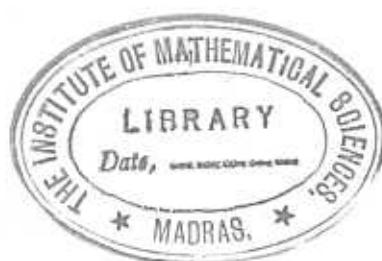




CLIFFORD ALGEBRA, ITS GENERALISATION  
AND THEIR APPLICATIONS  
TO  
SYMMETRIES AND RELATIVISTIC  
WAVE EQUATIONS

THESIS  
SUBMITTED TO  
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BY

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## PREFACE

This thesis titled 'Clifford algebra, its generalisation and their applications to symmetries and relativistic wave equations' consists of the work done by the author during 1967-71 for the Ph.D. degree (Phys.) of the University of Madras under the guidance of Professor Alladi Ramakrishnan, Director, The Institute of Mathematical Sciences, Madras-20, India. The author wishes to express his sincere thanks to him.

In this thesis, the author pursued the work initiated by Professor Alladi Ramakrishnan through the elegant paper, 'The Dirac Hamiltonian as a member of a Hierarchy of Matrices', *J.Math.Anal.Appl.*, 22, 9 (1967).

The nature and range of the work necessitated the author's collaboration with his colleagues and due acknowledgement of this is made at appropriate places.

Finally, the author would like to express his thankfulness to the members of MATSCIENCE for interesting discussions and to the Institute of Mathematical Sciences for the excellent facilities of research.

M. Chandrasekaran.  
(P.S.Chandrasekaran)

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### Introduction

In 1967, Ramakrishnan<sup>1)</sup> initiated a new programme of work starting with a mathematical procedure called the  $\circ$ -operation representing the transition from the  $2 \times 2$  Pauli<sup>2)</sup> matrices to higher dimensional anticommuting matrices. The  $\circ$ -operation defined by Ramakrishnan is a generalisation of the procedure adopted by Dirac<sup>3)</sup> to obtain the  $4 \times 4$  matrices occurring in his famous wave equation of the electron. If a linear combination of the higher dimensional matrices is considered then its eigenvectors turn out to be degenerate. To resolve this degeneracy, Ramakrishnan<sup>4)</sup> introduced the concept of helicity matrices in close analogy with the well known concept of helicity. It was also shown<sup>5)</sup> that energy and helicity belong to a hierarchy of eigenvalues.

In this thesis, we extend the above mentioned concepts which are related to the Clifford<sup>6)</sup> algebra of anticommuting matrices to the case of the generalised Clifford algebra of matrices which are the  $n$ th roots of the unit matrix

- 1) A. Ramakrishnan, 'Dirac Hamiltonian as a member of a hierarchy of matrices', *J. Math. Anal.* **21**, 2, (1967).
- 2) W. Pauli, 'Zur Quantenechanik des magnetischen Electrons', *Z. f. Physik*, **43**, 601 (1927).
- 3) P. A. M. Dirac, 'The quantum theory of the electron', *Proc. Roy. Soc. (A)* **117**, 610 (1928).
- 4) A. Ramakrishnan, 'Generalised helicity matrices', *J. Math. Anal. Appl.* **26**, 275, (1969).
- 5) A. Ramakrishnan, 'Helicity and energy as numbers of a hierarchy of eigenvalues', *J. Math. Anal.* **20**, 597, (1967).
- 6) W. K. Clifford, 'Applications of Grassmann's extensive algebra', *Amer. J. Math.* **1**, 350, (1878).

satisfying a generalised commutation rule, hereinafter called the  $\omega$ -commutation where  $\omega$  is a primitive  $n$ th root of unity. We then apply them to the study of the symmetry principles and elementary particle physics. To our surprise, we found that the generalised Clifford algebra was introduced as early as 1952 by Morinaga and Nono<sup>7)</sup> and independently by Yanasaki<sup>8)</sup>. The structure of the above generalisation of the Clifford algebra was studied by Morinaga and Nono<sup>7)</sup> and Morris<sup>9)</sup>. Our work essentially relates to the study of the eigenvector and eigenvalue structure of the matrices of the generalised Clifford algebra and their possible interpretation in elementary particle physics.

Instead of the anti-commuting property between the basic elements whose all possible products and their linear combinations thereof comprise the Clifford algebra,

$\omega$ -commutation relation between the basic elements is considered in the case of the generalised Clifford algebra (hereafter abbreviated as G.C.A.), where  $\omega$  is a primitive  $n$ th root of unity. In other words, the (-1) factor which is the square root of unity in anticommutation relations is replaced by  $\omega$ . The square condition which yields the unit

- 7) K.Morinaga and T.Nono, 'On the linearisation of a form of higher degree and its representation', J. Sci. Hiroshima Univ. Series A, 6, 15, (1952).
- 8) K.Yanasaki, 'On projective representation and ring extensions of finite groups', J. Fac. Sci. Univ. of Tokyo, Sect.1, 19, 147, (1964).
- 9) A.O.Morris, 'On a generalised Clifford algebra', Quart. J. Math. (Oxford), 18, 7, (1967).

matrix in the Clifford algebra is replaced in G.C.A. by the rule that each basic element is multiplied m-times to get the unit element.

It is important to note that even in the G.C.A. for each value of m there are only three  $\omega$ -commuting matrices in the lowest dimension, m. This is similar to the case of the set of the three Pauli matrices which is a representation of the ordinary Clifford algebra, in the lowest non-trivial dimension 2 (vide Chapter II).

In the first chapter of this thesis the method of Ramakrishnan<sup>1)</sup> called the  $\sigma$ -operation developed for the ordinary Clifford algebra is applied<sup>10)</sup> to the G.C.A. This chapter also contains a discussion<sup>11)</sup> of massless particles in higher dimensional spaces, using the Clifford algebra elements. Another method of obtaining irreducible representations using concept of ideals of G.C.A. is given<sup>12)</sup> in Chapter II, suitably modifying results of Rosevskii<sup>13)</sup>. The method<sup>14)</sup> of obtaining commuting sets of G.C.A. of the same type as and from a given G.C.A. is described in

- 10) A.Ramakrishnan, R.Vasudevan, N.R.Ranganathan and P.S.Chandrasekaran, 'On a generalisation of the L-matrix hierarchy', *J. Math. Anal. Appl.* 19, 10, (1969)
- 11) T.S.Santhanam and P.S.Chandrasekaran, 'Clifford algebra and massless particles', *Prog. Theor. Phys.* 41, 264, (1969)
- 12) A.Ramakrishnan, T.S.Santhanam and P.S.Chandrasekaran, 'On the representations of the generalised Clifford algebras', *J. Math. and phys. Sci.*, (Madras) 2, 307, (1969)
- 13) P.E.Rosevskii, 'The theory of spinors', *AM Translations (Series II)*, 5, 1, (1957).
- 14) A.Ramakrishnan, T.S.Santhanam, P.S.Chandrasekaran and A.Sundaram, 'Helicity matrices for the generalised Clifford algebra', *J. Math. Anal. Appl.* 26, 27, (1969).

Chapter III using the concept of helicity matrices introduced by Ramakrishnan<sup>5)</sup>.

Chapter IV and the succeeding ones constitute possible applications of the above ideas. If the Pauli matrices are understood through their Lie commutation relations they are recognised to be the elements of the Lie algebra of  $SU(2)$ , the special unitary matrices of dimension 2. In this thesis, we also show<sup>16)</sup> how the Lie algebra of  $3 \times 3$  special unitary matrices are connected with the Lie algebra of G.C.A. with  $n = 3$  (vide Chapter IV). As a consequence we also obtain a generalisation of the Gell-Mann-Nishijima relations. The above results are easily generalised to the  $SU(n)$  case. The Para-Fermi operators which were obtained by Green<sup>16)</sup> in his study generalising the Fermi-Dirac and Bose-Einstein quantisations are constructed<sup>17), 18)</sup> in Chapters V and VI using G.C.A. In Chapter VII, the elements of the Lie

- 15) A.Ramakrishnan, P.S.Chandrasekaran, N.R.Ranganathan, T.S.Ganthanayakam and R.Vasudevan, 'The generalised Clifford algebra and the unitary group', *J. Math. Anal. Appl.* **22**, 164, (1969).
- 16) H.S.Green, 'A generalised method of field quantisation', *Phys. Rev.* **92**, 270, (1953).
- 17) A.Ramakrishnan, R.Vasudevan, P.S.Chandrasekaran and N.R.Ranganathan, 'Kerner algebra and generalised Clifford elements', *J. Math. Anal. Appl.* **22**, 100, (1969).
- 18) A.Ramakrishnan, R.Vasudevan and P.S.Chandrasekaran, 'Representations of Para-Fermi operators and generalised Clifford algebra', *J. Math. Anal. Appl.* **52**, 1, (1975).

algebra of  $SU(n)$  are given<sup>19)</sup> as operator representation in terms of the Para-Fermi operators. In Chapter VIII, one of the two generalised Clifford conditions alone are insisted upon, that is, each element when multiplied  $n$  times yields the unit matrix and a method<sup>20)</sup> to obtain the matrix representation of such matrices is discussed.

- 19) A. Banerjeean, N. Vasudevan and P.G. Chandrasekaran, 'Para-Fermi operators and special unitary algebras', *J. Math. Anal. Appl.*, (to be published).
- 20) T.G. Santhanam, P.G. Chandrasekaran and Kalini D. Nenon, 'On general involitional matrices', *J. Math. Phys.*, (to be published).

## CHAPTER I

### Dirac Matrices And Their Generalisations

**1.1** In this chapter we give first the method of  $\sim$ -operation introduced by Ramakrishnan<sup>1)</sup> to get the representation of the Dirac<sup>2)</sup> matrices and their generalisations starting from the set of Pauli<sup>3)</sup> matrices.

The Pauli matrices are known to satisfy the Clifford algebra. Then the method is extended<sup>4)</sup> to find the representations of the generalised Clifford algebra<sup>5), 6), 7)</sup>

**1.2** The Clifford<sup>8)</sup> algebra,  $C_p^2$  is generated by a set of  $p$  elements  $L_i$ ,  $i=1, 2, \dots, p$  obeying the two

- 1) Alladi Ramakrishnan, 'The Dirac Hamiltonian as a member of a hierarchy of matrices', *J. Math. Anal. Appl.* **22**, 9, (1967).
- 2) P.A.M. Dirac, 'The quantum theory of the electron', *Proc. Roy. Soc. (A)* **117**, 620, (1928).
- 3) W. Pauli, 'Zur Quantemechanik des magnetischen electrons', *Z. f. Physik*, **43**, 601, (1927).
- 4) Alladi Ramakrishnan, R. Vasudevan, N.R. Ranganathan and P.S. Chandrasekaran, 'A generalisation of the L-matrix hierarchy', *J. Math. Anal. Appl.* **22**, 10, (1969).
- 5) K. Morinaga and T. Hono, 'On the linearisation of a form of higher degree and its representations', *J. of Sci. of Hiroshima Univ., Series A*, **14**, 15, (1954).
- 6) K. Yamazaki, 'On projective representations and ring extensions of finite groups', *J. Fac. Sci., Univ. of Tokyo, Sec. I*, **10**, 147, (1964).
- 7) A.O. Morris, 'On a generalised Clifford algebra', *Quart. J. Math. (Oxford)*, **18**, 7, (1967).
- 8) W.K. Clifford, 'Applications of Grassmann's extensive algebra', *Amer. J. Math.*, **1**, 350, (1876).

### Clifford conditions

$$\mathcal{L}_i \mathcal{L}_j = - \mathcal{L}_j \mathcal{L}_i ; i \neq j ; i, j = 1, 2, \dots, p \quad (1.1)$$

and

$$\mathcal{L}^2 = I \quad (1.2)$$

where  $I$  is the unit element. In  $\mathcal{C}_p^2$ ,  $p$  stands for the number of generating elements.  $\mathcal{L}_i$  and  $i$  the index to which each generating element has to be raised to get the unit element as in eq.(1.2). The well known examples of the algebra characterised by (1.1) and (1.2) are the Pauli and Dirac algebras.

1.3 Let us take a linear combination of the Pauli matrices  $\sigma_1, \sigma_2$  and  $\sigma_3$  and denote it by

$$\mathcal{L}_3 = \sum_{i=1}^3 \lambda_i \sigma_i = \sum_{i=1}^3 \lambda_i \mathcal{L}_i^3 = \begin{pmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix} \quad (1.3)$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.4)$$

where  $\lambda_i$ s are real or pure imaginary parameters.

Using the Clifford conditions (1.1) and (1.2) we have that

$$\mathcal{L}_3^2 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)I = A^2 I \quad (1.5)$$

where

$$\Delta_1 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2} \quad (1.6)$$

Now if we define a matrix,  $L_{2n+1}$  having  $(2n+1)$  parameters  $\lambda_1, \lambda_2, \dots, \lambda_{2n+1}$  obtained from  $L_{2n-1}$  by

$$L_{2n+1} = \begin{pmatrix} \lambda_{2n+1} I & L_{2n-1} - i\lambda_{2n} I \\ L_{2n-1} + i\lambda_{2n} I & -\lambda_{2n+1} I \end{pmatrix} \quad (1.7)$$

and  $L_{2n-1}$  is defined through  $L_{2n-3}$  and so on in the same way. Hence  $L_5$  is defined in terms of  $L_3$ .  
Let us also write

$$L_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \mathcal{L}_i^{2n+1} \quad (1.8)$$

and

$$L_{2n-1} = \sum_{i=1}^{2n-1} \lambda_i \mathcal{L}_i^{2n-1} \quad (1.9)$$

Then using (1.7), (1.8), (1.9) we get

$$\mathcal{L}_i^{2n+1} = \begin{pmatrix} 0 & \mathcal{L}_i^{2n-1} \\ \mathcal{L}_i^{2n-1} & 0 \end{pmatrix} \quad ; i = 1, 2, \dots, 2n-1 \quad (1.10)$$

$$\mathcal{L}_{2n}^{2n+1} = \begin{pmatrix} 0 - iI & \\ iI & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{L}_{2n+1}^{2n+1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (1.11)$$

Applying the above procedure to eq.(1.9) till we reach Pauli matrices, and using the anticommuting properties of the Pauli matrices we see that the  $\chi_i^{2n+1}$  ( $i=1, 2, \dots, 2n+1$ ) anticommute with one another and the square of each is the identity matrix.

The above method of obtaining  $L_{2n+1}$  from  $L_{2n-1}$

$$\text{i.e. } L_{2n+1} = \sigma(L_{2n-1}) \quad (1.12)$$

is called the  $\sigma$ -operation by Ramakrishnan<sup>1)</sup> which involves the addition of two parameters  $\lambda_{2n}$  and  $\lambda_{2n+1}$  to the set  $(\lambda_1, \lambda_2, \dots, \lambda_{2n-1})$  of  $L_{2n-1}$  and multiplying them by unit matrices of suitable dimensions.

The above procedure brings forth clearly that the dimension of  $\chi_i^{2n+1}$  ( $i=1, 2, \dots, 2n+1$ ) is  $2^n$  and  $(2n+1)$  is the maximal number of anticommuting matrices of dimension  $2^n$ . The linear combination  $L_{2n+1}$  containing this maximal set of  $(2n+1)$  matrices, will be called the saturated L-matrix. When  $L_{2n+1}$  contains only  $2n$  non-zero parameters then it is called an unsaturated matrix. We thus have the table connecting the number of parameters, dimension of the matrix and type:

| <u>Matrix</u> | <u>No. of parameters</u> | <u>Dimension</u> | <u>Character</u> |
|---------------|--------------------------|------------------|------------------|
| $L_1$         | 1                        | 1                | saturated (S)    |
| $L_2$         | 2                        | 2                | unsaturated (US) |
| $L_3$         | 3                        | 3                | S.               |
| $L_4$         | 4                        | 4                | US               |
| $L_5$         | 5                        | 4                | S                |
| .             | .                        | .                | .                |
| $L_{2n+1}$    | $2n+1$                   | $2^n$            | S                |

1.4 In the following we will consider the matrix

$$L_5 = \sum_{i=1}^5 \lambda_i L_i^5 \quad (1.13)$$

By the  $\sigma$ -operation it is also given by

$$L_5 = \sigma(L_3)$$

There are three ways of getting  $L_5$  starting from  $L_3$  by replacing any one of the three parameters in  $L_3$  by  $L_3$  itself and calling the other two as  $\lambda_4$  and  $\lambda_5$  and attaching unit matrices of dimension 2 to  $\lambda_4$  and  $\lambda_5$ .

$$L_5 = \begin{pmatrix} \lambda_5 I & L_3 - i\lambda_4 I \\ L_3 + i\lambda_4 I & -\lambda_5 I \end{pmatrix}; \quad L_5 = \begin{pmatrix} L_3 & (\lambda_4 - i\lambda_5)I \\ (\lambda_4 + i\lambda_5)I & -L_3 \end{pmatrix};$$

$$L_5 = \begin{pmatrix} \lambda_5 I & \lambda_4 I - iL_3 \\ \lambda_4 I + iL_3 & -\lambda_5 I \end{pmatrix} \quad (1.14)$$

All the three are connected to one another by similarity transformations. Let us consider the first of the three of eq.(1.14). If  $\alpha_x, \alpha_y, \alpha_z$  and  $\beta$  are the Dirac<sup>2)</sup> matrices i.e.

$$\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}; \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}; \alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix};$$

(1.15)

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \alpha_4 = \alpha_x \alpha_y \alpha_z \beta = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$$

(where  $\sigma_x, \sigma_y, \sigma_z$  are the Pauli matrices) and if  $\mathcal{L}_i^5$  ( $i = 1, 2, 3, 4, 5$ ) are the components of the matrix  $L_5$  then

$$\mathcal{L}_1^5 = \alpha_x; \mathcal{L}_2^5 = \alpha_y; \mathcal{L}_3^5 = \alpha_z; \mathcal{L}_4^5 = \alpha_4; \mathcal{L}_5^5 = \beta \quad (1.16)$$

where

$$\alpha_4 = + \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$$

1.5 If  $L$  is a linear combination of a set of anticommuting matrices with  $L^2 = \lambda^2 I$ , the eigenvalues of  $L$  are degenerate because the dimension of  $L$  is  $> 2$  and they are equal to  $\pm \lambda$ . Hence the following procedure is used to find the eigenvectors. If  $\Lambda$  is a diagonal matrix with half of its elements equal to  $\lambda$  and the rest  $-\lambda$  then

$$U = L + \Lambda I$$

(1.17)

satisfies the equation

$$LU = UA \quad (1.18)$$

i.e. the columns of the matrix  $U$  are the eigenvectors of  $L$  with eigenvalues  $\pm \lambda$ .

Hence the  $U$  matrix or the eigenvector matrix for  $L_5$  (of the first type in eq.(1.14) is

$$\begin{pmatrix} \lambda + \lambda_5 & 0 & \lambda_3 - i\lambda_4 & \lambda_1 - i\lambda_2 \\ 0 & \lambda + \lambda_5 & \lambda_1 + i\lambda_2 & -\lambda_3 - i\lambda_4 \\ \lambda_3 + i\lambda_4 & \lambda_1 - i\lambda_2 & -\lambda - \lambda_5 & 0 \\ \lambda_1 + i\lambda_2 & -\lambda_3 + i\lambda_4 & 0 & -\lambda - \lambda_5 \end{pmatrix} \quad (1.19)$$

corresponding to the eigenvalue  $+\lambda$  for the first two columns and  $-\lambda$  for the last two columns with

$$\lambda = +(\lambda_1^2 + \lambda_2^2 + \dots + \lambda_5^2)^{1/2} \quad (1.20)$$

In the above, if we put

$$\lambda_1 = p_x, \lambda_2 = p_y, \lambda_3 = p_z, \lambda_4 = 0, \lambda_5 = m, \lambda = E \quad (1.21)$$

where  $p_x, p_y, p_z$  are the components of the momenta of a particle of mass  $m = \lambda_5$ , then we see that  $L_5$  is simply the Dirac Hamiltonian for a particle with energy  $E$ .

If on the other hand we set

$$\lambda_1 = \beta_x, \lambda_2 = \beta_y, \lambda_3 = \beta_z, \lambda_4 = m, \lambda_5 = 0, \lambda = E \quad (1.22)$$

then instead of the Dirac equation

$$(\vec{\alpha} \cdot \vec{p} + \beta m) u = Eu \quad (1.23)$$

we have the new Dirac equation

$$(\vec{\alpha} \cdot \vec{p} + \alpha_4 m) u' = Eu' \quad (1.24)$$

However, there is a transformation<sup>9)</sup>

$$\left( \frac{1-\beta\alpha_4}{\sqrt{2}} \right) (\vec{\alpha} \cdot \vec{p} + \beta m) \left( \frac{1+\beta\alpha_4}{\sqrt{2}} \right) = \vec{\alpha} \cdot \vec{p} + \alpha_4 m \quad (1.25)$$

relating the two types of Hamiltonians.

The eigenvectors of  $L_{2m+1}$  can be given in terms of those of  $L_3$ . Denoting by  $u_{2m+1}$  an eigenvector of  $L_{2m+1}$  with dimension  $2^m$  it can be shown that

$$u_{2m+1} = \begin{pmatrix} a u_{2m-1} \\ b u_{2m-1} \end{pmatrix} \quad (1.26)$$

by evaluating a and b. Substituting the above form of the  $u_{2m+1}$  in the eigenvalue equation

$$L_{2m+1} u_{2m+1}^\pm = \pm \Lambda_m u_{2m+1}^\pm \quad (1.27)$$

9) This remark is due to T.S. Santhanam.

We obtain an eigenvalue equation for the two dimensional vector with components  $a, b$

$$\begin{pmatrix} \lambda_{2m+1} & \Delta_{m-1} - i\lambda_{2m} \\ \Delta_{m-1} + i\lambda_{2m} & -\lambda_{2m+1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \Delta_m \begin{pmatrix} a \\ b \end{pmatrix} \quad (1.28)$$

yielding two solutions

$$\frac{a}{b} = \frac{\pm \Delta_{m-1} - i\lambda_{2m}}{\pm \Delta_m - \lambda_{2m+1}} = \frac{\pm \Delta_m + \lambda_{2m+1}}{\Delta_{m-1} + i\lambda_{2m}} \quad (1.29)$$

In equation (1.27)  $u_{2m+1}^+$  is the eigenvector with the eigenvalue  $\Delta_m$  and  $u_{2m+1}^-$  corresponds to  $-\Delta_m$ . Similarly we have two more solutions if we replace by  $-\Delta_{m-1}$  in eq.(1.29).

1.6 In the following we apply the above considerations to the study<sup>4)</sup> of the generalised Clifford algebra ( $G, C, A$ ) introduced by K.Morinaga and T.Nono<sup>5)</sup> and independently by K.Tomasaki<sup>6)</sup>.

1.7 The generalised Clifford algebra,  $C_p^m$  is defined as the associative algebra generated by a set of  $p$  elements  $\mathcal{L}_i$  ( $i=1, 2, \dots, p$ ) obeying the conditions

$$\mathcal{L}_i \mathcal{L}_j = \omega \mathcal{L}_j \mathcal{L}_i \quad ; \quad i < j ; \quad i, j = 1, 2, \dots, p \quad (1.30)$$

and

$$\mathcal{L}_i^m = I \quad (1.31)$$

where  $\omega$  is a primitive  $n$ th root of unity. In  $C_p^m$ ,  $p$  stands for the number of  $\mathcal{L}$ -s and  $m$  the number of times each  $\mathcal{L}$  is multiplied by itself to get the unit matrix. The equations (1.30) and (1.31) are similar to the equations (1.1) and (1.2) respectively. The cross in  $\mathcal{L}$  indicates that we are working with the generators of G.C.A. When the number of generators is even :  $p=2n$  we can show that the algebra is simple (This will be done in Chapter II in a detailed fashion) so that the dimension of the algebra is  $m^{2n}$  and the matrix representations is of dimension  $m^n$ . When  $m=2$ ,  $p=2$  we have the Pauli algebra and the explicit derivation of the Pauli matrices themselves will be considered in Chapter II. In the next case when  $m=3$  and  $p=2$  we get the Pauli-like matrices

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ; Q = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } R = P^2 Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{pmatrix} \quad (1.32)$$

as the generators of the G.C.A.  $C_2^3$ . (These matrices were assumed apriori by A.O.Morris<sup>7)</sup> but in Chapter II we will actually derive them) The element corresponding to  $\sigma_z = i\sigma_x\sigma_y$  in  $C_2^2$  is  $P^2 Q$  in  $C_2^3$ .

In the remaining of this chapter we will concern ourselves with the application<sup>4)</sup> of the  $\sigma$ -operation to construct the representations of the algebra  $C_F^3$  having more number of generators starting with the linear combination of the matrices  $P$ ,  $Q$  and  $R = P^2Q$  corresponding to the case  $m = 3$  and  $n = 1$ . This procedure is easily applied to the general case of arbitrary  $n$ . Hereafter (in this chapter only) linear combinations of the generators in  $C_C\Lambda$  similar to eq.(1.3) will be denoted by  $L$ . Similar to eq.(1.3) we have

$$L_3 = \begin{pmatrix} \lambda_3 & \lambda_1 + \omega\lambda_2 & 0 \\ 0 & \omega\lambda_3 & \lambda_1 + \omega^2\lambda_2 \\ \lambda_1 + \lambda_2 & 0 & \omega^2\lambda_3 \end{pmatrix} \quad (1.33)$$

where  $\omega$  is a primitive cube root of unity. By the  $\sigma$ -operation one gets the matrix  $L_{2n+1}$  containing  $(2n+1)$  parameters by replacing in  $L_3$  the parameter  $\lambda_1$  by  $L_{2n-1}$  and calling  $\lambda_2$  and  $\lambda_3$  as  $\lambda_{2n}I$  and  $\lambda_{2n+1}I$  where  $I$  is the unit matrix of suitable dimension. In other words we can define

$$L_{2n+1} = \begin{pmatrix} \lambda_{2n+1}I & L_{2n-1} + \omega\lambda_{2n}I & 0 \\ 0 & \omega\lambda_{2n+1}I & L_{2n-1} + \omega^2\lambda_{2n}I \\ L_{2n-1} + \lambda_{2n}I & 0 & \omega^2\lambda_{2n+1}I \end{pmatrix} \quad (1.34)$$

Tracing  $L_{2n+1}$  to  $L_3$  backwards we note that the dimension of  $L_{2n+1}$  is  $3^n$  and the component matrices  $\omega$ -commute and their cubes give the unit matrix. In eq.(1.33) we could have replaced  $\lambda_2$  or  $\lambda_3$  by  $L_{2n+1}$  and named the other two as  $\lambda_{2n}I$  and  $\lambda_{2n+1}I$ . But this would mean that the matrices associated with  $\lambda_1, \lambda_2, \dots, \lambda_p$  with  $p=2n+1$  will not be ordered (vide eq.(1.30)) as the index value of  $\lambda$  increases. But this ordering can be made explicit by suitably redefining  $\lambda_i$  ( $i=1, 2, \dots, p$ ) as  $\lambda'_1, \lambda'_2, \dots, \lambda'_p$  so that  $\lambda'_i$  corresponds to  $L_i$  and so on. This is to be contrasted with the ordinary Clifford algebra where the above-mentioned reordering of  $\lambda$ 's is not at all necessary.

1.8 The matrix  $L_3$  given by eq.(1.33) has three eigenvalues  $\Lambda_1(1)$ ,  $\Lambda_1(2)$  and  $\Lambda_1(3)$  where

$$\Lambda_1(1) = \Lambda_1, \quad \Lambda_1(2) = \omega \Lambda_1, \quad \Lambda_1(3) = \omega^2 \Lambda_1$$

and

$$\Lambda_1 = (\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^{\frac{1}{3}} \quad (1.36)$$

Similar to eq.(1.27) with  $a$  and  $b$  given by eq.(1.29) we have the eigenvector corresponding to  $\Lambda_1(i)$ ; ( $i=1, 2, 3$ )

as

$$\begin{pmatrix} 1 \\ \frac{\Delta_1(i) - \lambda_3}{\lambda_1 + \omega \lambda_2} \\ \frac{(\Delta_1(i) - \lambda_3)(\Delta_1(i) - \omega \lambda_3)}{(\lambda_1 + \omega \lambda_2)(\lambda_1 + \omega^2 \lambda_2)} \end{pmatrix} \quad (1.36)$$

Since  $\mathbb{L}_{2n+1}$  is generated from  $\mathbb{L}_3$  we immediately recognise the structure of the eigenvectors of  $\mathbb{L}_{2n+1}$ . If  $\underline{\alpha}$  is an arbitrary vector of dimension  $3^{n+1}$  the vector of dimension  $3^n$  defined as

$$\begin{pmatrix} \underline{\alpha} \\ \underline{\alpha}' \\ \underline{\alpha}'' \end{pmatrix} = \begin{pmatrix} \underline{\alpha} \\ \frac{\Delta_n(i)\underline{\alpha} - \lambda_{2n+1}\underline{\alpha}}{\mathbb{L}_{2n-1} + \omega \lambda_{2n}\underline{\alpha}} \\ \frac{\mathbb{L}_{2n-1} + \lambda_{2n}\underline{\alpha}}{\Delta_n(i)\underline{\alpha} - \omega^2 \lambda_{2n}\underline{\alpha}} \end{pmatrix} \quad (1.37)$$

is an eigenvector of  $\mathbb{L}_{2n+1}$  with eigenvalues  $\Delta_n(i)$  where

$$\Delta_n^3 = (\lambda_1^3 + \lambda_2^3 + \dots + \lambda_{2n+1}^3) \quad (1.38)$$

and  $\Delta_n(i)$  for  $i = 1, 2, 3$ , stands for  $\Delta_n$ ,  $\omega \Delta_n$  and  $\omega^2 \Delta_n$  respectively.

It is to be noted that the matrix  $L_{2n+1}$  has only three eigenvalues  $\Delta_n$ ,  $\omega \Delta_n$  and  $\omega^2 \Delta_n$  though it has  $3^n$  eigenvectors. The U-matrix, defined in analogy with equation (1.19) for  $L_{2n+1}$  is

$$U = (L_{2n+1}^2 + L_{2n+1}\Delta + \Delta^2) \quad (1.39)$$

where

$$\Delta = \begin{pmatrix} \Delta_n I & & \\ & \omega \Delta_n I & \\ & & \omega^2 \Delta_n I \end{pmatrix} \quad (1.40)$$

where  $I$  is the  $3^{n-1} \times 3^{n-1}$  unit matrix.

For example the U-matrix of eigenvectors for  $L_3$  can be written as

$$\begin{pmatrix} \lambda_3^2 + \lambda_3 \Delta_1 + \Delta_1^2 & (\lambda_1 + \omega \lambda_2)(\omega \Delta_1 - \omega^2 \lambda_3) & (\lambda_1 + \omega \lambda_2)(\lambda_1 + \omega^2 \lambda_2) \\ (\lambda_1 + \omega^2 \lambda_2)(\lambda_1 + \lambda_2) & \omega^2(\lambda_3^2 + \Delta_1 \lambda_3 + \Delta_1^2) & (\lambda_1 + \omega^2 \lambda_2)(\omega^2 \Delta_1 - \lambda_3) \\ (\lambda_1 + \lambda_2)(\Delta_1 - \omega \lambda_3) & (\lambda_1 + \lambda_2)(\lambda_1 + \omega \lambda_2) & \omega(\lambda_3^2 + \lambda_3 \Delta_1 + \Delta_1^2) \end{pmatrix} \quad (1.41)$$

**1.9 Degeneracy is present even in the case of the eigenvectors of  $L_{2n+1}$  of the ordinary Clifford algebra characterised by eqs. (1.1) and (1.2).** This is because  $L_{2n+1}$  has got only two eigenvalues

$$\pm \Delta_n = \pm (\lambda_1^2 + \lambda_2^2 + \dots + \lambda_{2n+1}^2)^{1/2} \quad (1.42)$$

To resolve this degeneracy Ramakrishnan<sup>10)</sup> constructed a set of matrices

$$L_{2n+1}, \begin{pmatrix} L_{2n-1} & \\ & L_{2n-1} \end{pmatrix}, \dots, \begin{pmatrix} L_3 & & \\ & L_3 & \\ & & L_3 \end{pmatrix} \quad (1.43)$$

with eigenvalues

$$\pm \Lambda_n, \pm \Lambda_{n-1}, \dots, \pm \Lambda_1 \quad (1.44)$$

In eq.(1.43)  $L_{2k+1}$ ,  $k < \frac{p}{2} = 2n$  occurs  $2^{n-k}$  times on the diagonal so that all the matrices of eq.(1.43) are of dimension  $2^n$ .

Taking the particular case of  $L_5$  and taking

$$\lambda_1 = p_x, \lambda_2 = p_y, \lambda_3 = p_z, \lambda_4 = 0, \lambda_5 = m \text{ and } \Lambda_2 = E \quad (1.45)$$

we have

$$\Delta_1 = \sqrt{p_x^2 + p_y^2 + p_z^2} = |\vec{p}|^2$$

which is an eigenvalue of  $L_3$ . The set corresponding to eq.(1.39) is given by

$$L_5 \quad \text{and} \quad \begin{pmatrix} L_3 & & \\ & L_3 & \\ & & L_3 \end{pmatrix} \quad (1.46)$$

10) Alladi Ramakrishnan, 'Helicity and energy as members of a hierarchy of matrices', J. Math. Anal. Appl. 20 397, (1967).

Since helicity is the eigenvalue of the operator

$$\frac{\vec{e} \cdot \vec{p}}{|\vec{p}|} = \frac{\vec{e} \cdot \vec{p}}{\Lambda_1} \quad (1.47)$$

we have

$$\pm \Lambda_1 = \text{helicity} \cdot \Lambda_1 \quad (1.48)$$

Hence helicity multiplied by a positive numerical factor  $\Lambda_1$  and energy are members of an hierarchy of eigenvalues.

In a similar way the degeneracy of the eigenvectors of  $L_{2n+1}$  for the  $\omega^3=1$  case i.e.,  $C_{2n}^3$  can be resolved by constructing  $n$  matrices

$$L_{2n+1}, \begin{pmatrix} L_{2n-1} & & \\ & L_{2n-1} & \\ & & L_{2n-1} \end{pmatrix}, \dots, \begin{pmatrix} L_3 & & \\ & L_3 & \\ & & L_3 \end{pmatrix} \quad (1.49)$$

where  $L_{2m+1}$  ( $m < n$ ) occurs  $3^{n-m}$  times so that the dimension of all matrices in eq.(1.49) is  $3^n$ .

**1.10** The extension of the above procedure when  $m$  is any integer (i.e.  $\omega^m=1$ ) and when the number of generators equals  $p=2n$  can be made directly and the eigenvector matrix  $U$  has the form

$$U = [L_{2n+1}^{m-1} + L_{2n+1}^{m-2} \Lambda_1 + \dots + \Lambda_1^{m-1}] \quad (1.50)$$

where

$$\prod_{z=n+1}^m (\Delta_n)^m I = (\lambda_1^m + \lambda_2^m + \dots + \lambda_{z=n+1}^m) I \quad (1.51)$$

and

$$\Delta_n = \begin{pmatrix} \Delta_n(1) I & & \\ & \Delta_n(2) I & \\ & & \Delta_n(m) I \end{pmatrix} \quad (1.52)$$

where

$$\Delta_n(i) = w^{i-1} \Delta_n \quad (1.53)$$

and  $I$  is the  $m \times m$  unit matrix.

### 1.11 Clifford Algebra And Massless Particles<sup>11)</sup>

In the following we briefly describe the application of the ordinary Clifford algebra in the study of massless particles. The freedom one has to linearising the Klein-Gordon equation has been used to describe the massless spin-half particles (neutrino) in two different ways. The first, well known of course is to describe it through a way equation of the Dirac type in which the mass parameter is set equal to zero so that the equation describing it is

$$(-i)\gamma_\mu \partial^\mu \psi = 0 \quad (1.54)$$

$$\gamma_\mu \partial^\mu = \gamma_0 \partial_0 + \gamma_1 \partial_1 + \gamma_2 \partial_2 + \gamma_3 \partial_3 \quad (1.55)$$

where  $\gamma_\mu$  are the Dirac matrices and the  $\partial_\mu$  are the differential operator with respect to space and time and the summation convention for repeated indices has been used. The equivalence of this equation to the familiar two-component one is well known<sup>12)</sup>. An alternate

11) T.S.Santhanam and P.S.Chandrasekaran, 'Clifford algebra and massless particles', Prog. Theor. Phys. 42, 264, (1969).

12) T.D.Lee and C.N.Yang, 'Parity non-conservation and a two component theory of the neutrino', Phys. Rev. 106, 1671 (1957).

L.London, 'On the conservation laws for weak interactions', Nucl. Phys. 2, 127, (1957).

A.Salam, 'On parity non-conservation and neutrino mass', Nuovo Cimento, 5, 367, (1957).

K.H.Case, 'Reformulation of the Majorana theory of neutrino', Phys. Rev. 107, 307, (1957).

equivalent way is to use singular idempotent matrices without explicitly putting the mass parameter equal to zero. This has been done in the literature long back<sup>13)</sup>. Recently, attention has been drawn to this fact by Z.Tokuoka<sup>14)</sup> and N.D.Sengupta<sup>15)</sup>. In this case one uses all the five anticommuting matrices in four dimensions and the equation of motion can be written as

$$\left( (-i)\gamma_\mu \partial^\mu + m_1 (1 \pm \gamma_5) \right) \psi = 0 \quad (1.56)$$

using the Dirac matrices  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  and  $\gamma_5$ . It is clear that equations (1.54) and (1.56) are inequivalent. The parameter  $m_1$  (not connected with the mass as it does not make its appearance in the Klein-Gordon equation) has been interpreted as the degree of chirality.

In this chapter, we show that even in the case of a linear equation involving the complete set of  $(2n+1)$  mutually anticommuting matrices forming the basic elements of the Clifford algebra  $C_{2n}^2$ , of dimension  $2^{2n}$ , it is shown that this equation can be reduced to an equation

- 13) Harish-Chandra, 'On the correspondence between the particle and wave aspects of the meson and photon', Proc. Roy. Soc. A195, 552, (1946).
- 14) H.J.Bhabha, 'On the postulational basis of the theory of elementary particles', Rev. Mod. Phys., 22, 451, (1950).
- 15) Z.Tokuoka, 'A proposal of neutrino equation', Prog. Theor. Phys. 22, 603, (1967).
- 15) N.D.Sengupta, 'The wave-equation for zero rest-mass particles', Nucl. Phys. B4, 147, (1968).

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of the Dirac-type involving only four anticommuting matrices, when the particle is massive. On the other hand, in the case of the massless particles, the equation will involve five anticommuting matrices, one of them in a singular idempotent combination with the unit matrix. We show that even in this case we can reduce the equation to the original Dirac form without the mass term.

1.12 Now, we briefly review the work of N.D.Sengupta<sup>15).</sup> The equation of motion which uses all the five mutually anticommuting matrices in four dimensions can be written as

$$(-i\gamma_\mu \partial^\mu + m_1 \gamma_5 + m_2) \psi = 0 \quad (1.57)$$

where  $\gamma_\mu$  ~~matrices~~ are hermitian matrices. It follows that the square of the Hamiltonian

$$H^2 = |\vec{p}|^2 + (m_2^2 - m_1^2) \quad (1.58)$$

where  $|\vec{p}|^2$  is the square of the modulus of the momentum,  $\vec{p}$ .

If eq.(1.57) is multiplied by an operator

$$O = \frac{1}{\sqrt{m_2^2 - m_1^2}} (m_2 - m_1 \gamma_5) \quad (1.59)$$

we get

$$(-i\gamma_\mu' \partial^\mu + m') \psi = 0 \quad (1.60)$$

where

$$\gamma_{\mu}' = 0 \gamma_{\mu} \quad (1.61)$$

and

$$m' = (m_1^2 - m_2^2)^{\frac{1}{2}} \quad (1.62)$$

Now, the  $\gamma_{\mu}'$ 's are not Hermitian. However, by a transformation

$$\gamma_{\mu}' = e^{-\gamma_5 \varphi} \gamma_{\mu} e^{\gamma_5 \varphi} \quad (1.63)$$

with

$$\tan \varphi = \frac{m_1}{m_2} \quad (1.64)$$

eq.(1.63) reduces to

$$(-i \gamma_{\mu} \partial^{\mu} + m') \psi' = 0 \quad (1.65)$$

where

$$\psi' = e^{\gamma_5 \varphi} \psi \quad (1.66)$$

Hence, even if the complete set of anticommuting matrices were used in writing the equation of motion, by a suitable transformation, the equation can be brought to the standard Dirac form. The demonstration is true when  $m_1^2 \neq m_2^2$  but eq.(1.63) and (1.66) do not exist when  $m_1^2 = m_2^2$  in which case the equation of motion describing the massless spin-half particle splits

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into two, corresponding to the two signs + or - as

$$(-i\gamma_\mu \partial^\mu + m_1(1 \pm \gamma_5)) \psi_\pm = 0 \quad (1.67)$$

and the two equations are connected by a change conjugation operator. It is interesting to note that  $(1 \pm \gamma_5)$  is singular and idempotent. Arguments have been advanced that  $m_1$  can be interpreted as the degree of Chirality<sup>15)</sup>.

1.13 It is shown below that in the case of massive spin-half particles, even if we describe it by a wave function in  $2^n$  dimensions through an equation containing  $(2n+1)$  parameters, it could still be brought to the standard Dirac form involving only four anti-commuting matrices by a suitable transformation. In the case of massless particles, the equation can be reduced to the form of equation (1.67) involving five anti-commuting matrices. In this case, the equation involves a singular idempotent matrix.

It is known that there are  $(2n+1)$  mutually anti-commuting matrices of dimension  $2^n$  which form the complete set of elements satisfying the Clifford algebra  $C_{2n}^2$ . These matrices are easily constructed using the elegant method due to Ramakrishnan<sup>16)</sup>. The most general wave equation of a spinor particle which makes use of

all these  $(2n+1)$  matrices can be written as

$$(-i \Gamma_\mu \partial^\mu + m_1 \Gamma_4 + \dots + m_{2n-3} \Gamma_{2n} + m_{2n-2}) \psi = 0 \quad (1.68)$$

where

$$\Gamma_\mu = \Gamma_{0,1,2,3}$$

and

$$\begin{aligned} \Gamma_i \Gamma_j + \Gamma_j \Gamma_i &= 2g_{ij}; \quad i, j = 0, 1, 2, 3. \quad (1.69) \\ &= 2\delta_{ij}, \quad i, j = 4, 5, \dots, 2n \end{aligned}$$

with

$$\begin{aligned} \Gamma_\mu \partial^\mu &= \Gamma_0 \partial_0 + \Gamma_1 \partial_1 + \Gamma_2 \partial_2 + \Gamma_3 \partial_3 \quad (1.70) \\ g &= (1, -1, -1, -1) \end{aligned}$$

In this case we have

$$|\vec{p}|^2 = |\vec{p}|^2 + (m_{2n-2}^2 - m_1^2 - m_2^2 - \dots - m_{2n-3}^2) \quad (1.71)$$

If we multiply eq.(1.68) on the left by

$$O' = \frac{m_{2n-3} - (m_1 \Gamma_4 + \dots + m_{2n-3} \Gamma_{2n})}{[m_{2n-2}^2 - \sum_{i=1}^{2n-3} m_i^2]} \quad (1.72)$$

we get

$$(-i \Gamma_\mu' \partial^\mu + m') \psi = 0 \quad (1.73)$$

where

$$m' = (m_{2n-2}^2 - \sum_{i=1}^{2n-3} m_i^2)^{\frac{1}{2}} \quad (1.74)$$

and

$$\Gamma'_\mu' = 0 \Gamma_\mu$$

This can be brought to the normal Hermitian form by a transformation

$$e^{-\chi\varphi} \Gamma'_\mu e^{\chi\varphi}$$

where

$$\chi = \frac{m_1 \Gamma_4 + \dots + m_{2n-3} \Gamma'_{2n}}{\left( \sum_{i=1}^{2n-3} m_i^2 \right)^{1/2}}$$

and

$$\tan \varphi = \left( \sum_{i=1}^{2n-3} m_i^2 \right)^{1/2} / m_{2n-3} \quad (1.75)$$

so that

$$(-i \Gamma'_\mu \partial^\mu + m') \psi' = 0$$

with

$$\psi' = e^{\chi\varphi} \psi$$

The above eq.(1.75) is not valid when  $m_{2n-2}^2 = \sum_{i=1}^{2n-3} m_i^2$ .

In this case let us define

$$\Gamma' = \frac{1}{m_{2n-2}} (m_1 \Gamma_4 + m_2 \Gamma_5 + \dots + m_{2n-3} \Gamma'_{2n}) \quad (1.76)$$

so that

$$\Gamma'^2 = I$$

and

$$\Gamma \Gamma_\mu + \Gamma_\mu \Gamma = 0; \mu = 0, 1, 2, 3$$

Now eq.(3.14) takes form

$$(-i\Gamma_\mu \gamma^\mu + m_{2n-2} (I \pm \Gamma)) \psi = 0 \quad (3.77)$$

which has the same form as eq.(1.67).  $I \pm \Gamma$  are

again idempotent and singular. The parameters

$m_1, m_2, \dots, m_{2n-3}$  appear only in the representations of  $\Gamma$  matrix.

1.14 It is worth pointing out here that even in the case of massless particles, we can still describe them through an equation involving only four anticommuting matrices and in this case one gets a wave function which is an eigenstate of the chirality operator. For if we multiply equation (1.67) by the singular operator  $(I \mp \gamma_5)$  we get

$$(\gamma_\mu \gamma^\mu \psi'_\pm) = 0 \quad ; \quad \gamma_5^2 = I \quad (1.78)$$

$$\psi'_\pm = (I \pm \gamma_5) \psi$$

Equation (1.78) involves only four anticommuting matrices and this is just the Dirac equation, with zero mass.

It is interesting to note that (1.78) is  $\gamma_5$  invariant and  $\psi'_\pm$  are eigenstates of  $\gamma_5$ . This is also true in the case of (1.77) with the necessary changes:  $\gamma_5 \rightarrow \Gamma$ .

## CHAPTER II

On the Representations of the Generalised Clifford Algebra

2.1 In this chapter, we give a method<sup>1)</sup> generalising that Rasovskii<sup>2)</sup> for obtaining the representations of the G.C.A. We prove that the G.C.A. generated by an even number of elements is simple. When the number of generators is odd there exists several inequivalent representations.

2.2 The generalised Clifford algebra is defined as the associative algebra generated by  $\rho$  elements  $\mathcal{L}_i$  ( $i=1, 2, \dots, \rho$ ) over a field  $k$  containing the  $n$ th and  $2n$ th roots of unity such that

$$\begin{aligned}\mathcal{L}_i \mathcal{L}_j &= \omega \mathcal{L}_j \mathcal{L}_i ; i < j ; i, j = 1, 2, \dots, \rho \\ \mathcal{L}_i^n &= I\end{aligned}$$

where  $\omega$  is a primitive  $n$ th root of unity.

We call this algebra as  $C_{\rho}^n$  using the notation of Morris<sup>3)</sup>. A bar across  $\mathcal{L}$  has been introduced in the

- 
- 1) Alladi Ramakrishnan, T.S.Santhanam and P.S.Chandrasekaran, 'On the representations of the Generalised Clifford Algebra', J. Math. and Phys. Sci., (Madras), 2, 307, (1960).
  - 2) P.K.Rasovskii, 'The Theory of Spinors', A.M.S. Translations (Series II) 1, 1, (1957). See also the review article 'Lie-matrices and the fundamental theorem in the theory of spinors', by Alladi Ramakrishnan, T.S.Santhanam and P.S.Chandrasekaran in Symposia on Theoretical Physics and Maths. 10, 63, (1970) Published by Plenum Press, N.Y., USA.
  - 3) A.O.Morris, 'On a generalised Clifford algebra', Quart. J. Mathematics, (Oxford), 18, 7, (1967).

*Earlier refs. on G.C.A.:*  
 K. Morinaga and T. Nono, J. Sci. Hiroshima Univ. Series A, 6, 13, (1952).  
 K. Yamazaki, J. Fac. Sci. Univ. of Tokyo, Sect 1, 10, 147, (1964).

above equations to denote that we are working with the generators of  $C, C_A$ .

2.3 Let  $p = 2n$  (even). It is shown<sup>4)</sup> that the algebra is simple<sup>5)</sup>.

**Lemma.** For even values of  $p$ , the algebra  $C_p^m$  contains no two sided ideals except the null ideal and itself. In otherwords, it is simple.

**Proof.** Let  $\mathbb{I}$  be any two sided ideal in  $C_{2n}^m$  and let  $\mathbb{I}$  be different from the null ideal. Any element of the algebra will be a linear combination of the elements of the form

$$\mathcal{L}_1^{k_1} \mathcal{L}_2^{k_2} \dots \mathcal{L}_{2n}^{k_{2n}} \quad (2.1)$$

with  $0 \leq k_i \leq m-1; i=1, 2, \dots, 2^n$

The total number of elements given by (2.1) is  $m^{2n}$ . Any element of the ideal  $\mathbb{I}$  is also a linear combination of elements which is a subset of the elements given by (2.1). Let  $A$  be a linear combination of this subset belonging to  $\mathbb{I}$ . Without loss of generality we will assume that  $A$  contains constant terms also. Let ' $a$ ' be a member in the linear combination  $A$  and

$$a = \mathcal{L}_1^{p_1} \mathcal{L}_2^{p_2} \dots \mathcal{L}_{i_k}^{p_k} \quad (2.2)$$

4) P.S.Chandrasekaran (to be published).

5) Vide Appendix for definitions of ideal of algebra, simple algebra etc.

with

$$0 \leq p_j < m-1 \quad ; \quad j = 1, 2, \dots, n$$

$$i_1, i_2, \dots, i_k = 1, 2, \dots, 2n$$

For purposes of convenience let us assume, first, that there exists an  $\lambda_j$  such that  $j < i_1 < i_2 < \dots < i_k$ . Let  $d = p_1 + p_2 + \dots + p_k$ , such that  $d$  is not divisible by  $m$ . Then it follows that

$$\sum_{i=1}^{m-1} \lambda_j^i a \lambda_j^{m-i} = (\omega + \omega^2 + \dots + \omega^{m-1}) a \quad (2.3)$$

where

$$\omega = \omega^{p_1 + p_2 + \dots + p_k} \quad (2.4)$$

Since  $\omega$  is a primitive root we always have

$$\omega + \omega^2 + \dots + \omega^{m-1} = 1 \quad (2.5)$$

If the degree  $d$  is divisible by  $m$  then each factor in the sum (2.3) gives the factor 1 so that (2.3) reads

$$\sum_{i=1}^{m-1} \lambda_j^i a \lambda_j^{m-i} = (m-1)a \quad (2.6)$$

The above considerations still hold good even if  $j = i_1$ , or  $i_2, \dots$ , or  $i_k$  when we drop the  $j$  term in counting  $d$  defined above. The result of equations (2.3) and (2.6) is unaltered if  $j$  happens to be greater than  $i_1$  or  $i_2$  or ... or  $i_k$ .

Let us now consider the sum

$$A_1 = A + \sum_{i=1}^{m-1} \mathcal{L}_1^i A \mathcal{L}_1^{m-i} \quad (2.7)_1$$

Then because of (2.3), (2.5) and (2.6) some terms in  $A$  get cancelled with that arising from the second term and some terms get added to  $A$ . Again define

$$A_2 = A_1 + \sum_{i=1}^{m-1} \mathcal{L}_2^i A_1 \mathcal{L}_2^{m-i} \quad (2.7)_2$$

$$A_{2n} = A_{2n-1} + \sum_{i=1}^{m-1} \mathcal{L}_{2n}^i A_{2n-1} \mathcal{L}_{2n}^{m-i} \quad (2.7)_{2n}$$

Because of (2.3), (2.5) and (2.6)  $A_{2n}$  contains only constant terms.

The above is true for any element  $A \in \mathbb{I}$  and hence for  $\mathbb{I}$  itself. By the definition of the ideal it follows that

$$\begin{aligned} \mathbb{I} &= \sum_{i=1}^{m-1} \mathcal{L}_1^i \mathbb{I} \mathcal{L}_1^{m-i} + \sum_{i=1}^{m-1} \mathcal{L}_2^i \mathbb{I} \mathcal{L}_2^{m-i} + \dots \\ &\quad + \sum_{i=1}^{m-1} \mathcal{L}_{2n}^i \mathbb{I} \mathcal{L}_{2n}^{m-i} \end{aligned} \quad (2.7)$$

Also  $\mathbb{I} \cdot C_{2n}^m = \mathbb{I}$ . By  $(2.7)_1, (2.7)_2, \dots, (2.7)_{2n}$

we have that  $\mathbb{I} \cdot C_{2n}^m = \text{const. } C_{2n}^m$  so that  $C_{2n}^m$  is simple.

By Wedderburn's theorem<sup>6)</sup> then we have that there is only

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6) C. P. H. Boerner, 'Representations of Groups', North Holland Pub.

one irreducible representation of  $C_{2n}^m$  of dimension  $m^n$ .

We will illustrate the above results of equations (2.3), (2.5) and (2.6) by considering the example of  $C_4^3$ .

In this case  $1 + \omega + \omega^2 + \omega^3 = 0$  where  $\omega$  is a primitive 4th root of unity.

Let  $a = \mathbb{L}_2^2 \mathbb{L}_3$ . Here  $d = 3$  and  $n = 4$ , so that  $m$  does not divide  $d$ .

Case 1.  $j < i_1 < i_2 < \dots < i_k$  For the above  $a$ ,

$$i_1=2, i_k=3 \quad \text{Now consider } \sum_{i=1}^3 \mathbb{L}_1^i (\mathbb{L}_2^2 \mathbb{L}_3) \mathbb{L}_1^{4-i}$$

Here we have taken  $j = 1 < 2 < 3$ . Now,

$$\begin{aligned} \sum_{i=1}^3 \mathbb{L}_1^i (\mathbb{L}_2^2 \mathbb{L}_3) \mathbb{L}_1^{4-i} &= (\omega^3 + \omega^6 + \omega^9) \mathbb{L}_2^2 \mathbb{L}_3 \\ &= (-1) \mathbb{L}_2^2 \mathbb{L}_3 \quad (\text{video eqs. (2.3) and (2.5)}) \end{aligned}$$

Case 2.  $j$  is equal to one of the indices :  $i_1$  or  $i_2, \dots, i_{k-1}$  (for  $k=2$ ). Let us consider the sum

$$\sum_{i=1}^3 \mathbb{L}_2^i (\mathbb{L}_2^2 \mathbb{L}_3) \mathbb{L}_3^{4-i}$$

where we have taken  $j = 2 = i_1$ . The above sum is equal to (video eqns. (2.3) and (2.5))  $\mathbb{L}_2^2 \mathbb{L}_3 (\omega + \omega^2 + \omega^3) = -\mathbb{L}_2^2 \mathbb{L}_3$ .

Case 3.  $j$  is greater than some index in the set  $i_1, i_2, \dots, i_k$  ( $k=2$ ). To illustrate this case we consider a now

$$a = \mathbb{L}_1^2 \mathbb{L}_3$$

so that choosing  $j = 2$

$$\sum_{i=1}^3 \mathcal{L}_2^i (\mathcal{L}_1^2 \mathcal{L}_3) \mathcal{L}_2^{4-i} = (\omega^7 + \omega^{14} + \omega^{21}) \mathcal{L}_1^2 \mathcal{L}_3 \\ = -\mathcal{L}_1^2 \mathcal{L}_3$$

Let us illustrate the nature of the eqns. (2.7) to (2.7)<sub>2n</sub> in showing that  $C_2^3$  is simple. Here if  $\omega$  is a primitive cube-root of unity,  $1+\omega+\omega^2=0$ .

Let  $A = x_0 \cdot I + x_1 \mathcal{L}_1^2 \mathcal{L}_2 + x_2 \mathcal{L}_1 \mathcal{L}_2^2 + x_3 \mathcal{L}_1^2$  (say)

where  $x_0, x_1, x_2$  and  $x_3$  are scalars and  $I$  is the unit element. Then

$$A_1 = A + \sum_{i=1}^3 \mathcal{L}_1^i A \mathcal{L}_1^{3-i} \\ = A + 2x_0 + x_1(\omega + \omega^2) \mathcal{L}_1^2 \mathcal{L}_2 + x_2(\omega^2 + \omega) \mathcal{L}_1 \mathcal{L}_2^2 \\ + 2x_3 \mathcal{L}_1^2 \\ = 3x_0 I + (x_1 - x_2) \mathcal{L}_1^2 \mathcal{L}_2 + (x_2 - x_3) \mathcal{L}_1 \mathcal{L}_2^2 + 3x_3 \mathcal{L}_1^2 \\ = 3x_0 I + 3x_3 \mathcal{L}_1^2$$

$$A_2 = A_1 + x_3 \sum_{i=1}^3 \mathcal{L}_2^i \mathcal{L}_1^2 \mathcal{L}_2^{3-i} + 6x_0 I \\ = A_1 + 6x_0 \cdot I + 3(\omega + \omega^2) \mathcal{L}_1^2 \cdot x_3 \\ = 9x_0 \cdot I$$

Hence  $A_{2n}$  (here  $n=1$ ) contains only a constant term

2.4 When the number of elements is even, there exists a similarity transformation  $S$  which takes every generator  $\mathcal{L}_i$  to  $\omega \mathcal{L}_i$ . The transformation

$$S = \mathcal{L}_1^{m-1} \mathcal{L}_2 \mathcal{L}_3^{m-1} \mathcal{L}_4 \cdots \mathcal{L}_{2n-1}^{m-1} \mathcal{L}_{2n} \quad (2.8)$$

so that

$$S^{-1} \mathcal{L}_i S = \omega \mathcal{L}_i \quad ; \quad i=1, 2, \dots, 2n$$

Similarly

$$(S^{-1})^j \mathcal{L}_i (S)^j = \omega^j \mathcal{L}_i \quad (j=1, 2, \dots, m) \quad (2.9)$$

But when the number of generators is odd  $= 2n+1$ , the transformation is not possible. For in this case  $S \in C_{2n+1}^m$  is also a generator (say)

$$\mathcal{L}_{2n+1} = S \quad (2.10)$$

so that even though

$$(S^{-1})^j \mathcal{L}_i S^j = \omega^j \mathcal{L}_i \quad ; \quad \left\{ \begin{array}{l} i=1, 2, \dots, 2n \\ j=1, 2, \dots, m \end{array} \right\}$$

$$\begin{aligned} (S^{-1}) S (S) &= (S^{-1})(\mathcal{L}_{2n+1}) S \\ &= S = \mathcal{L}_{2n+1} \end{aligned} \quad (2.11)$$

which means that a transformation of the above type does not exist. This implies that if  $\mathcal{L}_i$  has an irreducible representation  $\omega \mathcal{L}_i, \omega^2 \mathcal{L}_i, \dots, \omega^{m-1} \mathcal{L}_i$  ( $i=1, 2, \dots, 2n+1$ ) give rise to inequivalent representations each of

dimension  $m^n$ . Hence  $C_{2n+1}^m$  is isomorphic to a sum of  $n$  irreducible matrix representation of the same dimension  $m^n$ .

The above result was proved by Morris<sup>7)</sup> using the method of primitive idempotents and the explicit construction of the primitive idempotents. Whereas here we have given a different proof based on the notion of ideals.

**2.6** In deriving the explicit matrix representation of  $C_p^m$  Morris<sup>7)</sup> uses the method of Brauer and Weyl<sup>8)</sup> starting with the representation of the generators of  $C_2^m$ . However, we explicitly derive the matrix form of the generators of  $C_2^m$  from  $C_1^m$ . In this section we generalise the method of Racovskii<sup>2)</sup> to the G.C.A. However, Racovskii assumes the form of the matrices which generate namely the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (2.12)$$

We derive those from Racovskii's approach itself applied to  $C_1^2$  the ordinary Clifford algebra generated by a single element.

<sup>7)</sup> Loc. cit.,

2.7 From the above analysis it is sufficient for us to consider the construction<sup>9)</sup> of representation for the algebra when the number of generators is even i.e.  $C_{2n}^m$ . An arbitrary element A of  $C_{2n}^m$  is given as

$$A = a_0 I + a_1 \mathcal{L}_1^{i_1} \mathcal{L}_2^{i_2} \cdots \mathcal{L}_n^{i_n} \quad (2.13)$$

$$0 \leq i_j \leq m-1 ; j=1, 2, \dots, n$$

where we have used in the above equation the summation convention for repeated indices and  $a_i$ 's are the scalars.

We divide the  $m^n$  elements of  $C_{2n}^m$  into n sets as

$$C_{2n}^m = (C_{2n}^m)_0 + (C_{2n}^m)_1 + \cdots + (C_{2n}^m)_{m-1} \quad (2.14)$$

where  $(C_{2n}^m)_i$  contains terms of degree  $i \bmod n$  in each having  $m^{n-1}$  elements. We now construct generalising Rasevskii's method<sup>10)</sup> the representation of  $C_{2n}^m$ , the algebra generated by  $2n$  generators satisfying the relations

$$\mathcal{L}_i' \mathcal{L}_j' = \omega \mathcal{L}_j' \mathcal{L}_i' ; i < j ; i, j = 1, 2, \dots, 2n \quad (2.15)$$

$$(\mathcal{L}_i')^m = I \quad (2.16)$$

9) Alladi Ramakrishnan, T.S.Santhanam and P.S.Chandrasekaran, 'On the representations of the generalized Clifford algebra', J. Math. and Phys. Sci., (Madras) 2, 307, (1969)

The first  $n$  elements of  $C_{2n}^m$  are obtained as the mapping

$$A \xrightarrow{\hat{E}} A\mathcal{L}_i ; \quad \mathcal{L}_i \in C_n^m ; \quad i=1, 2, \dots, n \quad (2.17)$$

where

$$A \in C_n^m$$

The other  $n$  elements of  $C_{2n}^m$  are obtained as the mapping

$$A \xrightarrow{\hat{E}_{n+i}} \mathcal{L}_i [\omega^{m-1}A_0 + \omega^{m-2}A_1 + \dots + A_{m-1}] \quad (2.18)$$

where

$$\begin{aligned} b &= 1 \text{ for } m \text{ odd} \\ &= \omega^{1/2} \text{ for } m \text{ even.} \end{aligned}$$

The factor  $b$  is necessary when  $m$  is even to ensure that  $(\mathcal{L}_i)^m = I$ . The mappings  $\hat{E}_1, \hat{E}_2, \dots, \hat{E}_{2n}$  are shown below to furnish a representation of  $\mathcal{L}_1', \mathcal{L}_2', \dots, \mathcal{L}_{2n}' \in C_{2n}^m$ .

Case 1. For the first  $n$  elements  $\hat{E}_1, \hat{E}_2, \dots, \hat{E}_n$  the proof is obvious since

$$A \xrightarrow{\hat{E}_i \hat{E}_j} A\mathcal{L}_i \mathcal{L}_j ; \quad i < j ; \quad i, j = 1, 2, \dots, n \quad (2.19)$$

but

$$A \xrightarrow{\hat{E}_j \hat{E}_i} A\mathcal{L}_j \mathcal{L}_i ; \quad i < j ; \quad i, j = 1, 2, \dots, n \quad (2.20)$$

$$\therefore \hat{E}_i \hat{E}_j = \omega \hat{E}_j \hat{E}_i ; i < j ; i, j = 1, 2, \dots, n \quad (2.21)$$

Also

$$A \xrightarrow{\hat{E}_i \hat{E}_i \dots \hat{E}_i \text{ (m times)}} A \not{E}_i \not{E}_i \dots \not{E}_i = A \quad (2.22)$$

$$\text{ie., } (\hat{E}_i)^m = I ; i = 1, 2, \dots, n \quad (2.23)$$

Case 2. For the next  $\hat{E}_{n+i}$  elements we use the construction in eq.(2.18). It has to be noted that since  $A_i$  is of degree  $i \bmod m$  and  $\not{E}_i A$  is of degree  $(i+1) \bmod m$ . Hence

$$A \xrightarrow{\hat{E}_{n+i} \hat{E}_{n+j}} \not{E}_i \not{E}_j [w^{2m-3} A_0 + w^{2m-5} A_1 + \dots + w^{m-1} A_{m-1}] \quad (2.24)$$

$$i < j ; i, j = 1, 2, \dots, n$$

and

$$A \xrightarrow{\hat{E}_{n+j} \hat{E}_{n+i}} \not{E}_j \not{E}_i [w^{2m-3} A_0 + w^{2m-5} A_1 + \dots + w^{m-1} A_{m-1}] \quad (2.25)$$

$$i < j ; i, j = 1, 2, \dots, n$$

since

$$\not{E}_i \not{E}_j = \omega \not{E}_j \not{E}_i ; i < j ; i, j = 1, 2, \dots, n$$

$$\hat{E}_{n+i} \hat{E}_{n+j} = \omega \hat{E}_{n+j} \hat{E}_{n+i} ; i < j ; i, j = 1, 2, \dots, n \quad (2.26)$$

It also follows that

$$(\hat{E}_{n+i})^m = I ; i = 1, 2, \dots, n \quad (2.27)$$

Case 3. Let us consider the elements  $\hat{E}_i$   
and  $\hat{E}_{n+j}$  when  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, n$

$$A \xrightarrow{\hat{E}_i \cdot \hat{E}_{n+j}} B \not\propto_j (\omega^{m-1} A_0 + \omega^{m-2} A_1 + \dots + A_{m-1}) \not\propto_i \quad (2.28)$$

$$A \xrightarrow{\hat{E}_{n+j} \hat{E}_i} B \not\propto_j (\omega^{m-2} A_0 + \omega^{m-3} A_1 + \dots + \omega^{m-1} A_{m-1}) \not\propto_i \quad (2.29)$$

$\times \not\propto_i$

where the eqns. (2.14) and (2.18) have been applied in eq. (2.29)  
to

$$A \not\propto_i \quad (2.30)$$

$$\therefore \hat{E}_i \hat{E}_{n+j} = \omega \hat{E}_{n+j} \hat{E}_i \quad ; i, j = 1, 2, \dots, n \quad (2.31)$$

Equations (2.21), (2.28), (2.30), (2.27) and (2.31)

describe a system of generators  $\hat{E}_1, \hat{E}_2, \dots, \hat{E}_{2n}$

or the algebra  $C_{2n}^m$  and the  $\hat{E}_1, \hat{E}_2, \dots, \hat{E}_{2n}$

constitute a  $m^n$  dimensional matrix representation

of the generators  $\not\propto_1, \not\propto_2, \dots, \not\propto_{2n}$  of  $C_{2n}^m$ .

2.8 The above considerations are now used to derive the matrix representations of the Pauli matrices in the ordinary Clifford algebra and the  $3 \times 3$  matrices assumed by Morris<sup>3)</sup>

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad Q = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 0^2 \\ 1 & 0 & 0 \end{pmatrix} \quad (2.32)$$

which are the analogues of the Pauli matrices in the case of the algebra  $C_2^3$ .

Let us consider the algebra  $C_1^2$  i.e., the algebra generated by a single element  $\mathcal{L}_1$  such that  $\mathcal{L}_1^2 = I$ . Any arbitrary element  $A$  of  $C_1^2$  is given by

$$A = \alpha_0 \cdot I + \alpha_1 \mathcal{L}_1 \quad (2.33)$$

where  $\alpha_0$  and  $\alpha_1$  are scalars. Now consider the transformation

$$A \xrightarrow{\hat{E}_1} A \mathcal{L}_1 = \alpha_1 \cdot I + \alpha_0 \mathcal{L}_1 \quad (2.34)$$

$$\hat{E}_1 \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \quad (2.35)$$

$$\text{i.e. } \hat{E}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_2 \quad (2.36)$$

Now consider the transformation

$$A \xrightarrow{\hat{E}_{1+1}} \mathcal{L}_1 (-\alpha_0 + \alpha_1) = \mathcal{L}_1 (-\alpha_0 \cdot I + \alpha_1 \mathcal{L}_1) \quad (2.37)$$

But

$$\mathcal{S} = \omega^{1/2} = i \quad (2.38)$$

$$\hat{E}_2 \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix} = i \begin{pmatrix} -\alpha_0 \\ \alpha_1 \end{pmatrix} \quad (2.39)$$

$$\text{i.e. } \hat{E}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_3 \quad (2.40)$$

It is also seen that

$$\hat{E}_1^2 = \hat{E}_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.41)$$

and

$$\hat{E}_1 \hat{E}_2 = -\hat{E}_2 \hat{E}_1 \quad (2.42)$$

i.e.  $\hat{E}_1$  and  $\hat{E}_2$  furnish the  $2 \times 2$  representation of the Pauli algebra  $C_2^2$ .

2.9 Let us now consider the algebra  $C_1^3$ , i.e. the algebra generated by a single element  $\mathcal{L}_1$  such that  $\mathcal{L}_1^3 = I$ . Any arbitrary element  $A$  of  $C_1^3$  is written as

$$A = a_0 \cdot I + a_1 \cdot \mathcal{L}_1 + a_2 \mathcal{L}_1^2 \quad (2.43)$$

Consider the transformations

$$\begin{aligned} A \xrightarrow{\hat{E}_1} A \mathcal{L}_1 &= (a_0 \cdot I + a_1 \mathcal{L}_1 + a_2 \mathcal{L}_1^2) \mathcal{L}_1 \\ &= a_0 \cdot I + a_0 \mathcal{L}_1 + a_1 \mathcal{L}_1^2 \end{aligned} \quad (2.44)$$

$$\hat{E}_1 \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_0 \\ a_2 \end{pmatrix} \quad (2.45)$$

$$\text{i.e., } \hat{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } (\hat{E}_1)^3 = I \quad (2.46)$$

Let us now consider the transformation

$$\begin{aligned} A &\xrightarrow{\hat{E}_{1+1}} \mathcal{L}_1 (\omega^2 a_0 + \omega a_1 \mathcal{L}_1 + a_2 \mathcal{L}_1^2) \\ &= 1 (\omega^2 a_0 \mathcal{L}_1 + \omega a_1 \mathcal{L}_1^2 + a_2 \mathcal{L}_1^3) \end{aligned} \quad (2.47)$$

$$\hat{E}_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}; (\hat{E}_2)^3 = I \quad (2.48)$$

The representations of  $\hat{E}_1$  and  $\hat{E}_2$  of the generators  $C_2^3$  are the ones assumed by Morris<sup>3)</sup>. Similarly the higher-dimensional representations can be worked for the algebra generated by even number of elements. The n-inequivalent representations in the case when the algebra is generated by an odd number of elements (whose nth power gives the identity) are obtained first by getting the representations, R of the algebra generated by the nearest even number (smaller than the given odd number) of generators and then multiplying R by  $\omega, \omega^2, \dots, \omega^{m-2}$  and  $\omega^{m-1}$ .

40  
Helicity Matrices<sup>1),2)</sup>

3.1 In this chapter we first give the results obtained by Ramakrishnan<sup>1)</sup>, for the ordinary Clifford algebra, on the construction of helicity matrices which will be defined below. Then the procedure is extended to the case<sup>2)</sup> of the generalised Clifford algebra.

3.2 The maximum number of anticommuting elements in an algebra with  $2n$  generators i.e.  $C_{2n}^2$  is  $(2n+1)$  and are of dimension  $2^n$ . These  $(2n+1)$  anticommuting matrices of dimension  $2^n$  have the product property:

$\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_{2n+1} = i^n I$  . Therefore, knowing  $2n$  of these we can construct the last one. From these  $2n$  matrices we can obtain  $n$  sets of generalised Pauli matrices, each set  $\{H^i\}$  consisting of three mutually anticommuting matrices

$$\{H^i\} = \{H_1^i, H_2^i, H_3^i\} ; i = 1, 2, \dots, n \quad (3.1)$$

Let us now define<sup>1)</sup>

$$\begin{aligned} H_1^n &= i \mathcal{L}_{2n} \mathcal{L}_{2n-1} \\ H_2^n &= \mathcal{L}_{2n} \\ H_3^n &= \mathcal{L}_{2n-1} \end{aligned} \quad (3.2)$$

1) Alladi Ramakrishnan, 'Generalised Helicity Matrices', J. Math. Anal. Appl. 26, 88, (1969).

2) Alladi Ramakrishnan, T.S.Sentharam, P.S.Chandrasekaran and A.Sundaram, 'Helicity matrices for generalised Clifford algebra', J. Math. Anal. Appl. 36, 275 (1971).

The set  $\{H_1^n, H_2^n, H_3^n\}$  has the required properties of Pauli matrices i.e.  $H_1^n, H_2^n$  and  $H_3^n$  anticommute with one another the square of each is the unit matrix. Let us now write

$$\mathcal{L}_{2n-2} = H_1^n \mathcal{L}_{2n-2}(1)$$

$$\mathcal{L}_{2n-3} = H_1^n \mathcal{L}_{2n-3}(+) \quad (3.3)$$

$$\mathcal{X}_1 = H_1^n \mathcal{X}_1(+)$$

Then  $\mathcal{L}_{2n-2}(1), \mathcal{L}_{2n-3}(1), \dots, \mathcal{L}_1(1)$  form a set of  $(2n-1)$  mutually anticommuting matrices each of whose square gives the unit matrix.

We next define another set

$$\begin{aligned} H_1^{n-1} &= i \mathcal{L}_{2n-2}(1) \mathcal{L}_{2n-3}(1) \\ H_2^{n-1} &= \mathcal{L}_{2n-2}(1) \\ H_3^{n-1} &= \mathcal{L}_{2n-3}(1) \end{aligned} \quad (3.4)$$

This set again consists of three anticommuting matrices each of whose square is the unit matrix i.e. it has the characteristics of a Pauli set. We now factor out

$H_1^{n-1}$  from  $\mathcal{L}_{2n-4}(1), \mathcal{L}_{2n-5}(1), \dots, \mathcal{L}_1(1)$  and write

$$\begin{aligned} \mathcal{L}_{2n-4}(1) &= H_1^{n-1} \mathcal{L}_{2n-4}(2) \\ \mathcal{L}_{2n-5}(1) &= H_1^{n-1} \mathcal{L}_{2n-5}(2) \\ \mathcal{X}_1(1) &= H_1^{n-1} \mathcal{X}_1(2) \end{aligned} \quad (3.5)$$

The above procedure can be iterated till we arrive at

$$\begin{aligned} H_1^1 &= i \mathcal{L}_2(n-1) \mathcal{L}_1(n-1) \\ H_2^1 &= \mathcal{L}_2(n-1) \\ H_3^1 &= \mathcal{L}_1(n-1) \end{aligned} \quad (3.6)$$

Any member of the set  $\{H^i\}$  commutes with any member of another set  $\{H^j\}$ . We now define the helicity matrix  $H^i$  as the linear combination of the set  $H_1^i, H_2^i$  and  $H_3^i$  and choose in particular

$$\begin{aligned} H^n &= \lambda_{2n+1} H_3^n + \lambda_{2n} H_2^n + \Delta_{n-1} H_1^n \\ H^i &= \lambda_{2i+1} H_3^i + \lambda_{2i} H_2^i + \Delta_{i-1} H_1^i \\ H^1 &= \lambda_3 H_3^1 + \lambda_2 H_2^1 + \lambda_1 H_1^1 \end{aligned} \quad (3.7)$$

where

$$\Delta_{i-1}^2 = \lambda_1^2 + \lambda_2^2 + \dots + \lambda_{2i-1}^2 \quad (3.8)$$

and

$$\Delta_{i-1}^2 = \lambda_{2i-1}^2 + \lambda_{2i-2}^2 + \Delta_{i-2}^2 \quad (3.9)$$

Then

$$\begin{aligned} (H^n)^2 &= \Delta_n^2 I = (\Delta_{n-1}^2 + \lambda_{2n}^2 + \lambda_{2n+1}^2) I \\ (H^i)^2 &= \Delta_i^2 I = (\Delta_{i-1}^2 + \lambda_{2i}^2 + \lambda_{2i+1}^2) I \\ (H^1)^2 &= \Delta_1^2 I = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) I \end{aligned} \quad (3.10)$$

Since  $\{H^i\}$  commutes with  $\{H^j\}$  for  $i \neq j$ , we find it is possible to obtain eigenvectors of  $L_{2n+1}$  (a linear combination  $(2n+1)$  anticommuting matrices\*) of dimension  $2^n$  which are simultaneous eigenvectors of the 'helicity' eigenvalues  $\Lambda_1, \Lambda_{n-1}, \dots, \Lambda_1$  respectively.

3.3 In fact in the above construction we might have started with any one of the  $2n$  matrices and called it as  $L_{2n}$  and any one of the remaining as  $L_{2n-1}$  and form the first set  $\{H^i\}$ . After factoring out  $H_i^0$  from the remaining  $(2n-2)$  might have taken  $L_{2n-2}$  to be any one of the remaining  $2n-2$  (as in eqn.(3.3)). This particular irrelevance in the choice was noted by Ramakrishnan<sup>3)</sup>.

3.4 In the following helicity matrices for generalised Clifford algebra are constructed<sup>3)</sup>. As in the ordinary Clifford algebra case we have triplets of elements such that one set commutes with another and among themselves they satisfy the conditions given by equations (1.30) and (1.31) but with  $i,j = 1, 2$  and the third is a product of the first two. The maximum number of matrices satisfying conditions (1.30) and (1.31) of dimension  $m^n$  is  $(2n+1)$  for the algebra  $C_{2n}^{(m)}$ .

\* Please see Chapter I.

3) Alladi Ramakrishnan, 'On the shell structure of an L-matrix', J. Math. Anal. Appl. (to be published).

If we now define

$$\mathcal{H}_1^n = \mathcal{L}_1, \quad \mathcal{H}_2^n = \mathcal{L}_2; \quad \mathcal{H}_3^n = \mathcal{E}(\mathcal{H}_1^n)^{m-1} \mathcal{H}_2^n \quad (3.11)$$

where

$$\mathcal{E} = 1 \quad \text{for } n \text{ odd}$$

$$= \omega^{1/2} \quad \text{for } n \text{ even.}$$

The set  $\{\mathcal{H}_i^n\}$ , ( $i=1, 2, 3$ ) satisfies the required generalised Clifford conditions, though  $\mathcal{H}_3^n$  is not a member of the set  $\mathcal{L}_i$  ( $i=1, 2, \dots, n$ ) which generate  $C_{2n}^m$ .

In a completely analogous way to eq.(3.3) we define

$$\mathcal{L}_i = (\mathcal{L}_2 \mathcal{L}_1^{m-1}) \mathcal{L}_1(i), \quad i=3, 4, \dots, 2n \quad (3.12)$$

i.e. we factor out  $(\mathcal{H}_3^n)^{m-1}$  from each one of the remaining matrices of the set  $\mathcal{L}_i$ . Then  $\mathcal{L}_3(1), \mathcal{L}_4(1), \mathcal{L}_5(1), \dots, \mathcal{L}_{2n}(1)$  form a set of  $(2n-1)$  matrices which obey the generalised Clifford conditions.

We next proceed to define

$$\begin{aligned} \mathcal{H}_1^{n-1} &= \mathcal{L}_3(1) = \mathcal{L}_1 \mathcal{L}_2^{m-1} \mathcal{L}_3 \\ \mathcal{H}_2^{n-1} &= \mathcal{L}_4(1) = \mathcal{L}_1 \mathcal{L}_2^{m-1} \mathcal{L}_4 \\ \mathcal{H}_3^{n-1} &= \mathcal{E} (\mathcal{H}_1^{n-1})^{m-1} (\mathcal{H}_2^{n-1}) \end{aligned} \quad (3.13)$$

This set  $\{\mathcal{H}_i^{n-1}\}$ ,  $i=1, 2, 3$  again satisfies the generalised Clifford conditions. The procedure can be iterated till we arrive at the set

$$\begin{aligned}\mathcal{H}_1^1 &= \mathcal{L}_{2n+1} (n-1) \\ \mathcal{H}_2^1 &= \mathcal{L}_{2n} (n-1) \\ \mathcal{H}_3^1 &= B (\mathcal{H}_1^1)^{m-1} (\mathcal{H}_2^1)\end{aligned}\quad (3.14)$$

Any member of the set  $\{\mathcal{H}^i\}$  commutes with any member of the set  $\{\mathcal{H}^j\}$  when  $i \neq j$ . We now define the helicity matrix as the linear combination of the members of the set  $\{\mathcal{H}^i\}$  and choose in particular

$$\begin{aligned}\mathcal{H}^n &= \lambda_{2n+1} \mathcal{H}_3^n + \lambda_{2n} \mathcal{H}_2^n + \lambda_{n-1} \mathcal{H}_1^n \\ \mathcal{H}^i &= \lambda_{2i+1} \mathcal{H}_3^i + \lambda_{2i} \mathcal{H}_2^i + \lambda_{i-1} \mathcal{H}_1^i \\ &\vdots \\ \mathcal{H}^1 &= \lambda_3 \mathcal{H}_3^1 + \lambda_2 \mathcal{H}_2^1 + \lambda_1 \mathcal{H}_1^1\end{aligned}\quad (3.15)$$

where

$$\Lambda_i = (\lambda_1^m + \lambda_2^m + \dots + \lambda_{2i+1}^m)^{1/m} \quad (3.16)$$

Since  $\{\mathcal{H}^i\}$  commutes with  $\{\mathcal{H}^j\}$  when  $i \neq j$  we find that it is possible to obtain the eigenvectors of

$$\mathcal{L}_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \mathcal{L}_i$$

which are the simultaneous eigenvectors of the 'helicity' matrices  $\mathcal{H}^1, \mathcal{H}^2, \dots, \mathcal{H}^n$  corresponding to the

eigenvalues  $f(j)\Lambda_i$  where

$$f(j) = 1, \omega, \omega^2, \dots, \omega^{m-1} ; j = 0, 1, 2, \dots, m-1$$

$$\text{and } \Lambda_i = (\lambda_1^m + \lambda_2^m + \dots + \lambda_{2i+1}^m)^{1/m} \quad (9.17)$$

$$i = 1, 2, \dots, n$$

## CHAPTER IV

The Generalised Clifford Algebra and the Special Unitary Algebra<sup>1)</sup>

4.1 The Pauli matrices which form the representation of the ordinary Clifford algebra in the lowest dimension, namely 2, under Lie multiplication give the  $SU(2)$  algebra. Starting with the associative algebra of the Pauli matrices we get the  $SU(2)$  algebra by defining a non-associative multiplication between the elements of the Pauli algebra. In this chapter we concern ourselves with the corresponding problem with regard to the associative structure of the algebra  $C_2^3$  and in general of  $C_2^m$ . It turns out that corresponding to we have the  $SU(3)$  algebra and in general to  $C_2^m$  corresponds the  $SU(m)$  algebra.

4.2 If  $\mathcal{U}$  is an associative algebra under the product  $\cdot$  and if  $a, b \in \mathcal{U}$  then  $a \cdot b \in \mathcal{U}$ . Now define a new product ' $*$ ', which is nonassociative, between the elements of  $\mathcal{U}$  and call the new algebra, equipped with the ' $*$ ' multiplication as  $\mathcal{U}^*$ . If the ' $*$ ' multiplication happens to be a commutator then  $\mathcal{U}^*$  becomes a Lie algebra. The famous Birkhoff-Poincaré-Witt lemma asserts

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- 1) Alaudin Ramakrishnan, P.S.Chandrasekaran, R.R.Ranganathan, T.S.Santhanam and R.Venkatesan, "The generalised Clifford Algebra and the Unitary Group", *J. Math. Anal. Appl.* 22 164 (1969).
  - 2) Cf.: L.J. Paige, Jordan Algebras, in *Studies in Modern Algebra* (A.H.A. Albert) published by the Mathematical Association of America (1963). Also N.Jacobson, *Lie Algebras*, Interscience Publishers.

that given any associative algebra one can always obtain a Lie algebra from it. Hence the existence of the Lie algebra obtained from the associative algebra  $C_2^3$  (for which we shall specialize) poses no problem.

4.3 The algebra  $C_2^3$  is generated by two elements  $\mathcal{L}_1, \mathcal{L}_2$  such that

$$\begin{aligned}\mathcal{L}_1 \mathcal{L}_2 &= \omega \mathcal{L}_2 \mathcal{L}_1 \\ \mathcal{L}_1^3 &= \mathcal{L}_2^3 = I\end{aligned}\tag{4.1}$$

where  $\omega$  is a primitive cube root of unity. The algebra has nine elements

$$I, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_1^2, \mathcal{L}_2^2, \mathcal{L}_1 \mathcal{L}_2, \mathcal{L}_1^2 \mathcal{L}_2, \mathcal{L}_1 \mathcal{L}_2^2, \mathcal{L}_1^2 \mathcal{L}_2^2\tag{4.3}$$

We rewrite the above nine elements into sets of three such that the degree of elements in each set is 0 mod 3, 1 mod 3 and 2 mod 3 respectively:

$$(I, \mathcal{L}_1^2 \mathcal{L}_2, \mathcal{L}_1 \mathcal{L}_2^2); (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_1^2 \mathcal{L}_2^2); (\mathcal{L}_1^2, \mathcal{L}_2^2, \mathcal{L}_1 \mathcal{L}_2)\tag{4.4}$$

It is also noticed that if we multiply an element with the degree  $i$  mod 3 with an element of degree  $j$  mod 3 then the product is of degree  $i+j$  mod 3. When we now impose commutation on the elements in (4.4) (leaving the unit, 1) they close under the operation of commutation

and give rise to a Lie algebra. In Cartan canonical form of a Lie algebra we have certain Cartan operators  $H_i \sim i = 1, 2, \dots, n$  and operators  $E_\alpha$  and  $E_{-\alpha} \sim$  which lower and raise the eigenvalues of the vectors of  $H_i \sim$ . These are called the ladder operators or lowering and raising operators and the commutation relations among the  $H_i \sim$  and  $E_\alpha \sim$  and  $E_{-\alpha} \sim$  read as

$$[H_i, H_j]_- = 0 \quad \text{for all } i, j \quad (4.5)$$

$$[H_i, E_\alpha]_- = \alpha^i E_\alpha \quad (4.6)$$

$$[E_\alpha, E_\beta]_- = N_{\alpha\beta} E_{\alpha+\beta} \quad (4.7)$$

where  $N_{\alpha\beta} \neq 0$  if  $\alpha + \beta$  is a root of the algebra

$$\text{and if } \alpha + \beta = 0 \quad [E_\alpha, E_{-\alpha}] = \sum_i \alpha^i H_i$$

The abovementioned property of the sets in (4.4) comes in handy in getting the operators  $H_i \sim$  and  $E_\alpha \sim$  and  $E_{-\alpha} \sim$  which have to satisfy (4.5), (4.6) and (4.7). For example, elements of the second and third sets have to be used in constructing  $E_\alpha \sim$  because by (4.6) on

the right hand side we have  $E_2$  again which is possible only if we multiply terms of the 1 mod 3 set or 2 mod 3 set with 0 mod 3 set. This is because the  $H_{i-2}$  are obtained from the 0 mod 3 set. Moreover we notice that elements of the 0 mod 3 set (leaving the unit, 1) commute with one another as do the  $H_{i-2}$  in equation (4.8).

With the definitions for  $H_{i-2} (i=1,2)$  and  $E_{i-2}$  and  $E_{-2}$  as

$$H_1 = (\omega \mathbb{Y}_1^2 \mathbb{Y}_2 - \mathbb{Y}_1 \mathbb{Y}_2^2) / \omega(1-\omega) \quad (4.8)$$

$$H_2 = -\frac{1}{\sqrt{3}} (\mathbb{Y}_1 \mathbb{Y}_2^2 + \omega \mathbb{Y}_1^2 \mathbb{Y}_2) \quad (4.9)$$

$$E_1 = \frac{1}{3} (\mathbb{Y}_1 + \omega^2 \mathbb{Y}_2 + \omega^2 \mathbb{Y}_1^2 \mathbb{Y}_2^2) \quad (4.10)$$

$$E_{-1} = \frac{1}{3} (\mathbb{Y}_1^2 + \omega \mathbb{Y}_2^2 + \mathbb{Y}_1 \mathbb{Y}_2) \quad (4.11)$$

$$E_2 = \frac{1}{3} (\mathbb{Y}_1^2 + \mathbb{Y}_2^2 + \omega \mathbb{Y}_1 \mathbb{Y}_2) \quad (4.12)$$

$$E_{-2} = \frac{1}{3} (\mathbb{Y}_1 + \mathbb{Y}_2 + \omega \mathbb{Y}_1^2 \mathbb{Y}_2^2) \quad (4.13)$$

$$E_3 = \frac{1}{3} (\mathbb{Y}_1 + \omega \mathbb{Y}_2 + \mathbb{Y}_1^2 \mathbb{Y}_2^2) \quad (4.14)$$

$$E_{-3} = \frac{1}{3} (\mathbb{Y}_1^2 + \omega^2 \mathbb{Y}_2^2 + \omega^2 \mathbb{Y}_1 \mathbb{Y}_2) \quad (4.15)$$

we recognise that the system of equations (4.8) - (4.15) represents the SU(3) algebra, special unitary algebra of  $3 \times 3$  matrices.

As in the connection between the ordinary Clifford algebra (associative) and the SU(2) algebra, the connection between the generalised Clifford algebra,  $C_2^3$  and SU(3) is only an algebraic isomorphism. As the various representations of the SU(2) algebra are not obtainable from the  $2 \times 2$  Pauli matrices the higher-dimensional representations of the SU(3) algebra are also not obtainable from the  $3 \times 3$  matrices  $\mathcal{L}_1$  and  $\mathcal{L}_2$  given as

$$\mathcal{L}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{L}_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \quad (4.16)$$

Using equations (4.8) - (4.15) and the representations of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  in (4.16) we can pass on to the Hermitian basis and we can obtain the Gell-Mann representations of the  $3 \times 3$  matrices of SU(3). For this first we list the product elements  $\mathcal{L}_1^k \mathcal{L}_2^l$  ( $k, l = 0, 1, 2$ ) (leaving the unit matrix).

$$\begin{aligned} \mathcal{L}_1 = & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathcal{L}_1^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad \mathcal{L}_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}; \\ \mathcal{L}_2^2 = & \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}; \quad \mathcal{L}_1 \mathcal{L}_2 = \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}; \quad \mathcal{L}_1^2 \mathcal{L}_2^2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ \omega^2 & 0 & 0 \end{pmatrix} \\ \mathcal{L}_1^2 \mathcal{L}_2 = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}; \quad \mathcal{L}_1 \mathcal{L}_2^2 = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \quad (4.17)$$

The Collision matrices  $\lambda_i$  ( $i=1, 2, \dots, 8$ ) now take the form

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{3} \left[ (\underline{x}_1^2 + \omega \underline{x}_2^2 + \underline{x}_1 \underline{x}_2) + (\underline{x}_1 + \omega^2 \underline{x}_2 + \omega^2 \underline{x}_1 \underline{x}_2) \right] \quad (4.18)$$

$$\lambda_2 = \begin{pmatrix} 0 & -1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{i}{3} \left[ (\underline{x}_1^2 + \omega \underline{x}_2^2 + \underline{x}_1 \underline{x}_2) - (\underline{x}_1 + \omega^2 \underline{x}_2 + \omega^2 \underline{x}_1 \underline{x}_2) \right] \quad (4.19)$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left[ (\omega \underline{x}_1^2 \underline{x}_2 - \underline{x}_1 \underline{x}_2^2) \right] / \omega(1-\omega) \quad (4.20)$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{1}{3} \left[ (\underline{x}_1^2 + \underline{x}_2^2 + \omega \underline{x}_1 \underline{x}_2) + (\underline{x}_1 + \underline{x}_2 + \omega \underline{x}_1^2 \underline{x}_2) \right] \quad (4.21)$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \frac{i}{3} \left[ ((\underline{x}_1^2 + \underline{x}_2^2 + \omega \underline{x}_1 \underline{x}_2)(-1)) + (\underline{x}_1 + \underline{x}_2 + \omega \underline{x}_1^2 \underline{x}_2) \right] \quad (4.22)$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{3} \left[ (\underline{x}_1 + \omega \underline{x}_2 + \underline{x}_1^2 \underline{x}_2^2) + (\underline{x}_1^2 + \omega^2 \underline{x}_2^2 + \omega^2 \underline{x}_1 \underline{x}_2) \right] \quad (4.23)$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = -\frac{i}{3} \left[ ((\underline{x}_1 + \omega \underline{x}_2 + \underline{x}_1^2 \underline{x}_2^2)) - (\underline{x}_1^2 + \omega^2 \underline{x}_2^2 + \omega^2 \underline{x}_1 \underline{x}_2) \right] \quad (4.24)$$

$$\lambda_8 = -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = -\frac{1}{\sqrt{3}} \left( \underline{x}_1 \underline{x}_2^2 + \omega \underline{x}_1^2 \underline{x}_2 \right) \quad (4.25)$$

(4.25)

4.4 In the last few years, attempts have been made to classify various elementary particles by assuming that they are composed of three fundamental objects called quarks (a triplet system) which are simultaneous eigenstates of  $\lambda_3$  and  $\lambda_8$  i.e. the operators corresponding to the third component of the isotopic spin  $I_z$  and hypercharge  $y$ . Since  $I_z$  and  $y$  can be expressed in terms of  $\gamma_1^2 \gamma_2$  and  $\gamma_1 \gamma_2^2$  we wish to suggest that the states of the triplets can be considered as the simultaneous eigenstates of  $\gamma_1^2 \gamma_2$  and  $\gamma_1 \gamma_2^2$ .

Relabelling for convenience of notation  $\gamma_1^2 \gamma_2$  and  $\gamma_1 \gamma_2^2$  as  $M_1$  and  $M_2$  and their eigenvalues as  $\mu_1$  and  $\mu_2$  respectively, we have the following table, giving the quark quantum numbers in terms of  $\mu_1$  and  $\mu_2$ .

| Quark | $I_z$      | $y$        | $\mu_1$ | $\mu_2$ |
|-------|------------|------------|---------|---------|
| A     | $\gamma_2$ | $\gamma_3$ | 1       | 1       |
| B     | $-1/2$     | $1/3$      | $w$     | $w^2$   |
| C     | 0          | $-2/3$     | $w^2$   | $w$     |

Further we have for charge

$$Q = \frac{1}{3} (\mu_1 + \mu_2) \quad (4.20)$$

and hypercharged

$$Y = -\frac{1}{3}(\omega \mu_1 + \omega^2 \mu_2) \quad (4.27)$$

4.5 The above type of analysis can be carried out for the case when we have two elements  $\mathcal{L}_1$  and  $\mathcal{L}_2$  whose  $n$ th power gives the unit matrix (In the above  $\mathcal{L}_1$  and  $\mathcal{L}_2$  each raised to the third power give the unit matrix) along with the  $\omega$  commutation ( $\omega$  being a primitive  $n$ th root of the unity). In this case there are  $(n-1)$  commuting operators

$$\mathcal{L}_1^{k_1} \mathcal{L}_2^{k_2}$$

with

$$k_1 + k_2 = m$$

Let us call the eigenvalues of these commuting matrices which act as  $H \rightarrow$  (in eqs. 4.5, 4.6) for the  $SU(m)$  algebra as

$$\lambda_1, \lambda_2, \dots, \lambda_{m-1}$$

i.e. they are the Clifford quantum numbers (as the  $\mu_1$  and  $\mu_2$  are in eq.(4.26) and (4.27) where  $\mu_1$  can take any value of the  $n$ th roots of unity and

$$\lambda_2 = \mu_1^2, \lambda_3 = \mu_1^3, \dots, \lambda_{m-1} = \mu_1^{m-1} \quad (4.28)$$



define  $n$  scalar quantum numbers (like  $Q$  and  $\chi$  as in eq.(4.26) and (4.27))  $s_1, s_2, \dots, s_{m-1}$ , with the restriction that

$$s_1 + s_2 + \dots + s_{m-1} = 0 \quad (4.29)$$

and they are given in terms of  $\mu^{-s}$  as

$$\begin{aligned} s_1 &= \frac{1}{m} (\mu_1 + \mu_2 + \dots + \mu_{m-1}) \\ &= \frac{1}{m} (\mu_1 + \mu_1^2 + \dots + \mu_1^{m-1}) \\ s_{m-1} &= \frac{1}{m} (\omega\mu_1 + \omega^2\mu_1^2 + \dots + \omega^{m-1}\mu_1^{m-1}) \\ s_{m-2} &= \frac{1}{m} (\omega^2\mu_1 + \omega^4\mu_1^2 + \dots + (\omega^{m-1}\mu_1)^{m-1}) \\ s_2 &= \frac{1}{m} [(\omega^{m-1}\mu_1) + (\omega^{m-1}\mu_1)^2 + \dots + (\omega^{m-1}\mu_1)^{m-1}] \end{aligned} \quad (4.30)$$

We can also define vector quantum numbers (like  $\vec{v}_j$  in the SU(3) case)

$$v_j = (s_j - s_{j+1})/2 \quad (4.31)$$

so that we have

$$v_1 + v_2 + \dots + v_{m-1} = 0$$

assuming that

$$s_m = s_1$$



In the particular case of  $SU(3)$  we can set

$$\begin{aligned} s_1 &= Q \\ s_2 &= -Q + Y \\ s_3 &= -Y \end{aligned} \quad (4.32)$$

where  $Q$  and  $Y$  are charge and hypercharge respectively and

$$\begin{aligned} v_1 &= I_Z, Z^- \text{ component of isospin} \\ v_2 &= U_Z, Z^- \quad " \quad U \text{ spin} \\ v_3 &= V_Z, Z^- \quad " \quad V \text{ spin} \end{aligned} \quad (4.33)$$

so that

$$\begin{aligned} I_Z &= \frac{Q}{2} - \frac{Y-Q}{2} \\ \text{i.e., } Q &= I_Z + \frac{Y}{2} \end{aligned} \quad (4.34)$$

The table for the values  $s_1, s_2, \dots, s_{m-1}$  for the basic multiplet  $SU(m)$  quark is

| Quark     | $s_1$           | $s_2$           | $\dots$  | $s_{m-1}$       |
|-----------|-----------------|-----------------|----------|-----------------|
| $A_1$     | $\frac{m-1}{m}$ | $-\frac{1}{m}$  | $\dots$  | $-\frac{1}{m}$  |
| $A_2$     | $-\frac{1}{m}$  | $\frac{m-1}{m}$ | $\dots$  | $-\frac{1}{m}$  |
| $\vdots$  | $\vdots$        | $\vdots$        | $\ddots$ | $\vdots$        |
| $A_{m-1}$ | $-\frac{1}{m}$  | $-\frac{1}{m}$  | $\dots$  | $\frac{m-1}{m}$ |

Similarly the table for the  $v$ 's for the basic representation is given as

| Quark     | $v_1$          | $v_2$         | $\dots$  | $v_{m-1}$      |
|-----------|----------------|---------------|----------|----------------|
| $A_1$     | $\frac{1}{2}$  | $0$           | $\dots$  | $-\frac{1}{2}$ |
| $A_2$     | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\dots$  | $0$            |
| $\vdots$  | $\vdots$       | $\vdots$      | $\ddots$ | $\vdots$       |
| $A_{m-1}$ | $0$            | $0$           | $\dots$  | $\gamma_2$     |

4.6 In the above analysis which establishes connection between the Generalised Clifford Algebra and the special unitary algebra is similar to that which exists between the Pauli algebra (the ordinary Clifford algebra with two elements) and the  $SU(2)$  algebra. As the higher representations of the  $SU(2)$  algebra are not attainable from the (unique) representation of the Pauli algebra, in the general case <sup>also</sup>, the other representations of  $SU(3)$  or  $SU(n)$  algebra are not obtainable from the (unique) representations of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  (vide Chapter II).

## CHAPTER V

Kemmer Algebra From Generalised Clifford Elements\*

5.1 In this chapter we first give the essentials of the Kemmer<sup>1)</sup> algebra and then later give a method of construction of the Kemmer elements using generalised Clifford elements.

5.2 The Dirac equation discussed in Chapter I corresponds to spin 1/2 particles. This suggests the question as to whether one could rewrite the equations for arbitrary spin in the form of a first order matrix differential equation.

$$(\beta_\mu \partial^\mu + x) \psi = 0 \quad (5.1)$$

To begin with let us consider the spin zero case and consider the Klein-Gordon equation

$$(\square - x^2) \psi = 0 \quad (5.2)$$

where  $\square$  is the de Alenbortian operator and  $x$  the mass of the particle and  $\psi$  the wave function. Let us

\* Alladi Ramakrishnan, R. Vasudevan, P.S. Chandrasekaran and H.R. Ranganathan, 'Kemmer algebra from generalised Clifford elements', J. Math. Appl. 22, 102-11, (1960).

1) R.J. Duffin, 'On the characteristic matrices of covariant systems', Phys. Rev., 54, 114, (1939).  
 N. Kemmer, 'The algebra of meson matrices', Proc. Camb. Phil. Soc. 39, 189, (1943).  
 G. Petiau, Thesis, Paris, (1936).

use for the gradient vector the symbol

$$\psi_u = -\frac{1}{x} \partial_u \psi \quad (5.3)$$

Using this (5.1) reads as

$$\partial_u \psi_u = -x \psi \quad (5.4)$$

The set (5.3), (5.4) is equivalent to (5.2) since any second order differential equation can be put into the form of a set of simultaneous equations, of first order. Writing (5.3) and (5.4) in detail gives

$$\begin{aligned} \partial_1 \psi + 0 + 0 + 0 + x \psi_1 &= 0 \\ 0 + \partial_2 \psi + 0 + 0 + x \psi_2 &= 0 \\ 0 + 0 + \partial_3 \psi + 0 + x \psi_3 &= 0 \\ 0 + 0 + 0 + \partial_4 \psi + x \psi_4 &= 0 \\ \partial_1 \psi_1 + \partial_2 \psi_2 + \partial_3 \psi_3 + \partial_4 \psi_4 + x \psi &= 0 \end{aligned} \quad (5.5)$$

Introducing the five-element column-matrix

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{x} \partial_1 \psi \\ -\frac{1}{x} \partial_2 \psi \\ -\frac{1}{x} \partial_3 \psi \\ -\frac{1}{x} \partial_4 \psi \\ \psi \end{pmatrix} \quad (5.6)$$

Eq.(5.5) can be given a compact matrix form

$$(\beta_u \partial_u + x \cdot I) \varphi = 0 \quad (5.7)$$

where

$$\beta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad \beta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix};$$

$$\beta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}; \quad \beta_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \quad \beta_5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(5.8)

Eq.(5.7) with the  $\beta$ -matrices given by (5.8) is called the Kemmer equation for spin zero.

By direct calculation one verifies that

$$\beta_\mu^3 = \beta_\mu \quad ; \quad \mu = 1, 2, 3, 4$$

$$\beta_\mu \beta_2^2 + \beta_2^2 \beta_\mu = \beta_\mu \quad ; \quad \mu \neq 2$$

$$\beta_\mu \beta_2 \beta_\mu = 0 \quad ; \quad \mu \neq 2 \text{ and no summation} \quad (5.9)$$

$$\beta_\mu \beta_2 \beta_\lambda + \beta_\lambda \beta_2 \beta_\mu = 0 \quad ; \quad \mu \neq \lambda \neq 2$$

$$\beta_\mu^2 \beta_2^2 = \beta_2^2 \beta_\mu^2$$

The set of equations (5.9) can be summarised as

$$\beta_\mu \beta_2 \beta_\lambda + \beta_\lambda \beta_2 \beta_\mu = \delta_{\mu\nu} \beta_\lambda + \delta_{\lambda\nu} \beta_\mu \quad ; \quad \mu, \nu, \lambda = 1, 2, 3, 4. \quad (5.10)$$

6.3 In the above the Kemmer algebra has four generating elements :  $\beta_1, \beta_2, \beta_3, \beta_4$ . It will, therefore, be denoted as  $K(4)$ . We will denote by  $K(r)$  the Kemmer

algebra generated by  $r$  elements with the commutation relation

$$\beta_l \beta_m \beta_n + \beta_n \beta_m \beta_l = \delta_{lm} \beta_n + \delta_{mn} \beta_l; l, m, n = 1, 2, \dots, r. \quad (5.11)$$

It is important to note that the algebra  $K(r)$  has always got an irreducible representation of dimension  $r+1$ . To obtain the basic  $r$  generators, let us consider the elements of the generalised Clifford algebra  $C_r^{r+1}$  i.e. generated by elements  $\mathcal{L}_1$  and  $\mathcal{L}_2$  such that

$$\begin{aligned} \mathcal{L}_1 \mathcal{L}_2 &= \omega \mathcal{L}_2 \mathcal{L}_1 \\ \mathcal{L}_1^{r+1} &= \mathcal{L}_2^{r+1} = I \end{aligned} \quad (5.12)$$

Let us consider the elements of  $C_r^{r+1}$  such as

$$E_{ij} = \frac{1}{r+1} \sum_{q=0}^r \omega^{q(i-j)} \mathcal{L}_1^q \mathcal{L}_2^{j-i}; i, j = 1, 2, \dots, r+1 \quad (5.13)$$

These entities possess an important property as shown by A.O.Morris<sup>3)</sup>

$$E_{ij} E_{kl} = \delta_{jk} E_{il} \quad (5.14)$$

3) A.O.Morris, 'On a generalised Clifford algebra', Quart. J. Math. (Oxford) 18, 7, (1969).

Let us now pick out of the  $E_{ij} - \delta_{ij}$  elements

$$E_{i,r+1} ; i=1,2,\dots,r$$

and define  $\beta_i - s$  such that

$$\beta_i = [E_{i,r+1} + E_{r+1,i}] ; i=1,2,\dots,r \quad (5.15)$$

It is easily checked that  $\beta_i - s$  satisfy the Kerner relations. In particular when  $r=4$ , we obtain the 6-dimensional representation of the Kerner matrices relating to spin zero particles.

## CHAPTER VI

### Representations of Para-Fermi Rings\*

**C.1** We first give a brief review of para-statistics which contains as special cases both Fermi-Dirac and Bose-Einstein statistics. Using the construction given in Chapter V for the Kerner algebra we get some of the reducible representations of <sup>1)</sup> Para-Fermi algebra which contains as a special case, the Fermi-Dirac algebra of anticommutation relations.

**C.2** Among the outstanding differences between present-day quantum mechanics and classical mechanics is the symmetrization postulate that quantum mechanical states of more than one particle must be either symmetric (bosons) or antisymmetric (fermions) under permutations. The question whether other types of symmetries under permutation can exist for a system of identical particles (or other statistics) can be pursued in two different frameworks, quantum mechanical and the field theoretical. The equivalence of the two approaches is well established for Bose and Fermi particles. For other types of particles the equivalence

\* Alladi Ramakrishnan, R. Vasudevan and P.S. Chandrasekaran, 'Representations of Para-Fermi rings and generalised Clifford elements', *J. Math. Anal. Appl.* 21, 1 (1970).

1) H.S. Green, 'A generalised method of field quantisation', *Phys. Rev.* 92, 870 (1953).

is not quite clear.

In the quantum mechanical framework, the symmetrisation postulate says that allowed states belong to one-dimensional representations of the permutation group. For general types of symmetry an obvious way would be to use the higher dimensional irreducible representations of the symmetric group.

6.3 For free Bose or Fermi particles, the observables such as the energy momentum and other operators can all be constructed in a simple ways starting from the number operators

$$n_k = a_k^* a_k \quad (6.1)$$

where  $a_k$  and  $a_k^*$  are the annihilation and creation operators for a particle in quantum state k. These operators obey the commutation relations

$$[a_k, a_\ell^*]_+ = \delta_{k\ell} ; [a_k, a_\ell]_+ = [a_k^*, a_\ell^*]_+ = 0 \quad (6.2)$$

where  $[A, B]_+ = AB - BA$  and the upper (lower) sign holds for Bose (Fermi) particles. The Fock representation which is used to represent the operators  $a_k$  and  $a_k^*$  is characterised by the vacuum state  $\emptyset$ , such that

$$a_k \emptyset_0 = 0 \quad (6.3)$$

where  $\vec{E}_0$  is the cyclic no-particle vector.<sup>10)</sup> From equations (6.1 and 6.2) we have the commutation relations

$$[n_k, a_\ell]_- = -\delta_{k\ell} a_\ell, [n_k, a_\ell^*]_- = \delta_{k\ell} a_\ell^* \quad (6.4)$$

and

$$[n_k, n_\ell]_- = 0 \quad (6.5)$$

From equations (6.2 and 6.3) and (6.4) it follows immediately that the  $n_k$  have integer eigenvalues  $0, 1, 2, \dots$  for Bose particles and 0 or 1 for Fermi particle. It is important to note that the definition eq.(6.1) and the commutation relations, eqns. (6.4) and (6.5) of the number operators are the same for both the Fermi and Bose cases. From the Bose and Fermi commutation relations it follows that many particle states can either be symmetric or anti-symmetric, respectively, under permutations.

Many particle states that do not obey the symmetrisation postulate which are due to different quantisation schemes were obtained by Green<sup>11)</sup>. He used the definition and commutation relations of the number operators as a clue and realised that the

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11) H.S. Green, 'A generalised method of field quantisation,' Phys. Rev. 92, 870, (1953).

properties of quantisation follow from eq.(4) so that any change in the definition of the number operators which preserves eq.(6.1) will lead to an interesting quantisation scheme. Green replaced the definition of the number operators definition (6.1) by

$$n_k = \frac{1}{2} [a_k^*, a_k]_{\pm} + \text{const}$$

where the upper (lower) sign refers to the generalisation of Bose (Fermi) quantisation which we call para-Bose (Para-Fermi). With eq.(6.4) and (6.6) Green obtained a stronger set of trilinear commutation relations that are known as the basic parafield commutation relations.

$$[[a_k^*, a_\ell]_{\pm}, a_m]_- = -2\delta_{km} a_\ell \quad (6.7)$$

$$[[a_k, a_\ell]_{\pm}, a_m]_- = 0 \quad (6.8)$$

and hence

$$[[a_k, a_\ell]_{\pm}, a_m^*]_- = 2\delta_{lm} a_k \pm 2\delta_{km} a_\ell \quad (6.9)$$

By taking hermitian adjoint of the above three equations we obtain still other relations.

Also from the above commutation relation it can be shown<sup>2)</sup> in general that the product  $a_k a_\ell^*$

<sup>2)</sup> A.H.L. Messiah and O.W. Greenberg, 'On the symmetrisation postulate and its experimental foundation', Phys. Rev. 136, B363, (1964).

acting on the cyclic vector  $|0\rangle$  yields

$$a_k a_\ell^* |0\rangle = p \delta_{k\ell} |0\rangle \quad (6.10)$$

for all  $k, \ell$  and  $p$  is a positive integer.

G.4 Green<sup>13)</sup> found an infinite set of solutions, labelled by integers  $p \geq 1$  for the equations (6.7) and (6.8). when  $p=1$  his solutions are the usual Bose and Fermi operators. He also showed that operators satisfying (6.7) and (6.8) can be decomposed as

$$a_k = \sum_{\alpha=1}^p b_k^{(\alpha)} \quad (6.11)$$

where  $b_k^{(\alpha)}$  are commuting Fermi operators in the case of Para-Fermi systems and anticommuting Bose operators in the case of Para-Bose systems. That is, in the former case for a given  $\alpha$ ,  $b_k^{(\alpha)}$  behave as Fermion operators but for different values of  $\alpha$  say  $\alpha_1$  and  $\alpha_2$ ,  $b_k^{(\alpha_1)}$  and  $b_k^{(\alpha_2)}$  commute. In the later case for the same value of  $\alpha$  the  $b_k^{(\alpha)}$  behave as Bose operators but for different values of  $\alpha$  they anticommute. In the Para-Fermi case characterised by  $p$  the Green's ansatz (i.e. eq. 6.11) yields the result

$$(a_k)^{p+1} = 0 \quad \text{for all } k \quad (6.12)$$

6.5 To proceed further let us suppose that there are  $v$  annihilation operators  $a_k$  ( $k = 1, 2, \dots, v$ ) and correspondingly creation operators  $a_k^*$  ( $k = 1, 2, \dots, v$ ) and define  $2v$  hermitian operators<sup>3), 4)</sup>

$$\beta_{2k-1} = \frac{1}{2} (a_k + a_k^*) ; \beta_{2k} = \frac{i}{2} (a_k - a_k^*); \\ k = 1, 2, \dots, v \quad (6.13)$$

Using eq.(6.7) and (6.8) for the para-hermitian we have that

$$[\beta_1, [\beta_\mu, \beta_\nu]_-]_- = \partial_\mu \lambda_\nu \beta_\nu - \partial_\nu \lambda_\mu \beta_\mu \quad (6.14)$$

The condition (6.10) (which is true only for the para-fermi case) becomes

$$(\beta_{2k-1} - i\beta_{2k})^{p+1} = 0 ; \quad k = 1, 2, \dots, v \quad (6.15)$$

and

$$(\beta_{2k-1} - i\beta_{2k})^j \neq 0, \quad \text{for } j < p \quad (6.16)$$

Ryan and Sudarshan<sup>3)</sup> obtained the algebra of the rotation group in  $(2v+1)$  dimensions using the  $\beta$ -s

- 3) C.Ryan and E.C.G.Sudarshan, 'Representations of Para-ferni rings', Nucl. Phys. **27**, 397, (1963).
- 4) S.Kamefuchi and Y.Takahashi, 'Field quantization and statistics', Nucl. Phys. **26**, 177, (1963).

defined above. It was again shown by them that when  $\beta=2, \beta=8$  defined by (6.13), (6.14), (6.15) and (6.16) gives raise to the Kerner algebra and a method of obtaining the basic representation of dimension  $(2v+1)$  for the Kerner algebra was given in Chapter V.

Now we shall give a method to construct the next higher irreducible representation of dimension

$N = 2v + {}^{2v}C_2$ . To obtain this, we take all commutators  $[\beta_m, \beta_n]_- = J_{mn}$  of the generating elements,  ${}^{2v}C_2$  in number where  $[\ , ]_-$  means commutation. If we add the  $2v$  generators  $\beta_m = J_{0m} = J_{m0}$  to the above we get a closed set under commutation. We can take the  $J_{mn}$ 's to constitute a vector space and consider transformations onto itself so that we can get a representation. Let us take an aggregate in this vector space which is any element of the algebra, say A

$$A = \sum a_{mn} J_{mn}; m+n; m, n = 0, 1, 2, \dots, 2v \quad (6.17)$$

Let us now define mappings  $\hat{E}_i \rightarrow$  such that

$$A \xrightarrow{\hat{E}_i} A' = [A, J_{0i}]_- = [A, J_{i0}]_-; i=1, 2, \dots, 2v \quad (6.18)$$

It is easily verified that

$$[\hat{E}_\lambda, [\hat{E}_\mu, \hat{E}_\nu]_-]_- = \delta_{\mu\nu} \hat{E}_\lambda - \delta_{\lambda\nu} \hat{E}_\mu \quad (6.19)$$

and

$$\hat{E}_i^3 = \hat{E}_i \quad ; \quad i = 1, 2, \dots, 2v \quad (6.20)$$

Thus we have the representation of the generators of the Kerner algebra  $K^{(2v)}$  which is of dimension  $2v + {}^{2v}C_2$ . It is to be noted that we need not know the actual matrix representation of the  $\beta^{-s}$  themselves to obtain  $\hat{E}_i$  matrices.

6.6 In this section a method obtaining representations of the  $\beta^{-s}$  satisfying (6.14), (6.15) and (6.16) is given for any order  $p$  of the Para-Dirac ring. To be specific we denote by  $\beta_k^{(p)} \quad (k = 1, 2, \dots, 2v)$  the generators of the  $p$ th order paraferm ring. Let  $\beta_k^{(p)}$  be constructed as

$$\beta_k^{(p)} = \frac{\gamma_k}{2} \otimes I + I \otimes \beta_k^{(p-1)} ; \quad p = 3, 4, \dots \text{etc} \quad (6.21)$$

where  $\gamma_k$ 's are the elements of the Clifford algebra  $C_{2v}^2$  i.e. the algebra of  $2v$  anticommuting elements each of whose square yields the unit of the algebra. Starting from the Pauli matrices, generators of  $C_{2v}^2$  can be obtained by the  $\sigma$ -operation, due to Ramakrishnan<sup>5)</sup> discussed in Chapter I. The dimension of  $\gamma_k \in C_{2v}^2$  is  $2^v$ . If we start with  $p = 3$  we have

$$\beta_k^{(3)} = \frac{\gamma_k}{2} \otimes I + I \otimes \beta_k^{(2)} \quad (6.22)$$

5) Alladi Ramakrishnan, 'The Dirac Hamiltonian as a member of a hierarchy of matrices', J. Math. anal. Appl. 22, 9, (1967).

It is easily seen that since  $\beta_k^{(2)}$  are the Kerner elements and  $\gamma_k$  are the Clifford elements that  $\beta_k^{(3)}$  the 3-order para-form operators obey condition (6.14), (6.15) and (6.16). Use has been made of the facts that

$$(\gamma_k - i\gamma_{k-1})^2 = 0 \quad (6.23)$$

and

$$(\beta_k^{(2)} - i\beta_{k-1}^{(2)})^j = 0 \quad \text{for } j=3 \text{ only} \quad (6.24)$$

$$\neq 0 \quad \text{for } j < 3 \quad (6.25)$$

Similarly defining

$$\beta_k^{(4)} = \frac{\gamma_k}{2} \otimes I + I \otimes \beta_k^{(3)} \quad (6.26)$$

that the relations corresponding to the para-form statistics for order  $p = 4$  are obeyed.

In general eq.(6.21) is found to be valid for all  $p$ . If we start with the  $2v+1$  dimensional representation of  $\beta_k^{(2)}$  ( $k = 1, 2, \dots, 2v$ ) then the dimensionally of  $\beta_k^{(p)}$  ( $k = 1, 2, \dots, 2v$ ) turns out to be

$$N = 2^{v(p-2)} (2v+1) ; p = 3, 4, \dots, \text{etc.} \quad (6.27)$$

Another type of construction starting again with the elements  $\beta_k^{(2)}$  ( $k = 1, 2, \dots, 2v$ ) (of the Kerner algebra)

would be to define

$$\beta_k^{2m} = \beta_k^{(m)} \otimes I + I \otimes \beta_k^{(m)} \quad \text{when } p = 2m, \text{ even} \quad (6.28)$$

and

$$\beta_k^{(2m+1)} = \frac{\gamma_k}{2} \otimes I + I \otimes \beta_k^{(2m)} \quad \text{when } p = 2m+1, \text{ odd}$$

$m = 1, 2, \dots$  etc

$$(6.29)$$

Again if we had started with the  $(2v+1)$  dimensional representation for  $\beta_k^{(2)}$  ( $k=1, 2, \dots, 2v$ ) the dimension  $N$  of the representation of the  $p$ th order  $\beta_k^{(p)}$  ( $k=1, 2, \dots, 2v$ ) is given by

$$N = (2v+1)^{2^{n-1}} 2^{(p-2)v} \quad (6.30)$$

where  $n$  is the biggest integer such that  $2^n < p$ .

However, if we had begun with

$$\beta_k^{(2)} = I \otimes \frac{\gamma_k}{2} + \frac{\gamma_k}{2} \otimes I \quad (6.31)$$

and proceeded along according eq.(6.31) the dimension of the representation at the  $p$ th stage would be  $2^{pv}$ .

## CHAPTER VII

Special Unitary Algebras Through Para-Fermi-Operators\*

7.1 Recently, different schemes<sup>1),2)</sup> have been built up with the object of arriving at para-fields by associating internal degrees of freedom with the ordinary statistics of elementary particles. Govorkov<sup>1)</sup> gives a construction to achieve the above said aim, in the case of second order  $\beta=2$  Para-Fermi algebra for the SU(2) generators. In this chapter the construction to get SU(n) generators from Para-Fermi operators of order  $\beta=n$  is given.

7.2 First a brief review of the work Govorkov<sup>1)</sup> is made. In addition our construction for the  $O(2,1)$  algebra is also given in terms of second order,  $\beta=2$  para-Bose operators. Extensive use of Green's ansatz (Chapter VI) is made in these derivations. It is important to note that in Green's construction (eq.(6.11)) if one attaches phase factors to the different Fermi operators  $b_k^{(\alpha)}$  ( $\alpha = 1, 2, \dots, b$ ) the commutation relations defining parastatistics,

\* Alladi Ramakrishnan, R. Vasudevan and P.S. Chandrasekaran, 'Para-Fermi operators and special unitary algebras', J. Math. Anal. Appl. (to be published).

- 1) Govorkov, 'Possibility of a parafield representation of internal degrees of freedom like isospin and strangeness', JETP 52, 960, (1960).
- 2) O. Steinmann, 'Symmetrization postulate and the Cluster property', Nuovo Cimento 44A, 753, (1960).

namely eqns. (6.7), (6.8) and (6.9) are unaltered. This particular property was first noticed by Feshbach and Tonlijanovich<sup>4)</sup>.

7.3 For constructing the SU(2) operators, let us consider the  $p = 2$  Para-Formi algebra obtained from the operators  $a_k$  and  $a_k^*$ ,  $k$  being the momentum index. By Green's ansatz<sup>3)</sup> we have for  $p = 2$  Para-Formi case that

$$a_k = b_k^{(1)} + b_k^{(2)} \quad (7.1)$$

where  $b_k^{(1)}$  and  $b_k^{(2)}$  are two sets of commuting Formi operators i.e. the operators of set 1 ( $i, k = 1$ ) commute with the operators of the set 2. i.e.,  $i = 2$ . Let us define<sup>1), 4)</sup> the operator

$$\tilde{a}_k = b_k^{(1)} - b_k^{(2)} \quad (7.2)$$

It is easily seen that  $\tilde{a}_k$  also like  $a_k$  satisfies the Para-Formi algebra of order 2. Denoting the generators of SU(2) as  $T_1, T_2$  and  $T_3$  and if they are defined<sup>1)</sup> as<sup>2)</sup>

$$T_1 = \frac{1}{4} \sum_k [\tilde{a}_k^*, a_k]_- \quad (7.3)$$

3) H.S. Green, 'A generalized method of field quantisation', Phys. Rev. 92, 870, (1953).

4) A. Feshbach and N. Tonlijanovich, 'Selection rules for para-particles', MIT Preprint (1967).

\* Here definitions differ from those ref. 1) slightly. Also  $[ \ ]_-$  indicates commutation and  $[ \ ]_+$  stands for anti-commutation.

$$T_2 = \frac{i}{4} \sum_k [\tilde{a}_k^*, a_k]_+ \quad (7.4)$$

$$T_3 = \frac{1}{4} \left\{ \sum_k [a_k^*, a_k]_+ - 2 \right\} \quad (7.5)$$

then it is easy to verify that

$$[T_i, T_j]_- = i \epsilon_{ijk} T_k; i, j, k = 1, 2, 3 \quad (7.6)$$

i.e. the algebra of  $SU(3)$  is obtained.

However, if we have started with Para-Bose operators  $a_k$ 's and using Green's ansatz for  $D=2$  case

$$a'_k = b'_k^{(1)} + b'_k^{(2)} \quad (7.7)$$

and

$$\tilde{a}'_k = b'_k^{(1)} - b'_k^{(2)} \quad (7.8)$$

where  $b'_k^{(1)}$  and  $b'_k^{(2)}$  are Bose operators,  $b'_k^{(1)}$  anticommuting with  $b'_k^{(2)}$ . Then defining

$$T'_1 = \frac{1}{4} \sum_k [\tilde{a}'_k^*, a'_k]_- \quad (7.9)$$

$$T'_2 = \frac{i}{4} \sum_k [\tilde{a}'_k^*, a'_k]_+ \quad (7.10)$$

and

$$T'_3 = \frac{1}{4} \left\{ \sum_k [a'_k^*, a'_k]_- - 2 \right\} \quad (7.11)$$

the  $T_i' - s$  ( $i=1,2,3$ ) satisfy the  $O(3,1)$  commutation relations:

$$\begin{aligned} [T_1', T_2']_- &= -iT_3'; [T_2', T_3']_- = iT_1'; \\ [T_3', T_1']_- &= iT_2' \end{aligned} \quad (7.12)$$

#### 7.4 SU(3) algebra and the Para-Fermi field of order three.

In this section we give an explicit construction of the generators of  $SU(3)$  making use of the Para-Fermi fields of order three defined through the modified Green's ansatz described in section 7.2.

Let us call the para-Fermi operator of order three as  $a_k$  ( $k$  denoting the momentum index) then, by Green's ansatz we have

$$a_k = b_k^{(1)} + b_k^{(2)} \quad (7.13)$$

Let us now define a  $\beta_k$  such that

$$\begin{aligned} \beta_k &= \frac{1}{2} a_k = \frac{1}{2} (b_k^{(1)} + b_k^{(2)} + b_k^{(3)}) \\ &= \frac{1}{2} (b_k^{(1)} + c_k^{(1)}) \end{aligned} \quad (7.14)$$

where

$$c_k^i = \sum_{j, i \neq j} b_k^{(j)} \quad ; \quad i, j = 1, 2, 3 \quad (7.15)$$

and  $\hat{\beta}_k^{(i)} = \frac{-b_k^{(i)} + c_k^{(i)}}{2}$  (7.16)

It is to be noted that the  $b_k^{(i)}$ 's are Fermi operators which commute for different values of  $i$ .

Let us again define objects  $E_j^i(k)$  such that

$$E_j^i(k) = \sum_r (\beta_r - \hat{\beta}_r^{(i)})^* (\beta_r - \hat{\beta}_r^{(j)}) x \\ \times (\beta_r - \hat{\beta}_r^{(k)}) (\beta_r - \hat{\beta}_r^{(ik)})^*, i+j+k \quad (7.17)$$

so that

$$E_j^i(k)^* = E_i^j(k) \quad (7.18)$$

Let us also define

$$H(k) = \frac{1}{2} [E_i^i(k) - E_j^j(k)]; i,j,k - \text{cycle} \quad (7.19)$$

so that  $\sum_{k=1}^3 H(k) = 0$

Hence only two of the  $H(k)$ 's are linearly independent and we thus have got six linearly independent operators which are closed under commutation. The three raising and lowering operators along with  $H(k)$  can be grouped into three  $I, U, V$  spin algebras of  $SU(3)$ . It is also noted that the non-diagonal operators have

the commutation rule

$$[E_j^i, E_\ell^k]_- = \delta_{jk} E_\ell^i - \delta_{ik} E_j^k \quad (7.20)$$

In terms of the three commuting  $H(k)$ 's the two Cartan operators are

$$H_1 \text{ and } \frac{2}{3}(H^{(2)} - H^{(3)}) \quad (7.21)$$

Hence we have the  $SU(3)$  algebra. It is also to be noted that various authors have constructed  $SU(3)$  algebra from Boson operators only or Fermion operators where the construction here uses essentially the Green's ansatz since Para-Fermi operators can be split up into commuting sets of Fermi algebras.

### 7.5 The generators of $SU(n)$ from Para-Fermi fields of order $n$ .

The above method is easily capable generalisation to the  $SU(N)$  case using Green's ansatz for the operators being Para-Fermi operators of the  $n$ th order. Extending the procedure of sec 7.4 we define

$$\beta_r = \frac{1}{2} \left( \sum_{i=1}^n b_r^{(i)} \right) \equiv \frac{1}{2} a_r \quad (7.22)$$

and

$$\hat{\beta}_r = \frac{1}{2} (c_r^{(i)} - b_r^{(i)}) \quad (7.23)$$

where

$$c_r^{(i)} = \sum_{j, i \neq j} b_r^{(j)} \quad (7.24)$$

The  $\frac{n^2-n}{2}$  raising and  $\frac{n^2-n}{2}$  lowering operators are defined as

$$E_{i_2}^{i_1}(i_3, i_4, \dots, i_n) = \sum_r (\beta_r - \hat{\beta}_r^{(i_1)})^* (\beta_r - \hat{\beta}_r^{(i_2)}) \prod_{k=3}^{i_n} (\beta_r - \hat{\beta}_r^{(i_k)}) (\beta_r - \hat{\beta}_r^{(i_k)})^* \\ k \neq l; i_k \neq i_l; (k, l = 1, 2, \dots, n) \quad (7.25)$$

Let us also define a quantity  $H$  as

$$H = \frac{1}{n} \sum_r \sum_p [(\beta_r - \hat{\beta}_r^{(i_1)})^* (\beta_r - \hat{\beta}_r^{(i_1)}) \prod_{i_k=i_2}^{i_n} (\beta_r - \hat{\beta}_r^{(i_k)}) (\beta_r - \hat{\beta}_r^{(i_k)})^*] \\ \text{where } p \text{ is a permutation of } [i_1, i_2, \dots, i_n] \quad (7.26)$$

and also  $(\beta_r - \hat{\beta}_r^{(i_k)})$  commutes with  $(\beta_s - \hat{\beta}_s^{(i_l)})$

The  $n$  commuting operators  $H(i_j)$  are defined to be

$$H(i_j) = \sum_r \left\{ (\beta_r - \hat{\beta}_r^{(i_j)})^* (\beta_r - \hat{\beta}_r^{(i_j)}) \prod_{i_k \neq i_j} (\beta_r - \hat{\beta}_r^{(i_k)}) (\beta_r - \hat{\beta}_r^{(i_k)})^* \right\} - H \quad (7.27)$$

where  $i_k$  takes all values from the set  $\{i_1, i_2, \dots, i_n\}$  except that of  $i_j$  which again is an element of the set  $i_1, i_2, \dots, i_n$ . With this definition of  $H(i_j)$  we have

$$\sum_{j=1}^n H(i_j) = 0 \quad (7.28)$$

Also we find that the  $H$ -s and  $E$ -s obey the

commutation relations

$$[H(i_j), H(i_k)]_- = 0; i_j, i_k = i_1, i_2, \dots, i_n \quad (7.29)$$

$$[H(i_\ell), E_{i_\ell}^{i_k}]_- = (\delta_{jk} - \delta_{j\ell}) E_{i_k}^{i_j}; \quad (7.30)$$

$$[E_{i_k}^{i_j}, E_{i_m}^{i_\ell}]_- = \delta_{k\ell} E_{i_m}^{i_j} - E_{i_k}^{i_\ell} \delta_{jm} \quad (7.31)$$

$$(i_j, i_k, i_\ell, i_m = i_1, i_2, \dots, i_n)$$

which are same other than the  $SU(n)$  commutation relations.  
Because of the condition (7.29) only  $(n-1)$  of  $H(i_j)$ 's  
are linearly independent.

## CHAPTER VIII

Induced Matrices for Matrices Satisfying  $A^m = I$ 

8.1 The algebraic properties of matrices which satisfy the conditions

$$\begin{aligned} A^m &= B^m = I \\ AB &= \omega B A ; \omega^m = 1 \end{aligned} \quad (8.1)$$

have received considerable attention in recent years.

The general mathematical formulation has been made by Horinaga and Nono<sup>1)</sup>, Yamazaki<sup>2)</sup> and Morris<sup>3)</sup> while its relation to physics through the study of their specific representations has been made systematically by Somakrishnan<sup>4)</sup> and collaborators.

<sup>1)</sup> T. N. Sentharam, P. S. Chandrasekaran and Balaji B. Naren, 'Induced matrices for matrices satisfying  $A^m = I$ ', *J. Math. Phys.* (to be published).

2) K. Horinaga and T. Nono, 'On the linearization of a form of higher degree and its representation', *J. Sci. Hiroshima Univ., Series A*, Vol. 2, No. 1, 13 (1952).

3) K. Yamazaki, 'On projective representations and ring extensions of finite groups', *J. Fac. Sci., Univ. of Tokyo, Section 1*, Vol. 19, 147, (1964).

4) A. G. Morris, 'On a generalised Clifford algebra', *Quart. J. Math. Oxford (2)*, Vol. 18, 7, (1967)

4) Alladi Somakrishnan, 'The Dirac Hamiltonian as a member of a hierarchy of matrices', *J. Math. Anal. Appl.* Vol. 21, 9 (1967).

Alladi Somakrishnan, R. Vasudevan, H. R. Banga and P. S. Chandrasekaran, 'On a generalization of the L-matrix hierarchy', *J. Math. Anal. Appl.*, Vol. 22, 360, (1968)

Alladi Somakrishnan, P. S. Chandrasekaran, H. R. Banga, T. S. Sentharam and R. Vasudevan, 'The generalised algebra and the Unitary groups', *J. Math. Anal. Appl.* Vol. 22, 164 (1968).

In this chapter we study matrices imposing only the first condition in eq.(1). The case when  $n = 2$  has been studied by Kim<sup>6)</sup> and here we extend these results for general  $n$ .

While in the case of matrices obeying both the conditions the lowest dimension of the matrices will be  $n \times n$  when they are irreducible, it turns out that in the case of the matrices satisfying condition (1) will have induced matrices<sup>6)</sup> of any dimension  $m$  which are all irreducible.

The induced matrix can be considered as a transformation acting on a vector whose components are of the  $q$ th degree homogeneous monomials in  $x_1, \dots, x_m$  where  $m$  is the dimension of the given matrix, the dimension of the  $q$ th induced matrix being  $\binom{m+q-1}{m-1}$ .

If  $A$ , the initially given matrix is involutory, so are the induced matrices. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.2)$$

- 6) S.K.Kim, 'Involutional matrices based on the representation theorem of  $GL(2)$ ', J. Math. Phys., Vol.10, 1929, (1929).
- 6) D.E. Littlewood, 'The theory of group characters and matrix representation of groups', Oxford Univ. Press 178, (1950).

is involutory then  $a = -d$  and  $ad-bc = 1$ . The first induced matrix is, for example, given by the transformation

$$\begin{pmatrix} x_1'^2 \\ x_1' x_2' \\ x_2'^2 \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{pmatrix} \quad (8.3)$$

Using  $a = -d$  and  $ad-bc = 1$ , the  $3 \times 3$  matrix in eq.(3) is verified to be also involutory. It has been pointed out by Ramakrishnan<sup>7)</sup> that the induced matrix can be arrived at by making use of helicity matrices<sup>8)</sup> defined by him when the matrix to be induced is involutory.

Denoting  $A \otimes I$  by  $H_1$  and  $I \otimes A$  by  $H_2$  and the vectors on which they act as  $(x_1, x_2) \otimes I$  and  $I \otimes (x_1, x_2)$  the combined effect of  $H_1 H_2$  is on so

that if

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \otimes \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = (A \otimes A) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

we have that

$$A \otimes A = \begin{pmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{pmatrix} \quad (8.4)$$

<sup>7)</sup> Alladi Ramakrishnan, Private communication.

<sup>8)</sup> Alladi Ramakrishnan, 'Generalised helicity matrices', J. Math. Anal. appl. Vol. 21, 88, (1969).

As pointed out by him eq. (8.3) is nothing but the 'contracted' form of eq.(8.4) by which is meant that the two middle columns and rows of the  $4 \times 4$  matrix in eq.(8.4) coalesce in a particular way to give the  $3 \times 3$  matrix in eq.(63). The involutory nature here follows from the direct product structure of  $H_1 H_2$  whose square yields a number times the identity. The same considerations apply to the general case as well.

Then, we summarise the work of Kim<sup>5)</sup> on the generating equation method of finding the induced matrix elements for the case  $n = 3$ . Later, the generating equations for the qth induced matrix of a  $(3 \times 3)$  matrix are calculated. It is further shown that a  $3 \times 3$  matrix whose cube is the unit matrix can be written in terms of a few generalised Clifford elements with the generalised hyperbolic functions as coefficients. Finally the eigenvalues of the above matrices are studied.

**8.2 Involutional transformations of  $GL(2)$ :** The complete set of qth degree polynomials in two variables  $x$  and  $y$

$$F_v(\Sigma) = x^{q-v} y^v \quad (8.6)$$

$$\Sigma = (x, y); v=0, 1, \dots, q$$

is taken as the basic set. An element  $R^{(2)}$  of  $GL(2)$  is given by the general  $(2 \times 2)$  matrix (non-singular)

$$R^{(2)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2); \quad ad - bc \neq 0 \quad (8.7)$$

where  $a, b, c, d$  are arbitrary parameters. The  $(q+1)$ -dimensional representation  $R^{(2)}$  is furnished by the  $q$ th induced matrix of  $R^{(2)}$  and is given by

$$\begin{aligned} F_q(R^{(2)})_v &= (ax + by)^{q-1} (cx + dy)^1 \\ &= \sum_{\mu=0}^q (R^{(2)}_{q\mu})_{\nu\mu} F_\mu(v) \end{aligned} \quad (8.8)$$

The explicit form of  $R^{(2)}_q$  is obtained by developing Eq.(8.8) in power series and one gets

$$(R^{(2)}_{q\mu})_{\nu\mu} = a^{q-\mu-1} b^\mu c^{\nu} \sum_k \binom{q}{k} \binom{q-1}{\mu-k} \left(\frac{ad}{bc}\right)^k \quad (8.9)$$

Now, an invariant matrix  $A_q$  of a matrix  $A$  is defined by the relation<sup>6</sup>

$$A_q A_q = (AA)_q \quad (8.10)$$

where  $A_q$  is the matrix whose entries are polynomials in the elements of the matrix  $A$ , from which it easily follows that

$$(A_q)^m = [(A)^m]_q = (A^m)_q = k^q I \quad (8.11)$$

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$$A^m = kI \quad (8.12)$$

where  $A_q$  is of dimension  $(q \times 1)$ .

Therefore the conditions on the four parameters of the  $(2 \times 2)$  matrix in order that Eq.(8.12) is satisfied, automatically leaves  $A_q^{(2)}$  involitional. When  $k = 1$ , the involitional matrix  $A^{(2)}$  involves only two parameters since in this case  $a + d = 0$  and  $bc = 1 - a^2$  and thus can be expressed as

$$A^{(2)}(\theta) = \begin{smallmatrix} & \\ \widehat{r}_Z & R(\theta) \end{smallmatrix} \quad (8.13)$$

where

$$\widehat{r}_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (8.14)$$

and  $R(\theta)$  is the rotation in two dimensions with defined by

$$(bc)^{\frac{1}{2}} = \sin\theta \quad (8.15)$$

**8.3 Generating Equations for the general Involutional Matrices:** By an exactly similar procedure as that utilised for the case of  $n = 2$ , we define the  $q$ th degree

homogeneous polynomials in three variables,  $x$ ,  $y$  and  $z$   
as

$$F_{(\alpha_1, \alpha_2)}(\underline{y}) = x^{\gamma - \alpha_1 - \alpha_2} y^{\alpha_1} z^{\alpha_2} \quad (8.16)$$

where the non-negative integers  $\gamma$ ,  $\alpha_1$ ,  $\alpha_2$  obey

$$\alpha_1 + \alpha_2 \leq \gamma \quad (8.17)$$

The linear homogeneous transformation  $R^{(3)}$  in three dimensions is given by a  $3 \times 3$

$$R^{(3)} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \quad (8.18)$$

where the  $a_{ij}$ 's are arbitrary parameters. The qth induced matrix of  $R^{(3)}$  involving nine parameters is then simply given by the equation

$$\begin{aligned} F_{(\alpha_1, \alpha_2)}(R^{(3)} \underline{x}) &= (a_{00}x + a_{01}y + a_{02}z)^{\gamma - \alpha_1 - \alpha_2} x \\ &\quad (a_{10}x + a_{11}y + a_{12}z)^{\alpha_1} x \\ &\quad (a_{20}x + a_{21}y + a_{22}z)^{\alpha_2} \end{aligned} \quad (8.19)$$

$$= \sum_{(\alpha'_1, \alpha'_2)} (R^{(3)})_{\gamma}^{(\alpha'_1, \alpha'_2)} F_{(\alpha'_1, \alpha'_2)}(\underline{x})$$

$$\underline{x} = (x, y, z)$$

where the matrix  $R_q^{(3)}$  is labelled by the different partitions of the non-negative integers  $(\alpha'_1, \alpha'_2)$  and  $(\alpha_1, \alpha_2)$  satisfying

$$\begin{aligned} \alpha'_1 + \alpha'_2 &\leq q \\ \alpha_1 + \alpha_2 &\leq q \end{aligned} \quad (8.20)$$

Hence the dimension of  $R_q^{(3)}$  is simply given by the number of solutions  $(\alpha'_1, \alpha'_2)$  of Eq.(8.20) which in this case is equal to  $\binom{q+2}{2}$ .

Obviously  $R_q^{(3)}$  reduces to  $R^{(3)}$  when  $q = 1$ . For convenience we can choose the partitions in decreasing order in  $\alpha'_1$  for a given value of  $\alpha'_1 + \alpha'_2$  and increasing order in  $\alpha_1 + \alpha_2$  for labelling the matrix.

The simple power series expansion of Eq.(8.19) yields an explicit expression for  $R_q^{(3)}$ , viz.

$$\begin{aligned} (R_q^{(3)})(\alpha_1, \alpha_2)(\alpha'_1, \alpha'_2) &= \left(\frac{\alpha_{11}}{\alpha_{20}}\right)^{\alpha'_1 + \alpha'_2} \left(\frac{\alpha_{22}}{\alpha_{21}}\right)^{\alpha'_2} \alpha_{00}^{\alpha_1 - \alpha'_1 - \alpha'_2} \alpha_{10}^{\alpha_1} \alpha_{20}^{\alpha_2} \times \\ &\times \sum_{\substack{\mu_1, \mu_2 \\ v_1, v_2}} \binom{\alpha_1 - \alpha'_1 - \alpha'_2}{\mu_1} \binom{\alpha'_1}{v_1} \binom{\alpha'_2}{\alpha'_1 + \alpha'_2 - \mu_1 - v_1} \binom{\mu_1}{\mu_2} \binom{v_1}{v_2} \\ &\times \binom{\alpha'_1 + \alpha'_2 - \mu_1 - v_1}{\alpha'_2 - \mu_2 - v_2} \left(\frac{\alpha_{01} \alpha_{20}}{\alpha_{00} \alpha_{21}}\right)^{\mu_1} \left(\frac{\alpha_{02} \alpha_{21}}{\alpha_{01} \alpha_{22}}\right)^{\mu_2} \\ &\times \left(\frac{\alpha_{11} \alpha_{20}}{\alpha_{10} \alpha_{21}}\right)^{v_1} \left(\frac{\alpha_{12} \alpha_{21}}{\alpha_{11} \alpha_{22}}\right)^{v_2} \end{aligned} \quad (8.21)$$

This is very similar to the Louricelle function<sup>9)</sup> except for a term.

The above procedure of generating the induced matrix can be easily generalised to the case of an arbitrary  $(n \times n)$  matrix  $R^{(n)}$  given by

$$R^{(n)} = \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \dots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \dots & a_{n-1,n-1} \end{pmatrix} \quad (8.22)$$

In this case we define the qth degree polynomials

$$\begin{aligned} F_{(\alpha_1, \alpha_2, \dots, \alpha_{n-1})}(\underline{x}) &\quad \text{in } n \text{ variables } x_1, \dots, x_n \\ F_{(\alpha_1, \alpha_2, \dots, \alpha_{n-1})}(\underline{x}) &= x_1^{\alpha_1 - \sum_{i=1}^{n-1} \alpha_i} x_2^{\alpha_2} \dots x_n^{\alpha_{n-1}} \\ \underline{x} &= (x_1, x_2, \dots, x_n) \end{aligned} \quad (8.23)$$

9) J. Slater, 'Generalised hypergeometric functions', Cambridge Univ. Press 27, (1966)

with non-negative integers  $\alpha_i$  satisfying the partition equation

$$\sum_{i=1}^{n-1} \alpha_i \leq q \quad (8.24)$$

The  $q$ th induced representation of  $R^{(n)}$  is given by the matrix  $R_q^{(n)}$  defined by

$$\begin{aligned} F_{(\alpha_1, \alpha_2, \dots, \alpha_{n-1})}(R^{(n)}\gamma) &= (a_{00}x_1 + a_{01}x_2 + \dots + a_{0,n-1}x_n)^{q - \sum_{i=1}^{n-1} \alpha_i} \times \\ &\quad \times \prod_{j=1}^{n-1} \left( \sum_{k=0}^{n-1} a_{jk} x_{k+1} \right)^{\alpha_j} \\ &= \sum_{(\alpha'_1, \alpha'_2, \dots, \alpha'_{n-1})} [R^{(n)}]_{(\alpha_1, \alpha_2, \dots, \alpha_{n-1})(\alpha'_1, \alpha'_2, \dots, \alpha'_{n-1})} \times \quad (8.25) \\ &\quad \times F_{(\alpha'_1, \alpha'_2, \dots, \alpha'_{n-1})}(\gamma) \end{aligned}$$

where the matrix is labelled by the distinct partitions given by Eq.(8.24). We can choose then in the decreasing order  $(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$  for a given value of  $(\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}) \leq q$ .

The dimension of  $R_q^{(n)}$  is just given by the number of solutions of the partitions equation Eq.(8.24) which is simply equal to  $\binom{n+q-1}{q} = \binom{n+q-1}{n-1}$ .

Let us now specialize the method of induction to the case of involutational matrices satisfying the equation

$$[R^{(n)}]^m = kI \quad (8.26)$$

As in the case of  $(3 \times 3)$  matrix the conditions on  $R^{(n)}$  so that it satisfies Eq.(8.26) leave its qth induced representation  $R_q^{(n)}$  to obey

$$[R_q^{(n)}]^m = R^q I \quad (8.27)$$

This follows directly from the property of induced matrices which form a special case of invariant matrices satisfying Eq.(8.10). The conditions on  $R^{(n)}$  implied by Eq.(8.26) when  $n = n$  are simply given by the characteristic equation of  $R^{(n)}$  which are

$$\text{Tr } R^{(n)} = \text{Tr } [R^{(n)}]^2 = \dots = \text{Tr } [R^{(n)}]^{n-1} = 0 \quad (8.28)$$

and

$$\det R^{(n)} = (-1)^n k$$

Let us consider the special case of a  $(3 \times 3)$  matrix satisfying the equation

$$(A^{(3)})^3 = I \quad (8.29)$$

The eigenvalues of  $A^{(3)}$  are then given by the cube roots of unity :  $1, \omega, \omega^2$ . As in the case of  $A^{(2)}$ ,  $A^{(3)}$

can be reduced to the form

$$F^{(3)}(\theta) = VA^{(3)}V^{-1} = \begin{pmatrix} f_1^{(3)} & \omega^2 f_2^{(3)} & \omega f_3^{(3)} \\ \omega^2 f_3^{(3)} & \omega f_1^{(3)} & f_2^{(3)} \\ \omega f_2^{(3)} & f_3^{(3)} & \omega^2 f_1^{(3)} \end{pmatrix} \quad (8.30)$$

where the  $f_i^{(3)}$  are the generalised hyperbolic function<sup>10)</sup> of order three with argument  $\lambda\theta$  with  $\lambda = \exp \frac{\pi i}{3} = \omega^{\frac{1}{2}}$ ,  $\omega$  being a primitive cube root of unity. It is immaterial to compute the matrix  $V$  whose existence can be inferred from the fact that both  $A^{(3)}$  and  $V^{(3)}$  are non-singular and satisfy Eq.(8.29). The  $f$ 's are functions of the entries of the matrix  $A^{(3)}$ . They satisfy the determinantal condition

$$\begin{vmatrix} f_1^{(3)} & f_2^{(3)} & f_3^{(3)} \\ f_2^{(3)} & f_1^{(3)} & f_2^{(3)} \\ f_3^{(3)} & f_2^{(3)} & f_1^{(3)} \end{vmatrix} = 1 \quad (8.31)$$

The  $f_i^{(3)}$  are related to the trigonometric functions of order three  $k_i^{(3)}$  through the relation

$$k_i^{(3)}(\theta) = \lambda^{(1-i)} f_i^{(3)}(\lambda\theta) \quad (8.32)$$

10) Louis A. Pipes, 'Cyclical functions and Permutation matrices', J. Franklin Inst. Vol. 222, 335 (1901).

Bateman Manuscript Project on Transcendental Functions Vol. III, p. 210, McGraw Hill Book Co., USA, 1953.

$F^{(3)}(\theta)$  can be expressed as

$$F^{(3)}(\theta) = B^{(3)} R^{(3)}(\theta) \quad (8.33)$$

where

$$B^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad (8.34)$$

and  $R^{(3)}(\theta)$  is the matrix

$$R^{(3)}(\theta) = \begin{pmatrix} f_1 & \omega^2 f_2 & \omega f_3 \\ \omega f_3 & f_1 & \omega^2 f_2 \\ \omega^2 f_2 & \omega f_3 & f_1 \end{pmatrix} \quad (8.35)$$

The interesting point is that  $R^{(3)}(\theta)$  can be expressed as

$$R^{(3)}(\theta) = f_1 \cdot I + f_2 \cdot \omega^2 P^{(3)} + f_3 \omega (P^{(3)})^2 \quad (8.36)$$

$$= f_1 I + f_2 (\omega^2 P^{(3)}) + f_3 (\omega^2 P^{(3)})^2 \quad (8.37)$$

where the matrix

$$P^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (8.38)$$

is a base element of the generalized Clifford algebra  $\mathbf{C}_3^3$  (Vide Chapter II).

The determinantal condition Eq.(8.31) can also be written

as

$$\begin{aligned} \det & \sum_{i=1}^3 f_i^{(3)}(\lambda\theta) (P^{(3)})^{i-1} \\ & = \det \sum_{i=1}^3 k_i^{(3)} \lambda^{i-1} (P^{(3)})^{i-1} \\ & = 1 \end{aligned} \quad (8.40)$$

From the above we have that

$$\begin{aligned} F^{(3)}(\theta) &= V A^{(3)} V^{-1} = B \left( f_1(\theta') \cdot I + f_2(\theta') P^{(3)} + f_3(\theta') (P^{(3)})^2 \right) \\ &= B \exp P^{(3)} \theta' ; \quad \theta' = \lambda \theta \end{aligned}$$

It has been pointed out by Ramkrishnan<sup>11)</sup> that if

$$T(x) = M e^{Nx}$$

with

$$MN = \omega NM ; \quad \omega^m = 1$$

and

$$M^m = N^m = I$$

where  $\omega$  is a primitive  $m^{\text{th}}$  root of unity, we always have that  $[T(x)]^m = I$ . So the above case of a  $3 \times 3$  matrix is a particular case of the general rule.

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11) Alladi Ramkrishnan, Private Communication.

The above discussion can now be carried for an arbitrary ( $n \times n$ ) involutonal matrix

$$[A^{(n)}]^n = I \quad (8.41)$$

which can be transformed to the form

$$VA^{(n)}V^{-1} = F^{(n)}(\theta) = B^{(n)}R^{(n)}(\theta) \quad (8.42)$$

where

$$B^{(n)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \\ 0 & 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix} \quad (8.43)$$

and

$$\omega^n = 1, \omega \text{ being a primitive } n^{\text{th}} \text{ root of unity} \quad (8.44)$$

where the matrix

$$P^{(n)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad (8.45)$$

is an element of the generalised Clifford algebra  $C_g^n$ .

The determinental condition on the hyperbolic and trigonometric functions of order  $n$  is simply given by

$$\begin{aligned} \det \sum_{i=1}^n f_i^{(n)}(\lambda\theta) (P^{(n)})^{i-1} \\ = \det \sum_{i=1}^n \lambda^{i-1} k_i^{(n)}(\theta) (P^{(n)})^{i-1} \\ = 1 \end{aligned} \quad (8.40)$$

The  $f$ 's are functions of the entries of the matrix  $A^{(n)}$  and the explicit relation is of little concern to us. The existence of  $v^{(n)}$  is again guaranteed by the fact that  $A^{(n)}$  and  $P^{(n)}(\theta)$  are both non-singular and satisfy Eq.(8.41).

**8.4 Eigenvalues of  $R_q^{(n)}$  and  $R_q^{(n)}$ :** In this section we first calculate the eigenvalues of the  $q$ th induced matrix  $R_q^{(n)}$  of the matrix  $R^{(n)}$  given by Eq.(3.10) and specialise to the case when  $R_q^{(n)}$  is involutorial. The calculation is based on the simple theorem that if the matrix  $R^{(n)}$  is triangular, then its induced matrix  $R_q^{(n)}$  is also triangular in shape similar to  $R^{(n)}$ . This theorem has been proved by Kim<sup>(4)</sup> for  $n = 2$ , and it is true even in the general case. Consider for example the case of  $n = 3$ . If  $R^{(3)}$  has the form

$$R_6^{(3)} = \begin{pmatrix} a_{00} & 0 & 0 \\ a_{10} & a_{11} & 0 \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \quad (8.47)$$

it follows from Eq.(8.49) that

$$[R_{\mathcal{V}}^{(3)}]_{(\alpha, \alpha')} = 0$$

unless  $\alpha'_2 \leq \alpha_2$  and  $\alpha'_1 + \alpha'_2 \leq \alpha_1 + \alpha_2$  where

$\alpha = (\alpha_1, \alpha_2)$  and  $\alpha' = (\alpha'_1, \alpha'_2)$ . These are simply the conditions for the matrix  $R_q^{(3)}$  to be triangular in shape similar to  $R^{(3)}$ . It is not hard to prove the same result for any  $n$ .

In fact it follows directly from Eq.(8.35) that if the matrix  $R^B$  is triangular then, since  $a_{ij} = 0$ ,  $i < j$ ,  $[R_{\mathcal{V}}^{(n)}] = 0$  unless

$$\sum_{i=k}^{n-1} \alpha_i \geq \sum_{i=k}^{n-1} \alpha'_i$$

$$k = 1, 2, \dots, n-1 \quad (8.48)$$

These are simply the conditions for  $R_q^{(n)}$  to be triangular and similar in shape to  $R_q^{(3)}$ . Eq.(8.48) incidentally suggests a more convenient labelling of  $R_q^{(n)}$  by  $(\beta_1, \beta_2, \dots, \beta_{n-1})$  satisfying

$$0 \leq \beta_{n-1} \leq \dots \leq \beta_1 \leq V,$$

$$\beta_1 + \beta_2 + \dots + \beta_{n-1} \leq V$$

where

$$\beta_j = \sum_{i=j}^{n-1} \alpha_i \quad ; \quad j = 1, 2, \dots, n-1 \quad (8.49)$$

The generating equation (8.26) for the induced matrix  $R_q^{(n)}$  then simply becomes

$$\prod_{j=0}^{n-1} \left( \sum_{k=0}^{n-1} a_{jk} x_{k+1} \right)^{\beta_j - \beta_{j+1}} = \sum_{\beta'} [R_q^{(n)}]_{\beta\beta'} F_{\beta'}(\gamma) \quad (8.50)$$

With

$$\beta_0 = q, \beta_n = 0 \quad (8.51)$$

Eqs.(8.26) and (8.50) are completely equivalent.

Now it is always possible to transform the matrix  $R^{(3)}$  into the triangular matrix  $R^{(3)T}$ ,

$$R^{(3)T} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ \xi_1 & \epsilon_2 & 0 \\ \xi_2 & \xi_3 & \epsilon_3 \end{pmatrix} \quad (8.52)$$

through a suitable unitary transformation. Here the  $\xi$ 's are constants and the  $\epsilon$ 's are the eigenvalues of  $R^{(3)}$ .

Substituting Eq.(8.52) in Eq.(8.21) we obtain

$$[R_q^{(3)T}]_{\alpha, \alpha'} = \epsilon_1^{\alpha - d_1 - d_2} \epsilon_3^{\alpha'_2} \sum_{\beta} \binom{d_1}{\beta} \binom{\alpha'_2}{d_1 + d_2 - \beta} \binom{d_1 + d_2 - \beta}{\alpha'_2} \times (\epsilon_2)^{\beta} \left( \frac{\xi_1}{\xi_2} \right)^{\beta - d_2} \left( \frac{\xi_2 - \alpha'_1 - \alpha'_2 + \beta}{\xi_2} \right) \left( \frac{\xi_3}{\xi_2} \right)^{\alpha'_1 - \beta}$$

if  $\xi_1, \xi_2, \xi_3 \neq 0$

$$= (\epsilon_1)^{q-\alpha_1-\alpha_2} (\epsilon_2)^{\alpha_1} (\epsilon_3)^{\alpha_2} \delta_{\alpha_1 \alpha_1'} \delta_{\alpha_2 \alpha_2'}$$

$$\text{if } \xi_1 = \xi_2 = \xi_3 = 0$$

(3.53)

The eigenvalues of  $R_q^{(3)}$  are then given by

$$\epsilon_1^{q-\alpha_1-\alpha_2} \epsilon_2^{\alpha_1} \epsilon_3^{\alpha_2}$$

$$\alpha_1 + \alpha_2 \leq q$$
(3.54)

The determinant of  $R_q^{(3)}$  is given by

$$\det R_q^{(3)} = \prod_{\alpha_1, \alpha_2} \epsilon_1^{q-\alpha_1-\alpha_2} \epsilon_2^{\alpha_1} \epsilon_3^{\alpha_2}$$

$$\quad \quad \quad \alpha_1 + \alpha_2 \leq q$$

$$= (\epsilon_1 \epsilon_2 \epsilon_3)^{\binom{q+2}{3}} = \Delta^{\binom{q+2}{3}}$$
(3.55)

where  $\Delta$  denotes the determinant of  $R^{(3)}$ .

The trace of  $R_q^{(3)}$  is given by

$$\text{Tr } R_q^{(3)} = \left( 1/(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)(\epsilon_1 - \epsilon_3) \right) \times$$

$$\times \left[ \epsilon_1 \epsilon_2 (\epsilon_1^{q+1} \epsilon_2^{q+1}) + \epsilon_2 \epsilon_3 (\epsilon_2^{q+1} \epsilon_3^{q+1}) \right.$$

$$\left. + \epsilon_3 \epsilon_1 (\epsilon_3^{q+1} - \epsilon_1^{q+1}) \right]$$

$$\text{if } \epsilon_1 \neq \epsilon_2 \neq \epsilon_3$$
(3.56)

$$\binom{q+2}{2} \epsilon^q ; \text{ if } \epsilon = \epsilon_1 = \epsilon_2 = \epsilon_3 \quad (3.57)$$

The above formula can be immediately generalised to yield

$$\det R_q^{(n)} = \Delta \binom{q+n-1}{n} \quad (3.58)$$

where

$$\Delta = \epsilon_1 \epsilon_2 \dots \epsilon_n \quad (3.59)$$

is the determinant of  $R^{(n)}$ . Further

$$\begin{aligned} \text{Tr } R_q^{(n)} &= \prod_{i < j} (\epsilon_i - \epsilon_j)^{-1} \sum_{\substack{k+l \\ k, l \text{ cyclic} \\ k, l = 1, 2, \dots, n}} \epsilon_k \epsilon_l (\epsilon_k^{q+1} - \epsilon_l^{q+1}) \\ &\quad \text{if } \epsilon_1 + \epsilon_2 + \dots + \epsilon_n \\ &= \binom{q+n-1}{n-1} \epsilon^q \quad \text{if } \epsilon = \epsilon_1 = \dots = \epsilon_n \end{aligned} \quad (3.60)$$

The eigenvalues of  $R_q^{(n)}$  are given by

$$\begin{aligned} \epsilon^{q - \sum_{i=1}^{n-1} d_i} \epsilon_2^{d_1} \epsilon_3^{d_2} \dots \epsilon_n^{d_{n-1}} ; \\ \sum_{i=1}^{n-1} d_i \leq q \end{aligned} \quad (3.61)$$

Another interesting property of  $A_q^{(n)}$  which can be easily verified from Eq.(8.25) is that

$$\sum_{ij=0}^{n-1} a_{ij} \frac{\partial R_q^{(n)}}{\partial a_{ij}} = q R_q^{(n)} \quad (8.63)$$

The above discussion can now be specialised to the case of general involutonal ( $n \times n$ ) matrix  $A^{(n)}$ . The eigenvalues of  $A^{(n)}$  are given by

$$\epsilon_1 = \epsilon, \quad \epsilon_2 = \omega \epsilon, \quad \dots, \quad \epsilon_n = \omega^{n-1} \epsilon$$

$$\omega^n = 1 \quad (8.64)$$

In this case we have

$$\text{Tr } A_q^{(n)} = \frac{(-1)^n \epsilon^q}{n} (1 + \omega^q + \omega^{2q} + \dots + \omega^{(n-1)q}) \quad (8.64)$$

so that

$$\begin{aligned} \text{Tr } A_q^{(n)} &= 0 \quad \text{for } q \neq 0 \bmod n \\ &= (-1)^n \epsilon \quad \text{for } q = 0 \bmod n \end{aligned} \quad (8.65)$$

The determinant of  $A_q^{(n)}$  is given by

$$\det A_q^{(n)} = (\epsilon^n \omega^{\binom{n}{2}})^{\binom{(n+n-1)}{n}} \quad (8.66)$$

Appendix

A.0 In the following we give the definitions of certain notions concerning algebras, ideals etc.

A.1 Definition of an Associative Algebra:

A set ' $\Lambda$ ' of elements  $\{a, b, \dots\}$  is an associative algebra over the field  $K$ , if

- i)  $\Lambda$  is a vector space over  $K$ ,
- ii) an operation of multiplication, ' $\cdot$ ', is defined in  $\Lambda$  and satisfies:

$$\alpha(a \cdot b) = (\alpha a) \cdot b$$

$$a \cdot (\alpha b) = \alpha(a \cdot b)$$

$$\alpha \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

$\forall a, b, c \in \Lambda$  and  $\forall \alpha \in K$

In the following when we say an algebra, we always mean an associative algebra.

A.2 Definition of Left Ideal:

A subset  $L$  of an algebra is called a left ideal if

- i)  $L$  is a vector subspace of  $\Lambda$

- ii)  $x \in L$  and  $a \in \Lambda$  implies that  $ax \in L$ .

In other words,  $AL \subseteq L$ .

A.3 Similarly a right ideal R is defined so that  
 $AR \subset R$ .

A.4 If  $I \subset A$  is both a left and right ideal then I is called a two-sided ideal or just an ideal.

A.5 Any algebra A contains the ideal A and  $\{0\}$ . An ideal different from A is called proper. It is important to note that the identity operator of the algebra A, cannot belong to any proper ideal.

#### A.6 Definition of a Simple Algebra:

An algebra with no proper two-sided ideals other than  $\{0\}$  is called simple.

#### A.7 Remark:

An algebra which is not simple is called semi-simple.

A.8 In the following we state without proof the famous theorem of Wedderburn<sup>1), 2)</sup> which we have used in Chapter 2.

Theorem: A simple algebra A of  $n^2$  elements is isomorphic to the complete matrix algebra of  $(n \times n)$  matrices over the field K.

- 1) J.H.M. Wedderburn, Proc. Lond. Math. Soc. (2), 5, 77, (1908).  
 2) See also H. Boerner, Representations of groups, North Holland Pub. Co., Amsterdam.

