TWISTED CONJUGACY CLASSES IN LATTICES IN SEMISIMPLE LIE GROUPS

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T. Mubeena

To my mother.

Abstract

Let Γ be a group and let $\phi : \Gamma \to \Gamma$ be an endomorphism. We define an action $g.x := gx\phi(g^{-1})$, for $g, x \in \Gamma$, of Γ on itself. The ϕ-twisted conjugacy class of an element $x \in \Gamma$ is the orbit of this action containing x. A group Γ has the *R_∞-property* if every automorphism ϕ of Γ has infinitely many ϕ-twisted conjugacy classes. In this thesis we show that any irreducible lattice in a non-compact connected semisimple Lie group with finite center and having real rank at least 2 has the R_∞-property. We also show that any countable abelian extensions Λ of Γ has the R_∞-property when (i) the lattice Γ is linear, (ii) Γ is a torsion free non-elementary hyperbolic group.

We also consider the R_∞-problem for S-arithmetic lattices.

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CONTENTS

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1

INTRODUCTION

1.1 OVERVIEW

Given a group Γ and an endomorphism $\phi : \Gamma \to \Gamma$, we can define an action of Γ on itself by $g.x = gx\phi(g^{-1})$. This is just the conjugation action when ϕ is the identity homomorphism. The orbits of this action are called *ϕ-twisted conjugacy classes*. The set of all ϕ-twisted conjugacy classes is denoted by $\mathcal{R}(\phi)$. The cardinality $R(\phi)$ of $\mathcal{R}(\phi)$ is called the *Reidemeister number* of ϕ. All these notions make sense for any group, not necessarily finitely generated. However, our main focus will be on automorphisms of countable groups. We say that Γ has the *R_∞-property* if there are infinitely many ϕ-twisted conjugacy classes for every automorphism ϕ of Γ. If Γ has the R_∞-property, we shall call Γ an *R_∞-group*.

There are countably infinite groups with only finitely many conjugacy classes (see [47] §1.4 or [31] Chapter 4, §3). Examples of such groups which are finitely generated have been constructed by S. Ivanov. Recently D. Osin [38] has constructed a finitely generated infinite group that has exactly one non trivial conjugacy class. The notion of twisted conjugacy in groups originated in Nielsen-Reidemeister fixed point theory ([28], [15]). It also arises in other areas of research such as number theory and representation theory. The problem of determining which class of groups have the R_∞-property was initiated by A. Fel'shtyn and R. Hill [19]. This is one of the principal problems in the theory of twisted conjugacy classes in infinite groups. The R_∞-property has its consequences in topology. We explain below its connection to fixed point theory.

Let X be a compact connected polyhedron and let $f : X \to X$ be a continuous map. Denote the fixed point set of f by $Fix(f) = \{x \in X \mid f(x) = x\}$. Let $p : \widetilde{X} \to X$ be the universal covering projection and let $\tilde{f} : \widetilde{X} \to \widetilde{X}$ be a lift of f to $\widetilde{X}$, that is $p \circ \tilde{f} = f \circ p$. Let $f_\# : \pi_1(X) \to \pi_1(X)$ be the induced homomorphism of f. Fix a base point $x_0 \in X$ such that $f(x_0) = x_0$. We say that two lifts $\tilde{f}, \tilde{f}'$ of f are equivalent if there is an element γ in the fundamental group $\Gamma := \pi_1(X, x_0)$ of X, which is identified with the deck transformation group of $\widetilde{X}$, such that $\tilde{f}' = \gamma\tilde{f}\gamma^{-1}$. This is an equivalence relation on the set of all lifts of f to $\widetilde{X}$. Denote by $[\tilde{f}]$ the equivalence class of $\tilde{f}$, called *lifting class* of f. Fix a lift $\tilde{f}$ and a base point $\widetilde{x_0} \in \widetilde{X}$. Let $\alpha \in \Gamma$. Then any lift of f is of the form $\alpha\tilde{f}$. Also for a lift $\tilde{f}$ and $\gamma \in \Gamma$, $p(Fix(\tilde{f})) = p(Fix(\gamma\tilde{f}\gamma^{-1}))$. Note that for any point $y \in \widetilde{X}$ such that $p(y) \in Fix(f)$, there is a unique map $\tilde{f}$ such that $p \circ \tilde{f} = f \circ p$ and $\tilde{f}(y) = y$. In particular, any fixed point of f is the projection of a fixed point of some lift $\tilde{f}$ of f. Hence we get an equivalence relation, denoted $\sim$, on the set $Fix(f)$ and we can write $Fix(f) = \bigsqcup_{[\tilde{f}]} p(Fix(\tilde{f}))$. The subset $p(Fix(\tilde{f})) \subseteq Fix(f)$ is called the *fixed point class of f* determined by the lifting class $[\tilde{f}]$ of f.

Again by fixing a lift $\tilde{f}$ of f together with an $\alpha \in \Gamma$, and an element $\tilde{x} \in \widetilde{X}$, we obtain a unique element $\tilde{\alpha} \in \Gamma$ such that $\tilde{\alpha}\tilde{f}(\tilde{x}) = \tilde{f}(\alpha(\tilde{x}))$. This gives a homomorphism $\phi : \pi_1(X) \to \pi_1(X)$ such that $\tilde{f}\alpha = \phi(\alpha)\tilde{f}$ for each $\alpha \in \Gamma$. Now for two lifts $\alpha\tilde{f}$ and $\beta\tilde{f}$ of f (where $\tilde{f}$ is fixed), $[\alpha\tilde{f}] = [\beta\tilde{f}]$ if and only if $\beta\tilde{f} = \gamma\alpha\tilde{f}\gamma^{-1}$ for some $\gamma \in \Gamma$, that is, if and only if $\beta\tilde{f} = \gamma\alpha\phi(\gamma^{-1})\tilde{f}$. By the uniqueness of lifts, we have $[\alpha\tilde{f}] = [\beta\tilde{f}]$ if and only if $\beta = \gamma\alpha\phi(\gamma^{-1})$ for some $\gamma \in \Gamma$. Furthermore, by choosing an appropriate base point, ϕ can be identified with the induced homomorphism $f_\# : \Gamma \to \Gamma$ of f. Thus it follows that there is a one-one correspondence between the set $\mathrm{Fix(f)}/\sim$ and $\mathcal{R}(f_\#)$, the Reidemeister classes of $f_\#$. If we choose a different lifting $\tilde{f}'$, we get a different homomorphism ϕ' and a bijection between the sets $\mathcal{R}(\phi)$ and $\mathcal{R}(\phi')$. Thus the number of fixed point classes of f is exactly the number $R(f_\#)$.

We continue with the same notations as in the section §1.2. Two points $x, y \in Fix(f)$ are said to be in the same Nielsen class of f if there is a path $c : [0,1] \to X$ joining x and y such that the paths c and $f \circ c$ are homotopic relative to the end points. This defines an equivalence relation on $Fix(f)$ and the equivalence class $[x]$ of x is called the *Nielsen class* of x. Let $x_1, x_2 \in X$ and let c be a path in X from x_1 to x_2 homotopic to $f \circ c$ with respect to the end points. Let $\tilde{f}$ be a lift of f and $\tilde{x}_1 \in Fix(\tilde{f})$ such that $p(\tilde{x}_1) = x_1$. Lift c to a path $\tilde{c}$ starting from $\tilde{x}_1$ and ending at some $\tilde{x}_2 \in \widetilde{X}$. Then p maps $\tilde{f} \circ \tilde{c}$ onto $f \circ c$ which is homotopic to c with respect to the end points. Thus $\tilde{f} \circ \tilde{c}$ also ends at $\tilde{x}_2$. Hence $\tilde{f}(\tilde{x}_2) = \tilde{x}_2$. In other words, both $p(\widetilde{x_1})$ and $p(\widetilde{x_2})$ lie in the same fixed point class. Conversely, let $\tilde{x}_1, \tilde{x}_2 \in Fix(\tilde{f})$ such that $p(\tilde{x}_1) = x_1 \neq x_2 = p(\tilde{x}_2)$. Let $\tilde{c} : I \to \widetilde{X}$ be a path from $\tilde{x}_1$ to $\tilde{x}_2$. Then $c = p \circ \tilde{c}$ is a path from x_1 to x_2 in X and $p(\tilde{f} \circ \tilde{c}) = f \circ p \circ \tilde{c} = f \circ c$. Hence p maps the loop $\tilde{c}(\tilde{f} \circ \tilde{c})^{-1}$ onto $c(f \circ c)^{-1}$. Since $\widetilde{X}$ is simply-connected, $\tilde{c}(\tilde{f} \circ \tilde{c})^{-1}$ is homotopic to the trivial loop. Thus c is homotopic to $f \circ c$. That is x_1 and x_2 are in the same Nielsen class. This shows that there is a one-one map from the set of Nielsen classes to the set $\mathcal{R}(f_\#)$. Thus, the number of Nielsen classes is a lower bound for $R(f_\#)$. A compact connected polyhedron has only finitely many Nielsen classes. Note that a lifting class $p(Fix(\tilde{f}))$ might be empty (since the lift $\tilde{f}$ may not have any fixed points), but a Nielsen class is always non-empty.

Assume the set $Fix(f)$ is non-empty. There is a notion of *index* which assigns a rational number to each Nielsen class of f. This index is homotopy invariant among maps in the same homotopy class of f. A Nielsen class is said to be *essential* if its index is non-zero and is called *inessential* otherwise. The *Nielsen number* $N(f)$ of f is defined to be the number of essential classes. We saw that the number $N(f)$ is an invariant among maps homotopic to f, always finite and a lower bound for the number of fixed points of f and hence for any map g homotopic to f. It also follows from the above discussion that $N(f) \leq R(f_\#)$. The number $R(f_\#)$ need not be finite. For example, if $f = 1_X$ then any two points are Nielsen equivalent, thus $N(f) \leq 1$ while $R(f)$ is the number of conjugacy classes in Γ. In particular, if Γ is abelian then $R(f) = |\Gamma|$, the cardinality of Γ. We refer the reader to [8] for more details on this.

1.4 THE JIANG SUBGROUP

The main tool to calculate $N(f)$ is the so-called Jiang subgroup $T(f) \leq \pi_1(X)$ introduced by B. Jiang in [28]. Fix a point $x_0 \in X$ and a self map f on X. We denote by $Map(X)$ the set of all continuous maps from X to itself with the supremum metric $d(f,g) = sup\{d(f(x), g(x)) \mid x \in X\}$, then it is a complete metric space. Let $p : Map(X) \to X$ be the map given by $p(g) = g(x_0)$. Then p induces a homomorphism $p_\pi : \pi_1(Map(X), f) \to \pi_1(X, f(x_0))$. The *Jiang subgroup* $T(f, x_0)$ with respect to f is the image of the homomorphism p_π. Equivalently, an element $\alpha \in \pi_1(X, f(x_0))$ is said to be in the Jiang subgroup $T(f, x_0)$ of f if there is a loop H in $Map(X)$ based at f such that the loop c in X defined by $c(t) = H(t)(x_0)$ is homotopic to α. A Jiang subgroup is independent of the base point and if f and g are homotopic maps then the corresponding Jiang subgroups are isomorphic. From now onwards, we use $T(f)$ instead of $T(f, x_0)$.

For any map $f : X \to X$, $T(X) \subset T(f)$, where $T(X) = T(1)$ the Jiang subgroup with respect to the identity map on X. We say that X is a *Jiang space* if $T(X) = \pi_1(X)$. For example, H-spaces satisfy this condition. For Jiang spaces, either $N(f) = R(f_\#)$ provided $R(f_\#)$ is finite or $N(f) = 0$ when $R(f_\#)$ is infinity. There are Jiang spaces in which if the fundamental group $\pi_1(X)$ has the R_∞-property then there is a fixed point free map homotopic to f. For any $n \geq 5$ a compact n-dimensional nilmanifold is constructed for which any homeomorphism is homotopic to a fixed point free map [22]. See [8], [16] and the references therein.

1.5 REPRESENTATION THEORY

We explain below the connection of twisted conjugacy with representation theory. Let G be any group and $\widehat{G}$ be the set of all equivalence classes of unitary irreducible representations of G. We will denote the equivalence class of a representation τ by $[\tau]$. The classical Burnside-Frobenius theorem says that if G is a finite group, the number of equivalence classes of irreducible representations is equal to the number of conjugacy classes of elements of G. Thus $\#\widehat{G} = R(Id)$. If $\phi \in Aut(G)$, it induces a map $\widehat{\phi} : \widehat{G} \to \widehat{G}$ given by $[\tau] \mapsto [\tau \circ \phi]$. When $\phi = Id$, $\#\widehat{G} = \#Fix(\widehat{\phi})$. It was later discovered in [19] by A. Fel'shtyn and Hill that $R(\phi) = \#Fix(\widehat{\phi})$ for any automorphism ϕ of a finite group G. Let $\widehat{G}_f$ be the set of equivalence classes of finite dimensional unitary irreducible representations of G and let $\widehat{\phi}_f : \widehat{G}_f \to \widehat{G}_f$ be the restriction of $\widehat{\phi}$ to $\widehat{G}_f$. A group G is said to be *ϕ-conjugacy separable* if any two non-ϕ-conjugate elements $g, h \in G$ are non-$\bar{\phi}$-conjugate in some finite quotient G/H of G, where $H \subset G$ is some subgroup such that $\phi(H) = H$. We say that G is *twisted conjugacy separable* if it is ϕ-conjugacy separable for any automorphism ϕ with $R(\phi) < \infty$ and we say that it is *strongly twisted conjugacy separable* if this condition is satisfied for any $\phi \in Aut(G)$. For example, polycyclic-by-finite groups are strongly twisted conjugacy separable. A. Fel'shtyn [16] has proved that if G is ϕ-conjugacy separable then $R(\phi) = \#Fix(\widehat{\phi}_f)$ whenever one of these numbers is finite. Recently, we heard from A. Fel'shtyn about a preprint [14] in which it is announced that for any finitely generated residually finite group, $R(\phi) = \#Fix(\widehat{\phi}_f)$, if $R(\phi)$ is finite.

1.6 KNOWN CLASSES OF GROUPS

We shall go through some classes of groups which are known to have (not have) the R_∞-property.

Hyperbolic Groups: A. Fel'shtyn and R. Hill [19] conjectured that any injective endomorphism of a finitely generated torsion free group with exponential growth would have infinitely many twisted conjugacy classes. D. Gonçalves and P. Wong [20] proved that Fel'shtyn-Hill conjecture is not true in general. They gave a non hyperbolic group of exponential growth having automorphisms with finite Reidemeister number. The notion of hyperbolicity of groups, which was introduced and developed by M. Gromov [23], are either virtually cyclic or have exponential growth. A hyperbolic group is called *elementary* if it is finite or if it contains an infinite cyclic subgroup of finite index. Otherwise it is called *non-elementary*. G. Levitt and M. Lustig [30] showed that this conjecture is true for automorphisms of any torsion free non-elementary hyperbolic group. That is, they have the R_∞-property. Later, A. Fel'shtyn [13] extended Levitt-Lustig's result to all non-elementary hyperbolic groups by removing the torsion free hypothesis. A lot of research has been done in this area since then and more groups have been found to have the R_∞-property. See section §2.3.2 for preliminaries.

Relatively Hyperbolic Groups: Relative hyperbolicity, a generalization of the concept of hyperbolicity in geometric group theory, was introduced by M. Gromov [23]. For example, there are non-uniform lattices in rank 1 real semisimple Lie groups which are relatively hyperbolic with respect to a set of subgroups but not hyperbolic. We shall discuss this in the preliminaries section §2.3.4. A. Fel'shtyn [16] showed that non-elementary relatively hyperbolic groups have the R_∞-property.

Baumslag-Solitar Groups: The Baumslag-Solitar group $BS(m,n) := \langle a, b \mid ab^m a^{-1} = b^n \rangle$, where $m \neq 0 \neq n \in \mathbb{Z}$, was introduced by G. Baumslag and D. Solitar [1] to provide examples of non-Hopfian groups. These groups act as counter examples for many results. For example, these groups disprove G. Higman's claim that every finitely generated one-relator group is Hopfian. Apart from this, these groups contain residually finite groups and Hopfian groups that are not residually finite. When $m = n = \pm 1$, $BS(m,n) \cong \mathbb{Z}^2$ and it is known that there is an automorphism of $\mathbb{Z}^2$ with only finitely many twisted conjugacy classes. A. Fel'shtyn and D. Gonçalves [17] showed that the Baumslag-Solitar groups have the R_∞-property except when $n = m = \pm 1$. The group $BS(1,m) = \langle a, t \mid tat^{-1} = a^m \rangle, m \neq 1$ is known as the *solvable Baumslag-Solitar group*. A solvable generalization of $BS(1,m)$ is the group $\Gamma(S) := \langle a, t_1, \cdots, t_k \mid t_i t_j = t_j t_i, i \neq j, t_i a t_i^{-1} = a^{n_i} \rangle$, where $S = \{n_1, n_2, \cdots, n_k\}$ are pairwise relatively prime positive integers. J. Taback and P. Wong [50] showed that the group $\Gamma(S)$ and any finitely generated group quasi-isometric to $\Gamma(S)$ have the R_∞-property. A *generalized Baumslag-Solitar group* is a finitely generated group which acts on a tree with all edge and vertex stabilizers being infinite cyclic. G. Levitt [29] showed that these groups have the R_∞-property.

Given a group G, the *restricted wreath product* of G with $\mathbb{Z}$ is the group $G \wr \mathbb{Z} := (\bigoplus_{i\in\mathbb{Z}} G_i) \rtimes_\alpha \mathbb{Z}$, where $G_i = G$ for all $i \in \mathbb{Z}$, and the $\mathbb{Z}$ -action α on $\bigoplus_{i\in\mathbb{Z}} G_i$ is given by the shift $\alpha(n)((x_i)) = (x_{i-n})$ for $(x_i) \in \bigoplus_{i\in\mathbb{Z}} G_i, n \in \mathbb{Z}$. D. Gonçalves and P. Wong [21] classified all finitely generated abelian groups G for which the group $G \wr \mathbb{Z}$ has the R_∞-property. In particular, the lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ have R_∞-property if and only if $2|n$ or $3|n$. For G_1, G_2 finite groups, $G_1 \wr \mathbb{Z}$ and $G_2 \wr \mathbb{Z}$ are quasi-isometric if and only if there exist positive integers d, r, s such that $|G_1| = d^r$ and $|G_2| = d^s$ (see [11]). Note that the groups $H_1 = \mathbb{Z}_4 \wr \mathbb{Z}$ and $H_2 = (\mathbb{Z}_2)^2 \wr \mathbb{Z}$ are quasi-isometric. The group H_1 has the R_∞-property while H_2 does not have the R_∞-property ([21]). Thus the R_∞-property is not preserved under quasi-isometries.

The groups $Aut(F_n)$ and $Out(F_n)$ of the free group F_n on n generators, the symplectic group $\mathrm{Sp}(2n, \mathbb{Z})$ for $n \geq 2$, mapping class group of an orientable closed surface S (not S^2) and its outer automorphism group have the R_∞-property [18].

Consider the free nilpotent group $N_{r,c}$ of rank r and nilpotency class c which is the quotient $N_{r,c} := F_r/\Gamma_{c+1}F_r$, where $\Gamma_1 F_r := F_r, \Gamma_2 F_r = [F_r, F_r]$ the commutator subgroup, and $\Gamma_{j+1}F_r := [F_r, \Gamma_j F_r]$. Here F_r denotes the free group of rank r. D. Gonąlves and P. Wong [22] showed that $N_{r,c}$ when $r = 2$ and $c \geq 9$ have the R_∞-property. V. Roman'kov [45] showed the following: $N_{r,c}$ when $r = 2$ or $r = 3$ and $c \geq 4r$ or $r \geq 4$ and $c \geq 2r$; any $N_{2,c}$ for $c \geq 4$ have the R_∞-property while $N_{2,2}, N_{2,3}$ and $N_{3,2}$ do not have the R_∞-property. It is known that the Heisenberg group of all 3×3 uppertriangular integer matrices has automorphisms with exactly $2N$ many twisted conjugacy classes for any given N [].

It is known that the R_∞-property is not inherited by finite index subgroups in general. This also shows that this property is not geometric, i.e, not invariant under quasi-isometry. For example, the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$, which contains the infinite cyclic group as an index 2 subgroup has the R_∞ -property [22] (whereas $R(-Id_{\mathbb{Z}}) = 2$). However there are classes of groups such as non-elementary hyperbolic groups within which this property is geometric. The R_∞-property is not preserved under quotients. For example, any finitely generated free group has this property while for example, their commutator quotient is free abelian which does not have this property.

It appears that the R_∞-problem for groups requires many different techniques which are peculiar to the class of groups under consideration. This makes the problem interesting and non-trivial. Some of them required geometric methods and some required algebraic methods.

1.7 THE MAIN RESULTS

The aim of this thesis is to study the R_∞-property of lattices in Lie groups. Our main results in the thesis address the R_∞-property for irreducible lattices in semisimple Lie groups of real rank at least 2 and the R_∞-property of their abelian extensions.

Theorem 1.7.1. *(Theorem 4.1.1) Let G be a connected semisimple non-compact Lie group of real rank at least 2 and having finite center. Let Γ be any irreducible lattice in G. Then Γ has the R_∞ property.*

When G has real rank 1, the above result is well-known. Indeed, assume that G has real rank 1. When the lattice Γ is cocompact, it is hyperbolic. When Γ is not cocompact, it is relatively hyperbolic. It is known that any non-elementary hyperbolic group as well as any non-elementary relatively hyperbolic group have the R_∞-property.

The linear groups $\mathrm{SL}(n,\mathbb{Z}), \mathrm{PSL}(n,\mathbb{Z}), \mathrm{PGL}(n,\mathbb{Z})$ and $\mathrm{PSp}(2n,\mathbb{Z})$ for $n \geq 2$ are a few examples of lattices. A separate elementary proof, showing the R_∞-property of these groups, is given in chapter 3 (see [34]). The R_∞-property for the group $GL(n,\mathbb{Z})$ follows from that of $\mathrm{PGL}(n,\mathbb{Z})$. The Theorem 1.7.1 when Γ is a principal congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$ is also shown in chapter 3 (see [34]). When $\Gamma = \mathrm{Sp}(2n,\mathbb{Z})$, the result was first proved by A. Fel'shtyn and D. Gonçalves [18]. We also give a proof for this result in chapter 3 (see [34]).

Our proof of the above theorem involves only elementary arguments, using some well-known but deep results concerning irreducible lattices in semi simple Lie groups. The main theorem is first established when G has no compact factors and has trivial centre. In this case, the proof uses the Zariski density property of Γ due to Borel as well as the strong rigidity theorem. When G has non-trivial compact factors, we need to use Margulis' normal subgroup theorem to reduce to the case when G has trivial centre and no compact factors.

We also consider abelian extensions of lattices in Lie groups. Consider a group extension:

$$1 \longrightarrow A \overset{j}{\hookrightarrow} \Lambda \overset{\eta}{\longrightarrow} \Gamma \longrightarrow 1.$$

We have the following result when A is abelian:

Theorem 1.7.2. *(Theorem 4.2.2) Let Γ and G be as in Theorem 1.7.1 and Λ be an extension of Γ by an arbitrary countable abelian group A. Assume that G is linear. Then Λ has the R_∞-property.*

The hyperbolic groups have many geometric properties. The following quasi-convexity properties of certain subgroups of (non-elementary) hyperbolic groups will be relevant for our purposes. The cyclic subgroup generated by an element of infinite order has finite index in its centralizer; infinite finitely generated subgroups which are quasi-convex have finite index in their normalizers. We have:

Theorem 1.7.3. *(Theorem 3.1.1) Let Λ be an extension of a group Γ by an arbitrary countable abelian group A. Then Λ has the R_∞-property in case any one of the following holds:*
(i) Γ is a torsion-free non-elementary hyperbolic group,
(ii) Γ is the fundamental group of a complete Riemannian manifold of constant negative sectional curvature and finite volume.

Remark 1.7.4. T. Nasybullov [35] has proved that $\mathrm{GL}(n,R)$ or $\mathrm{SL}(n,R)$, $n \geq 3$, where R is an infinite integral domain of characteristic zero and has no non-trivial automorphism, has R_∞-property.

In order to prove the above theorems, we needed to make use of the following well known results concerning lattices in Lie groups (See [42]):
Borel density theorem: Let G be a connected semisimple algebraic $\mathbb{R}$-group and let $G^0_{\mathbb{R}}$ be the identity component of the $\mathbb{R}$-points $G_{\mathbb{R}}$ of G. Let $\Gamma \subset G^0_{\mathbb{R}}$ be a lattice. If $G^0_{\mathbb{R}}$ has no compact factors, then Γ is Zariski dense in G.
Margulis' normal subgroup theorem: Let $\Gamma \subset G$ be an irreducible lattice where G is a connected semisimple Lie group of $\mathbb{R}$-rank ≥ 2 and with finite centre. If N is normal in Γ, then either N is of finite index in Γ or is a finite subgroup contained in the centre of G.
Strong rigidity theorem(Mostow-Margulis-Prasad) [51],[40]: Let G and G' be connected linear semisimple Lie groups with trivial centre and having no compact factors. Let $\Gamma \subset G$ and $\Gamma' \subset G'$ be irreducible lattices. Assume that G and G' are not locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$. Then any isomorphism $\phi : \Gamma \longrightarrow \Gamma'$ extends to an isomorphism $G \longrightarrow G'$ of Lie groups.

The R_∞-problem for abelian extensions of groups in the above theorems makes use of the following facts concerning normal subgroups of the group Γ. When Γ is hyperbolic, it uses the quasi-convexity property of infinite cyclic subgroups of Γ; in the case of the linear groups, non-central normal subgroups of $SL(n, \mathbb{Z}), n \geq 3$ and the fundamental group of complete Riemannian manifolds of constant negative sectional curvature of finite volume it uses the Margulis' normal subgroup theorem and Mostow-Margulis-Prasad strong rigidity theorem. Using the fact that Γ is hopfian, the R_∞-property for Λ can be deduced from the R_∞-property for Γ. We also consider the R_∞-problem for S-airthmetic lattices.

Our result contains as special cases the direct product $A \times \Gamma$ as well as the restricted wreath product $C \wr \Gamma := (\oplus_{\gamma \in \Gamma} C_\gamma) \rtimes \Gamma$, where $C_\gamma = C$ is any cyclic group. This leads to continuously many pairwise non-isomorphic abelian extensions of Γ which have the R_∞-property.

We describe below how this thesis is organized.

There are 5 chapters in this thesis including this chapter. We give a short chapter-wise description here.

Chapter 2 recalls necessary definitions and results concerning lattices in semisimple Lie groups. We state results such as Margulis normal subgroup theorem, Borel density theorem and superrigidity theorem which we use in later chapters. No proofs are given. Assuming some basic background in algebraic geometry, we recall algebraic groups, arithmetic lattices and the arithmeticity theorem of lattices in higher rank Lie groups. To bring geometry to some of the groups we consider in this thesis, we discuss some notions from geometric group theory such as quasi-isometry between geodesic metric spaces, hyperbolic groups and relatively hyperbolic groups. We use the residual finiteness and hopficity properties of groups in this thesis. We recall both the definitions and some results concerning such groups. The work of O. T. O'Meara played a role in one of our main results regarding linear groups. We briefly discuss the automorphism groups of the linear groups $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{GL}(n,\mathbb{Z})$ and $\mathrm{Sp}(2n,\mathbb{Z})$. At the end of this chapter, we make the necessary setup for our main results. We also introduce some elementary results concerning the R_∞-property of groups.

In Chapter 3 we study the R_∞-property for the groups $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{GL}(n,\mathbb{Z})$, $\mathrm{PSL}(n,\mathbb{Z})$, $\mathrm{PGL}(n,\mathbb{Z})$, all countable abelian extensions of these groups and non-central normal subgroups of $\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$.

Chapter 4 contains the main results of this thesis. We study the R_∞-property of lattices in semisimple Lie groups. We also investigate the R_∞-property for their (countable) abelian extensions when the Lie group is linear. We also consider the R_∞-problem for S-arithmetic lattices. At the end of this chapter, we conclude this thesis discussing some recent results in this area.

At the end of the thesis, there is an index which lists some of the important notations, definitions and terminology with references to the places where they are used.

Detailed chapter-wise description will be given in each chapter. A result with the mark $\square$ immediately following it means that no proof will be given.

2

PRELIMINARIES

As described in the introduction, the aim of the thesis is to study the R_∞-property for lattices in semisimple Lie groups. In this chapter, we recall certain definitions and results, concentrating on some important examples, which are necessary to understand the R_∞-problem for lattices.

The first two sections §2.1 and §2.2 of this chapter deal with lattices in semisimple Lie groups, algebraic groups and certain well known and deep results concerning lattices in semisimple Lie groups. Third section §2.3 of this chapter concentrates on quasi-isometry between geodesic metric spaces, the Švarc-Milnor Lemma which says that finitely generated groups are quasi-isometric to some length spaces on which the groups act properly cocompactly by isometries, the notion of Gromov's hyperbolicity for geodesic metric spaces and certain properties of hyperbolic groups together with some important examples which are relevant to our purpose, Gromov's hyperbolicity relative to a finite collection of subgroups and some of their properties. In the fourth section §2.4 we recall the definition of residually finite groups and hopfian groups. Section §2.5 deals with the automorphism groups of the linear groups $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{GL}(n,\mathbb{Z})$ and $\mathrm{Sp}(2n,\mathbb{Z})$. In the last section §2.6, we define and discuss the R_∞-property in groups and introduce some results which will be used in later chapters.

2.1.1 *Basic Definitions and Terminology*

Throughout this section we shall denote by G a locally compact Hausdorff topological group. A *Haar measure* μ on G is a regular Borel measure on G. A measure μ is *left invariant* if $\mu(gE) = \mu(E)$ for all Borel set $E \subset G$ and for all $g \in G$. We call a left invariant Haar measure on G a *left Haar measure*. A. Weil showed that every locally compact Hausdorff topological group G has at least one left Haar measure and if μ and ν are two left Haar measures on G then there exists a constant $c > 0$ such that $\nu = c\mu$, that is the left Haar measure is unique up to positive scalar multiples. In particular, any Lie group has a left Haar measure. Let $g \in G$ and let μ be a left Haar measure on G, then $\nu(E) := \mu(Eg)$, for each Borel set $E \subset G$, is another left Haar measure on G. By uniqueness of μ, there exists a continuous homomorphism $\chi_G : G \to \mathbb{R}_{>0}$, called *modular function*, such that $\nu(E) = \chi_G(g)\mu(E)$ for all Borel set $E \subset G$ and for all $g \in G$. A group is said to be *unimodular* if $\chi_G = 1$, the trivial homomorphism, equivalently, if the Haar measure is both left and right invariant. For example, groups G with $G = [G, G]$, abelian groups, connected nilpotent Lie groups and compact groups are unimodular.

Let $H \subset G$ be a closed subgroup of G. The quotient space G/H with the quotient topology is a locally compact Hausdörff topological space (not necessarily a group). Let χ_G and χ_H be the corresponding modular functions on G and H respectively. Then G/H has a left invariant measure if and only if $\chi_G(\gamma) = \chi_H(\gamma)$ for all $\gamma \in H$. The coset spaces G/H do not always have an invariant measure, for example, take $G = \mathrm{SL}(2, \mathbb{R})$ which is unimodular and $K = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \neq 0, b \in \mathbb{R}\right\}$, then G/K does not have a G-invariant measure. Let $\Gamma \subset G$ be a discrete subgroup of G. Then the counting measure on Γ is a left and right invariant Haar measure on Γ and hence Γ is unimodular and the quotient space G/Γ will have a left invariant Haar measure. If G is also unimodular then G/Γ has a left invariant Haar measure (see [42]).

2.1.2 *Lattices*

Let G be a Lie group. A subgroup $\Gamma \subset G$ is called a *lattice* if Γ is discrete and G/Γ has a left invariant measure ν such that $\nu(G/\Gamma) < \infty$. If G/Γ is compact we call Γ *uniform (or cocompact)* and *non-uniform (or non-cocompact)* otherwise. If G admits a lattice then G is unimodular but the converse is not true in general. For example there are simply connected nilpotent Lie groups which do not admit lattices. We refer to [42] for an example of such a Lie group. The upper triangular matrices in $\mathrm{SL}(2, \mathbb{R})$ is not unimodular and hence does not contain any lattice. Let Γ, Γ' be two closed subgroups of G such that $\Gamma' \subset \Gamma$. Note that the coset space G/Γ' admits a finite left invariant measure if and only if both the spaces G/Γ and Γ/Γ' admit a finite left invariant measure. Two subgroups Γ and Γ' of G are said to be *commensurable* if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. Being commensurable is an equivalence relation on the set of all subgroups of G. If one is discrete (resp. lattice) then so is the other if they are commensurable. In particular, any finite index subgroup of a lattice in G is a lattice in G.

2.1.3 *Semisimple Lie Groups*

Let G be a real Lie group with finitely many connected components and let $\mathfrak{g}$ be its Lie algebra. A Lie algebra is said to be *simple* if it is non abelian and has no proper non trivial ideals. A *semisimple* Lie algebra is a direct sum of simple ideals $\mathfrak{g}_i$, that is $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$, equivalently, $\mathfrak{g}$ has no nontrivial abelian ideals. We say that a Lie group is *semisimple* (resp. *simple*) if its Lie algebra is semisimple (resp. simple). A real semisimple Lie group G is said to have *no compact factors* if it has no compact normal subgroup which is not discrete. Note that when G is connected, a discrete normal subgroup N of G is contained in the center of G and it is finite if N is compact.

Let G be a non-compact semisimple Lie group. Then it admits both uniform and non-uniform lattices (see [42] and the references there in). Let G be a Lie group as above with finite center and finitely many connected components. Let Γ be a lattice in G. We say that Γ is *irreducible* if for any non-compact closed proper normal subgroup $H \subset G$, the image of Γ under the quotient map $G \to G/H$ is dense. If G has no compact factors, Γ is irreducible if and only if for any two closed normal subgroups H_1, H_2 of G such that $G = H_1.H_2$ and lattices $\Gamma_i \subset H_i$, the group $\Gamma_1.\Gamma_2$ is not commensurable with Γ. In particular, any lattice in G is irreducible if G is simple. We shall recall in the section §2.2 the definition of an arithmetic lattice in semisimple Lie group and some deep results concerning them which we will use in the next chapter.

Our definition of irreducible lattice differs from the one given in [51]. The two definitions agree when G has no non-trivial compact factors.

If $N \subset G$ is a compact normal subgroup of a connected Lie group G with finite centre and Γ a discrete subgroup of G, then Γ is a lattice in G if and only if the image of Γ under the quotient map $G \longrightarrow G/N$ is a lattice.

Let G be a connected Lie group and its Lie algebra $Lie(G) = \mathfrak{g}$. For $g \in G$, $i_g : G \to G$ is the inner automorphism $x \mapsto gxg^{-1}$. Its differential $Ad_g : \mathfrak{g} \to \mathfrak{g}$ gives a representation $Ad : G \to GL(\mathfrak{g})$ called the *adjoint representation* of G. The differential of Ad is $ad : \mathfrak{g} \to End(\mathfrak{g})$, called the *adjoint representation* of $\mathfrak{g}$, where $End(\mathfrak{g})$ denotes the group of all Lie algebra endomorphisms of $\mathfrak{g}$. Note that, when G is connected, $Ker(Ad) = Z(G)$ (resp. $Ker(ad) = Z(\mathfrak{g})$), $Lie(Z(G)) = Z(\mathfrak{g})$ and $Lie([G,G]) = [\mathfrak{g}, \mathfrak{g}]$ where $[G,G]$ is the commutator subgroup of G and $Z(G)$ is the center of the group G.

Let G be a connected semisimple Lie group. Denote by $\widetilde{G}$ the universal cover of G. If the center $Z(G)$ of G is trivial, then $Aut(G) \cong Aut(\mathfrak{g})$, where $Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$ is the group of all Lie algebra automorphisms of $\mathfrak{g}$. This is because, since $Z(G) = \{1\}$ and hence $\pi_1(G) = Z(\widetilde{G})$ we have $G \cong \widetilde{G}/Z(\widetilde{G})$ and their Lie algebras are isomorphic as $Z(\widetilde{G})$ is a discrete normal Lie subgroup. Also $Aut(G) \cong Aut(\widetilde{G})$. Since $\widetilde{G}$ is simply connected we get $Aut(\widetilde{G}) \cong Aut(Lie(\widetilde{G})) = Aut(\mathfrak{g})$. Hence $Aut(G) \cong Aut(\mathfrak{g})$. Since G is semisimple $Aut(\mathfrak{g})/Ad(G)$ is finite, where $Ad : G \to Aut(\mathfrak{g})$ is the adjoint representation of G. Note that if G is semisimple, then $Z(G)$ is discrete and the quotient group $G/Z(G)$ is a semisimple Lie group with trivial center.

Let G be a connected semisimple Lie group and $K \subset G$ a maximal compact subgroup. Note that every connected Lie group (indeed every connected locally compact group) admits a maximal compact subgroup and they are all conjugate to one another. Since the center $Z(G)$ is contained in every maximal compact subgroup of G, K contains $Z(G)$. For a semisimple Lie group G, *having no compact factor* is equivalent to having no maximal compact normal subgroups in G. Therefore if M is maximal compact normal in G, then G/M has no compact factors.

Definition 2.1.1. The *real rank* of G, denoted $\mathbb{R}$-rank, is the largest integer m such that the Euclidean space $\mathbb{R}^m$ can be imbedded as a totally geodesic submanifold of the symmetric space G/K. Equivalently, the real rank of G is the dimension of the largest abelian subalgebra contained in $\mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition. Here $\mathfrak{g} = Lie(G), \mathfrak{k} = Lie(K)$.

We say a subgroup $H \subset G$ of a group G has *finite index* in G if the quotient set G/H is finite. Let $\mathcal{P}$ be a property of groups. A group G is called *virtually* $\mathcal{P}$ if G has a subgroup of finite index with property $\mathcal{P}$.

The following well-known result will be needed in the proof of our main theorem.

Theorem 2.1.2. (Margulis' normal subgroup theorem [51]) *Let $\Gamma \subset G$ be an irreducible lattice where G is a connected semisimple Lie group of rank at least 2 and with finite centre. If N is normal in Γ, then either N is of finite index in Γ or is a finite subgroup contained in the centre of G.* □

Although our definition of an irreducible lattice differs from the one given in Zimmer [51], the above result is valid as stated. In fact one reduces to the case where G has no compact factors.

Remark 2.1.3. The linear group $\mathrm{SL}(n, \mathbb{Z})$ is a non-uniform irreducible lattice in the simple Lie group $\mathrm{SL}(n, \mathbb{R})$ for $n > 1$. The $\mathbb{R}$-rank of $\mathrm{SL}(n, \mathbb{R})$ is $n-1$. Thus the normal subgroup theorem holds for $n \geq 3$. For $n = 2$, let $\Gamma = F_2$, the free group of two generators. Note that Γ is contained in $\mathrm{SL}(2, \mathbb{Z})$ as a finite index subgroup and hence it is an irreducible lattice in $\mathrm{SL}(2, \mathbb{R})$. But the commutator $[\Gamma, \Gamma]$ of Γ is neither of finite index in $\mathrm{SL}(2, \mathbb{R})$ nor is contained in the center.

Margulis proved the normal subgroup theorem for a more general class of lattices such as lattices in Lie groups over local fields. For more details we refer to [[32], Chapter IV] and [[51], Chapter 8].

Next we state the strong rigidity for irreducible lattices.

Theorem 2.1.4. *(Strong rigidity [51]) Let G and G' be connected linear semisimple Lie groups with trivial centre and having no compact factors. Let $\Gamma \subset G$ and $\Gamma' \subset G'$ be irreducible lattices. Assume that G and G' are not locally isomorphic to $SL(2, \mathbb{R})$. Then any isomorphism $\phi : \Gamma \longrightarrow \Gamma'$ extends to an isomorphism $G \longrightarrow G'$ of Lie groups.* □

The strong rigidity theorem for cocompact lattices was obtained by Mostow [33]. Margulis showed that the result holds for G as above with real rank ≥ 2. The rank 1 case (when the lattice is non-cocompact) is due to Prasad [40], who extended the classical work of Mostow concerning rigidity of rank 1 compact locally symmetric manifolds. The proofs of the rigidity theorem for the case rank ≥ 2 and the Margulis' normal subgroup theorem can be found in [51].

2.2 ALGEBRAIC GROUPS

Throughout this section, K will be an algebraically closed field and k will be a subfield of K. All varieties considered will be affine varieties over K. We assume the reader is familiar with basic definitions. For details we refer the reader to [27], [32] and [2].

Let K be an algebraically closed field and let $R = K[x_1, x_2, \cdots, x_n]$ be the ring of polynomials in n variables with coefficients in K and let $\mathbb{A}^n := K^n$. Each $f \in R$ gives a map $f : K^n \to K$, namely the evaluation map defined by $x \mapsto f(x)$ for each $x \in K^n$. For an ideal $I \subset R$, the *zero set* of I is $V(I) := \{x \in K^n \mid f(x) = 0, \forall f \in I\}$. For example $V(\{0\}) = K^n$. An *affine algebraic variety* (or simply *affine variety*) is the set of common zeros in $\mathbb{A}^n$ of a finite collection of polynomials in R. By Hilbert basis theorem [27], every ideal I of R is finitely generated and hence $V(I)$ is an affine variety.

2.2.1 *Zariski Topology*

The *Zariski topology* on $\mathbb{A}^n$ is the topology in which the closed subsets are the affine varieties. Let $X \subset \mathbb{A}^n$ be an affine variety. A subset of X is said to be *closed* in X (resp. *open, dense*) if it is closed (resp. open, dense) in its Zariski topology. The *Zariski closure* of a subset of X is its closure in the Zariski topology of X. We say that X is *irreducible*, if it is not empty and cannot be represented as a union of two proper algebraic subvarieties. Note that every algebraic variety is a union of finitely many maximal algebraic irreducible subvarieties. These maximal irreducible subvarieties of X are called *irreducible components* of X. We say X is *Noetherian* if its Zariski closed subsets satisfy the descending chain condition in its Zariski topology. Note that any affine algebraic variety is Noetherian in the Zariski topology.

The *ideal* of a variety X is the ideal

$$I(X) := \{f \in R \mid f(x) = 0 \text{ for all } x \in X\} \subset R.$$

The *coordinate ring* of X is the ring $K[X] := R/I(X)$. This is the *algebra of regular functions* on X. It is clear that $X \subset V(I(X))$ (resp. $I \subset I(V(I))$), with equality if $X \subset \mathbb{A}^n$ is Zariski closed (resp. I is a radcal ideal).

2.2.2 *k-stuctures and k-morphisms*

Let $k \subset K$ be a subfield of K of char $k = 0$ and let X be an algebraic subvariety of the affine space K^n. The subvariety X is said to be *defined over k* if X is the set of common zeros of a finite system of polynomials with coefficients in k, in other words, if we denote $I_k(X) = I(X) \cap k[x_1, \cdots, x_n]$, X is defined over k if and only if $I(X) = K \otimes_k I_k(X)$. A *k-structure* on a variety X is an isomorphism $\alpha : X \to W$ onto an algebraic subvariety W of an affine space which is defined over k. A variety endowed with a k-structure is called a *k-variety*. A subvariety defined over k is called a *k-subvariety*. Denote $k[W] = k[x_1, \cdots, x_n]/I_k(W)$. A morphism $\phi : W \to V$ between two k-varieties is a *k-morphism* if the map $\Phi : K[V] \to K[W]$ defined by $\Phi(f) = f \circ \phi$ maps $k[V]$ into $k[W]$. If a morphism is a k-morphism and is an isomorphism, then its inverse is also a k-morphism. Let X be a k-variety of K^n. We denote by $W_k = W \cap k^n$. The set W_k is the *k-points* of X.

2.2.3 *k-Groups*

Let G be an affine algebraic variety over K which is also a group. Suppose that both the maps $m : (x, y) \mapsto xy$ and $i : x \mapsto x^{-1}$, for all $x, y \in G$, are morphisms of varieties. Then we say that G is an *algebraic group*. If G is defined over k and both the maps m and i are k-morphisms, then G is said to be *defined over k* (or *algebraic k-group* or simply *k-group*). We say an algebraic group is *connected* if it is connected in its Zariski topology. If G is connected, then it is irreducible.

Let G be an algebraic k-group. We denote by G^0 the connected (irreducible) component of the identity in the group G. Since the connected components are Zariski closed in G, we get G^0 is an algebraic group and defined over k. Since a variety is Noetherian, any variety will have only finitely many connected irreducible components. Hence G^0 has finite index in G, that is G/G^0 is finite.

An *algebraic group morphism* is a group homomorphism which at the same time is a variety morphism. A map $\phi : G \to G'$ is a *k-group k-morphism* if both G and G' are k-groups and ϕ is a k-morphism. If $\phi : G \to G'$ is a k-group morphism then $\phi(G)$ is a k-subgroup in G' and $\phi(G^0) = \phi(G)^0$. The irreducible component G^0 is a Zariski closed normal subgroup of G.

Example 2.2.1. Consider the linear group $G = \mathrm{GL}(n, K)$ of all $n \times n$ matrices over K. Its topology is induced from the Zariski topology of K^{n^2+1} with coordinates $(x_{ij}, z), 1 \leq i, j \leq n$. Then G is the zero set of the polynomial $det(x_{ij})z - 1$. Note that both maps multiplication m and inversion i of G are defined by polynomials and hence are morphisms. Thus G is an affine algebraic group. The special linear group $\mathrm{SL}(n, K)$, the orthogonal group $\mathrm{O}(n)$ of a symmetric non-degenerate bilinear form on K^n, the special orthogonal group $\mathrm{S}O(n)$ and the symplectic group $\mathrm{Sp}(n)$ are all affine algebraic groups.

2.2.4 *Linear Algebraic Groups*

Note that a closed (in Zariski topology) subgroup of $GL(n, K)$ is again an algebraic group. A *linear algebraic group* is a subgroup $G \subset \mathrm{GL}(n, K)$ which is algebraic in $\mathrm{G}L(n, K)$. For example, all groups in Example 2.2.1 are linear algebraic. Note that any affine k-group is k-isomorphic to a k-subgroup of $\mathrm{G}L(n, K)$ for some n sufficiently large. Thus now onwards, we identify G_k with its image in $\mathrm{GL}(n, k)$ for some n.

Lemma 2.2.2. *([2], Chapter I) If G is an algebraic k-group and H is a (abstract) subgroup of G_k, then the Zariski closure $\bar{H}$ of H is a k-subgroup of G.* □

An affine algebraic group is *connected* if it is irreducible as an affine variety. The connected (or irreducible) components are the cosets gG^0, where $g \in G$. Hence the quotient group is finite. For rest of this section, we restrict our attention to the case $K = \mathbb{C}$. Let k be a subfield of $\mathbb{C}$. Let G be an algebraic group, that is $G \subset GL(n, \mathbb{C})$. If G is defined over $\mathbb{R}$, then the $\mathbb{R}$-points $G_{\mathbb{R}}$ is a closed subgroup of $GL(n, \mathbb{R})$ in the usual Euclidean topology of $\mathbb{R}^{n^2}$, called the Hausdorff topology. Hence $G_{\mathbb{R}}$ gets a Lie group structure. The group $G_{\mathbb{R}}$ may not be connected even if G is connected, but it has only finitely many connected components. It is easy to see that the connected components of G in the Hausdorff topology and the irreducible components in its Zariski topology are same. Hence we simply use the expression connected components for both.

Recall that any affine algebraic group is a closed subgroup of $\mathrm{GL}_n(K)$ for some n. For a $g \in \mathrm{GL}_n(K)$, the Jordan decomposition is $g = g_s g_u$, where g_s is semisimple (i.e, g_s is diagonalizable over K) and g_u is unipotent. If $g \in G$ then $g_s, g_u \in G$, $g = g_s g_u$ is called the *Jordan decomposition* of G. An element $g \in G$ is called *unipotent* if $g = g_u$. For an algebraic homomorphism $\phi : G \to G'$ of algebraic groups, $\phi(g)_s = \phi(g_s)$ and $\phi(g)_u = \phi(g_u)$.

A *Borel subgroup* of G is a maximal connected solvable subgroup of G. For example, if $G = \mathrm{GL}_n(K)$, the subgroup of all upper triangular matrices is a Borel subgroup. Any two Borel subgroups are conjugate. A subgroup is called *parabolic* if it contains a Borel subgroup. Every element of G lies in a Borel subgroup. If $B \subset G$ is a Borel subgroup, then the center $Z(B) = Z(G)$.

A *torus* $T \subset G$ is a diagonalizable commutative algebraic group, that is, $T \cong (K^*)^n$ for some n. A torus T is called *maximal* if it is of maximum possible dimension. For example, the subgroup of all diagonal matrices in GL_n is a maximal torus. If G is defined over k, a *k-torus* of G is an algebraic subgroup of G isomorphic to $(k^*)^r$ for some r. The maximal such r is called the *k-rank* of G. If k-rank of G is equal to the K-rank of G, we say that G has a k-split torus. If G has a k-split torus, we say that G is *k-split*.

Let G be a connected semisimple real Lie group. Then we can identify its Lie algebra $\mathfrak{g}$ with $\mathbb{R}^n$ and its complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{C}^n$ via linear isomorphisms. Let $Aut(\mathfrak{g})$ denote the group of all Lie algebra automorphisms of $\mathfrak{g}$ and let $\mathrm{GL}(\mathfrak{g}) \cong \mathrm{GL}(n, \mathbb{R})$ be the linear isomorphisms of $\mathfrak{g}$. Let $H = Aut(\mathfrak{g}^{\mathbb{C}}) \subset GL(\mathfrak{g})$, then $Aut(\mathfrak{g})$ is the $\mathbb{R}$-points $H_{\mathbb{R}}$ of H.

We need the following theorem. We refer the reader to [4] for the notations used here. Also we refer to [25], Ch. XV, §3.

Theorem 2.2.3. *(Theorem 1.2, Ch VII, [4], [25]) Let G be a Lie group with finitely many connected components. Then*
1) Every compact subgroup of G is contained in a maximal compact subgroup. Any two maximal compact subgroups are conjugate under an inner automorphism $Int(x), (x \in G^0)$
2) If K is a maximal compact subgroup, there exists a multiexponential set E, invariant under K, such that $G = K.E$.
3) If K is a maximal compact subgroup, there exist closed, simply connected one-parameter subgroups $H_1, H_2, \cdots, H_s$, *where* $s = dimG - dimK$, *of G such that* $G = K.H_1 \cdots H_s$ □

A theorem of Borel and Harish-Chandra asserts that if G is a connected semisimple algebraic group defined over $\mathbb{Q}$ then $G_{\mathbb{Z}}$ is a lattice in $G_{\mathbb{R}}$. We say that a lattice $\Gamma \subset G$ is *arithmetic* if G is defined over $\mathbb{Q}$ and if Γ is commensurable with $G_{\mathbb{Z}}$. All lattices in nilpotent Lie groups (if they exist) are arithmetic.

Theorem 2.2.4. *[51] (Margulis' arithmeticity theorem) Let G be a connected semisimple Lie group with trivial center and no compact factors and $\mathbb{R}$-rank ≥ 2. Let $\Gamma \subset G$ be an irreducible lattice. Then Γ is arithmetic.*

□

We conclude this section with the following result.

Theorem 2.2.5. (Borel density theorem *[51]) Let* $\mathbf{G}$ *be a connected semisimple algebraic $\mathbb{R}$-group and assume $G := \mathbf{G}^0_{\mathbb{R}}$ has no compact factors. Let $\Gamma \subset G$ be any lattice in G. Then Γ is Zariski dense in* $\mathbf{G}$. □

2.3 GEOMETRIC GROUP THEORY

In geometric group theory, groups are studied via their actions on metric spaces via isometries. Let Γ be a group with a finite generating set $\mathcal{A}$. We always assume that $1 \notin \mathcal{A}$. Let $\gamma \in \Gamma$, write $\gamma = a_1 a_2 \cdots a_n$ with $a_i \in \mathcal{A} \cup \mathcal{A}^{-1}$. We call *length* of γ, denoted $l_{\mathcal{A}}(\gamma)$, length of the shortest such words representing γ. Equip Γ with the *word metric* $d_{\mathcal{A}}$ with respect to $\mathcal{A}$ defined by $d_{\mathcal{A}}(\gamma, \gamma') = l_{\mathcal{A}}(\gamma^{-1}\gamma')$ for any two elements $\gamma, \gamma' \in \Gamma$. This metric is invariant with respect to the action of Γ by left multiplication on itself. The *Cayley graph* $C_{\mathcal{A}}(\Gamma)$ is the graph whose set of vertices is Γ and there is an edge joining γ and γ' if and only if $\gamma' = \gamma a$ for some $a \in \mathcal{A} \cup \mathcal{A}^{-1}$. Note that the Cayley graph is connected, locally finite (that is the degree of every vertex is finite) since $\mathcal{A}$ is finite and Γ acts on it on the left by isomorphisms of graphs. By assigning each edge a metric of length 1, we can define a metric on $C_{\mathcal{A}}(\Gamma)$ which assigns to any pair of vertices (γ, γ') the length of a shortest path from γ to γ'. An expression $\gamma = a_1 a_2 \cdots a_n$ with $a_i \in \mathcal{A} \cup \mathcal{A}^{-1}$ is a *path* in $C_{\mathcal{A}}(\Gamma)$ from 1 to γ and this path is a *geodesic* if $l_{\mathcal{A}}(\gamma) = n$. Since $\Gamma \hookrightarrow C_{\mathcal{A}}(\Gamma)$, Γ gets a metric induced from the metric on $C_{\mathcal{A}}(\Gamma)$. This metric is exactly the word metric $d_{\mathcal{A}}$ on Γ. Word metrics associated to different finite generating sets $\mathcal{A}$ and $\mathcal{B}$ are *bi-Lipschitz* equivalent, that is, there exists a constant $\mu > 0$ such that $\frac{1}{\mu}d_{\mathcal{B}}(\gamma,\gamma') \leq d_{\mathcal{A}}(\gamma,\gamma') \leq \mu d_{\mathcal{B}}(\gamma,\gamma')$ for all $\gamma,\gamma' \in \Gamma$. We can see this by taking $\mu = max\{\mu_1,\mu_2\}$ where $\mu_1 = max\{l_{\mathcal{A}}(b) \mid b \in \mathcal{B}\}, \mu_2 = max\{l_{\mathcal{B}}(a) \mid a \in \mathcal{A}\}$. For details we refer to [7].

Throughout this section we regard Γ as a metric space with the word metric with respect to some finite generating set.

2.3.1 *Quasi-isometry*

Definition 2.3.1. Let (X_1,d_1) and (X_2,d_2) be metric spaces. A map $f : X_1 \to X_2$ is called a (λ,ϵ)-*quasi-isometric embedding* if there exist real numbers $\lambda \geq 1$ and $\epsilon \geq 0$ such that for all $x,y \in X_1$,

$$\frac{1}{\lambda}d_1(x,y) - \epsilon \leq d_2(f(x),f(y)) \leq \lambda d_1(x,y) + \epsilon.$$

If, in addition, there exists a constant $C \geq 0$ such that every point of X_2 lies in the C-neighbourhood of the image of f, then f is called a (λ,ϵ,C)-*quasi-isometry*.
The metric spaces X_1 and X_2 are said to be *quasi-isometric* if there exists a quasi-isometry $f : X_1 \to X_2$. In this case there is a *quasi-inverse* $g : X_2 \to X_1$, which is a (λ',ϵ',C')-quasi-isometry, for some $\lambda' \geq 1, \epsilon' \geq 0, C' \geq 0$, such that $d(f \circ g(x),x) \leq k$ and $d(g \circ f(y),y) \leq k$ for all $x \in X_2, y \in X_1$, for some $k \geq 0$.

For example, consider the group $(\mathbb{Z},d_{\mathcal{A}})$ with the word metric $d_{\mathcal{A}}$ with respect to the generating set $\mathcal{A} = \{1\}$ and $\mathbb{R}$ with the usual metric. Then the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a $(1,0,1/2)$-quasi-isometry. The map $x \mapsto [x]$ from $\mathbb{R} \to \mathbb{Z}$ is a quasi-inverse of the inclusion map. Note that $\mathbb{R}$ is the Cayley graph of $\mathbb{Z}$ with respect to $\mathcal{A}$. More generally, $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$.

We continue to assume that the group Γ is finitely generated. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two different finite generating sets of Γ, then the identity map is a quasi-isometry between $(\Gamma,d_{\mathcal{A}})$ and $(\Gamma,d_{\mathcal{B}})$ since these two word metrics are bi-Lipschitz equivalent and hence quasi-isometric to each other. It is clear that the inclusion $(\Gamma, d_{\mathcal{A}}) \hookrightarrow C_{\mathcal{A}}(\Gamma)$ is a $(1,0,1/2)$ quasi-isometry. We can also think of this inclusion as $\gamma \mapsto \gamma.1$, the natural action $\gamma.x = \gamma x, x \in \Gamma$, of Γ on $C_{\mathcal{A}}(\Gamma)$.

A metric space (X,d) is said to be a *length space* if the distance between any two points is the infimum of the length of rectifiable curves joining them. A *geodesic* joining two points x, y in X is the image of a continuous path $c : [0,l] \to X$ such that $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular $d(x,y) = l$. The metric space X is said to be a *geodesic space* if any two points in X are joined by a geodesic. For example, a complete Riemannian manifold is a geodesic metric space with respect to the metric which is defined as the infimum of the length of rectifiable curves between two points. If there is a group that acts properly and cocompactly by isometries on a length space, then X is complete and locally compact. So by the Hopf-Rinow theorem it is a proper (ie, all closed balls are compact) geodesic space. See [7]. Before going to examples, we recall the following result.

Proposition 2.3.2. *([7], Ch. I.8) (Švarc-Milnor Lemma): Let X be a length space. If Γ acts properly and cocompactly by isometries on X, then Γ is finitely generated and for any choice of base point $x_0 \in X$, the map $\gamma \mapsto \gamma.x_0$ from $\Gamma \to X$ is a quasi-isometry.*

Examples 2.3.3.

(i) Let

$$1 \longrightarrow \Gamma' \xrightarrow{i} \Gamma \xrightarrow{p} \Gamma'' \longrightarrow 1 \tag{1}$$

be a short exact sequence of groups, where Γ is finitely generated. Then i is a quasi-isometry if Γ'' is finite and p is a quasi-isometry if Γ' is finite. In general, a homomorphism $p : \Gamma \to \Gamma'$ between finitely generated groups is a quasi-isometry if and only if $ker(p)$ and $\Gamma'/\mathrm{Image(p)}$ are both finite.

(ii) Two groups Γ_1, Γ_2 are said to be *commensurable* if there is a group Γ which embeds in each $\Gamma_i, i = 1,2$ such that the index $[\Gamma_i : \Gamma] < \infty$. From the above example, it follows that any two commensurable groups are quasi-isometric.

(iii) Let M be a closed connected Riemannian manifold. Then M inherits the structure of a geodesic metric space from the Riemannian metric by defining the distance between two points to be the infimum of the length of rectifiable arcs joining them. Let $\widetilde{M}$ be the universal cover of M and let $\pi : \widetilde{M} \to M$ be the universal covering projection. Let Γ be the fundamental group of M. The group Γ acts on $\widetilde{M}$ properly discontinuously by deck transformations. Then $\widetilde{M}$ gets a unique Γ-invariant Riemannian metric for which the projection $\pi : \widetilde{M} \to M$ is a local isometry. Fix an element $x_0 \in \widetilde{M}$, and consider the map $\phi : \Gamma \to \widetilde{M}$ given by $\gamma \mapsto \gamma(x_0)$. Give Γ, which is finitely generated, a word metric with respect to some finite generating set. Then ϕ is a quasi-isometry between Γ and $\widetilde{M}$ by the Švarc-Milnor lemma 2.3.2. For example, the fundamental group $\pi_1(\Sigma_g) \subset PSL_2(\mathbb{R})$ of the closed surface Σ_g of genus $g \geq 2$ is quasi-isometric to the universal cover $\widetilde{\Sigma}_g = \mathbb{H}^2$, the Poincaré upper half space.

We shall recall some properties of groups which are invariant under quasi-isometries, i.e, if the finitely generated group Γ has the property, then every finitely generated group quasi-isometric to Γ also has this property: being finitely presented; being virtually infinite cyclic, virtually abelian, virtually nilpotent, virtually free; and being hyperbolic.

2.3.2 *Hyperbolic Space*

The notion of hyperbolicity of groups was introduced by Mikhail Gromov [23]. We use M. Gromov's formulation of hyperbolicity for geodesic metric spaces. There are a number of ways of formulating the hyperbolicity condition in a metric space. All these formulations are equivalent for geodesic metric spaces. For details we refer to [7].

Let (X, d) be a metric space. A *geodesic triangle* in X is a configuration of three points in X, its *vertices*, together with of three geodesic segments joining them pairwise, called its *sides*. For $\delta \geq 0$, a geodesic triangle is said to be δ-*slim* if each of its sides is contained in the δ-

neighbourhood of the union of the other two sides. A *quasi-geodesic* in X is a quasi-isometric embedding of an interval $I \subset \mathbb{R}$ to X.

Definition 2.3.4. Let X be a geodesic metric space and $\delta \geq 0$. We say X is *δ-hyperbolic* if every geodesic triangle in X is *δ-slim*. The exact value of δ is not often relevant for most purposes.

Hyperbolicily is preserved under quasi-isometry between geodesic spaces. See [7].

Definition 2.3.5. A finitely generated group is said to be *hyperbolic* if its Cayley graph with respect to some and hence any finite generating set is hyperbolic.

2.3.3 *Properties of Hyperbolic Groups*

We recall some of the important properties of hyperbolic groups and of its finitely generated subgroups which will be using in the thesis. For more details see [Chapter III.Γ, [7]].

Hyperbolicity is a quasi isometry invariant. A hyperbolic group is finitely presented. It is clear that being hyperbolic is not closed under subgroups or taking quotients. For example, the free group F_2 on 2 generators has its commutator a non-finitely generated subgroup and $\mathbb{Z}^2$ as its abelianization, which are not hyperbolic. If two groups Γ_1, Γ_2 are hyperbolic then their free product $\Gamma_1 * \Gamma_2$ is also hyperbolic. If Γ is infinite hyperbolic then it has an element of infinite order. Hence if Γ is a torsion group then it has to be finite. Also for any element $\gamma \in \Gamma$ of infinite order, the cyclic subgroup $\langle\gamma\rangle$ generated by γ has finite index in its centralizer $C(\gamma)$. Hence an infinite hyperbolic group connot contain $\mathbb{Z}^2$ as a subgroup. The map $n \mapsto \gamma^n$ from $\mathbb{Z} \to \Gamma$ is a quasi-geodesic, that is, $\gamma \mapsto \gamma^n$ is a quasi-isometric embedding, for any element γ of infinite order and this implies that Γ cannot have $BS(m,n) = \langle a, t \mid ta^n t^{-1} = a^m\rangle$ unless $|n| = |m|$ as an isometrically embedded subgroup. The groups $BS(m,n)$ provide examples of HNN-extensions of hyperbolic groups that are not hyperbolic.

Since the word metric with respect to different finite generating sets are quasi-isometry equivalent in groups and hyperbolicity is preserved under quasi-isometry this definition does not depend on the choice of generators for the group but the constant δ may depend on the generating set. Some of the examples of hyperbolic and non-hyperbolic spaces given below will be used in the thesis.

Examples 2.3.6. (1) Any metric space of finite diameter is hyperbolic. Any tree is 0-hyperbolic. Any free group of finite rank is 0-hyperbolic since its Cayley graph with respect to a basis is a tree. It follows from the Švarc-Milnor lemma 2.3.2 and the Hopf-Rinow theorem that any length space on which there is an action of a hyperbolic group which acts properly and cocompactly via isometries is hyperbolic. The Euclidean plane $\mathbb{R}^2$ cannot be hyperbolic. We can see this by considering isosceles triangles of arbitrary size. This implies the group $\mathbb{Z}^2$ which is quasi-isometric to $\mathbb{R}^2$ is also not hyperbolic. More generally, any group which contains $\mathbb{Z}^2$ as a subgroup cannot be hyperbolic.
(2) Consider the hyperbolic plane

$$\mathbb{H}^2 = \left\{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_0^2 = -1, x_0 > 0\right\}.$$

Note that any geodesic triangle here has its volume bounded by π and hence we can choose a $\delta > 0$ such that all the geodesic triangles are δ-slim and hence $\mathbb{H}^2$ is hyperbolic. In general, since any three (non-collinear) points of $\mathbb{H}^n$ lie in a hyperbolic plane it follows that $\mathbb{H}^n$ is hyperbolic for any $n > 2$. More generally, any complete simply connected Riemannian manifold having sectional curvature bounded above by a negative constant is a hyperbolic space. Since a connected homogeneous Riemannian manifold M of dimension n and constant curvature $\kappa < 0$ is isometric to the hyperbolic space $\mathbb{H}^n$, M is hyperbolic in the sense of Gromov. If M is compact the their fundamental group is also hyperbolic.
(3) Let Γ be a lattice in the Lie group $G = Isom(\mathbb{H}^2)$. If Γ is cocompact then it is hyperbolic by Švarc-Milnor lemma 2.3.2. Suppose Γ is non-cocompact. By the result of Selberg 2.4.6, any finitely generated linear group is virtually torsion free. Choose a finite index torsion free

subgroup $\Gamma' \subset \Gamma$. Then Γ' will act freely on $\mathbb{H}^2$ and $\mathbb{H}^2/\Gamma'$ will be homeomorphic to a manifold obtained by removing finitely many points from a compact orientable surface and hence is homotopic to a wedge of finitely many circles. This implies Γ' is a free group of finite rank. Since Γ' is quasi-isometric to Γ and free groups are hyperbolic we get Γ is hyperbolic. For $n > 2$, again if $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is a cocompact lattice then it is hyperbolic as above. On the other hand, if Γ is non-cocompact then it may not be hyperbolic in general. For example, the Bianchi group $\Gamma := PSL_2(\mathbb{Z}[\sqrt{-d}])$, where d is a positive square free integer, is a lattice in the real Lie group $PSL_2(\mathbb{C}) \cong SO^+(3,1) = Isom(\mathbb{H}^3)$. Since Γ has unipotent elements, by the Godement compactness criterion ([42], Ch. X), it cannot be cocompact. The subgroup generated by $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$, where $\omega = \sqrt{-d}$ if $d \equiv 2, 3 \bmod 4$ or $\omega = \frac{1+\sqrt{-d}}{2}$ when $d \equiv 1 \bmod 4$, is a free abelian group of rank 2 and hence Γ is not hyperbolic. Note that $G = Isom(\mathbb{H}^n)$ is a $\mathbb{R}$-rank 1 semisimple Lie group.
(4) A cocompact lattice Γ in a semisimple Lie group G of real rank ≥ 2 cannot be hyperbolic, since the symmetric space G/K has a copy of $\mathbb{R}^2$ isometrically embedded as a geodesic submanifold. The fundamental group $\pi_1(S^3 \backslash K)$ of nontrivial knot complement has its center either trivial or isomorphic to $\mathbb{Z}$. Hence if its center is non trivial, the knot group is not hyperbolic. Any group that contains a subgroup isomorphic to $BS(m,n)$ is not hyperbolic [7].

Let X be a geodesic space. A subspace C of X is said to be *convex* if for all $x, y \in C$ each geodesic joining x to y is contained in C. A subspace C of X is said to be *quasiconvex* if there exists a constant $k > 0$ such that for all $x, y \in C$ each geodesic joining x to y is contained in the k-neighbourhood of C. Let Γ be a group with finite generating set $\mathcal{A}$ and let $H \subset \Gamma$ be a subgroup. If H is quasiconvex in the Cayley graph $C_{\mathcal{A}}(\Gamma)$, then it is finitely generated and $H \hookrightarrow \Gamma$ is a quasi-isometric embedding (with respect to any choice of word metrics). The quasiconvex (equivalently, quasi-isometrically embedded) subgroups of hyperbolic groups are hyperbolic. If two subgroups $H_1, H_2 \subset \Gamma$ are quasiconvex in Γ, then so is

$H_1 \cap H_2$.
We conclude this section with the following result which will be used in the next chapter (see Cor. 3.10, III.Γ, [7]).

Lemma 2.3.7. *If Γ is a hyperbolic group, then*
1) The centralizer $C(\gamma)$ for every $\gamma \in \Gamma$ is a quasiconvex subgroup.
2) If a subgroup $H \subset \Gamma$ is infinite and quasiconvex, then it has finite index in its normalizer.

2.3.4 *Relatively Hyperbolic groups*

The notion of relatively hyperbolic groups was originally proposed by M. Gromov in his paper [23] in order to generalize the algebraic and geometric nature of groups such as hyperbolic groups, fundamental groups of non-compact complete Riemannnian manifolds of finite volume and having pinched negative sectional curvature, etc. There are two different approaches to the definition of the relative hyperbolicity of a group. One was by Bowditch, who elaborated Gromov's ideas in [5], and the other was due to Farb [12], who introduced an alternative approach in terms of the coset graphs with an additional condition, known as Bounded Coset Penetration property (or BCP). These two definitions were compared in [49], and proved there that if a group Γ is relatively hyperbolic with respect to a collection of subgroups in the sense of Bowditch then Γ is relatively hyperbolic with respect to the same set of subgroups in the sense of Farb, but not conversely. However, in ([9], [10]) it is shown that Bowditch's definition and Farb's definition with BCP are equivalent. Here we give Gromov's definition of a relatively hyperbolic group with respect to a finite collection of subgroups. (See [23]).

Let X be a complete, hyperbolic and locally compact geodesic space. One has a notion of boundary due to Gromov for a hyperbolic space. (See [7], or [23]). Denote the boundary of X by ∂X. Let Γ be a discrete group of isometries of X which acts on X freely such that the quotient $M = X/\Gamma$ is quasi-isometric to the union of k copies of $[0, \infty)$ joined at zero. Lift the k rays in X corresponding to the k points in ∂M to k rays $r_i : [0, \infty) \to$

$X, i = 1, \cdots, k$. Let h_i be the corresponding horofunctions and let $r_i(\infty) \in \partial X$ be the limit point of r_i. Denote by $H_i \subset \Gamma$ the isotropy subgroups of $r_i(\infty)$ for the action of Γ on ∂X and assume that H_i preserves h_i for all i. Denote by $B_i(\rho)$ the horoballs $h_i^{-1}(-\infty, \rho) \subset X$ and assume that for sufficiently small ρ the intersection $\gamma B_i(\rho) \cap B_j(\rho)$ is empty unless $i = j$ and $\gamma \in H_i$. Let $B(\rho) = \cup_{i=1}^{n} B_i(\rho)$, $\Gamma B(\rho)$ be its Γ translates and $X(\rho) = X \setminus \Gamma B(\rho)$. Assume the action of Γ on $X(\rho)$ is cocompact for all $\rho \in (-\infty, \infty)$.

Definition 2.3.8. A group Γ is said to be *relatively hyperbolic* with respect to the subgroups $H_1, .., H_k$ in Γ if Γ admits an action on some X with the above properties, where the isotropy subgroup of $r_i(\infty)$ is H_i, $1 \leq i \leq k$, in ∂X.

Examples 2.3.9. We list a few examples here which are relevant to our purpose:
(1) Let M be a complete non-compact Riemannian manifold of constant negative sectional curvature κ and having finite volume and let Γ be its fundamental group. Then Γ acts on the universal cover $\widetilde{M}$ of $M (= \widetilde{M}/\Gamma)$ freely. Denote by S the set of all $s \in \partial\widetilde{M}$ such that it is the unique fixed point of some $\gamma \in \Gamma$, called the cusps of Γ. Then Γ acts on S and since Γ has finite covolume, S will have only finitely many Γ-orbits. Let $\{H_1, H_2, \cdots, H_k\}$ be the stabilizers of distinct coset representatives. Then Γ is relatively hyperbolic with respect to these subgroups of Γ. More generally, let X be a complete simply connected Riemannian manifold with *pinched* negative sectional curvature $\kappa \in (-a^2, -b^2)$ for some $a, b < \infty$ and Γ be a discrete group of isometries of X with finite covolume. Then X/Γ is quasi-isometric to the wedge of several copies of $[0, \infty)$ and Γ is relatively hyperbolic with respect to the isotropy subgroups of $r_i(\infty) \in \partial X$. For example, any non-uniform lattice in a semisimple Lie group with $\mathbb{R}$-*rank* 1 is relatively hyperbolic with respect to the isotropy subgroups. See [23].
(2) Any hyperbolic group is relatively hyperbolic with respect to the trivial group.

A group Γ is said to be *hopfian* if any surjective endomorphism of Γ is injective and is called *co-hopfian* if any injective endomorphism is surjective. We say Γ is *residually finite* if for any $\gamma \in \Gamma$ there is a finite index normal subgroup $H \subset \Gamma$ such that $\gamma \notin H$. Equivalently, for each $(1 \neq)\gamma \in \Gamma$ there is a finite group F and a surjective homomorphism $\phi_\gamma : \Gamma \to F$ such that $\phi_\gamma(\gamma) \neq 1$. Hopf showed that the fundamental group of any closed orientable surface is hopfian. The first example of a finitely generated non-hopfian group was obtained by B.H Neumann. Shortly after, G. Higman found a finitely presented non-hopfian group. See [31] and the references therein for more details. The groups $\mathrm{BS(m,n)} := \langle a, t \mid t^{-1}a^m t = a^n \rangle, n, m \in \mathbb{Z}$, were introduced by G. Baumslag and D. Solitar [1] to provide examples of non-hopfian groups. They are non-hopfian provided $gcd(m,n) = 1, and\ |m| \neq 1 \neq |n|$. The simplest example of finitely presented non-hopfian group is $BS(2,3)$. We give a proof of non-hopficity of $BS(2,3)$ after Theorem 2.4.4. The group $BS(2,4)$ is hopfian but not residually finite. The Baumslag-Solitar groups contain examples of residually finite groups, hopfian groups and hopfian but not residually finite groups for suitable values of m, n. Miller and Schupp [31] have shown that every finitely presented group can be embedded in a finitely presented hopfian group.

The group $\mathbb{Z}^n$ and the general linear group $GL(n,\mathbb{Z})$ are residually finite for all $n \geq 1$. This can be seen by taking the kernel of the homomorphism $GL(n,\mathbb{Z}) \to GL(n,\mathbb{Z}/m\mathbb{Z})$ induced by the natural projection $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, where $m \in \mathbb{N}$ is such that for the given $\gamma = (a_{ij}) \in GL(n,\mathbb{Z})$, $m > max\{|a_{ij}|\}$.

We need the following lemma which is an immediate consequence of the definition of residual finiteness.

Lemma 2.4.1. *Let M be a set of finitely many nontrivial elements of a residually finite group G. Then there exists a finite index characteristic subgroup $N \subset G$ disjoint from M. N can be chosen so that the quotient map $\pi : G \to G/N$ sends M bijectively onto its image $\pi(M)$.* □

Any group having a finite index residually finite group is itself residually finite. Similarly any finitely generated infinite group having a finite index hopfian subgroup is hopfian. G. Baumslag proved that if Γ is any finitely generated residually finite group then so is its automorphism group $Aut(\Gamma)$. This is not true if we drop the finitely generated hypothesis.

Theorem 2.4.2. *(Mal'cev) [31] Any finitely generated subgroup of $GL(n, \mathbb{F})$, where $\mathbb{F}$ is any field and $n \geq 2$, is residually finite.* $\square$

This theorem implies that the cocompact lattices in linear Lie groups, which are finitely generated, are residually finite. It is unknown whether the fundamental groups of all negatively curved complete Riemannian manifolds are residually finite. The hyperbolic groups are in general not linear and it is still unknown that whether they are residually finite or not. Z. Sela [46] (see Theorem 2.4.3) showed that all torsion free hyperbolic groups are hopfian. The first one-relator non-linear group which is finitely generated residually finite is $\langle a, t \mid t^{-2}at^2 = a^2 \rangle$ by Sapir and Drutu. By Perelman's work, the fundamental group of any compact 3-manifold is residually finite.

Theorem 2.4.3. *[46] All torsion free hyperbolic groups are Hofpian.* $\square$

Most of the finitely generated groups which are not residually finite are from the following result by Mal'cev:

Theorem 2.4.4. *[31] Every finitely generated residually finite group is hopfian.* $\square$

The converse of Theorem 2.4.4 is not true. The group $BS(2,4)$ is a counter example. The group $\Gamma := BS(2,3)$ is not hopfian. To see this, let $\theta : \Gamma \to \Gamma$ be the map defined on the generators by $\theta : t \mapsto t, a \mapsto a^2$. Since a^2 is in the image of θ, it is clear that θ is a surjective homomorphism. Consider the commutator $x = [tat^{-1}, a]$, $\theta(x) = 1$ while $x \neq 1$. This implies that $Ker(\theta) \neq \{1\}$. Hence Γ is not hopfian. Therefore it is not residually finite.
It is clear from the definition that subgroups of a residually finite group and direct product of a family of residually finite groups are residually

finite. Hence $\mathrm{SL}(2,\mathbb{Z})$ and its subgroups F_2 and hence all F_n, free group of rank n, are residually finite. However, the quotient of a residually finite group need not be residually finite, for example, $BS(2,3)$ is a quotient of F_2 which, we saw, is not residually finite.

Lemma 2.4.5. *(Brauer [6]) Let G be a group of order n. Then the number of conjugacy classes in G is bounded below by* $\log\log n$. □

Theorem 2.4.6. *(Selberg) A finitely generated linear group over a field of characteristic zero is virtually torsion free.* □

We conclude this section with the following remarks.

Remark 2.4.7. (1) If Γ is infinite and residually finite, there exist finite quotients $\bar{\Gamma}$ of Γ having *arbitrarily large* (finite) order. By Lemma 2.4.5 (see also [41]), the number of conjugacy classes of a finite group of order n is bounded below by $\log\log n$. Thus Γ will have as many conjugacy classes as any of its quotients and it follows that Γ has infinitely many conjugacy classes.
(2) It is known, as a consequence of Thurston's geometrization theorem after Perelman's work, all knot groups Γ are residually finite. They are, in fact, extension of cyclic groups by $F = [\Gamma,\Gamma]$, their commutator subgroups. Hence Γ is residually finite if F is finitely generated non-abelian. Therefore all knot groups have infinitely many conjugacy classes.

2.5 AUTOMORPHISMS OF LINEAR GROUPS

In this section we recall the automorphisms of the following groups: (i) $\mathrm{GL}(n,\mathbb{Z})$, the $n\times n$ integral matrices of determinant ± 1; (ii) $\mathrm{SL}(n,\mathbb{Z})$, the $n\times n$ integral matrices of determinant 1; and (iii) $\mathrm{Sp}(2n,\mathbb{Z})$, the group of all integral $2n\times 2n$ matrices X satisfying $XJ'\,{}^tX = J'$ where $J' = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, I_n is the $n\times n$ identity matrix and ${}^tX$ represents the transpose of X. Let G be one of these groups. We denote the commutator subgroup $[G,G]$ of the group G by G' and the center of G by $Z(G)$. A *character* of G is a homomorphism $\chi : G \to \{\pm 1\}$.

2.5.1 *Automorphisms of* $SL(n,\mathbb{Z})$ *and* $GL(n,\mathbb{Z})$

For $n > 2$, the commutator subgroup of $\mathrm{GL}(n,\mathbb{Z})$ is $\mathrm{SL}(n,\mathbb{Z})$ and that of $\mathrm{SL}(n,\mathbb{Z})$ is $\mathrm{SL}(n,\mathbb{Z})$ itself, that is, $\mathrm{SL}(n,\mathbb{Z})$ is perfect for $n > 2$. When $n = 2$, the commutator subgroup of $\mathrm{SL}(2,\mathbb{Z})$ is an index two subgroup. L. K. Hua and I. Reiner [26] and O. T. O'Meara [37] have described the automorphism groups of these two groups. We recall their results here.

Theorem 2.5.1. *[26] Every automorphism of* $SL(2,\mathbb{Z})$ *is of one of the following forms:*
Let $g \in SL(2,\mathbb{Z})$,
(i) $g \mapsto aga^{-1}$, *for some* $a \in GL(2,\mathbb{Z})$,
(ii) $g \mapsto \chi(g)aga^{-1}$ *for some* $a \in GL(2,\mathbb{Z})$, *where* $\chi : SL(2,\mathbb{Z}) \to \{\pm 1\}$ *is a character.* □

The homomorphism χ occurs only for $n = 2$. Since $\mathrm{SL}(n,\mathbb{Z})$ is perfect for $n > 2$, there are no homomorphisms : $\mathrm{SL}(n,\mathbb{Z}) \to \{\pm 1\}$.

Theorem 2.5.2. *[26] For* $n > 2$, *every automorphism of* $GL(n,\mathbb{Z})$ *is of one of the following form:*
(i) $g \mapsto aga^{-1}$ *for some* $a \in GL(n,\mathbb{Z})$
(ii) $g \mapsto {}^tg^{-1}$
(iii) for even n only, $g \mapsto det(g)g$ □

When $n = 2$, the automorphism group of $\mathrm{GL}(2,\mathbb{Z})$ is generated by the automorphisms in $(1), (2)$, of Theorem 2.5.2, the automorphism $g \mapsto \chi(g)g$, $g \in \mathrm{SL}(2,\mathbb{Z})$, and the automorphism $g \mapsto \chi(Jg)g$ when $g \in \mathrm{GL}(2,\mathbb{Z})$ with $det(g) = -1$, where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The following theorem follows from §4B and Theorem A, §5 of [37].

Theorem 2.5.3. *[37] For* $n > 2$, *any automorphism of* $G = SL(n,\mathbb{Z})$ *is induced by an automorphism of* $GL(n,\mathbb{Z})$. □

It follows from Theorem 2.5.3 that the outer automorphism group $Out(G) \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ according as n is odd or even. The group $Out(G)$ is generated by σ and τ, where $\sigma : g \mapsto {}^tg^{-1}$ and τ :

$g \mapsto JgJ^{-1}$, $J = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$. That is $Out(G) \cong \langle\sigma\rangle$ or $Out(G) \cong \langle\sigma, \tau\rangle$ according as n is odd or even.

Theorem 2.5.4. *[36] Let Γ be a normal subgroup of $SL(n,\mathbb{Z}), n \geq 3$, containing a congruence subgroup. Then the normalizer of Γ in $SL(n,\mathbb{R})$ is $SL(n,\mathbb{Z})$.* □

Let Γ be a non-central normal subgroup of $\mathrm{SL}(n,\mathbb{Z}), n > 2$, and let $\phi \in Aut(\Gamma)$. Note that the $\mathbb{R}$-rank of the real semisimple Lie group $G := \mathrm{SL}(n,\mathbb{R})$ is $n-1 \geq 2$. Then by Mostrow-Margulis strong rigidity theorem 2.1.4 [[51], Chapter 5], ϕ extends to an automorphism $\widetilde{\phi} : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$. By Theorem 2.5.4 we have $N_G(\Gamma) = \mathrm{SL}(n,\mathbb{Z})$. Thus $\widetilde{\phi}$ restricts to an automorphism $\bar{\phi}$ of $\mathrm{SL}(n,\mathbb{Z})$. Hence ϕ is the restriction of an automorphism, namely $\bar{\phi}$, of $\mathrm{SL}(n,\mathbb{Z})$. Thus, for $n > 2$, every automorphism of a non-central normal subgroup of $\mathrm{SL}(n,\mathbb{Z})$ is induced by an automorphism of $\mathrm{SL}(n,\mathbb{Z})$.

2.5.2 *Automorphisms of $Sp(2n,\mathbb{Z})$*

We denote $C_n = \mathrm{Sp}(2n,\mathbb{Z})$ and let $\Gamma_n = \{X \in \mathrm{GL}(2n,\mathbb{Z}) \mid XJ'\,{}^tX = \pm J'\}$. When $n = 1$, $C_1 = \mathrm{SL}(2,\mathbb{Z})$ and we already know its automorphisms. It is shown by I. Reiner [44] that C_2 has exactly one non trivial character and C_n, for $n > 2$, has no non trivial characters.

We conclude this section with the following result which describes all the automorphisms of C_n apart from inner automorphisms.

Theorem 2.5.5. *[43] For $n > 2$, every automorphism of C_n is of one of the following forms:*
(i) $\tau : g \mapsto aga^{-1}$, $a \in \Gamma_n$, and
(ii) When $n = 2$, $\theta : g \mapsto \chi(g)aga^{-1}$, $a \in \Gamma_2$ and χ is the non trivial character of C_2. □

It follows from Theorem 2.5.5 that the outer automorphism group of $\mathrm{Sp}(2n,\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $n > 2$ and is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ when $n = 2$.

In this section we discuss some basic results on the R_∞-property. We briefly recall the definition of the R_∞-property here with the same notations as in the earlier chapters. Let Γ be an infinite group and let $\phi : \Gamma \longrightarrow \Gamma$ be an endomorphism. Consider the action of Γ on itself by $g.x = gx\phi(g^{-1})$. The orbits of this action are called the ϕ-twisted conjugacy classes. The ϕ-twisted conjugacy class containing $x \in \Gamma$ is denoted $[x]_\phi$ or simply $[x]$ when ϕ is clear from the context. If x and y are in the same ϕ-twisted conjugacy class, we write $x \sim_\phi y$. The set of all ϕ-twisted conjugacy classes is denoted by $\mathcal{R}(\phi)$. The cardinality $R(\phi)$ of $\mathcal{R}(\phi)$ is called the Reidemeister number of ϕ.

Definition 2.6.1. We say that Γ has the R_∞*-property* for automorphisms (more briefly, R_∞*-property*) if there are infinitely many ϕ-twisted conjugacy classes for every automorphism ϕ of Γ. If Γ has the R_∞-property, we shall call Γ an R_∞*-group*.

Consider the short exact sequence of groups

$$1 \longrightarrow N \overset{j}{\hookrightarrow} \Lambda \overset{\eta}{\longrightarrow} \Gamma \longrightarrow 1, \tag{1}$$

where η is the canonical quotient homomorphism and j is the inclusion. Let $x \in \Lambda$ and $\phi \in Aut(\Lambda)$ and let $\bar{\phi}$ be the induced endomorphism (if it exists) on Γ. The map $[x]_\phi \mapsto [\eta(x)]_{\bar{\phi}}$ is surjective from $\mathcal{R}(\phi) \to \mathcal{R}(\bar{\phi})$. Hence $R(\bar{\phi}) \le R(\phi)$, that is, if $\bar{\phi}$ is an automorphism and has infinitely many twisted conjugacy classes then so does ϕ.

Let A be a free abelian group of finite rank and $\phi \in Aut(A)$. The ϕ-twisted class of the identity element $0 \in A$ is $[0]_\phi = \{x - \phi(x) \mid x \in A\} = Image(1 - \phi)$, which is a subgroup, say H, of A. Let $x \in A$, then $[x]_\phi = x + H$, the coset of H in A containing x. Hence $R(\phi)$ is the number of disctinct cosets of H in A, that is $R(\phi) = \#(A/H)$. Equivalently, $R(\phi) = \infty$ if and only if 1 is an eigen value of ϕ or $\#Fix(\phi) > 1$. In particular $\mathbb{Z}^n, n \ge 1$ does not have the R_∞-property. We will see later in this section that any infinite finitely generated abelian group has an automorphism with only finitely many twisted conjugacy classes.

Let $\gamma \in \Gamma$ and let i_γ be the inner automorphism sending $x \mapsto \gamma x \gamma^{-1}$. Let $\phi \in Aut(\Gamma)$. Write $\phi_\gamma := i_\gamma \circ \phi$. For any two $x, y \in \Gamma$, $x \sim_\phi y$ if and only if $x\gamma^{-1} \sim_{\phi_\gamma} y\gamma^{-1}$. Hence the map $[x]_\phi \mapsto [x\gamma^{-1}]_{\phi_\gamma}$ (resp. $[x]_{\phi_\gamma} \mapsto [x\gamma]_\phi$) is a bijection from $\mathcal{R}(\phi) \to \mathcal{R}(\phi_\gamma)$ (resp. $\mathcal{R}(\phi_\gamma) \to \mathcal{R}(\phi)$). These two maps are inverse to each other. Hence we get $R(\phi) = R(i_\gamma \circ \phi)$ for all $\gamma \in \Gamma$. In particular, if a group has infinitely many (usual) conjugacy classes then any inner automorphism will have infinitely many twisted conjugacy classes. Using similar arguments, Γ has the R_∞-property if $R(\phi) = \infty$ for a set of distinct representatives of each element of the group $Out(\Gamma) = Aut(\Gamma)/Inn(\Gamma)$, where $Inn(\Gamma)$ is the group of all inner automorphisms of Γ. We recall a result here which will be used in later chapters.

Lemma 2.6.2. *[18] Let $H_i \subset \Gamma$, Γ and $\bar{\Gamma}_i = \Gamma/H_i, i \geq 1$, be groups so that*

$$1 \longrightarrow H_i \overset{j}{\hookrightarrow} \Gamma \xrightarrow{\eta_i} \bar{\Gamma}_i \longrightarrow 1 \tag{2}$$

is exact. Let $\phi \in Aut(\Gamma)$ and $\phi_i : \bar{\Gamma}_i \to \bar{\Gamma}_i$ be the induced homomorphism. Assume $\phi(H_i) \subset H_i$. If the sequence $\{R(\phi_i)\}$ is unbounded, then Γ has the R_∞-property. □

We shall recall here some facts concerning the R_∞-property. Let N, Λ and Γ be groups as in the exact sequence (1).

Lemma 2.6.3. *Suppose that N is characteristic in Λ and that Γ has the R_∞-property, then Λ also has the R_∞-property.*

Proof. Let $\phi : \Lambda \longrightarrow \Lambda$ be any automorphism. Since N is characteristic, $\phi(N) = N$ and so ϕ induces an automorphism $\bar{\phi} : \Gamma \longrightarrow \Gamma$. Since $R(\bar{\phi}) = \infty$, it follows that $R(\phi) = \infty$. □

The following proposition is perhaps well-known. We give a proof here.

Proposition 2.6.4. *Let Γ be a countably infinite residually finite group. Then $R(\phi) = \infty$ for any inner automorphism ϕ of Γ.*

Proof. Let $\phi = \iota_\gamma$, the inner conjugation $x \mapsto \gamma x \gamma^{-1}$, and let $x \sim_\phi y$. Thus $y = gx\gamma g^{-1}\gamma^{-1}$. Equivalently $x\gamma$ is conjugate to $y\gamma$. Hence it suffices to show that Γ has infinitely many conjugacy classes as we saw in Remark 2.4.7.

Since Γ is infinite and since Γ is residually finite, there exist finite quotients $\bar{\Gamma}$ of Γ having *arbitrarily large* (finite) order. The result of R. Brauer 2.4.5 ([6], see also [41]) says that the number of conjugacy classes of a finite group of order n is bounded below by $\log\log n$. Since Γ has at least as many conjugacy classes as any of its quotients, it follows that Γ has infinitely many conjugacy classes. □

Lemma 2.6.5. *Let N, Λ and Γ be groups as in* (1)*. Suppose that N is a characteristic subgroup of Λ. (i) If N is finite and Λ has the R_∞-property then Γ also has the R_∞ -property.*
(ii) If Γ is finite and N has the R_∞-proeperty, then Λ has the R_∞- property.

Proof. (i) Any automorphism $\phi : \Lambda \longrightarrow \Lambda$ maps N isomorphically onto itself and hence induces an automorphism $\bar{\phi} : \Gamma \longrightarrow \Gamma$ (where $\Gamma = \Lambda/N$).

It is readily seen that $x \sim_\phi y$ implies $\eta(x) \sim_{\bar{\phi}} \eta(y)$ for any $x, y \in \Lambda$. Therefore η induces a surjection $\widetilde{\eta} : \mathcal{R}(\phi) \longrightarrow \mathcal{R}(\bar{\phi})$ where $\widetilde{\eta}([x]_\phi) = [\eta(x)]_{\bar{\phi}}$. We need only show that the fibres of $\widetilde{\eta}$ are finite.

Suppose the contrary and let $x_k \in \Lambda, k \geq 0$, be such that $[x_k]_\phi \neq [x_l]_\phi$ for $k \neq l$ and that $[\eta(x_k)]_{\bar{\phi}} = [\eta(x_0)]_{\bar{\phi}}$. For each $k \geq 1$, there exists $g_k \in \Lambda$ such that $\eta(x_0) = \eta(g_k)\eta(x_k)\bar{\phi}(\eta(g_k^{-1})) = \eta(g_k x_k \phi(g_k^{-1}))$. Therefore there exists an $h_k \in N$ such that $x_0 h_k = g_k x_k \phi(g_k)^{-1}$. That is, for any $k \geq 1$, we have $x_k \sim_\phi x_0 h_k$ for some $h_k \in N$. Since N is finite, it follows that $x_k \sim_\phi x_l$ for some $k \neq l$, a contradiction.

(ii) Let $\phi : \Lambda \longrightarrow \Lambda$ be an automorphism and let $\theta = \phi|N$. Let $\widetilde{j} : \mathcal{R}(\theta) \longrightarrow \mathcal{R}(\phi)$ be the map defined as $[x]_\theta \mapsto [x]_\phi$. Suppose that $R(\phi) < \infty$ but that $R(\theta) = \infty$. Then there exist elements $x_k \in N, k \geq 0$, such that $[x_k]_\theta \neq [x_l]_\theta, k \neq l$, but $x_k \sim_\phi x_0$ for all $k \geq 0$. Choose $g_k \in \Lambda$ such that $x_k = g_k x_0 \phi(g_k^{-1}), k \geq 1$. Since $\Gamma = \Lambda/N$ is finite, there exist distinct positive integers k, l such that $h := g_k g_l^{-1} \in N$. Now $x_k = g_k x_0 \phi(g_k^{-1}) = g_k g_l^{-1} x_l \phi(g_l)\phi(g_k^{-1}) = h x_l \theta(h^{-1})$ and so $[x_k]_\theta = [x_l]_\theta$, a contradiction. This completes the proof. □

It follows from Lemma 2.6.5 (i) that any infinite finitely generated abelian group has an automorphism ϕ with only finitely many ϕ-conjugacy classes. On the other hand, there are virtually free abelian groups having the R_∞-property. For example, the fundamental group $\pi_1(K)$ of the Klein bottle K has the R_∞-property; $\pi_1(K)$ contains $\mathbb{Z}^2$ as a subgroup of finite index. This example also shows that R_∞-property is not invariant under quasi-isometry. Since any residually finite group has infinitely many conjugacy classes, by Lemma 2.6.5 (ii), any virtually residually finite group has infinitely many conjugacy classes. Therefore, by Proposition 2.6.4, all of its inner automorphisms have infinitely many twisted conjugacy classes. The following groups have infinitely many twisted conjugacy classes for all of its inner automorphisms.

1) Finitely generated infinite nilpotent groups as they are residually finite,
2) infinite groups having finitely generated residually finite subgroup of finite index,
3) infinite groups having finitely generated nilpotent subgroup of finite index, as they are residually finite, and
4) all knot groups other tha $\mathbb{Z}$ (see Remark 2.4.7).

Lemma 2.6.6. *Let N, Λ and Γ be groups as in (1). Suppose that there is no non-trivial homomorphism from N to Γ and that either Γ is hopfian or N is co-hopfian. If Γ has the R_∞-property, then so does Λ.*

Proof. Let $\phi : \Lambda \longrightarrow \Lambda$ be any automorphism. Consider the homomorphism $f : N \longrightarrow \Gamma$ defined as $f = \eta \circ \phi | N$ where $\eta : \Lambda \longrightarrow \Gamma$ is the quotient map as in (1). By our hypothesis f is trivial, and so it follows that $\phi(N) \subset \ker(\eta) = N$. If N is co-hopfian then $\phi(N) = N$ and so N is characteristic. In any case ϕ defines a homomorphism $\bar{\phi} : \Gamma \longrightarrow \Gamma$ where $\bar{\phi}(xN) = \phi(x)N,\ x \in \Lambda$. It is clear that $\bar{\phi}$ is surjective with kernel $\phi^{-1}(N)/N$. If Γ is hopfian, $\bar{\phi}$ is an isomorphism and it follows that $\phi(N) = N$. Thus our hypothesis implies that N is characteristic in Λ and the lemma now follows from Lemma 2.6.3. □

We conclude this section with the following results which will be used in later chapters. We refer the reader to [30] (see also [13]) and [16] for a proof.

Theorem 2.6.7. *[30], [13] Any non-elementary hyperbolic group is an R_∞-group.* $\square$

It follows from this result that the fundamental group of a Riemannian manifold of constant negative sectional curvature with finite covolume has the R_∞-property provided it is cocompact.

Theorem 2.6.8. *[16] Any non-elementary relatively hyperbolic group is an R_∞-group.* $\square$

Recall that $C_n = \mathrm{Sp}(2n, \mathbb{Z})$ has no non trivial character for $n > 2$ and when $n = 2$, C_2 has exactly one non trivial character. The following result is true for all C_n.

Lemma 2.6.9. *[18] Let $\phi : Sp(4, \mathbb{Z}) \to Sp(4, \mathbb{Z})$ be an automorphism and let $\chi : Sp(4, \mathbb{Z}) \to \{\pm 1\}$ be a character. Write $\chi\phi(x) := \chi(x)\phi(x)$. Then $R(\phi)$ is finite if and only if $R(\chi\phi)$ is finite.* $\square$

We have already described groups which have (or not have) the R_∞-property. We will see more examples in next chapters.

3

TWISTED CONJUGACY IN LINEAR GROUPS

3.1 INTRODUCTION

In the introduction, §1.6, we have listed some of the important class of groups which are known to have the R_∞-property. In this chapter we begin establishing new results obtained in this thesis. We investigate the R_∞-property for the following groups: $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{PSL}(n,\mathbb{Z})$, $\mathrm{GL}(n,\mathbb{Z})$, $\mathrm{PGL}(n,\mathbb{Z})$, $\mathrm{Sp}(2n,\mathbb{Z})$ and $\mathrm{P}Sp(2n,\mathbb{Z})$.

Recall that $\mathrm{Sp}(2n,\mathbb{Z})$ and $\mathrm{PSp}(2\mathrm{n},\mathbb{Z})$ were first established in [18]. We also show that any countable abelian extensions of Γ has the R_∞-property where Γ is any one of the groups: torsion free non-elementary hyperbolic group, $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{GL}(n,\mathbb{Z})$, $\mathrm{Sp}(2n,\mathbb{Z})$, a principal congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$, the fundamental group of a complete Riemannian manifold of constant negative sectional curvature.

We continue with the notations as in section §2.6. Let $j : A \hookrightarrow \Lambda$ be the inclusion and $\eta : \Lambda \longrightarrow \Gamma$ the canonical quotient map so that

$$1 \longrightarrow A \hookrightarrow \Lambda \longrightarrow \Gamma \longrightarrow 1 \tag{3}$$

is an exact sequence of groups, where A is a countable abelian group.

The main result of this chapter is the following:

Theorem 3.1.1. *Let Λ be an extension of a group Γ by an arbitrary countable abelian group A. Then Λ has the R_∞-property in case any one of the following holds:*
(i) Γ is a torsion-free non-elementary hyperbolic group,

(ii) $\Gamma = SL(n,\mathbb{Z}), PSL(n,\mathbb{Z}), GL(n,\mathbb{Z}), PGL(n,\mathbb{Z}), Sp(2n,\mathbb{Z}), PSp(2n,\mathbb{Z})$, $n \geq 2$,
(iii) Γ *is a normal subgroup of* $SL(n,\mathbb{Z}), n > 2$, *not contained in the centre, and,*
(iv) Γ *is the fundamental group of a complete Riemannian manifold of constant negative sectional curvature and have finite volume.*

Recall that a group Γ is said to be *non-elementary* if it is infinite and is not virtually infinite cyclic. A subgroup $H \subset \Gamma$ is said to have *finite index* in Γ if its quotient set Γ/H is finite.

For example, consider the group $\Lambda = \mathbb{Z}^2 \rtimes F_2$ as in the exact sequence (3) with $j : \mathbb{Z}^2 \to \Lambda$ is the inclusion and $\eta : \Lambda \to F_2$ is the projection. Let $\phi \in Aut(\Lambda)$ and let $f := \eta \circ \phi \circ j : \mathbb{Z}^2 \to F_2$. Note that $f(\mathbb{Z}^2)$ is an abelian normal subgroup of the free group F_2. Since F_2 has no such non trivial subgroups (see [31], Chapter 1), $f(\mathbb{Z}^2) = \{1\}$. This implies that the normal subgroup $\mathbb{Z}^2$ is characteristic in Λ (note that F_2 is hopfian and hence any surjective endomorphism is an isomorphism). Now by Lemma 2.6.3 we get $R(\phi) = \infty$. This is true if we replace F_2 by any $F_n, n \geq 2$, and $\mathbb{Z}^2$ by any abelian group. The group $F_n, n \geq 2$, is non-elementary hyperbolic (torsion free). In Theorem 3.1.1 we show this for Λ when Γ is any non-elementary torsion free hyperbolic group.

The free group $F_n, n > 1$, has no non trivial normal abelian subgroups. This property does not hold for non-elementary hyperbolic groups in general. For example, the hyperbolic group $\mathrm{SL}(2,\mathbb{Z})$ has its center $\{I, -I\}$. On the other hand hyperbolic groups have properties such as in Lemma 2.3.7, that is the centralizer $C(\gamma)$ for every $\gamma \in \Gamma$ is a quasiconvex subgroup and if a subgroup $H \subset \Gamma$ is infinite and quasiconvex, then it has finite index in its normalizer.

Our proofs involve straightforward arguments, using well-known results concerning the group Γ in each case. More precisely, in each of the cases, we show that A or a bigger subgroup $N \subset \Lambda$ in which A has finite index is *characteristic* in Λ. Proof of this requires some facts concerning normal subgroups of Γ. In the cases (ii), (iii) and (iv) we invoke the normal subgroup theorem of Margulis (see Theorem 2.1.2); in case (i) we use the quasi-convexity property (see Lemma 2.3.7) of infinite cyclic subgroups of Γ. Using the fact that Γ is hopfian, the R_∞-property for Λ is then deduced from the R_∞-property for Γ. That Γ has the R_∞-property when it is a non-elementary hyperbolic group is by Theorem 2.6.7. The R_∞-property for the groups $\mathrm{Sp}(2n, \mathbb{Z})$ was first established by Fel'shtyn and Gonçalves [18]. The R_∞-property for $\mathrm{SL}(n, \mathbb{Z})$ and $\mathrm{PGL}(n, \mathbb{Z})$ is established in §3.2. We show, in §3.2, the R_∞-property for non-central normal subgroups of $\mathrm{SL}(n, \mathbb{Z}), n > 2$, using the Mostow-Margulis strong rigidity theorem (see Theorem 2.1.4) and the *congruence subgroup property* of $\mathrm{SL}(n, \mathbb{Z})$ which states that every finite index normal subgroup of $\mathrm{SL}(n, \mathbb{Z}), n > 2$, contains a principal congruence subgroup of $\mathrm{SL}(n, \mathbb{Z})$. The proof of the main theorem is given in §3.3.

3.2 THE R_∞-PROPERTY FOR SPECIAL LINEAR GROUPS

In this section we give new examples of R_∞-groups which are subgroups of finite index in $\mathrm{SL}(n, \mathbb{Z})$. Recall that $\mathrm{Sp}(2n; \mathbb{Z})$ has been shown to have the R_∞-property.

The first result is the following.

Theorem 3.2.1. *The linear groups $SL(n, \mathbb{Z})$, $PSL(n, \mathbb{Z})$, $GL(n, \mathbb{Z})$ and $PGL(n, \mathbb{Z})$ have the R_∞-property for all $n \geq 2$.*

Proof. Since $SL(n, \mathbb{Z})$ is the commutator subgroup of $GL(n, \mathbb{Z})$ when $n > 2$, and a characteristic subgroup when $n = 2$, it follows from Lemma 2.6.5(ii) that the R_∞-property for $\mathrm{SL}(n, \mathbb{Z})$ implies that for $\mathrm{GL}(n, \mathbb{Z})$. Also the R_∞-property for $\mathrm{SL}(n, \mathbb{Z})$ (resp. $\mathrm{GL}(n, \mathbb{Z})$) implies that for $\mathrm{PSL}(n, \mathbb{Z})$ (resp. $\mathrm{PGL}(n, \mathbb{Z})$) in view of Lemma 2.6.5(i). Therefore we need only prove the theorem for $\mathrm{SL}(n, \mathbb{Z})$.

The group $\mathrm{SL}(2, \mathbb{Z})$ is non-elementary hyperbolic group and hence, by Theorem 2.6.7, has the R_∞-property. Let $n \geq 3$ and set $\Gamma := \mathrm{SL}(n, \mathbb{Z})$. Since Γ is residually finite, $R(\phi) = \infty$ for any inner automorphism by Proposition 2.6.4. In this case we can see this more directly: the set $\{tr(A) \mid A \in \Gamma\}$ is infinite and so there are infinitely many conjugacy classes in Γ.

It remains only to show that $\mathcal{R}(\phi)$ is infinite for a set of representatives of (the non-trivial) elements of the group $\mathrm{Out}(\Gamma)$ of all outer automorphisms of Γ. By Theorem 2.5.3 that $\mathrm{Out}(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ according as n is odd or even.

The group $\mathrm{Out}(\Gamma)$ is generated by a set S where $S = \{\tau\}$ when n is odd and when n is even, $S = \{\sigma, \tau\}$ where $\tau : \Gamma \longrightarrow \Gamma$ is defined as $X \mapsto {}^tX^{-1}$, and, when n is even, the involution $\sigma : \Gamma \longrightarrow \Gamma$ is defined as $X \mapsto JXJ^{-1} = JXJ$ where J is the diagonal matrix $diag(1, \dots, 1, -1)$. Thus $X \sim_\tau Y$ (resp. $X \sim_\sigma Y$) if and only if there exists a Z such that $Y = ZX({}^tZ)$ (resp. $Y = ZXJZ^{-1}J$)).

First we consider τ-twisted conjugacy classes. Let $k \geq 1$ and let $A(k)$ be the block diagonal matrix $A(k) = diag(B(k), I_{n-2})$ where $B(k) = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$. We shall show that $A(k) \sim_\tau A(l)$ implies $k = l$. This clearly implies that $R(\tau) = \infty$.

Let $X = (x_{ij}) \in \Gamma$ be such that

$$X.A(k).{}^tX = A(l). \tag{2}$$

We shall denote the i-th row and i-th column of X by r_i and c_i respectively. A straightforward computation shows that $X.A(k).{}^tX = X.{}^tX + kc_2.{}^tc_1$. Comparing the $(2,1)$-entries on both sides of (2) we get $r_2.{}^tr_1 + kx_{22}x_{11} = l$ whereas comparing the $(1,2)$-entries gives $r_1.{}^tr_2 + kx_{12}x_{21} = 0$. Therefore $r_2.{}^tr_1 = r_1.{}^tr_2 = -kx_{12}x_{21}$ and so $l = k(x_{11}x_{22} - x_{12}x_{21})$. Since $x_{i,j} \in \mathbb{Z}$, we obtain that $k|l$. Interchanging the roles of k, l we get $l|k$ and so we must have $k = l$ since $k, l \geq 1$.

Now consider σ-twisted conjugacy classes. Since $A \sim_\sigma B$ if and only if $AJ = X(BJ)X^{-1}$ for some $X \in \mathrm{SL}(n, \mathbb{Z})$. We need only show that the set $\{\mathrm{tr}(AJ) \mid A \in \mathrm{SL}(n, \mathbb{Z})\}$ is infinite. Let $A' \in \mathrm{SL}(n-1, \mathbb{Z})$ and let $A = diag(A', 1)$ where $A' \in \mathrm{SL}(n-1, \mathbb{Z})$. Then $AJ = diag(A', -1)$. Therefore $\mathrm{tr}(A) = \mathrm{tr}(A') - 1$. Since $n > 2$, the set $\{\mathrm{tr}(A') \mid A' \in \mathrm{SL}(n-1, \mathbb{Z})\}$ is infinite and we conclude that $R(\sigma) = \infty$.

Now we only have to consider the $\sigma\tau$- conjugacy classes. Set $\phi = \sigma\tau$. Note that $\phi(X) = J\,{}^tX^{-1}J$ for $X \in \Gamma$, where J is as above. Since $A \sim_\phi B$

if and only if $AJ = X(BJ)\,{}^tX$ for some $X \in \mathrm{SL}(n,\mathbb{Z})$, the proof of that $R(\sigma\tau) = \infty$ is similar and so we omit the details. $\square$

It is possible to give a more direct proof of the R_∞-property for $\mathrm{SL}(2,\mathbb{Z})$ as for $\mathrm{SL}(n,\mathbb{Z}), n > 2$, given above, using the description of the (outer) automorphism group of $\mathrm{SL}(2,\mathbb{Z})$ given in [26, Theorem 2]. (See 2.5.1).

Recall that the R_∞-property is not inherited by finite index subgroups in general. For example, the fundamental group $\pi_1(K)$ of the Klein bottle K has the R_∞-property. (See §2.6). However we have the following result.

Let Γ_m denote the principal level m congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$; thus Γ_m is the kernel of the surjection $\mathrm{SL}(n,\mathbb{Z}) \longrightarrow \mathrm{SL}(n,\mathbb{Z}/m\mathbb{Z})$ induced by the surjection $\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$.

Theorem 3.2.2. *Let $n \geq 3$. Let Λ be a non-central normal subgroup of $SL(n,\mathbb{Z})$. Then Λ has the R_∞-property.*

Proof. Let $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. We shall use the notations introduced in the proof of Theorem 3.2.1. It is known that any finite index subgroup of $\mathrm{SL}(n,\mathbb{Z})$ contains Γ_m for some $m \geq 2$. This is the congruence subgroup property for $\mathrm{SL}(n,\mathbb{Z}), n > 2$. See [48, §4.4].

Let $M = (m_{i,j}) \in \mathrm{SL}(n,\mathbb{Z})$ and let $\phi := \phi_M$ be the restriction to Λ of the inner automorphism ι_M of Γ. Then $X \sim_\phi Y$ if and only if $XM = Z(YM)Z^{-1}$ for some $Z \in \Lambda$. In particular $\mathrm{tr}(XM) = \mathrm{tr}(YM)$. To show that $R(\phi) = \infty$ we need only show that the set $\{\mathrm{tr}(AM) \mid A \in \Lambda\}$ is infinite for any $M \in \mathrm{SL}(n,\mathbb{Z})$. There are two cases to consider: (1) $m_{ii} \neq 0$ for some i, (2) $m_{ii} = 0$ for all i.

Case (1): Without loss of generality we may assume that $m_{11} \neq 0$. Let $k > 1$ and let $X(k)$ be the block diagonal matrix $X(k) = diag(C(k), I_{n-2})$ where $C(k) = \begin{pmatrix} k^2+1 & k \\ k & 1 \end{pmatrix}$. A straightforward computation shows that $\mathrm{tr}(X(k)M) = (k^2+1)m_{11} + k(m_{12}+m_{21}) + \sum_{2\leq j\leq n} m_{jj}$. Therefore $\mathrm{tr}(X(k)M) = \mathrm{tr}(X(l)M)$ if and only if $(k+l)m_{11} + m_{12} + m_{21} = 0$. Choose $k_0 > (m_{12}+m_{21})/m_{11}$. Then $X(mk), k \geq k_0$, belong to pairwise distinct ϕ-twisted conjugacy classes in this case.

Case (2): Without loss of generality assume that $m_{12} \neq 0$. Let $A(k)$ be as in the proof of Theorem 3.2.1. Then $\mathrm{tr}(A(k)M) = km_{1,2}$. Therefore

$\mathrm{tr}(A(k)M) = \mathrm{tr}(A(l)M)$ if and only if $k = l$. Since $A(mk) \in \Gamma_m \subset \Lambda$ for all k, it follows that $R(\phi) = \infty$ in this case as well.

Suppose that $\tau(\Lambda) = \Lambda$ where $\tau(X) = {}^tX^{-1}$ as in the proof of Theorem 3.2.1. We see that $R(\tau|\Lambda) = \infty$ arguing as we did to establish that $R(\tau) = \infty$ in the proof of Theorem 3.2.1 by considering the set of elements $A(mk) \in \Gamma_m \subset \Lambda, k \geq 1$. Similarly, we show that $R(\theta|\Lambda) = \infty$ for each representative θ of the outer automorphisms of Γ which leaves Λ invariant.

To complete the proof, we need only show that *every* automorphism of Λ extends to an automorphism of Γ. For this purpose we observe that the $\mathbb{R}$-rank of the semi simple Lie group $G := \mathrm{SL}(n, \mathbb{R})$ equals $n - 1 \geq 2$. Let $\theta : \Lambda \longrightarrow \Lambda$ be any automorphism. By the Mostow-Margulis strong rigidity theorem 2.1.4 (see also [51, Chapter 5]), θ extends to an automorphism $\widetilde{\theta} : \mathrm{SL}(n, \mathbb{R}) \longrightarrow \mathrm{SL}(n, \mathbb{R})$. By the result 2.5.4 of Newman we have $N_G(\Lambda) = \Gamma$. So $\widetilde{\theta}$ restricts to an automorphism $\bar{\theta}$ of Γ. Thus θ is the restriction of an automorphism of Γ, namely $\bar{\theta}$. This completes the proof. □

Remark 3.2.3. Recall that Fel'shtyn and Gonçalves [18] have shown that $\mathrm{Sp}(2n, \mathbb{Z})$ has the R_∞- property. One could also establish this result along the same lines as for $\mathrm{SL}(n, \mathbb{Z})$ given above. We assume that $n \geq 2$ as $\mathrm{Sp}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$. To fix notations, regard $\mathrm{Sp}(2n, \mathbb{Z})$ as the subgroup of $\mathrm{SL}(2n, \mathbb{Z})$ which preserves the skew symmetric form $\beta : \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \longrightarrow \mathbb{Z}$ defined as $\beta(e_{2i}, e_{2j}) = 0 = \beta(e_{2i-1}, e_{2j-1})$, $\beta(e_{2i-1}, e_{2j}) = \delta_{ij}, 1 \leq i \leq j \leq n$ (Kronecker δ). Equivalently $\mathrm{Sp}(2n, \mathbb{Z}) = \{X \in \mathrm{SL}(2n, \mathbb{Z}) \mid {}^tXJ_0X = J_0\}$ where $J_0 = diag(j_0, \ldots, j_0), j_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the matrix of β. Infinitude of (untwisted) conjugacy classes follows from the residual finiteness of $\mathrm{Sp}(2n, \mathbb{Z})$. (cf. Proposition 2.6.4). Alternatively, observe that $X(k) \in \mathrm{Sp}(2n, \mathbb{Z})$ where $X(k)$ is as in the proof of Theorem 3.2.2. This shows that the trace function is unbounded on $\mathrm{Sp}(2n, \mathbb{Z})$.

To complete the proof, we need only verify that $R(\phi) = \infty$ for representatives of the elements of $\mathrm{Out}(\mathrm{Sp}(2n, \mathbb{Z}))$. One knows from Theorem

2.5.5 that the outer automorphism group of $\mathrm{Sp}(2n,\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $n > 2$ and is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ when $n = 2$.

The generators of the outer automorphism groups may be described as follows. Let θ be the automorphism of $\mathrm{Sp}(2n,\mathbb{Z})$ which is conjugation by $J := diag(I', I_{2n-2}) \in \mathrm{GL}(2n,\mathbb{Z})$ where $I' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let ϕ be the automorphism of $\mathrm{Sp}(4,\mathbb{Z})$ defined as $\phi(X) = \chi(X)X$ where $\chi : \mathrm{Sp}(4,\mathbb{Z}) \longrightarrow \{1,-1\}$ is the (non-trivial) central character. Then $\mathrm{Out}(\mathrm{Sp}(2n,\mathbb{Z})) = \langle\theta\rangle, n > 2$, and $\mathrm{Out}(\mathrm{Sp}(4,\mathbb{Z})) = \langle\theta,\phi\rangle$.

To see that $R(\theta) = \infty$ we note that $\mathrm{tr}(X(k)J) = 2k + (2n-2)$. Therefore the $X(k)$, $k \geq 1$, belong to pairwise distinct θ-twisted conjugacy classes.

As observed already in Lemma 2.6.9, any ϕ-twisted conjugacy class of X is a union of the (untwisted) conjugacy class of X and of $-X$. Since the number of conjugacy classes in $\mathrm{Sp}(4,\mathbb{Z})$ is infinite, it follows that $R(\phi) = \infty$. Note that $\theta\phi(X) = \chi(X)JXJ^{-1}$. The proof that $R(\theta\phi) = \infty$ is similar and omitted. This completes the proof.

3.3 PROOF OF THE MAIN THEOREM

We now proceed to the proof of the main theorem. Let $j : A \hookrightarrow \Lambda$ be the inclusion and $\eta : \Lambda \longrightarrow \Gamma$ the canonical quotient map so that

$$1 \longrightarrow A \hookrightarrow \Lambda \longrightarrow \Gamma \longrightarrow 1 \tag{4}$$

is an exact sequence of groups, where A is a countable abelian group.

Proof of Theorem 3.1.1: Let $\phi : \Lambda \longrightarrow \Lambda$ be any automorphism and let $f : A \longrightarrow \Gamma$ be the composition $\eta \circ \phi \circ j$. Note that since A is normal in Λ, $\phi(A)$ is normal in Λ and hence $f(A)$ is normal in Γ.

Theorem 3.3.1. *Let Λ, Γ and A be groups as in (4). If Γ is a torsion-free non-elementary hyperbolic group, then Λ has the R_∞-property.*

Proof. In this case we claim that f is trivial. Suppose that $f(A)$ is *not* the trivial subgroup. Since Γ is a non-elementary hyperbolic group it does not contain a free abelian group of rank 2. Since Γ is torsion free, the centralizer of any non-trivial element of Γ is infinite cyclic. By Lemma 2.3.7 (i) $f(A)$ is quasi-convex. Hence by Lemma 2.3.7 (ii) the subgroup $f(A)$ has finite index in its normalizer, which is Γ. This contradicts the assumption that Γ is non-elementary. Therefore $f(A)$ must be trivial. This means that $\phi(A) \subset A$ and we have the following diagram in which the top horizontal sequence is exact:

$$\begin{array}{ccccc} A & \hookrightarrow & \Lambda & \longrightarrow & \Gamma \\ \phi|A \downarrow & & \phi \downarrow & & \downarrow \bar{\phi} \\ A & \hookrightarrow & \Lambda & \longrightarrow & \Gamma. \end{array}$$

Now $\bar{\phi}$ is a surjection since $\eta \circ \phi$ is. Since Γ is assumed to be torsion-free, by Sela's Theorem 2.4.3, Γ is hopfian and so $\bar{\phi}$ is an isomorphism. Therefore $\phi(A) = A$. Hence A is characteristic in Λ. Since Γ has the R_∞-property by Theorem 2.6.7, Lemma 2.6.3 now implies that Λ has the R_∞-property. □

Theorem 3.3.2. *Let Λ, Γ and A be groups as in (4). If Γ is one of the following groups:*
$SL(n,\mathbb{Z})$, $PSL(n,\mathbb{Z})$,$GL(n,\mathbb{Z})$, $PGL(n,\mathbb{Z})$, $Sp(2n,\mathbb{Z})$ or $PSp(2n,\mathbb{Z})$, $n \geq 2$, then Λ has the R_∞-property.

Proof. The group Γ is a lattice in one of the simple linear Lie groups $G = \mathrm{SL}(n,\mathbb{R})$, $\mathrm{PGL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{PSp}(2n,\mathbb{R})$. These Lie groups have centre a group of order at most 2. Also, Γ is hopfian. First we consider the case $\Gamma = \mathrm{SL}(n,\mathbb{Z}), \mathrm{PSL}(n,\mathbb{Z}), \mathrm{PGL}(n,\mathbb{Z}), n > 2$, or $\mathrm{Sp}(2n,\mathbb{Z})$, $\mathrm{PSp}(2n,\mathbb{Z})$, $n > 1$, so that the corresponding Lie group G has real rank at least 2. By the normal subgroup theorem of Margulis (see Theorem 2.1.2), the subgroup $f(A)$ being normal in Γ is either of finite index or is contained in the centre of G. Since A is abelian, $f(A)$ cannot be of finite index in Γ. Hence $f(A) \subset Z(\Gamma)$ the centre of Γ which is of order at most 2. First assume that $f(A)$ is trivial. Then we have $\phi(A) \subset A$. Using the fact that Γ is hopfian, we conclude as above, that A is characteristic. Now Γ has the R_∞-property by Theorem 3.2.1 in the case of $\mathrm{SL}(n,\mathbb{Z}), \mathrm{PSL}(n,\mathbb{Z}), \mathrm{PGL}(n,\mathbb{Z})$ and by Remark 3.2.3 (see also [18]) in the case of $\mathrm{Sp}(2n,\mathbb{Z}),\mathrm{PSp}(2n,\mathbb{Z})$ (cf. Lemma 2.6.5(i)). It follows as in case (i) that Λ also has the R_∞-property. Now assume that $f(A) = Z(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$. Set $\bar{\Gamma} = \Gamma/Z(\Gamma)$ which is the lattice $\mathrm{PSL}(n,\mathbb{Z})$ or $\mathrm{PSp}(2n,\mathbb{Z})$ in the corresponding Lie group of adjoint type. Let $N = \eta^{-1}(Z(\Gamma))$. Clearly $N/A \cong Z(\Gamma)$. Now we have the exact sequence $N \overset{\widetilde{j}}{\hookrightarrow} \Lambda \xrightarrow{\bar{\eta}} \bar{\Gamma}$ where $\bar{\eta}$ is the canonical quotient map. Now we claim that N is characteristic. Indeed, let $\widetilde{f} : N \longrightarrow \bar{\Gamma}$ be defined as $\bar{\eta} \circ \phi \circ \widetilde{j}$. Again using Margulis' normal subgroup theorem, the fact that N is virtually abelian forces $\widetilde{f}(N)$ to be contained in the centre of $\bar{\Gamma}$. Since $\bar{\Gamma}$ has trivial centre, we must have $\widetilde{f}(N) \subset N$. Now $\bar{\Gamma}$ is again hopfian (being finitely generated and linear). As before, we conclude that N is characteristic. By Lemma 2.6.3 applied to $\bar{\Gamma}$ we conclude that Λ has the R_∞-property.

We now consider the case $\mathrm{SL}(2,\mathbb{Z}) \cong \mathrm{Sp}(2,\mathbb{Z})$. Proceeding as above we see that $f(A)$ is a normal abelian subgroup of $\mathrm{SL}(2,\mathbb{Z})$. We need only show that $f(A) \subset \{I,-I\}$. Let $F \subset \mathrm{SL}(2,\mathbb{Z})$ be a free group of finite index which is normal. Then $F \cap f(A)$ is trivial since any normal subgroup of F is a non-abelian free group. Hence $f(A)$ is finite as it imbeds in the finite group $\mathrm{SL}(2,\mathbb{Z})/F$. Let C be the image of $f(A)$ in $\mathrm{PSL}(2,\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ under the natural quotient map. Any element in $\mathrm{PSL}(2,\mathbb{Z})$ of finite order is conjugate to the generator of $\mathrm{PSL}(2,\mathbb{Z})$ of order 2 or that of order 3 (see [31, Theorem 2.7, Chapter IV]). Since C is normal and finite, it follows easily that C is trivial. Hence $f(A) \subset \{I,-I\}$. □

Theorem 3.3.3. *Let Λ,Γ and A be groups as in (4). Let Γ be a normal subgroup of $SL(n,\mathbb{Z}), n > 2$, not contained in the centre. Then Λ has the R_∞-property.*

Proof. By Theorem 3.2.2 the R_∞-property holds for Γ. The rest of the proof is as in the proof of Theorem 3.3.2 above and hence omitted. □

Theorem 3.3.4. *If Γ is the fundamental group of a complete Riemannian manifold of constant negative sectional curvature and has finite volume, then Λ has the R_∞-property.*

Proof. Assume M is oriented. If M is compact, then Γ is a torsion-free hyperbolic group and our statement follows from Theorem 3.3.1. In any case, Γ is a lattice in G, the group of orientation preserving isometries of the universal cover of M. Thus G is a simple Lie group with trivial centre and real rank 1. In particular, G is linear and so Γ is residually finite. Indeed G is the identity component of the real points $\mathbf{G}_{\mathbb{R}}$ of the complex linear algebraic group $\mathbf{G}$ of adjoint type whose Lie algebra equals $\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$.

If M is non-compact then Γ is relatively hyperbolic (with respect to the family of stabilizers of the cusps of M). Such groups Γ have the R_∞-property by Theorem 2.6.8.

Next we show that $f(A)$ is trivial. Let $Z \subset \mathbf{G}_{\mathbb{R}}$ be the Zariski closure of $f(A)$ and let H be the normalizer of Z in $\mathbf{G}_{\mathbb{R}}$. Then H is an algebraic subgroup which contains Γ. Since Γ is Zariski dense in $\mathbf{G}_{\mathbb{R}}$ by the Borel density theorem 2.2.5, it follows that $H = \mathbf{G}_{\mathbb{R}}$ and so Z is normal in $\mathbf{G}_{\mathbb{R}}$. Since Z is abelian and since G is simple, it follows that Z is finite and is contained in the centre of $\mathbf{G}_{\mathbb{R}}$. Therefore $f(A)$ *equals* $Z \cap G$ and is contained in the centre of G. Since the centre of G is trivial, we conclude that $f(A) = \{1\}$. The rest of the proof is as in the previous cases above.
If M is not oriented, then the subgroup $\widetilde{\Gamma} \subset \Gamma$ of orientation preserving isometries of the universal cover $\widetilde{M}$ of M is an index two normal subgroup of Γ. Now by Lemma 2.6.3, it follows that Γ has the R_∞-property. □

We conclude this chapter with the following remarks.

Remark 3.3.5. *(i) Theorem 3.1.1 contains as special cases the direct product $A \times \Gamma$ as well as the restricted wreath product $C \wr \Gamma = (\oplus_{\gamma\in\Gamma} C_\gamma) \rtimes \Gamma$ where $C_\gamma = C$ is any cyclic group.*
(ii) Let P be any set of primes containing 2; thus any homomorphism $A(P) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is trivial. Let $A(P) = \mathbb{Z}[1/p|p \in P] \subset \mathbb{Q}$. Note that $i : A(P) \longrightarrow A(Q)$ is any non-trivial homomorphism, then $P \subset Q$. Set $\Lambda(P) := A(P) \wr \Gamma$ where Γ is as in Theorem 3.1.1. Suppose that $\theta : \Lambda(P) \longrightarrow \Lambda(Q)$ is an isomorphism. Then, as in the proof of Theorem 3.1.1, the composition $\oplus_{\gamma\in\Gamma} A(P) \hookrightarrow \Lambda(P) \xrightarrow{\theta} \Lambda(Q) \longrightarrow \Gamma$ is trivial. It follows that $\theta(\oplus_{\gamma\in\Gamma} A(P)) \subset \oplus_{\gamma\in\Gamma} A(Q)$ and so $P \subset Q$. Simi-

larly $Q \subset P$ and so $P = Q$. It follows that there are $2^{\aleph_0}$ many pairwise non-isomorphic countable groups Λ satisfying the R_∞-property for each Γ as in Theorem 3.1.1. The same conclusion can also be arrived at by considering the groups $A(P) \times \Gamma$.

4

TWISTED CONJUGACY IN LATTICES

4.1 INTRODUCTION

An important class of groups in algebraic topology is Lie groups and their lattices. We have already seen in Chapter 1, section §1.6 some of the important classes of groups which are known to have the R_∞-property. We established in Chapter 3, section §3.2 the R_∞-property for the groups $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{PSL}(n,\mathbb{Z})$, $\mathrm{PGL}(n,\mathbb{Z})$, $\mathrm{Sp}(2n,\mathbb{Z})$, etc. The R_∞-property for $\mathrm{Sp}(2n,\mathbb{Z})$ was first established by A. Fel'shtyn and D. Gonçalves [18]. In this chapter, we prove Theorem 4.1.1 establishing the R_∞-property for irreducible lattices in a connected non-compact semisimple Lie group having finite center and $\mathbb{R}$-rank at least 2. We also study the R_∞-property for their countable abelian extensions.

Recall that a group Γ has the R_∞-property for automorphisms if there are infinitely many ϕ-twisted conjugacy classes for every automorphism ϕ of Γ. If Γ has the R_∞-property, we shall call Γ an R_∞-group.

Let G be a non-compact semi simple Lie group with finite centre. We will continue using the same notations as in Chapter 2.

The main result of this chapter is the following. The proof of the main theorem is given in section §4.2.

Theorem 4.1.1. *Let Γ be any irreducible lattice in a connected semi simple non-compact Lie group G with finite centre. If the real rank of G is at least 2, then Γ has the R_∞ property.*

Remark 4.1.2. Recall that finitely generated residually finite groups are hopfian. A well-known class of residually finite groups is the class of finitely generated subgroups of $\mathrm{GL}(n, K)$ where K is any field. See [31]. This class includes, in particular, all lattices in linear Lie groups. An important unsolved problem is to decide whether hyperbolic groups are residually finite. It has been shown by Sela [46] that torsion-free hyperbolic groups are hopfian.

Recall that a discrete subgroup $\Gamma \subset G$ is called a *lattice* if G/Γ has a finite G-invariant measure. One says that Γ is *cocompact* if G/Γ is compact; otherwise Γ is non-cocompact. If, for any non-compact closed normal subgroup $H \subset G$, the image of Γ under the quotient map $G \longrightarrow G/H$ is dense, one says that Γ is irreducible. If G has no compact factors, Γ is irreducible if and only if for any two closed normal subgroups H_1, H_2 of G such that $G = H_1.H_2$ and lattices $\Gamma_i \subset H_i$, the group $\Gamma_1.\Gamma_2$ is not commensurable with Γ. In particular, any lattice in G is irreducible if G is simple. (See Chapter 2, §2.1).

Our proof of Theorem 4.1.1 involves only elementary arguments, using some well-known but deep results concerning irreducible lattices in semi simple Lie groups. The theorem is first established when G has no compact factors and has trivial centre. In this case, the proof uses the Zariski density property of Γ due to Borel as well as the strong rigidity theorem. We need to use Margulis' normal subgroup theorem to reduce to the case when G has trivial centre and no compact factors.

In Chapter 2, §2.1, and §2.2 we have recalled the results on lattices in semi simple Lie groups needed in the proof of Theorem 4.1.1 given in §4.2.

4.2 PROOF OF THEOREM 4.1.1

Proof of Theorem 4.1.1: First suppose that G has trivial centre and has no compact factors. Since the centre of G is trivial, the homomorphism $\iota : G \longrightarrow Aut(G)$ given by inner automorphism allows us to identify G with the group of inner automorphisms of G. Under this identification, G

is the identity component of $Aut(G)$ and $Aut(G)/G \cong Out(G)$ is finite. Also the group $Aut(G)$ is isomorphic to the linear Lie group $Aut(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ of the automorphisms of the Lie algebra $\mathfrak{g}$ of G under which $\phi \in Aut(G)$ corresponds to its derivative at the identity element. Thus we have a chain of monomorphisms $\Gamma \hookrightarrow G \xrightarrow{\iota} Aut(G) \cong Aut(\mathfrak{g}) \hookrightarrow \mathrm{GL}(\mathfrak{g})$. Furthermore, $Aut(G) \cong Aut(\mathfrak{g})$ is the $\mathbb{R}$-points of the complex algebraic group $\mathbf{H} := Aut(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$ and the identity component of $H_{\mathbb{R}}$ is $Aut(G)^0 = G$.

Suppose that $\phi : \Gamma \longrightarrow \Gamma$ is an automorphism. Clearly $\phi \circ \iota_\gamma = \iota_{\phi(\gamma)} \circ \phi$ where ι_γ denotes conjugation by γ. Now let $x, y \in \Gamma$ be such that $x \sim_\phi y$. Then there exists a $\gamma \in \Gamma$ such that $y = \gamma x \phi(\gamma^{-1})$; equivalently, $\iota_y = \iota_\gamma \iota_x \iota_{\phi(\gamma)^{-1}} = \iota_\gamma \iota_x \phi \iota_{\gamma^{-1}} \phi^{-1}$. Hence $\iota_y \phi = \iota_\gamma (\iota_x \phi) \iota_{\gamma^{-1}}$.

By the strong rigidity theorem (Theorem 2.1.4), $\phi \in Aut(\Gamma)$ extends to an automorphism of the Lie group G, again denoted $\phi \in Aut(G)$. For any $h \in H_{\mathbb{R}}$, consider the function $\tau_h : \mathbf{H} \longrightarrow \mathbb{C}$ defined as $\tau_h(x) = \mathrm{tr}(xh)$, the trace of $xh \in \mathbf{H} \subset \mathrm{GL}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. Clearly this is a morphism of varieties defined over $\mathbb{R}$. We have that, if $x, y \in \Gamma$, $x \sim_\phi y$, then $\tau_\phi(y) = \tau_\phi(x)$ since $\iota_y \phi$ and $\iota_x \phi$ are conjugates in $\mathbf{H}$.

Assume that the Reidemeister number of ϕ is finite. Then, by what has been observed above, τ_ϕ assumes only finitely many values on $\Gamma \subset H^0_{\mathbb{R}} = G$. Since, by the Borel density theorem 2.2.5, Γ is Zariski dense in $\mathbf{H}^0$, it follows that τ_ϕ is constant on $\mathbf{H}^0$. Since $\tau_{h\phi}(x) = tr(xh\phi) = \tau_\phi(xh), \forall x \in \mathbf{H^0}$, this clearly implies that $\tau_{h\phi}$ is constant for *any* $h \in H^0_{\mathbb{R}}$.

Let K be a maximal compact subgroup of $H_{\mathbb{R}} = Aut(G)$. Since $Aut(G)$ has only finitely many components, by a well-known result of Mostow, K meets *every* connected component of $Aut(G)$. (See Theorem 2.2.3). Thus K contains representatives of every element of $Out(\Gamma)$ and so we may choose an $h \in H^0_{\mathbb{R}}$ such that $\theta := h\phi \in K$. The automorphism $Ad(\theta)$ on the Lie algebra $Lie(K^0)$ fixes a regular (semi simple) element $X \in Lie(K^0)$ by §3.2, Ch. VII of [4]. Hence the one-parameter subgroup $S := \{\exp(tX) \mid t \in \mathbb{R}\} \subset K^0$ is contained in the centralizer $C_{H_{\mathbb{R}}}(\theta) = \{x \in H_{\mathbb{R}} \mid \theta x = x\theta\}$. Note that θ is also semi simple since K is compact subgroup of $\mathrm{GL}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. It follows that θ and $\exp(tX), t \in \mathbb{R}$, are simultaneously diagonalizable (over $\mathbb{C}$). It is now readily seen that τ_θ

is not constant on $S \subset H^0_{\mathbb{R}}$, a contradiction to our earlier observation that $\tau_{h\phi}$ is a constant function for any $h \in H^0_{\mathbb{R}}$. This implies that $R(\phi) = \infty$.

Next suppose that G has no compact factors but possibly with non-trivial centre, Z. By our hypothesis Z is finite. Clearly $Z \cap \Gamma \subset Z(\Gamma)$ the centre of Γ. Since $\bar{\Gamma} := \Gamma/(Z \cap \Gamma)$ is Zariski dense in G/Z, and since G/Z has trivial centre, we see that $\Gamma/(Z \cap \Gamma)$ has trivial centre. It follows that $Z(\Gamma) = Z \cap \Gamma$. Consider the exact sequence

$$1 \to Z \cap \Gamma \to \Gamma \to \bar{\Gamma} \to 1. \tag{5}$$

Since $Z \cap \Gamma = Z(\Gamma)$ is a finite characteristic subgroup of Γ, the R_∞ property for Γ follows from that for $\bar{\Gamma}$ by Lemma 2.6.3.

Finally let G be any Lie group as in the theorem. Let M be the maximal compact normal subgroup of G. Note that M contains the centre Z of G. Now $M \cap \Gamma$ is a *finite* normal subgroup of Γ. We invoke Theorem 2.1.2 to conclude that $M \cap \Gamma$ is contained in the centre of G. Also $Z(\Gamma)$ is contained in Z since, otherwise, by Theorem 2.1.2 again, Γ would be virtually abelian. Since G is a non-compact semi simple Lie group, this is impossible. Since M contains Z, we see that $M \cap \Gamma = Z \cap \Gamma$ *equals* the centre of Γ and hence is characteristic in Γ. Now $\bar{\Gamma} := \Gamma/(M \cap \Gamma)$ is an irreducible lattice in G/M, which has trivial centre and no compact factors. Using the exact sequence (5) again, we see that $R(\phi) = \infty$. This completes the proof. □

When G has real rank 1, the above result is well-known. Indeed, assume that G has real rank 1. When the lattice Γ is cocompact, it is hyperbolic. When Γ is not cocompact, it is relatively hyperbolic. Both these groups have the R_∞-property (see Chapter 1, §1.6).

Remark 4.2.1. Suppose that G is not locally isomorphic to $SL(2, \mathbb{R})$ and that the real rank of G equals 1. When G has no compact factors, the above proof can be repeated verbatim to show that Γ has the R_∞ property. When G has compact factors and Γ is residually finite (for example when G is linear), by Lemma 2.4.1, one can find a finite index characteristic subgroup Γ' of Γ such that $\Gamma' \cap M = \{1\}$ where M is as in the above proof.

Now $\Gamma' \cong \Gamma'/M$ and so has the R_∞ property. It follows from Lemma 2.6.3 that Γ has the R_∞ property.

Let $j : A \hookrightarrow \Lambda$ be the inclusion and $\eta : \Lambda \longrightarrow \Gamma$ the canonical quotient map so that

$$1 \longrightarrow A \hookrightarrow \Lambda \longrightarrow \Gamma \longrightarrow 1 \tag{6}$$

is an exact sequence of groups.

Theorem 4.2.2. *Let Γ and G be as in Theorem 4.1.1. Assume that G is linear. Let Λ be an extension of Γ by an arbitrary countable abelian group A as in (6). Then Λ has the R_∞-property.*

Proof. Since Γ is finitely generated and linear, it follows that Γ is residually finite and hence Hopfian. Let $1 \to A \overset{j}{\hookrightarrow} \Lambda \overset{\eta}{\to} \Gamma \to 1$ be an exact sequence of groups where A is any countable abelian group. Let $\phi : \Lambda \longrightarrow \Lambda$ be any automorphism and let $f : A \longrightarrow \Gamma$ be the composition $\eta \circ \phi \circ j$. Note that since A is normal in Λ, $\phi(A)$ is normal in Λ and hence $f(A)$ is normal in Γ. By the normal subgroup theorem 2.1.2 of Margulis, the subgroup $f(A)$ being normal in Γ is either of finite index or is contained in the center of G. Since A is abelian and Γ is not virtually abelian, $f(A)$ cannot be of finite index in Γ. Therefore $f(A) \subset Z(G)(\subset Z(\Gamma))$ and hence is finite. First assume that $f(A)$ is trivial. This means that $\phi(A) \subset A$ and we have the following diagram in which the top horizontal sequence is exact:

$$\begin{array}{ccccc} A & \hookrightarrow & \Lambda & \longrightarrow & \Gamma \\ \phi|A \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} \\ A & \hookrightarrow & \Lambda & \longrightarrow & \Gamma. \end{array}$$

Now $\bar{\phi}$ is a surjection since $\eta \circ \phi$ is surjective. Since Γ is hopfian $\bar{\phi}$ is an isomorphism. Therefore $\phi(A) = A$. Hence A is characteristic in Λ. Since Γ has the R_∞-property by Theorem 4.1.1, Lemma 2.6.3 now implies that Λ has the R_∞-property.

Now assume that $f(A) \subseteq Z(\Gamma)$ Let $\bar{\Gamma} := \Gamma/Z(\Gamma)$ which is a lattice in $\Gamma/Z(\Gamma)$. Let $N = \eta^{-1}(Z(\Gamma))$. Clearly $N/A \cong Z(\Gamma)$. Now we have the exact sequence $N \overset{\widetilde{j}}{\hookrightarrow} \Lambda \overset{\bar{\eta}}{\to} \bar{\Gamma}$, where $\bar{\eta}$ is the canonical quotient map. Now we claim that N is characteristic in Λ. Indeed, let $\widetilde{f} : N \to \bar{\Gamma}$ be defined as $\bar{\eta} \circ \phi \circ \widetilde{j}$. Again using Margulis' normal subgroup theorem 2.1.2, the fact that N is virtually abelian forces $\widetilde{f}(N)$ to be contained in the centre of $\bar{\Gamma}$. Since $\bar{\Gamma}$ has trivial centre, we must have $\widetilde{f}(N) \subset N$. Now $\bar{\Gamma}$ is again hopfian (being finitely generated and linear). As before, we conclude that N is characteristic. By Lemma 2.6.3 applied to $\bar{\Gamma}$ we conclude that Λ has the R_∞-property. This completes the proof. □

4.3 S-ARITHMETIC LATTICES

In this section we recall the definition of S-arithmetic groups, referring the reader to [39] and [32].

Let k be an algebraic number field, that is a finite extension of $\mathbb{Q}$ and $\mathcal{O}_k$ (or $\mathcal{O}$) be the ring of integers of k. Let V (resp. V_∞) be the set of all valuations (resp. archimedean valuations) of k. Let S be a finite subset of V containing V_∞. Let $\mathcal{O}(S) = \{x \in k \mid |x|_v \leq 1, v \notin S\}$ denote the ring of S-integers in k.

For example, when $k = \mathbb{Q}$, the set V corresponds to the set of all primes $\{p_1, \cdots, p_n, \cdots\} \cup \{\infty\}$, where ∞ corresponds to the archimedean valuation of $\mathbb{Q}$. When $V \supset S = \{p_1, \cdots, p_n, \infty\}$, we have $\mathcal{O}(S) = \mathbb{Z}[1/p_1, 1/p_2, \cdots, 1/p_n]$ which we denote by $\mathbb{Z}(S)$.

Denote by k_v the completion of k with respect to the metric defined by the valuation v. When v is archimedean, k_v is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

Suppose that $\mathbf{G}$ is a linear algebraic group defined over k (or more briefly a k-group). We write G_ℓ for the ℓ-points of $\mathbf{G}$ where ℓ is an extension field of k and denote by the same symbol $\mathbf{G}$ the $\mathbb{C}$-points of $\mathbf{G}$. Set $G_v := G_{k_v}$, $G_S := \prod_{v \in S} G_v$, $G_\infty := G_{V_\infty}$. The group G_∞ is a Lie group whereas G_v is a locally compact totally disconnected group when v is non-archimedean. Thus G_S is a locally compact topological group and has a left invariant Haar measure, which is also right invariant if $\mathbf{G}$ is semisimple.

A subgroup $\Gamma \subset G_k$ is called an *S-arithmetic group* if there is a faithful k-morphism $r : \mathbf{G} \to \mathbf{GL}_n$ such that $r(\Gamma)$ is commensurable with $r(G)_{\mathcal{O}_S} := r(G_k) \cap GL(\mathcal{O}_k(S))$. It is known that any S-arithmetic group

$\Gamma \subset G_k$ in a semisimple group $\mathbf{G}$ defined over k is finitely generated ([39, Theorem 5.7]) and hence residually finite and hopfian. It is evident that $G_{O(S)}$ contains G_O. When $\mathbf{G}$ is connected and semisimple, the image of Γ under the diagonal imbedding $G_k \to G_S$ is a *lattice* in G_S defined by the imbedding of $k \to \prod_{v \in S} k_v$. That is, G_S/Γ has a finite G_S-invariant (regular) measure.

Suppose that $\mathbf{G}$ is defined over $\mathbb{Q}$ and is $\mathbb{Q}$-split, that is, $G_\mathbb{Q}$ has a $\mathbb{Q}$-torus $T_\mathbb{Q} \cong (\mathbb{Q}^\times)^l$ where $l = rank(\mathbf{G})$. Then $\mathbf{G}$ is k-split. We consider only S-arithmetic subgroups of the form $\Gamma = G_{O(S)}$. If σ is any automorphism of the field k which stabilizes S, then σ induces an automorphism of G_k which stabilizes $G_{O(S)}$. The group $Aut(\mathbf{G})$ is the semi-direct product of the group of inner automorphisms of $\mathbf{G}$ and the group $Out(\mathbf{G})$ of outer automorphisms. Thus $Out(\mathbf{G})$ may be viewed as a subgroup of $Aut(\mathbf{G})$. Furthermore, it may be arranged so that $Out(\mathbf{G})$ preserves the O-structure of G_k and so $Out(G)$ acts on $G_{O(S)}$. See [3] for details. For example if $\mathbf{G} = SL(n), n \geq 3$, then $Out(\mathbf{G}) \cong \mathbb{Z}/2\mathbb{Z}$ generated by $g \mapsto (g^t)^{-1}$.

We need the following theorem due to Borel [3, Theorem 4.3] which describes the automorphisms of $G_{O(S)}$. Let $Aut(k, S)$ denote the set of all automorphisms σ of the field k such that $\sigma(S) = S$.

Theorem 4.3.1. *(Borel [3]) Suppose that $\mathbf{G}$ is a connected simple group, defined and split over $\mathbb{Q}$. Suppose that $rank(\mathbf{G}) \geq 2$ or that $card(S) \geq 2$.*
(i) Suppose that the centre of $\mathbf{G}$ is trivial. Then $Aut(G_{O(S)})$ is generated by $Out(\mathbf{G})$, $Aut(k, S)$, and the inner automorphisms of $G_{O(S)}$.
(ii) Suppose that $\mathbf{G}$ is simply connected. Then $Aut(G_{O(S)})$ is generated by $Aut(k, S), Out(\mathbf{G})$, and automorphisms $\theta_{f,v}$ of the form $x \mapsto f(x)vxv^{-1}$, where $v \in N_\mathbf{G}(G_{O(S)})$ and $f : G_{O(S)} \to Z(G_k)$ is a homomorphism. □

Using Theorem 4.3.1 we obtain

Theorem 4.3.2. *We keep the notations and hypotheses of Theorem 4.3.1. Assume that $Out(\mathbf{G}) = 1$. Then the group $G_{O(S)}$ has the R_∞-property.*

Proof. First assume that $\mathbf{G}$ has trivial centre. By the residual finiteness of $\Gamma := G_{O(S)}$ in view of Proposition 2.6.4 it suffices to show that $\mathcal{R}(\phi)$ is infinite for a set of representatives ϕ of the outer automorphisms of Γ.

By the above theorem of Borel and our hypothesis that $Out(\mathbf{G})$ is trivial, and so it suffices to show that $R(\sigma) = \infty$ for $\sigma \in Out(\Gamma) \cong Aut(k, S) \subset Aut(k) < \infty$.

Let $n = o(\sigma)$, the order of σ. Suppose that $x, y \in \Gamma$ are fixed by σ and that $y = z^{-1}x\sigma(z)$ for some $z \in \Gamma$. Applying σ to both sides successively and using $\sigma(x) = x, \sigma(y) = y$, we obtain $y = \sigma^j(z^{-1})x\sigma^{j+1}(z)$ for $0 \leq j < n$. Multiplying these equations we obtain $y^n = z^{-1}x^n\sigma^n(z) = z^{-1}x^n z$. Thus y^n and x^n are conjugate in Γ.

To complete the proof that $R(\sigma) = \infty$ we need only show the existence of a sequence (x_m) of elements of Γ such that $\sigma(x_m) = x_m$ and x_r^n and x_s^n are pairwise non-conjugate in Γ.

Since $\sigma \in Aut(k, S) \subset Aut(k)$ restricts to the identity automorphism of $\mathbb{Q}$, it is clear that σ viewed as an element of $Aut(\Gamma)$ restricts to the identity automorphism of $\Gamma \cap G_{\mathbb{Q}}$. In particular $\sigma(x) = x$ for all x in $G_{\mathbb{Z}}$. Clearly $G_{\mathbb{Z}}$ is a lattice in $\mathbf{G}$. Consider the morphism $\psi : \mathbf{G} \to \mathbb{C}$ defined as $\psi(x) = \mathrm{tr}(Ad(x^n))$. Then $\psi(x) = \psi(y)$ if x^n and y^n are conjugate in $\mathbf{G}$. This morphism is clearly non-constant. Since $G_{\mathbb{Z}}$ is Zariski dense in $\mathbf{G}$, its image under ψ cannot be finite. Any sequence of elements $x_m \in G_{\mathbb{Z}} \subset \Gamma$ having pairwise distinct images under ψ clearly have the property that the x_m^n belong to pairwise distinct conjugacy classes in Γ.

It remains to consider the case when the centre $Z(\mathbf{G})$ is non-trivial. Since $\mathbf{G}$ is simple, $Z(\mathbf{G})$ is finite. Note that $Z(\Gamma)$, the centre of Γ, equals $Z(\mathbf{G}) \cap \Gamma$. This follows from the density property of $\Gamma \subset \mathbf{G}$. Hence $\bar{\Gamma} := \Gamma/Z(\Gamma) = \Gamma/(\Gamma \cap Z(\mathbf{G}) \subset \mathbf{G}/Z(\mathbf{G}) =: \bar{\mathbf{G}}$. It follows that $\bar{\Gamma} = \bar{\mathbf{G}}_{O(S)}$. Hence by what has been established already $\bar{\Gamma}$ has the R_∞ property. It follows by Lemma 2.6.3 that Γ has the R_∞-property. □

It is known that the outer automorphism group is trivial in the following cases (cf. [24, §6, Chapter X]: $\mathbf{SL}_2$, $\mathbf{Spin}(2n+1), n \geq 2, \mathbf{Sp}_n, n \geq 3$, and the exceptional groups $\mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4$ and $\mathbf{G}_2$. Theorem 4.3.2 is applicable to these groups $\mathbf{G}$ provided it is defined and split over $\mathbb{Q}$. However it leaves out the important case of special linear group $\mathbf{G} = \mathrm{SL}(n, \mathbb{C})$ as they have non-trivial outer automorphisms. We treat this case separately.

Theorem 4.3.3. *Let* $\Gamma = \mathbf{G}(\mathbb{Z}(S))$ *where* $\mathbf{G} = \mathbf{SL}_n/Z$ *where* $Z \subset Z(\mathbf{G})$. *Then* Γ *has the* R_∞*-property.*

Proof. Leaving out the case $n = 2$ which is already covered by Theorem 4.3.2, we have $Out(\mathbf{G}) \cong \mathbb{Z}/2\mathbb{Z}$ generated by the involution σ defined as $g \mapsto {}^t g^{-1}, g \in \mathbf{G}$. A direct verification (indicated below) as in Theorem 3.2.1 shows that $R(\sigma) = \infty$ when $\mathbf{G} = \mathbf{PSL}_n$. Since $Aut(\mathbb{Q}, S) = 1$, this already establishes the theorem in the case $\mathbf{G} = \mathbf{PSL}_n$. Note that $Z(\mathbf{G})$ is cyclic and is characteristic. Therefore any subgroup Z of $Z(\mathbf{G})$ is also characteristic. Again invoking Lemma 2.6.3 we get the result for all $\mathbf{G} = \mathbf{SL}/Z$.

To complete the proof, we exhibit matrices $[A_r] \in \Gamma = \mathrm{PSL}_n(\mathbb{Z}(S))$, $r \in \mathbf{N}$, which are in pairwise distinct σ-conjugacy classes. For convenience we work with matrices in $\mathrm{SL}_n(\mathbb{Z}(S))$.

As in Chapter 3, §3.2, consider the matrix $A_p = I_n + pE_{2,1}$ where E_{ij} is the matrix with (i, j)-th entry 1 and others zero. For any $X \in \Gamma$ and non-zero S-integers, the relation $A_p = XA_q\sigma(X^{-1}) = XA_q{}^tX$ implies, on comparing the $(2, 1)$-entry, that $p = q(x_{11}x_{22} - x_{12}x_{21})$ where $X = (x_{ij}) \in \Gamma$ that is $x_{ij} \in O(S)$. Reversing the roles of A_p and A_q we obtain $q = p(y_{11}y_{22} - y_{12}y_{21})$ for some $y_{ij} \in O(S)$. Since $x_{ij}, y_{ij} \in O(S)$, we conclude that p/q is an invertible element in $O(S)$. Now choose $p_r \in O_k$ such that $v(p_r) = r$ where $v \in S$, $r \in \mathbb{N}$. (Here $v(x)$ denotes the additive valuation of x). Then A_{p_r} and A_{p_s} are not σ-conjugate for $r \neq s$. This shows that $R(\sigma) = \infty$. □

We do not know how to extend Theorem 4.3.2 to *arbitrary* S-arithmetic groups in semisimple algebraic groups over an arbitrary number field. As mentioned in the introduction, the work of Fel'shtyn and Troitsky [14] establishes the R_∞ property for *all* S-arithmetic groups.

4.4 CONCLUSION

The aim of this thesis has been to study the R_∞-property for lattices in Lie groups. We achieved it for all irreducible lattices in non compact connected semisimple Lie groups with finite center and having real rank at least 2. Chapter 1 provided an overview of all of our results in this thesis.

After this research was completed, we came across an unpublished paper by A. Fel'shtyn and E. Troitsky [14] in which they announce that every finitely generated non-amenable residually finite groups has the R_∞-property. This result covers most of the known classes of groups having the R_∞-property including most of the lattices. Their proof uses the theory of C^*-algebra, which is entirely different from our techniques.

Recently T. Nasybullov [35] has established the R_∞-property for the groups $\mathrm{SL}(n,K)$ and $GL(n,K)$ where K is any infinite integral domain such that either (i) characteristic of K is zero and $Aut(K)$ is torsion, or, (ii) K has arbitrary characteristic and $Aut(K)$ is the trivial group.

We conclude this thesis with the following two open problems.

1) *Nilpotent groups:* Finitely generated nilpotent group of class 1 are abelian and are known to have automorphisms with only finitely many twisted conjugacy classes. Free nilpotent groups $N_{r,c}$ of finite rank r and nilpotency class c are studied by V. Roman'kov [45], D. Gonçalves and P. Wong [22]. An interesting example is the multiplicative group of $n \times n$ upper triangular matrices of integer entries with $1's$ on the diagonal. It is known that the Heisenberg group has automorphisms with exactly $2n$ many twisted conjugacy classes for any given n. Since finitely generated nilpotent groups are residually finite, they will have infinitely many twisted conjugacy classes for all inner automorphisms. To study the R_∞-problem of nilpotent groups, we only have to look for the infiniteness of twisted conjugacy classes for a set of representatives of each element of the group of outer automorphisms. The group of automorphisms of upper triangular matrices over integers has been determined. By a theorem of Mal'cev, a finitely generated torsion free nilpotent group (not virtually $\mathbb{Z}$) can be embedded in a nilpotent Lie group N in such a way that it is discrete and cocompact in N.

2) *Kähler groups:* For a Kähler group G (assuming not virtually abelian), that is, the fundamental group of a connected compact Kähler manifold, the abelianization $G/[G,G]$ is of even rank (Hodge). In particular, any inner automorphism has infinitely many twisted conjugacy classes and any

$\phi \in Aut(G)$ with $\#(Fix(\bar{\phi})) > 1$ will have infinitely many twisted conjugacy classes, where $\bar{\phi}$ is the induced automorphism of its abelianization. Thus we only have to check for a set of representatives of outer automorphisms.
Two theorems by Benson and Gordon say that if a nilmanifold G/Γ, where G is a nilpotent Lie group and Γ is a cocompact discrete subgroup, admits a Kähler structure, then G is abelian and G/Γ is diffeomorphic to a torus. Therefore finitely generated nilpotent groups are not covered under Kähler groups.

While nilpotent groups are amenable, it is well known that there exist non-residually finite Kähler groups.

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