square backets in (3.13) can be shown to be independent of the angle varinble.

$$\left[\frac{\partial}{\partial \omega_\ell}(\hat{e}_{\lambda'}\cdot\vec{P}_\ell)\frac{\partial \tilde{w}}{\partial J_\ell} - \frac{\partial}{\partial J_\ell}(\hat{e}_{\lambda'}\cdot\vec{P}_\ell)\frac{\partial \tilde{w}}{\partial \omega_\ell} - e_{\lambda'_2}\frac{\partial \tilde{w}}{\partial q_{\ell 2}} \right]$$

$$= 3\sqrt{\frac{m_\ell \varpi_\ell}{4\hbar}}\left[(\hat{a}\cdot\hat{e}_{\lambda'})\sin\varpi_\ell T' - (\hat{b}\cdot\hat{e}_{\lambda'})\cos\varpi_\ell T' \right] \tag{3.15}$$

Thus M_1, as given by (3.13), may be rewritten as

$$M_1 = \frac{3}{2^n n!\sqrt{\pi}}\sqrt{\frac{m_\ell \varpi_\ell}{4\hbar}}\left[(\hat{a}\cdot\hat{e}_{\lambda'})\sin\varpi_\ell T' - (\hat{b}\cdot\hat{e}_{\lambda'})\cos\varpi_\ell T' \right]$$

$$\int_0^\infty dJ_\ell\, \delta(J_\ell - (n+\tfrac{1}{2})\hbar)\int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell 3}\int d\vec{q}_\ell\, (\hat{e}_{\lambda'}\cdot\vec{P}_\ell)$$

$$\cos(\vec{k}_\lambda\cdot\vec{q}_\ell)\cos(\vec{k}_{\lambda'}\cdot\vec{\tilde{q}}_\ell)\frac{\partial}{\partial \bar{W}}\left[\exp(-\bar{W}^2)H_n^2(\bar{W})\right]\cdot F_n \tag{3.16}$$

where

$$\bar{W} = \exp[-I_\ell \tau]$$

$$= \sqrt{\frac{m_\ell \varpi_\ell}{4\hbar}}\left\{ q_{\ell_2} + \sqrt{\frac{2J_\ell}{m_\ell \varpi_\ell}}\left[\cos\omega_\ell - 3\cos(\omega_\ell - \varpi_\ell T)\right]\right\} \tag{3.17}$$

and

$$\vec{\tilde{q}}_\ell = \exp[-I_\ell \tau]\,\vec{q}_\ell$$

$$= \hat{a}\left[q_{\ell_1} + \sqrt{\frac{2J_\ell}{m_\ell \varpi_\ell}}\left\{ (\sin(\omega - \varpi_\ell \tau) - \sin\omega)\right\}\right]$$

$$+ \hat{b}\left[q_{\ell_2} + \sqrt{\frac{2J_\ell}{m_\ell \varpi_\ell}}\left\{\cos\omega - \cos(\omega - \varpi_\ell \tau)\right\}\right] + \hat{c}\left[q_{\ell_3} - \frac{p_{\ell_3}\tau}{m_\ell}\right] \tag{3.18}$$

Adding the $(-\lambda)$ part of the interaction to (3.16)

$$M_1 = \frac{3}{2^n n! \sqrt{\pi}} \sqrt{\frac{m_\ell \varpi_\ell}{4\hbar}} \left[(\hat{a}\cdot\hat{e}_{\lambda'}) \sin \varpi_\ell T' - (\hat{b}\cdot\hat{e}_{\lambda'}) \cos \varpi_\ell T' \right]$$

$$\int_0^\infty dJ_\ell \, \delta[J_\ell - (n+\tfrac{1}{2})\hbar] \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell 3} \int d\vec{q}_\ell$$

$$\cos[(\vec{k}_{\lambda'} - \vec{k}_\lambda)\cdot\vec{q}_\ell + \sqrt{J_\ell}\,\phi - \gamma] \frac{\partial}{\partial \bar{w}} [H_n^2(\bar{w}) \exp(-\bar{w}^2)]$$

$$F_n \tag{3.19}$$

where

$$\phi = \sqrt{\frac{2}{m_\ell \omega_\ell}} \left[k_{\lambda'_1} (\sin(\omega_\ell - \varpi_\ell \tau) - \sin \omega_\ell) + k_{\lambda'_2} (\cos \omega_\ell - \cos(\omega_\ell - \varpi_\ell \tau)) \right] \tag{3.20}$$

and

$$\gamma = k_{\lambda_3} p_{\ell 3} \tau / m_\ell \tag{3.21}$$

The integrals over q_{ℓ_1} and q_{ℓ_3} yield delta function $\delta(k_{\lambda'_1} - k_{\lambda_1})$ and $\delta(k_{\lambda'_3} - k_{\lambda_3})$. To do the integration over q_{ℓ_2} it is convenient to change the variable to $\bar{w}$.

$$q_{\ell_2} = \sqrt{\frac{4\hbar}{m_\ell \varpi_\ell}}\, \bar{w} - \sqrt{\frac{2J_\ell}{m_\ell \varpi_\ell}} \left[\cos \omega_\ell - 3 \cos(\omega_\ell - \varpi_\ell T) \right] \tag{3.22}$$

Thus

$$M_1 = 3K \sqrt{\frac{4\hbar}{m_\ell \varpi_\ell}} \left[(\hat{a}\cdot\hat{e}_{\lambda'}) \sin \varpi_\ell T' - (b\cdot e_{\lambda'}) \cos \varpi_\ell T' \right] \delta(k_{\lambda'_1} - k_{\lambda_1}) \, \delta(k_{\lambda'_3} - k_{\lambda_3})$$

(contd ...)

(... Contd)

$$\frac{1}{2^n n!\sqrt{\pi}} \int_0^\infty dJ_\ell \, \delta[J_\ell-(n+\tfrac{1}{2})\hbar] \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dP_{\ell 3} \int d\vec{q}_\ell$$

$$(\vec{P}_\ell \cdot \hat{e}_\lambda) \sin\left[K\sqrt{\frac{4\hbar}{m_\ell \varpi_\ell}}\,\overline{W} + \sqrt{J_\ell}\,(\phi+\phi') - \gamma\right]$$

$$\exp(-\overline{W}^2)\, H_n^2(\overline{W})\, F_n$$

(3.23)

where

$$\phi' = -K\sqrt{\frac{2}{m_\ell \varpi_\ell}}\left[\cos\omega_\ell - 3\cos(\omega_\ell - \varpi_\ell T)\right] \tag{3.24}$$

and

$$K = k_{\lambda'_2} - k_{\lambda_2} \tag{3.25}$$

As the magnetic field is assumed to be a weak one K is taken to be independent of the mode λ of the electromagnetic field. Thus

$$M_1 = 3K\sqrt{\frac{4\hbar}{m_\ell \varpi_\ell}}\left[(\hat{a}\cdot\hat{e}_{\lambda'})\sin\varpi_\ell T' - (\hat{b}\cdot\hat{e}_{\lambda'})\cos\varpi_\ell T'\right]$$

$$L_n\left(2\frac{\hbar K^2}{m_\ell \varpi_\ell}\right)\exp\left(-\frac{\hbar K^2}{m_\ell \varpi_\ell}\right)\delta(k_{\lambda'_1}-k_{\lambda_1})\,\delta(k_{\lambda'_3}-k_{\lambda_3})$$

$$\int_0^\infty dJ_\ell\, \delta[J_\ell-(n+\tfrac{1}{2})\hbar] \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dP_{\ell 3}$$

$$(\vec{P}_\ell\cdot\hat{e}_{\lambda'})\sin\left[\sqrt{J_\ell}\,(\phi+\phi') - \gamma\right] F_n$$

(3.26)

where L_n are the Laguerre Polynomials of order n. The other part of (3.8) may be denoted by M_2.

$$M_2 = -k_{\lambda'_3} \frac{1}{2^n n! \sqrt{\pi}} \int_0^\infty d\sigma_\ell \, \delta[\sigma_\ell - (n+\tfrac{1}{2})\hbar] \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell_3} \int d\vec{q}_\ell$$

$$(\hat{e}_\lambda \cdot \vec{p}_\ell) \quad \cos(\vec{k}_\lambda \cdot \vec{q}_\ell) \quad \exp[-I_\ell \tau]$$

$$(\hat{e}_{\lambda'} \cdot \vec{p}_\ell) \quad \sin(\vec{k}_{\lambda'} \cdot \vec{q}_\ell)$$

$$\frac{\partial}{\partial p_{\ell_3}} \left\{ \exp[-I_\ell T'] \exp[-W^2] \; H_n^2(W) \right\} F_n$$

(3.27)

$$= -k_{\lambda'_3} \frac{1}{2^n n! \sqrt{\pi}} \int_0^\infty d\sigma_\ell \, \delta[\sigma_\ell - (n+\tfrac{1}{2})\hbar] \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell_3} \int d\vec{q}_\ell$$

$$(\hat{e}_\lambda \cdot \vec{p}_\ell) \; \cos(\vec{k}_\lambda \cdot \vec{q}_\ell) \; (\hat{e}_{\lambda'} \cdot \vec{\tilde{p}}_\ell) \; \sin(\vec{k}_{\lambda'} \cdot \vec{\tilde{q}}_\ell)$$

$$\exp(-\bar{W}^2) \; H_n^2(\bar{W}) \; \frac{\partial F_n}{\partial p_{\ell_3}}$$

(3.28)

where

$$\vec{\tilde{p}}_\ell = \sqrt{2 m_\ell \varpi_\ell \sigma_\ell} \left[\hat{a} \; \cos(\omega_\ell - \varpi_\ell \tau) + \hat{b} \; \sin(\omega_\ell - \varpi_\ell \tau) \right] - \hat{c} \; p_{\ell_3} \tau / m_\ell$$

(3.29)

$\vec{\tilde{q}}_\ell$ and $\bar{W}$ are defined in (3.18) and (3.17) respectively. Following essentially the same procedure employed in the evaluation of M_1, the following equation is obtained for M_2.

$$M_2 = -k_{\lambda_3}\sqrt{\frac{4\hbar}{m_\ell\varpi_\ell}}\int_0^\infty dJ_\ell\,\delta[J_\ell-(n+\tfrac{1}{2})\hbar]\int_0^{2\pi}d\omega_\ell\int_{-\infty}^{\infty}dp_{\ell_3}$$

$$L_n\left(2\frac{\hbar K^2}{m_\ell\varpi_\ell}\right)\exp\left(-\frac{\hbar K^2}{m_\ell\varpi_\ell}\right)(\hat{e}_\lambda\cdot\vec{P}_\ell)\,(\hat{e}_{\lambda'}\cdot\vec{\tilde{P}}_\ell)$$

$$\sin\left[\sqrt{J_\ell}(\phi+\phi')-\gamma\right]\frac{\partial F_n}{\partial p_{\ell_3}}$$

(3.30)

Thus $E(T,T')$ may be written as

$$E(T,T') = \frac{8\pi}{\Omega\nu_{\lambda'}}\left(\frac{e_\ell}{m_\ell}\right)^2\sqrt{\frac{4\hbar}{m_\ell\varpi_\ell}}\,L_n\left(2\frac{\hbar K^2}{m_\ell\varpi_\ell}\right)\exp\left(-\frac{\hbar K^2}{m_\ell\varpi_\ell}\right)$$

$$\delta(k_{\lambda'_1}-k_{\lambda_1})\,\delta(k_{\lambda'_2}-k_{\lambda_2})\int_0^\infty dJ_\ell\,\delta[J_\ell-(n+\tfrac{1}{2})\hbar]$$

$$\int_0^{2\pi}d\omega_\ell\int_{-\infty}^{\infty}dp_{\ell_3}\,(\hat{e}_\lambda\cdot\vec{P}_\ell)\,\sin\left[\sqrt{J_\ell}(\phi+\phi')-\gamma\right]$$

$$\Big\{3K\left[(\hat{a}\cdot\hat{e}_{\lambda'})\sin\varpi_\ell T-(\hat{b}\cdot\hat{e}_{\lambda'})\cos\varpi_\ell T\right]F_n$$

$$-k_{\lambda'_3}(\hat{e}_{\lambda'}\cdot\vec{\tilde{P}}_\ell)\frac{\partial F_n}{\partial p_{\ell_3}}\Big\}$$

(3.31)

ω_ℓ integration can be performed by putting

$$\phi+\phi' = R\,\sin(\omega_\ell+\theta)$$

(3.32)

R and θ are determined by the relations

$$R\cos\theta = \sqrt{\frac{2}{m_\ell\varpi_\ell}}\Big\{k_{\lambda'_1}(\cos\varpi_\ell\tau-1)-k_{\lambda'_2}\sin\varpi_\ell\tau$$

$$+3K\sin\varpi_\ell T\Big\}$$

(3.33)

$$R \sin\theta = \sqrt{\frac{2}{m_\ell \omega_\ell}} \left[- k_{\lambda'_1} \sin \omega_\ell \tau + (1 - \cos \omega_\ell \tau) k_{\lambda'_2} - K (1 - 3 \cos \omega_\ell T) \right] \tag{3.34}$$

Making use of the following relations[8)]

$$\sin [R\sqrt{J_\ell} \sin (\omega_\ell + \theta)] = 2 \sum_{r=0}^{\infty} J_{2r+1} (R \sqrt{J_\ell}) \sin [(2r+1)(\omega_\ell + \theta)] \tag{3.35}$$

$$\cos [R\sqrt{J_\ell} \sin (\omega_\ell + \theta)] = J_0 (R \sqrt{J_\ell}) + 2 \sum_{r=0}^{\infty} J_{2r} (R\sqrt{J_\ell}) \sin (2r (\omega_\ell + \theta)) \tag{3.36}$$

These relations are useful in seperating the ω_ℓ and J_ℓ dependence of the functions appearing in (3.31). Further in the equation (3.31) there is a summation over the medium particles. This gives a quantity N, which when taken along with the Ω^{-1} factor, yields the concentration d in the thermodynamic limit. Thus it is possible to get the plasma frequency $\omega_{p\ell}$ in the expression given by (3.31).

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8) W.Magnus and F.Oberhettinger, "Formulas and Theorems for functions of Mathematical Physics" Chelsa Publishing Company, New York (1949).

Thus

$$E(\tau,\tau') = \left(2\frac{\omega_{pl}^2}{m_l \nu_{\lambda'}}\right)\left(2\frac{\hbar K^2}{m_l \varpi_l}\right)^{\frac{1}{2}} L_n\left(2\frac{\hbar K^2}{m_l \varpi_l}\right) \exp\left[-\frac{\hbar K^2}{m_l \varpi_l}\right]$$

$$\Big\{ 3K\left[(\hat{e}_{\lambda'}\cdot\hat{a})\sin\varpi_l\tau' - (\hat{e}_{\lambda'}\cdot\hat{b})\cos\varpi_l\tau'\right]$$

$$\Big[2\pi\sqrt{2m_l\varpi_l(n+\tfrac{1}{2})\hbar}\; J_1\left(R\sqrt{(n+\tfrac{1}{2})\hbar}\right) C_0 \left((\hat{e}_\lambda\cdot\hat{a})\sin\theta + (\hat{e}_\lambda\cdot\hat{b})\cos\theta\right)$$

$$-2\pi(e_\lambda\cdot c)\, S_1\, J_0\left(R\sqrt{(n+\tfrac{1}{2})\hbar}\right)\Big]$$

$$-2\pi k_{\lambda_3}\left[2m_l\varpi_l(n+\tfrac{1}{2})\hbar\right] S_0'$$

$$\Big[J_0\left(R\sqrt{(n+\tfrac{1}{2})\hbar}\right)\Big\{\big((\hat{e}_\lambda\cdot\hat{a})(\hat{e}_{\lambda'}\cdot\hat{a}) + (\hat{e}_\lambda\cdot\hat{b})(\hat{e}_{\lambda'}\cdot\hat{b})\big)\cos\varpi_l\tau + \big((\hat{e}_\lambda\cdot\hat{b})(\hat{e}_{\lambda'}\cdot\hat{a}) - (\hat{e}_\lambda\cdot\hat{a})(\hat{e}_{\lambda'}\cdot\hat{b})\big)\sin\varpi_l\tau\Big\}$$

$$+ J_2\left(R\sqrt{(n+\tfrac{1}{2})\hbar}\right)\Big\{\big((\hat{e}_\lambda\cdot\hat{a})(\hat{e}_{\lambda'}\cdot\hat{a}) - (\hat{e}_\lambda\cdot\hat{b})(\hat{e}_{\lambda'}\cdot\hat{b})\big)\cos(\varpi_l\tau+2\theta) - \big((\hat{e}_\lambda\cdot\hat{b})(\hat{e}_{\lambda'}\cdot\hat{a}) + (\hat{e}_\lambda\cdot\hat{a})(\hat{e}_{\lambda'}\cdot\hat{b})\big)\sin(\varpi_l\tau+2\theta)\Big\}\Big]$$

$$+ 2\pi k_{\lambda_3}\sqrt{2m_l\varpi_l(n+\tfrac{1}{2})\hbar}\; C_1'\, J_1\left(R\sqrt{(n+\tfrac{1}{2})\hbar}\right)$$

$$\Big[(\hat{e}_\lambda\cdot\hat{c})(\hat{e}_{\lambda'}\cdot\hat{a})\sin(\varpi_l\tau+\theta) + (\hat{e}_\lambda\cdot\hat{c})(\hat{e}_{\lambda'}\cdot\hat{b})\cos(\varpi_l\tau+\theta) + (\hat{e}_{\lambda'}\cdot\hat{c})(\hat{e}_\lambda\cdot\hat{a})\sin\theta + (\hat{e}_{\lambda'}\cdot\hat{c})(\hat{e}_\lambda\cdot\hat{b})\cos\theta\Big]$$

$$-2\pi k_{\lambda_3}(\hat{e}_\lambda\cdot\hat{c})(\hat{e}_{\lambda'}\cdot\hat{c})\; S_2'\, J_0\left(R\sqrt{(n+\tfrac{1}{2})\hbar}\right)\Big\}$$

(3.37)

where

$$C_{2n} = \int_{-\infty}^{\infty} dp_{\ell_3} \, (p_{\ell_3})^{2n} \, (\cos\gamma) \, \tilde{F}_n(p_{\ell_3}) \tag{3.38a}$$

$$S_{2n+1} = \int_{-\infty}^{\infty} dp_{\ell_3} \, (p_{\ell_3})^{2n+1} \, (\sin\gamma) \, \tilde{F}_n(p_{\ell_3}) \tag{3.38b}$$

$$C'_{2n+1} = \int_{-\infty}^{\infty} dp_{\ell_3} (p_{\ell_3})^{2n+1} (\cos\gamma) \frac{d}{dp_{\ell_3}} \left(\tilde{F}_n(p_{\ell_3})\right) \tag{3.38c}$$

$$S'_{2n} = \int_{-\infty}^{\infty} dp_{\ell_3} \, (p_{\ell_3})^{2n} \, (\sin\gamma) \, \frac{d}{dp_{\ell_3}} \left(\tilde{F}_n(p_{\ell_3})\right) \tag{3.38d}$$

with

$$\tilde{F}_n(p_{\ell_3}) = \left\{ 1 + \exp\left[\beta\left((n+\tfrac{1}{2})\hbar\omega_\ell + p_{\ell_3}^2/2m_\ell\right)\right]\right\}^{-1} \tag{3.39}$$

The summation over the Landau levels may now be performed.

Let

$$\sigma = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \int_{-\infty}^{\infty} dp_{\ell_3} \, (p_{\ell_3})^2 \, (\sin\gamma)$$

$$\frac{d}{dp_{\ell_3}} \left\{ \exp\left[-(j+1)\beta\left((n+\tfrac{1}{2})\hbar\omega_\ell + (p_{\ell_3}^2/2m_\ell)\right)\right]\right\}$$

$$L_n\left(2\frac{\hbar K^2}{m_\ell \omega_\ell}\right) J_0\left(R\sqrt{(n+\tfrac{1}{2})\hbar}\right) \tag{3.40}$$

It may be noted that in (3.40) the function F_n has been expandend binamially.

Making use of the relation[9)]

$$J_0(2\sqrt{x}) = \int_0^\infty ds\; J_0(s)\; J_0\left(\frac{x}{s}\right) \tag{3.41}$$

and expressing $J_0\left(\frac{x}{s}\right)$ in terms of its generating function[9)] it can be readily seen that

$$\sigma = \sum_{j=0}^{\infty} (-1)^j\; I_2(j) \int_0^\infty ds\; J_0(s)$$

$$\frac{1}{\pi}\int_0^\pi d\alpha\; \frac{t^{1/2}}{1-t}\; \exp\left[-\frac{2\hbar K^2}{m_\ell \omega_\ell}\;\frac{t}{1-t}\right] \tag{3.42}$$

where

$$I_2(j) = \int_{-\infty}^{\infty} dp_{\ell 3}\,(p_{\ell 3})^2\,(\sin\gamma)\;\frac{d}{dp_{\ell 3}}\left\{\exp\left[-\beta(j+1)\cdot\left(\left(p_{\ell 3}^2/2m_\ell\right)-\mu\right)\right]\right\} \tag{3.43}$$

and

$$t = \exp\left\{-(j+1)\,\beta\hbar\omega_\ell + i\,R^2\hbar\,(\cos\alpha)\,/4s\right\} \tag{3.44}$$

No approximation procedure has been employed in deriving the equation (3.42). It is, however, very difficult to evaluate

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9) I.S.Gradshteyn and I.M. Ryzhik, Tables of integrals, series and products, Academic Press (New York) (1965)
The generating function for the Laguerre polynomials has also to be used to derive (3.42).

$$\sum_n L_n(x)\, y^n = (1-y)^{-1}\exp\left[-xy(1-y)^{-1}\right],\quad |y|<1.$$

the integrals over α and S . For this purpose the following approximations are invoked. The perturbation series $\sum_{r=0}^{\infty} a_r (e^2 d)^r$ has been summed under assumption of a weak external magnetic field. Further the temperature is also assumed very low so that $\beta\hbar\omega_\ell >> 1$. In general $|t| < 1$ Therefore for weak external fields and for low temperatures, is neglected in comparison with unity. This simplifies the integral to a very great extent. The α integration can be performed by expanding the exponential function and the S integration is also done.

$$\sigma = \sum_{j=0}^{\infty} (-1)^j I_2(j) \sum_{r=0}^{\infty} \frac{1}{r!} \left(2\frac{\hbar K^2}{m_\ell \omega_\ell}\right)^r$$

$$\exp\left[-(r+\tfrac{1}{2})(j+1)\beta\hbar\omega_\ell\right] J_0\left[R\sqrt{(n+\tfrac{1}{2})\hbar}\right] \tag{3:45}$$

Taking the $r=0$ term alone, the j summation can be done

$$\sigma = J_0\left(R\frac{\sqrt{\hbar}}{2}\right)\int_{-\infty}^{\infty} dp_{\ell_3} (p_{\ell_3})^2 (\sin\gamma)$$

$$\frac{d}{dp_{\ell_3}}\left\{\frac{1}{1+\exp\left[\beta\left((p_{\ell_3}^2/2m_\ell)+\frac{\hbar\omega_\ell}{2}-\mu\right)\right]}\right\} \tag{3.46}$$

Considering all the terms in (3.37), it is not a very bad approximation to assume that σ gives the dominant contribution for weak fields and low temperatures. The non conservation of wave vectors in the y direction, given by the

quantity K , arises only because of the magnetic field. In the absence of the external field this quantity is zero. Further the present treatment is quasiclassical in which the electromagnetic waves act as momentum carriers. The non conservation of the wave vectors in the y direction is essentially due to the existence of currents in the system. For weak external fields the currents may be assumed be very weak also and hence the non-conservation may also be taken to be small. Under these circumstances products of K, ω_l and higher powers of these quantities may be neglected. This justifies to some extent the assumption that σ gives the dominant contribution.

4. <u>The dielectric function</u>:

From the discussion given in the previous section, it can be shown that from the fourth order contribution, the second order graph can be factored out. From equations (3.7) and (3.8) it is clear that

$$W_{P\ell\ell P} = \left(-\pi \frac{e_p^2}{c^2} \frac{8c^2}{\Omega}\right) \sum_{\lambda} \sum_{\lambda'} \frac{1}{\nu_\lambda} \int_0^t dt_1 \int_0^{t_1} dt_2 \exp[-L_p(t-t_1)]$$

$$\theta_p^\lambda \exp[-L_p(t_1-t_2)] \, (\hat{e}_{\lambda'} \cdot \vec{P}_p) \cos(\vec{k}_{\lambda'} \cdot \vec{q}_p) \exp[-L_p t_2]$$

$$\cdot \rho_p^{(0)}$$

$$\int_{t_2}^{t_1} d\tau \int_{t_2}^{\tau} d\tau' \sin[\nu_\lambda(t_1-\tau)] \sin[\nu_{\lambda'}(\tau'-t_2)] \; E(\tau,\tau')$$

(4.1)

Comparing (3.6) with (4.1) it is seen that

$$W_{P\ell\ell P} = W_{PP}\, \mathcal{E}(z) \tag{4.2}$$

where $\mathcal{E}(z)$ is the Laplace transform of the leading term in (3.37). For weak fields $\mathcal{E}(z)$ has the representation[10)]

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10) In getting χ given by (4.5), the polarisation vectors have to be eliminated by summation over the two possible directions.

$$(\hat{e}_\lambda \cdot \vec{k}_\lambda) = 0 \quad \text{(i)} \quad (\hat{e}_{\lambda'} \cdot \vec{k}_{\lambda'}) = 0 \quad \text{(ii)} \quad \text{and } \vec{k}_{\lambda'} = \vec{k}_\lambda + \hat{b}K \quad \text{(iii)}$$

Therefore the following expansion is possible

$$e_\lambda = \frac{1}{2}\frac{\vec{k}_\lambda \times \vec{H}}{|\vec{k}_\lambda \times \vec{H}|} + \frac{1}{2}\frac{\vec{k}_\lambda \times (\vec{k}_\lambda \times \vec{H})}{|\vec{k}_\lambda \times (\vec{k}_\lambda \times \vec{H})|} \tag{iv}$$

and

$$e_{\lambda'} = \frac{1}{2}\frac{\vec{k}_{\lambda'} \times \vec{H}}{|\vec{k}_{\lambda'} \times \vec{H}|} + \frac{1}{2}\frac{\vec{k}_{\lambda'} \times (\vec{k}_{\lambda'} \times \vec{H})}{|\vec{k}_{\lambda'} \times (\vec{k}_{\lambda'} \times \vec{H})|} \tag{v}$$

Using (iii) it is seen that

$$|\vec{k}_{\lambda'} \times \vec{H}| \leq |\vec{k}_\lambda \times \vec{H}| + K\,|\hat{b} \times \vec{H}| \tag{vi}$$

Expanding to first order in K

$$\hat{e}_{\lambda'} = \hat{e}_\lambda + \frac{K}{2}\left[\frac{\hat{b} \times \vec{H}}{|\hat{b} \times \vec{H}|} + \frac{\vec{k}_\lambda \times (\hat{b} \times \vec{H}) + (\hat{b} \times (\vec{k}_\lambda \times \vec{H}))}{|\vec{k}_\lambda \times (\vec{k}_\lambda \times \vec{H})|}\right]$$

$$= \hat{e}_\lambda + u \quad (\text{say}) \tag{vii}$$

$$\mathcal{E}(z) = \frac{\chi}{(z^2-\nu_{\lambda'}^2)(z^2-\bar{\gamma}^2)} \tag{4.3}$$

where

$$\bar{\gamma} = k_{\lambda_3} p_0 / m_\ell \tag{4.4}$$

$$\chi = -\omega_{p\ell}^2 \left(32\pi^3/m_\ell\right) \left[v_F/\varpi_\ell (3\pi^2 c)^{1/3}\right]^{\frac{1}{2}} \left(k_{\lambda_3} p_0/m_\ell\right)^2$$

$$J_0\left(R\sqrt{\bar{\gamma}}/2\right) G(\vec{k}_\lambda, \vec{H}) \frac{d}{dp_0}\left[\tilde{F}_0(p_0)\right] \tag{4.5}$$

The integration over p_{ℓ_3} occurring in (3.46) is trivially done by replacing the integral by the value of the integrand at the point p_0 where the momenta in the z -direction are assumed to be peaked. Further

$$G(\vec{k}_\lambda, \vec{H}) = \left[4H^2 |\vec{k}_\lambda \times (\vec{k}_\lambda \times \vec{H})|^2\right]^{-1}$$
$$\left\{ \left[\vec{k}_\lambda \times (\vec{k}_\lambda \times \vec{H}) \cdot \vec{H}\right]^2 + K \left[\vec{k}_\lambda \times (\vec{k}_\lambda \times \vec{H}) \cdot \vec{H}\right] \cdot \left[\vec{k}_\lambda \times (\hat{b} \times \vec{H}) \cdot \vec{H}\right] + \left[\hat{b} \times (\vec{k}_\lambda \times \vec{H}) \cdot \vec{H}\right] \right\} \tag{4.6}$$

In a similar fashion the sixth order term can be simplified.

$$W_{p\ell\ell\ell'\ell'p} = W_{pp}\, \mathcal{E}^2(z) \tag{4.7}$$

The infinite series can be summed in the following fashion

$$\rho(\vec{p}_p, \vec{q}_p; t) = \exp\left[-L_p t\right] \left\{\rho(0) + W_{pp}\left[1 + \mathcal{E}(z) + \mathcal{E}^2(z) + \cdots\right]\right\}$$

$$= \exp[-L_p t]\Big\{ \rho_p(0) + \frac{8\pi e^2 \alpha_p^2}{\Omega \nu_\lambda} \int_0^t dt_1 \int_0^{t_1} dt_2$$

$$\exp[L_p t_1]\ \theta_p^\lambda\ \exp[-L_p(t_1-t_2)]$$

$$\left[\oint dz \exp[-iz(t_1-t_2)] \frac{1}{(z^2-\nu_\lambda^2)(1-\varepsilon(z))} \right]$$

$$[(\hat{e}_\lambda + u)\cdot \vec{P}_p) \cos(\vec{k}_\lambda \cdot \vec{q}_p + K q_{p_2})$$

$$\exp[-L_p t_2]\ \rho_p(0) \Big\} \tag{4.8}$$

The inverse transform occurring in (4.8) is written as

$$\oint dz \exp[-iz(t_1-t_2)] \frac{(z^2-\nu_{\lambda'}^2)(z^2-\bar{\gamma}^2)}{(z^2-\nu_\lambda^2)[z^2-(\psi+\bar{\psi})][z^2-(\psi-\bar{\psi})]} \tag{4.9}$$

where

$$\psi = (\bar{\gamma}^2 + \nu_{\lambda'}^2)/2 \tag{4.10}$$

$$\bar{\psi} = (\bar{\gamma}^2 - \nu_{\lambda'}^2)/2 + \chi \tag{4.11}$$

with the assumption $\nu = \nu_{\lambda'} - \nu_\lambda << \nu_\lambda$, the integrand (4.9) can be written as a sum of partial fractions and then the contour integral is evaluated to obtain

$$-\frac{1}{2}\left[1 + \frac{\bar{\gamma}^2 - \nu_{\lambda'}^2}{2\bar{\psi}}\right](\psi-\bar{\psi})^{-1/2} \sin[(\psi-\bar{\psi})(t_1-t_2)]$$

$$-\frac{1}{2}\left[1 - \frac{\bar{\gamma}^2 - \nu_{\lambda'}^2}{2\bar{\psi}}\right](\psi+\bar{\psi})^{-1/2} \sin[(\psi+\bar{\psi})(t_1-t_2)] \tag{4.12}$$

5. Conclusions:

The dielectric function for a system of charged particles in an external magnetic field has been evaluvated. In this formalism the dynamical role played by the ions has been suppressed and is valid at low temperatures only. The inverse transform of the function $\mathcal{E}(z)$ involves sin function with arguments of the form $(\psi+\bar{\psi})^{1/2}(t_1-t_2)$. Thus if these functions and hence the distribution function have to be real the following condition has to be satisfied.

$$(\psi+\bar{\psi}) \geq 0 \tag{5.1}$$

This condition will be called the reality condition. Equation (5.1) may be explicitly stated as

$$\nu_\lambda^2+\nu^2 \geq 32\pi^3 p_0 \omega_{pl}^2 \left[v_F/\omega_l (3\pi^2 c)^{1/3}\right]^{1/2}$$

$$J_0\left(2\sqrt{\frac{\hbar K^2}{m_l \omega_l}}\right) G(\vec{k}_\lambda, \vec{H}) \frac{d}{dp_0}\left(\tilde{F}(p_0)\right) \tag{5.2}$$

Using the relation[11)]

$$F = -\beta \log \rho \tag{5.3}$$

the oscillation of the free energy is obtained. It is seen that these oscillations are essentially of the Bessel function type. Bessel function of order zero in the inverse of the magnitude of the applied magnetic field. Analogous behaviour

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11) See for instance: L.D.Landau and E.M.Lifshitz, "Statistical Physics", Pergamon Press (New York)(1965).

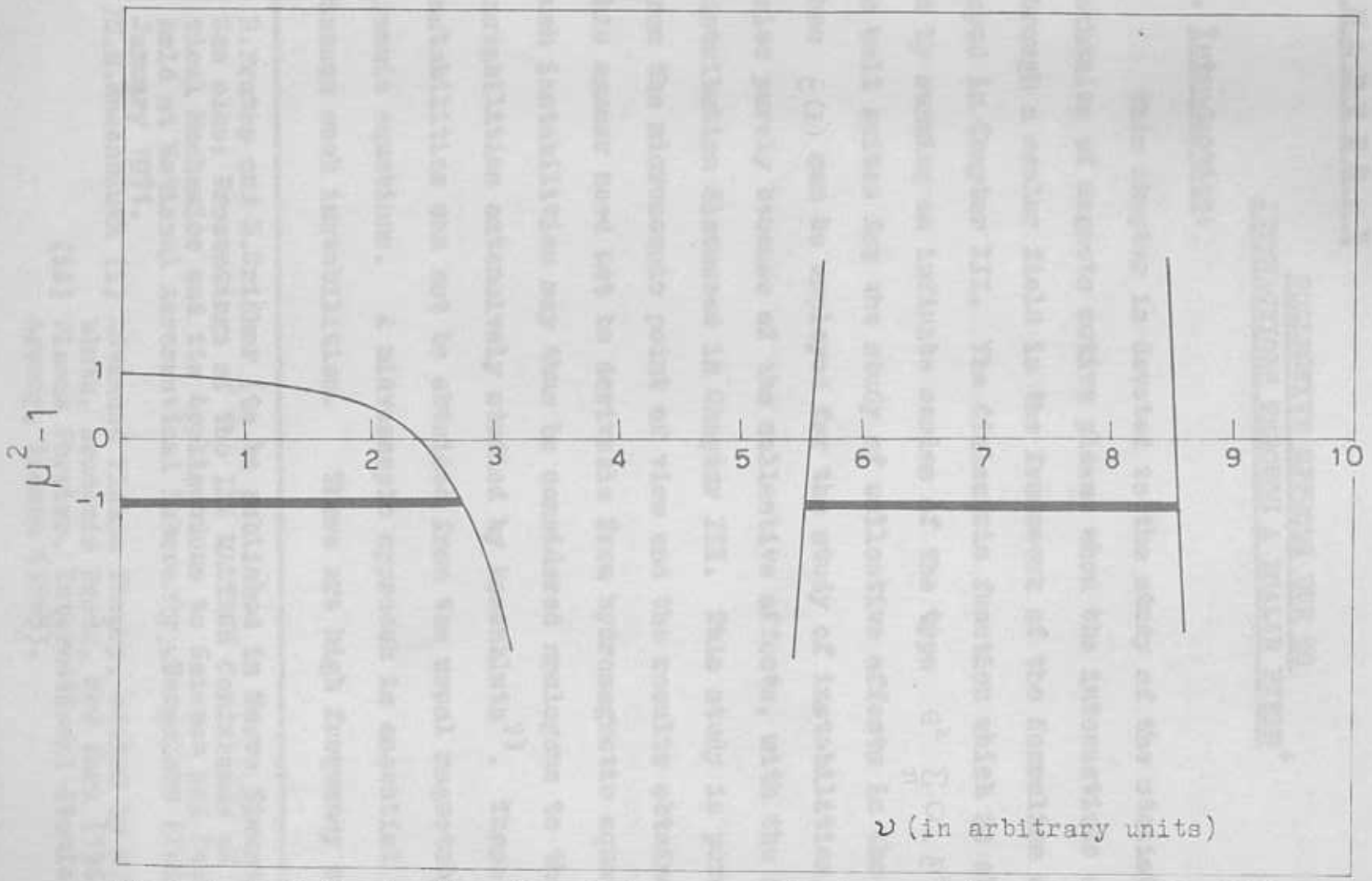
can, therefore, be expected in quantities like magnetic susceptibility.

From Maxwell's equations the refractive index can be computed from the dielectric function through the relation

$$\bar{\mu}^2 = 1 - \varepsilon(z) \tag{5.4}$$

It is seen that if the concentration d goes to zero, the plasma frequency also goes to zero and hence the dielectric function also vanishes. This implies the absence of collective effects in the system. In this situation the electromagnetic waves travel freely and hence there is no correlation. The same mathematical result could however be obtained whenever the Bessel function appearing in χ goes to zero - that is, for those values of K for which $2\hbar K^2/m_\ell \omega_\ell$ are zeros of the Bessel function. For these frequencies ν_{oi}, where is the order of the zero, the right hand side of (5.2) reduces to zero and electromagnetic waves behave as if there is no correlation. Thus the electromagnetic waves, which are the momentum carriers, will have discrete frequencies and because of the collective effects, the emission of radition from the system is allowed only for a certain band of frequencies. The plot of $\mu^2 - 1$ against the frequency, in arbitrary units as shown in figure 1, reveals the allowed frequencies. Such a technique is wellknown in plasma physics[12)]

12) I.B.Bernstein, Phys.Rev. 109 (1958) 10.



Thick bars indicate the allowed frequency domains and the rest are forbidden zeros.

FIG. 1

CHAPTER. V

COLLECTIVE EFFECTS DUE TO INTERACTIONS THROUGH A SCALAR FIELD+

1. Introduction:

This chapter is devoted to the study of the statistical mechanics of magneto active plasma when the interactions are through a scalar field in the framework of the formalism developed in Chapter III. The dielectric function which is obtained by summing an infinite series of the type $e^2 \sum_n a_n (e^2 d)^n$ is well suited for the study of collective effects in the system. Thus $\varepsilon(z)$ can be employed for the study of instabilities, which arise purely because of the collective effects, with the initial distribution discussed in Chapter III. This study is purely from the microscopic point of view and the results obtained in this manner need not be derivable from hydromagnetic equations. Such instabilities may thus be considered analogous to the micro instabilities extensively studied by Rosenbluth[1]. These micro instabilities can not be obtained from the usual magnetohydrodynamic equations. A microscopic approach is essential to discuss such instabilities. These are high frequency or

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+ R.Pratap and R.Sridhar (to be published in Nuovo Cimento (1971)) See also: Proceedings of the III MASTECH Conference on "Statistical Mechanics and its Applications to Science and Technology" held at National Aeronautical Laboratory, Bangalore (India) January 1971.

1) M.N.Rosenbluth (i) Advanced Plasma Theory, edited by M.N.Rosenbluth, Academic Press, New York (1964).
(ii) Plasma Physics, International Atomic Energy Agency, Vienna (1965).

small wavelength effects and may not lead to a gross disassembly of the plasma. However, these may lead to such phenomena as turbulance, enhanced resistivity etc.

Harris[2)] has studied the stability criteria for an electron plasma with an anisotropic velocity distribution. He investigated the small amplitude oscillations of a fully ionised quasi neutral plasma in a uniform, time independent, external magnetic field. In this procedure the motion of the ions and the perturbations of the magnetic field were neglected and the line arised Vlasov equation was considered.

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \frac{\partial \rho}{\partial \vec{q}} - \frac{e}{m}\left[\vec{E} + \frac{1}{c}\,\vec{v} \times \vec{B}\right] \cdot \frac{\partial \rho}{\partial \vec{v}} = 0 \qquad (1.1)$$

where the electric field E has to satisfy the condition

$$\nabla \cdot \vec{E} = -\nabla^2 \Phi = -4\pi e \int \rho \, d\vec{v} \qquad (1.2)$$

In solving this set of coupled equations, the distribution ρ function ρ is assumed to change only slightly from ρ_0, the zeroth order distribution. Equations (1.1) and (1.2) are written in a linearised from with an additional assumption that the spatial dependence of the perturbation is of the type $\exp(ik.q)$. Harris showed that the plasma, under these conditions, is unstable against the excitation of cyclotron

2) E.G.Harris (i) Phys.Rev.Letters 2 (1959) 34
(ii) J.Nucl.Energy, Part C: Plasma Physics 2 (1961) 138.

waves. His result agrees with the result of Bernstein[3] when k_z, the propagation vector in the z direction is zero. Harris further observed that there is neither instability nor damping when $k_z = 0$. Also, there can be no instability when $k_\perp = 0$ though there may be Landau damping.

The present treatment of magneto active plasma differs considerably from the treatment outlined by Balescu[4]. Here the initial distribution is quantised in the plane perpendicular to the applied magnetic field. Thus it leads to a formalism in the quasi classical framework. This peculiar feature profoundly alters the dynamics of correlations.

Further, the initial distribution employed here also turns out to be a new one. Usually the unstable waves have been investigated through the use of the following distributions[5].

(i) The anisotropic Maxwellian distribution

$$\rho_\perp(0) = \left(\frac{m}{2\pi}\right)^{3/2} \frac{1}{T_\perp T_\parallel^{1/2}} \exp\left\{-\frac{m v_\perp^2}{2T_\perp} - \frac{m v_\parallel^2}{2T_\parallel}\right\} \tag{1.3}$$

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3) I.B.Bernstein, Phys.Rev. 109 (1958) 10.
4) R.Balescu, "Statistical Mechanics of Charged Particles" Interscience Publishers, New York (1963)
5) See for instance the review article by A.V.Timofeev and V.I. Pistunovich, Reviews of Plasma Physics, Consultants Bureau, New York, Vol.5 (1970).

(ii) The delta function distribution

$$\rho_2(0) = \frac{1}{2\pi v_\perp} \delta(v_\perp - v_0)\, \delta(v_\parallel) \tag{1.4}$$

where the subscripts $\parallel$ and $\perp$ denote the components parallel and perpendicular to the applied magnetic field.

In section 2 the problem is formulated following the ideas discussed in Chapter III. Section 3 deals with the evaluvation of the perturbation series. The important result of this section being the Laplace transformed dielectric function which leads to the dispersion relation connecting the frequency and the wave vector. The results are discussed in section 4.

2. The Formulation:

The hamiltonian consists of three parts: the hamiltonian of the test particle, $\mathcal{H}_p$, the hamiltonian of the medium, $\mathcal{H}_M$ and that of the interactions $\mathcal{H}_F$. The test particle is correlated with the medium through the scalar field

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_M + \mathcal{H}_F \tag{2.1}$$

$$\mathcal{H}_p = \frac{1}{2m_p}\left[\vec{p}_p - \frac{e_p}{2c}\vec{q}_p \times \vec{H}\right]^2 + e_p\, \Phi(\vec{q}_p) \tag{2.2}$$

$$\mathcal{H}_M = \sum_\ell \frac{1}{2m_\ell}\left[\vec{p}_\ell - \frac{e_\ell}{2c}\vec{q}_\ell \times \vec{H}\right]^2 + e_\ell\, \Phi(\vec{q}_\ell) \tag{2.3}$$

$$\mathcal{H}_F = \sum_{0\sigma} \nu_{0\sigma} \left[J_{0\sigma} + J_{-0\sigma} \right] \tag{2.4}$$

(0σ) is the index for the scalar field and $\Phi(\vec{q})$ has the expansion[6)]

$$\Phi(\vec{q}) = \sum_{0\sigma} \left(\frac{8c^2}{\Omega \nu_{0\sigma}} \right)^{1/2} \left\{ \sqrt{J_{0\sigma}} \sin(\vec{k}_\sigma \cdot \vec{q}) \cos\omega_{0\sigma} - \sqrt{J_{-0\sigma}} \cos(\vec{k}_\sigma \cdot \vec{q}) \cos\omega_{-0\sigma} \right\} \tag{2.5}$$

where $\nu_{0\sigma}$ and $\vec{k}_\sigma$ denote respectively the frequency and the propagation vector for the field. $\Phi(\vec{q}_p)$ and $\Phi(\vec{q}_\ell)$ are obtained by attaching the proper suffixes to $\vec{q}$ in the expansion (2.5). Following the method adopted in the previous chapter, it is convenient to introduce the action and angle variables to describe the motion in the plane perpendicular to the direction of the externally produced, time independent magnetic fifi

Let $$\vec{P} = \left(\vec{p} - \frac{e}{2c} \vec{q} \times \vec{H} \right) \tag{2.6}$$

The action and angle variables are defined as follows:

$$P_x = \sqrt{2m\omega J} \cos\omega \quad ; \quad P_y = \sqrt{2m\omega J} \sin\omega$$

$$P_z = p_z \tag{2.7}$$

6) W.Heitler, Quantum Theory of Radiation, Oxford (1954)
F.Henin, Physica 29 (1963) 1233.

The Liouville equation is written in the form

$$\frac{\partial \rho}{\partial t} + i\mathcal{L}_0 \rho = (i\delta L)\rho \tag{2.8}$$

where

$$i\mathcal{L}_0 = L_p + L_\ell + L_{0\sigma}$$

$$L_p = \omega_p \frac{\partial}{\partial \omega_p} + \frac{P_p}{m_p} \cdot \frac{\partial}{\partial \vec{q}_p} \quad ; \quad L_{0\sigma} = \nu_\sigma \frac{\partial}{\partial \omega_{0\sigma}} \tag{2.9}$$

L_ℓ has a structure similar to L_p . The interaction part consists of four parts

$$(i\delta L) = \mathcal{A}_p + \mathcal{B}_p + \mathcal{A}_\ell + \mathcal{B}_\ell \tag{2.10}$$

with

$$\mathcal{A}_p = \alpha_p \left(\frac{8c^2}{\Omega \nu_{0\sigma}}\right)^{1/2} \sum_{0\sigma} k_{\sigma_3} \cos(\vec{k}_\sigma \cdot \vec{q}_p) \sqrt{J_{0\sigma}} \, (\cos \omega_{0\sigma}) \frac{\partial}{\partial p_{p_3}} \tag{2.11}$$

$$\mathcal{B}_p = \alpha_p \left(\frac{8c^2}{\Omega \nu_{0\sigma}}\right)^{\frac{1}{2}} \sum_{0\sigma} \sin(\vec{k}_\sigma \cdot \vec{q}_p) \left[\frac{\partial}{\partial \omega_{0\sigma}} \left(\sqrt{J_{0\sigma}} \cos \omega_{0\sigma}\right) \frac{\partial}{\partial J_{0\sigma}} - \frac{\partial}{\partial J_{0\sigma}} \left(\sqrt{J_{0\sigma}} \cos \omega_{0\sigma}\right) \frac{\partial}{\partial \omega_{0\sigma}} \right] \tag{2.12}$$

$\mathcal{A}_\ell$ and $\mathcal{B}_\ell$ have exactly similar structures with the only replacement of the suffix p by ℓ. The (-0σ) part of $\mathcal{A}$ and $\mathcal{B}$ can be obtained by substituting Sin $(\vec{k}_\sigma \cdot \vec{q})$ by $-$ Cos $(\vec{k}_\sigma \cdot \vec{q})$ and $\mathrm{Cos}(\vec{k}_\sigma \cdot \vec{q})$ by $\mathrm{Sin}(\vec{k}_\sigma \cdot \vec{q})$

3. Evaluvation of the series:-

The lowest order contribution - of order e^2 - has the following form

$$W_{pp} = e^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \; \exp[-i\mathcal{L}(t-t_1)]\, \mathcal{A}_p^{0\sigma}$$

$$\exp[-i\mathcal{L}(t_1-t_2)]\, \mathcal{B}_p^{0\sigma} \exp[-i\mathcal{L}t_2]\, \rho(0)$$

$$= -\left(\frac{8\pi e^2 \alpha_p^2}{\Omega}\right) \sum_{0\sigma} \frac{k_{\sigma 3}}{\nu_\sigma} \int_0^t dt_1 \int_0^{t_1} dt_2 \; \exp[-L_p(t-t_1)]$$

$$\mathrm{Cos}(\vec{k}_\sigma \cdot \vec{q}_p) \frac{\partial}{\partial p_{p_3}} \exp[-L_p(t_1-t_2)]\, \mathrm{Sin}(\vec{k}_\sigma \cdot \vec{q}_p)$$

$$\exp[-L_p t_2]\, \rho_p(0) \oint dz \, \exp\left[-iz(t_1-t_2)\right]$$

$$\cdot \left[-\frac{\nu_\sigma}{z^2-\nu_{0\sigma}^2}\right] \qquad (3.1)$$

The integrations over $J_{0\sigma}$ and $\omega_{0\sigma}$ have been performed in getting (3.1). The term $\mathrm{Sin}[\nu_{0\sigma}(t_1-t_2)]$ which arises because of this integration is represented as a contour integral over its laplace transform $\left[-\nu_{0\sigma}/(z^2-\nu_{0\sigma}^2)\right]$

The next higher order term $W_{p\ell\ell p}$ is given by the following equation

$$W_{p\ell\ell p} = -\left(\frac{8\pi e^2 \alpha_p^2}{\Omega}\right) \sum_{o\sigma} \sum_{o\sigma'} \left(k_{\sigma_3}/\nu_\sigma\right) \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4$$

$$[\sin \nu_\sigma (t_1 - t_2)] \; [\sin \nu_{\sigma'} (t_3 - t_4)] \; \exp[-L_p (t - t_1)]$$

$$\cos(\vec{k}_\sigma \cdot \vec{q}_p) \frac{\partial}{\partial p_{p_3}} \exp[-L_p (t_1 - t_4)]$$

$$\left\{ \sum_\ell \left[-\frac{e^2 \alpha_\ell^2}{c^2} \frac{8\pi c^2}{\Omega} \left(k_{\sigma'_3}/\nu_{o\sigma'}\right)\right] \int_0^\infty d\tau_\ell \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell_3} \right.$$

$$\int d\vec{q}_\ell \; \exp[-L_\ell (t - t_2)] \; \sin(\vec{k}_\sigma \cdot \vec{q}_\ell) \; \exp[-L_\ell (t_2 - t_3)]$$

$$\left. \cos(\vec{k}_{\sigma'} \cdot \vec{q}_\ell) \frac{\partial}{\partial p_{\ell_3}} \exp[-L_\ell t_3] \; \rho_\ell(0) \right\}$$

$$\sin(\vec{k}_\sigma \cdot \vec{q}_p) \; \exp(-L_p t_4) \; \rho_p(0)$$

(3.2)

Following the procedure employed in the previous chapter $W_{p\ell\ell p}$ **may be written as**

$$W_{p\ell\ell p} = -\left(\frac{8\pi \alpha_p^2 e^2}{\Omega}\right) \sum_{o\sigma} \sum_{o\sigma'} \int_0^t dt_1 \int_0^{t_1} dt_2 \quad E(t_1, t_2)$$

$$\exp[-L_p (t - t_1)] \; \cos(\vec{k}_\sigma \cdot \vec{q}_p) \frac{\partial}{\partial p_{p_3}}$$

$$\exp[-L_p (t_1 - t_2)] \; \sin(\vec{k}_{\sigma'} \cdot \vec{q}_p)$$

$$\exp[-L_p t_2] \; \rho_p(0)$$

(3.3)

Where

$$E = \sum_n \frac{1}{2^n n! \sqrt{\pi}} \left[- \frac{e^2 \alpha_\ell^2}{c^2} \frac{8\pi c^2}{\Omega \nu_{0\sigma'}} k_{\sigma' 3} \right]$$

$$\int_{t_2}^{t_1} dT \int_{t_2}^{T} dT' \; \sin[\nu_{0\sigma}(t_1 - T)] \sin[\nu_{0\sigma'}(T' - t_2)]$$

$$\int_0^\infty d J_\ell \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} d p_{\ell_3} \int d\vec{q}_\ell \; \sin(\vec{k}_\sigma . \vec{q}_\ell) \cos(\vec{k}_{\sigma'} . \vec{\tilde{q}}_\ell)$$

$$\exp(-\bar{W}^2) \; H_n^2(\bar{W}) \frac{\partial F_n}{\partial p_{\ell_3}} \; \delta[J_\ell - (n + \tfrac{1}{2})\hbar] \tag{3.4}$$

The Baker-Hausdorff decomposition formula has been made use of to decouple the propagator exp $[-I_\ell t]$ (See Appendix 1) Further in (3.4)

$$\vec{\tilde{q}}_\ell = \vec{q}_\ell + \left(\frac{2J_\ell}{m_\ell \varpi_\ell} \right)^{\frac{1}{2}} \left\{ \hat{a} \left[\sin(\omega_\ell - \varpi_\ell \tau) - \sin \omega_\ell \right] \right.$$

$$\left. + \hat{b} \left[\cos \omega_\ell - \cos(\omega_\ell - \varpi_\ell \tau) \right] - \hat{c} \, p_{\ell_3} \tau / m_\ell \right\} \tag{3.5}$$

with $\tau = T - T'$

$$\bar{W} = \sqrt{\frac{m_\ell \varpi_\ell}{4\hbar}} \left\{ q_{\ell_2} + \sqrt{\frac{2J_\ell}{m_\ell \varpi_\ell}} \left[\cos \omega_\ell - 3 \cos(\omega_\ell - \varpi_\ell t_2) \right] \right\} \tag{3.6}$$

$$F_n = \left\{ 1 + \exp\left[\beta \left(E_n + p_{\ell_3}^2 / 2m_\ell \right) \right] \right\}^{-1} \tag{3.7}$$

Adding (-0σ) part of the interaction, (3.4) may be written as

$$E = \sum_n \frac{1}{2^n n!\sqrt{\pi}} \left[-\frac{e^2\alpha_\ell^2}{c^2} \frac{8\pi c^2}{\Omega \nu_{0\sigma'}} k_{\sigma'3} \right]$$

$$\int_{t_2}^{t_1} dT \int_{t_2}^{T} dT' \sin[\nu_{0\sigma}(t_1 - T)] \sin[\nu_{0\sigma'}(T' - t_2)]$$

$$\int_0^\infty d\mathcal{J}_\ell \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell_3} \int d\vec{q}_\ell \sin\left[-(k_{\sigma'} - k_\sigma)\cdot\left(\vec{q}_\ell - \sqrt{\frac{2\mathcal{J}_\ell}{m_\ell \omega_\ell}}\phi + \gamma\right)\right]$$

$$\exp[-\overline{W}^2] \, H_n^2(\overline{W}) \frac{\partial F_n}{\partial p_{\ell_3}} \tag{3.8}$$

where

$$\phi = \sqrt{\frac{2}{m_\ell \omega_\ell}} \left[k_{\sigma'_1} \left(\sin(\omega_\ell - \omega_\ell \tau) - \sin\omega_\ell \right) + k_{\sigma'_2} \left(\cos\omega_\ell - \cos(\omega_\ell - \omega_\ell \tau) \right) \right] \tag{3.9}$$

and $\gamma = k_{\sigma'_3} p_{\ell_3} \tau / m_\ell$ (3.10)

The integrations over q_{ℓ_1} and q_{ℓ_3} yield delta functions $\delta(k_{\sigma'_1} - k_{\sigma_1})$ and $\delta(k_{\sigma'_3} - k_{\sigma_3})$. The integration over q_{ℓ_2} is performed by changing the variable to $\overline{W}$ which is given by (3.7).

Thus $$E(t_1,t_2) = -\frac{e_\ell^2}{\nu_{\sigma'}} \frac{8\pi}{\Omega} k_{\sigma_3} \int_{t_2}^{t_1} dT \int_{t_2}^{T} dT'$$

$$\sin[\nu_\sigma(t_1 - T)] \sin[\nu_{\sigma'}(T' - t_2)] \; S_0'$$

$$\exp\left[-\frac{\hbar K^2}{m_\ell \omega_\ell}\right] L_n\left(2\frac{\hbar K^2}{m_\ell \omega_\ell}\right) J_0\left[R\sqrt{(n+\tfrac{1}{2})\hbar}\right] \tag{3.11}$$

where

$$S_0' = \int_{-\infty}^{\infty} dp_{\ell_3} (\sin\gamma) \frac{\partial F_n}{\partial p_{\ell_3}} \tag{3.12}$$

Further

$$K = k_{\sigma_2'} - k_{\sigma_2} \tag{3.13}$$

It is assumed that K is independent of the mode of the electromagnetic field, as the external magnetic field is sufficiently weak. It may also be noted that J_ℓ and ω_ℓ integrations have been performed by using the relations.

$$\sin[R\sqrt{J_\ell}\,\sin(\omega_\ell + \theta)]$$

$$= 2\sum_{\gamma=0}^{\infty} J_{2\gamma+1}(R\sqrt{J_\ell})\,\sin[(2\gamma+1)(\omega_\ell+\theta)] \tag{3.14a}$$

$$\cos[R\sqrt{J_\ell}\,\sin(\omega_\ell+\theta)]$$

$$= \cos(R\sqrt{J_\ell}) + 2\sum_{r=0}^{\infty} J_{2r}(R\sqrt{J_\ell})\,\sin[2r(\omega_\ell+\theta)] \tag{3.14b}$$

R and θ are defined by the relations (3.15)

$$R\cos\theta = \sqrt{\frac{2}{m\omega}}\,[k_{\sigma'_1}(\cos\omega_\ell\tau - 1) - k_{\sigma'_2}\sin\omega_\ell\tau + 3K\sin\omega_\ell T] \tag{3.15a}$$

$$R\sin\theta = \sqrt{\frac{2}{m\omega}}\,[-k_{\sigma'_1}\sin\omega_\ell\tau + k_{\sigma'_2}(1-\cos\omega_\ell\tau) - K(1-3\cos\omega_\ell T)] \tag{3.15b}$$

For sufficiently weak fields $(\omega\tau << 1)$, R^2 turns out to be independent of time

$$R^2 \sim 8K^2/m_\ell\omega_\ell \tag{3.16}$$

The sum over Landau levels may now be performed. For this, it is to be noted that J_0 can be expressed in terms of Laguerre polynomials[7)]

$$J_0(\sqrt{(4n+2)x}) = \exp(-x/2)\,L_n(x) \tag{3.17}$$

7) A.Erdelyi et al (Bateman Project) Higher Transcendental functions (Mac Graw Hill) (1953) Vol.2 Page 199.

Expressing the Fermi function as a power series and making use of the relationship[8)]

$$\sum_{n=0}^{\infty} L_n(x)\, L_n(y)\, z^n = \frac{1}{1-z} \exp\left(-z\,\frac{x+y}{1-z}\right) \cdot I_0\left(2\,\frac{\sqrt{xyz}}{1-z}\right) \qquad (|z|<1) \tag{3.18}$$

where I_0 is the modified Bessel function of order zero. Finally,

$$E = \left(\frac{2m}{\nu_{\sigma\sigma'}}\, k_{\sigma'3}\, \omega_{P\ell}^2\right) \int_{t_2}^{t_1} dT \int_{t_2}^{T} dT' \; \sin[\nu_{\sigma\sigma}(t_1-T)]\, \sin[\nu_{\sigma\sigma'}(T'-t_2)]$$
$$\exp\left[-\frac{\hbar K^2}{m_\ell \omega_\ell} - \frac{R^2}{8}\right] \sum_{r=0}^{\infty} (-1)^r\, \tilde{\mathfrak{G}}(r)$$
$$\frac{\exp\left[-\frac{1}{2}(r+1)\beta\hbar\omega_\ell\right]}{1-\exp\left[-(r+1)\beta\hbar\omega_\ell\right]}$$
$$\exp\left\{-e^{-(r+1)\beta\hbar\omega_\ell/2}\, \frac{2\frac{\hbar K^2}{m_\ell\omega_\ell} + \frac{R^2}{4}}{1-\exp\left[-(r+1)\beta\hbar\omega_\ell\right]}\right\}$$
$$I_0\left[2\,\frac{\sqrt{\frac{2\hbar K^2}{m_\ell\omega_\ell}\,\frac{R^2}{4}}\,\exp\left[-(r+1)\beta\hbar\omega_\ell/2\right]}{1-\exp\left[-(r+1)\beta\hbar\omega_\ell\right]}\right] \tag{3.19}$$

8) I.S.Gradshteyn and I.M.Ryzhik, Tables of integrals, series and products, Academic Press (1965).

where

$$\tilde{\mathcal{T}}(r) = \int_{-\infty}^{\infty} dp_{\ell_3} \, (\sin\gamma) \frac{d}{dp_{\ell_3}} \left\{ \exp\left[-\beta(r+1)\left(\frac{p_{\ell_3}^2}{2m_\ell} - \mu \right) \right] \right\}$$

(3.20)

p_{ℓ_3} integration is trivially done by replacing the integral by the value of the integrand at the point p_0 where the momenta in the z -direction are assumed to be peaked. Plasma frequency $\omega_{p\ell}$ has found a place in (3.19) because of the summation over the medium particles and the appearance of the volume factor Ω in the denominator. This yields the concentration d in the thermodynamic limit. By resorting to the technique of Laplace transform the fourth order term may be written as

$$W_{p\ell\ell p} = \left(\frac{e}{c}\right)^2 \left(\pi \alpha_p^2 \frac{8c^2}{\Omega \nu_{0\sigma'}} k_{\sigma_3} \right) \int_0^t dt_1 \int_0^{t_1} dt_2$$

$$\exp[-L_p(t-t_1)] \cos(\vec{k}_\sigma \cdot \vec{q}_p) \frac{\partial}{\partial p_{p_3}} \exp[-L_p(t_1-t_2)]$$

$$\sin[\vec{k}_\sigma \cdot \vec{q}_p + K q_{p_2}] \exp[-L_p t_2] \, \rho_p(0)$$

$$\left\{ \oint dz \, \exp[-iz(t_1-t_2)] \, \mathcal{E}(z) \right\}$$

(3.21)

with $\mathcal{E}(z)$ given by

$$\mathcal{E}(z) = \frac{\nu_{0\sigma'}}{z^2 - \nu_{0\sigma'}^2} \frac{\chi}{z^2 - \bar{\gamma}^2}$$

(3.22)

where

$$\chi = -\left(k_{\sigma 3}/\nu_{0\sigma'}\right)\,\omega_{P\ell}^{2}\,\exp\left[-2\,\frac{\hbar K^{2}}{m_{\ell}\varpi_{\ell}}\right]\left(p_{0}^{2}/2m_{\ell}\right)$$

$$\sum_{r=0}^{\infty}(-1)^{r}\,\beta(r+1)\,\exp\left[-\beta(r+1)\left(\left(p_{0}^{2}/2m_{\ell}\right)-\mu\right)\right]$$

$$\frac{1}{\sinh Y}\,\exp\left[-2\,\frac{\hbar K^{2}}{m_{\ell}\varpi_{\ell}}\,\frac{\exp Y}{\sinh Y}\right]$$

$$I_{0}\left[2\,\frac{\hbar K^{2}}{m_{\ell}\varpi_{\ell}}\,\frac{1}{\sinh Y}\right] \tag{3.23}$$

with

$$Y = \beta(r+1)\hbar\,\varpi_{\ell}/2 \tag{3.24}$$

and

$$\bar{\gamma} = k_{\sigma 3}\,p_{\ell 3}/m_{\ell} \tag{3.25}$$

with $\mathcal{E}(z)$ given by (3.22) the series can be summed exactly.

$$\rho(\vec{q}_{p},\vec{p}_{p};t) = \exp[-L_{p}t]\,\Big\{\rho(0) + \sum\left(\frac{8\pi\alpha_{p}^{2}e^{2}}{\Omega}\right)\left(k_{\sigma' 3}/\nu_{0\sigma'}\right)$$

$$\int_{0}^{t}dt_{1}\int_{0}^{t_{1}}dt_{2}\,\exp(L_{p}t_{1})\,\cos(\vec{k}_{\sigma}\cdot\vec{q}_{p})\,\frac{\partial}{\partial p_{p_{3}}}$$

$$\exp[-L_{p}(t_{1}-t_{2})]\oint dz\,\exp[-iz(t_{1}-t_{2})]$$

$$\left(-\frac{\nu_{0\sigma}}{z^{2}-\nu_{0\sigma}^{2}}\right)\left[1-\frac{\chi\,\nu_{0\sigma'}}{(z^{2}-\nu_{0\sigma'}^{2})(z^{2}-\bar{\gamma}^{2})}\right]^{-1}$$

$$\sin\left(\vec{k}_{\sigma}\cdot\vec{q}_{p} + K q_{p_{2}}\right)\exp[-L_{p}t_{2}]\,\rho_{p}(0)\Big\} \tag{3.26}$$

The inverse Laplace transform occurring in (3.26) has to be evaluvated now. After retionalising the integrand with the assumption that $\nu = \nu_{0\sigma'} - \nu_{0\sigma}$ is much smaller than $\nu_{0\sigma}$ the inverse transform may be written as

$$\oint dz \, \exp[-iz(t_1-t_2)] \left\{ \frac{-\nu_{0\sigma}(z^2-\bar{\gamma}^2)}{(z^2-\nu_{0\sigma'}^2)(z^2-\bar{\gamma}^2)-\nu_{0\sigma'}\chi} \right\} \tag{3.27}$$

The approximation procedure employed here is valid only for weak fields. (3.27) may now be rewritten as

$$-\nu_{0\sigma} \oint dz \, \exp[-iz(t_1-t_2)] \left\{ \frac{1}{2}\left(1-\frac{\bar{\gamma}^2-\nu_{0\sigma'}^2}{2\tilde{\Omega}}\right) \frac{1}{z^2-\left(\frac{\nu_{0\sigma'}^2+\bar{\gamma}^2}{2}+\tilde{\Omega}\right)} + \frac{1}{2}\left(1+\frac{\bar{\gamma}^2-\nu_{0\sigma'}^2}{2\tilde{\Omega}}\right) \frac{1}{z^2-\left(\frac{\nu_{0\sigma'}^2+\bar{\gamma}^2}{2}-\tilde{\Omega}\right)} \right\} \tag{3.28}$$

with $\tilde{\Omega}^2 = \left(\frac{\nu_{0\sigma'}^2+\bar{\gamma}^2}{2}\right)^2 + \nu_{0\sigma'}\chi$

After this, it is a straightforward procedure to take the inverse transform which yields the result

$$\frac{1}{2}\nu_{0\sigma}\left[\frac{\nu_{0\sigma'}^2+\bar{\gamma}^2}{2}+\tilde{\Omega}\right]^{-1/2}\left[1-\frac{\bar{\gamma}^2-\nu_{0\sigma'}^2}{2\tilde{\Omega}}\right] \sin\left[\left(\frac{\nu_{0\sigma'}^2+\bar{\gamma}^2}{2}+\tilde{\Omega}\right)^{\frac{1}{2}}(t_1-t_2)\right]$$
$$+\frac{1}{2}\nu_{0\sigma}\left[\frac{\nu_{0\sigma'}^2+\bar{\gamma}^2}{2}-\tilde{\Omega}\right]^{-\frac{1}{2}} \cdot \left[1-\frac{\bar{\gamma}^2-\nu_{0\sigma'}^2}{2\tilde{\Omega}}\right] \sin\left[\left(\frac{\nu_{0\sigma'}^2+\bar{\gamma}^2}{2}-\tilde{\Omega}\right)^{\frac{1}{2}}(t_1-t_2)\right] \tag{3.29}$$

4. Conclusions:

An important outcome of the calculation presented in the previous section is the Laplace transformed dielectric function $\varepsilon(z)$. The inverse transform has been taken under the assumption that the externally applied magnetic field is a weak one and that $\nu_{0\sigma'} - \nu_{0\sigma} << \nu_{0\sigma}$. This leads to the one particle reduced distribution function given by equations (3.26) and (3.29). It is to be noted that equation (3.29) involves terms of the type

$$\sin\left[\left(\frac{\nu_{0\sigma'}^2 + \bar{\gamma}^2}{2} \pm \tilde{\Omega}\right)^{1/2} (t_1 - t_2)\right]$$

Thus if the one particle reduced distribution function has to be real then the term with in the square root (in the argument of the sin function) has to be positive or equal to zero. This leads to the condition

$$\frac{\nu_{0\sigma'}^2 + \bar{\gamma}^2}{2} - \tilde{\Omega} \geq 0 \qquad (4.1)$$

This condition is called the reality condition. The instabilities that owe their origins to collective effects may be discussed on the basis of this inequality. (4.1) may be rewritten as

$$\frac{1}{\lambda_D^2}\frac{1}{\nu_{00'}p_0}\exp\left(-2\frac{\hbar K^2}{m_\ell\omega_\ell}\right)\sum_{r=0}^{\infty}(-1)^r(r+1)\frac{1}{\sinh Y}$$

$$\exp\left\{-\beta(r+1)\left[\frac{p_0^2}{2m_\ell}-\mu\right]-2\frac{\hbar K^2}{m_\ell\omega_\ell}\frac{\exp(-Y)}{\sinh Y}\right\}$$

$$I_0\left\{2\frac{\hbar K^2}{m_\ell\omega_\ell}\frac{1}{\sinh Y}\right\}\leq 1 \tag{4.2}$$

This condition represents the criterion for instability of the electron plasma with respect to perturbations characterised by frequency ν . In the above the Debye length is given by

$$\lambda_D=\left(\frac{1}{4\pi\beta d e^2}\right)^{\frac{1}{2}} \tag{4.3}$$

As the magnetic field is assumed to be a weak one, the following approximations may be made

$$\sinh Y = Y \quad ; \quad \frac{\exp(-Y)}{\sinh Y}\sim\frac{1}{Y} \tag{4.4}$$

On the basis of this approximation, (4.2) may be written as

$$\frac{1}{\lambda_D^2}\frac{1}{\nu_{00'}p_0}\exp\left(-2\frac{\hbar K^2}{m_\ell\omega_\ell}\right)\sum_{r=0}^{\infty}(-1)^r(r+1)\;{}^1/_Y$$

$$\exp\left\{-\beta(r+1)\left[\left(\frac{p_0^2}{2m_\ell}\right)-\mu\right]-\frac{4}{(r+1)}\frac{K^2}{\beta m_\ell\omega_\ell^2}\right\}$$

$$I_0\left[\frac{4}{(r+1)}\frac{K^2}{\beta m_\ell\omega_\ell^2}\right]\leq 1 \tag{4.5}$$

It is easily seen that if K is zero, equation (4.2) is trivially satisfied so that no instability is possible. Further if k_{σ_3} is zero then from the calculation presented in the previous section it is seen that the entire series to all orders of e^2 reduces to zero so that there is no instability due to collective effects. Thus instabilities can occur only if both K and k_{σ_3} are different from zero.

In the asymptotic time limit $(z \to 0)$ the dispersion relation can be written down as

$$1 - \varepsilon(0) = 0 \tag{4.6}$$

or equivalently

$$1 + \frac{1}{\lambda_D^2} \frac{2}{\beta \omega_\ell^2 \hbar \nu_{0\sigma'}^2} \exp\left(-2 \frac{\hbar K^2}{m_\ell \omega_\ell}\right) \sum_{r=0}^{\infty} (-1)^r$$

$$\exp\left[-\beta(r+1)\left(\left(p_0^2/2m_\ell\right) - \mu\right)\right] \exp\left[-\frac{4}{r+1} \frac{K^2}{\beta m_\ell \omega_\ell^2}\right]$$

$$I_0\left[\frac{4}{(r+1)} \frac{K^2}{m_\ell \omega_\ell^2 \beta}\right] = 0 \tag{4.7}$$

Introducing the dimensionless variable

$$S = \frac{K^2}{\beta m_\ell \omega_\ell^2} \frac{4}{(r+1)} \tag{4.8}$$

and using the asymptotic expansion

$$\lim_{S \to \infty} \exp[-S]\, I_0(S) = (4\pi S)^{-1/2} \tag{4.9}$$

the dispersion relation can be written down in a solvable form. Though the results derived here are to some extent analogous to the already existing ones, the differences are essentially due to the special choice of the initial distribution and the summation of an infinite series to all orders in $e^2 d$. Further the discussion is based on a quasi classical frame work.

To sum up, the statistical mechanics of an electron plasma has been developed in a systematic fashion. The important developement is the mathematical formulation of the motion of the system in the phase space when the interactions are through a scalar potential including a quantisation perpendicular to the magnetic field. The Laplace transformed dielectric function for this system is calculated and the stability criterion is obtained as a reality condition. The case of asymptotic time limit is discussed and the relevant dielectric function is obtained.

CHAPTER . VI

LONGITUDINAL OSCILLATIONS IN A MAGNETO ACTIVE PLASMA+

1. Introduction:

In Chapters IV and V the statistical mechanics of magneto active electron plasma has been studied when the interactions are through a transverse (purely magnetic in character) and a scalar field, respectively. This chapter deals with the inclusion of longitudinal interactions and the consequent modifications in the dynamics of correlations. This attempt, of course, is made under the assumption that the longitudinal modes can be decoupled from the transverse ones. Such an assumption is not very unfamiliar in plasma physics[1]. Baldwin, Bernstein and Weenink[2] have given an excellent account of the recent developments in this direction which mostly depend upon the linearisation of the Vlasov equation together with Maxwell's equations. As the present attempt depends upon the choice of an infinite series of terms of the order $(e^2 d)$ and in its exact sum, it includes the collective effects in a consistent fashion and it is therefore worthwhile to study the longitudinal oscillations through this approach.

+ R.Sridhar, MATSCIENCE Preprint (1971)
1)V.D.Shafranov, Review of Plasma Physics, Consultants Bureau (N.Y) (1967) Vol.3 Page 1.
2)D.E.Baldwin, I.B.Bernstein and M.P.H.Weenink, Advances in Plasma Physics, Interscience Publishers (New York) (1969) Vol.3, Page 1.

2. Evaluvation of the dielectric function:

The formulation of the problem is analogous to the one used in chapters IV and V. The hamiltonian is the sum of the hamiltonians of the test particle, $\mathcal{H}_p$, the medium $\mathcal{H}_M$ and the longitudinal interaction, $\mathcal{H}_F$: $\mathcal{H} = \mathcal{H}_p + \mathcal{H}_M + \mathcal{H}_F$, where

$$\mathcal{H}_p = \frac{1}{2m_p}\left\{ \vec{p}_p - \frac{e}{2c}\vec{q}_p \times \vec{H} - \frac{e}{c}\vec{A}(\vec{q}_p)\right\}^2, \quad \mathcal{H}_M = \sum_{\ell} \mathcal{H}_\ell \tag{2.1}$$

$$\mathcal{H}_F = \sum_{\sigma} \nu_\sigma (J_\sigma + J_{-\sigma}) \tag{2.2}$$

$\vec{A}$ denotes the longitudinal electromagnetic field which has the following expansion

$$\vec{A} = \left(\frac{8c^2}{\Omega}\right)^{1/2} \sum_{\sigma,\vec{k}_\sigma} \frac{\hat{e}_\sigma}{\sqrt{\nu_\sigma}} \left\{ \sqrt{J_\sigma}\, \cos\omega_\sigma\, \cos(\vec{k}_\sigma \cdot \vec{q}) + \sqrt{J_{-\sigma}}\, \cos\omega_{-\sigma}\, \sin(\vec{k}_\sigma \cdot \vec{q}) \right\} \tag{2.3}$$

with the condition

$$(\hat{e}_\sigma \times \vec{k}_\sigma) = 0 \tag{2.4}$$

The effective momentum $\vec{P}$ is defined as

$$\vec{P} = \left[\vec{p} - \frac{e}{2c}\vec{q} \times \vec{H} \right] \tag{2.5}$$

The x and y components of $\vec{P}$ are expressed in terms of action and angle variables as is done in the earlier

chapters. (See equations (III 2.9) to (III 2.11). The Roman numeral III denotes chapter III). The Liouville equation is written in the form given by equation (III 2.14) and the free propagators L_p, L_M and L_F are analogous to equations (III 2.16) while the interaction part $(i\delta L)$ has a form analogous to (III 2.17), the initial condition is also analogous to (III 3.16) and (III 3.17). (In the equations of Chapter III referred to above, λ, the index for transverse polarisation, has to be replaced by σ, the index for longitudinal polarisation.

The lowest order diagram W_{pp} has the following form

$$W_{pp} = \left(\frac{e}{c}\right)^2 \left(\frac{8\pi\alpha_p^2 c^2}{\Omega}\right) \int_0^t dt_1 \int_0^{t_1} dt_2 \; \exp[-L_p(t-t_1)] \; \theta_p^\sigma$$

$$\exp[-L_p(t_1-t_2)] \, (\hat{e}_\sigma \cdot \vec{P}_p) \cos(\vec{k}_\sigma \cdot \vec{q}_p) \exp[-L_p t_2] \, \rho_{p(0)}$$

$$\left\{ \oint dz \; \exp[-iz\tau] \; \frac{1}{z^2 - \nu_\sigma^2} \right\}$$

$$(\tau = t_1 - t_2) \tag{2.6}$$

with

$$\theta_p^\sigma = \cos(\vec{k}_\sigma \cdot \vec{q}_p) \left[\frac{\partial}{\partial \omega_p} (\hat{e}_\sigma \cdot \vec{P}_p) \frac{\partial}{\partial \vec{p}_p} - \frac{\partial}{\partial \vec{p}_p} (\hat{e}_\sigma \cdot \vec{P}_p) \frac{\partial}{\partial \omega_p} - \hat{e}_\sigma \cdot \frac{\partial}{\partial \vec{q}_p} \right]$$

$$- k_{\sigma_3} \, (\hat{e}_\sigma \cdot \vec{P}_p) \sin(\vec{k}_\sigma \cdot \vec{q}_p) \frac{\partial}{\partial p_{p_3}} \tag{2.7}$$

The fourth order graph is given by

$$W_{p\ell\ell p} = \frac{8\pi e^2 \alpha_p^2}{m_p^2} \frac{1}{\Omega \nu_\sigma} \int_0^t dt_1 \int_0^{t_1} dt_2 \; \exp[-L_p(t-t_1)]\, \theta_p^\sigma$$

$$\exp[-L_p(t_1-t_2)] \; \bar{E}(t_1, t_2)\, (\hat{e}_{\sigma'} \cdot \vec{P}_p)$$

$$\cos[\vec{k}_{\sigma'} \cdot \vec{q}_p] \; \exp[-L_p t_2] \; \rho_p(0)$$

(2.8)

where

$$\bar{E}(t_1, t_2) = \int_{t_2}^{t_1} dT \int_{t_2}^{T} dT' \; \sin[\nu_\sigma (t_1 - T)] \; \sin[\nu_{\sigma'}(T' - t_2)]$$

$$E(T, T')$$

(2.9)

and

$$E(T, T') = \frac{8\pi}{\Omega \nu_{\sigma'}} \frac{e_\ell^2}{m_\ell^2} \int_0^\infty d\bar{\nu}_\ell \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dp_{\ell_3} \int d\vec{q}_\ell \, (\hat{e}_\sigma \cdot \vec{P}_\ell)$$

$$\cos[\vec{k}_\sigma \cdot \vec{q}_\ell] \; \exp[-L_\ell \tau] \; \theta_\ell^{\sigma'} \exp[-L_\ell T']$$

$$\frac{1}{2^n n! \sqrt{\pi}} \exp[-W^2] \, H_n^2(W) \; F_n$$

(2.10)

$$F_n = \{1 + \exp[\beta(E_n - \mu)]\}^{-1} \qquad \textbf{(2.11)}$$

$$E_n = (n + \tfrac{1}{2})\hbar\omega_\ell + p_{\ell_3}^2 / 2m_\ell \qquad \textbf{(2.12)}$$

$$W = \sqrt{\frac{m_\ell \omega_\ell}{4\hbar}} \left[q_{\ell_2} - \sqrt{\frac{8\bar{\nu}_\ell}{m_\ell \omega_\ell}} \cos\omega_\ell \right] \qquad \textbf{(2.13)}$$

The evaluation of this term is exactly similar to that of the fourth order term for the case of transverse electromagnetic interactions. After some manupulations $E(T,T')$ can be shown to be of the following form[3)]

$$E(T,T') = \frac{8\pi e_\ell}{m_\ell^2} \frac{1}{\Omega \nu_{\sigma'}} \sqrt{\frac{4\hbar}{m_\ell \omega_\ell}} \; L_n\left(2\frac{\hbar K^2}{m_\ell \omega_\ell}\right) \exp\left(-\frac{\hbar K^2}{m_\ell \omega_\ell}\right)$$

$$\delta(k_{\sigma'_1} - k_{\sigma_1})\, \delta(k_{\sigma'_3} - k_{\sigma_3}) \left[|\vec{k}_\sigma|\,|k_{\sigma'}|\right]^{-1}$$

$$\int_0^\infty d\mathcal{J}_\ell\, \delta[\mathcal{J}_\ell - (n+\tfrac{1}{2})\hbar] \int_0^{2\pi} d\omega_\ell \int_{-\infty}^{\infty} dP_{\ell_3} (\vec{k}_\sigma \cdot \vec{P}_\ell) \sin[\sqrt{\mathcal{J}_\ell}(\phi+\phi') - \gamma]$$

$$\Big\{ 3K \left[(\hat{a}\cdot\vec{k}_{\sigma'}) \sin\omega_\ell T' - (\hat{b}\cdot\vec{k}_{\sigma'}) \cos\omega_\ell T' \right] F_n$$

$$- k_{\sigma'_3} (\vec{k}_{\sigma'} \cdot \vec{P}_\ell) \frac{\partial F_n}{\partial P_{\ell_3}} \Big\} \qquad (2.14)$$

R and ω_ℓ are defined by equations identical to (IV 3.35) and (IV 3.36). Performing the $\mathcal{J}_\ell$ and ω_ℓ integrations following the procedure employed in the earlier chapters it is readily seen that

$$E(T,T') = \frac{2\omega_{p\ell}^2}{m_\ell \nu_{\sigma'}} \sqrt{\frac{4\hbar}{m_\ell \omega_\ell}} \; L_n\left(2\frac{\hbar K^2}{m_\ell \omega_\ell}\right) \exp\left(-\frac{\hbar K^2}{m_\ell \omega_\ell}\right) \left[|\vec{k}_\sigma|\,|\vec{k}_{\sigma'}|\right]^{-1}$$

$$\Big\{ 3K \left[(\hat{a}\cdot\vec{k}_{\sigma'}) \sin\omega_\ell T' - (\hat{b}\cdot\vec{k}_{\sigma'}) \cos\omega_\ell T' \right]$$

$$\Big\{ 2\pi\sqrt{2 m_\ell \omega_\ell (n+\tfrac{1}{2})\hbar}\; J_1\left(R\sqrt{(n+\tfrac{1}{2})\hbar}\right) C_0$$

$$\left[(\hat{a}\cdot\vec{k}_\sigma) \sin\theta + (\hat{b}\cdot\vec{k}_\sigma) \cos\theta \right] \qquad \text{(Contd...)}$$

---

3) It may be noted that the polarization vectors $\hat{e}_\sigma$ and $\hat{e}_{\sigma'}$ have been eliminated in (2.14). This procedure is not so complicated as the one needed for the case of transverse interactions. $\hat{e}_{\sigma'} = \vec{k}_{\sigma'}/|\vec{k}_{\sigma'}|$; $\hat{e}_\sigma = k_\sigma/|\vec{k}_\sigma|$

$$-2\pi\,(\hat{c}\cdot\vec{k}_\sigma)\;S_1\;J_0\left[R\sqrt{(n+\tfrac{1}{2})\hbar}\right]\Big\}$$

$$-2\pi k_{\sigma_3}\left[2m_\ell\omega_\ell(n+\tfrac{1}{2})\hbar\right]S_0'$$

$$\Big\{J_0\left[R\sqrt{(n+\tfrac{1}{2})\hbar}\right]\Big[\Big((\hat{a}\cdot\vec{k}_\sigma)(\hat{a}\cdot\vec{k}_{\sigma'})+(\hat{b}\cdot\vec{k}_\sigma)(\hat{b}\cdot\vec{k}_{\sigma'})\Big)\cos\omega_\ell\tau$$
$$+\Big((\hat{b}\cdot\vec{k}_\sigma)(\hat{a}\cdot\vec{k}_{\sigma'})-(\hat{a}\cdot\vec{k}_\sigma)(\hat{b}\cdot\vec{k}_{\sigma'})\Big)\sin\omega_\ell\tau\Big]$$
$$+J_2\left[R\sqrt{(n+\tfrac{1}{2})\hbar}\right]\Big[\Big((\hat{a}\cdot\vec{k}_\sigma)(\hat{a}\cdot\vec{k}_{\sigma'})-(\hat{b}\cdot\vec{k}_\sigma)(\hat{b}\cdot\vec{k}_{\sigma'})\Big)$$
$$\cdot\cos(\omega_\ell\tau+2\theta)$$
$$-\Big((\hat{b}\cdot\vec{k}_\sigma(\hat{a}\cdot\vec{k}_{\sigma'})+(\hat{a}\cdot\vec{k}_\sigma)(\hat{b}\cdot\vec{k}_{\sigma'})\Big)\sin(\omega_\ell\tau+2\theta)\Big]\Big\}$$

$$+2\pi k_{\sigma_3}\sqrt{2m_\ell\omega_\ell(n+\tfrac{1}{2})\hbar}\;C_1'\;J_1\left[R\sqrt{(n+\tfrac{1}{2})\hbar}\right]$$

$$\Big\{(\hat{c}\cdot\vec{k}_\sigma)(\hat{a}\cdot\vec{k}_{\sigma'})\sin(\omega_\ell\tau+\theta)$$
$$+(\hat{c}\cdot\vec{k}_\sigma)(\hat{b}\cdot\vec{k}_{\sigma'})\cos(\omega_\ell\tau+\theta)$$
$$+(\hat{c}\cdot\vec{k}_{\sigma'})(\hat{a}\cdot\vec{k}_\sigma)\sin\theta+(\hat{c}\cdot\vec{k}_{\sigma'})(\hat{b}\cdot\vec{k}_\sigma)\cos\theta\Big\}$$

$$-2\pi k_{\sigma_3}(\hat{c}\cdot\vec{k}_\sigma)(\hat{c}\cdot\vec{k}_{\sigma'})\;S_2'\;J_0\left[R\sqrt{(n+\tfrac{1}{2})\hbar}\right]\Big\}\Big\}$$

(2.15)

Equations (IV 3.38) and (IV 3.39) may be referred to for the definitions of C_{2n}, S_{2n+1}, C_{2n+1}', S_{2n}' **and** $\tilde{F}_n(P_{\ell_3})$ **The summation over Landau levels is performed in a form analo-**

gous to the one employed in chapter IV. Here also the assumption of weak external magnetic field and low temperature is invoked to perform this summation. Because of this assumption products of the quantities K and ω_ℓ as well as their higher powers are neglected. In this manner the following term alone is assumed to be the dominant one.

$$\sigma = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \int_{-\infty}^{\infty} dp_{\ell 3} \, (p_{\ell 3})^2 \sin\gamma$$

$$\frac{d}{dp_{\ell 3}} \left\{ \exp\left[-(j+1)\beta \left[(n+\tfrac{1}{2})\hbar\omega_\ell + (p_{\ell 3}^2/2m_\ell) - \mu \right] \right] \right\}$$

$$L_n\left(2\frac{\hbar K^2}{m_\ell \omega_\ell}\right) J_0\left[R\sqrt{(n+\tfrac{1}{2})\hbar} \right] \tag{2.16}$$

$$= J_0\left(R\sqrt{\hbar/2}\right) \int_{-\infty}^{\infty} dp_{\ell 3} \, (p_{\ell 3})^2 (\sin\gamma)$$

$$\frac{d}{dp_{\ell 3}} \left\{ 1 + \exp\left[\beta \left((p_{\ell 3}^2/2m_\ell) + \frac{\hbar\omega_\ell}{2} - \mu \right] \right\}^{-1} \tag{2.17}$$

With these calculations it is possible to write the fourth order contribution as

$$W_{P\ell\ell P} = W_{PP} \, \mathcal{E}(Z) \tag{2.18}$$

where

$$\mathcal{E}(Z) = \frac{\chi}{(Z^2 - \nu_\sigma^2)(Z^2 - \bar{\gamma}^2)} \tag{2.19}$$

and

$$\bar{\gamma} = k_{\sigma_3} p_{\ell_3}/m_\ell \tag{2.20}$$

$$\chi = -\omega_{p\ell}^2 \left(32\pi^3/m_\ell\right) \left[v_F/\omega_\ell (3\pi^2 C)^{1/3} \right]^{1/2} \left(k_{\sigma_3} p_\nu/m_\ell\right)^2$$

$$\left[|\vec{k}_\sigma| |\vec{k}_\sigma| \right]^{-1} J_0\left(R\frac{\sqrt{3}}{2}\right) \frac{d}{dp_0}\left(\tilde{F}(p_0)\right) \tag{2.21}$$

The one particle distribution function may now be written down in a form analogous to (IV 4.8). The inverse Laplace transform may also be evaluvated easily under the assumption of weak magnetic field and low temperatures.

3.-Conclusions:

Longitudinal oscillations in a magneto active electron plasma have been studied in the framework of irreversible statistical mechanics as formulated by Prigogine and co-workers. It is found that the oscillations are essentially of the Bessel function type.

The inverse transform of the quantity $\mathcal{E}(z)$ can be taken by resolving the right side of equation (2.19) into partial fractions. This procedure is identical to the one used in chapter IV. This leads to the reality condition

$$\nu_\sigma^2 + \nu^2 \geq 32\pi^3 P_0 \, \omega_{Pl}^2 \left[v_F / \omega_l (3\pi^2 C)^{1/3} \right]^{1/2}$$

$$J_0 \left[2\sqrt{\frac{\hbar K^2}{m_l \omega_l}} \right] \left[|\vec{k}_l| \, |\vec{k}_l| \right]^{-1} \frac{d}{dP_0} \left[\tilde{F}_0(P_0) \right] \tag{3.1}$$

It can be seen that if this condition is not satisfied then the distribution function develops an imaginary part and hence the expectation values evaluvated with this distribution function will exhibit instability in the system. The study of optics of the problem also leads to similar results as is obtained in chapter IV.

APPENDIX .1

THE BAKER - HAUSDORFF RELATION+

The free propagator used in chapters III to VI consists of two non commuting parts which prevents its straight forward use as a displacement operator. The two non commuting parts have to be decoupled if the propagator has to be applied on the initial distribution. The derivation presented here is due to Karplus and Schwinger[1].

Introduce the operator

$$F(\lambda) = \exp\left[\lambda(A+B)\right] \tag{A1.1}$$

$F(\lambda)$ satisfies the following differential equation

$$\frac{dF(\lambda)}{d\lambda} = (A+B)\exp\left[\lambda(A+B)\right] \tag{A1.2}$$

the initial condition being

$$F(0) = I \tag{A1.3}$$

Then the following transformation is made

$$F(\lambda) = \exp(\lambda A)\, G(\lambda) \tag{A1.4}$$

Hence

$$G(0) = I \tag{A1.5}$$

+ R. Karplus and J.Schwinger, Phys.Rev. 73 (1948) 1020.

Differentiate (A1.4) with respect to λ

$$\frac{dF(\lambda)}{d\lambda} = A\exp(\lambda A)\left[A\,G(\lambda) + \frac{dG(\lambda)}{d\lambda}\right] \tag{A1.6}$$

On substituting (A1.1) on the left side of (A1.6) and using (A1.2) and (A1.4)

$$(A+B)\exp(\lambda A)\;G(\lambda) = \exp(\lambda A)\left[A\,G(\lambda) + \frac{dG(\lambda)}{d\lambda}\right]$$

which implies

$$B\exp(\lambda A)\;G(\lambda) = \exp(\lambda A)\;\frac{dG(\lambda)}{d\lambda}$$

multiplying both sides by $\exp(-\lambda A)$ on the left, the following differential equation is obtained for $G(\lambda)$

$$\frac{dG(\lambda)}{d\lambda} = \exp(-\lambda A)\;B\exp(\lambda A)\,G(\lambda) \tag{A1.7}$$

which can be converted into an integral equation with the use of (A1.5)

$$G(\lambda) = I + \int_0^{\lambda} d\lambda'\, e^{-\lambda' A}\;B\;e^{-\lambda' A}\;G(\lambda') \tag{A1.8}$$

Making successive iterations

$$\begin{aligned} G(\lambda) = I &+ \int_0^{\lambda} d\lambda'\, e^{-\lambda' A}\;B\;e^{\lambda' A} \\ &+ \int_0^{\lambda} d\lambda'\, e^{-\lambda' A}\;B\;e^{\lambda' A} \int_0^{\lambda'} d\lambda''\, e^{-\lambda'' A}\;B\;e^{\lambda'' A} \\ &+ \cdots \end{aligned} \tag{A1.9}$$

It may be noted that the operators A and B are independent of the parameter λ . This fact is used to simplify the right side of (A1.9)

$$G(\lambda) = \exp\left\{ \int_0^{\lambda} d\lambda' \, e^{-\lambda' A} \, B \, e^{\lambda' A} \right\} \qquad \text{(A1.10)}$$

From chapter III equation (14)

$$L = \omega \frac{\partial}{\partial \omega}$$

$$+ \left\{ \sqrt{2\omega J/m} \left[\cos\omega \frac{\partial}{\partial q_1} + \sin\omega \frac{\partial}{\partial q_2} \right] + \frac{p_3}{m} \cdot \frac{\partial}{\partial q_3} \right\}$$

The operator $(p_3/m)\,(\partial/\partial q_3)$ commutes with all the other operators and hence can be decoupled trivially. Let

$$A = \omega t \,(\partial/\partial\omega)$$

$$B = \sqrt{\frac{2\omega J}{m}} \cdot t \left[\cos\omega \frac{\partial}{\partial q_1} + \sin\omega \frac{\partial}{\partial q_2} \right] \qquad \text{(A1.11)}$$

Then,

$$[A, B]_- = (AB - BA) = \omega t \, B^*$$

where $B^* = \sqrt{\frac{2\omega J}{m}} \cdot t \left[-\sin\omega \frac{\partial}{\partial q_1} + \cos\omega \frac{\partial}{\partial q_2} \right]$ (A1.12)

$$[A, B^*]_- = -(\omega t) B .$$

Thus the commutators from a closed set. (A1.10) can be expanded as

$$G(\lambda) = \exp\left\{ \int_0^\lambda d\lambda' \left(B - \lambda' [A,B]_- + \frac{\lambda'^2}{2!} [A,[A,B]_-]_- - \cdots \right) \right\}$$

$$= \exp\left\{ \int_0^\lambda d\lambda' \left(B - \lambda' \omega t\, B^* + \frac{(\lambda'\omega t)^2}{2!} B - \frac{(\lambda'\omega t)^3}{3!} B^* + \cdots \right) \right\}$$

$$= \exp\left\{ \int_0^\lambda d\lambda' \left(B \cos \omega\lambda t - B^* \sin \omega\lambda t \right) \right\}$$

For the present case $\lambda = 1$. Thus

$$G(1) = \exp\left\{ \sqrt{\frac{2J}{m\omega}} \left[-\left(\sin(\omega - \omega t) - \sin\omega\right) \frac{\partial}{\partial q_1} + \left(\cos(\omega - \omega t) - \cos\omega\right) \frac{\partial}{\partial q_2} \right] \right\} \qquad \text{(A1;13)}$$

It may be noted that the propagator satisfies the group property

$$\exp(Lt_1)\, \exp(Lt_2)\, \rho(0) = \exp(Lt_2) \exp(Lt_1)\, \rho(0)$$
$$= \rho(t_1 + t_2)$$

For the sake of simplicity ρ **can be taken as a function of** q_1 **only.**

$$\exp(Lt_2)\, \rho(q_1) = \exp\left(\omega t_2 \frac{\partial}{\partial \omega}\right)$$
$$\exp\left[-\sqrt{\frac{2J}{m\omega}} \left(\sin(\omega - \omega t_2) - \sin\omega\right) \frac{\partial}{\partial q_1} \right] \rho(q_1)$$
$$= \rho\left[q_1 - \sqrt{\frac{2J}{m\omega}} \left(\sin\omega - \sin(\omega + \omega t_2)\right) \right]$$

$$\exp(Lt_1)\, \exp(Lt_2)\, \rho(q_1)$$
$$= \rho\left[q_1 - \sqrt{\frac{2J}{m\omega}} \left(\sin\omega - \sin(\omega + \omega t_1 + \omega t_2)\right) \right]$$
$$= \exp(Lt_2)\, \exp(Lt_1)\, \rho(q_1).$$