FORMALITY OF CERTAIN CW COMPLEXES AND APPLICATIONS

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MATH10200804004

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A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

April, 2014

Homi Bhabha National Institute

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Prateep Chakraborty

ABSTRACT

Let X be a simply connected space having rational homology of finite type. Suppose X is a formal space in the sense that its minimal model can be constructed from $(H^*(X;\mathbb{Q}),0)$. Let $Y = X \cup_\alpha e^n$ where $\alpha\colon \mathbb{S}^{n-1} \to X$ is a continuous map. In this thesis we obtain a criterion for formality of Y in terms of $[\alpha] \in \pi_{n-1}(X)$. In another direction, we consider maps between grass manifolds. Let $G_{n,k}$ denote the complex Grassmann manifold of all k-dimensional vector subspaces of $\mathbb{C}^n$. Using the fact that any map $f\colon G_{n,k} \to G_{m,l}$ is formal, we shall show that the set $[G_{m,l}, G_{n,k}]$ of homotopy classes of maps is finite if $1 \leq k \leq \lfloor n/2 \rfloor$,$1 \leq l \leq \lfloor m/2 \rfloor$,$k < l$, $m - l > n - k$ and $m - l > 2k^2 - k - 1$ or $1 \leq k \leq 3$. We obtain some applications of this result.

we now give more precise statements of the main results of the thesis.

Let η_k (or more briefly η) denote the Hurewicz homomorphism $\pi_k^{\mathbb{Q}}(X) \to H_k(X;\mathbb{Q})$ and let $\widetilde{u} \in H^n(Y,X;\mathbb{Z}) \cong H^n(D^n, \mathbb{S}^{n-1};\mathbb{Z})) \cong \mathbb{Z}$ be the canonical generator. Let $j : Y \hookrightarrow (Y,X)$ be the inclusion map, and set $u := j^*(\widetilde{u})$.

Theorem 0.1. *Suppose that X is a simply connected space and is formal. Let $\mathcal{M}_X = \Lambda(V)$ and suppose that $V = \oplus_{k\geq 0} V_k$ is a standard lower gradation of V. Let $Y = X \cup_\alpha e^n$. Suppose that $\eta([\alpha]) = 0$ so that $j^*(\widetilde{u}) =: u \neq 0$. (i) If $[\alpha] \in \pi_{n-1}(X)$ is a torsion element then u is indecomposable and Y is formal. (ii) Let $[\alpha] \neq 0$ in $\pi_{n-1}^{\mathbb{Q}}(X)$. Suppose that $\langle v, [\alpha] \rangle = 0$ for all $v \in V_k \cap V^{n-1}, k \neq 1$, and that u is decomposable in $H^*(Y;\mathbb{Q})$. Then Y is formal. (iii) If $[\alpha] \in \pi_{n-1}(X)$ is not a torsion element and u is not decomposable, then Y is not formal. In cases (i) and (ii), the inclusion $i\colon X \hookrightarrow Y$ is formal.*

Theorem 0.2. *Let $1 \leq k \leq \lfloor n/2 \rfloor$, $1 \leq l \leq \lfloor m/2 \rfloor$ and $k < l$, where m, n are positive integers such that $m - l > n - k$. Suppose that (i) $m - l > 2k^2 - k - 1$ or $1 \leq k \leq 3$. Then the set $[(G_{m,l})_0, (G_{n,k})_0]$ of homotopy classes of continuous maps consists of only the class of constant maps and consequently the set $[G_{m,l}, G_{n,k}]$ of homotopy classes of maps is finite.*

ACKNOWLEDGEMENTS

At first I would like to thank my advisor Prof. Parameswaran Sankaran for introducing me to the Rational Homotopy Theory and teaching the other stuff of Mathematics, for his most fruitful discussions and for his unending encouragement. I would like to thank Prof. D. S. Nagaraj, Prof. K. N. Raghavan, Prof. A. R. Shastri and Prof. Samik Basu for their most valuable and helpful discussions. I am also grateful to all the academic and administrative members of the IMSc.

I have also learnt many from the discussions with Thakur, Dubey, Sarbeswarda, Panchu, Suratno, Anil, Krishanu, Chandan, Arghya, Priyavrat and many others. I thank them all. I thank my office-mates and room-mates (Joyjit and others).

I am also grateful to all my friends from and outside the IMSc for their joyful company.

I would also like to thank all my family members for their unending inspiration, encouragement and support. At last but not the least my soleful gratefulness is to my parents and I dedicate this thesis to these two visible gods of the earth.

PRATEEP

CONTENTS

NOTE:

List of Publications

1)Prateep Chakraborty and Parameswaran Sankaran, Formality of certain CW complexes and applications to Schubert varieties and torus manifolds, *J. Ramanujan Math. Soc.* Special Issue in honour of C. S. Seshadri, **28A** (2013), 55-74.

2) Prateep Chakraborty and Parameswaran Sankaran, Maps between certain complex Grassmann manifolds; *Topology and its Applications*, **170** (2014), 119-123, (arXiv:1312.4743).

3) Prateep Chakraborty and Parameswaran Sankaran, Errata: " Formality of certain CW complexes and applications to Schubert varieties and torus manifolds ", to apprear in *J. Ramanujan Math. Soc.* (arXiv:1301.5421).

1. INTRODUCTION

The main results of this thesis are summarised here: Let X be a simply connected space having rational homology of finite type. Suppose X is a formal space in the sense that its minimal model can be constructed from $(H^*(X;\mathbb{Q}), 0)$. Let $Y = X \cup_\alpha e^n$ where $\alpha\colon \mathbb{S}^{n-1} \to X$ is a continuous map. In this thesis we obtain a criterion for formality of Y in terms of $[\alpha] \in \pi_{n-1}(X)$. We obtain a necessary condition for the formality of a map. We shall apply these two results to prove formality of union of Schubert varieties of low dimensions in complex Grassmann manifolds. Let $G_{n,k}$ denote the complex Grassmann manifold of all k-dimensional vector subspaces of $\mathbb{C}^n$. Using the fact that any map $f\colon G_{n,k} \to G_{m,l}$ is formal, we shall show that the set $[G_{m,l}, G_{n,k}]$ of homotopy classes of maps is finite if $1 \le k \le \lfloor n/2 \rfloor$,$1 \le l \le \lfloor m/2 \rfloor$,$k < l$, $m-l > n-k$ and $m-l > 2k^2-k-1$ or $1 \le k \le 3$. We obtain some applications of this result.

1.1. **Sullivan algebra and formality.** A differential graded commutative algebra over $\mathbb{Q}$ (abbreviated *dgca*) (M,d) is called a *Sullivan algebra* if the following conditions hold: (i) *Freeness*: There exists a graded $\mathbb{Q}$-vector space $V = \oplus_{q\ge 1} V^q$ such that M is freely generated by V, that is, $M = \Lambda V := S^*(V^{even}) \otimes E^*(V^{odd})$ where $V^{even} = \oplus_{q\ge1} V^{2q}$, $V^{odd} = \oplus_{q \ge 1} V^{2q-1}$.(ii) *Nilpotence:* There is an increasing filtration $V = \cup_{k\ge 0} V(k)$, such that $d(V(k)) \subset \Lambda(V(k-1)), k \ge 1$, and $d(V(0)) = 0$. This gradation V_k is referred to as *the lower gradation* of V. A Sullivan algebra (M,d) is called *minimal* if $d(M)$ is contained in the ideal $M^+.M^+$ of decomposable elements. A *minimal model* $(\mathcal{M}_A, d)$ for a dgca (A,d) is a minimal Sullivan algebra $(\mathcal{M}_A, d)$ together with a dgca morphism $\phi : (\mathcal{M}_A, d) \to (A,d)$ which is a *quasi-isomorphism*, that is, ϕ induces an isomorphism $\phi^* : H^*(\mathcal{M}_A, d) \to H^*(A,d)$. A dgc algebra A with $H^0(A) = \mathbb{Q}$ has a minimal model $\rho_A : (\mathcal{M}_A, d) \to (A,d)$ where $(\mathcal{M}_A, d)$ is unique up to isomorphism. There is a notion of *standard lower gradation* for a minimal model $(\mathcal{M}_A, d)$ of a dgca $(A,0)$ (see §4.2).

Given a path connected topological space X, one has a functorial differential graded commutative algebra (dgca) $(A_{PL}(X), d)$ over $\mathbb{Q}$, called complex of *polynomial differential forms* on X, such that $H^*(A_{PL}(X), d)$ is naturally isomorphic to $H^*(X;\mathbb{Q})$, a minimal model $(\mathcal{M}_X, d)$ which is a dgca, and a dgc algebra morphism $\rho_X : \mathcal{M}_X \to A_{PL}(X)$ which induces isomorphism in cohomology. We say that X is *formal* if there exists a dgca morphism $\Phi : (\mathcal{M}_X, d) \to (H^*(X), 0)$ which is a quasi-isomorphism. Similarly, a map $f : X \to Y$ is *formal* if $A_{PL}(f) : A_{PL}(Y) \to A_{PL}(X)$ is connected to $H^*(f) : H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ by a chain of quasi-isomorphisms. For detailed definitions, see chapters 2 and 3.

1.2. **Main results.** Let $(\mathcal{M}_X, d) = (\Lambda V, d)$ be the minimal model of a simply-connected space X, then we have a pairing $\langle -, - \rangle : V^k \times \pi_k^{\mathbb{Q}}(X) \to \mathbb{Q}$. Let η_k (or more briefly η) denote the Hurewicz homomorphism $\pi_k^{\mathbb{Q}}(X) \to H_k(X;\mathbb{Q})$ and let $\widetilde{u} \in H^n(Y, X;\mathbb{Z}) \cong H^n(D^n, \mathbb{S}^{n-1};\mathbb{Z})) \cong \mathbb{Z}$ be the canonical generator. Let $j : Y \hookrightarrow (Y, X)$ be the inclusion map, and set $u := j^*(\widetilde{u})$.

Theorem 1.1. *Suppose that X is a simply connected space and is formal. Let $\mathcal{M}_X = \Lambda(V)$ and suppose that $V = \oplus_{k\geq 0} V_k$ is a standard lower gradation of V. Let $Y = X \cup_\alpha e^n$. Suppose that $\eta([\alpha]) = 0$ so that $j^*(\widetilde{u}) =: u \neq 0$. (i) If $[\alpha] \in \pi_{n-1}(X)$ is a torsion element then u is indecomposable and Y is formal. (ii) Let $[\alpha] \neq 0$ in $\pi_{n-1}^{\mathbb{Q}}(X)$. Suppose that $\langle v, [\alpha] \rangle = 0$ for all $v \in V_k \cap V^{n-1}, k \neq 1$, and that u is decomposable in $H^*(Y;\mathbb{Q})$. Then Y is formal. (iii) If $[\alpha] \in \pi_{n-1}(X)$ is not a torsion element and u is not decomposable, then Y is not formal. In cases (i) and (ii), the inclusion $i\colon X \hookrightarrow Y$ is formal.*

As a corollary to Theorem 1 we obtain the following.

Suppose X is formal space and suppose $(\Lambda V, d)$ is the minimal model of X with standard lower gradation. If $(V_0 \oplus (\oplus_{k\geq 2} V_k)) \cap V^{n-1} = 0, \alpha\colon \mathbb{S}^{n-1} \to X$ represents a non-torsion element and $\eta([\alpha]) = 0$ (where η is as defined above), then $Y = X \cup_\alpha e^n$ is formal.

One of the questions that stimulated our research is the following: *Suppose that X is a simply connected finite CW complex with cells only in even dimensions. Is X a formal space?* The complex Grassmann manifolds $G_{n,k}$, complex flag varieties [6], quasi-toric manifolds [24] etc., lend support to an affirmative answer to the above question. In this context we show that Schubert varieties of low dimensions in $G_{n,k}$ are formal. However, we obtain the following examples:

(1) a finite CW complex Y with only even dimensional cells, which is not formal.
(2) a cell attachment $Y = X \cup_\alpha e^n$ which is not formal, but X is formal and $u \in H^n(Y;\mathbb{Q})$ decomposable (with u as in Theorem 1.1),
(3) a finite CW complex Y with only even dimensional cells and a subcomplex X of Y, such that Y is formal but X is not formal.

Let X be a simply connected formal space having minimal model with standard lower gradation. An element $[\alpha] \in \pi_{n-1}(X)$ is said to be *special* if $\langle v, [\alpha]\rangle = 0$ for all $v \in V_k \cap V^{n-1}, k \neq 1$.

Let $f : Z \to X$. Let $\beta : \mathbb{S}^{n-1} \to Z, n \geq 2$, be continuous. Suppose that both Z and X are simply connected and formal spaces and that f is a formal map. Let $\mathcal{M}_Z = (\Lambda U, d)$ and $\mathcal{M}_X = (\Lambda V, d)$ be minimal models of Z and X respectively with standard lower gradation.

Theorem 1.2. *Suppose that $[\beta] \in \pi_{n-1}(Z)$ is special and $f : Z \to X$ is formal. Then $f_*([\beta])$ is special.*

The Theorem 1.1 has been divided into two different theorems, namely Theorem 7.2 and 8.2 and have been proved in chapter 8. The theorem 1.2 has been restated in chaper 7 as Theorem 7.5 and has been proved in chapter 8.

1.3. **Rational homotopy classification of maps between complex Grassmann manifolds.** Let X be any simply connected CW complex and let X_0 denote its rationalization. Thus $\widetilde{H}^*(X_0;\mathbb{Z}) \cong \widetilde{H}^*(X;\mathbb{Q})$. If $f : X \to Y$ is a continuous map of such spaces, then there exists a rationalization of f, namely a continuous map $f_0 : X_0 \to Y_0$ such that $f_0^* : \widetilde{H}^*(Y_0;\mathbb{Z}) \to \widetilde{H}^*(X_0;\mathbb{Z})$ is the same as $f^* : \widetilde{H}^*(Y;\mathbb{Q}) \to \widetilde{H}^*(X;\mathbb{Q})$. Denoting the minimal model of X by $\mathcal{M}_X$, one has a bijection $[X_0, Y_0] \cong [\mathcal{M}_Y, \mathcal{M}_X]$ where on the left we have homotopy classes of continuous maps $X_0 \to Y_0$ and on the right we have homotopy classes of dgca homomorphisms between Sullivan algebras $\mathcal{M}_Y \to \mathcal{M}_X$. In the case when X is a complex Grassmann manifold, one knows that $\mathcal{M}_X$ can be computed directly from its rational cohomology algebra. When X and Y are complex Grassmann manifolds, $[X_0, Y_0]$ is in bijection with the set of graded $\mathbb{Q}$-algebra homomorphisms $H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$. If $X = Y$ and both are same complex Grassmannian, and if $h\colon X_0 \to X_0$ with $h^*(c_1) = 0$, where h^* is the cohomology algebra morphism induced by h and c_1 is the degree 2 generator of $H^*(X;\mathbb{Q})$, then h is null-homotopic if $2k \leq n, n \geq 2k^2 - 1$ or $k \leq 3$. It has been conjectured that the result is true without any restriction on n, k. See [9]. In this thesis, we study $[X, Y]$ when X and Y are different complex Grassmann manifolds.

Theorem 1.3. *Let $1 \leq k \leq \lfloor n/2 \rfloor$, $1 \leq l \leq \lfloor m/2 \rfloor$ and $k < l$, where m, n are positive integers such that $m - l > n - k$. Suppose that (i) $m - l > 2k^2 - k - 1$ or $1 \leq k \leq 3$. Then the set $[(G_{m,l})_0, (G_{n,k})_0]$ of homotopy classes of continuous maps consists of only the class of constant maps and consequently the set $[G_{m,l}, G_{n,k}]$ of homotopy classes of maps is finite.*

As a corollary to the above theorem we obtain the following result.

Theorem 1.4. *Let $f : G_{m,l} \to G_{n,k}$ be any continuous map where l, k, m, n are as in Theorem 1.3. Then there exists an element $x \in G_{m,l}$ such that $f(x) \subset x$.*

These theorems have been restated as Theorem 10.2 and 10.3 and proved in chapter 10.

2. MINIMAL MODEL AND FORMALITY

In this chapter we discuss the notion of a Sullivan algebra, formality, the Sullivan model for a cell attachment, and the stepwise construction of the minimal Sullivan model for a differential graded commutative cochain algebra. The reader is referred to [7] for a comprehensive treatment of rational homotopy theory.

2.1. **Sullivan algebra.** A *differential graded commutative algebra* (dgca for short) is a graded $\mathbb{Q}$ - algebra (A, d_A) together with a differential d_A of degree 1, that is a derivation, ie $d_A(a.b) = d_A a.b + (-1)^{|a|}a.d_A b$ and $a.b = (-1)^{|a||b|}b.a$. The *cohomology algebra* $H^*(A, d_A)$ is the graded algebra where $H^k(A, d_A)$ = kernel of $d_A(k)/$ Image of $d_A(k-1)$. A morphism $f : (A, d_A) \to (B, d_B)$ is a morphism of graded algebras of degree 0 satisfying $f \circ d_A = d_B \circ f$. It induces a morphism $H^*(f) : H^*(A, d_A) \to H^*(B, d_B)$. A morphism $f : (A, d) \to (B, d)$ is said to be a *quasi-isomorphism* (denoted by $f : (A, d) \xrightarrow{\simeq} (B, d)$) if $H^*(f) : H^*(A, d) \to H^*(B, d)$ is an isomorphism. Any graded commutative algebra may be regarded as a dgca with zero differential. In particular, the cohomology algebra $H^*(A, d_A)$ may be regarded as a dgca $(H^*(A, d_A), 0)$. A dgc algebra (A, d) is *connected* if $A^0 = \mathbb{Q}$ and *simply connected* if $A^0 = \mathbb{Q}$ and $A^1 = 0$. Henceforth, we shall denote the differential of any dgc algebra by d unless there is a danger of confusion.

A differential graded commutative algebra (M, d) is called a *Sullivan algebra* if the following hold: (i) *Freeness*: There exists a graded $\mathbb{Q}$-vector space $V = \oplus_{q\geq 1} V^q$ such that M is freely generated by V, that is, $M = \Lambda V := S^*(V^{even}) \otimes E^*(V^{odd})$ where $V^{even} = \oplus_{q\geq 1} V^{2q}$, $V^{odd} = \oplus_{q\geq 1} V^{2q-1}$. Here $S^*(V)$ denotes the symmetric algebra on V and $E^*(V)$ denotes the exterior algebra on V. (ii) *Nilpotence:* There is a well-ordering on a basis $\{v_\alpha\}$ of V consisting of homogeneous elements such that for each α, $d(v_\alpha)$ is a polynomial in the $v_\beta, \beta < \alpha$.

The nilpotence condition can be restated as follows: There is an increasing filtration $V = \cup_{k\geq 0} V(k)$, such that $d(V(k)) \subset \Lambda(V(k-1))$ and there exists a subspace $V_k \subset V(k)$ such that $d(V(0)) = 0$ and $d(V_k) \subset \Lambda(V(k-1))$ and $\Lambda(V(k)) = \Lambda(V_k) \otimes \Lambda(V(k-1))$. This filtration is referred to as *the lower filtration* of M. A Sullivan algebra (M, d) is called a *minimal algebra* if $d(M) \subset M^+.M^+$ (here M^+ is $\oplus_{k\geq 1} M^k$), the ideal of decomposable elements. If $\theta : (\Lambda V, d) \to (\Lambda W, d)$ be a dgca morphism between Sullivan algebras, then the *linear part* $Q(\theta) : V \to W$ is a linear map of degree 0, defined by $(\theta - Q(\theta))(v) \in \Lambda^{\geq 2} W$ for $v \in V$.

2.2. **Minimal model and Formality of dgc algebra.** A *Sullivan model* (M, d) for a dgca (A, d) is a Sullivan algebra (M, d) together with a dgca morphism $\rho : M \to A$ which is a quasi-isomorphism. Thus, ρ is a dgca morphism which induces an isomorphism $\rho^* : H^*(M, d) \to H^*(A, d)$. Similarly, a *minimal model* (M, d) for a dgca (A, d) is a minimal algebra (M, d) together with a quasi-isomorphism $\rho : M \to A$. A dgc algebra A with $H^0(A) = \mathbb{Q}$ has a unique minimal model up to isomorphism. We denote this by $(\mathcal{M}_A, d)$. The dgc algebra A is called *formal* if there exists a dgc morphism $\Phi : (\mathcal{M}_A, d) \to (H^*(A), 0)$ which is a quasi-isomorphism.

Notations. If V is a graded vector space, then $V^{\leq k}$ (resp. $V^{<k}$) denotes the subspace consisting of elements of degree at most (resp. less than) k. If A is a differential graded algebra, $A^{\leq k}$ (resp. $A^{<k}$) denotes the differential graded subalgebra of A generated by elements of degree at most (resp. less than) k.

If A is a dgc algebra, we denote by $\mathcal{D}(A)$ (or simply $\mathcal{D}$ if A is clear from the context) the ideal of decomposable elements i.e. $A^+.A^+$ (here $A^+ = \oplus_{k\geq 1} A^k$) in A. By abuse of notation we write $A^n/\mathcal{D}$ to mean $A^n/\mathcal{D} \cap A^n \subset A/\mathcal{D}$.

Two dgc algebras (A,d) and (B,d) are *weakly equivalent* if there is a finite sequence of dgc morphisms $f := \{f_i\}$ where $A_0 \overset{f_0}{\rightarrow} A_1 \overset{f_1}{\leftarrow} A_2 \overset{f_2}{\rightarrow} \cdots \overset{f_{2n-1}}{\leftarrow} A_{2n}$ with $(A_0,d) = (A,d), (A_{2n},d) = (B,d)$ such that induced morphisms in cohomology are all isomorphisms. In this case we write $(A,d) \overset{f}{\leftrightarrow} (B,d)$ or $(A,d) \simeq (B,d)$. We denote by $f^* : H^*(A,d) \to H^*(B,d)$ the composition of isomorphisms $(f^*_{2n-1})^{-1} \cdots \circ f^*_0$. A dgc algebra (A,d) (graded algebra A) is of finite type if each A^k is finite dimensional vector space over $\mathbb{Q}$.

2.3. **Homotopy between dgca morphisms.** Next we recall the notion of homotopy between two dgca morphisms from a Sullivan algebra. One has the dgc algebra $\Lambda(t,dt)$ where degree of t is zero and the differential of t is dt. The augmentations $\epsilon_j : \Lambda(t,dt) \to \mathbb{Q}$, where $\epsilon_j(t) = j, j = 0,1$ are dgca morphisms. For any dgca A, the algebra morphism $\eta_j : A \otimes \Lambda(t,dt) \to A$ defined by $a \otimes t \mapsto \epsilon_j(t)a$ is a dgca morphism for $j = 0,1$. Two dgca morphisms $\phi_0, \phi_1 : M \to A$ from a Sullivan algebra $M = \Lambda V$ to a dgca A are *homotopic* (denoted by $\phi_0 \simeq \phi_1$) if there exists a dgca morphism $H : M \to A \otimes \Lambda(t,dt)$ such that $\eta_j \circ H = \phi_j, j = 0,1$. The set of homotopy classes of dgca morphisms from a Sullivan algebra (M,d) to a dgca (A,d) is denoted by $[(M,d),(A,d)]$.

2.4. **Formality of dgca morphism.** If $f : (A,d) \to (B,d)$ is a quasi-isomorphism and (M,d) is a Sullivan algebra, then the natural map $\rho_\sharp : [(M,d),(A,d))] \to [(M,d),(B,d)]$ is a bijection. Suppose that $\rho_A : (M_A,d) \to (A,d)$ and $\rho_B : (M_B,d) \to (B,d)$ are Sullivan models of A and B respectively. Using the last bijection, for each morphism $f : (A,d) \to (B,d)$, we get ϕ_f (or ϕ) : $(M_A,d) \to (M_B,d)$ yielding the following homotopy commutative diagram:

$$\begin{array}{ccc} (A,d) & \stackrel{f}{\to} & (B,d) \\ \rho_A \uparrow & & \uparrow \rho_B \\ (M_A,d) & \stackrel{\phi_f}{\to} & (M_B,d). \end{array}$$

If A and B are dgca with with $H^0(A,d) = H^0(B,d)) = \mathbb{Q}$, then there are quasi-isomorphisms $\rho_A : \mathcal{M}_A \to (A,d)$ and $\rho_B : \mathcal{M}_B \to (B,d)$, where $\mathcal{M}_A$ and $\mathcal{M}_B$ are minimal models. For a dgca morphism $f : (A,d) \to (B,d)$, we have Sullivan representative $\phi : \mathcal{M}_A \to \mathcal{M}_B$. Now, suppose that A and B are formal dgca, then we have dgca morphisms $\mathcal{M}_A \stackrel{\Phi}{\to} H^*(A)$ and $\mathcal{M}_B \stackrel{\Psi}{\to} H^*(B)$, which are quasi-isomorphisms inducing ρ_A^* and ρ_B^* in cohomology. The dgca morphism f is said to be a *formal map* if ϕ satisfies $\Psi \circ \phi \simeq f^* \circ \Phi$ for some choice of $\mathcal{M}_A$, $\mathcal{M}_B$, ρ_A, ρ_B, Φ, Ψ (See [1]). But when we shall talk of a formal map, we shall always fix these minimal models and dgca morphisms. Thus f is formal if ϕ is a common Sullivan representative for both f and f^*. See diagram below.

$$\begin{array}{ccc} (A,d) & \stackrel{f}{\to} & (B,d) \\ \rho_A \uparrow & & \uparrow \rho_B \\ (\mathcal{M}_A,d) & \stackrel{\phi}{\to} & (\mathcal{M}_B,d) \\ \Phi \downarrow & & \downarrow \Psi \\ H^*(A) & \stackrel{f^*}{\to} & H^*(B). \end{array}$$

It is evident that if $g : (B,d) \to (C,d)$ is another formal map, then their composition $h = g \circ f$ is also a formal map.

3. RATIONAL SPACES AND RATIONAL HOMOTOPY THEORY

3.1. **Rationalization.** A simply connected space X_0 is said to be a *rational space* if $\pi_k(X_0)$ (or equivalently $H_k(X_0, pt; \mathbb{Z})$) are rational vector spaces for all $k \geq 2$. A *rationalization* of a simply connected space X is a map $\iota : X \to X_0$ to a simply connected rational space X_0 such that ι induces any one of the following isomorphisms

(a) $\pi_k(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \pi_k(X_0)$ for all k,

(b) $H_*(X; \mathbb{Q}) \xrightarrow{\cong} H_*(X_0; \mathbb{Q})$.

The equivalence of (a) and (b) follows from Whitehead-Serre Theorem (See [7]). We obtain the existence and uniqueness of rationalization X_0 of a simply connected space and the universal property of a rationalization from the following theorem

Theorem 3.1. *([7, Ch. 9]) (1) For each simply connected space X there is a relative CW complex (X_0, X) with no zero-cell and no one-cells such that the inclusion ι : $X \to X_0$ is a rationalization.*

(2) If (X_0, X) is as in (1), then any continuous map f from X to a simply connected rational space Z extends to a map $g : X_0 \to Z$. If $g' : X_0 \to Z$ extends $f' : X \to Z$, then any homotopy from f to f' extends to a homotopy from g to g'.

(3) In particular, the rationalization of (1) are unique up to homotopy equivalence relative X (i.e. the maps of the homotopy equivalence are restricted to identity on X).

A continuous map $f : Z_1 \to Z_2$ is a *weak homotopy equivalence* if $\pi_0(f) : \pi_0(Z_1) \to \pi_0(Z_2)$ is a bijection and each

$$\pi_n(f) : \pi_n(Z_1, z_1) \to \pi_n(Z_2, f(z_1)), z_1 \in Z_1, n \geq 1,$$

is an isomorphism.

A continuous map $f : Z_1 \to Z_2$ between simply connected spaces Z_1 and Z_2 is a *rational homotopy equivalence* if it satisfies the following equivalent conditions:

(a) $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism.

(b) $H_*(f; \mathbb{Q})$ is an isomorphism.

(c) $H^*(f; \mathbb{Q})$ is an isomorphism.

Two spaces X and Y have the *same weak homotopy type* if they are connected by a chain of weak homotopy equivalences

$$X \leftarrow Z(0) \rightarrow \cdots \leftarrow Z(n) \rightarrow Y.$$

The weak homotopy type of X_0 (the rationalization of X) is the *rational homotopy type* of X. A *rational cellular model* for a simply connected space Y is a rational homotopy equivalence $f : X \to Y$ from a CW complex X such that $X^{(1)} = X^{(0)}$ = point. (Here $X^{(i)}$ is the i-th skeleton of X.) We have the following theorem regarding rational cellular model.

Theorem 3.2. *([7, Theorem 9.11]) Every simply connected space Y is rationally modelled by a CW complex X for which the differential in the integral cellular chain complex is identically zero.*

As corollaries we get the following facts for a simply connected space Y.

(1) $H_*(Y; \mathbb{Q})$ has finite type (i.e. finite dimensional in each dimension) $\Leftrightarrow$ Y is rationally modelled by a CW complex of finite type.

(2) $H_*(Y; \mathbb{Q})$ is finite dimensional and concentrated in degrees $\leq N \Leftrightarrow Y$ is rationally modelled by a CW complex of dimension $\leq N$.

Note: Rational homotopy theory is the study of properties of spaces and maps that depend only on rational homotopy type; i.e., are invariant under rational homotopy equivalence.

3.2. **Formality of spaces and maps.** Any path-connected topological space X has an associated naturally defined differential graded commutative algebra (dgca) $A_{\mathrm{PL}}(X)$ over $\mathbb{Q}$ and natural cochain algebra morphisms which are quasi-isomorphisms (See [7])

$$C^*(X;\mathbb{Q}) \xrightarrow{\simeq} D(X) \xleftarrow{\simeq} A_{PL}(X).$$

where $C^*(X;\mathbb{Q})$ denotes cochain algebra of normalized singular cochain on X and $D(X)$ is a third natural cochain algebra. These quasi-isomorphisms define a natural isomorphism of graded algebras,

$$H^*(X;\mathbb{Q}) = H^*(A_{PL}(X)).$$

The dgc algebra $A_{PL}(X)$ is the cochain algebra of *polynomial differential forms on* X *with coefficients in* $\mathbb{Q}$. This association

$$X \rightsquigarrow A_{PL}(X)$$

gives a contravariant functor from the category of path-connected topological spaces to the category of differential graded commutative algebras. A *commutative cochain algebra model* for a space X (or simply a *commutative model* for X) is a cochain algebra (A, d) together with a weak equivalence

$$(A, d) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} A_{PL}(X).$$

The construction of $A_{PL}(X)$, due to Sullivan [30], is inspired from C^∞ differential forms, while reflecting the combinatorial nature of how the singular simplices of X fit together.

The dgc algebra $A_{PL}(X)$ can be constructed over any field of characteristic zero, in particular over $\mathbb{R}$, which we denote by $A_{PL}(X;\mathbb{R})$. For a smooth manifold M the classical de Rham algebra $A_{DR}(M)$ of smooth differential forms is weakly equivalent to $A_{PL}(M;\mathbb{R})$.

Suppose that X is a path connected topological space. A *Sullivan model* (resp. *minimal model*) for X is by definition a Sullivan model (resp. minimal model) for $A_{\mathrm{PL}}(X)$. Any path connected space X has a unique minimal model $\mathcal{M}_X$. See [7, Proposition 12.1 and corollary of Theorem 14.12]. If X_1 and X_2 are simply connected spaces with rational homology of finite type and have the same rational homotopy type, then their minimal models are isomorphic (as dgc algebras). Assume that X and Y are simply connected and with rational homology of finite type. Then we have an isomorphism of sets: $[X_0, Y_0] \to [\mathcal{M}_Y, \mathcal{M}_X]$ where X_0 denotes the rationalization of X, $[X_0, Y_0]$ denotes the set of homotopy classes of maps between X_0 and Y_0, $[\mathcal{M}_Y, \mathcal{M}_X]$ denotes the homotopy classes of dgca morphisms between Sullivan algebras $\mathcal{M}_Y \to \mathcal{M}_X$ and $\mathcal{M}_X$ is the minimal model of X. Observe that $\mathcal{M}_X$ and $\mathcal{M}_{X_0}$ are naturally isomorphic since $X \subset X_0$ is a rational homotopy equivalence. The isomorphism is obtained by sending $[f] \in [X_0, Y_0]$ to the homotopy class of a Sullivan representative of $\phi_f : \mathcal{M}_Y \to \mathcal{M}_X$ of $A_{\mathrm{PL}}(f) : A_{\mathrm{PL}}(Y_0) \to A_{\mathrm{PL}}(X_0)$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_Y & \xrightarrow{\phi_f} & \mathcal{M}_Y \\ \rho_Y \downarrow & & \downarrow \rho_X \\ A_{\mathrm{PL}}(Y_0) & \xrightarrow{A_{\mathrm{PL}}(f)} & A_{\mathrm{PL}}(X_0). \end{array}$$

In fact we have the following bijections, where we restrict to simply connected CW complexes with rational homology of finite type and to simply connected minimal Sullivan algebras of finite type.

$$\{\text{rational homotopy types}\}$$

$$\updownarrow$$

$$\{\text{isomorphism classes of minimal algebras over } \mathbb{Q}\}$$

and

$$\{\text{homotopy classes of continuous maps of rational spaces}\}$$

$$\updownarrow$$

$$\{\text{homotopy classes of morphisms of minimal Sullivan algebras over } \mathbb{Q}.\}$$

By the above discussion, any path connected space X has a unique minimal model $\rho_X : \mathcal{M}_X \to A_{PL}(X)$. A space X is said be a *formal space* if $A_{PL}(X)$ is a formal dgc algebra. Then we have a dgca morphism $\Phi : \mathcal{M}_X \to H^*(X;\mathbb{Q})$, which is a quasi-isomorphism. A continuous map $f : X \to Y$ between two formal spaces is called a *formal map* if $A_{PL}(f) : A_{PL}(Y) \to A_{PL}(X)$ is a formal dgca morphism. Here we set our notations as follows.

For a formal space X, we have the minimal model $\rho_X : \mathcal{M}_X \to A_{PL}(X)$ and $\Phi : \mathcal{M}_X \to H^*(X;\mathbb{Q})$ with $\rho_X^* = \Phi^*$. We shall write $\mathcal{M}_X = \Lambda V$ and similarly for Y we have $\mathcal{M}_Y = \Lambda W$, $\Psi : \mathcal{M}_Y \to H^*(Y;\mathbb{Q})$.

If X is a simply connected space with rational homology of finite type, then there is a bilinear map $\langle -,- \rangle : V^k \times \pi_k^{\mathbb{Q}}(X) \to \mathbb{Q}$ (where $\pi_k^{\mathbb{Q}}(X) := \pi_k(X) \otimes \mathbb{Q}$), which is non-degenerate. This gives that $V^k \cong Hom(\pi_k(X);\mathbb{Q})$. Recall that if $\theta : (\Lambda V, d) \to (\Lambda W, d)$ is a dgca morphism, then the linear part $Q(\theta) : V \to W$ is a linear map of degree 0, defined by $(\theta - Q(\theta))(v) \in \Lambda^{\geq 2} W$ for $v \in V$. Suppose $f : X \to Y$ is a continuous map between simply connected spaces, $\rho_X : (\Lambda V, d) \to A_{PL}(X)$ and $\rho_Y : (\Lambda W, d) \to A_{PL}(Y)$ be minimal models of X and Y respectively. Let $\phi_f : (\Lambda W, d) \to (\Lambda V, d)$ be a Sullivan representative of f. Then through the bilinear map $\langle -,- \rangle : V \times \pi_*^{\mathbb{Q}}(X) \to \mathbb{Q}$, we get that $Q(\phi_f)$ is the transpose of $f_* : \pi_*^{\mathbb{Q}}(X) \to \pi_*^{\mathbb{Q}}(Y)$.

4. CONSTRUCTION OF MINIMAL MODEL

Here we give the construction of minimal models in two cases, first for the general case and then for a dgca with zero differential. In both cases, we assume that $H^0(A,d) = \mathbb{Q}$ and $H^1(A,d) = 0$. In the special case, when $d = 0$, the second construction yields a description of the lower gradation.

4.1. **Minimal model for a simply connected dgc algebra.** We briefly recall the construction of the minimal model $(\mathcal{M}_A, d)$ for a dgca (A,d) with $H^0(A,d) = \mathbb{Q}$, in the special case when $H^1(A,d) = 0$. See [7], p.144 for details.

Choose cocycles $z_\alpha \in A^2, \alpha \in J_2$, so that the cohomology classes $[z_\alpha] \in H^2(A)$ form a $\mathbb{Q}$-basis. Let $V^2 \subset A^2$ be the $\mathbb{Q}$-span of $\{z_\alpha\}, \alpha \in J_2$. Then the inclusion map $V^2 \to A$ extends to a dgca morphism $\rho_2 : (\Lambda V^2, 0) \to (A,d)$ inducing an isomorphism in degree 2. Since $H^1(A) = 0$, and $H^3(\Lambda V^2) = 0$ we see that ρ_2 induces isomorphism in degree ≤ 2 and a monomorphism in degree 3.

Inductively assume that we have constructed $V^j, j \leq k$, and a dgca morphism $\rho_k : (\Lambda(\oplus_{j\leq k} V^j), d) \to (A,d)$ which induces isomorphism in degree $\leq k$ and a monomorphism in degree $k+1$. We now explain how to construct V^{k+1} and ρ_{k+1}.

Choose cocycles $z_\alpha \in A^{k+1}, \alpha \in J_{k+1}$, such that the images of the $[z_\alpha] \in H^{k+1}(A)$ form a $\mathbb{Q}$-basis for the cokernel of $H^{k+1}(\Lambda(\oplus_{j\leq k} V^j), d) \to H^{k+1}(A)$. Also choose cocycles $u_\beta \in \Lambda(\oplus_{j\leq k} V^j)^{k+2}, \beta \in J'_{k+1}$, so that the $[u_\beta] \in H^{k+2}(\Lambda(\oplus_{j\leq k} V^j), d)$ form a $\mathbb{Q}$-basis for the kernel of $H^{k+2}(\rho_k)$. Choose $w_\beta \in A^{k+1}$ such that $\rho_k(u_\beta) = dw_\beta$.

Set $V^{k+1} := \oplus_{\alpha\in J_{k+1}} \mathbb{Q} z_\alpha \oplus (\oplus_{\beta\in J'_{k+1}} \mathbb{Q} v_\beta)$ and extend the derivation on $\Lambda(V^{\leq k})$ to a derivation on $\Lambda(V^{\leq k+1})$ by setting $dz_\alpha = 0, dv_\beta = u_\beta$. The morphism ρ_k is extended to $\Lambda(V^{\leq k+1})$ by setting $\rho_{k+1}(z_\alpha) = z_\alpha, \alpha \in J_{k+1}$, and $\rho_{k+1}(v_\beta) = w_\beta, \beta \in J'_{k+1}$. Then ρ_{k+1} induces isomorphism in degree $\leq k+1$ and a monomorphism in degree

$k+2$. The required minimal model $(\mathcal{M}_A, d)$ is $(\Lambda V, d)$ with $V := \oplus_{k\geq 2} V^k$ and the morphism $\rho : \mathcal{M}_A \to A$ is defined by $\rho|_{V^k} = \rho_k|_{V^k}, k \geq 2$.

4.2. **Minimal model for simply connected dgca with d=0.** Let (A, d) be a dgc algebra with $A^0 = \mathbb{Q}$ and $A^1 = 0$. Then we have a minimal model $\rho_A : (\mathcal{M}_A, d) \to (A, d)$ where $\mathcal{M}_A = \Lambda V$ with the following properties:

a) $\Lambda V_0 \xrightarrow{\rho} A$ is surjective,

b) $V_{\geq 1} \xrightarrow{\rho} 0$.

(See [13, §3]).

Here we give the construction.

We set $V_0 = A^+/A^+.A^+$, the space of indecomposables for A. We define $d = 0$ in V_0 and $\rho : \Lambda V_0 \to A$ so its restriction to V_0 splits the projection $A^+ \to V_0$. Thus V_0 is a "space of generators for A". Then $\rho : \Lambda V_0 \to A$ is surjective. Let K be its kernel. Then $K^0 = K^1 = 0$.

We define $V_1 = K/K.\Lambda^+ V_0$ with gradation defined as

$$V_1^p = (K/K.\Lambda^+ V_0)^{p+1}$$

The space V_1 is the "space of generators for the relations in A". Since $K^0 = K^1 = 0$, $V_1 = \oplus_{p\geq 1} V_1^p$. We extend d on V_1 by requiring that it be a linear map $V_1 \to K$ splitting the projection. Then $d : V_1 \to \Lambda V_0$ and we define ρ to be 0 on V_1.

Suppose $V(n) = V_0 \oplus V_1 \oplus \dots V_n$ has been constructed for some $n \geq 1$. Now, define V_{n+1}^p to be $(H^*(\Lambda V(n), d)/(H^*(V(n), d).\Lambda^+ V_0)^{p+1}$. We define d so that $d : V_{n+1} \to (\Lambda V(n)) \cap (\ker d)$ splits the projection of $\Lambda V(n) \cap \ker d$ onto V_{n+1}. We define $\rho : V(n+1) \to A$ where $V(n+1) = V_0 \oplus V_1 \oplus \cdots \oplus V_{n+1}$ by $\rho|_{V_{n+1}} = 0$.

Now, let $(\Lambda V, d) \xrightarrow{\rho} (A, 0)$ be the dgca morphism constructed in this way, with $V = \oplus_{n=0}^{\infty} V_n$.

This is the required minimal model of $(A, 0)$. For our purpose it is important to have a lower gradation that satisfies a further property. We have the following lemma which achieves this.

Lemma 4.1. *Let $(\mathcal{M}_A, d) = (\Lambda V, d)$ be a minimal model of a dgca $(A, 0)$ with zero differential. Then there exists a lower gradation $V = \oplus_{k\geq 0} V_k$ such that (i) $\rho(V_k) = 0$ for all $k \geq 1$, and, (ii) $d(V_k) \subset \Lambda(V_0).\Lambda^+(V_1 \oplus \cdots \oplus V_{k-1})$ for $k \geq 2$.*

Proof. The existence of a lower gradation $V_k, k \geq 1$, such that $\rho(V_k) = 0$ is given in 4.2. We start such a lower gradation $V_k, k \geq 0$ and modify this to obtain a new lower gradation V'_k so as to meet both our requirements. We set $V_k^n = V_k \cap V^n$.

Let $\{y_\gamma\}_{\gamma \in J_{k,2}}$ be a basis for V_2^k. Write $dy_\gamma = u_0 + u_1$ where $u_0 \in \Lambda(V_0)^{k+1}$ and $u_1 \in \Lambda(V_0).\Lambda^+(V_1)$. Then $\rho(u_1) = 0$ by Leibniz rule using $\rho(V_1) = 0$. Therefore, $0 = d\rho(w) = \rho(dw) = \rho(u_0)$ implies that $u_0 = \sum f_i.dv_i = d(\sum f_i v_i)$ where $f_i \in \Lambda(V_0), v_i \in V_1$ since $u_0 \in \Lambda(V_0)$. Now let $y'_\gamma = y_\gamma - \sum f_i v_i$. Then $dy'_\gamma = u_1 \in \Lambda(V_0).\Lambda^+(V_1)$ and $\rho(y'_\gamma) = \rho(y_\gamma) - \sum \rho(f_i)\rho(v_i) = 0$ as $\rho(V_2) = 0 = \rho(V_1)$. We define $V_2'^k \subset \Lambda(V_0 \oplus V_1)^k \oplus V_2^k$ to be the space spanned by $y'_\gamma, \gamma \in J_{k,2}$. Set $V'_2 = \oplus_{k\geq 3} V_2'^k$. Note that $V'_2 \cap (V_0 \oplus V_1) = 0$, $V(2) = V_0 + V_1 + V'_2$, $\rho(V'_2) = 0$ and $d(V'_2) \subset \Lambda(V_0).\Lambda^+(V_1)$.

We now proceed by induction. Assume that $V'_j, 2 \leq j < n$, have been constructed satisfying (i) and (ii) such that $V_0 + V_1 + V'_2 + \cdots + V'_{n-1} = V(n-1)$. Let $\{y_\gamma\}_{\gamma \in J_{k,n}}$ be a basis for V_n^k. Write $dy_\gamma = z_0 + z_1$ where $z_0 \in \Lambda(V_0)^k$ and $z_1 \in \Lambda(V_0).\Lambda^+(V_1 \oplus V'_2 \oplus \cdots \oplus V'_{n-1})$. Then $\rho(z_1) = 0$ by Leibniz rule using $\rho(V'_j) = 0, j \geq 1$. Therefore, $0 = d\rho(y_\gamma) = \rho(dy_\gamma) = \rho(z_0)$ implies that $z_0 = d(\sum f_j.x_j)$ where $f_j \in \Lambda(V_0), x_j \in \Lambda^+(V_1)$ since $u_0 \in \Lambda(V_0)$. Set $y'_\gamma := y_\gamma - \sum f_j x_j$. Then $dy'_\gamma = z_1 \in \Lambda(V_0).\Lambda^+(V_1 \oplus \cdots \oplus V'_{n-1})$ and $\rho(y'_\gamma) = 0$ as $\rho(y_\gamma) = 0$ and $\rho(V_j) = 0, 1 \leq j < n$. Then $V'_n := \oplus_{k\geq 3}(\oplus_{\gamma \in J_{k,n}} \mathbb{Q} y'_\gamma)$ satisfies (i) and (ii). Furthermore $V(n) = V(n-1) + V'_n$, $V_n \cap V(n-1) = 0$. This completes the induction step and we see that $V_0, V_1, V'_j, j \geq 2$ yield a lower gradation for V that meets our requirements. $\square$

Definition 4.2. *Let $\mathcal{M}_A$ be a minimal model of $(A, 0)$. We say that a lower gradation $V = \oplus_{k\geq 0} V_k$ of $\mathcal{M}_A = \Lambda(V)$ is* standard *if it satisfies conditions (i) and (ii) of Lemma 4.1.*

4.3. **Minimal models and inverse limits.** We shall obtain a simple criterion for the minimal model of a simply connected dgc algebra (A,d) to be isomorphic to an inverse limit of dgc algebras. Let $f_n : A \to A_n, f_{m,n} : A_n \to A_m$, $n \geq m \geq 1$, be dgca morphisms such that $f_{n,n} = id$ and $f_{m,n} \circ f_n = f_m$. Suppose that $f_m^* : H^q(A) \to H^q(A_m), f_{m,n}^* : H^q(A_n) \to H^q(A_m)$ are isomorphisms for $q \leq m$. We assume that $H^0(A) = \mathbb{Q}, H^1(A) = 0$. We denote by $\mathcal{M} := \mathcal{M}_A = \Lambda(V), V = \oplus_{q\geq 2} V^q$, the minimal model of A.

Let $z_\alpha, \alpha \in J_q, v_\beta, u_\beta, w_\beta, \beta \in J'_q$ have the same meaning as in the above description of V^q in 4.1 . The quasi-isomorphism $\rho : \mathcal{M} \to A$ is chosen to be as described above. We set $z_{n,\alpha} := f_n(z_\alpha) \in A_n, \alpha \in J_q$, for $q \leq n$ and $w_{n,\beta} := f_n(w_\beta), \beta \in J'_q$ for all $q \leq n-1$. Then $f_{m,n}(z_{n,\alpha}) = z_{m,\alpha}$ and $f_{m,n}(w_{n,\beta}) = w_{m,\beta}$. Moreover, setting $V_n^q := \oplus_{\alpha\in J_q} \mathbb{Q} z_{n,\alpha} \oplus_{\beta \in J'_q} \mathbb{Q} v_{n,q}, q \leq n$, we obtain a dgca $\Lambda(\oplus_{2\leq q\leq n} V_n^q)$ where the differential is defined as $dz_{n,\alpha} = 0, dv_{n,\beta} = u_{n,\beta}$. Also, we have a dgca morphism $\mu_n : \Lambda(\oplus_{2\leq q \leq n} V_n^q) \to A_n$ defined as $\mu_n(z_{n,\alpha}) = z_{n,\alpha} = f_n \circ \rho(z_\alpha)$ and $\mu_n(v_{n,\beta}) = w_{n,\beta}$. Since $f_n^* : H^q(A) \to H^q(A_n)$ is an isomorphism for $q \leq n$, it follows that μ_n induces isomorphism in dimensions $q < n$. Denoting the minimal model of A_n by $\mathcal{M}_n$, it is clear from the stepwise construction of $\mathcal{M}_n$ that $\mathcal{M}^{<n} := \Lambda(\oplus_{q<n} V_{n,q})$ and that the quasi-isomorphism $\rho_n : \mathcal{M}_n \to A_n$ may be chosen to agree with μ_n on $\mathcal{M}_n^{<n}$. Thus we have a commuting diagram of dgca morphisms

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Phi_n} & \mathcal{M}_n \\ \rho\downarrow & & \downarrow\rho_n \\ A & \xrightarrow{f_n} & A_n \end{array}$$

in which $\Phi_n|_{V^q}$ is an isomorphism for $q < n$ where $z_\alpha \mapsto z_{n,\alpha}, \alpha \in J_q$, and $v_\beta \mapsto v_{n,\beta}, \beta \in J'_q$. Furthermore when $m \leq n$ there are dgca morphism $\Phi_{m,n} : \mathcal{M}_n \to \mathcal{M}_m$ covering $f_{m,n} : A_n \to A_m$ such that $\Phi_{m,n} \circ \Phi_n|_{V^q} = \Phi_m|_{V^q}$ for $q < m \leq n$ and $\Phi_n|_{\mathcal{M}^{<n}}, \Phi_{m,n}|_{\mathcal{M}_n^{\leq m}}$ are *isomorphisms* of dgc algebras onto $\mathcal{M}_n^{<n}$ and $\mathcal{M}_m^{<m}$ respectively. We denote by $\Phi_{m,n}^{\leq q}$ the dgca morphism $\Phi_{m,n}|_{\mathcal{M}_n^{\leq q}} : \mathcal{M}_{m,n}^{\leq q} \to \mathcal{M}_m^{\leq q}$.

Lemma 4.3. *Let* $A, \{A_n, f_{m,n}\}, \{\mathcal{M}_n, \Phi_{m,n}\}$ *be as above. Suppose that* $H^0(A) = \mathbb{Q}, H^1(A) = 0$. *Then: (i) The minimal model* $\mathcal{M}$ *is the inverse limit of* $\{\mathcal{M}_n, \Phi_{m,n}\}$. *(ii) If each* A_n *is formal, then so is* A.

Proof. (i) From the above discussion it is clear that $\mathcal{M}^{\leq q}$ is isomorphic to the inverse limit of $\{\mathcal{M}_n^{\leq q}, \Phi_{m,n}^{\leq q}\}$ as a dgc algebra. It follows that $\mathcal{M}$ is isomorphic to the inverse limit of $\{\mathcal{M}_n, \Phi_{m,n}\}$.

(ii) Choose dgca morphisms $\phi_n : \mathcal{M}_n \to H^*(A_n)$ so that ϕ_n induces isomorphism in cohomology and $\phi_n(z_{n,\alpha}) = [z_{n,\alpha}], \phi_n(v_\beta) = 0$ for $\alpha \in J_q, \beta \in J'_q, q < n$. Then we have a diagram

$$\begin{array}{ccc} \mathcal{M}_n & \overset{\Phi_{m,n}}{\to} & \mathcal{M}_m \\ \phi_n \downarrow & & \downarrow \phi_m \\ H^*(A_n) & \overset{f^*_{m,n}}{\to} & H^*(A_m) \end{array}$$

which commutes for all $q < m \leq n$. In view of the assumption that $f_n^* : H^q(A) \to H^q(A_n)$ and $f^*_{m,n} : H^q(A_n) \to H^q(A_m)$ are isomorphisms for $q \leq m$, we see that the first assertion of the lemma implies the second. □

In our applications, A will equal $A_{PL}(X)$ for a simply connected CW complex X, $(A_n, d) = A_{PL}(X^{(n)})$, $A_n \overset{f_{m,n}}{\to} A_m$, $n \geq m$, $A \overset{f_n}{\to} A_n$, where $f_{m,n}$ is $A_{PL}(i_{m,n}) : A_{PL}(X^{(n)}) \to A_{PL}(X^{(m)})$ ($i_{m,n}$ is the inclusion of $X^{(m)}$ into $X^{(n)}$) and f_n is $A_{PL}(i_n) : A_{PL}(X) \to A_{PL}(X^{(n)})$ (i_n is the inclusion of $X^{(n)}$ into X). Therefore, if X is a simply connected CW complex such that each skeleton $X^{(n)}$ is formal, then X is also formal.

5. EXAMPLES OF FORMAL AND NON-FORMAL SPACES AND MAPS

In this chapter we briefly recall some well-known examples of formal and non-formal spaces and maps.

5.1. **Examples of formal spaces.** The following spaces are well-known to be formal.

(1) Product of two formal spaces is formal if at least one of them has the rational homology of finite type. An arbitrary wedge of formal spaces is formal. (See [7])

(2) Spheres $\mathbb{S}^n$ are formal. So their minimal models can be computed from the cohomology algebras, which are as follows:
For n is odd, $\mathcal{M}_{\mathbb{S}^n} = \Lambda(e)$, deg $e = n$, $d(e) = 0$.
For n is even, $\mathcal{M}_{\mathbb{S}^n} = \Lambda(e, e')$, deg $e = n$, deg $e' = 2n-1$, $d(e) = 0$, $d(e') = e^2$.
(See [7]).

(3) A path-connected H-space X with rational homology of finite type (in particular Eilenberg-MacLane space $K(\pi, n)$, we assume π is abelian when $n = 1$) is formal. The minimal model of X is as follows:
We choose a graded vector space $V \subset H^+(X)$ so that $H^+(X) = V \oplus H^+(X).H^+(X)$.
Then $(\Lambda V, 0)$ will be the minimal model of X. (See [7]).

(4) Suppose that τ is an involution of a compact connected Lie group G, and that K is the connected component of the identity in the subgroup of elements fixed by τ. Then G/K is called a symmetric space of compact type. E. Cartan showed that a symmetric space of compact type is formal.

(5) Suppose that M is a complex manifold with almost complex structure $J : TM \to TM$ so that $J^2 = -1$. Let $\langle , \rangle$ be a Riemannian metric such that $\langle J\xi, J\eta \rangle = \langle \xi, \eta \rangle$. Define $\omega \in A^2_{DR}(M)$ by $\omega(\xi, \eta) = \langle J\xi, \eta \rangle$. The manifold M is called *Kähler*

if $\langle,\rangle$ can be chosen so that $d\omega = 0$. Deligne, Griffiths, Morgan and Sullivan [6] showed that compact Kähler manifolds are formal. In particular, all smooth complex projective varieties are formal. Examples of such spaces are the complex projective spaces, complex Grassmann manifolds $G_{n,k}$, complex flag varieties and complex projective toric varietis. The minimal models of complex projective spaces as follows.
For $\mathbb{CP}^n$, $\mathcal{M}_{\mathbb{CP}^n} = \Lambda(u,v)$, deg $u = 2$, deg $v = 2n+1$, $d(u) = 0$, $d(v) = u^{n+1}$.
For $\mathbb{CP}^\infty$, $\mathcal{M}_{\mathbb{CP}^\infty} = \Lambda(u)$, deg $u = 2$, $d(u) = 0$.

(6) Simply connected compact manifolds of dimension ≤ 6 are formal (See [22]).

(7) There is a natural action of $(\mathbb{S}^1)^n$ on $(\mathbb{R}^2)^n$, called the *standard representation*. Let M be a $2n$-dimensional connected compact C^∞ manifold and $(\mathbb{S}^1)^n$ acts on M. A local isomorphism of M with the standard representation consists of:
a) $(\mathbb{S}^1)^n$ stable open sets U_1 in M and U_2in $(\mathbb{R}^2)^n$ and
b) an automorphism $\theta : (\mathbb{S}^1)^n \to (\mathbb{S}^1)^n$,
c) a θ-equivariant homeomorphism $f : U_1 \to U_2$ (i.e.$f(tx) = \theta(t)f(x)$) $\forall t \in (\mathbb{S}^1)^n$, $\forall x \in U_1$.
Now, M is said to be a *quasitoric manifold* if each point of M is in the domain of a local isomorphism and the orbit space of this action is homeomorphic to a simple polytope P. The manifold M can be reconstructed from the polytope P and a combinatorial data that codes the action of $(\mathbb{S}^1)^n$ on M. This is the topological analogue of a smooth projective toric variety. Panov and Ray proved that quasitoric manifolds are formal (See [24]).

(8) A dgc algebra (A,d) is said to be *intrinsically formal* if any dgc algebra (B,d) with cohomology algebra isomorphic to $H^*(A,d)$ is formal. For example, $(A,d) = \Lambda(x_1,x_2,\ldots,x_n;\, y_1,y_2,\ldots,y_m), |x_i| = even, |y_j| = odd, dx_i = 0, dy_j = P_j$, where $\{P_j\}$ form a *regular* sequence in $\mathbb{Q}[x_1,x_2,\ldots,x_n]$ (i.e. P_i is not a zero divisor in $\mathbb{Q}[x_1,x_2,\ldots,x_n]/\langle P_1,P_2,\ldots,P_{i-1}\rangle$) is intrinsically formal. A complex Grassmann manifold is intrinsically formal as it has the cohomology algebra of the above form (See [30], p.317). (The cohomology algebra of complex Grassmann manifold will

be discussed in §9.2 and §9.3 of this thesis.) Naturally, an intrinsically formal dgc algebra is formal. The minimal model of a complex Grassmann manifold $G_{n,k}$ ($2k \leq n$) is the following.

$\mathcal{M}_{G_{n,k}} = \Lambda(c_1, c_2, \cdots, c_k; y_1, y_2, \cdots, y_k)$, deg $c_i = 2i$, deg $y_j = 2(n-k+j)-1$, $d(c_i) = 0$, $d(y_j) = P_j$ $\forall i, j \leq k$, where the P_j are certain elements which form a regular sequence in $\Lambda(c_1, c_2, \cdots, c_k)$.

The minimal model for $BU(n)$ is

$\mathcal{M}_{BU(n)} = \Lambda(c_1, c_2, \cdots, c_n)$, where deg $c_i = 2i$, $d(c_i) = 0$.

(9) A skeleton of a formal connected CW complex is formal (See Lemma 6.5, [18]).

5.2. **Examples of non-formal spaces.** (1) For formal spaces all Massey triple products vanish. It is known that if $X = \mathbb{S}^3 - \mathbb{B}$, where $\mathbb{B}$ is Borromean ring, then there is a certain Massey triple product in $H^*(X)$ which does not vanish. This shows that X is not a formal space.

(2) Not all simply connected homogeneous spaces are formal. It is shown in ([23], Proposition 4.15) that $M = SU(6)/(SU(3) \times SU(3))$ is not formal by exhibiting a non-vanishing Massey triple product.

(3) Recall that a symplectic manifold is a 2n dimensional manifold M equipped with a $\omega \in A^2_{DR}(M)$ satisfying $d\omega = 0$ and $\omega^n_x \neq 0$, $x \in M$. I.K. Babenko and I.A.Taimanov [2] have constructed, for any $n \geq 5$, infinitely many pairwise non homotopy equivalent non formal simply connected symplectic manifolds of dimension 2n.

(4) We shall construct CW complexes which are non-formal spaces in §8.3. In our examples all Massey triple products vanish; in fact they have non-zero cohomology groups in even degrees only.

5.3. **Examples of formal maps.** Here spaces are considered to be simply connected, path connected and with rational homology of finite type. Without giving details of proofs we give the following examples.

(1) If $f : X \to Y$ and $g : X \to Z$ be two formal maps, then $h : X \to Y \times Z$ defined by $h(x) = (f(x), g(x)), x \in X$ is formal, where all of X, Y, Z.

(2) If $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ are formal maps, $f : X_1 \bigvee X_2 \to Y$ defined by $f|X_i = f_i$, then f is again a formal map. If X and Y are formal maps, then the projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are formal maps.

(3) Every continuous map in $[X, Y]$, where Y is $\mathbb{S}^n$ (n is odd), path-connected H-space with rational homology of finite type (in particular Eilenberg-MacLane space $K(\pi, n)$, where we assume π is abelian when $n = 1$) and X is any space, is formal. Maps between complex Grassmann manifolds are formal (We shall prove this in §10.2).

5.4. **Examples of non formal map.** (1) Hopf fibration $f : \mathbb{S}^3 \to \mathbb{S}^2$ is not a formal. More generally, any $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$, which represents a non-torsion element in $\pi_{4n-1}(\mathbb{S}^{2n})$, is not formal.

(2) Let X be a simply connected finite CW complex. Suppose dim $X = d$. If $n > d$ is such that $\pi_n(X) \otimes \mathbb{Q} \neq 0$, then there is non-formal maps $f : \mathbb{S}^n \to X$. If X is of rationally hyperbolic type, these non-formal maps exist for infinitely many n.

6. A MODEL FOR CELL ATTACHMENT

Let X be a simply connected topological space. Let $Y = X \cup_\alpha e^n$ where $\alpha : \mathbb{S}^{n-1} \to X$ represents an element $[\alpha] \in \pi_{n-1}(X)$. We assume that $n \geq 2$ so that Y is also simply connected. We recall the following proposition which will play a crucial role in our proofs. Let $\rho_X : (\mathcal{M}_X, d) \to (A_{\mathrm{PL}}(X), d)$ be a minimal Sullivan model for X. Suppose that $\mathcal{M}_X = \Lambda(V)$ so that $V = \oplus_{k\geq 2} V^k$. (Note that $V^1 = 0$ since X is simply connected.) We have the pairing $\langle -, - \rangle : V^k \times \pi_k^{\mathbb{Q}}(X) \to \mathbb{Q}$ defined by evaluation.

Let $n \geq 3$. Let $M_\alpha = \Lambda(V_\alpha)$ be the dgca defined as follows: $V_\alpha := V \oplus \mathbb{Q}u_\alpha$, $\deg(u_\alpha) = n$, $u_\alpha^2 = u_\alpha.v = 0, v \in V$, with differential d_α where $d_\alpha(u_\alpha) = 0$ and $d_\alpha(v) = dv + \langle v, \alpha \rangle u_\alpha, \quad v \in V$.

Proposition 6.1. *The dgca (M_α, d_α) defined above is a model for $Y = X \cup_\alpha e^n$. Moreover, one has the following diagram of dgc algebras in which the rows are exact and the vertical arrows are quasi-isomorphism:*

$$
\begin{array}{ccccccc}
0 \to & \mathbb{Q}u_\alpha & \hookrightarrow & M_\alpha & \overset{\lambda}{\to} & \mathcal{M}_X & \to 0 \\
& \updownarrow\simeq & & \updownarrow \mu & & \downarrow \rho_X & \\
0 \to & A_{PL}(Y, X) & \overset{A_{PL}(j)}{\to} & A_{PL}(Y) & \overset{A_{PL}(i)}{\to} & A_{PL}(X) & \to 0 \\
& \downarrow\simeq & & & & & \\
& A_{PL}(D^n, \mathbb{S}^{n-1}) & & & & &
\end{array}
$$

where $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow (Y, X)$ are inclusions and λ is induced by projection $V_\alpha \to V$. The induced diagram

$$\begin{array}{ccccccc}
0 \to & \mathbb{Q}u_\alpha & \hookrightarrow & H^*(M_\alpha) & \xrightarrow{\lambda^*} & H^*(\mathcal{M}_X) & \to 0 \\
& \downarrow\cong & & \downarrow \mu^* & & \downarrow \rho_X^* & \\
0 \to & H^*(Y,X) & \xrightarrow{j^*} & H^*(Y) & \xrightarrow{i^*} & H^*(X) & \to 0 \\
& \downarrow\cong & & & & & \\
& H^*(D^n, \mathbb{S}^{n-1}) & & & & &
\end{array} \tag{1}$$

is commutative with exact rows in which the vertical arrows are all isomorphisms.

□

We refer the reader to [7, Chapter 13] for a proof.

Remark 6.2. *The dgca M_α is not a minimal model for Y most often. Indeed it is not free except in the case $V = 0$ and n odd, since $u_\alpha^2 = 0$ and the relation $u_\alpha.v = 0$ holds for $v \in V$.*

7. FORMALITY OF CELL ATTACHMENT

We come to one of the main results of this thesis, namely theorem 7.2. Recall the notion of a standard lower gradation of simply connected minimal Sullivan algebra $(\Lambda V, d)$ (See §4.2). We introduce the notion of a special element:

Definition 7.1. *(1) Let $(\Lambda V, d)$ be a minimal Sullivan algebra with standard lower gradation $V = \oplus_{k\geq 0} V_k$. We say that an element $\alpha \in Hom(V^n;\mathbb{Q})$ is special if $\langle v, \alpha\rangle = 0$ $\forall v \in V_k, k \neq 1$.*

(2) Let X be a simply connected space with rational homology of finite type. Recall from §3.2, $\pi_n^{\mathbb{Q}}(X) \cong Hom(V^n;\mathbb{Q})$. We say $\alpha \in \pi_n^{\mathbb{Q}}(X)$ is special if $\langle v, \alpha\rangle = 0$ $\forall v \in V_k, k \neq 1$, where $\mathcal{M}_X = \Lambda V$ and $V = \oplus_{k\geq 0} V_k$ is standard lower gradation.

Let X be a simply connected topological space with rational homology of finite type, which is formal. Let $\alpha : \mathbb{S}^{n-1} \to X$ represent an element in the kernel of the Hurewicz homomorphism $\eta : \pi_{n-1}^{\mathbb{Q}}(X) \to H_{n-1}(X;\mathbb{Q})$. Let $Y = X \cup_\alpha e^n$. Let $i : X \hookrightarrow Y$ be the inclusion. We have the inclusion map $j : Y \hookrightarrow (Y, X)$ and the characteristic map $(D^n, \mathbb{S}^{n-1}) \to (Y, X)$. Then $j^* : H^n(Y, X;\mathbb{Z}) \to H^n(Y;\mathbb{Z})$ maps the positive generator $\widetilde{u} \in H^n(Y, X;\mathbb{Z}) \cong H^n(D, \mathbb{S}^{n-1}) \cong \mathbb{Z}$ to a non-zero element u in $H^n(Y;\mathbb{Z})$.

Our main result is the following. Recall that a minimal model of a simply-connected space is isomorphic as a graded algebra to ΛV where V is a graded $\mathbb{Q}$-vector space $V = \oplus_{k\geq 2} V^k$ and ΛV stands for the free graded-commutative algebra over V. Here we give conditions for formality (and non-formality) of Y.

Theorem 7.2. *Suppose that X is a simply connected CW complex and is formal. Let $\mathcal{M}_X = \Lambda(V)$ and suppose that $V = \oplus_{k\geq 0} V_k$ is a standard lower gradation of V. Let $Y = X \cup_\alpha e^n$. Suppose that $\eta([\alpha]) = 0$ so that $j^*(\widetilde{u}) =: u \neq 0$. (i) If $[\alpha] \in \pi_{n-1}(X)$ is a torsion element then u is indecomposable and Y is formal. (ii)*

Let $[\alpha] \neq 0$ in $\pi_{n-1}^{\mathbb{Q}}(X)$. Suppose that α is special and that u is decomposable in $H^*(Y;\mathbb{Q})$. Then Y is formal. (iii) If $[\alpha] \in \pi_{n-1}(X)$ is not a torsion element and u is not decomposable, then Y is not formal.

As a corollary we shall prove,

Theorem 7.3. *Let X be a connected finite CW complex having cells only in even dimensions. If $H^*(X)$ is generated by $H^{2k}(X)$ and $\dim X \leq 4k$, then X is formal.*

Remark 7.4. *(i) By a result of Halperin and Stasheff* [13, Theorem 1.5] *a nilpotent finite CW complex with only odd dimensional cells in positive dimensions is formal. Such a CW complex is in fact rationally equivalent to a bouquet of odd-dimensional spheres. Halperin and Stasheff point out that this result has also been obtained independently by Baus.*
(ii) Papadima [25] *has obtained a criterion for the formality of cell attachments. He also considers spaces X whose cohomology algebra is generated by degree k elements and remarks that formality of such spaces can be obtained under the hypothesis that $k \geq c$ (resp. $c-1$) where c is the rational cup-length of X (resp. when X is a Poincaré duality space).*

Let $\alpha : \mathbb{S}^{n-1} \to Z$, where $n \geq 2$. Suppose that both Z and X are simply connected, formal spaces with rational homology of finite type. Let $\mathcal{M}_Z = (\Lambda U, d)$ and $\mathcal{M}_X = (\Lambda V, d)$ be minimal models of Z and X respectively with standard lower gradation.

Theorem 7.5. *Suppose that $[\alpha] \in \pi_{n-1}^{\mathbb{Q}}(Z)$ is special and $f : Z \to X$ is formal. Then $f_*([\alpha])$ is special.*

8. PROOFS OF MAIN RESULTS

8.1. **Proof of Theorem 7.2.** Here $H_*(X)$ (and $H^*(X)$) denote the singular homology (and cohomology respectively) of X with $\mathbb{Q}$ -coefficients. We keep the notations and set-up of §6. It is understood that a base point for X is chosen and fixed and it serves as the point for $Y = X \cup_\alpha e^n$ as well; the homotopy groups are defined with respect to this choice and will be suppressed in the notation $\pi_k(X)$, etc. Recall that i denotes the inclusion $X \hookrightarrow Y$. Also V and W are graded vector spaces so that $\mathcal{M}_X = \Lambda(V)$ and $\mathcal{M}_Y = \Lambda(W)$. We shall use the construction of minimal model of $(H^*(X), 0)$ as given in §4.2 as here X is a formal space. We assume that $V = \oplus V_k$ and $W = \oplus W_k$ are standard lower gradations. One has a morphism of dgca $\phi : \mathcal{M}_Y \to \mathcal{M}_X$ which is a lift of $A_{\mathrm{PL}}(i) : A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$ (i.e. ϕ is the Sullivan representative of $A_{PL}(i)$). The linear part $Q(\phi) : W \to V$ of ϕ is defined by the requirement that $\phi(w) - Q(\phi(w)) \in \Lambda^{\geq 2} V$; it induces $i^* : \mathrm{Hom}(\pi_k(Y), \mathbb{Q}) \to \mathrm{Hom}(\pi_k(X), \mathbb{Q})$ for all k under the isomorphisms $V^k \cong \mathrm{Hom}(\pi_k(X), \mathbb{Q})$ and $W^k \cong \mathrm{Hom}(\pi_k(Y), \mathbb{Q})$.

Recall that X is simply connected. By the relative Hurewicz theorem, we obtain that $\eta : \pi_n(Y, X) \cong H_n(Y, X; \mathbb{Z}) \cong \mathbb{Z}$. The group $\pi_n(Y, X) = \mathbb{Z}$ is generated by the homotopy class of the characteristic map $\widetilde{\alpha} : (D^n, \mathbb{S}^{n-1}) \to (Y, X)$ of the cell e_α. The homomorphism $\partial : \pi_n(Y, X) \to \pi_{n-1}(X)$ maps $[\widetilde{\alpha}]$ to $[\alpha]$. Denoting by $\pi_k^{\mathbb{Q}}$ the rational homotopy group functor $\pi_k(-) \otimes \mathbb{Q}$, we have the commuting diagram

$$
\begin{array}{ccccccc}
 & & \pi_n^{\mathbb{Q}}(Y) & \xrightarrow{j_*} & \pi_n^{\mathbb{Q}}(Y, X) & \xrightarrow{\partial} & \pi_{n-1}^{\mathbb{Q}}(X) \\
 & & \eta \downarrow & & \eta \downarrow & & \downarrow \eta \\
H_n(X) & \to & H_n(Y) & \xrightarrow{j_*} & H_n(Y, X) & \xrightarrow{\partial} & H_{n-1}(X).
\end{array} \tag{2}
$$

where η denotes the Hurewicz homomorphism, with the middle one being an isomorphism.

Suppose that $\eta([\alpha]) \neq 0$. Then $\partial(\eta[\widetilde{\alpha}]) = \eta([\alpha]) \neq 0$ and since $H_n(Y,X) \cong \mathbb{Q}$, we conclude that $j_* = 0$ and $i_* : H_n(X) \to H_n(Y)$ is an isomorphism since $H_{n+1}(Y,X) = 0$. Therefore $i^* : H^n(Y) \to H^n(X)$ is also an isomorphism.

Suppose that $\eta([\alpha]) = 0$. This happens, for example, when α is a torsion element or when $H_{n-1}(X) = 0$. (However $\eta[\alpha] = 0$ does not imply that α is of finite order. For example, one can choose α to be an element of infinite order in $\pi_{4m-1}(\mathbb{S}^{2m})$.) There exists an element $\gamma \in H_n(Y)$ such that $j_*(\gamma) = \eta([\widetilde{\alpha}])$. Let $\widetilde{u}$ denote the generator of $H^n(Y,X) = \mathrm{Hom}(H_n(Y,X),\mathbb{Q}) \cong \mathbb{Q}$ such that $\langle \widetilde{u}, \eta([\widetilde{\alpha}])\rangle = 1$. Then $j^*(\widetilde{u}) = u$ is a non-zero element of $H^n(Y)$ and we have $\langle u, \gamma\rangle = 1$. Therefore, using the exact sequence $H^{n-1}(Y,X) \xrightarrow{j^*} H^{n-1}(Y) \xrightarrow{i^*} H^{n-1}(X) \to H^n(Y,X) \xrightarrow{j^*} H^n(Y) \xrightarrow{i^*} H^n(X) \to H^{n+1}(Y,X) = 0$, we have

$$H^n(Y) \cong \begin{cases} H^n(X) \oplus \mathbb{Q}u & \text{if } \eta([\alpha]) = 0, \\ H^n(X) & \text{if } \eta([\alpha]) \neq 0, \end{cases} \tag{3}$$

and

$$H^{n-1}(X) \cong \begin{cases} H^{n-1}(Y) & \text{if } \eta([\alpha]) = 0, \\ H^{n-1}(Y) \oplus \mathbb{Q}\widetilde{u} & \text{if } \eta([\alpha]) \neq 0. \end{cases} \tag{4}$$

Since $\mathrm{Hom}(\pi_{n-1}(X),\mathbb{Q}) \cong V^{n-1}$, using the exactness of the sequence $\pi_n^{\mathbb{Q}}(Y,X) \xrightarrow{\partial} \pi_{n-1}^{\mathbb{Q}}(X) \xrightarrow{i_*} \pi_{n-1}^{\mathbb{Q}}(Y) \to \pi_{n-1}^{\mathbb{Q}}(Y,X) = 0$ we see that

$$V^{n-1} \cong \begin{cases} W^{n-1} \oplus \mathbb{Q} & \text{if } [\alpha] \neq 0, \\ W^{n-1} & \text{if } [\alpha] = 0, \end{cases} \tag{5}$$

via the restriction of $Q(\phi)$.

Summarizing the above discussion we obtain the following.

Lemma 8.1. *(i) Suppose that $\eta[\alpha] = 0$. Then $j^*(\widetilde{u}) = u \neq 0$ and $H^n(Y) \cong H^n(X) \oplus \mathbb{Q}u$, $H^k(Y) \cong H^k(X), k \neq n$. If $[\alpha] \neq 0$ in $\pi_{n-1}^{\mathbb{Q}}(X)$, then $V^{n-1} \cong W^{n-1} \oplus \mathbb{Q}$.*
(ii) Suppose that $\eta[\alpha] \neq 0$. Then $j^*(\widetilde{u}) = 0$ and $H^k(Y) \cong H^k(X), k \neq n-1$, $H^{n-1}(X) \cong H^{n-1}(Y) \oplus \mathbb{Q}[\widetilde{u}]$. Moreover, $V^{n-1} \cong W^{n-1} \oplus \mathbb{Q}$. $\square$

We now establish Theorem 7.2.

Proof of Theorem 7.2. (i) If $[\alpha] \in \pi_{n-1}(X)$ is a torsion element then Y is rational homotopically equivalent to $X_0 \vee \mathbb{S}^n$. Hence Y is formal.

(ii) In this case $Q(\phi) : W^k \to V^k$ is an isomorphism for $k \leq n-2$ and is a monomorphism when $k = n-1$. Moreover, $Q(\phi)(W^{n-1}) = \ker([\alpha]) \subset V^{n-1}$ has codimension 1. Write $Q(\phi)(W^{n-1}) \oplus \mathbb{Q}v_\alpha = V^{n-1}$ where $v_\alpha \in V^{n-1}$ is an element such that $\langle v_\alpha, [\alpha]\rangle = 1$. (We shall presently make a more specific choice of v_α.) Write $u = P(\overline{v}_1, \ldots, \overline{v}_r)$ with $\overline{v}_q \in H^{<(n-1)}(Y) \cong H^{<(n-1)}(X)$ where $\overline{v}_q$ are indecomposable elements. Since $\Lambda(V_0) \to H^*(X)$ is onto, we choose cocycles $v_q \in V_0 \subset \mathcal{M}_X$ so that $v_q \mapsto \overline{v}_q$. We set $w = P(v_1, \ldots, v_r) \in \mathcal{M}_X$. Since X is formal we have a quasi-isomorphism $\Phi : (\mathcal{M}_X, d) \to (H^*(X), 0)$. Since $i^*(u) = 0$ we have $\Phi(P(v_1, \ldots, v_r)) = 0$ in $H^n(X)$. That is, $P(v_1, \ldots, v_r) =: w \in \ker(\Phi)$. Note that $w \in \mathcal{M}_X$ is a cocyle. Since Φ^* is a monomorphism, $w = d_X(v_\alpha)$ for some $v_\alpha \in (\Lambda V(1))^{n-1} \subset \mathcal{M}_X^{n-1}$. We claim that $\langle v_\alpha, [\alpha]\rangle \neq 0$. Indeed, since $\mu : (M_\alpha, d_\alpha) \leftrightarrow A_{\mathrm{PL}}(Y)$ is a quasi-isomorphism we have, using the commutative diagram (1) of 6.1, that $\mu^*([w]) = P(\overline{v}_1, \ldots, \overline{v}_r) = u \in H^n(Y)$ is non-zero. So $[w] \neq 0$ in $H^n(M_\alpha)$. If $\langle v_\alpha, [\alpha]\rangle = 0$, then $d_\alpha(v_\alpha) = d_X(v_\alpha) = w$, whence $\mu^*([w]) = 0$ in $H^n(M_\alpha)$, a contradiction. Therefore $\langle v_\alpha, [\alpha]\rangle \neq 0$. Now this implies that $v_\alpha \notin Q(\phi)(W) = \ker([\alpha])$. We can write $v_\alpha = v_1 + v_2$ where $v_1 \in V_1$ and $v_2 \in \Lambda^+ V_0.V_1$. If $\langle v_1, [\alpha]\rangle = 0$, then

$$d_\alpha(v_\alpha) = d_\alpha(v_1) + d_\alpha(v_2) = d_X(v_1) + d_X(v_2) = d_X(v_1 + v_2) = d_X(v_\alpha)$$

showing that $d_\alpha(v_\alpha) = d_X(v_\alpha) = \omega$, hence $\mu^*([\omega]) = 0$ in $H^n(\mathcal{M}_\alpha)$, a contradiction. Therefore, we can choose $v_\alpha \in V_1^{n-1}$ so that $\langle v_\alpha, [\alpha]\rangle = 0$.

By hypothesis $\langle v, [\alpha]\rangle = 0 \; \forall v \in V_0^{n-1} \oplus (\oplus_{k\geq 2} V_k^{n-1})$.

The surjective homomorphism $i^* : H^n(Y) \to H^n(X)$ induces an isomorphism $H^n(Y)/\mathcal{D} \to H^n(X)/\mathcal{D} \cong V_0^n$. (Here $\mathcal{D}$ stands for the space of decomposable elements.) Choose a linear map $\theta' : V_0^n \to H^n(Y)$ such that $\Phi(v) = i^*(\theta'(v)), v \in V_0^n$ and extend it to a linear map $\theta : V^n \to H^n(Y)$ by setting $\theta(v) = 0$ for $v \in \oplus_{k\geq 1} V_k^n$.

Define a vector space homomorphism $\psi : V \oplus \mathbb{Q}u_\alpha \to H^*(Y)$ as follows:

$$\psi(v) = \begin{cases} (i^*)^{-1} \circ \Phi(v) & \text{if } \ v \in V^k, \quad k \neq n, \\ \theta(v) & \text{if } \ v \in V^n, \\ -u/\langle v_\alpha, [\alpha]\rangle & \text{if } \ v = u_\alpha. \end{cases}$$

This extends to a homomorphism $M_\alpha \to H^*(Y)$, again denoted by ψ, of the graded commutative algebra M_α because the relations $u^2 = 0, u.z = 0$ for all $z \in H^*(Y)$ hold. Note that $\psi(w) = u$.

We claim that ψ is a dgca morphism, that is, $\psi \circ d_\alpha = 0$. Clearly $\psi(d_\alpha(u_\alpha)) = \psi(0) = 0$. Let $v \in V^k, k \neq n-1$. Then $d_\alpha(v) = d_X v$ and so $\psi(d_\alpha(v)) = (i^*)^{-1}(\Phi(d_X v)) = 0$. If $v \in V_j^{n-1}, j \neq 1$, then $d_\alpha(v) = d_X v$ since $v \in \ker([\alpha])$. If $j = 0$, then $d_X(v) = 0$ whence $\psi(d_\alpha v) = 0$. Assume that $j > 1$. Since $V = \oplus_{k\geq 0} V^k$ is a standard lower gradation, we see that $d_X v$ is a sum of monomials in each of which *there is a factor belonging to* $V_i, 1 \leq i < j$, *present* by Lemma 4.1. Therefore $\psi(d_X v) = 0$.

Finally, let $v \in V_1^{n-1} = V_1^{n-1} \cap \ker([\alpha]) \oplus \mathbb{Q}v_\alpha$. Suppose $v \in \ker([\alpha])$. Then $d_\alpha(v) = d_X v = f(w_1, \ldots, w_s)$. Since $\mu^* : H^*(M_\alpha) \to H^*(Y)$ is an isomorphism of algebras which agrees with $H^*(\mathcal{M}_X) \to H^*(X)$ in degrees less than $n-1$, we see that $0 = [d_\alpha v]$ in $H^*(M_\alpha)$ implies that $[d_X v] = 0$ in $H^n(Y)$. On the other hand $\psi(d_X v) = \psi(f(w_1, \ldots, w_s)) = f(\psi(w_1), \ldots, \psi(w_s))$ is the image of the element $f(w_1, \ldots, w_s)$ under μ^*. As $f(w_1, \ldots, w_s) = d_\alpha v$, we conclude that $\psi(d_\alpha v) = 0$.

It remains to consider the case $v = v_\alpha$. Then $d_\alpha(v_\alpha) = d_X v_\alpha + \langle v_\alpha, [\alpha]\rangle u_\alpha = w + \langle v_\alpha, [\alpha]\rangle u$. It follows that $\psi(d_\alpha v_\alpha) = \psi(w) - u = 0$. It is clear that ψ induces isomorphism in cohomology.

Since $M_\alpha \simeq \mathcal{M}_Y$, there exists a dgca morphism $h : \mathcal{M}_Y \to M_\alpha$ which induces isomorphism in cohomology. Then $\psi \circ h : \mathcal{M}_Y \to H^*(Y)$ induces isomorphism in cohomology.

(iii) Let $[\alpha] \neq 0$ in $\pi_{n-1}^{\mathbb{Q}}(X)$. Assume that $j^*(\widetilde{u}) = u$ is *not* decomposable, and that Y is formal. We shall arrive at a contradiction. Recall that by Proposition 6.1, $\mu : M_\alpha \leftrightarrow A_{\mathrm{PL}}(Y)$ is a quasi-isomorphism.

Let $\nu : \mathcal{M}_Y \to M_\alpha$ and $\Psi : \mathcal{M}_Y \to H^*(Y)$ be quasi-isomorphisms so that $\mu^* \circ \nu^* = \Psi^*$. Let $\lambda : M_\alpha \to \mathcal{M}_X$ be the dgca morphism considered in Proposition 6.1. Then $\phi = \lambda \circ \nu : \mathcal{M}_Y \to \mathcal{M}_X$ is a lift of $A_{\mathrm{PL}}(i) : A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$ and induces $i^* : H^*(Y) \to H^*(X)$. From (5), we have an isomorphism $V^{n-1} \cong W^{n-1} \oplus \mathbb{Q}$ given by $Q(\phi)$. Also $V^k \cong W^k$ if $k \leq n-2$.

Consider the dgca morphism $\iota : \Lambda(W^{\leq n-1}) \to M_\alpha$. Then ι^* is an isomorphism in dimension $\leq n-2$ and the cokernel of $\iota^* : H^n(\Lambda(W^{\leq n-1})) \to H^n(M_\alpha) \cong H^n(Y)$ is isomorphic to $\ker(d_Y) \cap W^n$. Since $[dv_\alpha] = u$ and since $dv_\alpha \in \Lambda(V^{\leq n-2}) = \Lambda(W^{\leq n-2})$, we see that u belongs to the image of (ι^*). Hence, if we try to construct a minimal model of $\mathcal{M}_\alpha$ following §4.1, we see that $\dim(coker(\iota^*)^n) = \dim(ker(d_Y) \cap W^n)$. Since u is indecomposable we see that $\dim(coker(\iota^*))^n < \dim(H^n(M_\alpha)/\mathcal{D}) = \dim(H^n(Y)/\mathcal{D})$. Therefore $\dim(\ker(d_Y) \cap W^n) < \dim(H^n(Y)/\mathcal{D})$. On the other hand, since Y is formal, $\dim(H^n(Y)/\mathcal{D}) = \dim(\ker(d_Y) \cap W^n)$. Therefore we conclude that Y cannot be formal. $\square$

Now, we shall define a morphism $\psi : \Lambda V \oplus \mathbb{Q}u_\alpha \to H^*(Y)$ for case (i). Choose a linear map $\theta' : V_0^n \to H^n(Y)$ such that $\Phi(v) = i^*(\theta'(v)), v \in V_0^n$ and extend it to a linear map $\theta : V^n \to H^n(Y)$ by setting $\theta(v) = 0$ for $v \in \oplus_{k \geq 1} V_k^n$.

Define a vector space homomorphism $\psi : V \oplus \mathbb{Q}u_\alpha \to H^*(Y)$ as follows:

$$\psi(v) = \begin{cases} (i^*)^{-1} \circ \Phi(v) & \text{if } \ v \in V^k, \quad k \neq n, \\ \theta(v) & \text{if } \ v \in V^n, \\ u & \text{if } \ v = u_\alpha. \end{cases}$$

This extends to a homomorphism $M_\alpha \to H^*(Y)$, again denoted by ψ, of the graded commutative algebra M_α because the relations $u^2 = 0, u.z = 0$ for all $z \in H^*(Y)$ hold. Now, we can show that ψ is a quasi-isomorphism as in case (ii).

Both in case (i) and (ii), we have the following diagram:

$$\begin{array}{ccccccc}
A_{\mathrm{PL}}(Y) & \stackrel{\rho_Y}{\leftarrow} & \mathcal{M}_Y & \stackrel{\mu}{\to} & M_\alpha & \stackrel{\psi}{\to} & H^*(Y) \\
A_{\mathrm{PL}}(i)\downarrow & & \phi\downarrow & & \lambda\downarrow & & \downarrow i^* \\
A_{\mathrm{PL}}(X) & \stackrel{\rho_X}{\leftarrow} & \mathcal{M}_X & \stackrel{id}{\to} & \mathcal{M}_X & \stackrel{\Phi}{\to} & H^*(X).
\end{array}$$

The left and middle squares are homotopy commutative, whereas the right square is commutative.

Thus we have proved the following.

Theorem 8.2. *Let X and $\alpha : \mathbb{S}^{n-1} \to X$ are as in theorem 7.2. Then in cases (i) and (ii), the inclusion map $i : X \hookrightarrow Y$ is formal.*

As an application of Theorem 7.2 we shall prove the following theorem.

Theorem 8.3. *Let X be a connected finite CW complex having cells only in $2ik$ dimensions ($i = 0, 1, 2$) and $X^{(0)} = \{pt\}$. If $H^*(X)$ is generated by $H^{2k}(X)$ and $\dim X \leq 4k$, then X is formal.*

Proof. The $2k$-skeleton $X^{(2k)}$ of X is bouquet of $2k$-dimensional spheres and hence is formal. We may assume that X is of dimension $4k$. The minimal model $\mathcal{M} = \Lambda V$ of $X^{(2k)}$ has the property that for any standard lower gradation $V = \oplus_{j\geq 0} V_j$, we have $V_0 = V^{2k}, V_{\geq 2} \cap V^{4k-1} = 0$. So any element in $\pi_{4-1}(X^{2k})$ is special. (See Defn. 7.1). (Note that the Hurewicz homomorphism in dimension $4k-1$ vanishes since $H^{4k-1}(X) = 0$.) It follows that attaching any $4k$-cell to $X^{(2k)}$ results in a formal space. Note that any sub complex of X again has the property that its rational cohomology algebra is generated by degree $2k$ elements. We may regard $X = X^{(4k)}$ as obtained from successive cell-attachments $X_1, \ldots, X_s$ where X_{j+1} is obtained from

X_j (with $X_0 := X^{(2k)}$) by attaching a $4k$-cell, s being the number of $4k$-cells in X. We have just shown that X_1 is formal.

Inductively assume that X_j is formal. Writing $X_{j+1} = X_j \cup_\alpha e^{4k}$ for a suitable $\alpha \in \pi_{4k-1}(X_j)$, we need only to show that α is special. We shall again write $\mathcal{M} = \Lambda V$ for the minimal model of X_j. Once again $V_0 = V^{2k}$. We claim that $V_2 \cap V^{4k-1} = 0$ (and consequently $V_j \cap V^{4k-1} = 0$ for all $j > 2$). Indeed for dimension reasons $V_1 \cap V^p = 0$ for all $p < 4k-1$. It follows that there are no relations involving elements of $V_0 = V^{2k}$ and $V_1 \cap V^{4k-1}$ in dimensions less than $6k-1$. Thus any element of $\pi_{4k-1}(X_j)$ is special. It follows by Theorem 7.2 that X_{j+1} is special. $\square$

Now, if Y is a simply connected space with cohomology generated by $H^{2k}(Y)$ and $H^{>4k}(Y) = 0$ and *dim* $H^{2k}(Y) < \infty$, then by theorem 3.2, we have a rational homotopy equivalence $\iota : X \to Y$ with $X^{(0)} = X^{(1)} = \textit{point}$ and the differential in the integral cellular chain complex is identically zero. Then obviously X is a finite CW complex with cells only in dimensions $2k, 4k$ and $H^*(X)$ is generated by H^{2k}. Therefore by Theorem 8.3, X is formal. So, Y is formal too. We get the following corollary.

Corollary 8.4. *Let Y be a simply connected space with $H^*(Y)$ is generated by $H^{2k}(Y)$, $H^{>4k}(Y) = 0$ and dim $H^{2k}(Y) < \infty$ then Y is formal.*

Here we give a necessary criterion for formality of a map. Recall the definition of $Q(\phi) : W \to V$ for a dgca morphism $\phi : (\Lambda W, d) \to (\Lambda V, d)$ (See §2.1).

Lemma 8.5. *Suppose that $f : (B, 0) \to (A, 0)$ is a dgca morphism where B is of finite type. Let $\mathcal{M}_B = (\Lambda W, d)$ and $\mathcal{M}_A = (\Lambda V, d)$ be minimal models of B and A respectively with standard lower gradations. Then there is a Sullivan representative $\phi : \mathcal{M}_B \to \mathcal{M}_A$ of f such that*

$$\phi(W_0) \subset \Lambda V_0, \tag{6}$$

$$\phi(W_1) \subset \Lambda^+ V_{\geq 1}.\Lambda V_0, \tag{7}$$

$$Q(\phi)(W_{\geq 2}) \subset V_{\geq 2}. \tag{8}$$

Proof. Any Sullivan representative $\phi : \mathcal{M}_B \to \mathcal{M}_A$ is determined by its restriction to $\oplus_{k\geq 0} W_k$. We shall construct linear maps $\phi_k : W_k \to \Lambda V$ inductively for $k \geq 0$ and extend ϕ_k to an algebra homomorphism $\Lambda W(k) \to \Lambda V$, again denoted ϕ_k, extending $\phi_{k-1} : \Lambda W(k-1) \to \Lambda V$. From our construction it will follow that ϕ_k is a morphism of dgc algebras. It will be shown that the dgca morphism $\phi : \Lambda W = \cup_{k\geq 0} \Lambda W(k) \to \Lambda V$, defined as the union of the morphisms $\phi_k, k \geq 0$, meets the requirements of the lemma.

Fix a basis $\{w_j\}, j \in J_k$, of $W_k, k \geq 0$, and let $J = \cup_{k\geq 0} J_k$.

For $j \in J_0$ and choose $v_j \in \Lambda V_0$ such that $\rho_A(v_j) = f(\rho_B(w_j))$; such a v_j exists since $\Lambda V_j \to A = H^*(A, d)$ is surjective. Here $\rho_A : \mathcal{M}_A \to A$ and $\rho_B : \mathcal{M}_B \to B$ are quasi-isomorphisms inducing the identity map in cohomology. We define $\phi_0 : W_0 \to \Lambda V_0$ by $\phi_0(w_j) = v_j, j \in J_0$. Note that ϕ_0 satisfies (6). Clearly $d\phi_0(w_j) = dv_j = 0 = \phi_0(dw_j)$. The linear map ϕ_0 extends to a dgca morphism $\phi_0 : \Lambda W_0 \to \Lambda V$. Note that $f\rho_B = \rho_A \phi_0$.

Extend ϕ_0 to a dgca morphism $\psi_0 : \Lambda W \to \Lambda V$. We will use it to construct ϕ_1. Let $j \in J_1$. Then $dw_j \in \Lambda^+ W_0$. Write $\psi_0(w_j) = u_j + v_j$ so that $u_j \in \Lambda^+ V_0$, $v_j \in \Lambda^+ V_{\geq 1}.\Lambda V_0$. Note that $d\psi_0(w_j) = dv_j$ as $du_j = 0$. Now define $\phi_1(w_j) = v_j \forall j \in J_1$ to obtain an algebra morphism $\phi_1 : \Lambda W(1) \to \Lambda V$ that extends ϕ_0. We have $d\phi_1(w_j) = dv_j = d\psi_0(w_j) = \psi_0(dw_j) = \phi_0(dw_j) = \phi_1(dw_j) \forall j \in J_1$, since $dw_j \in \Lambda W_0$. Hence ϕ_1 is a dgca morphism. Note that ϕ_1 satisfies (7), that is, $\phi_1(W_1) \subset \Lambda V_0.\Lambda^+ V_{\geq 1}$ by the very definition of ϕ_1.

Next we shall construct, inductively, dgca morphisms $\phi_k : \Lambda W(k) \to \Lambda V$, $k \geq 2$, such that ϕ_k extends ϕ_{k-1}, and has the property that $\phi_k(w)$ *contains no monomial belonging to* $\Lambda^{\leq 1} V_1.\Lambda V_0$ that is, for all $w \in W_k$,

$$\phi_k(w) \in (\Lambda^{\geq 2} V_{\geq 1} + \Lambda^+ V_{\geq 2}).\Lambda V_0. \tag{9}$$

Let $j \in J_2$ so that $dw_j \in \Lambda W(1)$. Now $d(\phi_1(dw_j)) = \phi_1(ddw_j) = 0$ and so $\phi_1(dw_j)$ is closed. Since $dw_j \in \Lambda^+ W_1.\Lambda W_0$ (as W has standard lower gradation) and since ϕ_1 satisfies condition (7), it follows that $\phi_1(dw_j)$ is *exact*. Write $\phi_1(dw_j) = dz_j$ where $z_j = u_j + v_j$ with $u_j \in \Lambda^{\leq 1} V_1.\Lambda V_0$ and every monomial present in v_j having at least two factors from V_1 or a factor from $V_{\geq 2}$. Since ϕ_1 satisfies (7), and since $du_j \in \Lambda V_0$, we see that $du_j = 0$ as $\phi_1(dw_j) = du_j + dv_j$. We set $\phi_2(w_j) := v_j$ $\forall j \in J_2$ which defines a dgca morphism $\phi_2 : \Lambda W(2) \to \Lambda V$ extending ϕ_1. Note that ϕ_2 satisfies (9).

Having obtained, inductively, a dgca morphism $\phi_k : \Lambda W(k) \to \Lambda V$ satisfying (7) and (9), we extend it to a dgca morphism $\phi_{k+1} : \Lambda W(k+1) \to \Lambda V$.

Let $j \in J_{k+1}, k \geq 2$. Note that $d\phi_k(dw_j) = \phi_k(ddw_j) = 0$. It follows that, since $\phi_k(dw_j) \in \Lambda^+ V_{\geq 1}.\Lambda V_0$ (using induction and the fact that W has standard lower gradation), it is exact. Write $\phi_k(dw_j) = dz_j$. Since ϕ_k satisfies (7) (already proved) and (9), dz_j has no monomial from ΛV_0. Write $z_j = u_j + v_j$ where $u_j \in \Lambda^{\leq 1} V_1.\Lambda V_0$, $v_j \in (\Lambda^{\geq 2} V_1 + \Lambda^+ V_{\geq 2}).\Lambda V_0$. Then $du_j \in \Lambda V_0$ and $dv_j \in \Lambda V_0.\Lambda^+ V_{\geq 1}$. It follows that $du_j = 0$ and so $\phi_k(dw_j) = dv_j$. Set $\phi_{k+1}(w_j) := v_j$, $\forall j \in J_{k+1}$. This leads to algebra homomorphism $\Lambda W(k+1) \to \Lambda V$ that extends ϕ_k. Note that $d\phi(w_j) = dv_j = dz_j = \phi(dw_j)$ $\forall j \in J_{k+1}$. Hence ϕ_{k+1} is a dgca morphism. It is clear that ϕ_{k+1} satisfies (7) and (9). This completes the induction step. Thus we obtain a dgca morphism $\phi : \Lambda W \to \Lambda V$ defined by the condition that $\phi|_{\Lambda V(k)} = \phi_k$. It is clear that ϕ is a Sullivan representative for f and satisfies (7), and (9) for all $w \in W(k), k \geq 2$. Now property (8) follows immediately as (9) holds for ϕ. □

8.2. **Proof of theorem 7.5.** Now, if $\alpha \in \pi_{n-1}^{\mathbb{Q}}(Z)$ be special, as $\mathcal{M}_Z = (\Lambda U, d)$ then $\langle u, \alpha \rangle = 0$ for $u \in U_k$, $k \neq 1$.

Now, if $f : Z \to X$ is formal map, then its Sullivan representative $\phi : (\Lambda V, d) \to (\Lambda U, d)$ can be built from $f^* : H^*(X) \to H^*(Z)$. Using lemma 8.5, we choose a Sullivan representative $\phi : (\Lambda V, d) \to (\Lambda U, d)$ so that $Q(\phi)(V_{\geq 2}) \subset U_{\geq 2}$ and

$Q(\phi)(V_0) \subset U_0$.

$$\langle Q(\phi)(v), \alpha\rangle = \langle v, f_*(\alpha)\rangle$$

So, $v \in V_k$, $k \neq 1$ implies $Q(\phi)(v) \in U_k$, $k \neq 1$ so α is special leads to the fact that $\langle Q(\phi)(v), \alpha\rangle = 0$. So, $\langle v, f_*(\alpha)\rangle = 0$ for $v \in V_k$, $k \neq 1$, so $f_*(\alpha)$ is a special element.

8.3. **Examples.** In this section we construct various illustrative examples. We first give an example of a finite CW complex with only even dimensional cells which is not formal.

Example 8.6. (i) Let $X = \mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2$. Then X is formal. Computing the minimal model $\mathcal{M}_X = \Lambda(V)$ of X up to degree five we have the following table.

deg i	dim V^i	basis	differential
2	3	a_1, a_2, a_3	$da_i = 0$
3	6	b_{11}, b_{22}, b_{33} b_{12}, b_{23}, b_{13}	$db_{ii} = a_i^2$ $db_{ij} = a_i a_j$
4	6	$f_{ij}, i \neq j$	$df_{ij} = b_{ii}a_j - a_i b_{ij}$
5	3	k_{12}, k_{23}, k_{13}	$dk_{ij} = a_j f_{ij} - a_i f_{ji} + b_{ii}b_{jj}$

Let $\alpha \in \pi_5^{\mathbb{Q}}(X)$ be such that $\langle k_{12}, \alpha\rangle = 1$. Let $Y = X \cup_\alpha e^6$. Then Y is not formal by Theorem 7.2 because the class $[u] \in H^6(Y; \mathbb{Q}) \cong \mathbb{Q}$ is indecomposable.
(ii) Consider the same space Y regarded as a subcomplex of $\widetilde{Y} = \mathbb{C}P^3 \times \mathbb{C}P^3 \times \mathbb{C}P^3 \cup_\alpha e^6$ where we regard X as the 2-skeleton of $\widetilde{X} := \mathbb{C}P^3 \times \mathbb{C}P^3 \times \mathbb{C}P^3$. Note that $\pi_5(\widetilde{X}) = 0$ and so $[\alpha] = 0$. It follows that $\widetilde{Y}$ is formal. Since Y is not formal we see that a subcomplex of a formal space with only even-dimensional cell is not necessarily formal.

Our next example shows that merely assuming u (as in Theorem 7.2) to be decomposable is not sufficient to conclude formality of Y.

Example 8.7. Let $X = X_0 \vee X_1$ where $X_0 = \mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2, X_1 = \{(x,y,z) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \mid \ x \text{ or } y \text{ or } z \text{ equals } *\}$ where $*$ denotes the base point of $\mathbb{S}^2$. Then X_0 is formal, being a wedge of formal spaces. The space X_1 is formal since it is the 4-skeleton of the formal $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ where the 2-sphere is given the cell structure with one 0-cell and one 2-cell. Therefore X is formal. One has $H^*(X_0;\mathbb{Q}) = \mathbb{Q}[a_1,a_2,a_3]/\langle a_1^2, a_2^2, a_3^2, a_1a_2, a_2a_3, a_3a_1\rangle, |a_i| = 2$ and $H^*(X_1;\mathbb{Q}) = \mathbb{Q}[x_1,x_2,x_3]/\langle x_1^2, x_2^2x_3^2, x_1x_2x_3\rangle, |x_i| = 2$. The cohomology algebra of X is readily computed from this. The minimal model $(\mathcal{M}_X, d) = (\Lambda(V), d)$ of X can be computed from the description of $H^*(X;\mathbb{Q})$ since X is formal. We obtain that $V_0 = V^2 = H^2(X;\mathbb{Q})$ is six dimensional, with basis a_1,a_2,a_3,x_1,x_2,x_3 all in degree 2 where $d(a_i) = d(x_i) = 0$, $V^3 \subset V_1$ has basis $b_{ij}(1 \le i \le j \le 3), v_i, w_{ij}(1 \le i,j \le 3)$, where $db_{ij} = a_ia_j, dv_i = x_i^2, dw_{ij} = a_ix_j$. Also note that there are elements $f_{ij} \in V_2 \cap V^4, i \ne j$ such that $df_{ij} = a_ib_{ij} - a_jb_{ii}$, elements $k_{ij} \in V_3 \cap V^5, i \ne j$, such that $dk_{ij} = a_jf_{ij} - a_if_{ji} + b_{ii}b_{jj}$ and there is an element $z \in V_1, |z| = 5$, with $dz = x_1x_2x_3$. Let $\alpha : \mathbb{S}^5 \to X$ be such that $\langle k_{12}, [\alpha]\rangle = l, \langle z, [\alpha]\rangle = m$ where l, m are non-zero integers. We observe that α is in the kernel of the Hurewicz homomorphism. Then $y := mk_{12} - lz$ vanishes on $[\alpha]$, $d_\alpha y = mdk_{1,2} - lx_1x_2x_3, d_\alpha z = x_1x_2x_3 + mu_\alpha$. So, in $H^*(M_\alpha) \cong H^*(Y)$ we obtain $[u_\alpha] = u = (-1/m)x_1x_2x_3$, a decomposable element. *Note that* $k_{12} \in V_3 \cap V^5$ *and* $\langle -, [\alpha]\rangle : V_3 \to \mathbb{Q}$ *is non-zero.*

We claim that Y is not formal. Suppose that Y is formal then it is readily seen that, writing $\mathcal{M}_Y = \Lambda W$, $W^j = V^j, j \le 4$, and $V^5 \cong W^5 \oplus \mathbb{Q}z$ where W^5 is identified with the kernel of $[\alpha]$. In particular $y \in W^5$. Let $\Psi : \mathcal{M}_Y \to H^*(Y)$ be a dgca morphism that induces isomorphism in cohomology. Then $\Psi|W^j = \Phi|W^j, j \le 3$, where $\Phi : \mathcal{M}_X \to H^*(X)$ is a suitable quasi-isomorphism. Since $\Psi(a_i) = \Phi(a_i) = a_i$ and since $a_i.H^4(Y;\mathbb{Q}) = 0$ we get $0 = \Psi(d_Y(y)) = \Psi(mdg_{12} - lx_1x_2x_3) = \Psi(m(a_jf_{ij} - a_if_{ji} + b_{ii}b_{jj}) - lx_1x_2x_3) = -lx_1x_2x_3 \ne 0$, a contradiction. Hence Y is not formal.

9. COHOMOLOGY OF GRASSMANN MANIFOLD

9.1. **Cellular structure of Grassmann manifold.** Let $G_{n,k}$ denote the complex Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. Then $G_{n,k} = \mathrm{SL}(n, \mathbb{C})/P_k$ where P_k is the parabolic subgroup consisting of those linear transformations of $\mathbb{C}^n$ which stabilize $\mathbb{C}^k \subset \mathbb{C}^n$, the coordinate subspace spanned by the first k standard basis vectors $e_j, 1 \leq j \leq k$. Let $U(n)$ denote the unitary group. Then $G_{n,k} = U(n)/U(k) \times U(n-k)$. Thus we see that $G_{n,k}$ is a compact smooth manifold of complex dimension $k(n-k)$. In fact $G_{n,k}$ has the structure of a smooth projective variety. The variety $G_{n,k}$ has an *algebraic* cell decomposition given by Schubert cells, the labeling set for which is the coset space $I_{n,k} = S_n/(S_k \times S_{n-k})$, where S_n denotes the symmetric group. The set $I_{n,k}$ may be identified with the set of all sequences $\mathbf{i} = i_1 < \cdots < i_k$ where $1 \leq i_r \leq n$ for all $r \leq k$. The Schubert variety $X(\mathbf{i})$ corresponding to $\mathbf{i}$ is

$$X(\mathbf{i}) = \{A \in G_{n,k} \mid \dim_{\mathbb{C}} A \cap \mathbb{C}^{i_r} \leq r,\ 1 \leq r \leq k\}.$$

This is the closure of the Schubert cell $C(\mathbf{i}) = \{A \in G_{n,k} \mid \dim_{\mathbb{C}} A \cap \mathbb{C}^{i_r - 1} = r-1, \dim_{\mathbb{C}} A \cap \mathbb{C}^{i_r} = r, 1 \leq r \leq k\} \subset G_{n,k}$ which is isomorphic to the affine space $\mathbb{C}^d$ where $d = \dim X(\mathbf{i})$. It is equal to the orbit $B.e_{\mathbf{i}}$ of the point $e_{\mathbf{i}} := \mathbb{C}e_{i_1} + \cdots + \mathbb{C}e_{i_r} \in G_{n,k}$ where B is the Borel subgroup that fixes the standard flag $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \mathbb{C}^n$. Writing matrices with respect to the standard basis, B is the group of upper triangular matrices in $\mathrm{SL}((n, \mathbb{C})$. The (complex) dimension of $X(\mathbf{i})$ equals $\sum_{1 \leq r \leq k}(i_r - r)$. The Chevalley-Bruhat order on $I_{n,k}$ is obtained as $\mathbf{i} \leq \mathbf{j}$ if $i_q \leq j_q$ for $1 \leq q \leq k$. In particular, $X(\mathbf{i}) \cap X(\mathbf{j}) = X(\mathbf{t})$ where $t_q = \min\{i_q, j_q\}, 1 \leq q \leq k$.

9.2. **Cohomology of $G_{n,k}$.** The homology classes of Schubert varieties $X(\mathbf{i})$ form a $\mathbb{Z}$-basis $z_{\mathbf{i}}$ of $H_*(G_{n,k}; \mathbb{Z})$. The dual basis elements of $H^*(G_{n,k}; \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(H_*(G_{n,k}; \mathbb{Z}), \mathbb{Z})$

will be denoted $[X(\mathbf{i})], \mathbf{i} \in I_{n,k}$. Thus $[X(\mathbf{i})](z_{\mathbf{j}}) = \delta_{\mathbf{i},\mathbf{j}}$. We shall denote by X_i the *special Schubert variety* corresponding to $(1, 2, \ldots, k-1, k+i) \in I_{n,k}$ having (complex) dimension i. The elements $[X_i] \cong \mathbb{P}^i, 1 \leq i \leq n-k$, generate the cohomology ring $H^*(G_{n,k}; \mathbb{Z})$. Note that $[X_0] = 1 \in H^0(G_{n,k}; \mathbb{Z})$.

The structure constants $[X(\mathbf{i})].[X_j] = \sum_{\mathbf{r}} c^{\mathbf{r}}_{\mathbf{i},\mathbf{j}}.[X(\mathbf{r})]$ are determined by the Pieri formula which expresses $[X(\mathbf{i})].[X_j]$ in terms of the basis elements and the Giambelli formula which expresses $[X(\mathbf{i})]$ as a polynomial in the $[X_j], 0 \leq j \leq k$. In order to describe these formulae we associate to each $\mathbf{i} \in I_{n,k}$ a partition $\nu(\mathbf{i}) = \nu = \nu_1 \geq \nu_2 \geq \cdots \geq \nu_k \geq 0$ where $\nu_{k-p+1} := i_p - p$ of $\dim_{\mathbb{C}} X_{\mathbf{i}} = \sum \nu_p =: |\nu|$ into at most k parts. Denote by X_ν the Schubert variety $X(\mathbf{i})$. Thus the partition corresponding to X_j is j. We recall below the Pieri and the Giambelli formulae. See [11], [8] for details.

- *Pieri formula.* $[X_\nu].[X_j] = \sum_\mu [X_\mu]$ where the sum is over all partitions μ of $|\nu| + j$ such that $\mu_k \geq \nu_k \geq \cdots \geq \mu_1 \geq \nu_1$ where $\mu_i - \nu_i \leq j$ for all i.

- *Giambelli formula.* The element $[X_\nu]$ equals the determinant of the $k \times k$ matrix $([X_{\nu_i+j-i}])$ where it is understood that $[X_r] = 0$ if $r < 0$.

Let $\gamma_{n,k}$ denote the tautological bundle over $G_{n,k}$ whose fibre over $V \in G_{n,k}$ is the vector space V. Then $\gamma_{n,k}$ is naturally a subbundle of the trivial complex vector bundle $n\epsilon$ of rank n over $G_{n,k}$. Let $\gamma^\perp_{n,k}$ denote the orthogonal complement of $\gamma_{n,k}$ with respect to the standard hermitian product on $\mathbb{C}^n$. Thus the fibre of $\gamma^\perp_{n,k}$ over $V \in G_{n,k}$ is $V^\perp \subset \mathbb{C}^n$. (Note that one has an isomorphism of complex vector bundles: $\gamma^\perp_{n,k} \cong n\epsilon/\gamma_{n,k}$, where $n\epsilon$ denotes the trivial bundle of rank n.) Since $\gamma_{n,k} \oplus \gamma^\perp_{n,k} \cong n\epsilon$, the following relation holds in $H^*(G_{n,k}; \mathbb{Z})$:

$$c(\gamma_{n,k}).c(\gamma^\perp_{n,k}) = 1 \tag{10}$$

where $c(\xi) = 1 + c_1(\xi) + \ldots + c_m(\xi)$ is the total Chern class of a complex vector bundle of rank m, $c_i(\xi)$ being the i-th Chern class of ξ. Using (10) one can express

$c_i(\gamma_{n,k}^{\perp})$ as a polynomial $h_i(c_1(\gamma_{n,k}),\ldots,c_k(\gamma_{n,k}))$ for all i. In particular, $h_j = 0$ if $j > n-k$.

It is well-known that the integral cohomology algebra of $G_{n,k}$ is generated by $c_j := c_j(\gamma_{n,k}), 1 \leq j \leq k$, subject only to relations arising (10). More precisely, $H^*(G_{n,k};\mathbb{Z})$ is the quotient of the polynomial ring in the 'indeterminates' $c_i(\gamma_{n,k}), 1 \leq i \leq k$, modulo the ideal generated by $h_j(c_1(\gamma_{n,k}),\ldots,c_k(\gamma_{n,k})), j > n-k$. It follows that there are no algebraic relations among $c_1(\gamma_{n,k}),\ldots,c_k(\gamma_{n,k})$ in dimensions up to $2(n-k)$. When $n-k < j \leq k$, it can be seen using the relation $h_i = 0, n-k < i \leq j$, that $c_j(\gamma_{n,k})$ is expressible as a polynomial in $c_1(\gamma_{n,k}),\ldots,c_{j-1}(\gamma_{n,k})$. The cohomology class $[X_i]$ equals $c_i(\gamma_{n,k}^{\perp})$, and, consequently there are no algebraic relations among $[X_i], 1 \leq i \leq k$, in dimensions up to $2(n-k)$ and, when $k > n-k$, $[X_j] \in H^*(G_{n,k};\mathbb{Z})$ is decomposable for $n-k < j \leq k$. Using the Giambelli formula one obtains a formula for the class of any Schubert variety $[X_\nu]$ in terms of the Chern classes $c_i(\gamma_{n,k})$. See [8, §14].

9.3. **Facts concerning cohomology ring of $G_{n,k}$.** The cohomology ring of $G_{n,k}$ has a presentation

$$H^*(G_{n,k};\mathbb{Z}) = \mathbb{Z}[c_1,\ldots,c_k]/\langle h_{n-k+1},\ldots,h_n\rangle$$

as the quotient of the polynomial ring modulo the ideal generated by the elements $h_j, n-k+1 \leq j \leq n$, where $|c_i| = 2i$; here h_j is defined as the $2j$-th degree term in the expansion of $(1+c_1+\cdots+c_k)^{-1}$. Under the above isomorphism, c_i corresponds to the element $c_i(\gamma_{n,k}) \in H^{2i}(G_{n,k};\mathbb{Z}), 1 \leq i \leq k$.

The following are well-known facts concerning the cohomology ring:

(1) The cohomology group $H^r(G_{n,k};\mathbb{Z})$ is a free abelian group for any r; it is zero when r is odd. This follows from the fact that $G_{n,k}$ admits a cell-structure with cells only in even dimensions. See, for example, p.196, [11].

(2) The elements $h_j, n-k+1 \leq j \leq n$, form a *regular sequence* in the polynomial algebra $R_k := \mathbb{Z}[c_1, \ldots, c_k]$ for any $n \geq 2k$. That is, $h_{n-k+1} \neq 0$ and h_{n-k+r} is a not a zero divisor in $R_k/\langle h_{n-k+1}, \cdots, h_{n-k+r-1}\rangle, 2 \leq r \leq k$. See for example [3].

(3) The element $c_1^d \neq 0$ where $d = \dim_{\mathbb{C}} G_{n,k} = k(n-k)$. This follows immediately from the fact that $G_{n,k}$ has the structure of a Kähler manifold with second Betti number 1. In fact it is known that $H^{2d}(G_{n,k};\mathbb{Z})$ is generated by the element c_k^{n-k} and that $c_1^d = Nc_k^{n-k}$ where $N = (d!1!2! \cdots (k-1)!)/((n-k)! \cdots (n-1)!)$. See [8, §14].

(4) The natural embeddings $i : G_{n,k} \subset G_{n+1,k}$ and $j : G_{n,k} \subset G_{n+1,k+1}$, induce surjections $i^* : H^*(G_{n+1,k};\mathbb{Z}) \to H^*(G_{n,k},\mathbb{Z})$ and $j^* : H^*(G_{n+1,k+1};\mathbb{Z}) \to H^*(G_{n,k};\mathbb{Z})$ where $i^*(c_r(\gamma_{n+1,k})) = c_r(\gamma_{n,k}), 1 \leq r \leq k$ and $j^*(c_r(\gamma_{n+1,k+1})) = c_r(\gamma_{n,k})$ when $r \leq k$ and $j^*(c_{k+1}(\gamma_{n+1,k+1})) = 0$. The homomorphism i^* induces isomorphisms in cohomology in dimensions up to $2(n-k)$ and j^* induces isomorphisms in cohomology in dimensions up to $2k$.

9.4. **Indecomposable Schubert classes.** Suppose that X is a union of Schubert varieties in $G_{n,k}$. We determine the Schubert classes $[X_\nu] \in H^*(X;\mathbb{Z})$ which are indecomposable in the cohomology ring $H^*(X;\mathbb{Z})$. We need only consider the problem when $\dim X_\nu = \dim X$ since otherwise we may replace X by its $2|\nu|$-dimensional skeleton $X^{(2|\nu|)}$. Also we may assume, without loss of generality, that X is a equidimensional variety.

First we consider the case where $X = G_{n,k}$ and assume, without loss of generality, that $2k \leq n$.

Lemma 9.1. *Let $1 \leq k \leq \lfloor n/2 \rfloor$. The Schubert class $[X_\nu]$ is indecomposable in $H^*(G_{n,k};\mathbb{Z})$ if and only if $0 \leq \nu_2 \leq 1$ and $|\nu| \leq k$. Furthermore, if $[X_\nu]$ is indecomposable, then $[X_\nu] - (-1)^{|\nu|-\nu_1}[X_{|\nu|}]$ is decomposable.*

Proof. As remarked above, there are no algebraic relations among the $[X_j], 1 \leq j \leq k$.

Let $0 \leq \nu_2 \leq 1$ and $|\nu| \leq k$. Put $r := |\nu| - \nu_1 + 1$. Expanding the determinant on the right hand side of the Giambelli formula for $[X_\nu]$ we obtain the term

$$(-1)^{r+1} a_{1,r} a_{21} \dots a_{r-2,r-1} a_{r+1,r+1} \dots a_{k,k}$$

where $(a_{i,j}) = ([X_{\nu_i+j-i}])$. We see that $a_{1,r} = [X_{|\nu|}], a_{i-1,i} = [X_0] = 1$ if $2 \leq i \leq r$ and $a_{i,i} = [X_0] = 1$ when $i > r$. Hence the term $(-1)^{r+1}[X_{|\nu|}]$ occurs in the formula for $[X_\nu]$. (Note that there can be no cancellation of terms.) Since $[X_j]$ is indecomposable if and only if $j \leq k$ we see that $[X_\nu]$ is indecomposable and that $[X_\nu] - (-1)^{|\nu|-\nu_1}[X_{|\nu|}]$ is decomposable.

It is easily seen, again using the Giambelli formula, that $[X_\nu]$ is decomposable if $\nu_2 \geq 2$ or if $|\nu| > k$. $\square$

Let X be a union of Schubert varieties. The inclusion map $X \hookrightarrow G_{n,k}$ induces a surjection in cohomology whose kernel is generated (as an abelian group) by those Schubert cells which are not contained in X. The ring structure of $H^*(X;\mathbb{Z})$ is again obtained from the Pieri and Giambelli formulae by simply setting equal to zero the Schubert classes $[X_{\mathbf{j}}]$ which occur in these formulae whenever $X_{\mathbf{j}}$ is not contained in X. We shall be particularly concerned with such an X when it is equidimensional, that is, every irreducible component of X has the same dimension. As an immediate corollary, we obtain the following criterion for the irreducibility of $[X_{\mathbf{i}}] \in H^*(X;\mathbb{Z})$. We make no notational distinction between the Schubert class $[X_{\mathbf{i}}] \in H^*(G_{n,k};\mathbb{Z})$ and its image in $H^*(X;\mathbb{Z})$.

Proposition 9.2. *Let $2 \leq r \leq k \leq \lfloor n/2 \rfloor$. Suppose that $X \subset G_{n,k}$ is a union of Schubert varieties all having dimension r over $\mathbb{C}$.*

(i) Suppose that X_ν is not contained in X for some ν such that $\nu_2 \leq 1$ and $|\nu| = r$. Then there are no indecomposable classes in $H^{2r}(X;\mathbb{Z})$.

(ii) If X contains all the Schubert varieties X_ν with $\nu_2 \le 1$ where $|\nu| = r$, then each such $[X_\nu] \in H^{2r}(X;\mathbb{Z})$ is indecomposable.

Proof. (i) Suppose that $[X_\mu] \in H^{2r}(G_{n,k};\mathbb{Z})$ is indecomposable. By Lemma 9.1, it is clear that $[X_\mu] - \epsilon[X_\nu] \in H^{2r}(G_{n,k};\mathbb{Z})$ is decomposable where $\epsilon = 1$ or -1. As $[X_\nu]$ maps to zero in $H^{2r}(X;\mathbb{Z})$ it follows that $[X_\mu]$ maps to a decomposable element.

(ii) Our hypothesis, together with Lemma 9.1, implies that $H^{2r}(X;\mathbb{Z})$ isomorphic to the quotient of $H^{2r}(G_{n,k};\mathbb{Z})$ by a subgroup generated by certain decomposable elements. It follows that any indecomposable element in $H^{2r}(G_{n,k};\mathbb{Z})$ is mapped to an indecomposable element in $H^{2r}(X;\mathbb{Z})$. This completes the proof. $\square$

Application 9.3. Let $G_{6,3}$ denote the complex Grassmann manifold of 3-dimensional vector subspaces of $\mathbb{C}^6$. Now recall some facts regarding cell structure and cohomology ring of $G_{6,3}$ which we have already seen for general $G_{n,k}$ in §9.1 and §9.2.
The manifold $G_{6,3}$ is a compact smooth manifold of complex dimension 9 and it has the structure of a smooth projective variety. The variety $G_{6,3}$ has an *algebraic* cell decomposition given by Schubert cells, the labeling set for which is the coset space $I_{6,3} = S_6/(S_3 \times S_3)$, where S_6 and S_3 denote the symmetric groups. The set $I_{6,3}$ may be identified with the set of all sequences $\mathbf{i} = i_1 < i_2 < i_3$ where $1 \le i_r \le 6$ for all $r \le 3$. The Schubert variety $X(\mathbf{i})$ corresponding to $\mathbf{i}$ is

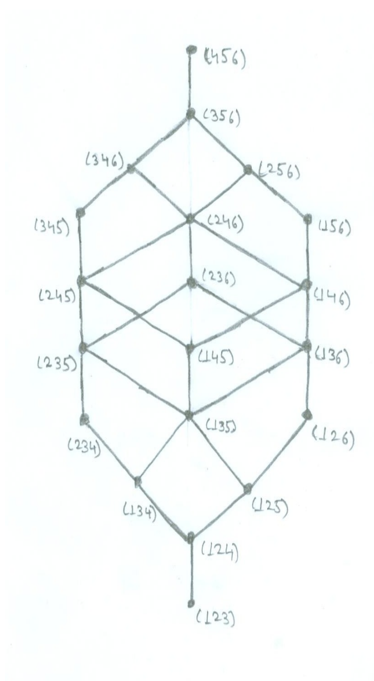
$$X(\mathbf{i}) = \{A \in G_{6,3} \mid \dim_{\mathbb{C}} A \cap \mathbb{C}^{i_r} \le r,\ 1 \le r \le 3\}.$$

This is the closure of the Schubert cell $C(\mathbf{i}) = \{A \in G_{6,3} \mid \dim_{\mathbb{C}} A \cap \mathbb{C}^{i_r - 1} = r-1, \dim_{\mathbb{C}} A \cap \mathbb{C}^{i_r} = r, 1 \le r \le 3\} \subset G_{6,3}$. The complex dimension of $X(\mathbf{i})$ equals $\sum_{1\le r\le 3}(i_r - r)$. The Chevalley-Bruhat order on $I_{6,3}$ is obtained as $\mathbf{i} \le \mathbf{j}$ if $i_q \le j_q$ for $1 \le q \le 3$. In particular, $X(\mathbf{i}) \cap X(\mathbf{j}) = X(\mathbf{t})$ where $t_q = \min\{i_q, j_q\}, 1 \le q \le 3$.

The integral cohomology algebra of $G_{6,3}$ is $\mathbb{Z}[c_1, c_2, c_3]/\langle h_4, h_5, h_6\rangle$ where deg c_i is $2i$ and deg h_j is $2j$ and h_1, h_2, h_3 forms a regular sequence (see example (8), §5.1). Similarly, the rational cohomology algebra of $G_{6,3}$ is $\mathbb{Q}[c_1, c_2, c_3]/\langle h_4, h_5, h_6\rangle$. And also using Giambelli formula (see §9.2), we shall be able to compute cohomology class corresponding to each cell.

Using Theorems 7.2 and 7.5, we shall prove the formality of union of same dimensional Schubert varieties in $G_{6,3}$.

The following is the CW complex structure of $G_{6,3}$



(1) The Schubert varieties of $G_{6,3}$ of complex dimension ≤ 2, are smooth complex projective varieties, so are formal.

(2) The union of these two Schubert varieties $X(125) \cup X(134)$ is also formal as we have $X(125) \cup X(134) = X(134) \cup C(125)$. The minimal model $\mathcal{M}_{X(134)} = \mathcal{M}_{\mathbb{CP}^2}$ with standard lower gradation (abbreviated *s.l.g.*) does not contain any non-zero element in degree 3, so $C(125)$ is attached by the trivial element in $\pi_3(X(134)) = \pi_3(\mathbb{CP}^2) = 0$, which is in the kernel of the Hurewicz map. Therefore, by (i) of Theorem 7.2, $X(134) \cup C(125)$ is a formal space.

(3) For complex dimension 3, there are three Schubert varieties, namely $X(234)$, $X(135)$, $X(126)$. The Schubert variety $X(1, 2, 6)$ and $X(234)$ are homeomorphic to

$\mathbb{CP}^3$, so are formal.

Let $X = X(135)$. In (3), we have seen that $X^{(4)} = X(134) \cup X(125)$ which is formal. So we can compute its minimal model from its cohomology algebra, which is $\mathcal{M}^{\leq 5}_{X^{(4)}} = (\Lambda(x_1, x_2, y_1, y_2))^{\leq 5}$, where deg x_i is $2i$, deg y_j is odd, $d(x_i) = 0$, $d(y_1) = x_1^3$, $d(y_2) = x_1 x_2$. Therefore, we have constructed a minimal model $(\Lambda V, d)$ up to degree 5 with s.l.g. such that $x_i \in V_0, y_j \in V_1$. Then $V_j \cap V^5 = 0$, $j \neq 1$. This shows that the attaching map of $C(1,3,5)$ to $X^{(4)}$ is special and also is in the kernel of the Hurewicz homomorphism. Also the cohomology class u, as defined in Theorem 7.2 is decomposable. Thus, applying (ii) of Theorem 7.2, we see that $X = X(135)$ is formal.

(4) Now consider $X = X(234) \cup X(135)$. We have already shown that $X(135)$ is formal. Now, we can compute its minimal model from its cohomology algebra. We can show that the attaching map of $C(234)$ to $X(135)$ is special and in the kernel of the Hurewicz map, so satisfies the criterion of (ii) of Theorem 7.2, so $X = X(234) \cup X(135)$ is formal.

Similarly we can show that $X(135) \cup X(126)$ and $X(234) \cup X(126)$ is formal. Though here we shall give a different proof of the formality of $X(234) \cup X(126)$.

First, I shall prove that the inclusion of $i : X(125) \hookrightarrow X(125) \cup X(234)$ is formal. This is because i is the composition of i_1 and i_2, i.e. $i = i_2 \circ i_1$ where $i_1 : X(125) \hookrightarrow X(125) \cup X(134)$ and $i_2 : X(125) \cup X(134) \hookrightarrow X(125) \cup X(234)$. The inclusion i_1 is formal as we have already seen in (3) that the attaching map of $C(125)$ is the trivial element in $\pi_3(X(134)) = \pi_3(\mathbb{CP}^2) = 0$ and in the kernel of Hurewicz homomorphism. So by Theorem 8.2, i_1 is formal.

In (4) we have constructed the minimal model of $X(1,3,4) \cup X(1,2,5)$ with s.l.g., such that $V_j \cap V^5 = 0$, $j \neq 1$. Therefore, the attaching map of $C(234)$ to $X(134) \cup X(125)$ is special and in the kernel of the Hurewicz homomorphism and the cohomology class u is decomposable. So, by Theorem 8.2, the inclusion i_2 is formal. Then $i = i_2 \circ i_1$ is formal (see §2.4).

The Schubert variety $X(125)$ is already proven to be formal and its minimal model with s.l.g. is as follows:

$\mathcal{M}_{(X(125))} = (\Lambda(V), d)$, where $V = V_0 \oplus V_1$, $V_0 = \mathbb{Q}v_1$, $V_1 = \mathbb{Q}v_2$, deg $v_1 = 2$, deg v_2=5, $d(v_1) = 0$, $d(v_2) = v_1^3$.

Evidently, $V_j \cap V^5 = 0$, $j \neq 1$ and so the attaching map α of $C(126)$ to $X(125)$ is special. Therefore, by Theorem 8.2, $i_*(\alpha)$ is special too. So, $C(126)$ is attached to $X(234) \cup X(125)$ by a special homotopy element and the attaching map is in the kernel of the Hurewicz homomorphism and the cohomology class u is decomposable. Thus, by Theorem 7.2, $X(234) \cup X(126)$ is formal.

(5) Let X be a union of complex k-dimensional Schubert varieties with the property that $X^{2(k-1)} = G_{6,3}^{2(k-1)}$, then we shall prove that X is formal. We shall prove this by induction on k. The $2(k-1)$-st skeleton $X^{2(k-1)} = X_1$ has same property, i.e. $X_1^{2(k-2)} = G_{6,3}^{2(k-2)}$. By induction, X_1 is formal. Then we can compute its minimal model expliciyely from its cohomology algebra up to dimension $2k-1$. Its minimal model with s.l.g. is of the form $(\Lambda V, d)$, where $V^{\leq 2k-1} = V_0^{2k-1} \oplus V_1^{2k-1}$. This happens because the cohomology algebra of $G_{6,3}$ is the quotient of polynomial algebra by an ideal, generated by a regular sequence. Then we can show that the attaching maps of cells will be special (torsion or free element), and in the kernel of the Hurewicz homomorphism in both cases. Therefore, X will be formal. This will prove most of the union of Schubert varieties of same complex dimension ≥ 4 will be formal. For remaining few cases, we can prove formality using above techniques repeatedly.

10. MAPS BETWEEN GRASSMANN MANIFOLDS

In this chapter we prove the following theorem.

10.1. **Statement of theorems.**

Theorem 10.1. *Let $1 \leq k \leq \lfloor n/2 \rfloor$, $1 \leq l \leq \lfloor m/2 \rfloor$ and $k < l$, where m, n are positive integers such that $m - l > n - k$. Suppose that $m - l \geq 2k^2 - k - 1$ or $1 \leq k \leq 3$. Then any homomorphism of graded rings $\phi : H^*(G_{n,k}; \mathbb{Z}) \to H^*(G_{m,l}; \mathbb{Z})$ vanishes in positive dimensions.*

As a corollary to the above theorem we obtain the following result on the homotopy classification of maps between the complex Grassmann manifolds.

Theorem 10.2. *Let l, k, m, n be as in the above theorem. Then the set $[G_{m,l}, G_{n,k}]$ of homotopy classes of maps is finite and moreover each homotopy class is rationally null-homotopic.*

As another application of Theorem 10.1 we obtain the following *invariant subspace theorem.* See [20] for an analogous result for real Grassmann manifolds. We shall regard $\mathbb{C}^n$ as a subspace of $\mathbb{C}^m$ consisting of vectors with last $m - n$ coordinates zero. Thus, if $y \in G_{n,k}$ and $x \in G_{m,l}$ it is meaningful to write $y \subset x$.

Theorem 10.3. *Let $f : G_{m,l} \to G_{n,k}$ be any continuous map where l, k, m, n are as in Theorem 10.1. Then there exists an element $x \in G_{m,l}$ such that $f(x) \subset x$.*

We point out that the classification of self-maps of a complex Grassmann manifold has been studied in terms of their induced endomorphisms of the cohomology algebra by several authors. See [21], [5], [9], [14], [15]. Similar study of maps between two *distinct* (real) Grassmann manifolds seems to have been initiated in [17]. Sankaran and Sarkar [28] have studied the existence (or non-existence) of maps of non-zero degree between two different complex (resp. quaternionic) Grassmann manifolds of the same dimension. The same problem for oriented real Grassmann manifolds has been settled by Ramani and Sankaran [27].

Our methods are straightforward. To prove Theorem 10.1, we reduce the problem to one about endomorphisms of the cohomology ring of a certain Grassmann manifold and appeal to a result of Glover and Homer [9]. Theorem 10.2 is proved using a result due to Glover and Homer [10], namely, any map between any two complex Grassmann manifolds—indeed complex flag manifolds—is formal. Our approach to the proof of Theorem 10.3 is similar in spirit to that of [20, Theorem 1.1].

It has been conjectured that if ϕ is any endomorphism of the graded $\mathbb{Z}$-algebra $H^*(G_{n,k};\mathbb{Z})$ which vanishes on $H^2(G_{n,k};\mathbb{Q})$, then ϕ vanishes in all positive degrees. See [9]. Our proof shows that the conjecture, if true, implies the validity of Theorems 10.1 and 10.2 hold without the restriction $m - l \geq 2k^2 - k - 1$.

10.2. **Proofs of the above theorems.** Recall that the cohomology ring of $G_{n,k}$ has a presentation

$$H^*(G_{n,k};\mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_k]/\langle h_{n-k+1}, \ldots, h_n\rangle$$

as the quotient of the polynomial ring modulo the ideal generated by the elements $h_j, n-k+1 \leq j \leq n$, where $|c_i| = 2i$; here h_r is defined as the $2r$-th degree term in the expansion of $(1 + c_1 + \cdots + c_k)^{-1}$. Under the above isomorphism c_i corresponds to the element $c_i(\gamma_{n,k}) \in H^{2i}(G_{n,k};\mathbb{Z}), 1 \leq i \leq k$. We shall denote the polynomial ring $\mathbb{Z}[c_1, c_2, \cdots, c_k]$ by R_k.

Proof of Theorem 10.1: One has an inclusion $U(m - l + k) \subset U(m)$ where a matrix $X \in U(m - l + k)$ corresponds to the matrix in block diagonal form with diagonal blocks X, I_{k-l}. (Here I_{k-l} denotes the identity matrix.) This induces an imbedding $G_{m-l+k,k} \subset G_{m,l}$. Similarly, since $m - l > n - k$, we have the inclusion $U(n) \subset U(m-l+k)$ which induces an imbedding $G_{n,k} \subset G_{m-l+k,k}$. These inclusions are merely compositions of appropriate inclusions considered in Fact (4) (§9.3) above.

Let $\alpha : H^*(G_{m,l}, \mathbb{Z}) \to H^*(G_{m-l+k,k}; \mathbb{Z})$ and $\beta : H^*(G_{m-l+k,k}; \mathbb{Z}) \to H^*(G_{n,k}; \mathbb{Z})$ be the inclusion-induced homomorphisms. It follows from Fact (4) (§9.3) that $\beta(c_i(\gamma_{m-l+k,k})) = c_i(\gamma_{n,k}), i \leq k$. Also, $\alpha(c_i(\gamma_{m,l})) = c_i(\gamma_{m-l+k,k}), i \leq k$. Then we obtain an endomorphism $\alpha \circ \phi \circ \beta$ of the graded ring $H^*(G_{m-l+k,k})$ where $\phi : H^*(G_{n,k}; \mathbb{Z}) \to H^*(G_{m,l}; \mathbb{Z})$ is any graded ring homomorphism. Note that our hypothesis on k, l, m, n implies that $\dim G_{n,k} < \dim G_{m,l}$. Hence by Fact (3) (§9.3) above, $\phi(c_1(\gamma_{n,k})) = 0$. Therefore $\alpha \circ \phi \circ \beta(c_1(\gamma_{m-l+k,k})) = 0$. Our hypothesis that $m - l \geq 2k^2 - k - 1$ or $k \leq 3$ implies, by [9], that this endomorphism is zero in positive dimensions. $\square$

We remark that Theorem 10.1 and the above proof hold when the coefficient ring $\mathbb{Z}$ is replaced by any subring of $\mathbb{Q}$ throughout. If ϕ is induced by a continuous map f, then f^* is zero for any commutative ring R.

Before taking up the proof of Theorem 10.2, we recall a relation between the homotopy class of a map and the homomorphism it induces in cohomology with rational coefficients.

Let X be any simply connected finite CW complex and let X_0 denote its rationalization. Denoting the minimal model of X by $\mathcal{M}_X$, one has a bijection $[X_0, Y_0] \cong [\mathcal{M}_Y, \mathcal{M}_X], [h] \mapsto [\Phi_h]$ where on the left we have homotopy classes of continuous maps $X_0 \to Y_0$ and on the right we have homotopy classes of differential graded commutative algebra homomorphism of the minimal models $\mathcal{M}_Y \to \mathcal{M}_X$ (as described in §3.2).

We say that a graded $\mathbb{Q}$-algebra is "good" if it is of the form

$$\mathbb{Q}[u_1, u_2, \cdots, u_k]/I$$

where I is generated by polynomials $f_1, f_2, \cdots, f_r$ which form a regular sequence, i.e., f_l does not represent a zero divisor in $\mathbb{Q}[u_1, u_2, \cdots, u_k]/\langle f_1, f_2, \cdots, f_{l-1}\rangle$ for $l = 1, 2, \cdots, r$. If X is a simply connected space with "good" cohomology algebra, then X is a formal space and the minimal model of X is of the form $\mathcal{M}_X = (\Lambda(V_0 \oplus V_1), d)$

where V_0 is a vector space generated by $x_1, x_2, \cdots, x_k$, degree of x_i is even and d on V_0 is 0; V_1 is a vector space generated by $y_1, y_2, \cdots, y_r$, degree of y_l is odd and $d(y_l) = f_l$ and as X is a formal space, so there is a quasi-isomorphism $\rho_X : \mathcal{M}_X \to H^*(X; \mathbb{Q})$ defined by $\rho_X(x_i) = u_i$ and $\rho_X(y_j) = 0$.

For any morphism $\phi : \mathcal{M}_Y \to \mathcal{M}_X$ between minimal models of simply connected spaces with "good" cohomology algebra, we have the following diagram which is evidently commutative

$$\begin{array}{ccc} (\mathcal{M}_Y, d) & \stackrel{\phi}{\to} & (\mathcal{M}_X, d) \\ \rho_Y \downarrow & & \downarrow \rho_X \\ H^*(Y) & \stackrel{\phi^*}{\to} & H^*(X). \end{array}$$

Therefore, if X and Y are simply connected spaces with "good" cohomology algebra, then we have the following bijection

$$[\mathcal{M}_Y, \mathcal{M}_X] \cong Hom_{alg}(H^*(Y; \mathbb{Q}), H^*(X; \mathbb{Q})).$$

Thus, if X and Y are simply connected spaces with "good" cohomology algebra, then we have the following bijection (see §16, [4])

$$[X_0, Y_0] \cong Hom_{alg}(H^*(Y; \mathbb{Q}), H^*(X; \mathbb{Q})).$$

It is known that complex Grassmann manifold have "good" cohomology algebra. (See 9.2 and 9.3).

Thus we have

Theorem 10.4. *Let X, Y be complex Grassmann manifolds. Then $[h] \mapsto H^*(h; \mathbb{Q})$ establishes a bijection from $[X_0, Y_0]$ to the set of graded $\mathbb{Q}$-algebra homomorphisms $Hom_{alg}(H^*(Y; \mathbb{Q}), H^*(X; \mathbb{Q}))$.* □

We now turn to the proof of Theorem 10.2.

Proof of Theorem 10.2: By Theorem 10.1 we know that any such f^* is the trivial homomorphism (which is identity in degree zero and is zero in positive dimensions). By the above theorem f_0 is null-homotopic. This proves the second statement of Theorem 10.2. The first statement follows from the second since there exist, up to homotopy, at most finitely many continuous maps $f : G_{m,l} \to G_{n,k}$ having the *same* rationalization f_0. (See [30, §12].) This completes the proof of Theorem 10.2. $\square$

Next we turn to the proof of Theorem 10.3; we use cohomology with rational coefficients although one may use integer coefficients.

We shall write M, N respectively for $G_{m,l}$ and $G_{n,k}$. Suppose that $1 \leq k < l$, $m - l \geq n - k$. As usual we assume that $2k \leq n, 2l \leq m$. Let $V \subset M \times N$ be the subspace $V := \{(x, y) \in M \times N \mid y \subset x\} \subset M \times N$. (Recall that $\mathbb{C}^n = \mathbb{C}^n \oplus 0 \subset \mathbb{C}^m$.) One has a map $\pi : V \to N$ that sends $(x, y) \in V$ to $y \in N$. This is the projection of a fibre bundle over N with fibre space $G_{m-k,l-k}$. To see this, regard V as a submanifold of the complex flag manifold $F = U(m)/U(k) \times U(l-k) \times U(m-l) = \{(A, B) \mid \dim_{\mathbb{C}} A = k, \dim_{\mathbb{C}} B = l - k, A \perp B, A, B \subset \mathbb{C}^m\}$ where a point $(x, y) \in V$ is identified with the point $(y, x') \in F$ where x' is the orthogonal complement of y in x so that $x' \perp y$ and $x = x' + y$. The projection map $p : F \to G_{m,k}$, defined as $(A, B) \mapsto A \in G_{m,k}$, of the $G_{m-k,l-k}$-bundle θ over $G_{m,k}$ maps V onto $G_{n,k} \subset G_{m,k}$. In fact $V = p^{-1}(G_{n,k})$ and so $\pi : V \to G_{n,k}$ is the projection of the bundle $\theta|_{G_{n,k}}$.

As usual we denote by $[N]$ the generator of the top cohomology group $H^{2k(n-k)}(N; \mathbb{Q})$.

Lemma 10.5. *Let $c = \operatorname{codim}_{M \times N} V = 2k(m - l)$. Let $v \in H^c(M \times N; \mathbb{Q})$ denote the cohomology class dual to $j : V \hookrightarrow M \times N$. Then $v \cup [N] \neq 0$ in $H^*(M \times N; \mathbb{Q})$.*

Proof. The cohomology class $[N]$ is dual to the submanifold $i : M \hookrightarrow M \times N$ where $i(x) = (x, \mathbb{C}^k), x \in M$. First we shall show that $i(M)$ intersects V transversely. Note that $i(M) \cap V = \{(x, \mathbb{C}^k) \mid \mathbb{C}^k \subset x \subset \mathbb{C}^m\} \cong G_{m-k,l-k}$, which is the fibre over the point $\mathbb{C}^k \in N$ of the bundle projection $\pi : V \to N$. Therefore $T_{i(x)}V/T_{i(x)}(V) \cap T_{i(x)}i(M)) \cong T_{\mathbb{C}^l}N$. Since $T_{i(x)}(M \times N)/T_{i(x)}M \cong T_{\mathbb{C}^l}N$, follows that $i(M)$ intersects

V transversely. Therefore $v \cup [N]$ is dual to the submanifold $V \cap i(M) \subset M \times N$. Since $V \cap i(M) \cong G_{m-k,l-k} \subset G_{m,l} = M$ represents a non-zero homology class in $H_{2(l-k)(m-l)}(M;\mathbb{Q}) \cong H_{2(l-k)(m-l)}(M;\mathbb{Q}) \otimes H_0(N;\mathbb{Q}) \subset H_{2(l-k)(m-l)}(M \times N;\mathbb{Q})$, its Poincaré dual, which equals $v \cup [N]$, is a non-zero cohomology class in $H^{2d}(M \times N;\mathbb{Q})$ where $d = k(m-l) + k(n-k)$. $\square$

Proof of Theorem 10.3: Consider the map $\phi := id \times f : M \times M \to M \times N$ defined as $\phi(x,y) = (x, f(y))$. Denote by $\delta : M \to M \times M$ the diagonal map.

We need to show that $\phi(\delta(M)) \cap V \neq \emptyset$.

Let $v \in H^*(M \times N;\mathbb{Q})$ denote the cohomology class dual to the manifold $V \subset M \times N$ and let $\Delta \in H^*(M \times M;\mathbb{Q})$ denote the diagonal class, i.e., the class dual to $\delta(M) \subset M \times M$. As is well-known v is in the image of the inclusion-induced homomorphism $H^*(M \times N, M \times N \setminus V;\mathbb{Q}) \to H^*(M \times N;\mathbb{Q})$. (See for example [19, Chapter 11].) By the naturality of cup-products and by considering the bilinear map $H^*(M \times N, M \times N \setminus \phi(\delta(M));\mathbb{Q}) \otimes H^*(M \times N, M \times N \setminus V;\mathbb{Q}) \stackrel{\cup}{\to} H^*(M \times N, M \times N \setminus (V \cap \phi(\delta(M)));\mathbb{Q})$ induced by the inclusion maps, it follows that if $V \cap \phi(\delta(M)) = \emptyset$, then $v \cup w = 0$ for any $w \in H^+(M \times N, M \times N \setminus \phi(\delta(M));\mathbb{Q})$. (See [29, §6, Chapter 5].) In particular, this holds for the class w that maps to the cohomology class α_f dual to the submanifold $\phi(\delta(M)) \hookrightarrow M \times N$ under the inclusion-induced map $H^{2k(n-k)}(M \times N, M \times N \setminus \phi(\delta(M));\mathbb{Q}) \to H^{2k(n-k)}(M \times N;\mathbb{Q})$. Thus $v \cup \alpha_f = 0$.

At the same time, $\mu_{M \times N} \cap \alpha_f = \phi_*(\delta_*(\mu_M))$. Our hypothesis on k, l, m, n implies, by Theorem 10.1, that ϕ_* does not depend on f. In particular, taking $f = c$, the constant map sending M to $\mathbb{C}^k \in N$, we obtain $\phi \circ \delta = i : M \hookrightarrow M \times N$ considered in the previous lemma. So $\phi_* \delta_*(\mu_M) = i_*(\mu_M)$ and we have $\alpha_f = [N]$. By the above lemma we have $v \cup \alpha_f = v \cup [N] \neq 0$, a contradiction. This completes the proof. $\square$

Suppose that $\dim(G_{n,k}) \leq \dim G_{m,l}$ and let $f : G_{m,l} \to G_{n,k}$ be a holomorphic map where we assume that $k \leq n/2, l \leq m/2$. When $\dim(G_{n,k}) = \dim G_{m,l}$ and $k > 1$, so that $G_{n,k}$ is not the projective space, it was proved by Paranjape and

Srinivas [26] that if f is not a constant map, then $(n,k) = (m,l)$ and f is an *isomorphism* of varieties.

Suppose that $\dim G_{n,k} < \dim G_{m,l}$. We claim that any holomorphic map $f : G_{m,l} \to G_{n,k}$ is a constant map. Indeed, the Picard group $Pic(G_{m,l})$ of $G_{m,l}$ of the isomorphism classes of complex (equivalently algebraic or holomorphic) line bundles is isomorphic to $H^2(G_{m,l};\mathbb{Z}) \cong \mathbb{Z}$ via the first Chern class. It is generated by the bundle $\xi_{m,l} := \det(\gamma_{m,l})$. The dual bundle $\xi_{m,l}^{\vee}$ is a very ample bundle (or a positive line bundle in the sense of Kodaira). Note that any holomorphic map between non-singular complex projective manifolds is a morphism of algebraic varieties. Now our claim is a consequence of the following more general observation.

Lemma 10.6. *Let $f : X \to Y$ be a morphism between two complex projective varieties where $Pic(X)$, the group of isomorphism class of algebraic line bundles over X, is isomorphic to the infinite cyclic group. If $\dim X > \dim Y$, then f is a constant morphism.*

Proof. Suppose that f is a non-constant morphism. Then there exists a projective curve $C \subset X$ such that $f|_C$ is a finite morphism. Let ξ be a very ample line bundle over Y and let $\eta = f^*(\xi)$. Since ξ is very ample, it is generated by its (algebraic) sections and so it follows that η is also generated by sections. Since $f|_C$ is a finite morphism, we see that $\eta|_C$ is ample, that is, some positive tensor power of $\eta|_C$ is very ample. In particular η is not trivial. Denote by ω the ample generator of $Pic(X) \cong \mathbb{Z}$ and let $\eta = \omega^{\otimes r}$ for some r. Since η is generated by its sections, we have $r \geq 0$. Since η is non-trivial, $r \neq 0$. It follows that $r > 0$ and η is ample.

On the other hand, since $\dim X > \dim Y$, some fibre Z of f is positive dimensional and the bundle $\eta|_Z$ is trivial. This is a contradiction since the restriction of an ample bundle to a positive dimensional subvariety is ample and non-trivial. $\square$

Here we have another application of 10.1.

Let X be a finite CW complex, then we can associate the set of all equivalent classes of complex vector bundles over it and we denote it by $K(X)$. Then $K(X)$ will form a $\mathbb{Z}$ -algebra. We have the Chern character map $ch : K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{even}(X; \mathbb{Q})$, defined by

$ch([\xi]) = Ch(c_1(\xi), c_2(\xi), \cdots, c_k(\xi))$,
where ξ is a complex bundle of rank k over X, $[\xi]$ is the class of ξ, $c_1(\xi), c_2(\xi), \cdots, c_k(\xi)$ are Chern classes of the bundle ξ and $Ch(c_1(\xi), c_2(\xi), \cdots, c_k(\xi))$ is defined as follows: If ξ is a line bundle, then

$$Ch(\xi) = \sum_{k\geq 0} c_1(\xi)^k/k! \ = \ exp(c_1(\xi))$$

If $\xi \cong \xi_1 \oplus \cdots \oplus \xi_k$ is a Whitney sum of line bundles, then $Ch(\xi) = \sum_{j=1}^{k} Ch(\xi_j)$. These properties define $Ch(\xi)$ uniquely by the splitting principle.

This map $ch : K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{even}(X; \mathbb{Q})$ is always a natural isomorphism of rings (See §3, chapter V, [16]).

When X has cells only in even dimensions, then $K(X) \cong \mathbb{Z}^{\chi(X)}$ (where $\chi(X)$ is the Euler characteristic of X). In particular, $K(X)$ is a free abelian group. Let $f : G_{m,l} \to G_{n,k}$ be a map, then by naturality we have the following diagram

$$\begin{array}{lcl} K(G_{m,l}) \otimes_{\mathbb{Z}} \mathbb{Q} & \stackrel{ch}{\to} & (H^*(G_{m,l}; \mathbb{Q}) \\ f^* \uparrow & & \uparrow f^* \\ K(G_{n,k}) \otimes_{\mathbb{Z}} \mathbb{Q} & \stackrel{ch}{\to} & H^*(G_{n,k}; \mathbb{Q}). \end{array}$$

As $G_{n,k}$ and $G_{m,l}$ have cells only in even dimensions, so $K(G_{n,k})$ and $K(G_{m,l})$ are free abelian groups, so we have the following diagram

$$\begin{array}{lcl} K(G_{m,l}) & \stackrel{ch}{\to} & (H^*(G_{m,l}; \mathbb{Q}) \\ f^* \uparrow & & \uparrow f^* \\ K(G_{n,k}) & \stackrel{ch}{\to} & H^*(G_{n,k}; \mathbb{Q}). \end{array}$$

where the horizontal maps are injective.

Now, if m, l, k, n are as in 10.1, then $f^* : H^*(G_{n,k}; \mathbb{Q}) \to H^*(G_{m,l}; \mathbb{Q})$ is the trivial map between graded $\mathbb{Q}$ -algebras. Then, $f^* : K(G_{n,k}) \to K(G_{m,l})$ is again a trivial map, so we get the following result

Theorem 10.7. *Let $1 \leq k \leq \lfloor n/2 \rfloor$, $1 \leq l \leq \lfloor m/2 \rfloor$ and $k < l$, where m, n are positive integers such that $m - l > n - k$. Suppose that $m - l \geq 2k^2 - k - 1$ or $1 \leq k \leq 3$. Let $f : G_{m,l} \to G_{n,k}$ be a map. Then $f^*(\xi)$ is stably trivial for any complex vector bundle ξ over $G_{n,k}$.*

Remark 10.8. *Now. let $G_{n,k}$ be the quaternionic Grassmann manifold of k-dimensional subspaces of $\mathbb{H}^n$. If ϕ is any endomorphism of the graded $\mathbb{Q}$-algebra $H^*(G_{n,k}; \mathbb{Q})$ which vanishes on $H^2(G_{n,k}; \mathbb{Q})$, then ϕ vanishes in all positive degree if $k \leq 3$ or $n \geq 2k^2 - 1$ (analogous to the complex Grassmann manifold). See* [9]. *All other tools we have used for complex Grassmann are valid for quaternionic Grassmann manifolds, namely the natural imbedding and the formality of maps between Grassmann manifolds. So, we observe that Theorem 10.1, 10.2, 10.3, 10.7 are valid for quatrenionic Grassmann manifolds also.*

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