Attractor Mechanism in Gauged Supergravity

By

Inbasekar Karthik

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The Institute of Mathematical Sciences, Chennai

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Inbasekar Karthik

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Inbasekar Karthik

DEDICATIONS

To my parents...

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SYNOPSIS

Theme of Thesis: One of the significant successes of string theory as a quantum theory of gravity is that it can give a statistical description of the thermodynamic black hole entropy via microstate counting and a macroscopic description via the attractor mechanism. In this thesis we explore both the descriptions. In the microscopic side we study the counting of certain class of BPS states in string theory. In the macroscopic side, we study a possible generalisation of the attractor mechanism suitable for extremal black brane horizons in gauged supergravity. We construct generalised attractors in gauged supergravity and investigate their stability.

String theory: [1] Superstring theory is one of the leading candidates for a quantum theory of gravity. The low energy limit of string theory gives effective theories of gravity coupled to matter fields called as supergravities. In Chapter 1, we discuss the basics of string theory and the recent developments.

Black holes microstate counting in string theory: In Chapter 2 of the thesis, we discuss black hole microstate counting in string theory. String theory has successfully given a statistical understanding of the thermodynamic Bekenstein-Hawking area law by counting microscopic degrees of freedom of certain supersymmetric extremal black holes [2]. We studied microstate counting for a class of states called twisted half-BPS states within the framework of CHL orbifold models when the twist generating group does not commute with the orbifold group [3] We find that the leading contribution to the degeneracy comes from the untwisted sector of the orbifold partition function.

Attractor mechanism in supergravity: In Chapter 3, we study the attractor mechanism in supergravity. String theory also gives a macroscopic understanding of the Bekenstein-Hawking entropy in the supergravity regime via the attractor mechanism [4–6]. Supergravity theories contain scalar fields known as moduli. In a given black hole background, the moduli fields in the theory may take arbitrary values at asymptotically flat spatial infinity and vary continuously. Nevertheless, the extremal black hole entropy is still given by the area law, with the area being a function of black hole charges. The independence of the Bekenstein-Hawking entropy of extremal black holes from the asymptotic values of moduli fields is explained by the attractor mechanism.

Black holes in AdS: In Chapter 4, we discuss black holes in AdS space, black brane limits and the Bianchi classification of five dimensional homogeneous extremal black brane horizons [7]. Bianchi type geometries have generalised translation symmetries which do not commute as opposed to usual translation symmetries for branes. The metrics are written in terms of invariant one forms as a result of which the geometries have constant anholonomy coefficients.

Gauged supergravity: In Chapter 5, we discuss the background material in gauged supergravity. Gauged supergravities are supersymmetry preserving deformations of ungauged supergravity. The deformations are implemented by promoting some of the global symmetries of the ungauged theory to local symmetries. Gaugings are usually done by coupling the symmetry generators to corresponding gauge fields. More recently, Gauged supergravities are understood as low energy effective theories that describe flux compactifications of string theory [8–10]. In the context of the AdS/CFT correspondence [11], gauged supergravity generically describes the supergravity regime of the bulk theory. In this thesis, we focus mainly on five dimensional $\mathcal{N} = 2$ gauged supergravity coupled to arbitrary vector, tensor and hyper multiplets [12, 13].

Generalised attractors in gauged supergravity: In chapter 6, we discuss the generalised attractors in gauged supergravity and construct explicit examples of Bianchi attractors from specific models. The generalised attractors are defined as solutions to equations of motion that reduce to algebraic equations, when all fields and curvature tensor components are constants in tangent space. The attractor geometries are characterised by constant anholonomy coefficients and include planar solutions such as black branes and domain walls. A general analysis of attractors with constant anholonomy coefficients in $\mathcal{N} = 2$ gauged supergravity in four dimensions has been carried out in [14]. Such gauged supergravity theories are also known to admit Lifshitz [15] as well as Schrodinger [16] type solutions which belong to Bianchi type I geometry.

In [17] we have studied generalised attractors in $\mathcal{N} = 2$ gauged supergravity theories in five dimensions coupled to arbitrary number of vector, hyper and tensor multiplets. We analysed the equations of motion of the theory and showed that the field equations become algebraic at the attractor points. We obtained an attractor potential from the scalar field equations and further showed that the attractor potential can be independently constructed from generalised fermionic shifts. The generalised attractors in five dimensional gauged supergravity include near horizon geometries of extremal black branes with homogeneity in spatial directions [7]. We considered a simple gauged supergravity model [18, 19] with one vector multiplet and constructed some explicit examples of such Bianchi attractors. In particular, we constructed a $z = 3$ Lifshitz solution, Bianchi type II and Bianchi Type VI solutions and argued that Bianchi type III and type V geometries do not exist in the model considered. In the thesis, we also present an additional example of a two charge Bianchi type I ($AdS_2 \times \mathbb{R}^3$) solution not discussed in the paper. In [20], we explore different gauged supergravity models, including models in the hypermultiplet sector to embed all the Bianchi type metrics in gauged supergravity.

Stability of generalised attractors in gauged supergravity: In Chapter 7, we discuss the stability of generalised attractors. We have considered gauge invariant scalar fluctuations about the attractor value and investigated the stability of electrically charged generalised attractors. We find that the stress energy tensor in gauged supergravity linearly depends on scalar fluctuations even at the first order perturbation due to the interaction terms of the theory. Stable attractors in this theory would be those with scalar fluctuations which die out as one approaches the horizon. In particular, if the fluctuations diverge as one approaches the horizon the corresponding geometry would suffer infinite backreaction, thereby signalling an instability. We find that the maximally symmetric black brane geometry, namely $AdS_2 \times \mathbb{R}^3$ is the most stable generalised attractor. The result of the stability analysis [21] is reported in the thesis.

Conclusion: In Chapter 8, we conclude and discuss future directions. In this thesis, we focussed on the microscopic state counting in string theory and a generalisation of the attractor mechanism to gauged supergravity. In the microscopic side, we studied the counting of a class of twisted BPS states in CHL models. In the macroscopic side we have provided some evidence that universal features of the attractor mechanism in supergravity also extend to gauged supergravity.

List of Publications:

This thesis is based on the following publications.

- **Published**

  1. Karthik Inbasekar and P. K. Tripathy, "Generalized Attractors in Five Dimensional Gauged Supergravity," *JHEP* **1209** (2012) 003, arXiv:1206.3887 [hep-th].

- **Submitted**

  1. Karthik Inbasekar and P. K. Tripathy, "Stability of Bianchi attractors in Gauged Supergravity," arXiv:1307.1314 [hep-th].

- **Preprints**

  1. S. Govindarajan and Karthik Inbasekar, "A non-commuting twist in the partition function," arXiv:1201.1628 [hep-th].

Contents

Chapter 1

Introduction

It is an exciting time in the particle world. The missing piece in the standard model of particle physics [22–25], the Higgs boson is almost certainly found [26], nearly fifty years after its original proposal. With this the standard model describes successfully, the electromagnetic, strong and weak interactions of the particle world. No wonder the standard model of particle physics is sometimes called as the theory of almost everything.

There are several phenomena the standard model fails to explain or account for like matter antimatter asymmetry, neutrino masses, and no explanation for the expanding universe. Then there are theoretical inconsistencies and expectations. One glaring missive is that the standard model does not include gravitational interactions. On one hand it is justifiable to not include gravitational interactions in the standard model, since gravity is much weaker than the other fundamental forces at the Gev scale. However if one does try to include gravitational interactions, one finds that the standard model is incompatible with general relativity(GR)–our best understood theory of classical gravity.

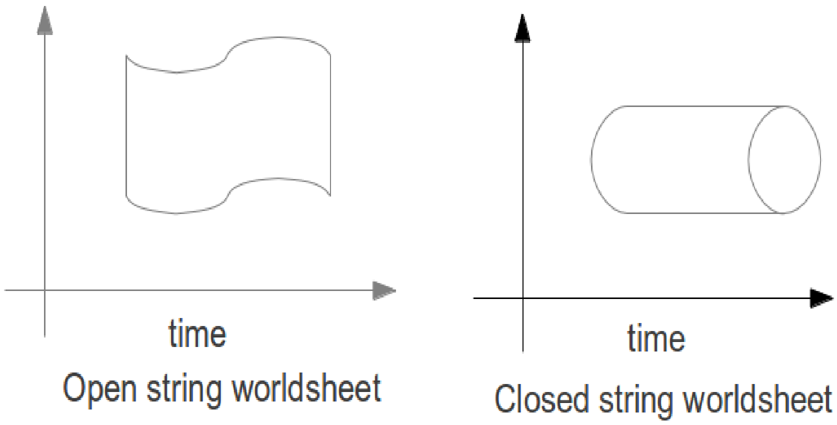
The question is why should one include gravitational interactions in the standard model. The physical situations where one expects gravitational effects to be dominant over the other fundamental forces are when one deals with high energies and small distances such as the origin of the universe or the study of black holes. In these situations the quantum effects of gravity are expected to play a major role. Interestingly, gravity in four dimensions by itself cannot be quantised in the framework of quantum field theory since it is non-renormalisable. This is easy to see as the Newtons coupling constant G is dimensionful unlike the coupling constants of the other fundamental forces. Thus, we need a new framework compatible with quantum mechanics and general relativity to describe unified interactions of the fundamental forces. This framework should also necessarily be a quantum theory of gravity.

1.1 String theory

One of the leading candidates for a quantum theory of gravity is string theory [1, 27–29]. Here the fundamental objects are not point particles but one dimensional strings. There are two kinds of strings, open and closed. The size of the strings l_s is of the order of Planck length $10^{-35}m$. Therefore, strings are not observable at the energy scales at which the current particle accelerators can operate. Note that this minimum length scale acts as a UV cutoff, and thus stringy interactions are free from short distance divergences which plague point particle interactions.

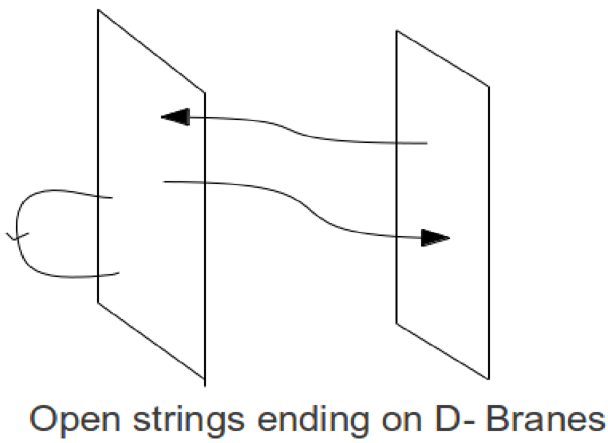
A point particle is zero dimensional and hence its motion in time is described by a one dimensional world line. While the string is a one dimensional object, and its propagation in space time is described by a two dimensional surface called as the worldsheet (fig 1.1). One then writes a worldsheet action for the strings and studies the quantum mechanics in a Poincáre invariant way. One of the surprising results of the quantisation of the closed string is that the low energy massless excitation contains a spin two particle, which is consistent with the properties of a graviton–the force carrier for the gravitational interaction. Thus studying the quantum mechanics of a relativistic string has already made a prediction–Gravity!

Figure 1.1: World Sheet description



Open strings can end on solitonic objects in string theory known as D-branes (fig 1.2). The D stands for the Dirichlet boundary conditions one puts on the end points of the string and brane is a short for membrane like higher dimensional objects.

Figure 1.2: Strings ending on branes. The arrows represent the orientation of the string



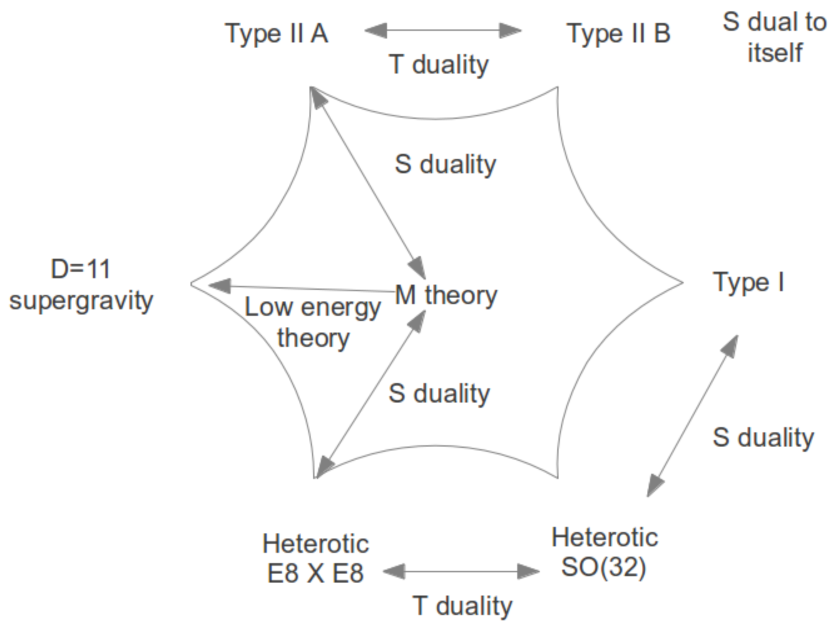
The low energy massless excitations of the open strings contain gauge fields. Each string starting and ending on the same brane gives rise to one massless $U(1)$ gauge field. All other string states are massive. Interestingly the masses of the other gauge fields are proportional to the distance of separation of the branes. For example in figure 1.2, initially the gauge symmetry is $U(1) \times U(1)$, but when the branes coincide the gauge symmetry is enhanced to $U(2)$. Similarly, when there are N coincident branes one gets a $U(N)$ gauge group. Inverting the above argument, starting from N coincident D branes, separating them makes some of the gauge fields massive reducing the gauge group to $U(1)^N$. This is a stringy realisation of the Higgs mechanism in particle physics. In addition to the excitations discussed above, all string theories have infinite tower of massive excitations which will become relevant at high energies of the order of Planck scale. At low energies, these excitations are invisible to modern day experiments. All the features that we have discussed so far are common to bosonic and superstrings, which we will discuss shortly. However bosonic string exists in 26 space-time dimensions and has a tachyonic particle in its spectrum. It is not known whether the bosonic string is stable by itself, while superstrings are free from tachyonic instabilities.

Fermions can be obtained in string theory by adding fermionic coordinates to the world-sheet theory. These string theories are supersymmetric and are called superstrings. Supersymmetry, which was also studied independently of string theory relates the bosons and fermions in a theory. It turns out that there are five consistent superstring theories named as type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. All these theories exist in ten dimensions. To obtain a realistic four dimensional theory, six of the spatial dimensions are assumed to be compact with size of the order of Planck scale. Of the five superstring theories: Type I, Type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$ contain chiral fermions in the spectrum. They also have different degree of worldsheet supersymmetry. For example, type I, heterotic $SO(32)$, and heterotic $E_8 \times E_8$ have $\mathcal{N} = (1, 0)$, type IIB has $\mathcal{N} = (2, 0)$ while type IIA has $\mathcal{N} = (1, 1)$ supersymmetry.

At the first sight, all the different superstring theories appear to be very different from one another. However, they are related to each other by dualities in string theory. For example, T-duality [30] (T for target space) relates a theory compactified on a circle of radius R and a circle of radius l_s^2/R. Under this symmetry, the perturbative spectrum of these two theories become identical and hence they are one and the same. Type IIA and type IIB theories are T dual to each other, so are the two heterotic theories.

Another more intriguing duality is the S duality (strong-weak duality) [31]. String theories have a coupling constant g_s, defined as the expectation value of the dilaton $g_s = exp < \phi >$. S duality relates a theory at strong coupling g_s, to a theory at weak coupling $1/g_s$. The well known examples are the strongly coupled Type I theory which is dual to the weakly coupled heterotic $SO(32)$ theory and the Type IIB theory which is dual to itself. As one can see this duality is non-perturbative in nature. Type IIA theory and heterotic theory each at strong coupling g_s, grow an extra dimension of size $g_s l_s$, which suggests that they are S dual to a mysterious 11 dimensional theory called as M theory (fig 1.3) .

Figure 1.3: The bigger picture



This theory is non-perturbative, far from well understood and is a subject of active research. The low energy effective theory of M theory is the well studied 11 dimensional supergravity [32].

1.2 AdS/CFT correspondence

One of the most recent and exciting developments in string theory is the AdS/CFT correspondence [11, 33, 34]. It is a concrete realisation of the holographic principle which is a property expected of any quantum theory of gravity [35–37]. Holography means that the information describing a volume of space is encoded in the boundary of the volume. The AdS/CFT conjecture does this by relating a string theory on anti de-Sitter (AdS) space and a conformal field theory (CFT) living on the boundary of AdS. More precisely, the AdS/CFT conjecture states that the following two theories are equivalent.

- $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory in four dimensions with a $SU(N)$ gauge group.
- Type IIB string theory on $AdS_5 \times S^5$.

The rank of the $U(N)$ gauge group is related to the integral flux of the five form field strength in the string theory by $N = \int_{S^5} F_5$. The coupling constant of the SYM theory g_{YM}^2 is related to the string coupling constant g_s through $g_{YM}^2 = g_s$. The curvature R of the AdS space is related to N by $R^4 = 4\pi g_s N \alpha'^2$, where $\alpha' = l_s^2$. This is an example of a open-close duality, the fields in the $SU(N)$ SYM theory are massless open string modes, while the five form field strength is generated by massless closed string modes.

Actually, the above statement is the "strong" form of the conjecture. String theories on Ramond-Ramond sectors are poorly understood as yet. A more refined and practically useful form of the conjecture is obtained by taking two further limits. The first one is the 't Hooft limit, by keeping $\lambda = g_{YM}^2 N = g_s N$ fixed and taking the large N, $N \to \infty$

limit. In this limit, the SYM theory reduces to the planar limit as non-planar diagrams are suppressed by a factor of $1/N^g$, where g is the genus of the surface. In the 't Hooft limit, the full string theory reduces to a classical string theory since $g_s \to 0$.

In addition to the 't Hooft limit, a further simplification can be made by taking $\lambda \to \infty$. In this limit the SYM becomes strongly coupled and is in the non-perturbative regime, whereas the string theory reduces to supergravity. This is because the low energy effective action has terms proportional to $\alpha'^2 = (1/\lambda)$. In the large λ limit, the stringy terms do not survive and the theory reduces to the massless modes described by the supergravity action.

In this "weak" but practically more useful form, the AdS/CFT conjecture states that: $\mathcal{N} = 4, d = 4, SU(N)$ SYM in the limits $N \to \infty, \lambda \to \infty$ is dual to Type IIB supergravity on $AdS_5 \times S^5$. This strong-weak version of the duality allows one to map difficult to solve strongly coupled problems in the field theory side to weakly coupled gravity side where it may be solved.

1.3 Black holes and String theory

Black holes are regions in space time where the gravitational fields are strong enough to not let any particle or signal escape the region. The boundary of this region is called an event horizon. Black holes are very much physical and can be formed from collapse of stars, galactic collisions or possibly even during the big bang [38].

Theoretically, black holes are interesting objects to study as they are the situations where one expects quantum effects of gravity to play a dominant role. Due to the existence of an event horizon, any information pertaining to the interior of the black hole is inaccessible by an external observer. Only extrinsic quantities such as the total mass M, charge Q and angular momentum J are observable. Therefore for every specific choice of these variables there are several possible ways in which the black hole could have formed. The situation is completely analogous to the thermodynamic (macroscopic) description of a physical system with many internal (microscopic) states. Since a black hole hides information about the various possible microstates behind the horizon, it is natural to associate an entropy and temperature with the black hole [39] given by (in Planck units),

$$T_{BH} = \frac{\kappa}{2\pi}\,, \quad S_{BH} = \frac{A}{4}\,. \tag{1.3.1}$$

where κ is the surface gravity of the black hole and A is the area of the event horizon. Since the only information accessible to an observer is the mass, charge and angular momentum, the area and surface gravity should be a function of these variables. Both these quantities are properties of the full black hole, but calculated using quantities defined at the boundary (event horizon). Thus the information of the various possible ways in which the black hole could have formed is actually embedded in the boundary of the black hole. This is a realisation of the holographic principle.

Thus, we have analogous laws of black hole thermodynamics [40]. The zeroth law states that for stationary black holes κ is constant over the horizon. This is reminiscent of the similar law in thermodynamics, which states that a system in thermodynamic equilibrium has uniform temperature. The first law of thermodynamics is a statement of conservation of energy. The analogous statement, for example in an electrically charged, rotating black hole is ,

$$dM = \frac{\kappa}{8\pi}dA + \Omega dJ + \Phi dQ\,, \tag{1.3.2}$$

where the extrinsic variables are the Area A, charge Q, mass M ,and angular momentum J. The locally defined quantities κ, angular velocity Ω and electrostatic potential Φ are constants over the horizon. As can be seen from (1.3.2) the Area A of the event horizon plays the role of entropy and consequently the analogue of the second law in black hole thermodynamics is that the area of the black hole is an increasing monotonic function in time. This is Hawking's area increase theorem which assumes the weak energy condition and is valid classically. Quantum mechanically black holes can radiate via pair production at the horizon [41] ,thereby decreasing both their mass and entropy. The analogue of the third law for black holes is not well understood, and it roughly states that a black hole with vanishing surface gravity cannot be achieved in a finite number of steps.

The black holes with vanishing surface gravity have a special name. They are called extremal black holes. For a given charge Q and angular momentum J, the extremal black holes have the smallest possible mass. A general electrically charged, rotating black hole has the surface gravity and area as a function of the extrinsic variables given by,

$$\kappa = \frac{4\pi\mu}{A}, \quad A = 4\pi(2M(M+\mu) - Q^2)\,, \quad \mu = \sqrt{M^2 - Q^2 - \left(\frac{J}{M}\right)^2}\,. \tag{1.3.3}$$

The parameter μ is called an extremality parameter. For the value $\mu = 0$, we see that the temperature of the black hole vanishes, but there is still finite non zero entropy! Another interesting feature of extremal black holes is the near horizon geometry. In particular, for four dimensional extremal black holes, the near horizon geometry factorises into $AdS_2 \times S^2$ [42,43]. As it turns out, extremal black holes play a very important role in the understanding of black hole entropy in string theory.

One of the significant successes of string theory is the understanding of the Bekenstein-Hawking area law (1.3.1), for extremal black holes both from a macroscopic and microscopic perspective. The macroscopic perspective helps us understand the thermodynamics and the microscopic study gives a statistical description, both of which is necessary for a complete understanding of black hole physics.

The first example of microstate counting in string theory was the calculation of the Bekenstein - Hawking entropy for five dimensional extremal black hole solutions by counting the microscopic degeneracy of BPS states [2]. BPS states are short representations of extended supersymmetry algebras, and it can be shown that the BPS condition implies extremality $\mu = 0$. The BPS property of the supersymmetric black hole ensures that the microstates states do not jump discontinuously as one smoothly varies the moduli or coupling constants of the theory [44]. Hence, one can reliably count the microscopic states having a fixed charge at weak coupling and analytically continue to the strong coupling regime, where the same set of states are described by a black hole solution in the theory. Since the advent of the Strominger-Vafa computation, the microscopic counting has been carried out in a variety of supersymmetric theories and entropy agrees with the black hole entropy in the large charge limit [45].

String theory also gives the macroscopic understanding of the Bekenstein Hawking area law through the attractor mechanism in supergravity [4–6]. The moduli fields in a given black hole background flow radially to a fixed charge dependent value at the horizon, regardless of their asymptotic values. As a result, the entropy of the black hole is just a function of the black hole charges and is independent of the asymptotic value of the moduli. We saw earlier that the asymptotic observer has access only to the charge, mass and angular momentum, and hence any quantity associated with the black hole must be functions of these variables. The attractor mechanism achieves precisely this remarkable feature.

1.4 Scope of the Thesis

The broad area of study in this thesis is an exploration of both the microscopic and macroscopic description of black holes in string theory. In the microscopic side, we study the counting of certain class of BPS states called twisted BPS states in string theory. In the macroscopic side, we explore a generalisation of the attractor mechanism in gauged supergravity that may aid the understanding of asymptotically AdS black branes.

In the microscopic side, we have explored the microstate counting of a special class of $1/2$ BPS states in $\mathcal{N} = 4$ supersymmetric theories known as CHL models [17]. These special class of states are called twisted BPS states, where the twist is generated by a group which does not commute with the orbifolding group. Orbifolding reduces the number of states due to the group invariant projections and the twist generating group further reduces the invariant states. Therefore one would expect the weights of the generating function for the degeneracy of twisted BPS states to be lesser than the untwisted case. We find that this is indeed true. This work may be useful in understanding counting of states in a non-supersymmetric setting as it is possible to break supersymmetry using the twist generators.

The microscopic and macroscopic understanding of black holes in string theory has been limited to black holes in asymptotically flat spaces. An interesting extension of this study would be to pursue the generalisation to curved spaces, in particular AdS space. This would be very interesting from the AdS/CFT perspective and may shed light on the understanding of string theory on AdS spaces. Since perturbative string theory on $AdS_5 \times S^5$ is still a work in progress, microstate counting of string theories on these background is quite a challenging task. However, one can still pursue the macroscopic study since supergravities with AdS vacuua have been well understood. The supergravities which support AdS vaccua are known as gauged supergravities and they describe the supergravity regime of the AdS/CFT correspondence [11]. Gauged supergravity theories are also of interest in string theory because of their connection to the low energy effective theory that describes string compactifications in the presence of fluxes [9, 10].

Black holes in AdS spaces are interesting in their own regard. In the AdS/CFT correspondence, AdS black holes are dual to field theories at finite temperature [11]. In the application of AdS/CFT to condensed matter theories, charged extremal black branes are duals to the zero temperature phases of the field theory. These zero temperature states are very interesting from the field theory perspective as they exhibit phase transitions due to quantum fluctuations [46]. Typical examples are black holes with Lifshitz like near horizon geometries and AdS asymptotics that are duals to field theories with a violation of Lorentz symmetry [47–52].

Recently a classification of such black brane horizons which are homogeneous but not isotropic has also been carried out in [7, 53]. These near horizon geometries, known as the Bianchi attractors, have a richer structure, exhibit scale invariance and are characterised by constant anholonomy coefficients. Some of these geometries have been shown to numerically interpolate to AdS_5. In this thesis, our goal is to generalise the attractor mechanism to gauged supergravity with the end points of the attractor flow being the Bianchi type geometries.

There are various recent approaches to generalise the attractor mechanism in gauged supergravity [54–60]. A prescription has been given to obtain some generalised attractor geometries such as the Lifshitz solutions from $\mathcal{N} = 2, d = 4$ gauged supergravity [14]. In this framework, one sets all the fields and the curvature components to constants in the tangent space. Following this prescription, we extended the study of generalised attractors to $\mathcal{N} = 2, d = 5$ gauged supergravities with arbitrary matter content. We also considered an explicit gauged supergravity model and embedded some of the Bianchi attractors in five dimensional gauged supergravity [17]. Our approach did not rely on supersymmetry but rather on extremization of an attractor potential. This method is generic and in principle could include non-supersymmetric attractor solutions. Hence, we have investigated the stability of the generalised attractors under scalar fluctuations about the attractor value and obtained conditions for stability [21].

Our analysis indicates that there are several possible end points for an attractor flow in five dimensional gauged supergravity. Our stability criteria points out that a sub class of Bianchi attractors, whose symmetry groups factorise into a direct product form stable attractors. One of the most important things which we would have liked to include is to construct and solve the full flow equation thereby proving the attractor mechanism in gauged supergravity. However, this requires analytical black brane solutions that interpolate between AdS_5 and the Bianchi horizons, which seems to be a much harder task and is beyond the scope of this thesis. Nevertheless we do find that the generalised attractor procedure in gauged supergravity captures many important features of the attractor mechanism that occurs in ungauged supergravity. The attractor equations are determined by extremising an attractor potential. The field equations are algebraic at the attractor points and the moduli are determined in terms of the charges. The symmetry groups of the stable attractors split into a direct product form and exhibit scale invariance only along the radial and time directions. This parallels the situation for near horizon geometries of four dimensional extremal black holes.

1.5 Organisation of the Thesis

In chapter 2, we present our work on microstate counting of twisted 1/2 BPS states in $\mathcal{N} = 4$ supersymmetric theory [3].

Chapters 3, 4, 5 are review material on the attractor mechanism in supergravity, black holes in AdS spaces including the Bianchi attractors and gauged supergravity respectively.

In chapter 6, we present our work on generalised attractors in five dimensional gauged supergravity [17] and in chapter 7, we present our results on the stability analysis [21].

Finally, we summarise and conclude with some open questions in chapter 8.

Chapter 2

Black hole microstate counting in string theory

2.1 Introduction

In this chapter, we give a flavor of microscopic state counting in string theory. In addition to the prediction of black hole entropy, microscopic counting in four-dimensional string theories with $\mathcal{N} = 4$ supersymmetry has turned out to have a surprisingly rich structure [61, 62]. This has provided connections to modular forms, Lie algebras [63, 64] as well as sporadic groups [65, 66]. Due to the large amount of supersymmetry, these theories work as "laboratories" for us to test ideas that presumably should continue to work in situations with fewer supersymmetries. In this chapter, we do a simple counting of microscopic states called as twisted BPS states in string theory. We set up the counting problem in theories with $\mathcal{N} = 4$ supersymmetry, where the twist does not commute with the orbifolding group.

We consider four dimensional CHL $\mathbb{Z}_n$-orbifolds with $\mathcal{N} = 4$ supersymmetry [67, 68] . These models are asymmetric orbifolds [69, 70] constructed by starting with a heterotic string compactified on a $T^4{\times}S^1{\times}\tilde{S^1}$ and then quotienting the theory by a $\mathbb{Z}_n$ transformation which involves a $1/n$ shift along the $\tilde{S^1}$. The $\mathbb{Z}_n$ symmetry has a non-trivial action on the internal conformal field theory coordinates describing the heterotic compactification on T^4. A large class of such models were constructed in [71, 72] and were shown to be dual to a type II description compactified on $K3 \times S^1 \times \tilde{S^1}$ via string-string duality [73, 74].

By construction, CHL models possess maximal supersymmetry and fewer massless vector multiplets at generic points in the moduli space. The requirement of maximal supersymmetry restricts one to consider symplectic automorphisms on $K3$. Symplectic automorphisms leave the holomorphic $(2,0)$ forms invariant and hence preserve supersymmetry. The action of these symmetries have fixed points on the $K3$ surface and is accompanied by translations on the circle to avoid quotient singularities. So the allowed groups must faithfully represent translations in $\mathbb{R}^2$ which implies that the quotienting group has to be abelian [75]. The possible abelian groups that act symplectically on $K3$ were classified and the action of the group on the $K3$ cohomology was calculated [76]. Once the action on the cohomology is determined one uses string-string duality to map the action to the Heterotic side. The map is allowed provided the supergravity side is free from fixed points, i.e the action on $K3$ must be accompanied by shifts on the torus.

The work of Mukai [77], opened up the possibility that non-abelian groups can act as symplectic automorphisms on the $K3$ surface. Recently, Garbagnati [78] constructed elliptic $K3$ surfaces that admit dihedral group as symplectic automorphisms. These automorphisms are constructed by combining automorphisms which act both on the base and the fiber such that the resulting action is symplectic. In particular , [78] determined the ranks of the invariant sublattice and the orthogonal complement and identified the orthogonal complement to the invariant sublattice with the lattices in [79]. However, for compactifications down to four dimensions one cannot quotient by a non-abelian group since these groups do not represent translations faithfully. However, one can consider the theory to be on special points in the moduli space that admit non-abelian symmetries and quotient by the commutator subgroup, which is abelian.

We consider the CHL $\mathbb{Z}_n$-orbifold models ($3 \leq n \leq 6$) at special points in the moduli space where they admit dihedral $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ symmetry[1].The $\mathbb{Z}_n$ subgroup is the commutator subgroup of the D_n group and may be quotiented. The special points in moduli space are specified by the elliptic $K3$ surfaces that admit $D_n, 3 \leq n \leq 6$ symmetries constructed in [78]. Since the action of $\mathbb{Z}_n$ group is known on the $K3$ side, we map it to the heterotic string using the string-string duality. We then construct the CHL $\mathbb{Z}_n$ orbifold in the heterotic picture and let the additional $\mathbb{Z}_2$ symmetry act as a twist in the partition function of the orbifolded theory. These twist symmetries are identical to the ones considered in [71,72] but without shifts. For $\mathcal{N} = 4$ supersymmetry to be preserved these twists must commute with all the unbroken supersymmetries of the theory. Such twists have been considered in the $g \in \mathbb{Z}_n$ twisted partition function [80] for unorbifolded theories, which counts the index/degeneracy[2] of elementary string states when the theory is restricted to special points in moduli space. The g-twisted helicity index is defined as,

$$B^g_{2m} = \frac{1}{2m!}\mathrm{Tr}[g(-1)^{2\ell}(2\ell)^{2m}]\,, \tag{2.1.1}$$

where g generates a symmetry of finite order, ℓ is the third component of angular momentum of a state in the rest frame, and the trace is taken over all states carrying a given set of charges. States which break less than or equal to $4m$ g-invariant supersymmetries give non-vanishing contributions to B^g_{2m} [80]. For the case of $1/2$ BPS states that we consider in this chapter, the relevant index is B^g_4.

For our case, the choice of the moduli space that has dihedral symmetry is compatible with the $g \in \mathbb{Z}_2$ twist. The other requirement that the physical charges have to be g invariant is met by requiring the charges Q to take values from lattices invariant under Dihedral symmetry [78, 79]. This choice is also compatible with the orbifold action, since these

[1] In our notation, D_n is the dihedral group of order 2n, see §2.3.

[2] Both are identical for the cases considered in this chapter.

lattices possess invariance under both $\mathbb{Z}_2$ and $\mathbb{Z}_n$ actions. Thus one meets the requirements for the twist and orbifold action to be well defined.

We count the degeneracy of electrically charged $1/2$ BPS elementary string states for a fixed charge Q in these theories following the method described in [81]. The $\mathbb{Z}_2$ twisted partition function in the $\mathbb{Z}_n$ orbifold theories receives contribution only from the orbifold untwisted sector for odd n and additionally from the orbifold sector twisted by the element $h^{n/2}$ for even n. From the point of view of the dihedral group, for even n, the element $h^{n/2}$ is a nontrivial center of the group and commutes with every element. We derive a generating function for these degeneracies and find that it has the expected asymptotic limit.

The Chapter is organised as follows. In section §2.2, we discuss the relation between the twisted index and the black hole entropy for the abelian twists. Subsequently, in §2.3, we give a pedagogical introduction to non-abelian orbifolds and define the twisted partition function to indicate the contributing orbifold twisted sectors. We discuss the construction of CHL $\mathbb{Z}_n$ orbifolds in the heterotic picture and the derivation of the half-BPS degeneracies of $g \in \mathbb{Z}_2$ twisted BPS states in §2.4. We conclude with a summary of our results in §2.5.

2.2 Twisted index and black hole entropy

In this section, we briefly review the relation between the twisted index and black hole entropy for abelian twists. We consider type IIB theory compactified on $K3 \times S^1 \times \tilde{S}^1$ which gives rise to $\mathcal{N} = 4$ supersymmetric theory in four dimensions. As described in the introduction of this chapter we go to special points on the moduli space where the theory has enhanced discrete $\mathbb{Z}_N$ symmetries such that $g^N = 1$. These symmetries are assumed to leave the holomorphic $(2, 0)$ form on $K3$ invariant and hence commute with the supersymmetries. In other words, these twists preserve the supersymmetry.

We are interested in the counting of dyonic supersymmetric states which preserve $1/4$ of the $\mathcal{N} = 4$ supersymmetry. The index (2.1.1) captures information of g invariant states which break $4m$ supersymmetries. The $1/4$ BPS states preserve 4 of the 16 supersymmetries in the $\mathcal{N} = 4$ theory and the relevant index is then B_6. The index is usually written as a Fourier transform of the partition function. Remember that we are in the weak coupling regime where the states in question have not formed a black hole yet. In the weakly coupled type IIB regime, the low energy physics is dominated by [45],

- Excitation modes of the Kaluza-Klein (KK) monopole,
- Center of mass motion of the D1-D5 brane system in the KK background,
- Motion of the D1 branes relative to the D5 brane.

The full partition function of the theory is a direct product of all the above contributions [80],

$$Z^g(\rho, \sigma, v) = Z_{KK} Z_{cm} Z_{D1D5} = \frac{1}{\Phi(\rho, \sigma, v)} , \tag{2.2.1}$$

where Φ is a Siegel modular form given by,

$$\Phi(\rho, \sigma, v) = e^{2\pi i(\rho+\sigma+v)} \prod_{b=0}^{1} \prod_{r=0}^{N-1} \prod_{\substack{k,l\in\mathbb{Z}, j=2\mathbb{Z}+b, \\ k,l\geq 0, j<0 \text{ for } k=l=0}} (1 - e^{2\pi i r/N} e^{2\pi i(k\sigma+l\rho+jv)})^{\sum_{s=0}^{N-1} e^{-2\pi i rs/N} c_b^{(0,s)}(4kl-j^2)} , \tag{2.2.2}$$

where c_b are Fourier coefficients, N is the order of the orbifold group. The index is expressed as a complex integral of the partition function as,

$$B_6^g(Q, P) = (-1)^{Q.P} \int_C d\rho d\sigma dv e^{-\pi i(P^2\rho+Q^2\sigma+2(Q.P)v)} Z^g(\rho, \sigma, v) , \tag{2.2.3}$$

where Q and P are electric and magnetic charges of the dyonic states. The combinations $(Q^2, P^2, Q.P)$ are the only T duality invariants of the theory. This can be seen as follows, the type IIB string theory on $K3 \times S^1 \times \tilde{S}^1$ is dual to $E_8 \times E_8$ heterotic string theory on T^6 [75]. The heterotic theory has 28 $U(1)$ gauge fields from the Cartan generators of the $E_8 \times E_8$ group, and from the metric and the antisymmetric B field along the six compact directions. A generic state in the theory is characterised by a $(28,28)$ dimensional charge vector pair $(\vec{Q}, \vec{P})$. These charges transform as vectors under the T-duality group $O(22,6,\mathbb{Z})$, and are restricted to take integer values such that [82],

$$gcd(Q_i P_j - Q_j P_i) = 1 \;, \quad 1 \leq i, j \leq 28 \;. \tag{2.2.4}$$

The integral (2.2.3) is over the complex plane and gets leading contributions from poles of the partition function or equivalently the zeroes of the Siegel modular form (2.2.2) [81],

$$n_2(\rho\sigma - v^2) - m_1\rho + n_1\sigma + m_2 + jv = 0 \;, \tag{2.2.5}$$

where,

$$m_1, n_1, m_2 \in \mathbb{Z} \;, \quad n_2 \in N\mathbb{Z} \;, \quad j \in 2\mathbb{Z} + 1 \;, \quad m_1 n_1 + m_2 n_2 + \frac{1}{4} j^2 = \frac{1}{4} \;. \tag{2.2.6}$$

The asymptotic behaviour of the index (2.2.3) for large charges is controlled by the zeroes (2.2.6) of the Siegel modular form for $n_2 \geq 0$. The smallest of which is given by $n_2 = N$, for this value the logarithm of the index has the form,

$$\ln |B_6^g(Q,P)| = \frac{\pi}{N} \sqrt{Q^2 P^2 - (Q.P)^2} = \frac{S_{BH}}{N} \;, \tag{2.2.7}$$

where S_{BH} is the entropy of a dyonic black hole [81].

2.3 Non-abelian orbifolds

In this section, we describe the standard CFT approach for constructing the twisted partition function in non-abelian orbifold theories. For a general description of orbifolds in string theory see [83–89]. For some phenomenological model building approaches based on non-abelian orbifold string theories see [90, 91]. Orbifold CFT's are generally constructed by considering a modular invariant theory $\mathcal{T}$, whose Hilbert space admits a finite discrete symmetry G consistent with the allowed interactions of the theory, and constructing a quotiented theory $\mathcal{T}/G$ that is also modular invariant. When G is an abelian symmetry, the quotient theory can be constructed by modding out the full group. Whereas, when G is non abelian, the quotient group corresponds to the stabiliser group of G, which contains only the commuting elements of G. For example, consider Dihedral groups D_n of order $2n$. The quotienting group is the cyclic group $\mathbb{Z}_n$ of order n.

Before proceeding further, it is useful to define some notations. Let us denote the world-sheet coordinate as $X(\tau, \sigma)$, with τ and σ being the "space" and "time" directions of the torus. By,

$$g \underset{h}{\square} \equiv \mathrm{Tr}_{\mathcal{H}_h}(g\, q^H) \,, \tag{2.3.1}$$

we mean the following closed string boundary conditions are applied simultaneously.

$$\begin{aligned} X(\tau + 2\pi, \sigma) &= g \cdot X(\tau, \sigma) \,, \\ X(\tau, \sigma + 2\pi) &= h \cdot X(\tau, \sigma) \,. \end{aligned} \tag{2.3.2}$$

$\mathrm{Tr}_{\mathcal{H}_h}$ denotes the trace taken in a Hilbert space sector $\mathcal{H}_h$ corresponding to a spatial twist element h. We also denote $|G|$ as the order of the group G. The module $g \underset{h}{\square}$ is not well defined for $gh \neq hg$ as we will explain below.

For the CFT to be well defined, the states of the theory must be invariant under the action of the group. Therefore one projects onto G-invariant states by defining a projection operator,

$$P = \frac{1}{|G|} \sum_{g \in G} g \,. \tag{2.3.3}$$

The projection is implemented by including g in the trace and then by summing over all twists in the time direction. The inclusion of g in the trace amounts to twisting the fields by g along the time direction, i.e $g \cdot X(\tau, \sigma) = X(\tau + 2\pi, \sigma)$. The contribution to the partition function from the spatially untwisted sector of the orbifold CFT is then given by,

$$Z_{\mathcal{H}_e} = \frac{1}{|G|} \sum_{g \in G} \ . \tag{2.3.4}$$

Modular invariance under $SL(2, \mathbb{Z})$ transformations requires the addition of spatially twisted sectors $e \underset{h}{\square}$, i.e sectors where fields satisfy $h \cdot X(\tau, \sigma) = X(\tau, \sigma + 2\pi)$. Each of these spatially h-twisted sectors corresponds to a distinct Hilbert space $\mathcal{H}_h$ and one must project onto the group invariant states within every Hilbert space. This would mean that the fields would have simultaneous boundary conditions due to the action of g and h.

$$\begin{aligned}
X(\tau, \sigma + 2\pi) &= hX(\tau, \sigma) \ , & X(\tau + 2\pi, \sigma) &= gX(\tau, \sigma) \ , \\
gX(\tau, \sigma + 2\pi) &= ghX(\tau, \sigma) \ , & hX(\tau + 2\pi, \sigma) &= gX(\tau, \sigma) \ , \\
gX(\tau, \sigma + 2\pi) &= ghg^{-1}gX(\tau, \sigma) \ , & hX(\tau + 2\pi, \sigma) &= hgh^{-1}hX(\tau, \sigma) \ , \\
X(\tau + 2\pi, \sigma + 2\pi) &= ghX(\tau, \sigma) \ , & X(\tau + 2\pi, \sigma + 2\pi) &= hgX(\tau, \sigma) \ .
\end{aligned} \tag{2.3.5}$$

From the above equations, one can see that the action of g takes the string in the Hilbert space $\mathcal{H}_h$ to the Hilbert space $\mathcal{H}_{ghg^{-1}}$. When g and h do not commute these Hilbert spaces are different. The elements h and $h' = ghg^{-1}$ are in the same conjugacy class and hence the projection operator would mix Hilbert spaces corresponding to elements that belong to a given conjugacy class. Thus, one is unable to do a full group invariant projection within the Hilbert spaces in the spatially twisted sectors. In the operator language, the presence of a time twist g that doesn't commute with the spatial twist element h would not allow simultaneous diagonalization of their respective matrix representations. Nevertheless one can choose a basis for g such that it acts on the oscillators and eventually on the vacuum. As explained above, the vacuum is not left invariant and the vacuum in $\mathcal{H}_h$ taken to the vacuum in $\mathcal{H}_{ghg^{-1}}$. So the trace would be over an off-diagonal matrix with diagonal entries zero and hence would vanish. Or equivalently, the path integral vanishes due to the inconsistent boundary condition (2.3.5). Since the spatially twisted sectors are not invariant under the full group, for a given spatially twisted sector $\mathcal{H}_h$ one identifies the little group N_h consisting of elements that commute with h and project onto states invariant under the little group ,

$$Z_{\mathcal{H}_h} = \frac{1}{|N_h|} \sum_{g \in N_h} g \underset{h}{\square} \, . \tag{2.3.6}$$

The various spatially twisted sectors in a given conjugacy class are treated in equal footing and hence are labelled by their conjugacy class C_i instead of the group element itself. This follows from “naive” modular invariance [3],

$$Z_{C_i} = \frac{1}{|C_i|} \sum_{h \in C_i} Z_{\mathcal{H}_h} = \frac{1}{|C_i|} \sum_{h \in C_i} \Bigg(\frac{1}{|N_h|} \sum_{g \in N_h} g \underset{h}{\square} \Bigg) \, . \tag{2.3.7}$$

The group invariant states in the theory are formed by taking a linear combination of states from a sector twisted by a group element g and all other sectors conjugate to it. The full partition function is then given by summing over all the conjugacy classes,

$$Z_{\mathcal{T}/G} = \sum_{C_i} Z_{C_i} \, . \tag{2.3.8}$$

Since for any group G, the order of the little group N_h is the same for every element $h \in C_i$[4] , we have $|G| = |N_h||C_i|$ for every conjugacy class C_i. Thus the full CFT partition function for a general non-abelian orbifold theory can also be written as,

$$Z_{\mathcal{T}/G} \equiv \frac{1}{|G|} \sum_{\substack{g,h \in G \\ gh = hg}} g \underset{h}{\square} \, . \tag{2.3.9}$$

We will compute the twisted partition function in CHL $\mathbb{Z}_n$ orbifold models at special points in the moduli space that admit dihedral symmetry $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$. Hence, we

[3] modular invariance under $PSL(2,\mathbb{Z})$ transformations, It is naive because the modular transformation $\tau \to \tau + n$ can introduce anomalous phases that could spoil modular invariance.

[4] this is because every element in a conjugacy class has the same order, a group element h is of order n if $h^n = 1$.

summarise some properties of Dihedral groups which will be useful later. The dihedral group denoted as D_n is of order $2n$. One has the representation,

$$D_n \cong \langle h, g | h^n = e, g^2 = e, ghg = h^{-1} \rangle \, . \tag{2.3.10}$$

where h and g generate $\mathbb{Z}_n$ and $\mathbb{Z}_2$ symmetries respectively. The group elements are given by $D_n = \{e, h, h^2, \ldots, h^{n-1}, g, gh, gh^2, \ldots, gh^{n-1}\}$. The $\mathbb{Z}_2$ generator acts as an inversion on the axes of reflection, all the elements of the form gh^j are of order 2, i.e $(gh^j)^2 = 1$. The properties of dihedral group depend on whether n is even or odd. For odd n, D_n has $\lfloor n/2 \rfloor + 2$ conjugacy classes are given by (the little groups N_{c_i} for each element c_i in C_i are indicated beside),

$$\begin{aligned}
&C_0 = \{e\} \, , && N_e = D_n \, , \\
&C_1 = \{g, gh, gh^2, \ldots, gh^{n-1}\} \, , && N_{c_1} = \{e, c_1\} \, , \\
&C_k = \{h, h^{n-1}\}, \{h^2, h^{n-2}\}, \ldots, \{h^{\lfloor n/2 \rfloor}, h^{\lfloor n/2 \rfloor + 1}\} \, , && N_{c_k} = \mathbb{Z}_n \, .
\end{aligned} \tag{2.3.11}$$

For even n ,D_n has $n/2 + 3$ conjugacy classes which are given by,

$$\begin{aligned}
&C_0 = \{e\} \, , && N_e = D_n \, , \\
&C_1 = \{h^{n/2}\} \, , && N_{c_1} = D_n \, , \\
&C_2 = \{g, gh^2, gh^4, \ldots, gh^{n-2}\} \, , && N_{c_2} = \{e, c_2, h^{n/2}, c_2 h^{n/2}\} \, , \\
&C_3 = \{gh, gh^3, gh^5, \ldots, gh^{n-1}\} \, , && N_{c_3} = \{e, c_3, h^{n/2}, c_3 h^{n/2}\} \, , \\
&C_k = \{h, h^{n-1}\}, \{h^2, h^{n-2}\}, \ldots, \{h^{n/2-1}, h^{n/2+1}\} \, , && N_{c_k} = \mathbb{Z}_n \, .
\end{aligned} \tag{2.3.12}$$

The group invariant projection operator for D_n has the property,

$$\begin{aligned}
P_{D_n} =& \frac{1}{2n}\Big(\sum_{j=0}^{n-1} h^j + \sum_{j=0}^{n-1} gh^j\Big) , \\
=& \frac{1}{2}\sum_{k=0}^{1} g^k\Big(\frac{1}{n}\sum_{j=0}^{n-1} h^j\Big) , \\
=& P_{\mathbb{Z}_2}.P_{\mathbb{Z}_n} ,
\end{aligned} \tag{2.3.13}$$

which follows from the property of the group elements (2.3.10). Even though the element g does not commute with elements $h \in \mathbb{Z}_n$, it commutes with the projector of $\mathbb{Z}_n$. Thus if we take g to be a twist, it *commutes with the orbifold projection*. The $\mathbb{Z}_n$ partition function is given by,

$$Z_{T/\mathbb{Z}_n} = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} h^j \underset{h^k}{\square} . \tag{2.3.14}$$

Twisting the partition function by $g \in \mathbb{Z}_2$ amounts to insertion of g in the trace,

$$\mathrm{Tr}_{\mathcal{H}_h}(g\, q^H) . \tag{2.3.15}$$

By the arguments given in (2.3.5) only the following terms contribute to the trace,

$$Z^g_{T/\mathbb{Z}_n} = \frac{1}{n}\Big[\sum_{j=0}^{n-1} gh^j \underset{e}{\square} + \delta_{\frac{n}{2},[\frac{n}{2}]}\sum_{j=0}^{n-1} gh^j \underset{h^{n/2}}{\square}\Big] . \tag{2.3.16}$$

The second sets of terms are there only for even n as can be seen from (2.3.12). We refer to this partition function as the "twisted" partition function. Since the twist generating group $\mathbb{Z}_2$ does not commute with the orbifold group $\mathbb{Z}_n$, we refer to it as a non-commuting twist. In the following sections, we discuss the orbifold action and then evaluate (2.3.16) for the CHL $\mathbb{Z}_n$-orbifolds.

2.4 Computing the Twisted Partition Function

We adapt the half-BPS counting method of Sen [81] to compute the twisted partition function. In the notation $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2 = H \rtimes G$, H is the commutator subgroup of D_n which is also the orbifolding group. G represents an additional symmetry of the theory that appears at special points in the moduli spaces. The CHL $\mathbb{Z}_n$-orbifold can be described as an asymmetric orbifold of the heterotic string compactified on $T^4 \times T^2$. The $\mathbb{Z}_n$ symmetry acts as a shift on one of the circles in the T^2 and as a symmetry transformation on the rest of the CFT involving the T^4 coordinates and the 16 left-moving world-sheet scalars associated with the $E_8 \times E_8$ gauge group. The action of a group element h of the orbifold group H is the combination of a shift a_h and a rotation R_h acting on the Narain Lattice $\Gamma^{(22,6)}$. The action of the twist $g \in \mathbb{Z}_2$ on the $K3$ side is known [71, 76] and has been used to compute twisted indices in [80]. g leaves 14 of the 22 2-cycles of $K3$ invariant, in other words it exchanges the two E_8's. Furthermore g is not accompanied by shifts. The $g \in \mathbb{Z}_2$ insertion in trace requires the physical charges Q to be g-invariant and the orbifolding requires it to be compatible with the $\mathbb{Z}_n$ orbifold projection. Hence, we let Q takes values in the lattices that are invariant under $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ symmetry [79]. For the rest of the computation we fix the value of Q. Once this is done the twist g has no further action on the lattice.

The set of $R_h \; \forall \; h \in H$ forms a group that describes the rotational part of H and is represented as R_H. To preserve $\mathcal{N} = 4$ supersymmetry both R_H and g must act trivially on the right movers. In the $K3$ side this is enforced by requiring the respective automorphisms to be symplectic. The group H leaves $22 - k$ of the 22 left moving directions invariant, where k is the number of directions that are not invariant under H. Then, R_H can be characterised by $k/2$ phases $\phi_j(h)$ with $j = 1, 2, \ldots, k/2$. The complex coordinates X^j represent the planes of rotation and the effect of the rotation R_H is to multiply the complex oscillators by phases.

The groups also act on the Narain lattice $\Gamma^{(22,6)}$ and leave a sublattice $\Lambda_\perp$ invariant. The

G	rank($\Lambda_{\|\|}$)	rank($\Lambda_{\perp L}$)
$\mathbb{Z}_2$	8	14
$\mathbb{Z}_3$	12	10
$\mathbb{Z}_4$	14	8
$\mathbb{Z}_5$	16	6
$\mathbb{Z}_6$	16	6
$\mathbb{Z}_7$	18	4
$\mathbb{Z}_8$	18	4
$D_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$	12	10
$\mathbb{Z}_2 \times \mathbb{Z}_4$	16	6
$\mathbb{Z}_2 \times \mathbb{Z}_6$	18	4

Table 2.1: For the abelian groups the ranks of the invariant sublattice and the orthogonal complement are given in [92].

orthogonal complement to $\Lambda_\perp$ is denoted as $\Lambda_{||}$. To preserve $\mathcal{N} = 4$ supersymmetry the right movers take their charge values only from the invariant part of the lattices and the non-invariant part of the lattice is only due to the k left moving directions that are not invariant under the action of the group.. Thus rank$(\Lambda_{\perp L}) = 22 - k$, rank$(\Lambda_{||}) = k$ and rank$(\Lambda_{\perp R}) = 6$.[5] The total number of $U(1)$ gauge fields in the theory is given by rank$(\Lambda_\perp) = 22 + 6 - k$. For the $\mathbb{Z}_n$ groups, the values of k can be read off from Table 2.1.

We recollect some lattice definitions from [81] for convenience. Let V be the $22 + 6$ dimensional vector space in which the Narain lattice $\Gamma^{(22,6)}$ is embedded. The action of a given group element $h \in \mathbb{Z}_n$ on V leaves a subspace $V_\perp(h)$ invariant. The planes of rotation lie along a subspace denoted as $V_{||}(h)$. It is clear that $V_{||}(h)$ and $V_\perp(h)$ are mutually orthogonal to each other. The action of the entire group thus separates the vector space V into an invariant subspace $V_\perp$ and its orthogonal complement $V_{||}$ which are defined as[6],

$$V_\perp = \bigcap_{h \in \mathbb{Z}_n} V_\perp(h) \quad , \quad V_{||} = \bigcup_{h \in \mathbb{Z}_n} V_{||}(h) \, . \tag{2.4.1}$$

[5] This corresponds to the six graviphotons that arise from the toroidal compactification.
[6] The sublattice that is invariant under a group G acting on a lattice, Λ, is usually denoted by Λ^G and its orthogonal complement by Λ_G.

The invariant sublattice $\Lambda_\perp$ and its orthogonal complement $\Lambda_{||}$ are defined as,

$$\Lambda^{\mathbb{Z}_n} := \Lambda_\perp = \Gamma \bigcap V_\perp \quad , \quad \Lambda_{\mathbb{Z}_n} := \Lambda_{||} = \Gamma \bigcap V_{||} \, . \tag{2.4.2}$$

and,

$$\Lambda_\perp(h) = \Gamma \bigcap V_\perp(h) \quad , \quad \Lambda_{||}(h) = \Gamma \bigcap V_{||}(h) \, , \tag{2.4.3}$$

where $\Lambda_\perp(h)$ is the lattice component left invariant by a group element h and $\Lambda_{||}(h)$ is the orthogonal complement. The ranks of these lattices are the dimensions of their respective vector spaces.

In the following, we describe the heterotic construction of the counting [81] in the untwisted sector as the non-commuting twist obtains no contribution from the twisted sectors. The projection is unto states invariant under the orbifold group $\mathbb{Z}_n$. For individual elements, $h \in \mathbb{Z}_n$ there will be a non-trivial shift vector along with the rotation. In order to obtain expressions for $g \in \mathbb{Z}_2$ one has to just put the shift vectors a_g to zero. For composite elements like gh one has a rotation due to h followed by a reflection on the axes of rotation by g and there is also a shift on the lattice due to h, this follows from the group multiplication law. However one does not need such explicit details in the computation as we will show later.

As is known, the momenta and windings in the compact directions of the theory takes values in the Narain lattice $\Gamma^{(22,6)}$. The (left,right) components of the momentum vector are denoted as $\vec{P} = (\vec{P}_L, \vec{P}_R)$. Let N_L, N_R be the total level of left moving and right moving oscillator excitations respectively. For a BPS state, the right movers are kept at the lowest eigenvalue allowed by GSO projection, i.e $N_R = 0$. The level matching condition in the untwisted sector is,

$$N_L - 1 + \frac{1}{2}(\vec{P}_L^2 - \vec{P}_R^2) = 0 \, . \tag{2.4.4}$$

Let $Q = (\vec{Q}_L, \vec{Q}_R)$ denote the projection of $\vec{P}$ along $V_\perp$ and $P_{||} = (\vec{P}_{||L}, 0)$ the projection of $\vec{P}$ along $V_{||}$. In an orbifold theory such as this one, only the components of P along $V_\perp$ can act as sources for electric fields. Since $\mathcal{N} = 4$ supersymmetry requires the right-moving momenta to take values only from the invariant sublattice, $\vec{P}_R$ lies entirely along $V_\perp$, we deduce $\vec{P}_R = \vec{Q}_R$. It is then clear that $\vec{P}_L$ has the projection $\vec{Q}_L$ along $V_\perp$ and $\vec{P}_{\parallel L}$ along $V_\parallel$. Thus $\vec{P}_L$ has a orthogonal decomposition,

$$\vec{P}_L = \vec{Q}_L + \vec{P}_{\parallel L} \, . \tag{2.4.5}$$

Writing $N = \frac{1}{2}(\vec{Q}_R^2 - \vec{Q}_L^2)$ the level matching condition in the untwisted sector (2.4.4) reads,

$$N_L - 1 + \frac{1}{2}\vec{P}_{\parallel L}^2 = N \, . \tag{2.4.6}$$

Note that, the information that the charge vector should take values on some specific lattice has gone into N, and the orbifold projection proceeds in the usual way. The counting of the number of $\mathbb{Z}_n$-invariant BPS states for a given charge Q is then done by implementing the group invariant projection. The contribution to the trace with a group element $h \in \mathbb{Z}_n$ inserted comes only from those $\vec{P}_{\parallel L}$ which are invariant under the action of h, i.e from those $\vec{P}_{\parallel L}$ which satisfy the condition,

$$\vec{P}_{\parallel L} \in V_\perp(h) \, . \tag{2.4.7}$$

Furthermore, two vectors P and P' in Λ which may correspond to the same charge vector Q would differ by a constant vector. Hence the allowed values of $\vec{P}_{\parallel L}$ for a given charge vector $\vec{Q}$ are of the form,

$$\vec{P}_{\parallel L} = \vec{K}(Q) + \vec{p} \, , \tag{2.4.8}$$

where $\vec{p} \in \Lambda_\parallel$ and $\vec{K}(Q) \in (\Lambda_\parallel^*/\Lambda_\parallel)$ is a constant vector that lies in the unit cell of $\Lambda_\parallel$. The total momentum vector can thus be decomposed as,

$$\vec{P} = \vec{P}_L + \vec{P}_R = (\vec{Q}_L + \vec{P}_{\parallel L}) + \vec{Q}_R = \vec{Q} + (\vec{p} + \vec{K}(Q)) \, . \tag{2.4.9}$$

When a group element h acts on the vacuum carrying such a momentum $\vec{P}$ it will produce a phase [69],

$$h\,|P\rangle = e^{2\pi i \vec{a}_h\cdot\vec{Q}} e^{-2\pi i \vec{a}_{hL}\cdot(\vec{p}+\vec{K}(Q))}\,|P\rangle\,, \tag{2.4.10}$$

where $\vec{a}_h$ is the shift vector on the lattice associated with the group element h and $\vec{a}_{hL}$ is its left moving component. Note that there no phases associated with g since $a_g = 0$. The negative sign is due to the signature of the lattice. Thus we can now express the degeneracy of BPS states in the untwisted sector of the orbifold carrying a charge $\vec{Q} \in \Gamma_\perp$ as,

$$\begin{aligned} d(Q) =& \frac{16}{|\mathbb{Z}_n|}\sum_{h\in\mathbb{Z}_n}\sum_{N_L=0}^{\infty} d^{osc}(N_L,h) e^{2\pi i \vec{a}_h\cdot\vec{Q}} \\ & \sum_{\substack{\vec{p}\in\Lambda_\parallel \\ \vec{p}+\vec{K}(Q)\in V_\perp(h)}} e^{-2\pi i \vec{a}_{hL}\cdot\vec{p}}\ \delta_{N_L-1+\frac{1}{2}(\vec{p}+\vec{K}(Q))^2,N}\,, \end{aligned} \tag{2.4.11}$$

where $d^{osc}(N_L,h)$ is the number of ways one can construct oscillator level N_L from the 24 left-movers weighted by the action of h. The factor of 16 accounts for the degeneracy of a single BPS multiplet. The $\vec{Q}$-dependent phase in the above equation prevents us from directly computing the generating function of the degeneracies. Sen [81] evaluates the degeneracy treating $\vec{Q}$ and N as independent variables in the right hand side of the above equation and calling it $F(Q,\hat{N})$. Of course, setting $\hat{N} = N = \frac{1}{2}Q^2$ in $F(Q,\hat{N})$, one recovers $d(Q)$. The symbol $\hat{N}$ is used to indicate that N is treated as an independent variable.

$F(Q,\hat{N})$ counts the number of states in the CFT which carry a given charge Q, with right-movers in the ground state. The CFT has $\bar{L}_0 - L_0$ eigenvalue $\hat{N} - \frac{1}{2}Q^2$ which takes integer values from one-loop modular invariance. The integer condition for level matching is satisfied only after summing over all the h in the trace. A partition function can be defined as follows:

$$\tilde{F}(Q,\mu) = \sum_{\hat{N}} F(Q,\hat{N}) e^{-\mu\hat{N}}\,, \tag{2.4.12}$$

where $\hat{N}$ runs over values for which $F(Q,\hat{N})$ is non-zero.

$\tilde{F}(Q,\mu)$ acts as a generating function for the degeneracy of electrically charged 1/2 BPS states in the theory. Substituting for $F(Q,\hat{N})$ from equation (2.4.11) one obtains,

$$\tilde{F}(Q,\mu) = \frac{16}{|\mathbb{Z}_n|} \sum_{\hat{N}} \Bigg[\sum_{h\in\mathbb{Z}_n} \sum_{N_L=0}^{\infty} d^{osc}(N_L,h) e^{2\pi i \vec{a}_h\cdot\vec{Q}} e^{-2\pi i \vec{a}_{hL}\cdot\vec{K}(Q)} \sum_{\substack{\vec{p}\in\Lambda_{\|} \\ \vec{p}+\vec{K}(Q)\in V_{\perp}(h)}} e^{-2\pi i \vec{a}_{hL}\cdot\vec{p}} \; \delta_{N_L-1+\frac{1}{2}(\vec{p}+\vec{K}(Q))^2,\hat{N}} \Bigg] e^{-\mu\hat{N}} \, . \tag{2.4.13}$$

The sum over $\hat{N}$ can be carried out and it gets rid of the Kronecker delta function to give,

$$\tilde{F}(Q,\mu) = \frac{16}{|\mathbb{Z}_n|} \sum_{h\in\mathbb{Z}_n} e^{2\pi i \vec{a}_h\cdot\vec{Q}} \; e^{-2\pi i \vec{a}_{hL}\cdot\vec{K}(Q)} \; \tilde{F}^{osc}(h,\mu) \; \tilde{F}^{lat}(Q,h,\mu) \, . \tag{2.4.14}$$

where the oscillator and lattice contribution to the partition function as,

$$\begin{aligned} \tilde{F}^{osc}(h,\mu) &= \sum_{N_L=0}^{\infty} d^{osc}(N_L,h) e^{-\mu(N_L-1)} \, , \\ \tilde{F}^{lat}(Q,h,\mu) &= \sum_{\substack{\vec{p}\in\Lambda_{\|} \\ \vec{p}+\vec{K}(Q)\in V_{\perp}(h)}} e^{-2\pi i \vec{a}_{hL}\cdot\vec{p}} e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2} \, . \end{aligned} \tag{2.4.15}$$

Note that $\tilde{F}^{osc}$ has *no* dependence on Q while $\tilde{F}^{lat}$ depends weakly on $\vec{Q}$ only through $\vec{K}(Q)$.

The inverse of the partition function gives the degeneracy,

$$F(Q,\tilde{N}) = \frac{1}{2\pi i} \int_{\epsilon-i\pi}^{\epsilon+i\pi} d\mu \; \tilde{F}(Q,\mu) \; e^{\mu\tilde{N}} \, , \tag{2.4.16}$$

where $\mu = 2\pi\tau/i$ and ϵ is a real positive number. It has been argued in [81] that this integral receives its dominant contribution from a small region around the origin. Hence, we will take the $\mu \to 0$ limit later. The oscillator contribution is calculated easily by noting that the upon the action of a group element h the oscillator acquires a phase $e^{2\pi\phi_j(h)}$

[7],

$$\tilde{F}^{osc}(h,\mu) = q^{-1}\Big(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\Big)^{24-k}\prod_{j=1}^{k/2}\Big(\prod_{n=1}^{\infty}\frac{1}{1-e^{2\pi i\phi_j(h)}q^n}\frac{1}{1-e^{-2\pi i\phi_j(h)}q^n}\Big)\,, \tag{2.4.17}$$

where k is the number of non-invariant directions under $\mathbb{Z}_n$. When g is inserted into the trace, It will act on the oscillators. The phase and number of directions of rotation due to the elements in $\tilde{F}^{osc}(h,\mu)$ depends only on the order of the group element. In evaluating the oscillator contribution for,

$$g\underset{e}{\square} + gh\underset{e}{\square} + gh^2\underset{e}{\square} + \ldots + gh^{n-1}\underset{e}{\square}\,. \tag{2.4.18}$$

One notices that all the elements $g, gh, \ldots, gh^{n-1}$ are of order 2. Hence all of their oscillator contributions are identical to $g\underset{e}{\square}$. Since g exchanges the E_8 co-ordinates, the number of directions that are rotated (2.1) $k = 8$ and non zero phases $\phi_j(g) = 1/2$. Upon simplification, the oscillator contribution becomes,

$$\tilde{F}^{osc}(g,\mu) = \frac{1}{\eta(\tau)^8\eta(2\tau)^8}\,, \tag{2.4.19}$$

where,

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \quad \text{with } q = e^{2\pi i\tau} = e^{-\mu}\,. \tag{2.4.20}$$

To write down the generating function, we need the lattice contribution due a particular group element h which is given by,

$$\tilde{F}^{lat}(Q,h,\mu) = \sum_{\substack{\vec{p}\in\Lambda_{\|} \\ \vec{p}+\vec{K}(Q)\in V_{\perp}(h)}} e^{-2\pi i a_{\vec{h}L}\cdot\vec{p}}\, e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2}\,. \tag{2.4.21}$$

We have already restricted the charges to take values on the D_n invariant lattices, hence g insertion has no further action on the lattice. When h is identity the conditions on

[7]Note that the elements $h \in \mathbb{Z}_n$ are of cyclic type, i.e $h^n = 1$ for some $n \in \mathbb{Z}$, so the phases are all of type $\frac{p}{n}$ for some $p \in \mathbb{Z}$

$\vec{P}_{\|L} = \vec{P} + \vec{K}(Q) \in V_\perp(h)$ is trivially satisfied since $V_\perp(e) = V$. For any other h, since we have $\dim V_\perp(h) < \dim(V)$, it follows that,

$$\tilde{F}^{lat}(Q,h,\mu) \leq \tilde{F}^{lat}(Q,e,\mu)\ . \tag{2.4.22}$$

Therefore the dominant contribution is when $h = e$,

$$\tilde{F}^{lat}(Q,e,\mu) = \sum_{\vec{p}\in\Lambda_\|} e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2}\ . \tag{2.4.23}$$

where the phase has disappeared as the identity element doesn't shift the vectors. As mentioned earlier, this lattice theta function depends on Q, only through the $\vec{K}(Q) \in \Lambda^*_\|/\Lambda = \Lambda^*_\perp/\Lambda_\perp$ – thus there are only a finite number of lattice sums to consider.

Thus, combining the oscillator and lattice contributions (2.4.19) and (2.4.23) we get the result,

$$\boxed{\tilde{F}(Q,\mu) \sim \frac{16}{|\mathbb{Z}_n|}\frac{\tilde{F}^{lat}(Q,e,\mu)}{\eta(\tau)^8\eta(2\tau)^8}}\ , \tag{2.4.24}$$

with $\tau = i\mu/2\pi$. The nice thing about the right hand side of the above equation is that it depends only on $\vec{K}(Q)$. Thus, up to exponentially smaller terms corresponding to $h \neq e$, the right hand side is the generating function of g-twisted half-BPS states in the charge sector $\vec{K}(Q)$. This is the main result of this section.

This $g \in \mathbb{Z}_2$ twisted partition function counts g-twisted half-BPS states in a $\mathbb{Z}_n$ orbifold theory, so naturally we expect these modular forms to have weights smaller than the ones obtained for the untwisted orbifold theories. We will check that this is indeed the case by taking the asymptotic limit of (2.4.24). The $\mu \to 0$ limit of Dedekind eta function ,

$$\eta(\mu) \simeq e^{-\frac{\pi^2}{6\mu}}\sqrt{\frac{2\pi}{\mu}}\ , \tag{2.4.25}$$

and the lattice contribution (2.4.23) after doing a Poisson resummation is,

$$\tilde{F}^{lat}(e,\mu) \simeq \frac{1}{\mathrm{vol}_{\Lambda_{\|}}}\left(\frac{\mu}{2\pi}\right)^{-\frac{k_{\mathbb{Z}_n}}{2}}, \tag{2.4.26}$$

up to exponentially suppressed terms. Thus (2.4.24) has $\mu \to 0$ limit,

$$\lim_{\mu\to 0}\tilde{F}(\mu) \simeq \frac{16}{|\mathbb{Z}_n|}\frac{1}{\mathrm{vol}_{\Lambda_{\|}}}e^{2\pi^2/\mu}\left(\frac{\mu}{2\pi}\right)^{8-\frac{k_{\mathbb{Z}_n}}{2}}. \tag{2.4.27}$$

We compare the weights of the modular forms for the half-BPS states in $\mathbb{Z}_n$ orbifolds [65, 81] and the modular forms for g twisted half-BPS states in $\mathbb{Z}_n$ orbifolds,

Group	$12-\frac{k_{\mathbb{Z}_n}}{2}$	$8-\frac{k_{\mathbb{Z}_n}}{2}$	$k_{\mathbb{Z}_n}$ =rank$(\Lambda_{\|})$
$\mathbb{Z}_3$	6	2	12
$\mathbb{Z}_4$	5	1	14
$\mathbb{Z}_5$	4	0	16
$\mathbb{Z}_6$	4	0	16

One can see from the above table that the weights for the g twisted half-BPS states are indeed smaller.

The other contribution for even n

For the even n, as noted in the end of §2.3, we will get additional contribution from the orbifold twisted sector due to the element $h^{n/2}$.

$$g\underset{h^{n/2}}{\square} + gh\underset{h^{n/2}}{\square} + gh^2\underset{h^{n/2}}{\square} + \ldots + gh^{n-1}\underset{h^{n/2}}{\square}. \tag{2.4.28}$$

Here again, the oscillator contribution from each module is identical since the elements have the same order. The $\mathbb{Z}_n$ groups, for n even have $\mathbb{Z}_2$ as a subgroup which would commute with the g twist in the partition function to give a $\mathbb{Z}_2\times\mathbb{Z}_2$. This case was already computed in [93] (see Appendix A) in the context of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the result is,

$$\tilde{F}^{osc}(\mu) = \frac{1}{\eta(2\tau)^{12}} \,. \tag{2.4.29}$$

We need to compute the lattice contribution in this $h^{n/2}$ sector. Here $\vec{P} \in \Lambda^{\mathbb{Z}_2}$ the lattice invariant under the $\mathbb{Z}_2$ generated by $h^{n/2}$ unlike the untwisted sector where it was in Λ. The charge vectors Q take value in the projection of $\vec{P}$ along $V_\perp$. Thus, we have the lattice contribution given by,

$$\tilde{F}^{lat}(Q,h,\mu) = \sum_{\substack{\vec{p}\in\Lambda_{\parallel}^{\mathbb{Z}_2} \\ \vec{p}+\vec{K}(Q)\in V_\perp(h)}} e^{-2\pi i \vec{a}_{hL}\cdot\vec{p}} e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2} \,, \tag{2.4.30}$$

where $\Lambda_{\parallel}^{\mathbb{Z}_2} = \Lambda^{\mathbb{Z}_2} \cap V_\parallel$ and $\vec{K}(Q) \in \Lambda_{\parallel}^{\mathbb{Z}_2 *}/\Lambda_{\parallel}^{\mathbb{Z}_2}$. Again, the dominant contribution to the lattice sum occurs when $h = e$. The weight of the relevant modular form is now $6 - [k/2]$ where k is the rank of the lattice $\Lambda_{\parallel}^{\mathbb{Z}_2}$. We estimate k using the relevant cycle shapes for the $\mathbb{Z}_4$ and $\mathbb{Z}_6$ orbifolds to be 6 and 8 respectively. when $n = 4$, the cycle shape for the element h is $1^4 2^2 4^4$. The invariant lattice has dimension $12 = 4 + 2 + 4$ and thus $\dim V_\parallel = 24 - 12 = 12$. Elements that belong to $\Lambda_{\parallel}^{\mathbb{Z}_2}$ are those that correspond to an h-eigenvalue equal to -1. There are precisely six of them, two coming from the two-cycles and four from the four cycles. A similar analysis for the cycle shape $1^2 2^2 3^2 6^2$ for $n = 6$ shows that each three- and six-cycle contribute 2 elements with h^3-eigenvalue equal to unity but h-eigenvalue not equal to unity and hence $k = 8$. A simple asymptotic counting as we did earlier then shows that this contribution is *larger* than the contribution from the untwisted sector given in Eq. (2.4.24).

2.5 Summary

In this chapter, we have computed generating functions for non-commuting $\mathbb{Z}_2$ twists for CHL $\mathbb{Z}_n$ orbifolds ($3 \leq n \leq 6$). The generating functions turn out be ratios of the theta functions for the $\mathbb{Z}_n$ group and eta products associated with the $\mathbb{Z}_2$ group. When $n = 4$ and 6, we find additional contributions also arise. We then verified the consistency of the computation by considering the asymptotic expansion of the degeneracy and found that it has the expected limit.

Our computations did make use of the properties of the dihedral group. It would be interesting to extend this method to other nonabelian groups as well. On another note, this computation may also be extended to 1/4 BPS states. One can use the symplectic automorphisms that act on the elliptic $K3$ directly in the Type IIA theory [80]. It will also be useful to consider twists that break supersymmetry, which means we would have to consider non-symplectic automorphisms on $K3$. Such twists will provide a controlled way to count BPS states in $\mathcal{N} = 2$ string theories.

Chapter 3

Attractor mechanism in supergravity

3.1 Introduction

In chapter 2, we studied the counting of microscopic states of bound states of D-Branes in a string theory. We saw that the BPS nature of the bound state configurations played an important role in the calculation of the microscopic degeneracy. In this chapter, we will focus on the macroscopic side. Here the BPS nature of the black hole simplifies the analysis of the Killing spinor equations which arise from the vanishing of fermionic supersymmetry transformations. Exact black hole solutions can often be found by solving the Killing spinor equations rather than the second order Einstein field equations. Once again, it is the BPS nature of the black hole that allows the comparison of the statistical entropy calculated at the weak coupling limit of the theory with the Bekenstein-Hawking entropy of the black hole in the strong coupling limit.

The attractor mechanism explains the macroscopic entropy of extremal black holes in supergravity [4–6]. The moduli fields for a given extremal black hole, flow radially to a fixed value at the horizon regardless of their asymptotic values. The corresponding black hole solution is called an attractor and the mechanism has been named as the attractor mechanism. Solving the attractor equations relates the fixed values of the moduli in terms of the quantised charges of the black hole. As a result the entropy of the black hole is determined completely in terms of its charges. The attractor mechanism works not mainly because of supersymmetry but due to extremality of the black hole [94–96] and hence it can also be extended to the case of non-supersymmetric black holes [95, 97–100]. In the non-supersymmetric case, one can no longer use the Killing spinor equations to study the attractor. For single centered, extremal non-supersymmetric black holes the attractor mechanism is understood in terms of an effective black hole potential. The attractor point corresponds to an extremum of this black hole potential. Some review articles covering the subject are [101–103].

The near horizon geometry of an extremal blackhole in spacetime corresponds to the attractor point in the moduli space. The attractor geometry for black holes preserving supersymmetry is always stable. For the non-supersymmetric case, the attractors are stable when the critical point is an absolute minima of the effective black hole potential. In the asymptotically flat case this is strictly true [95, 97]. Thus, for the stable attractors, the matrix of second derivatives of the effective potential should have positive eigenvalues.

The organisation of this chapter is as follows. In §3.2 we review some essential material in $\mathcal{N} = 2$ supergravity related to special geometry. We then discuss the supersymmetry conditions that give rise to the attractor behaviour and black hole entropy in §3.3. In the next section §3.4, we demand regularity of the horizon, and consequent analysis reduces the scalar field equations to extremization of an effective potential. This leads to the discussion on non supersymmetric attractors and their stability in §3.5. We then summarise in §3.6.

3.2 Preliminaries

The $\mathcal{N} = 2, d = 4$ supergravity coupled to vector and hyper multiplets has the following field contents. The gravity multiplet consists of,

$$\{e^a_\mu, \psi^A, A^0_\mu\} \,, \tag{3.2.1}$$

where e^a_μ is the vielbein with $a = 0, 1, 2, 3$, ψ_A are the gravitinos with $A = 1, 2$ and A^0_μ is the graviphoton. The chirality conditions are given by $\gamma_5 \psi_A = 1 = -\gamma_5 \psi^A$. The vector multiplet consists of,

$$\{A^i_\mu, \lambda^{iA}, z^i\} \,, \tag{3.2.2}$$

where A^i_μ are the gauge bosons with $i = 1, 2, \ldots n_V$, the gauginos are denoted by λ^{iA} and the complex scalars are written as z^i, where $i = 1, 2, \ldots n_V$. The graviphoton and the gauge bosons A^i_μ coming from the n_V vector multiplets are together denoted by A^Λ_μ, with $\Lambda = 0, 1, \ldots n_V$. The scalars in the vector multiplet, parametrise a special Kähler manifold.

A Kähler manifold has mutually compatible complex structure, Riemannian structure and a symplectic structure [104]. The metric on a Kähler manifold is Ricci flat, hermitian and is derived from a Kähler potential,

$$g_{ij^*} = \partial_i \partial_{j^*} K \,. \tag{3.2.3}$$

A Kähler manifold is special Kähler when there exists local holomorphic sections (X^Λ, F_Λ) which can be used to express the the Kähler potential as,

$$K = -\ln(i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)) \,. \tag{3.2.4}$$

Since the Kähler manifold is also symplectic one can introduce the symplectic sections

$(L^\Lambda(z,\bar{z}), M_\Lambda(z,\bar{z}))$, with $\Lambda = 0, 1, \ldots n_V$ that satisfy the relation,

$$i(\bar{L}^\Lambda M_\Lambda - L^\Lambda \bar{M}_\Lambda) = 1 \ . \tag{3.2.5}$$

The above condition is satisfied by choosing L^Λ and M_Λ in terms of holomorphic coordinates X^Λ and a holomorphic prepotential F,

$$L^\Lambda = e^{K/2} X^\Lambda \ , \quad M_\Lambda = e^{K/2} F_\Lambda \ , \quad F_\Lambda = \frac{dF}{dX^\Lambda} \ . \tag{3.2.6}$$

Once the sections are specified, all the matter couplings are completely determined in terms of them. For example, the scalars in the vector multiplet couple to the gauge fields through a period matrix $N_{\Lambda\Sigma}(z,\bar{z})$ defined through the symplectic sections as,

$$M_\Lambda = N_{\Lambda\Sigma} L^\Sigma \ . \tag{3.2.7}$$

For future reference, we also define the Kähler covariant derivatives as,

$$\begin{aligned} D_i V &= (d_i + \frac{1}{2} d_i K) V \ , \quad D_i \bar{V} = 0 \ , \\ D_{i^*} \bar{V} &= (d_{i^*} - \frac{1}{2} d_{i^*} K) \bar{V} \ , \quad D_{i^*} V = 0 \ , \end{aligned} \tag{3.2.8}$$

for any holomorphic V.

The hypermultiplet consists of,

$$\{\zeta^\alpha, q^u\} \ , \tag{3.2.9}$$

where q^u are the scalars in the hypermultiplet with $u = 1, 2, \ldots 4n_H$ and ζ^α are the hyperinos with $\alpha = 1, 2, \ldots 2n_H$. The quaternions q^u parametrise a quaternionic manifold of dimension $4n_H$. The quaternionic manifold is also an example of a Kähler manifold with mutually compatible Riemannian, complex and a symplectic structures [105, 106]. The metric on the quaternion Kähler manifold is defined by,

$$ds^2 = h_{uv} dq^u \otimes dq^v \ . \tag{3.2.10}$$

The manifold is called quaternionic as the three complex structures J^x that exist on the manifold satisfy a quaternionic identity,

$$(J^x)_u{}^w (J^y)_w{}^v = -\delta^{xy} (Id)_u{}^v + \epsilon^{xyz} (J^z)_u{}^v \ , \tag{3.2.11}$$

where $x = 1, 2, 3$. The metric h_{uv} is hermitian with respect to J^x,

$$(J^x)_v{}^u (J^x)_w{}^t h_{ut} = h_{vw} \ , \tag{3.2.12}$$

as expected for a Kähler manifold.

With the preliminaries in hand, the bosonic part of the Lagrangian is given by,

$$\mathcal{L} = \sqrt{-g}[R + g_{ij^*} \partial^\mu z^i \partial_\mu \bar{z}^{j^*} + h_{uv} \partial^\mu q^u \partial_\mu q^v + i(\bar{N}_{\Lambda\Sigma} \mathcal{F}^{-\Lambda}_{\mu\nu} \mathcal{F}^{-\Sigma|\mu\nu} - N_{\Lambda\Sigma} \mathcal{F}^{+\Lambda}_{\mu\nu} \mathcal{F}^{+\Sigma|\mu\nu})] \ , \tag{3.2.13}$$

where g_{ij^*} is the metric on the special Kähler manifold, h_{uv} is the metric on the quaternion manifold. The self dual and anti-self dual form field strengths of the gauge fields are defined as,

$$\mathcal{F}^{\mp}_{\mu\nu} = \frac{1}{2}(\mathcal{F}_{\mu\nu} \mp \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma}) \ , \tag{3.2.14}$$

where $\epsilon_{0123} = 1$.

3.3 Supersymmetry, attractors and black hole entropy

In this section, we illustrate the emergence of the attractor mechanism from supersymmetry considerations. The supersymmetry transformations of the fermions in the theory are given by,

$$
\begin{aligned}
\delta\psi_{A\mu} &= \mathcal{D}_\mu\epsilon_A + \epsilon_{AB}T^-_{\mu\nu}\gamma^\nu\epsilon^B \ , \\
\delta\lambda^{iA} &= i\gamma^\mu\partial_\mu z^i\epsilon^A + \frac{1}{2}\mathcal{F}^{i-}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\epsilon^{AB} \ , \\
\delta\zeta_\alpha &= i\mathcal{U}^{B\beta}_u\partial_\mu q^u\gamma^\mu\epsilon^A\epsilon_{AB}C_{\alpha\beta} \ .
\end{aligned} \tag{3.3.1}
$$

where $\mathcal{D}_\mu\epsilon_A = \partial_\mu\epsilon_A + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}$, $\mathcal{U}^{B\beta}_u$ are the quaternionic vielbein and $T_{\mu\nu}$ and $\mathcal{F}^{i-}_{\mu\nu}$ are symplectic invariant combinations of the field strength defined by,

$$
\begin{aligned}
T^-_{\mu\nu} &= M_\Lambda\mathcal{F}^{-\Lambda}_{\mu\nu} - L^\Lambda\bar{N}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma}_{\mu\nu} \ , \\
\mathcal{F}^{-i}_{\mu\nu} &= g^{ij^*}(D_{j^*}\bar{M}_\Lambda\mathcal{F}^{-\Lambda}_{\mu\nu} - D_{j^*}\bar{L}_\Lambda\bar{N}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma}_{\mu\nu}) \ .
\end{aligned} \tag{3.3.2}
$$

We are interested in static, spherically symmetric, charged, supersymmetric black hole solutions of the type,

$$
ds^2 = -e^{2U}dt^2 + e^{-2U}(dr^2 + r^2 d\Omega_2^2) \ , \quad U \equiv U(r) \ , \tag{3.3.3}
$$

which asymptote to Minkowski space. In supersymmetric theories, the ADM mass is given by the central charge of the supersymmetry algebra. In general, the central charge is a function of the moduli (z^i) and the physical charges of the black hole. Extremization of the central charge relates the moduli to the charges and the black hole entropy is then given by the value of the central charge at the extremum values [6]. The principle of extremization of the central charge follows from the requirement that the near horizon geometry and the asymptotic geometry represent maximally supersymmetric solutions of (3.3.4).

It is easy to see that the Minkowski space satisfies the conditions,

$$
\delta\psi_{A\mu} = 0 \ , \quad \delta\lambda^{iA} = 0 \ , \quad \delta\zeta_\alpha = 0 \ , \tag{3.3.4}
$$

for arbitrary ϵ_A, when there are no vector fields and when all scalars in the theory take arbitrary constant values,

$$T^-_{\mu\nu} = 0 \,, \quad \mathcal{F}^{-i}_{\mu\nu} = 0 \,, \quad z^i = z^i_0 \,, \quad q^u = q^u_0 \,. \tag{3.3.5}$$

The flat space solution thus preserves the full $\mathcal{N} = 2$ supersymmetry of the theory. The other solution is the the near horizon geometry of the black hole solution when,

$$e^{-2U} \to \frac{M^2}{r^2} \quad \text{as} \quad r \to 0 \,. \tag{3.3.6}$$

where $M^2 = \frac{A}{4\pi}$ is the ADM mass of the black hole. The near horizon metric takes the form $AdS_2 \times S^2$ [1],

$$ds^2 = -\frac{r^2}{M^2}dt^2 + \frac{M^2}{r^2}dr^2 + M^2 d\Omega_2^2 \,, \tag{3.3.7}$$

also known as the Bertotti-Robinson universe. This is a solution [6] of the supersymmetry equations (3.3.4) with,

$$\mathcal{F}^{-i}_{\mu\nu} = 0 \,, \quad \partial_\mu z^i = 0 \,, \quad \partial_\mu q^u = 0 \,. \tag{3.3.8}$$

This solves the gaugino and hyperino conditions. The Killing spinor integrability condition from the gravitino variation (3.3.1) gives terms proportional to $\gamma^\mu, \gamma^{\mu\nu}$. The coefficients of each of these terms must identically vanish as these matrices form a complete basis. This gives,

$$\begin{aligned} &\frac{1}{4}R_{\mu\nu}{}^{\lambda\sigma} - 2T^{-\lambda}_{\mu}T^{-\sigma}_{\nu} = 0 \,, \\ &\mathcal{D}_\nu T^-_{\mu\lambda} = 0 \,, \end{aligned} \tag{3.3.9}$$

which are the Einstein equations and the condition for a covariantly constant graviphoton field strength respectively. These conditions are necessary for the solution to exist and to

[1] We have used Planck units $G = 1, \hbar = 1, c = 1, k_B = 1, k = 1$.

preserve the $\mathcal{N} = 2$ supersymmetry.

The central charge of the supersymmetry algebra is given by [107],

$$Z = -\frac{1}{2}\int_{S^2} T^- = L^\Lambda q_\Lambda - M_\Lambda p^\Lambda \ ,$$
$$Z_i = D_i Z = -\frac{1}{2}\int_{S^2} \mathcal{F}^{+j^*} g_{ij^*} \ . \tag{3.3.10}$$

The gaugino condition together with $\mathcal{F}^{-i}_{\mu\nu} = 0$ and $\frac{d}{dr}z^i(r) = 0$ implies,

$$D_i Z = 0 \ , \tag{3.3.11}$$

which is solved by,

$$p^\Lambda = i(\bar{Z}L^\Lambda - Z\bar{L}^\Lambda) \ , \quad q_\Lambda = i(\bar{Z}M_\Lambda - Z\bar{M}_\Lambda) \ , \tag{3.3.12}$$

where we have used (3.2.5) and the fact that $N_{\Lambda\Sigma}$ is Kähler covariant. Then (3.3.12) determine the sections X^Λ completely in terms of the charges up to Kähler gauge transformations, which are fixed by choosing the gauge $X^0 = 1$ in (3.2.6). Defining,

$$|Z_c(q,p)| = |Z|_{D_iZ=0} \ , \tag{3.3.13}$$

we see from (3.3.12) that the central charge is purely a function of the charges carried by the black hole. Since we are looking at BPS solutions, $|M| = |Z_c(q,p)|$ and hence the black hole entropy in Planck units is given by,

$$S_{BH} = \frac{A}{4} = \pi M^2 = \pi |Z_c(q,p)|^2 \ . \tag{3.3.14}$$

This result is also arrived at by studying the flow equations for the scalar fields in the background of the full black hole solution. For the magnetically charged black hole, this was arrived at by requiring the gaugino supersymmetry transformations to vanish resulting in a first order equation which relates the moduli in terms of the ratio of the magnetic charges [4]. [2] This result was further generalised to include electrical and dyonic black holes in [5].

3.4 Regularity

In this section, we discuss the approach of [94], where the radial equations are obtained from an effective one dimensional action. Regularity of the metric and moduli fields on the horizon gives rise to the $AdS_2 \times S^2$ near horizon geometry and an extremization condition on the effective black hole potential.

A general static, spherically symmetric, non-extremal black hole solution in a Einstein-Maxwell-Dilaton theory is specified by the ansatz [94],

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left(\frac{c^4 d\rho^2}{\sinh^4 c\rho} + \frac{c^2}{\sinh^2 c\rho}d\Omega_2^2\right), \tag{3.4.1}$$

where c is the extremality parameter defined by $c^2 = 2ST$, with S being the entropy and T being the temperature of the black hole. The extremal limit corresponds to $c \to 0$ and we get,

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left(\frac{d\rho^2}{\rho^4} + \frac{1}{\rho^2}d\Omega_2^2\right). \tag{3.4.2}$$

In order to have a regular area for the horizon we require the condition,

$$e^{-2U} \to \frac{A}{4\pi}\rho^2\,, \tag{3.4.3}$$

as $\rho \to -\infty$. In this limit, the metric becomes the direct product form, $AdS_2 \times S^2$ after the change of variables to $r = -\frac{1}{\rho}$,

$$ds^2 = -\frac{4\pi}{A}r^2dt^2 + \frac{A}{4\pi}\left(\frac{dr^2}{r^2} + d\Omega_2^2\right). \tag{3.4.4}$$

[2]The solution for the moduli fields usually occur as ratios of the charges as is evident from (3.3.12).

Thus we get the Bertotti-Robinson metric which we assumed earlier to be the near horizon geometry by demanding regularity near the horizon.

Similarly we will determine a condition on the black hole potential by requiring regularity of the moduli near the horizon. The black hole potential is defined as,

$$V(p, q, z, \bar{z}) = |Z|^2 + |D_i Z|^2 \ , \tag{3.4.5}$$

and the effective one dimensional Lagrangian reads as,

$$\mathcal{L}_{eff}(z(\rho), \bar{z}(\rho), U(\rho)) = \left(\frac{dU}{d\rho}\right)^2 + g_{ij^*}\frac{dz^i}{d\rho}\frac{d\bar{z}^{j^*}}{d\rho} + e^{2U}V(p, q, z, \bar{z}) \ , \tag{3.4.6}$$

together with the constraint equation given by,

$$\left(\frac{dU}{d\rho}\right)^2 + g_{ij^*}\frac{dz^i}{d\rho}\frac{d\bar{z}^{j^*}}{d\rho} - e^{2U}V(p, q, z, \bar{z}) = 0 \ . \tag{3.4.7}$$

The second order field equations are,

$$\begin{aligned} e^{-2U}\frac{d^2U}{d\rho^2} &= V(p, q, z, \bar{z}) \ , \\ e^{-2U}\frac{d}{d\rho}\left(g_{ij^*}\frac{d\bar{z}^{j^*}}{d\rho}\right) &= 2\frac{d}{dz^i}(V(p, q, z, \bar{z})) \ , \end{aligned} \tag{3.4.8}$$

where we have used the constraint equation (3.4.7) for simplification. The field equations (3.4.8) obtained from the effective one dimensional Lagrangian along with the constraints are equivalent to the Einstein equations. The scalar field equation for z^i can be further simplified using,

$$\partial_k g_{ij^*} = \partial_k \partial_i \partial_{j^*} K = \partial_i g_{kj^*} \ , \tag{3.4.9}$$

and using the constraint equation (3.4.7) to get,

$$\frac{d^2 z^i}{d\rho^2} = e^{2U}\partial^j V \ . \tag{3.4.10}$$

We have already seen that near the horizon regularity requires (3.4.3). The above equation becomes,

$$\frac{d^2 z^i}{d\rho^2} = \frac{4\pi}{A\rho^2}\partial^j V \,, \tag{3.4.11}$$

and is solved by,

$$z^i = \frac{4\pi}{A}\partial^i V \ln\frac{1}{\rho} + z_c^i \,. \tag{3.4.12}$$

In this coordinate system, the horizon of the black hole is located at $\rho = -\infty$. We see that the scalars have a regular behaviour near the horizon only if $\partial^i V = 0$. This also implies that the scalars become constants at the horizon. Thus, demanding regularity near the horizon reduces the scalar field equations to an extremization condition on the black hole potential:

$$\frac{\partial V(z, \bar{z}, q, p)}{\partial z^i} = 0 \,. \tag{3.4.13}$$

Solving the above equation relates the moduli in terms of the charges $z^i = z_c^i(q, p)$. The equation for U evaluated at the horizon gives the entropy as,

$$\frac{A}{4\pi} = V(q, p, z_c^i(q, p), \bar{z}_c^i(q, p)) \,, \tag{3.4.14}$$

which is the value of the black hole potential evaluated at the critical points. The extremization of the black hole potential is compatible with the condition (3.3.11) obtained in the supersymmetric case. To see this we use the identities [107],

$$\bar{D}_{j^*} Z = 0 \,, \quad D_i \bar{D}_{j^*} \bar{Z} = g^{ij^*} \bar{Z} \,, \quad D_i D_j Z = i c_{ijk} g^{kk^*} \bar{D}_{k^*} \bar{Z} \,, \tag{3.4.15}$$

where c_{ijk} is symmetric in all indices and satisfies $\bar{D}_{l^*} c_{ijk} = 0$. Using the above it can be shown that,

$$\partial_i V = \partial_i (|Z|^2 + |D_i Z|^2) = 2\bar{Z} D_i Z + i c_{ijk} g^{jl^*} g^{kk^*} \bar{D}_{l^*} \bar{Z} \bar{D}_{m^*} \bar{Z} \,. \tag{3.4.16}$$

Thus $D_i Z = 0 = \bar{D}_{i^*} \bar{Z}$ implies the condition $\partial_i V = 0$. Note that we have not used any supersymmetry in this discussion. Also note that $\partial_i V = 0$ does not always imply $D_i Z = 0$, which is valid only for supersymmetric attractors. This suggests that the procedure of extremization of an effective potential is generic to capture attractors in non-supersymmetric theories as well. We explore some aspects of non-supersymmetric attractors and their stability conditions in the next section.

3.5 Non-supersymmetric attractors and stability

In the previous section, we studied some features of non-supersymmetric attractors in $\mathcal{N} = 2$ supergravity theory. However, the attractor mechanism is much more general and all it requires is an extremal black hole with Minkowski asymptotics in any theory of gravity with generic matter content. In this section, we take cue from the previous discussion on the effective potential approach and review the non-supersymmetric attractors and their stability conditions [95]. We consider generic four dimensional Einstein-Maxwell-dilatonic theories with abelian gauge fields given by the Lagrangian,

$$\mathcal{L} = R - 2\partial_\mu\phi\partial^\mu\phi - a_{IJ}(\phi)F^I_{\mu\nu}F^{J\mu\nu} \,, \tag{3.5.1}$$

where I refers to the number of $U(1)$ gauge fields. The function a_{IJ} is similar to the period matrix $N_{\Lambda\Sigma}$ and we consider the dilatonic couplings to be $a_{IJ} = e^{\beta_i\phi}\delta_{IJ}$. We consider the magnetically charged black holes of Reissner-Nordstrom type for this discussion. The black hole ansatz is of the form,

$$ds^2 = -a(r)^2 dt^2 + \frac{dr^2}{a(r)^2} + b(r)^2 d\Omega_2^2 \,. \tag{3.5.2}$$

with the magnetic field strength,

$$F^I_{\theta\phi} = p^I \sin\theta \,, \tag{3.5.3}$$

where p^I are the magnetic charges. The independent components of the Einstein field equations read,

$$\partial_r^2(a^2b^2) - 2 = 0 \,,$$
$$\partial_r^2 b + b(\partial_r\phi)^2 = 0 \,, \tag{3.5.4}$$

together with a constraint,

$$a^2b^2(\partial_r\phi)^2 + V_{eff}(\phi) = b^2(1 - a^2(\partial_r b)^2 - \frac{1}{2}(\partial_r a)^2(\partial_r b)^2) \,, \tag{3.5.5}$$

where $V_{eff}(\phi, p) = a_{IJ}(\phi)p^I p^J$ is the effective potential. The scalar field equations are given by,

$$2b^2\partial_r(a^2b^2\partial_r\phi) - \frac{\partial V_{eff}}{\partial\phi} = 0 \,. \tag{3.5.6}$$

As expected from our discussion in the previous section, all the field equations can be derived from an effective one dimensional Lagrangian,

$$\mathcal{L}_{eff} = \partial_r b\partial_r(a^2b) - a^2b^2(\partial_r\phi)^2 - \frac{V_{eff}(\phi)}{b^2} \,, \tag{3.5.7}$$

together imposing the constraint (3.5.5). For the double extreme Reissner-Nordstrom black hole $a(r) = (1 - \frac{r_h}{r})$ and $b(r) = r$ and we can see that the field equations near the horizon give,

$$\left.\frac{\partial V_{eff}}{\partial\phi}\right|_{\phi_c} = 0 \,, \quad V_{eff}(\phi_c, p) = r_h^2 \,. \tag{3.5.8}$$

The entropy of the black hole is then given by,

$$S = \frac{A}{4\pi} = r_h^2 = V_{eff}(\phi_c, p) \,, \tag{3.5.9}$$

which agrees with the discussions in previous sections.

To discuss the stability condition, we consider for simplicity two gauge fields such that the effective potential becomes,

$$V_{eff} = e^{\beta_1 \phi} p_1^2 + e^{\beta_2 \phi} p_2^2 \, , \tag{3.5.10}$$

The condition $\frac{\partial V_{eff}}{\partial \phi} = 0$ determines the critical point ϕ_c at the horizon,

$$\phi_c = \frac{1}{\beta_1 - \beta_2} \ln\Big(- \frac{\beta_2 p_2^2}{\beta_1 p_1^2} \Big) \, , \tag{3.5.11}$$

which makes sense only if one of the β_i are negative. Now, consider small perturbations of the scalar field values $\delta\phi = \phi_c + \delta\phi$ about the critical points. For this discussion, we will ignore the back reaction of the scalar field on the attractor geometry. The scalar field equations for the perturbations take the form,

$$2r^2 \partial_r((r - r_h)^2 \partial_r \delta\phi) - \frac{\partial^2 V_{eff}}{\partial \phi^2}\Big|_{\phi_c} \delta\phi = 0 \, , \tag{3.5.12}$$

where we have expanded the effective potential about the critical point. For the simple model we consider the double derivative evaluated at the critical point is,

$$\frac{\partial^2 V_{eff}}{\partial \phi^2}\Big|_{\phi_c} = -\beta_1 \beta_2 \, . \tag{3.5.13}$$

Substituting the above, the fluctuation equations become,

$$(r - r_h)^2 \partial_r^2 \delta\phi + 2(r - r_h)\partial_r \delta\phi + \frac{\beta_1 \beta_2}{2r^2} \delta\phi = 0 \, , \tag{3.5.14}$$

The solutions for the fluctuations are easily determined as,

$$\delta\phi = C_\pm \Big(\frac{r - r_h}{r} \Big)^{\frac{1}{2}(\pm \sqrt{1 - 2\beta_1 \beta_2 / r_h} - 1)} \, . \tag{3.5.15}$$

We see that there is a regular solution which vanishes as one approaches the horizon,

$$\delta\phi = C_+\left(\frac{r-r_h}{r}\right)^{\frac{1}{2}(\sqrt{1-2\beta_1\beta_2/r_h}-1)}, \tag{3.5.16}$$

and becomes constant asymptotically provided $\beta_1\beta_2 < 0$. Thus the existence of a constant solution at infinity allows one to vary the scalar values by changing the constant C_+. While at the horizon, the fluctuations vanish and scalar values are attracted to a fixed value ϕ_c. Note that under the requirement $\beta_1\beta_2 < 0$, the double derivative of the effective potential (3.5.13) is positive which implies that the attractor geometry corresponds to an absolute minimum of the effective potential.

Using these conditions [95], have shown by perturbative analysis including backreaction that the near horizon attractor solution is stable under scalar perturbations about the attractor values. In chapter 7, we do the scalar perturbation analysis for black brane solutions in gauged supergravities and determine analogous conditions for stability.

3.6 Summary

In this chapter, we studied the attractor mechanism in supergravity theories. We saw that requiring maximal supersymmetry near the horizon led to an extremization condition on the central charge. The moduli values at the horizon are completely determined in terms of the charges carried by the black hole. The BPS nature of the extremal solution required the ADM mass of the black hole to be same as the central charge, which then determined the extremal black hole entropy in terms of black hole charges. Later we saw that regularity near the horizon is sufficient to determine the moduli in terms of the charges and that the effective potential approach agrees with the conditions obtained from supersymmetry. We then discussed a simple magnetically charged extremal black hole solution in a non-supersymmetric Einstein-Maxwell-Dilaton theory and the conditions for a stable attractor solution.

Chapter 4

Black holes in AdS

4.1 Introduction

In the previous chapter, we studied the attractor mechanism for asymptotically flat black hole solutions in supergravity. We saw that the near horizon geometry played an important role in determining the attractor behavior. In this chapter, we give a small introduction to black holes in Anti de-Sitter spaces (AdS) with particular focus on the near horizon geometries of extremal black branes. Historically, black holes in AdS spaces gained attention when the positive energy theorem, which states that the energy of an asymptotically flat space time is non zero, was proven for asymptotically AdS spaces [108, 109]. This result was also extended to supergravities and gauged supergravities where one can often find AdS vacuum solutions [110, 111].

It is well known that a Schwarzschild black hole in asymptotically flat spacetime has negative specific heat and is thermodynamically unstable. However, in AdS spacetime the system undergoes a first order phase transition from a radiation dominated low temperature phase to a black hole dominated high temperature phase. Hence, the AdS schwarzschild black hole can exist with a positive specific heat and is thermodynamically stable at high temperature. This is the famous Hawking-Page transition [112]. In the context of the AdS/CFT correspondence [11], the Hawking page transition is equivalent to a confinement-deconfinement phase transition in a quark-gluon system in the dual theory [34].

Charged black branes play an important role in the correspondence as holographic duals to field theories at finite temperature and chemical potential. Extremal black branes, in particular, correspond to the zero temperature ground states of the dual field theory. Even at zero temperature, several systems in condensed matter theory display novel behaviour such as phase transitions due to quantum fluctuations [46]. Field theories which describe such systems often show a wide variety of phases while the corresponding dual black brane solutions are not as many. Also, many of the condensed matter systems have non-relativistic symmetry groups and it would be interesting to explore extremal black branes with such symmetries to map the study of quantum phase transitions to the gravity side.

Metrics which display symmetries of non-relativistic condensed matter systems such as Galilean [113] and Lifshitz [47] symmetries have been constructed, and can sometimes be embedded in string theory [15, 16, 114]. Interestingly, some charged dilatonic black branes with Lifshitz-like near horizon geometry and asymptotic AdS can also exhibit attractor behaviour [115, 116]. More recently, a large class of extremal homogeneous anisotropic black brane horizons have been extensively studied [7, 53]. These metrics have generalised translational symmetries which do not commute, as opposed to the usual translational symmetries along the brane directions. The generators of these symmetries form an algebra which is isomorphic to the three dimensional real Lie algebras given by the Bianchi classification. In this chapter, we review the construction of metrics with the Bianchi type symmetries. This will form a useful background for chapter 6, where we realise some of the Bianchi type metrics as generalised attractors. The most useful references for this chapter are [7, 53, 117–120].

The organisation of this chapter is as follows. In §4.2, we discuss the physics of the AdS schwarzchild black hole and the AdS Reissner Nordstrom black hole , followed by description of black brane limits of these configurations and their near horizon geometries in §4.3. Taking the lead from the study of AdS Reissner-Nordstrom black brane, we study geometries with constant anholonomy coefficients and explore the connection with homogeneous spaces on §4.4. We then give a detailed description of five dimensional homogeneous extremal black brane horizons belonging to the Bianchi classification in §4.5. Then, we summarise the contents of this chapter in §4.6.

4.2 Schwarzschild and Reissner-Nordstrom black holes in AdS space

In this section, we will describe the Schwarzschild black hole in four dimensional AdS space followed by a discussion on the five dimensional AdS Reissner-Nordstrom black hole. First, we recall the definition of AdS spaces and describe some well known coordinate systems which will be useful later. AdS_4 space is defined as the hyperboloid,

$$- X_0^2 - X_4^2 + X_1^2 + X_2^2 + X_3^2 = -R^2 \ , \tag{4.2.1}$$

embedded in a 4 + 1 dimensional flat space with the metric,

$$ds^2 = -dX_0^2 - dX_4^2 + dX_1^2 + dX_2^2 + dX_3^2 \ . \tag{4.2.2}$$

It has the isometry group $SO(2, 3)$ generated by the 10 Killing vectors,

$$J_{\alpha\beta} = X_\alpha \partial_\beta - X_\beta \partial_\alpha \ . \tag{4.2.3}$$

Using the following global coordinates,

$$X_0 = R\cosh\rho\cos\tau\ , \quad X_4 = R\cosh\rho\sin\tau\ ,$$
$$X_i = R\sinh\rho\Omega_i\ , \quad ,\sum_{i=1}^{3}\Omega_i = 1\ , \tag{4.2.4}$$

the metric (4.2.2) can be expressed as,

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2)\ . \tag{4.2.5}$$

where $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$. This coordinate system is called a global coordinate since it covers the entire hyperboloid (4.2.1). Another commonly used set of coordinates are the Poincaré coordinates which cover one half of the hyperboloid. These coordinates are given by,

$$X_0 = \frac{1}{2r}(1 + r^2(R^2 + \vec{x}^2 - t^2))\ , \quad X_4 = Rrt\ ,$$
$$X_i = Rrx_i\ , \quad i = 1, 2\ ,$$
$$X_3 = \frac{1}{2r}(1 - r^2(R^2 - \vec{x}^2 + t^2))\ , \tag{4.2.6}$$

and the metric takes the form,

$$ds^2 = R^2(-r^2dt^2 + \frac{dr^2}{r^2} + r^2d\vec{x}^2)\ . \tag{4.2.7}$$

The AdS metric is a solution to Einstein's equation with negative cosmological constant. There also exists another vacuum solution, the AdS Schwarzschild black hole which we discuss next.

4.2.1 AdS Schwarzschild Black hole

It is well known that the familiar Schwarzschild black hole in an asymptotically flat space time is thermodynamically unstable due to negative specific heat. However, black holes in AdS spaces have positive specific heat at high temperatures and thus thermodynamically stable [112]. The AdS schwarzschild black hole is a vacuum solution to Einstein's equations with a negative cosmological constant. The black hole metric in four dimensions is given by,

$$\begin{aligned} ds^2 &= -Vdt^2 + \frac{dr^2}{V} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,, \\ V &= 1 - \frac{2M}{r} + \frac{\Lambda r^2}{3} \,, \end{aligned} \tag{4.2.8}$$

where Λ is the cosmological constant [1]. We have also set the four dimensional Newtons constant $G_4 = 1$. For large r the black hole approaches the form,

$$ds^2 = -(1 + \frac{\Lambda r^2}{3})dt^2 + \frac{dr^2}{(1 + \frac{\Lambda r^2}{3})} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{4.2.9}$$

which is nothing but the AdS_4 metric, which can be obtained from (4.2.5) by the coordinate choice $\tau = t\sqrt{\frac{\Lambda}{3}}$, $\sinh\rho = r\sqrt{\frac{\Lambda}{3}}$ and setting $R = \sqrt{\frac{3}{\Lambda}}$. In the asymptotic limit, the schwarzchild AdS black hole approaches AdS space. The horizon of the black hole is located at $r = r_h$, where r_h is the largest root of $V(r) = 0$.

Writing $\tau = it$ and expanding the metric (4.2.8) near the horizon we find,

$$ds^2 = (r - r_h)V'(r_h)d\tau^2 + \frac{dr^2}{(r - r_h)V'(r_h)} + r_h^2 d\Omega_2^2 \,. \tag{4.2.10}$$

Rewriting $r = r_h + \frac{V'(r_h)}{4}\rho^2$ we get,

$$ds^2 = \frac{V'(r_h)^2}{4}\rho^2 d\tau^2 + d\rho^2 + r_h^2 d\Omega_2^2 \,. \tag{4.2.11}$$

[1] For convenience we have chosen the conventions $\Lambda > 0$ for AdS spaces.

We see that the conical singularity at $r = r_h$ is resolved by regarding τ as an angular coordinate with a period,

$$\beta = \frac{4\pi}{V'(r_h)} = \frac{4\pi r_h}{(1 + \Lambda r_h^2)} \ . \tag{4.2.12}$$

The temperature is the inverse of β and has a minimum value,

$$T_{min} = \frac{\sqrt{\Lambda}}{2\pi} \ , \tag{4.2.13}$$

at $r_0 = 1/\sqrt{\Lambda}$. The mass of the black hole can be expressed in terms of the horizon radius r_h,

$$M = \frac{r_h}{2}\left(1 + \frac{r_h^2}{R^2}\right). \tag{4.2.14}$$

As one can see, the temperature no longer decreases with the mass, but attains a minimum value T_{min} below which only radiation exists. For $T > T_{min}$ there are two black hole solutions, one for $r_h < r_0$ and other for $r_h > r_0$. The former is called a small black hole, has negative specific heat and is thermodynamically unstable. While the black hole with $r_h > r_0$ has positive specific heat and is thermodynamically stable. The entropy of the AdS schwarzschild black hole calculated using euclidean path integral methods is given by,

$$S_{BH} = \pi r_h^2 = \frac{A_{BH}}{4} \ , \tag{4.2.15}$$

where A_{BH} is the area of the black hole horizon.

4.2.2 *AdS* Reissner-Nordstrom Black hole

Another well known black hole solution in AdS space is the Reissner-Nordstrom black hole. This black hole solution is obtained from theories with gravity coupled to massless gauge fields. For the purpose of future reference, we will consider the five dimensional black hole given by [7, 121, 122],

$$
\begin{aligned}
ds^2 &= -Vd\tilde{t}^2 + \frac{d\tilde{r}^2}{V} + \tilde{r}^2 d\Omega_3^2 \,, \\
V &= 1 + \frac{Q^2}{12\tilde{r}^4} + \frac{\tilde{r}^2\Lambda}{12} - \frac{M}{\tilde{r}^2} \,, \\
A &= -Q\Big(\frac{1}{2\tilde{r}^2} - \frac{1}{2\tilde{r}_h^2}\Big)d\tilde{t} \,.
\end{aligned}
\tag{4.2.16}
$$

Where Q is the electric charge and M is the mass of the black hole. We have also set the five dimensional Newtons constant $G_5 = 1$. As before the horizon radius $\tilde{r}_h$ is determined by the largest root of $V(\tilde{r}) = 0$. The temperature of the black hole is determined as before by euclidean rotation. We expand the metric near the horizon and resolve the conical singularity to get,

$$
T = \frac{V'(\tilde{r}_h)}{4\pi} = \frac{12M\tilde{r}_h^2 + \tilde{r}_h^6\Lambda - 2Q^2}{24\pi\tilde{r}_h^5} \,. \tag{4.2.17}
$$

From the above equation, we can see that the temperature vanishes when,

$$
Q_c^2 = 2\tilde{r}_h^6\Lambda, \quad M_c = \frac{\tilde{r}_h^4\Lambda}{4} \,, \tag{4.2.18}
$$

Since for these values both $V(\tilde{r}_h)$ and $V'(\tilde{r}_h)$ vanish, the black hole becomes extremal. To understand the regime in which the extremal black hole is stable it is convenient to rewrite (4.2.17) as,

$$
T = \frac{12 + 2\tilde{r}_h^2\Lambda - \Phi^2}{24\pi\tilde{r}_h} \,, \quad \Phi = \frac{Q}{\tilde{r}_h^2} \,, \tag{4.2.19}
$$

where we have used $V(\tilde{r}_h) = 0$ for simplification. It is clear that Φ plays the role of an electrostatic potential. In the large $\tilde{r}_h$ regime, the temperature vanishes when $\Phi \geq \sqrt{12}$ and,

$$
\tilde{r}_h^2 = \frac{\Phi^2 - 12}{2\Lambda} \,. \tag{4.2.20}
$$

Thus, the extremal black hole is also stable in the large $\tilde{r}_h$ regime just as the Schwarzschild AdS black hole. For $\Phi < \sqrt{12}$ in the small $\tilde{r}_h \to 0$ regime the black hole is unstable and has negative specific heat. The entropy of the black hole can be calculated once again using Euclidean path integral methods and we get,

$$S_{BH} = \frac{2\pi^2 r_h^3}{4} = \frac{A}{4}\,. \tag{4.2.21}$$

4.3 Black branes and near horizon limits

In this section we will discuss black branes in AdS. First we describe the Schwarzschild black brane and subsequently the Reissner-Nordstrom black brane in AdS.

4.3.1 AdS Schwarzschild black brane

A black p brane is a generalisation of a black hole with additional translational symmetries along p spatial directions. In particular, this implies that for these objects the horizon does not have the spherical topology of black holes in the p brane directions. Instead, the topology of the horizon is planar. We illustrate this by considering the black brane limit of the AdS schwarzschild black hole (4.2.8) studied in the previous section. Consider the following rescaling of the co-ordinates [123],

$$r = \left(\frac{2M}{R}\right)^{\frac{1}{3}}\rho\,, \quad t = \left(\frac{2M}{R}\right)^{-\frac{1}{3}}\tau\,. \tag{4.3.1}$$

The function $V(r)$ takes the form,

$$V(r) = 1 - \left(\frac{2M}{R}\right)^{\frac{2}{3}}\left[\frac{\rho^2}{R^2} - \frac{R}{\rho}\right], \tag{4.3.2}$$

and the metric (4.2.8) looks like,

$$ds^2 = -\left(\frac{2M}{R}\right)^{-\frac{2}{3}} V(r)d\tau^2 + \left(\frac{2M}{R}\right)^{\frac{2}{3}}\frac{d\rho^2}{V(r)} + \left(\frac{2M}{R}\right)^{\frac{2}{3}}\rho^2 d\Omega_2^2\,. \tag{4.3.3}$$

In the limit $M \to \infty$ the radius of the S^2 becomes infinite and the sphere appears locally as $\mathbb{R}^2$. This is the familiar idea that a sphere $d\Omega_i = \sum_{i=1}^{2}(dy^i)^2$, is just a plane with a point at infinity. By changing the sphere coordinates locally into $y^i = \left(\frac{2M}{R}\right)^{-\frac{1}{3}} x^i$ and considering the large M limit we get,

$$ds^2 = -\left[\frac{\rho^2}{R^2} - \frac{R}{\rho}\right]d\tau^2 + \frac{d\rho^2}{\left[\frac{\rho^2}{R^2} - \frac{R}{\rho}\right]} + \rho^2(dx^i)^2 \,, \tag{4.3.4}$$

we see that the (4.2.8) horizon has a planar topology in the black brane limit. The five dimensional analogue of the AdS Schwarzschild black brane is realised as the near horizon geometry of extremal $D3$ branes in type IIB string theory [124].

4.3.2 Reissner-Nordstrom black brane

The black brane limit of the Reissner-Nordstrom solution (4.2.16) is obtained in the same way as in the Schwarzschild black brane and is given by,

$$\begin{aligned} ds^2 &= -V(\tilde{r})d\tilde{t}^2 + \frac{d\tilde{r}^2}{V(\tilde{r})} + \tilde{r}^2(d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2) \,, \\ V(\tilde{r}) &= \frac{Q^2}{12\tilde{r}^4} + \frac{\tilde{r}^2\Lambda}{12} - \frac{M}{\tilde{r}^2} \,, \end{aligned} \tag{4.3.5}$$

We saw that in the extremal limit $V(\tilde{r}_h) = 0,\ V'(\tilde{r}_h) = 0$ and we get,

$$M = \frac{\tilde{r}_h^4\Lambda}{4}, \quad Q = \sqrt{2\tilde{r}^6\Lambda} \,. \tag{4.3.6}$$

The function $V(\tilde{r})$ then takes the form,

$$V(\tilde{r}) = \frac{(\tilde{r} - \tilde{r}_h)^2(\tilde{r} + \tilde{r}_h)^2(\tilde{r}^2 + 2\tilde{r}_h^2)\Lambda}{12\tilde{r}^4} \,. \tag{4.3.7}$$

We expand the metric near the horizon using the coordinates,

$$\frac{\tilde{r}-\tilde{r}_h}{\lambda} = r, \quad \tilde{t} = \frac{t}{\lambda}, \quad \tilde{x}^i = \frac{x^i}{\tilde{r}_h}\,, \tag{4.3.8}$$

to get,

$$\begin{aligned} ds^2 =& - r^2\Lambda dt^2 + \frac{dr^2}{\Lambda r^2} + (dx^2+dy^2+dz^2) \\ &+ \lambda\Big[\frac{7\Lambda}{3\tilde{r}_h} r^3 dt^2 + \frac{7}{3\tilde{r}_h}\frac{dr^2}{\Lambda r} + 2\frac{r}{\tilde{r}_h}(dx^2+dy^2+dz^2)\Big]\,. \end{aligned} \tag{4.3.9}$$

The gauge field expands as,

$$A = -r\sqrt{2\Lambda}dt\,. \tag{4.3.10}$$

We get a one parameter λ worth of solutions, which looks locally like $AdS_2 \times \mathbb{R}^3$,

$$ds^2 = -r^2\Lambda dt^2 + \frac{dr^2}{\Lambda r^2} + (dx^2+dy^2+dz^2)\,, \tag{4.3.11}$$

for the special value $\lambda = 0$. On first thought, the value $\lambda = 0$ appears to be singular. In the limit $\tilde{r} \to \tilde{r}_h$ and $\lambda \to 0$ such that r is kept fixed, the "near horizon" geometry of the full Reissner-Nordstrom black brane approaches a geometry which is isomorphic to $AdS_2 \times \mathbb{R}^3$. It can be checked that the metric (4.3.11) itself is an independent solution of the Einstein equation with the gauge field (4.3.10) and is valid for any r. It is a feature of extremal black holes that the "near horizon" geometry independently solves the equations of motion and is often easier to find than the full black hole solution itself.

We will now explore the symmetries preserved along the spatial directions of the $AdS_2 \times \mathbb{R}^3$ metric. This will lead us into the discussion of homogeneous spaces and Bianchi classification. For this purpose we introduce the vielbein of the $AdS_2 \times \mathbb{R}^3$ metric as [2],

$$e^t_0 = r\Lambda\,, \quad e^r_1 = \frac{1}{r\Lambda}\,, \quad e^x_2 = 1\,, \quad e^y_3 = 1\,, \quad e^z_4 = 1\,. \tag{4.3.12}$$

[2]The notations and conventions for tangent space are summarised in Appendix A.

The corresponding vector fields $\tilde{e}_a \equiv e_a^\mu \partial_\mu$ satisfy the algebra,

$$[\tilde{e}_a, \tilde{e}_b] = c_{ab}{}^c \tilde{e}_c \,, \tag{4.3.13}$$

with,

$$c_{10}^0 = \Lambda = -c_{01}^0 \,, \tag{4.3.14}$$

being the only non vanishing anholonomy coefficients. Note that the anholonomy coefficients are constants independent of the spacetime coordinates. We also note that the sub-algebra generated by,

$$\tilde{e}_2 = \partial_x \,, \quad \tilde{e}_3 = \partial_y \,, \quad \tilde{e}_4 = \partial_z \,, \tag{4.3.15}$$

is isomorphic to the three dimensional Lie algebra,

$$[\tilde{e}_2, \tilde{e}_3] = 0 \,, \quad [\tilde{e}_2, \tilde{e}_4] = 0 \,, \quad [\tilde{e}_3, \tilde{e}_4] = 0 \,, \tag{4.3.16}$$

which belongs to the Bianchi I class in the classification of real Lie algebras of dimension three [117–119]. In the next section, we explore the connection between constant anholonomy and homogeneous spaces.

4.4 Constant Anholonomy and Homogeneity

Towards the end of the previous section, we saw that the $AdS_2 \times \mathbb{R}^3$ solution has constant anholonomy coefficients and that the vector fields $\tilde{e}_a$ along the spatial directions are generators of a Lie algebra belonging to Bianchi Type I. Since we will be studying the attractors characterised by constant anholonomy in chapter 6, we would like to emphasise the relation between constant anholonomy and metrics with homogeneous symmetries.

First, we explain the concept of homogeneous symmetries through a simple example.

Homogeneous symmetries are those which connect two different points on a manifold by a continuous transformation. In general, the generators of such symmetries do not commute which leads to a Lie algebraic structure [118]. For example, consider the following vector fields on a three dimensional euclidean space [7],

$$\xi_1 = \partial_y \,, \quad \xi_2 = \partial_z \,, \quad \xi_3 = \partial_x + y\partial_z - z\partial_y \,. \tag{4.4.1}$$

This corresponds to the helical motion of a particle with translation along the x direction and rotations in the (y, z) plane. The vector fields close to form an algebra,

$$[\xi_1, \xi_2] = 0 \,, \quad [\xi_1, \xi_3] = \xi_2 \,, \quad [\xi_2, \xi_3] = -\xi_1 \,. \tag{4.4.2}$$

Note the Lie algebraic structure and that the structure constants (anholonomy coefficients) are independent of space time coordinates. We will see later in §4.5 that this is isomorphic to the Bianchi VII Lie algebra. If there is a d dimensional metric with a subset of three Killing vectors that generate the above symmetries, the metric has a three dimensional homogeneous subspace with Bianchi VII symmetry.

We would also like to explain that homogeneity follows from the assumption of constant anholonomy. This will lead to an understanding of why metrics with homogeneous subspaces arise as generalised attractors characterised by constant anholonomy coefficients [14]. In this section, we will assume a generic ansatz for the metric belonging to Bianchi type I, impose constant anholonomy and then determine the restrictions it puts on the form of the metric. We will focus on five dimensional metrics with three dimensional homogeneous subspaces as we expect these geometries to be attractors in five dimensional gauged supergravity.

Let us consider a black brane metric of the form,

$$ds^2 = -a(\tilde{r})^2 d\tilde{t}^2 + \frac{d\tilde{r}^2}{b(\tilde{r})^2} + c(\tilde{r})^2 d\tilde{x}^2 + d(\tilde{r})^2 d\tilde{y}^2 + e(\tilde{r})^2 d\tilde{z}^2 \,, \tag{4.4.3}$$

where $a(\tilde{r}), b(\tilde{r}), c(\tilde{r}), d(\tilde{r})$ and $e(\tilde{r})$ are all functions of $\tilde{r}$. The fünfbein for the metric are given by,

$$e^0_{\tilde{t}} = a(\tilde{r}) \ , \ e^1_{\tilde{r}} = \frac{1}{b(\tilde{r})} \ , \ e^2_{\tilde{x}} = c(\tilde{r}) \ , \ e^3_{\tilde{y}} = d(\tilde{r}) \ , \ e^4_{\tilde{z}} = e(\tilde{r}). \tag{4.4.4}$$

The only independent non-vanishing anholonomy coefficients (A.0.2) are,

$$c_{01}{}^{0} = b(\tilde{r})\frac{a'(\tilde{r})}{a(\tilde{r})} \ , \quad c_{21}{}^{2} = b(\tilde{r})\frac{c'(\tilde{r})}{c(\tilde{r})} \ , \quad c_{31}{}^{3} = b(\tilde{r})\frac{d'(\tilde{r})}{d(\tilde{r})} \ , \quad c_{41}{}^{4} = b(\tilde{r})\frac{e'(\tilde{r})}{e(\tilde{r})} \ , \tag{4.4.5}$$

where the prime indicates derivative with respect to $\tilde{r}$. Demanding constant anholonomy coefficients leads to the following equations,

$$\frac{a'(\tilde{r})}{a(\tilde{r})} = \frac{C_0}{b(\tilde{r})} \ , \ \frac{c'(\tilde{r})}{c(\tilde{r})} = \frac{C_2}{b(\tilde{r})} \ , \ \frac{d'(\tilde{r})}{d(\tilde{r})} = \frac{C_3}{b(\tilde{r})} \ , \ \frac{e'(\tilde{r})}{e(\tilde{r})} = \frac{C_4}{b(\tilde{r})} \ , \tag{4.4.6}$$

where C_0, C_2, C_3, C_4 are the constant values of the anholonomy coefficients. Since we have assumed all the unknown functions to be pure functions of $\tilde{r}$, we may treat the above partial differential equations as ordinary differential equations.

Let us consider some specific cases to simplify the problem. The first case, $b(\tilde{r}) = a(\tilde{r})$ leads to the near horizon geometry of the extremal AdS Reissner-Nordstrom black hole.

case i) $b(\tilde{r}) = a(\tilde{r})$: The metric takes the following form,

$$ds^2 = -C_0^2 r^2 dt^2 + \frac{dr^2}{C_0^2 r^2} + r^{2\frac{C_2}{C_0}} dx^2 + r^{2\frac{C_3}{C_0}} dy^2 + r^{2\frac{C_4}{C_0}} dz^2 \ . \tag{4.4.7}$$

where $r = \tilde{r} + \frac{a_0}{C_0}$ and $(x, y, z) = (a_2\tilde{x}, a_3\tilde{y}, a_4\tilde{z})$. Here all the a_i are integration constants. The metric (4.4.7) is the near horizon geometry of the extremal Reissner-Nordstrom black brane (4.3.11) with the identifications $C_0 = \sqrt{\Lambda}, C_2 = C_3 = C_4 = 0$.

case ii) $b(\tilde{r}) = c(\tilde{r})$: Solving for the other functions, the metric takes the form,

$$ds^2 = -r^{2\frac{C_0}{C_2}} dt^2 + \frac{dr^2}{C_2^2 r^2} + r^2 C_2^2 dx^2 + r^{2\frac{C_3}{C_2}} dy^2 + r^{2\frac{C_4}{C_2}} dz^2 \ . \tag{4.4.8}$$

where $r = \tilde{r} + \frac{a_2}{C_2}$ and $(t, x, y, z) = (a_0\tilde{t}, \tilde{x}, a_3\tilde{y}, a_4\tilde{z})$ and the a_i's are all integration constants. The metric (4.4.8) is called the anisotropic Lifshitz metric [125, 126] and can be put in a familiar form by choosing $C_0 = \frac{u_0}{L}, C_2 = \frac{1}{L}, C_3 = \frac{u_1}{L}, C_4 = \frac{u_2}{L}$ and $a_0 = \frac{1}{L^{u_0}}, a_3 = \frac{1}{L^{u_1}}, a_4 = \frac{1}{L^{u_2}}$ to get,

$$ds^2 = L^2\Big[- \hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2 d\hat{x}^2 + \hat{r}^{2u_1} d\hat{y}^2 + \hat{r}^{2u_2} d\hat{z}^2\Big] . \tag{4.4.9}$$

The isotropic Lifshitz metric [47] can be obtained by choosing $C_0 = \frac{u_0}{L}, C_2 = C_3 = C_4 = \frac{1}{L}$ and $a_0 = \frac{1}{L_0^u}, a_3 = a_4 = \frac{1}{L}$, where L is the size of the spacetime. Redefining $\hat{t} = Lt$, $(\hat{r}, \hat{x}, \hat{y}, \hat{z}) = \frac{1}{L}(r, x, y, z)$, one gets the standard Lifshitz metric,

$$ds^2 = L^2\Big[- \hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2)\Big] . \tag{4.4.10}$$

Thus, one can see that constant anholonomy requires the extremal black brane metric (4.4.3) to have a specific form such as (4.4.7) or (4.4.8). We now argue that metrics with constant holonomy are homogeneous spaces.

The hypersurfaces on which the algebra of vectors $\tilde{e}_i$, $i = 1, 2, 3$, (see Appendix A) have constant anholonomy coefficients are called surfaces of transitivity and the vectors $\tilde{e}_i$ generate a simply transitive group. It is known that for homogeneous spacetimes with space-like hypersurfaces of dimension three, there exists Lie groups of symmetries that act simply transitively on the surfaces [127]. Thus the algebra of the invariant vectors (A.0.2) can be shown to be isomorphic to the real Lie algebras of dimension three, which were classified by Bianchi [117]. The three dimensional real Lie algebras are of nine types labelled Bianchi I-IX. These symmetries are realised in homogeneous spaces where the Killing vectors generate an isomorphic Lie algebra.

Consider a basis of Killing vectors that generate a simply transitive group of dimension three. These Killing vectors have the algebra,

$$[\xi_\mu, \xi_\nu] = \tilde{C}_{\mu\nu}{}^{\lambda} \xi_\lambda . \tag{4.4.11}$$

For each of the Bianchi classes, one can go to a suitable basis and construct invariant vector fields $\tilde{e}_i$ that commute with the Killing vectors,

$$[\xi_\mu, \tilde{e}_i] = 0 \,. \tag{4.4.12}$$

Now, the Jacobi identity between $(\tilde{e}_i, \xi_\mu, \xi_\nu)$ implies $\tilde{C}_{\mu\nu}{}^{\lambda}$ are constants in spacetime. These are the structure constants of the three dimensional real Lie algebras given by the Bianchi classification. The Jacobi identity between $(\tilde{e}_i, \tilde{e}_j, \xi_\mu)$ together with (4.4.12) imply that the anholonomy coefficients $c_{ij}{}^{k}$ are constants on the surface of transitivity.

Alternatively, given that the invariant one form have an algebra (A.0.2) with constant anholonomy coefficients, [119] have shown that (4.4.12) is satisfied by three independent Killing vectors, provided the following conditions are satisfied:

$$c_{0i}{}^{0} = c_{ij}{}^{0} = 0 \,. \tag{4.4.13}$$

A quick look at the metric (4.4.3), its vielbeins and non-vanishing anholonomy coefficients shows that both the conditions hold good for all $i, j = 1, 2, 3$. This implies (4.4.12) is satisfied for the spatial directions (x, y, z), which means that these directions are homogeneous. We have used a simple class belonging to Bianchi type I to illustrate the connection between constant anholonomy coefficients and homogeneous spaces. This argument equally applies to all the Bianchi classes. In the next section, we list the various Bianchi type algebras, their structure constants, and briefly give an overview of the construction of metrics with these symmetry groups along the spatial directions [7].

4.5 Bianchi classification

In this section, we illustrate the construction of the five dimensional black brane horizons with homogeneous symmetries in the spatial directions [7]. To ensure that the metric has the required symmetries it is written in terms of invariant one forms ω^i dual to the invariant vectors $\tilde{e}_i$. The invariant one forms satisfy the relation,

$$d\omega^k = \frac{1}{2} c_{ij}{}^k \omega^i \wedge \omega^j \, . \tag{4.5.1}$$

The rest of the metric has to be fixed by demanding additional symmetries. Assuming time translational symmetries requires the metric to be time independent and requiring scaling symmetries of the form,

$$\hat{r} \to \lambda \hat{r} \, , \quad \hat{t} \to \lambda^{-u_0} \hat{t} \, , \quad \omega^i \to \lambda^{-u_i} \omega^i \, , \tag{4.5.2}$$

fixes the metric to be of the form,

$$ds^2 = L^2 \Big[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2(u_i+u_j)} \eta_{ij} \omega^i \otimes \omega^j \Big] \, , \tag{4.5.3}$$

where u_0, u_i are positive in order to have a regular horizon, $i = 1, 2, 3$ corresponds to the $\hat{x}, \hat{y}, \hat{z}$ directions and η_{ij} is a constant diagonal metric independent of spacetime coordinates. Note that the scaling symmetries in ω^i are determined by scaling $(\hat{x}, \hat{y}, \hat{z})$. The nature of the one forms will dictate what powers of $\hat{r}$ that will appear to have the required scale invariance of the metric. In fact, this can be determined just by looking at the Killing vectors that generate the homogeneous symmetries and we explain this below.

Bianchi I

We have already discussed this class in the previous section, here we get the general form of the metric (4.4.7) from symmetry considerations. The symmetry group of the Bianchi I class is isomorphic to the three dimensional translational group. It is also the symmetry group of the flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The Bianchi I class is also the most simplest among the Bianchi classes and is generated by the Killing vectors ξ^i which commute with each other. The Killing vectors, invariant vector fields, invariant one forms and structure constants are,

$$\begin{aligned} c_{ij}{}^k &= 0 \quad , \quad d\omega^i = 0 \, , \\ \xi_1 &= \partial_{\hat{x}} = \tilde{e}_1 \, , \quad \omega^1 = d\hat{x} \, , \\ \xi_2 &= \partial_{\hat{y}} = \tilde{e}_2 \, , \quad \omega^2 = d\hat{y} \, , \\ \xi_3 &= \partial_{\hat{z}} = \tilde{e}_3 \, , \quad \omega^3 = d\hat{z} \, . \end{aligned} \tag{4.5.4}$$

As one can see from above, demanding scale invariance in the directions is possible with the weights,

$$(\hat{x}, \hat{y}, \hat{z}) \to (\lambda^{-u_1} \hat{x}, \lambda^{-u_2} \hat{y}, \lambda^{-u_3} \hat{z}) \, , \tag{4.5.5}$$

and the one forms scale as,

$$(\omega^1, \omega^2, \omega^3) \to (\lambda^{-u_1} \omega^1, \lambda^{-u_2} \omega^2, \lambda^{-u_3} \omega^3) \, . \tag{4.5.6}$$

Hence the most general metric of Bianchi type I with the scale invariance (4.5.2) along all the directions is given by,

$$ds^2 = L^2 \Big[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_1} (\omega^1)^2 + \hat{r}^{2u_2} (\omega^2)^2 + \hat{r}^{2u_3} (\omega^3)^2 \Big] \, . \tag{4.5.7}$$

We see that symmetries are quite powerful and the most general form of the Bianchi I type has been determined by requiring homogeneous symmetries and scale invariance. The AdS metric is a special example of this type, with $u_0 = u_1 = u_2 = u_3 = 1$. We also saw earlier that the Lifshitz and $AdS_2 \times \mathbb{R}^3$ are examples of this class.

Bianchi II

The Bianchi II group is called the Heisenberg group of symmetries. The Killing vectors, invariant vector fields, invariant one forms and structure constants are,

$$
\begin{aligned}
& c_{23}{}^{1} = 1 = -c_{32}{}^{1} \ , \\
& \xi_1 = \partial_{\hat{y}} \ , && \tilde{e}_1 = \partial_{\hat{y}} \ , && \omega^1 = d\hat{y} - \hat{x} d\hat{z} , && d\omega^1 = \omega^2 \wedge \omega^3 \ , \\
& \xi_2 = \partial_{\hat{z}} \ , && \tilde{e}_2 = \hat{x}\partial_{\hat{y}} + \partial_{\hat{z}} \ , && \omega^2 = d\hat{z} \ , && d\omega^2 = 0 \ , \\
& \xi_3 = \partial_{\hat{x}} + \hat{z}\partial_{\hat{y}} \ , && \tilde{e}_3 = \partial_{\hat{x}} \ , && \omega^3 = d\hat{x} \ , && d\omega^3 = 0 \ .
\end{aligned}
\tag{4.5.8}
$$

This time we see that the scale invariance is different. The coordinates scale as,

$$
(\hat{x}, \hat{y}, \hat{z}) \to (\lambda^{-u_1}\hat{x}, \lambda^{-(u_1+u_3)}\hat{y}, \lambda^{-u_3}\hat{z}) \ , \tag{4.5.9}
$$

so that the invariant one forms scale as,

$$
(\omega^1, \omega^2, \omega^3) \to (\lambda^{-(u_1+u_3)}\omega^1, \lambda^{-u_3}\omega^2, \lambda^{-u_1}\omega^3) \ , \tag{4.5.10}
$$

which fixes the form of the metric to be,

$$
ds^2 = L^2 \Big[- \hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2(u_1+u_3)}(\omega^1)^2 + \hat{r}^{2u_3}(\omega^2)^2 + \hat{r}^{2u_1}(\omega^3)^2 \Big] \ . \tag{4.5.11}
$$

Bianchi VI_h, V and III

We discuss the Bianchi VI_h group of symmetries in detail, the Bianchi VI_h algebra is labelled by one arbitrary unfixed parameter $h \neq 0, 1$, which is constant and independent of the coordinates. The Bianchi III algebra is recovered when $h = 0$ and Bianchi V is recovered when $h = 1$. The Killing vectors, invariant vector fields, invariant one forms and structure constants are,

$$
\begin{aligned}
&c_{13}{}^{1} = 1\,, \quad c_{23}{}^{2} = h\,, \\
&\xi_1 = \partial_{\hat{y}}\,, && \tilde{e}_1 = e^{\hat{x}}\partial_{\hat{y}}\,, && \omega^1 = e^{-\hat{x}}d\hat{y}\,, && d\omega^1 = \omega^1 \wedge \omega^3\,, \\
&\xi_2 = \partial_{\hat{z}}\,, && \tilde{e}_2 = e^{h\hat{x}}\partial_{\hat{z}}\,, && \omega^2 = e^{-h\hat{x}}d\hat{z}\,, && d\omega^2 = h\omega^2 \wedge \omega^3\,, \\
&\xi_3 = \partial_{\hat{x}} + \hat{y}\partial_{\hat{y}} + h\hat{z}\partial_{\hat{z}}\,, && \tilde{e}_3 = \partial_{\hat{x}}\,, && \omega^3 = d\hat{x}\,, && d\omega^3 = 0\,.
\end{aligned}
\tag{4.5.12}
$$

As one can see from above, there is no scaling possible in the x direction for all the three classes. The scaling in the other directions are,

$$
(\hat{x}, \hat{y}, \hat{z}) \to (\hat{x}, \lambda^{-u_2}\hat{y}, \lambda^{-u_3}\hat{z})\,, \tag{4.5.13}
$$

and the one forms scale as,

$$
(\omega^1, \omega^2, \omega^3) \to (\lambda^{-u_2}\omega^1, \lambda^{-u_3}\omega^2, \omega^3)\,, \tag{4.5.14}
$$

which fix the metric to be of the form,

$$
ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_2}(\omega^1)^2 + \hat{r}^{2u_3}(\omega^2)^2 + (\omega^3)^2\Big]\,. \tag{4.5.15}
$$

The Bianchi III and V classes are recovered by taking $h = 0$ and $h = 1$ respectively. The Bianchi V class has a cosmological significance, it is the symmetry group of an open FLRW universe.

Bianchi IV

This is yet another class where scale invariance is not present in the $\hat{x}$ direction. The structure constants are given by,

$$
c_{13}{}^{1} = c_{23}{}^{1} = c_{23}{}^{2} = 1\,. \tag{4.5.16}
$$

The Killing vectors and the invariant vector fields are,

$$\begin{aligned} \xi_1 &= \partial_{\hat{y}} \,, & \tilde{e}_1 &= e^{\hat{x}}\partial_{\hat{y}} \,, \\ \xi_2 &= \partial_{\hat{z}} \,, & \tilde{e}_2 &= \hat{x}e^{\hat{x}}\partial_{\hat{y}} + e^{\hat{x}}\partial_{\hat{z}} \,, \\ \xi_3 &= \partial_{\hat{x}} + (\hat{y} + \hat{z})\partial_{\hat{y}} + \hat{z}\partial_{\hat{z}} \,, & \tilde{e}_3 &= \partial_{\hat{x}} \,. \end{aligned} \tag{4.5.17}$$

The invariant one forms are,

$$\begin{aligned} \omega^1 &= e^{-\hat{x}}d\hat{y} - \hat{x}e^{-\hat{x}}d\hat{z} \,, & d\omega^1 &= \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 \,, \\ \omega^2 &= e^{-\hat{x}}d\hat{z} \,, & d\omega^2 &= \omega^2 \wedge \omega^3 \,, \\ \omega^3 &= d\hat{x} \,, & d\omega^3 &= 0 \,. \end{aligned} \tag{4.5.18}$$

The scaling symmetries in the other directions are,

$$(\hat{x}, \hat{y}, \hat{z}) \to (\hat{x}, \lambda^{-u_2}\hat{y}, \lambda^{-u_2}\hat{z}) \,, \tag{4.5.19}$$

and the one forms scale as,

$$(\omega^1, \omega^2, \omega^3) \to (\lambda^{-u_2}\omega^1, \lambda^{-u_2}\omega^2, \omega^3) \,, \tag{4.5.20}$$

which fix the metric to be of the form,

$$ds^2 = L^2\Big[- \hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_2}((\omega^1)^2 + (\omega^2)^2) + (\omega^3)^2\Big] \,. \tag{4.5.21}$$

Bianchi VII_0

This class is a favourite example since its symmetries have a nice physical description. The algebra of the Bianchi VII_h class has an arbitrary unfixed constant parameter as in the previous case. We consider the $h = 0$ case in this section. The structure constants are,

$$c_{32}{}^{1} = c_{13}{}^{2} = 1 \, . \tag{4.5.22}$$

The Killing vectors and invariant vector fields are given by,

$$\begin{aligned}
\xi_1 &= \partial_{\hat{y}} \, , & \tilde{e}_1 &= \cos(\hat{x})\partial_{\hat{y}} + \sin(\hat{x})\partial_{\hat{z}} \, , \\
\xi_2 &= \partial_{\hat{z}} \, , & \tilde{e}_2 &= -\sin(\hat{x})\partial_{\hat{y}} + \cos(\hat{x})\partial_{\hat{z}} \, , \\
\xi_3 &= \partial_{\hat{x}} - \hat{z}\partial_{\hat{y}} + \hat{y}\partial_{\hat{z}} \, , & \tilde{e}_3 &= \partial_{\hat{x}} \, .
\end{aligned} \tag{4.5.23}$$

The invariant one forms are given by,

$$\begin{aligned}
\omega^1 &= \cos(\hat{x})d\hat{y} + \sin(\hat{x})d\hat{z} \, , & d\omega^1 &= -\,\omega^2 \wedge \omega^3 \, , \\
\omega^2 &= -\sin(\hat{x})d\hat{y} + \cos(\hat{x})d\hat{z} \, , & d\omega^2 &= \omega^1 \wedge \omega^3 \, , \\
\omega^3 &= d\hat{x} \, , & d\omega^3 &= 0 \, .
\end{aligned} \tag{4.5.24}$$

We see again that there is no scaling in the $\hat{x}$ direction and the $\hat{y}, \hat{z}$ directions scale uniformly as,

$$(\hat{x}, \hat{y}, \hat{z}) \to (\hat{x}, \lambda^{-u_2}\hat{y}, \lambda^{-u_2}\hat{z}) \, , \tag{4.5.25}$$

and the one forms scale as,

$$(\omega^1, \omega^2, \omega^3) \to (\lambda^{-u_2}\omega^1, \lambda^{-u_2}\omega^2, \omega^3) \, , \tag{4.5.26}$$

which fix the metric to be of the form,

$$ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_2}(\alpha(\omega^1)^2 + (\omega^2)^2) + (\omega^3)^2\Big] \, . \tag{4.5.27}$$

We have put an arbitrary constant parameter $\alpha \neq 0$ just to highlight the difference with the Bianchi type I. As one can see from the invariant one forms (4.5.24), $(\omega^1)^2 + (\omega^2)^2 =$

$d\hat{y}^2 + d\hat{z}^2$ and when $\alpha = 1$, this becomes a special case of the Bianchi I types (4.5.7). The physical description of the symmetry group is as follows. The Killing vector ξ_3 generates translations along $\hat{x}$ and rotations along the $(\hat{y}, \hat{z})$ plane. One full rotation in the $(\hat{y}, \hat{z})$ plane corresponds to a translation of $2\pi L$ along the $\hat{x}$ direction giving rise to a helical motion. In this case, the size of the spacetime L has a physical description as the pitch of the helix.

Bianchi VIII

This is the first class that we deal with which gives rise to a metric with no scaling symmetry along all the three spatial directions! The structure constants are,

$$c_{32}{}^{1} = c_{31}{}^{2} = c_{12}{}^{3} = 1 \ .$$

The Killing vectors and the invariant vector fields are given by,

$$\begin{aligned}
\xi_1 &= \frac{1}{2}e^{-\hat{z}}\partial_{\hat{x}} + \frac{1}{2}(e^{\hat{z}} - \hat{y}^2 e^{-\hat{z}})\partial_{\hat{y}} - \hat{y}e^{-\hat{z}}\partial_{\hat{z}} \ , & \tilde{e}_1 &= \frac{1}{2}(1 + \hat{x}^2)\partial_{\hat{x}} + \frac{1}{2}(1 - 2\hat{x}\hat{y})\partial_{\hat{y}} - \hat{x}\partial_{\hat{z}} \ , \\
\xi_2 &= \partial_{\hat{z}} \ , & \tilde{e}_2 &= -\hat{x}\partial_{\hat{x}} + \hat{y}\partial_{\hat{y}} + \partial_{\hat{z}} \ , \\
\xi_3 &= \frac{1}{2}e^{-\hat{z}}\partial_{\hat{x}} - \frac{1}{2}(e^{\hat{z}} + \hat{y}^2 e^{-\hat{z}})\partial_{\hat{y}} - \hat{y}e^{-\hat{z}}\partial_{\hat{z}} \ , & \tilde{e}_3 &= \frac{1}{2}(1 - \hat{x}^2)\partial_{\hat{x}} + \frac{1}{2}(-1 + 2\hat{x}\hat{y})\partial_{\hat{y}} + \hat{x}\partial_{\hat{z}} \ .
\end{aligned} \tag{4.5.28}$$

The invariant one forms are given by,

$$\begin{aligned}
\omega^1 &= d\hat{x} + (1 + \hat{y}^2)d\hat{y}^2 + (\hat{x} - \hat{y} - \hat{x}^2\hat{y})d\hat{z} \ , & d\omega^1 &= -\omega^2 \wedge \omega^3 \ , \\
\omega^2 &= 2\hat{x}d\hat{y} + (1 - 2\hat{x}\hat{y})d\hat{z} \ , & d\omega^2 &= \omega^3 \wedge \omega^1 \ , \\
\omega^3 &= d\hat{x} + (-1 + \hat{x}^2)d\hat{y}^2 + (\hat{x} + \hat{y} - \hat{x}^2\hat{y})d\hat{z} \ , & d\omega^3 &= \omega^1 \wedge \omega^2 \ .
\end{aligned} \tag{4.5.29}$$

This class has no scaling along any of the $\hat{x}, \hat{y}, \hat{z}$ directions with homogeneous symmetries. The metric then takes the form,

$$ds^2 = \Big[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 \Big] . \tag{4.5.30}$$

The interesting thing about this metric is it factorises as $Lif_2(u_0) \times M_{VIII}$, where $Lif_2(u_0)$ is the two dimensional Lifshitz metric and M_{VIII} is the metric written in terms of invariant one forms respecting the symmetry group of Bianchi VIII.

Bianchi IX

The symmetry group of this class is isomorphic to the three dimensional rotational group $SO(3, \mathbb{R})$. The structure constants are,

$$c_{23}{}^1 = c_{31}{}^2 = c_{12}{}^3 = 1 . \tag{4.5.31}$$

The Killing vectors and invariant vector fields are given by,

$$\begin{aligned}
\xi_1 &= \partial_{\hat{y}} , & \tilde{e}_1 &= -\sin(\hat{z})\partial_{\hat{x}} + \frac{\cos(\hat{z})}{\sin(\hat{x})}\partial_{\hat{y}} - \cot(\hat{x})\cos(\hat{z})\partial_{\hat{z}}, \\
\xi_2 &= \cos(\hat{y})\partial_{\hat{x}} - \cot(\hat{x})\sin(\hat{y})\partial_{\hat{y}} + \frac{\sin(\hat{y})}{\sin(\hat{x})}\partial_{\hat{z}} , & \tilde{e}_2 &= \cos(\hat{z})\partial_{\hat{x}} + \frac{\sin(\hat{z})}{\sin(\hat{x})}\partial_{\hat{y}} - \sin(\hat{z})\cot(\hat{x})\partial_{\hat{z}} , \\
\xi_3 &= -\sin(\hat{y})\partial_{\hat{x}} - \cot\hat{x}\cos(\hat{y})\partial_{\hat{y}} + \frac{\cos(\hat{y})}{\sin(\hat{x})}\partial_{\hat{z}} , & \tilde{e}_3 &= \partial_{\hat{z}} .
\end{aligned} \tag{4.5.32}$$

The invariant one forms are given by,

$$\begin{aligned}
\omega^1 &= -\sin(\hat{z})d\hat{x} + \sin(\hat{x})\cos(\hat{z})d\hat{y}, & d\omega^1 &= \omega^2 \wedge \omega^3 , \\
\omega^2 &= \cos(\hat{z})d\hat{x} + \sin(\hat{x})\sin(\hat{z})d\hat{y}, & d\omega^2 &= \omega^3 \wedge \omega^1 , \\
\omega^3 &= \cos(\hat{x})d\hat{y} + d\hat{z}, & d\omega^3 &= \omega^1 \wedge \omega^2 .
\end{aligned} \tag{4.5.33}$$

This time again there are no scaling symmetries in the $(\hat{x}, \hat{y}, \hat{z})$ directions and the metric factorises into $Lif_2(u_0) \times M_{IX}$,

$$ds^2 = \Big[- \hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 \Big] . \tag{4.5.34}$$

Metrics with Bianchi IX symmetries have the symmetry group $SO(3, \mathbb{R})$. In $3 + 1$ dimensions metrics of such type have been extensively studied in cosmology. In fact, the closed FRW universe has the Bianchi IX symmetry and describes an anisotropic universe with rotating matter.

So far, we discussed the various symmetry classes in the Bianchi classification and used simple scaling symmetry requirements to arrive at metrics which respect these symmetries. It is well known that the near horizon geometries of black holes independently solve the field equations. It has been shown in [7] that many of these solutions can be recovered from simple matter systems like gravity coupled to massive gauge fields. In chapter 6, we will use this information to construct some of the Bianchi type solutions from gauged supergravity.

We end this chapter with an observation. If we did not demand scale invariance along the $(\hat{x}, \hat{y}, \hat{z})$ directions, but keep the scale invariance along $(\hat{r}, \hat{t})$ directions the metric (4.5.3) splits into a direct product form as,

$$ds^2 = L^2 \Big[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \eta_{ij} \omega^i \otimes \omega^j \Big] . \tag{4.5.35}$$

We call this subset of metrics as $Lif_{u_0}(2) \otimes M$, where $Lif_{u_0}(2)$ is the two dimensional Lifshitz metric and M corresponds to the spatial part of the metrics that display homogeneous symmetries labelled as $M_I, M_{II}, \ldots M_{IX}$.

4.6 Summary

In this chapter, we studied black holes in Anti de-Sitter space. We started by studying the Schwarzschild and Reissner-Nordstrom black holes in this background. We then studied the black brane limit of the Schwarzschild black hole and the near horizon geometry of the extremal Reissner-Nordstrom black brane. We also observed that the near horizon geometry of the extremal Reissner-Nordstrom black brane takes the form $AdS_2 \times \mathbb{R}^3$, and in addition has constant anholonomy coefficients.

We then studied some of the properties of the near horizon geometry. Especially we observed starting with the assumption of constant anholonomy in a simple example and obtained the most general metric type consistent with scale invariance and the symmetries of Bianchi I class. We then explained the relation between constant anholonomy and homogeneous spaces. We saw that the various Bianchi classes indeed have constant anholonomy coefficients and discussed the homogeneous extremal black brane horizons classified by the Bianchi classification in detail.

We saw that a simple requirement demanding scale invariance and homogeneity along various directions was sufficient to fix the most general form of the metric. We observed that when we demand scale invariance only along the radial and time directions, the most general metric consistent with these symmetries split into a direct product form which is reminiscent of near horizon geometries of extremal black holes.

As an end note, we feel it is important to mention that not all of the Bianchi class metrics have been realised as near horizon geometries as extremal Black branes. Numerically, solutions interpolating between one of the Bianchi classes and AdS_5 have been found in [7]. An analytic interpolating black hole solution is still lacking except for familiar cases like $AdS_2 \times \mathbb{R}^3$. Such interpolating solutions would be very valuable both for understanding the attractor mechanism in AdS spaces as well as for studying field theory renormalisation group flows in the dual gravity side.

Chapter 5

Gauged supergravity

5.1 Introduction

In chapter 4, we studied five dimensional homogeneous brane geometries classified by the Bianchi classification. In a recent work, Bianchi I type geometries such as the Lifshitz solution where embedded in $\mathcal{N} = 2, d = 4$ gauged supergravity using the generalised attractors procedure [14]. In this approach, one sets the bosonic fields in the theory to be constants in tangent space, which results in solvable algebraic field equations. We wish to extend the study to five dimensions and also realise the Bianchi type metrics as generalised attractor solutions. This requires some background in five dimensional gauged supergravity which we provide in this chapter.

Gauged supergravities are supersymmetry preserving deformations of ungauged supergravity. The deformations are implemented by promoting some of the global symmetries of the ungauged theory to local symmetries. Gaugings are usually done by coupling the symmetry generators to corresponding gauge fields. The first example of gauged supergravity was obtained by gauging the $SO(8)_R$ global symmetry of $\mathcal{N} = 8$ supergravity [128]. Gauged supergravities with non-compact gaugings were constructed in [129–131] and generalisation to higher dimensions were constructed in [132–134]. More recently, gauged supergravities are understood as low energy effective theories that describe flux compactifications of string theory. For example, the low energy theory from Type IIB string theory compactified on a Calabi-Yau manifold in the presence of Ramond-Ramond and Neveu-Schwarz fluxes for the three form fields is a $\mathcal{N} = 2, d = 4$ gauged supergravity [8–10].

Ungauged supergravity contains free scalar fields called moduli that take values on a moduli space. The moduli parametrise a non-linear sigma model that defines a manifold. For example, the non-linear sigma model for the scalars in the vector multiplet of $\mathcal{N} = 2, d = 4$ supergravity defines a Kähler manifold [135]. While in $\mathcal{N} = 2, d = 5$ supergravity, the corresponding scalar manifold is real and very special [136, 137]. The scalars in the hypermultiplet parametrise a quaternionic manifold in both cases [138]. When the symmetries of the scalar manifold leave the non-linear sigma model invariant, they often extend to symmetries of the full Lagrangian. For example, in four dimensions the symmetries of the scalar manifold always extend to symmetries of the full supersymmetric Lagrangian, whereas in five dimensions symmetries of the scalar manifold can sometimes be broken by supergravity interactions [139]. The R-symmetry group, which is an automorphism of the Poincaré superalgebra is another global symmetry of the theory. Gauged supergravity is obtained by gauging some or all of the global symmetries of supergravity.

In the context of the AdS/CFT correspondence [11], gauged supergravity generically describes the supergravity regime of the bulk theory. This is due to the fact that many gauged supergravities support an AdS vaccum due to the presence of non-trivial potentials for the scalar fields in the theory. The potential terms are of the order $O(g^2)$, where g is the gauge coupling constant. Supersymmetry requires the presence of the potential term to compensate for the additional terms that appear in the covariant derivatives of gauged supergravity. When the scalar fields take their extremum, the value of the potential sets the cosmological constant of the theory. For example, the $\mathcal{N} = 8, d = 5$ $SO(6)$ gauged supergravity [134] describes type IIB supergravity compactified on $AdS_5 \times S^5$. According to the AdS/CFT correspondence [11] this theory is dual to the four dimensional $SU(N), \mathcal{N} = 4$ super Yang-Mills theory as discussed in the introduction of the thesis.

In this chapter, we describe the necessary background in five dimensional gauged supergravity. We begin with a discussion on the global symmetries of the ungauged theory in §5.2. We discuss the global symmetries of the very special, quaternionic manifolds and the R symmetry. We then follow up with a discussion of gauged supergravity in §5.3. In particular, we focus on $\mathcal{N} = 2, d = 5$ gauged supergravity coupled to vector, tensor and hypermultiplets [12, 13, 140]. Towards the end of the chapter we discuss a simple gauged supergravity model with one vector multiplet in §5.4. Useful supplementary material is provided in §C.

5.2 $\mathcal{N} = 2, d = 5$ supergravity

5.2.1 Field content

The $\mathcal{N} = 2, d = 5$ ungauged supergravity, often called as Maxwell-Einstein supergravity was constructed in [132, 134, 136]. The field contents of the theory are the following.

- The gravity multiplet contains the graviton e_μ^a, two gravitinos ψ_μ^i and a graviphoton.
- The vector multiplet contains a vector field A_μ, $SU(2)_R$ doublet of fermions (gauginos) λ^i and a real scalar field ϕ.
- The hyper multiplet contains a doublet of fermions (hyperinos) ζ^A with $A = 1, 2$ and four real scalars q^X with $X = 1, \ldots, 4$.

The n_V vector multiplets together with the graviphoton constitute $n_V + 1$ vectors $A_\mu^I, I = 0, \ldots, n_V$. The vector multiplet contains n_V scalars $\phi^x, x = 1, 2, \ldots, n_V$ and the hyper multiplet contains $4n_H$ scalars q^X, with $X = 1, 2, \ldots 4n_H$. The bosonic part of the Lagrangian is given by,

$$\hat{e}^{-1}\mathcal{L}^{\mathcal{N}=2}_{Bosonic} = -\frac{1}{2}R - \frac{1}{4}a_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}g_{XY}\partial_\mu q^X\partial^\mu q^Y - \frac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y \\ + \frac{\hat{e}^{-1}}{6\sqrt{6}}C_{IJK}\epsilon^{\mu\nu\rho\sigma\tau}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\tau\ , \tag{5.2.1}$$

where $\hat{e} = \sqrt{-detg_{\mu\nu}}$, a_{IJ} is the ambient metric used to raise and lower the vector indices, g_{xy} is the metric on the scalar manifold and g_{XY} is the metric on the quaternionic manifold. The coefficients C_{IJK} that appear with the Chern-Simons term are constant symmetric tensors.

5.2.2 Global Symmetries

The scalars in the theory parametrise a manifold that factorises into direct product of a very special and a quaternionic manifold,

$$\mathcal{M}_{scalar} = \mathcal{S}(n_v) \otimes \mathcal{Q}(n_H)\ . \tag{5.2.2}$$

Some important references for this section are [12, 140–143].

Very special Manifold

The scalars in the vector multiplet are real and parametrise a very special manifold in five dimensions. A very special manifold $\mathcal{S}$ is a real n dimensional manifold defined by the hypersurface,

$$N \equiv C_{IJK}h^I h^J h^K = 1\ , \tag{5.2.3}$$

where the $h^I \equiv h^I(\phi)$ are co-ordinates in $\mathbb{R}^{n+1}$. The metric on the very special manifold is given by the pullback of the metric on $\mathbb{R}^{n+1}$,

$$
\begin{aligned}
ds^2_{\mathbb{R}^{n+1}} &= a_{IJ} dh^I \otimes dh^J \ , \\
a_{IJ} &= -\frac{1}{2}\frac{\partial}{\partial h^I}\frac{\partial}{\partial h^J} \ln N \Big|_{N=1} \ .
\end{aligned} \tag{5.2.4}
$$

In a given co-ordinate frame, the metric on the scalar manifold g_{xy} is then defined as,

$$
\begin{aligned}
g_{xy} &= h^I_x h^J_y a_{IJ} \ , \\
a_{IJ} &= h_I h_J + h^x_I h^y_J g_{xy} \ ,
\end{aligned} \tag{5.2.5}
$$

where h^I_x are defined by,

$$
\frac{\partial h_I}{\partial \phi^x} \equiv h_{I,x} = \sqrt{\frac{2}{3}} h_{Ix}, \quad \frac{\partial h^I}{\partial \phi^x} \equiv h^I{}_{,x} = -\sqrt{\frac{2}{3}} h^I{}_x \ . \tag{5.2.6}
$$

In supergravity one often works in the frame language and the following relations are useful,

$$
\begin{aligned}
f^a_x f^b_y \eta_{ab} &= g_{xy} \ , \\
f^a_{[x,y]} + \Omega^{ab}_{[y} f^b_{x]} &= 0 \ .
\end{aligned} \tag{5.2.7}
$$

Here f^a_x and Ω^{ab}_y are the n_V-bein and the spin connection on $\mathcal{S}$ respectively. The indices a, b are flat indices and η_{ab} is the flat metric with signature $\{-, +, \ldots\}$.

The symmetries of the scalar manifold are the transformations that leave (5.2.3) invariant. These symmetries can be made manifest when the kinetic term of the scalars in the vector multiplet,

$$
-\frac{1}{2} g_{xy} \partial_\mu \phi^x \partial^\mu \phi^y \ , \tag{5.2.8}
$$

is written in terms of h^I. The completeness relations (5.2.5) can be used to rewrite the kinetic term as,

$$-\frac{1}{2}a_{IJ}h^I_x h^J_y \partial_\mu\phi^x\partial^\mu\phi^y = -\frac{3}{4}a_{IJ}h^I{}_{,x}h^I{}_{,y}\partial_\mu\phi^x\partial^\mu\phi^y = -\frac{3}{4}a_{IJ}\partial_\mu h^I\partial^\mu h^J \ . \tag{5.2.9}$$

The definition of the ambient metric (5.2.4) can be simplified to obtain $a_{IJ} = \frac{3}{2}C_{IJK}h^K$, using the relations $C_{IJK}h^J h^K = \frac{1}{\alpha}h_I$ and $C_{IJK}h^K = \frac{1}{\alpha^2}h_I h_J$, where $\alpha = h^I h_I$. Using the above relations the scalar kinetic term takes the form,

$$-\frac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y = -\frac{9}{8}C_{IJK}h^I\partial_\mu h^J\partial^\mu h^K \ . \tag{5.2.10}$$

In this form, the scalar kinetic term manifestly exhibits the symmetries of the scalar manifold. Consider a group of linear transformations,

$$h^I \to B^I{}_J h^J \ . \tag{5.2.11}$$

These are symmetries of the scalar manifold if (5.2.3) is invariant, which requires the C_{IJK} to transform as,

$$B^M_I B^N_J B^P_K C_{MNP} = C_{IJK} \ . \tag{5.2.12}$$

From (5.2.10) and (5.2.12), it is clear that (5.2.11) are symmetries of the sigma model. The transformations (5.2.11) extend to the full Lagrangian (5.2.1) provided the gauge fields transform as,

$$A^I \to B^I{}_J A^J \ . \tag{5.2.13}$$

This can be seen by using the relation $a_{IJ} = \frac{3}{2}C_{IJK}h^K$ and (5.2.12) in the kinetic term for the gauge fields. Thus the symmetries of the scalar manifold are global symmetries of the Lagrangian. Note that so far the C_{IJK} are unspecified and arbitrary. Due to this, for a fixed number of vector multiplets several target manifolds are possible. In fact, from (5.2.12) it is evident that the classification of the C_{IJK} (with (5.2.3) satisfied) is equivalent to classification of the very special manifolds. This approach has been pursued in the literature and the classification of symmetric very special manifolds was done in [136, 144, 145]. This was extended to include very special manifolds that are homogeneous spaces in [141].

As a simple example, consider the symmetric very special manifold which belongs to the "generic Jordan class" in the classification with a coset structure [136, 137],

$$\mathcal{M} = \frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)} \,, \quad n \geq 1 \,. \tag{5.2.14}$$

The symmetry group of this manifold is given by $G = SO(n-1,1) \times SO(1,1)$. This symmetry group can be made manifest by choosing a suitable parametrisation to satisfy the constraint (5.2.3). For example, in terms of co-ordinates ξ^I on $\mathbb{R}^{n+1}$, (5.2.3) takes the form [146],

$$\begin{aligned} N(\xi) &= \left(\frac{2}{3}\right)^{\frac{3}{2}} C_{IJK}\xi^I\xi^J\xi^K \,, \\ &= \sqrt{2}\xi^0[(\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - \ldots - (\xi^n)^2] \,. \end{aligned} \tag{5.2.15}$$

One can see that the symmetry group G is manifest in this parametrisation. Of course the parametrisation also has to satisfy $N(\xi) = 1$, which can be solved in terms of the scalar fields in the Lagrangian by choosing $\xi \equiv \xi(\phi)$ as,

$$\begin{aligned} \xi^0 &= \frac{1}{\sqrt{2}||\phi||^2} \,, \\ \xi^1 &= \phi^1 \,, \\ &\vdots \\ \xi^n &= \phi^n \,, \end{aligned} \tag{5.2.16}$$

where,

$$||\phi||^2 = (\phi^1)^2 - (\phi^2)^2 - \ldots - (\phi^n)^2 \,, \tag{5.2.17}$$

In any parametrisation, the scalar fields must be restricted to suitable domains such that the metric on scalar manifold g_{xy} and a_{IJ} are positive definite. This ensures that the kinetic terms in the Lagrangian (5.2.1) have proper sign. In principle, the above information is sufficient to determine the metrics a_{IJ} and g_{xy} completely using (5.2.4) and (5.2.5) respectively. We will show this in §5.4, where we consider a simple gauged supergravity model with one vector multiplet and the very special manifold is an example of the symmetric spaces discussed above.

Quaternionic Kähler manifold

The hypermultiplet in a $\mathcal{N} = 2$ supergravity theory contains four real scalars which are locally considered as components of a quaternion q^X. For n_H hypermultiplets, the q^X parametrise a quaternionic Kähler manifold[1] $\mathcal{Q}$ endowed with a metric,

$$ds^2 = g_{XY}(q)dq^X \otimes dq^Y \quad , \quad X, Y = 1, 2, \dots 4n_H \,, \tag{5.2.18}$$

and three complex structures J^x that satisfy the quaternionic identity,

$$(J^x)_X^{\ Z}(J^y)_Z^{\ Y} = -\delta^{xy}(Id)_X^{\ Y} + \epsilon^{xyz}(J^z)_X^{\ Y} \,, \tag{5.2.19}$$

where $x = 1, 2, 3$. The metric on $\mathcal{Q}$ is hermitian with respect to J^x,

$$(J^x)_V^{\ X}(J^x)_W^{\ Y} g_{XY} = g_{VW} \,. \tag{5.2.20}$$

The existence of a hermitian metric together with a complex structure defines a natural two form on the manifold. This can be seen by multiplying the above equation with $(J^x)_Z^{\ V}$,

[1]Hypermultiplets can appear in rigid supersymmetric Yang-Mills theories as well as supergravity theories. In the former case, the scalar manifold is HyperKähler, while in the latter it is Quaternionic. The difference in the two cases lies in the curvature of the principal bundle.

using (5.2.19) and defining $K^x_{XY} = g_{YZ}(J^x)_X^{\ Z}$ to get,

$$K^x_{WZ} = -K^x_{ZW} \quad , \quad K^x = K^x_{XY} dq^X \wedge dq^Y \ . \tag{5.2.21}$$

The natural two form K^x on the quaternionic manifold is called a HyperKähler form. It also implies that the manifold Q has a symplectic structure. Supersymmetry requires the existence of an $SU(2)$ principal bundle $\mathcal{SU} \rightarrow Q$ and ω^x is the connection on such a bundle. The HyperKähler form is covariantly closed with respect to the connection ω^x,

$$\nabla K \equiv dK^x + \epsilon^{xyz}\omega^y \wedge K^z = 0 \ . \tag{5.2.22}$$

The curvature on the $SU(2)$ bundle can then be defined as,

$$\Omega^x \equiv d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z \ . \tag{5.2.23}$$

Hence, for quaternionic manifolds the curvature on the principal bundle is proportional to the HyperKähler form,

$$\Omega^x = \lambda K^x \ , \tag{5.2.24}$$

where λ is real number related to the scale of the Quaternionic manifold. Since the tangent space is not flat, the holonomy group of Q is $SU(2) \otimes \mathcal{H}$ with $\mathcal{H} \subset Sp(2n_H)$. One can introduce Quaternionic vielbeins f^X_{iA} (with $i \in SU(2)$ and $A \in Sp(2n_H)$) as follows,

$$\begin{aligned} f^X_{iC} f^{YC}_j + f^Y_{iC} f^{XC}_j &= g^{XY}\epsilon_{ij} \ , \\ g_{XY} f^X_{iA} f^Y_{jB} &= \epsilon_{ij} C_{AB} \ , \\ f^X_{iA} f^{Yi}_B + f^Y_{iA} f^{Xi}_B &= \frac{1}{n_H} g^{XY} C_{AB} \ , \end{aligned} \tag{5.2.25}$$

where ϵ_{ij} and C_{AB} are $SU(2)$ and $Sp(2n_H)$ invariant tensors respectively [13].

As discussed above, the quaternionic manifold is Riemannian, has a complex structure and a compatible symplectic structure. Thus the quaternionic manifold is also a Kähler manifold [105, 106]. The classification of homogeneous Quaternionic manifolds first appeared in the mathematics literature in [147], and is further discussed in [141–143].

We now describe a simple example of a Quaternionic Kähler manifold [54, 105, 106],

$$\frac{SU(2,1)}{SU(2)\times U(1)}\,, \tag{5.2.26}$$

and illustrate its symmetries. As argued before, the quaternionic manifold is also Kähler and hence the metric can be derived from a suitable Kähler potential. Following [54], let us denote the quaternion $q^X = \{V, \sigma, \theta, \tau\}$ and define the variables,

$$S = V + \theta^2 + \tau^2 + i\sigma\,, \quad C = \theta - i\tau\,. \tag{5.2.27}$$

The Kähler potential has the form,

$$\mathcal{K} = -\frac{1}{2}\log(S + \bar{S} - 2C\bar{C})\,. \tag{5.2.28}$$

The metric on Q is defined by,

$$g_{z_a\bar{z}_b} = \frac{\partial^2\mathcal{K}}{\partial z_a \partial\bar{z}_b}\,, \quad z_a = S, C\,. \tag{5.2.29}$$

and simplifies to,

$$ds^2 = e^{4\mathcal{K}}\left[\frac{1}{2}dS\,d\bar{S} + 2C\bar{C}dCd\bar{C} - \bar{C}dCd\bar{S} - Cd\bar{C}dS\right] + e^{2\mathcal{K}}dCd\bar{C}\,. \tag{5.2.30}$$

This can be rewritten in terms of the original co-ordinates $q^X = \{V, \sigma, \theta, \tau\}$ as,

$$ds^2 = \frac{1}{2V^2}(dV^2 + (d\sigma + 2\theta d\tau - 2\tau d\theta)^2) + \frac{2}{V}(d\theta^2 + d\tau^2)\,. \tag{5.2.31}$$

The symmetries of this metric are the symmetries of the sigma model. The full set of

Killing vectors k_α^X given below generate an $SU(2,1)$ algebra [54].

$$k_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 \\ 2\theta \\ 0 \\ 1 \end{pmatrix}, \quad k_3 = \begin{pmatrix} 0 \\ \text{-}2\tau \\ 1 \\ 0 \end{pmatrix}, \quad k_4 = \begin{pmatrix} 0 \\ 0 \\ \text{-}\tau \\ \theta \end{pmatrix}, k_5 = \begin{pmatrix} \mathrm{V} \\ \sigma \\ \theta/2 \\ \tau/2 \end{pmatrix},$$

$$k_6 = \begin{pmatrix} 2V\sigma \\ \sigma^2 - (V + \theta^2 + \tau^2)^2 \\ \sigma\theta - \tau(V + \theta^2 + \tau^2) \\ \sigma\tau + \theta(V + \theta^2 + \tau^2) \end{pmatrix}, \quad k_7 = \begin{pmatrix} -2V\sigma \\ -\sigma\theta + V\tau + \tau(\theta^2 + \tau^2) \\ (V - \theta^2 + 3\tau^2)/2 \\ -2\theta\tau - \sigma/2 \end{pmatrix},$$

$$k_8 = \begin{pmatrix} -2V\tau \\ -\sigma\tau - V\theta - \theta(\theta^2 + \tau^2) \\ -2\theta\tau + \sigma/2 \\ (V + 3\theta^2 - \tau^2)/2 \end{pmatrix}. \tag{5.2.32}$$

The Killing vectors (k_1, k_2, k_3) generate translations in the (σ, θ, τ) respectively, k_4 generates rotations in (σ, τ), while k_5 corresponds to dilatations and (k_6, k_7, k_8) generate other isometries of (5.2.31). The manifold (5.2.26) has the symmetry group $SU(2) \times U(1)$, this can be seen by rewriting the Killing vectors as,

$$\begin{aligned}
T_1 &= \frac{1}{4}(k_2 - 2k_8)\,, & T_5 &= \frac{-i}{2}(k_1 - k_6)\,, \\
T_2 &= \frac{1}{4}(k_3 - 2k_7)\,, & T_6 &= \frac{-i}{4}(k_3 + 2k_7)\,, \\
T_3 &= \frac{1}{4}(k_1 + k_6 - 3k_4)\,, & T_7 &= \frac{-i}{4}(k_2 + 2k_8)\,, \\
T_4 &= ik_5, & T_8 &= \frac{\sqrt{3}}{4}(k_1 + k_4 + k_6)\,.
\end{aligned} \tag{5.2.33}$$

Here, T_1, T_2, T_3 generate the $SU(2)$ algebra, T_8 is the $U(1)$ generator and T_4, T_5, T_6, T_7 generate $\frac{SU(2,1)}{U(2)}$.

R symmetry

The R-symmetry group, which is an automorphism of the Poincaré superalgebra is a global symmetry of the supergravity theory. In this section, we motivate the $SU(2)$ action of the R symmetry group by describing the $\mathcal{N} = 2$ superconformal algebra in $d = 5$ [148, 149]. The spinors in 5d are symplectic majorana and the conventions are described in the appendix (B). According to Nahm's classification [150], the bosonic subgroup of the superconformal group is a direct product of the conformal group and the R symmetry group. For $\mathcal{N} = 2, d = 5$, the bosonic subgroup is $SO(5,2) \times SU(2)_R$. The $SO(5,2)$ conformal group generated by translations P_μ, Lorentz transformations $M_{\mu\nu}$, dilatations D and special conformal transformations K_μ is given by,

$$\begin{aligned}
&[M_{\mu\nu}, M_{\lambda\sigma}] = \eta_{\mu[\lambda} M_{\sigma]\nu} - \eta_{\nu[\lambda} M_{\sigma]\mu} \ , \\
&[P_\mu, M_{\nu\lambda}] = \eta_{\mu[\nu} P_{\lambda]}, \quad [K_\mu, M_{\nu\lambda}] = \eta_{\mu[\nu} K_{\lambda]} \ , \\
&[D, P_\mu] = P_\mu \ , \quad [D, K_\mu] = -K_\mu \ , \\
&[P_\mu, K_\nu] = 2(\eta_{\mu\nu} D + 2M_{\mu\nu}) \ .
\end{aligned} \tag{5.2.34}$$

The R symmetry group $SU(2)_R$ is generated by,

$$U_i{}^j = i(U_1\sigma_1 + U_2\sigma_2 + U_3\sigma_3)_i{}^j, \quad U_i{}^i = 0, \quad U_i{}^j = -(U_i{}^j)^* \ , \tag{5.2.35}$$

where U_i are real and σ_i are the usual Pauli matrices. The $SU(2)$ algebra is given by,

$$[U_i{}^j, U_k{}^l] = \delta_i{}^l U_k{}^j - \delta_k{}^j U_i{}^l \ . \tag{5.2.36}$$

The fermionic part is generated by the usual supersymmetry Q^i and special supersymmetry S^i,

$$\{Q_{i\alpha}, Q^{j\beta}\} = -\frac{1}{2}\delta_i^{\ j}(\gamma^\mu)_\alpha^\beta P_\mu, \quad \{S_{i\alpha}, S^{j\beta}\} = -\frac{1}{2}\delta_i^{\ j}(\gamma^\mu)_\alpha^\beta K_\mu \ ,$$
$$\{Q_{i\alpha}, S^{j\beta}\} = -\frac{i}{2}(\delta_i^{\ j}\delta_\alpha^\beta D + \delta_i^{\ j}(\gamma^{\mu\nu})_\alpha^\beta M_{\mu,\nu} + 3\delta_\alpha^\beta U_i^{\ j}) \ . \tag{5.2.37}$$

The action of the conformal group on the supersymmetries is given by,

$$[M_{\mu\nu}, Q_\alpha^i] = -\frac{1}{4}(\gamma_{\mu\nu}Q^i)_\alpha \ , \quad [M_{\mu\nu}, S_\alpha^{\ i}] = -\frac{1}{4}(\gamma_{\mu\nu}S^i)_\alpha \ ,$$
$$[D, Q_\alpha^i] = \frac{1}{2}Q_\alpha^{\ i} \ , \quad [D, S_\alpha^{\ i}] = -\frac{1}{2}S_\alpha^{\ i} \ ,$$
$$[K_\mu, Q_\alpha^i] = i(\gamma_\mu S^i)_\alpha \ , \quad [P_\mu, S_\alpha^{\ i}] = -i(\gamma_\mu Q^i)_\alpha \ . \tag{5.2.38}$$

Finally, The action of the R symmetry group on the fermionic generators is given by,

$$[U_i^{\ j}, Q_\alpha^k] = \delta_i^{\ k}Q_\alpha^j - \frac{1}{2}\delta_i^{\ j}Q_\alpha^{\ k} \ ,$$
$$[U_i^{\ j}, S_\alpha^{\ k}] = \delta_i^{\ k}S_\alpha^{\ j} - \frac{1}{2}\delta_i^{\ j}S_\alpha^{\ k} \ , \tag{5.2.39}$$

which is an $SU(2)$ rotation. Thus in the $\mathcal{N} = 2$ theories the R symmetry acts as an $SU(2)$ rotation on the fermions of the theory (the gravitino $\psi_{\mu i}$, gaugino λ_i and the hyperino ξ_i).

So, far we have discussed the global symmetries of five dimensional $\mathcal{N} = 2$ supergravity. In the next section, we gauge the symmetries and describe gauged supergravity.

5.3 $\mathcal{N} = 2, d = 5$ gauged supergravity

In the previous section, we saw the global symmetries of five dimensional supergravity. This group of global symmetries is a direct product of the symmetry group of the very special manifold and the quaternionic manifold. Let us call the isometry group of the scalar manifold as G. In addition, we also saw that there is an $SU(2)_R$ symmetry group. The global symmetry group of $\mathcal{N} = 2, d = 5$ supergravity is of the form $G \times SU(2)_R$. In general one has various possibilities for constructing a gauged supergravity from an ungauged supergravity.

Firstly, one can just gauge a subgroup of the R symmetry group. Note that the gauge fields are inert under the R symmetry group as the group acts only on the fermions. Theories of this type are called Maxwell-Einstein supergravity theories and possess a scalar potential [137]. One can also gauge a subgroup of the symmetries of the scalar manifold $K \subset G$. A subset of the gauge fields from the ungauged theory has to transform in the adjoint representation of K so that they can act as Yang-Mills gauge fields. If such a group K exists, the gauge fields can in general transform under K as,

$$Gaugefields \to Adj(K) + Singlets(K) + Nonsinglets(K)\ . \tag{5.3.1}$$

For the singlets the structure constants of K are assumed to be zero and if K is abelian the presence of singlets do not change anything. If some of the gauge fields are charged under K, they lead to mass terms and break supersymmetry. This issue is resolved by dualising the charged vectors to tensor fields satisfying self dual field equations [151]. We have given some background on the origin of tensor multiplets in Appendix C. Finally, One can do the most general gauging of the subgroups of $SU(2)_R$ and $K \subset G$ simultaneously. This leads to the most general gauged supergravity in five dimensions [13].

In this section, we review the most general gauged supergravity in five dimensions with n_V vector multiplets, n_T tensor multiplets and n_H hypermultiplets with a generic gauging of the symmetries of the scalar manifold and gauging of $U(1)_R \subset SU(2)_R$ R symmetry group. Before proceeding further, we highlight some important differences of the five dimensional theory as compared with the four dimensional theory. Usually, the gauging can be described in terms of what is called as the momentum map associated with the scalar manifold. For the $d = 4, \mathcal{N} = 2$ theories the scalar manifold in the vector multiplet is special Kähler and there exists a momentum map for the isometries [140]. Whereas in the case of $d = 5$ the scalar manifold in the vector multiplet is very special, real and non-symplectic. Hence momentum maps do not exist for the isometries. However, the possible symmetry groups for the very special manifold have been classified for the homogeneous cases [141–143] and one need to identify the subgroup of these group of symmetries for gauging. Another significant difference in the $\mathcal{N} = 2, d = 5$ theory is the presence of tensor multiplets which originate due to the gauging. [2] It is interesting to observe that the quaternion structure is the same in both $d = 4$ and $d = 5$ theories. Consequently there exist Killing prepotentials (i.e. there exist Killing vectors which are given in terms of the derivatives of prepotentials) as in the case of four dimensional gauged supergravity. We now discuss the gauging of global symmetries of the five dimensional supergravity (5.2.1).

5.3.1 Gauging the symmetries

In the previous section, we studied the global symmetries if five dimensional $\mathcal{N} = 2$ supergravity. The global symmetry group G is a direct product of the group of symmetries on the very special manifold $\mathcal{S}$, the quaternionic Kähler manifold $\mathcal{Q}$ and the $SU(2)_R$ symmetry group. One then identifies a subgroup of symmetries K for gauging. The gauging of symmetries on scalar manifolds is done by introducing Killing vectors $K_I^{\tilde{x}}(\phi)$ and $K_I^X(q)$ that act on $\mathcal{S}$ and $\mathcal{Q}$,

$$
\begin{aligned}
\phi^{\tilde{x}} &\to \phi^{\tilde{x}} + \epsilon^I K_I^{\tilde{x}}(\phi) \, , \\
q^X &\to q^X + \epsilon^I K_I^X(q) \, ,
\end{aligned}
\tag{5.3.2}
$$

where ϵ^I are infinitesimal parameters. Then one replaces the ordinary derivatives on scalar and fermions by the K-covariant derivatives. The bosonic part of the theory then gets the

[2]See Appendix C for more details.

following replacements [13, 18]:

$$
\begin{aligned}
\partial_\mu \phi^{\tilde{x}} &\to \mathcal{D}_\mu \phi^{\tilde{x}} \equiv \partial_\mu \phi^{\tilde{x}} + g A_\mu^I K_I^{\tilde{x}}(\phi) , \\
\partial_\mu q^X &\to \mathcal{D}_\mu q^X \equiv \partial_\mu q^X + g A_\mu^I K_I^X(q) , \\
\nabla_\mu B_{\nu\rho}^M &\to \mathcal{D}_\mu B_{\nu\rho}^M \equiv \nabla_\mu B_{\nu\rho}^M + g A_\mu^I \Lambda_{IN}^M B_{\nu\rho}^N ,
\end{aligned} \tag{5.3.3}
$$

where g is the gauge coupling and ∇_μ is the Lorentz covariant derivative. The Λ_{IN}^M are constant matrices which are valued in certain representations of K. The derivatives acting on the fermions are also modified, but they have additional terms due to the gauging of R symmetry. In this case, the $SU(2)_R$ connection is replaced by,

$$
\omega_j^i \to \omega_j^i + g_R A^I P_{Ij}{}^i(q) , \tag{5.3.4}
$$

where g_R is the $SU(2)_R$ gauge coupling and $P_{Ii}{}^j(q)$ are Killing prepotentials that exist due to the quaternionic structure on the hypermultiplet sector. The fermions get the replacements,

$$
\begin{aligned}
\nabla_\mu \psi_{\mu i} &\to \nabla_\mu \psi_{\nu i} + g_R A_\mu^I P_{Ii}{}^j(q) \psi_{\nu j} , \\
\nabla_\mu \lambda_i^{\tilde{a}} &\to \nabla_\mu \lambda_i^{\tilde{a}} + g_R A_\mu^I P_{Ii}{}^j(q) \lambda_j^{\tilde{a}} + g A_\mu^I L_I^{\tilde{a}\tilde{b}} \lambda_i^{\tilde{b}} , \\
\nabla_\mu \zeta^A &\to \nabla_\mu \zeta^A + g A_\mu^I \omega_{IB}{}^A(q) \zeta^B ,
\end{aligned} \tag{5.3.5}
$$

where g_R is the gauge coupling constant associated with gauging of R symmetry and,

$$
L_I^{\tilde{a}\tilde{b}} \equiv \partial^{\tilde{b}} K_I^{\tilde{a}}, \quad \omega_{IB}{}^A(q) \equiv K_{IX;Y} f_i^{XA} f_B^{Yi} . \tag{5.3.6}
$$

We have defined $K_I^{\tilde{a}} = K_I^{\tilde{x}} f_{\tilde{x}}^{\tilde{a}}$ using the vielbein on scalar manifold (5.2.7), and the covariant derivative on K_{IX} is with respect to the metric g_{XY} on the quaternionic Kähler manifold.

5.3.2 Field content

The five dimensional supergravity with a generic gauging of the symmetries of the scalar manifold and the $SU(2)$ R symmetry was constructed by Ceresole and Dall'Agata [13]. The theory contains gravity coupled to vector, tensor and hyper multiplets. The gravity multiplet contains the graviton e^a_μ, two gravitinos ψ^i_μ and a graviphoton A_μ. The hypermultiplet contains a doublet of spin $1/2$ fermions (hyperinos) ζ^A with $A = 1,2$ and four real scalars q^X with $X = 1,\ldots,4$. The vector multiplet contains a vector field A_μ, $SU(2)_R$ doublet of fermions (gauginos) λ^i and a real scalar field ϕ. The tensor multiplet contains a massive antisymmetric self-dual tensor field $B_{\mu\nu}$, $SU(2)_R$ doublet of fermions λ^i and a real scalar field ϕ.

To summarise, for n_V vector, n_T tensor and n_H hypermultiplets the field content is given by,

$$\{e_\mu^{\ a}\ ,\psi^i_\mu\ ,A^I_\mu\ ,B^M_{\mu\nu}\ ,\lambda^{i\tilde{a}}\ ,\zeta^A\ ,\phi^{\tilde{x}}\ ,q^X\}\ . \tag{5.3.7}$$

The scalars in the vector and tensor multiplets are collectively denoted by $\phi^{\tilde{x}}$, where $\tilde{x} = 1,2,\ldots,n_v + n_T$. The constraint equation (5.2.3) on the scalar fields is now written as ,

$$C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}h^{\tilde{J}}h^{\tilde{K}} = 1\ , \quad h^{\tilde{I}} \equiv h^{\tilde{I}}(\phi^{\tilde{x}})\ , \tag{5.3.8}$$

where only C_{IJK} and C_{IMN} are non zero as required by supersymmetry. The vector field index is $I = 0,1,\ldots,n_V$ and $I = 0$ refers to the graviphoton. The index $M = 1,2,\ldots,n_T$ counts the number tensor multiplets. The vector and tensor field strengths are collectively written as $\mathcal{H}^{\tilde{I}}_{\mu\nu} = (F^I_{\mu\nu}, B^M_{\mu\nu})$ where $\tilde{I} = (I, M)$.

The gauginos $\lambda^{i\tilde{a}}$ in the vector and tensor multiplets transform as vectors under $SO(n_V + n_T)$ and $\tilde{a} = 1,2\ldots,n_V + n_T$ is a flat index. The quaternions $q^X, X = 1,2,\ldots 4n_H$ are the scalars in the n_H hypermultiplets. The hyperinos $\zeta^A, A = 1,2,\ldots,2n_H$ form fundamental representations of $USp(2n_H)$ and $USp(2) \simeq SU(2)$. The conventions on the $SU(2)$ tensor ϵ^{ij} are summarised in Appendix B.

5.3.3 Lagrangian

The bosonic part of the five dimensional $\mathcal{N} = 2$ gauged supergravity is given by,

$$\begin{aligned}
\hat{e}^{-1}\mathcal{L}^{\mathcal{N}=2}_{Bosonic} =& -\frac{1}{2}R - \frac{1}{4}a_{\tilde{I}\tilde{J}}\mathcal{H}^{\tilde{I}}_{\mu\nu}\mathcal{H}^{\tilde{J}\mu\nu} - \frac{1}{2}g_{XY}\mathcal{D}_{\mu}q^{X}\mathcal{D}^{\mu}q^{Y} - \frac{1}{2}g_{\tilde{x}\tilde{y}}\mathcal{D}_{\mu}\phi^{\tilde{x}}\mathcal{D}^{\mu}\phi^{\tilde{y}} \\
&+ \frac{\hat{e}^{-1}}{6\sqrt{6}}C_{IJK}\epsilon^{\mu\nu\rho\sigma\tau}F^{I}_{\mu\nu}F^{J}_{\rho\sigma}A^{K}_{\tau} + \frac{\hat{e}^{-1}}{4g}\epsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}B^{M}_{\mu\nu}\mathcal{D}_{\rho}B^{N}_{\sigma\tau} \\
&- \mathcal{V}(\phi, q) ,
\end{aligned} \tag{5.3.9}$$

where $\hat{e} = \sqrt{-detg_{\mu\nu}}$ and Ω_{MN} is a constant real symplectic matrix that satisfies the following conditions,

$$\Omega_{MN} = -\Omega_{NM} , \quad \Omega_{MN}\Omega^{NP} = \delta^{P}_{M} . \tag{5.3.10}$$

Gauging the supergravity introduces a non-trivial scalar potential which is given by,

$$\mathcal{V}(\phi, q) = 2g^2 W^{\tilde{a}}W^{\tilde{a}} - g_R^2[2P_{ij}P^{ij} - P^{\tilde{a}}_{ij}P^{\tilde{a}ij}] + 2g^2\mathcal{N}_{iA}\mathcal{N}^{iA} , \tag{5.3.11}$$

where,

$$\begin{aligned}
P_{ij} &\equiv h^{I}P_{Iij} , \\
P^{\tilde{a}}_{ij} &\equiv h^{\tilde{a}I}P_{Iij} , \\
W^{\tilde{a}} &\equiv \frac{\sqrt{6}}{4}h^{I}K^{\tilde{x}}_{I}f^{\tilde{a}}_{\tilde{x}} = -\frac{\sqrt{6}}{8}\Omega^{MN}h^{\tilde{a}}_{M}h_{N} , \\
\mathcal{N}^{iA} &\equiv \frac{\sqrt{6}}{4}h^{I}K^{X}_{I}f^{Ai}_{X} .
\end{aligned} \tag{5.3.12}$$

The bosonic part of the supersymmetry transformation rules are:

$$\begin{aligned}
\delta_{\epsilon}\psi_{\mu i} =& \sqrt{6}\nabla_{\mu}\epsilon_i + \frac{i}{4}h_{\tilde{I}}(\gamma_{\mu\nu\rho}\epsilon_i - 4g_{\mu\nu}\gamma_{\rho}\epsilon_i)\mathcal{H}^{\nu\rho\tilde{I}} + ig_R\gamma_{\mu}\epsilon^{j}P_{ij} , \\
\delta_{\epsilon}\lambda^{\tilde{a}}_{i} =& -\frac{i}{2}f^{\tilde{a}}_{\tilde{x}}\gamma^{\mu}\epsilon_i\mathcal{D}_{\mu}\phi^{\tilde{x}} + \frac{1}{4}h^{\tilde{a}}_{\tilde{I}}\gamma^{\mu\nu}\epsilon_i\mathcal{H}^{\tilde{I}}_{\mu\nu} + g_R\epsilon^{j}P^{\tilde{a}}_{ij} + gW^{\tilde{a}}\epsilon_i , \\
\delta_{\epsilon}\zeta^{A} =& -\frac{i}{2}f^{A}_{iX}\gamma^{\mu}\epsilon^{i}\mathcal{D}_{\mu}q^{X} + g\epsilon^{i}\mathcal{N}^{A}_{i} .
\end{aligned} \tag{5.3.13}$$

The terms that are proportional to the gauge coupling constants in the supersymmetry transformation are called fermionic shifts. These appear due to supersymmetric completion of the additional terms that appear due to gauging. A supersymmetric ward identity relates the potential $\mathcal{V}(\phi, q)$, the gravitino mass matrix P_{ij} and the fermionic shifts [12, 54, 152–154]. As one can see from (5.3.13) the scalar potential (5.3.11) can be written in terms of the squares of the gravitino mass matrix and the fermion shifts in the supersymmetry transformations that appear due to the gauging.

Each term in (5.3.11) has its origin from different sectors in the theory. The terms proportional to g^2 arise due to gauging of the symmetries of the scalar manifold. In particular the terms $W^{\tilde{a}}W^{\tilde{a}}$ arise due to tensor multiplets and $\mathcal{N}_{iA}\mathcal{N}^{iA}$ appear due to hypermultiplets. The terms proportional to g_R^2 occur due to gauging the R symmetry.

We now discuss the possibilities of AdS vacuum in this theory which occurs whenever,

$$\mathcal{V}(\phi, q)' = 0\,, \quad \mathcal{V}(\phi_c, q_c) < 0\,, \tag{5.3.14}$$

where the derivative is with respect to the scalars and ϕ_c, q_c are the critical points of the potential. The metrics $g_{\tilde{x}\tilde{y}}, g_{XY}$ are positive definite and the term in the potential which can contribute to an AdS vacuum is $P_{ij}P^{ij}$. For example, consider the case $n_V = n_T = n_H = 0$ there is only the gravity multiplet with a single graviphoton. Since K_X^I are zero, the prepotentials P_{Iij} are either zero or $SU(2)$ valued constants. If we choose to gauge a subgroup $U(1)_R \subset SU(2)_R$, then $P_{Iij} = V\delta_{ij}$. [3] Thus the potential (5.3.11) becomes,

$$\mathcal{V} = -4V^2\,, \tag{5.3.15}$$

which acts as a cosmological constant. The corresponding theory is referred in the literature as Anti de-Sitter supergravity [155]. The important point here is that one definitely need to gauge some of the R symmetry in the theory to get an AdS vacuum. Of course,

[3] In general we could choose $P_{Iij} = V_I\delta_{ij}$. The parameters V_I appear in the R symmetry gauging as $A_\mu(U(1)_R) = V_I A^I_\mu$. In this case, we have only one graviphoton and we have chosen $V_I = V$.

we could simultaneously gauge the other symmetries simultaneously, as the terms $W^{\tilde{a}}$ and $\mathcal{N}_{iA}$ in (5.3.11) can at the most change the shape of the critical point.

5.4 Gauged supergravity with one vector multiplet

In this section, we describe a simple gauged supergravity model in some detail [18, 19]. We will use this model for constructing generalised attractors in a later chapter. This gauged supergravity model consists of one vector multiplet ($A^I, I = 0$ corresponds to the graviphoton.) and two tensor multiplets. The field content is summarised as,

$$\{e_\mu^a\, , \psi_\mu^i\, , A_\mu^I\, , B_{\mu\nu}^M\, , \lambda^{i\tilde{a}}\, , \phi^{\tilde{x}}\}\ . \tag{5.4.1}$$

We use the same notations as in the previous section. The very special manifold, which is parametrised by the scalars in the vector and tensor multiplets has the coset structure given by,

$$\mathcal{S} = SO(1,1) \times \frac{SO(2,1)}{SO(2)}\ . \tag{5.4.2}$$

This model is an example of a symmetric space discussed in the introduction of this chapter. In this model, the symmetry group of the scalar manifold is $G = SO(1,1) \times SO(2,1)$. The gauging we consider is an $SO(2) \subset SO(2,1)$ subgroup of the $SO(2,1)$ in G and the gauging of the $U(1)_R \subset SU(2)_R$. The symmetries of the scalar manifold can be made manifest by going to a suitable basis such that the condition (5.2.3) is satisfied. Note that the index I in (5.2.3) is replaced with $\tilde{I} = (I, M)$ to collectively label the scalars in the vector/tensor multiplets. The scalar constraint now reads as,

$$N \equiv C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1\ . \tag{5.4.3}$$

As always, we need to choose a suitable parametrisation to satisfy the above equation. We choose,

$$h^{\tilde{I}} = \sqrt{\frac{2}{3}}\xi^{\tilde{I}}\Big|_{N=1} , h_{\tilde{I}} = \frac{1}{\sqrt{6}}\frac{\partial}{\partial \xi^{\tilde{I}}}N\Big|_{N=1} , \quad (5.4.4)$$

such that the constraint takes the form,

$$N(\xi) = \sqrt{2}\xi^0[(\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2] = 1 , \quad (5.4.5)$$

where,

$$\xi^0 = \frac{1}{\sqrt{2}||\phi||^2} , \ \xi^1 = \phi^1 , \ \xi^2 = \phi^2 , \ \xi^3 = \phi^3 , \quad (5.4.6)$$

and,

$$||\phi||^2 = (\phi^1)^2 - (\phi^2)^2 - (\phi^3)^2 , \quad (5.4.7)$$

is assumed to be positive so that $a_{\tilde{I}\tilde{J}}$ and $g_{\tilde{x}\tilde{y}}$ are positive definite. The regions $\phi^1 > 0$ and $\phi^1 < 0$ are equivalent in the moduli space due to the above relation. However for our purposes we will stick to the region $\phi^1 > 0$ in the moduli space.

The $h^{\tilde{I}}$ are related to the fields ϕ in the Lagrangian through the following relations,

$$h^0 = \frac{1}{\sqrt{3}||\phi||^2} , \qquad h^1 = \sqrt{\frac{2}{3}}\phi^1 , \qquad h^2 = \sqrt{\frac{2}{3}}\phi^2 , \qquad h^3 = \sqrt{\frac{2}{3}}\phi^3 . \quad (5.4.8)$$

$$h_0 = \frac{1}{\sqrt{3}}||\phi||^2 , \qquad h_1 = \frac{2}{\sqrt{6}}\frac{\phi^1}{||\phi||^2} , \qquad h_2 = -\frac{2}{\sqrt{6}}\frac{\phi^2}{||\phi||^2} , \qquad h_3 = -\frac{2}{\sqrt{6}}\frac{\phi^3}{||\phi||^2} . \quad (5.4.9)$$

Using the above relations and the scalar constraint (5.2.3) we can read off the non-vanishing $C_{\tilde{I}\tilde{J}\tilde{K}}$ as,

$$C_{011} = \frac{\sqrt{3}}{2} , C_{022} = C_{033} = -\frac{\sqrt{3}}{2} . \quad (5.4.10)$$

The metric $a_{\tilde{I}\tilde{J}}$ computed using the relation (5.2.4), (5.4.8) and (5.4.9) is given by,

$$a_{\tilde{I}\tilde{J}} = \begin{pmatrix} ||\phi||^4 & 0 & 0 & 0 \\ 0 & 2(\phi^1)^2||\phi||^{-4} - ||\phi||^{-2} & -2\phi^1\phi^2||\phi||^{-4} & -2\phi^1\phi^3||\phi||^{-4} \\ 0 & -2\phi^1\phi^2||\phi||^{-4} & 2(\phi^2)^2||\phi||^{-4} + ||\phi||^{-2} & 2\phi^2\phi^3||\phi||^{-4} \\ 0 & -2\phi^1\phi^3||\phi||^{-4} & 2\phi^2\phi^3||\phi||^{-4} & 2(\phi^3)^2||\phi||^{-4} + ||\phi||^{-2} \end{pmatrix} . \tag{5.4.11}$$

The metric on the scalar manifold $g_{\tilde{x}\tilde{y}}$ is then computed using the completeness relations (5.2.5) and is given by,

$$g_{\tilde{x}\tilde{y}} = \begin{pmatrix} 4(\phi^1)^2||\phi||^{-4} - ||\phi||^{-2} & -4\phi^1\phi^2||\phi||^{-4} & -4\phi^1\phi^3||\phi||^{-4} \\ -4\phi^1\phi^2||\phi||^{-4} & 4(\phi^2)^2||\phi||^{-4} + ||\phi||^{-2} & 4\phi^2\phi^3||\phi||^{-4} \\ -4\phi^1\phi^3||\phi||^{-4} & 4\phi^2\phi^3||\phi||^{-4} & 4(\phi^3)^2||\phi||^{-4} + ||\phi||^{-2} \end{pmatrix} . \tag{5.4.12}$$

Recollect that the scalar manifold has the symmetry group $G = SO(1,1) \times SO(2,1)$. We consider the gauging of a compact subgroup $SO(2)$ for our purposes. Since it is an abelian group and has only one generator, the Killing vector which generates this symmetry can couple to one vector field. In this case, the gauge field is the graviphoton A^0_μ. The Killing vector that generates the $SO(2)$ symmetry is found by solving the Killing vector equation,

$$\tilde{\nabla}^{\tilde{x}} K_0^{\tilde{y}} + \tilde{\nabla}^{\tilde{y}} K_0^{\tilde{x}} = 0 \,, \tag{5.4.13}$$

where the covariant derivative $\tilde{\nabla}$ is with respect to the metric $g_{\tilde{x}\tilde{y}}$ on the scalar manifold. It can be checked that the following vector,

$$K_0^{\tilde{x}} = \left\{ -\frac{\phi^1}{||\phi||^2} \, , \frac{\phi^2}{||\phi||^2} \, , \frac{\phi^3}{||\phi||^2} \right\} , \tag{5.4.14}$$

is indeed a solution to the Killing equations. The $SO(2)$ symmetry which rotates the ϕ^2, ϕ^3 directions is manifest in (5.4.14).

The $U(1)_R$ gauging is done using a linear combination of gauge fields in the theory given by,

$$A_\mu(U(1)_R) = V_I A^I_\mu\,, \quad I = 0, 1, \tag{5.4.15}$$

where V_I are constant parameters. For a general gauging of a non abelian R symmetry the parameters V_I are constrained by the relation,

$$V_I f^I{}_{JK} = 0\,, \tag{5.4.16}$$

where $f^I{}_{JK}$ are structure constants of the gauge group. For abelian gauging such as the one considered here, the parameters V_I are free since the structure constants vanish.

Now, we can calculate the scalar potential using (5.3.11) by setting $\mathcal{N}_{iA}$ to zero since there are no hypermultiplets. We also have $P_{Iij} = V_I\delta_{ij}$ for $U(1)_R$ gauging, as discussed in the previous section. Using the relations (5.3.12) the potential (5.3.11) can be expressed as,

$$\mathcal{V} = \frac{3}{16}g^2\Omega^{MN}\Omega^{PQ}h_N h_Q h^{\tilde{x}}_M h^{\tilde{y}}_P g_{\tilde{x}\tilde{y}} - g_R^2[4h^I h^J V_I V_J - 2g_{\tilde{x}\tilde{y}}h^{\tilde{x}I}h^{\tilde{y}J}V_I V_J]\,, \tag{5.4.17}$$

where the $h^{\tilde{x}I}$ are as defined in (5.2.9), and the conventions for Ω_{MN} are,

$$\Omega_{MN} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.4.18}$$

After some simplifications using (5.4.8) and (5.4.9) the scalar potential of this model can be written as,

$$\mathcal{V}(\phi) = \frac{g^2}{8}\left[\frac{[(\phi^2)^2 + (\phi^3)^2]}{||\phi||^6}\right] - 2g_R^2\left[2\sqrt{2}\frac{\phi^1}{||\phi||^2}V_0V_1 + ||\phi||^2V_1^2\right]. \tag{5.4.19}$$

The critical points of this potential have been investigated in great detail in [19]. We will consider the case where the AdS vacuum preserves $\mathcal{N} = 2$ supersymmetry. This requires the condition [13],

$$W^{\tilde{a}}|_{\phi_c} = P^{\tilde{a}}_{ij}|_{\phi_c} = 0\,. \tag{5.4.20}$$

We will derive this condition in the next chapter by studying the Killing spinor integrability conditions for generalised attractors. The critical point which satisfies (5.4.20) is given by,

$$\phi^2 = 0\,, \quad \phi^3 = 0\,, \quad \phi^1_c = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}}\,, \tag{5.4.21}$$

together with the constraints,

$$V_0 V_1 > 0\,, \quad 32\frac{g_R^2}{g^2}V_0^2 \leq 1\,. \tag{5.4.22}$$

These constraints determine the nature of the critical point. In this case, the critical point is a saddle point with a maximum in the ϕ^1 direction and minima in ϕ^2, ϕ^3 directions. The value of the potential (5.4.19) at the critical point (5.4.21),

$$\mathcal{V}(\phi_c) = \Lambda = -6g_R^2(\phi^1_c)^2 V_1^2\,, \tag{5.4.23}$$

is the value of the AdS cosmological constant of the theory.

5.5 Summary

In this chapter, we have provided some background material in five dimensional $\mathcal{N} = 2$ gauged supergravity. First we studied the global symmetries of ungauged supergravity. We saw that the scalars in the vector multiplet parametrise a very special manifold and the scalars in the hypermultiplet parametrise a quaternionic Kähler manifold. We also studied the $\mathcal{N} = 2$ superconformal algebra and discussed global $SU(2)_R$ symmetry group.

We then discussed the gauging of a subgroup of the global symmetries of the ungauged theory. We saw that the symmetries can be gauged by introducing Killing vectors acting on the scalar manifold. These Killing vectors generate the group of symmetries which are to be gauged. The various gauge fields in the theory couple to the Killing vectors and the ordinary derivatives get replaced by gauge covariant derivatives. Due to the additional terms that appear because of the gauging, supersymmetric closure requires the existence of a potential term in gauged supergravity. We saw that the value of the potential at its critical point gives the cosmological constant of the theory. In particular, we saw that a subgroup of the R symmetry group has to be gauged to get a negative cosmological constant and hence AdS vacuum.

Then, we studied an example of a simple gauged supergravity with one vector multiplet. We demonstrated that by choosing an explicit parametrisation for the scalar constraint (5.2.3) the symmetries of the manifold can be made manifest. In particular, we showed that the metric on the scalar manifold can be written down explicitly and constructed the Killing vector which generates an abelian symmetry group. We also discussed the gauging of a $U(1)$ component of the $SU(2)_R$ symmetry group and explained that the parameters involved in the gauging are unconstrained for abelian gauging of the R symmetry group. We saw that the potential in this model has critical points which gave rise to an AdS vacuum.

Chapter 6

Generalised attractors in gauged supergravity

6.1 Introduction

In chapter 3, we saw that the attractor mechanism explains the macroscopic entropy of extremal black holes in supergravity [4–6]. The moduli fields in a given black hole background flow radially to a fixed value at the horizon. The constant values of the scalars are obtained by solving a set of algebraic equations called the attractor equations and are given in terms of the quantised charges of the black hole. The entropy of the black hole is then given in terms of the charges and is independent of the asymptotic values of the moduli fields. The attractor mechanism works not mainly because of supersymmetry but due to extremality of the black hole [94] and it also extends to non-supersymmetric cases [95, 97].

Recently, there is much interest in generalising the attractor mechanism to gauged supergravities [14, 54–60, 156]. The main motivation of these works is to understand the attractor mechanism for black holes in AdS spaces. Gauged supergravities are ideal grounds for this study due to the availability of AdS vacuum. Another motivation from the AdS/CFT correspondence is that AdS black holes are dual to field theories at finite temperature [11]. In the application of AdS/CFT to condensed matter theories, charged extremal black branes are duals to the zero temperature phases of the field theory. Typical examples are black holes with Lifshitz like near horizon geometries and AdS asymptotics that are duals to field theories with a violation of Lorentz symmetry [47]. The attractor behaviour for such solutions has been studied for charged black branes interpolating between the two geometries in [115, 116]. More recently the classification of homogeneous anisotropic extremal black brane horizons has been studied in five dimensions and examples were constructed in Einstein-Maxwell theories with massive gauge fields [7, 53].

We discussed these Bianchi attractors in Chapter 4 and saw that they possess an important common property, i.e., they have constant anholonomy coefficients. A prescription has been given to obtain generalised attractors characterised by constant anholonomy, from gauged supergravity [14]. In this framework one sets all the fields and the curvature components to constants in the tangent space. The generalised attractor geometries are characterised by constant anholonomy coefficients. At the attractor point, the equations of motion become algebraic and the attractor potential takes a simple form. Moreover, due to constant anholonomy the components of the Riemann tensor become constants independent of spacetime coordinates in tangent space. It follows that the curvature invariants are constant and hence the attractor geometries characterised by constant anholonomy coefficients are regular.

In this chapter, we generalise the analysis of [14] to $\mathcal{N} = 2$ gauged supergravity in five dimensions coupled to arbitrary number of hyper, vector and tensor multiplets and derive the attractor potential. We then consider a simple gauged supergravity model (see §5.4) in five dimensions with one vector and two tensor multiplets [19] and show that some of the Bianchi type solutions considered recently in [7] can be realised as generalised attractor solutions in this model. We construct explicit solutions of Bianchi I, Bianchi

II , Bianchi VI types in this model [17]. In a $U(1)_R$ gauged version of this model, we construct $Lif_{u_0} \times M_I$, $Lif_{u_0} \times M_{II}$ solutions whose symmetry groups factorise as direct products [21].

The organisation of this chapter is as follows. In §6.2, we define the generalised attractor ansatz and show that the gauge field, tensor field, Einstein and scalar field equations become algebraic at the attractor points. We also derive the attractor potential from the scalar field equations. In §6.3, we analyse the attractor potential, construct it from generalised fermionic shifts and give the conditions for maximal supersymmetry. Then we use the generalised attractor procedure to construct explicit examples within a simple gauged supergravity model in §6.4 and §6.5. We then summarise the results in §6.6.

6.2 Generalised Attractors

In this section, we consider the $\mathcal{N} = 2, d = 5$ gauged supergravity coupled to vector, tensor and hypermultiplets discussed in chapter 5 and show that the equations of motion reduce to algebraic equations in the tangent space. For convenience, we recall the Lagrangian of the five dimensional gauged supergravity (5.3.9) ,

$$\begin{aligned}\hat{e}^{-1}\mathcal{L}^{N=2}_{Bosonic} = & -\frac{1}{2}R - \frac{1}{4}a_{\tilde{I}\tilde{J}}\mathcal{H}^{\tilde{I}}_{\mu\nu}\mathcal{H}^{\tilde{J}\mu\nu} - \frac{1}{2}g_{XY}\mathcal{D}_\mu q^X \mathcal{D}^\mu q^Y - \frac{1}{2}g_{\tilde{x}\tilde{y}}\mathcal{D}_\mu \phi^{\tilde{x}}\mathcal{D}^\mu \phi^{\tilde{y}} \\ & + \frac{\hat{e}^{-1}}{6\sqrt{6}}C_{IJK}\epsilon^{\mu\nu\rho\sigma\tau}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\tau + \frac{\hat{e}^{-1}}{4g}\epsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}B^M_{\mu\nu}\mathcal{D}_\rho B^N_{\sigma\tau} \\ & - \mathcal{V}(\phi, q)\,,\end{aligned} \tag{6.2.1}$$

where $\mathcal{V}(\phi, q)$ is the scalar potential.

We consider attractors with the following ansatz for the scalar, vector and tensor fields at the attractor point,

$$\phi^{\tilde{z}} = \text{const}\,, q^Z = \text{const}\,, A^I_a = \text{const}\,, B^M_{ab} = \text{const}\,, c_{bc}{}^a = \text{const}\,. \tag{6.2.2}$$

Here, in addition to the assumptions considered in [14], we take the tensor fields to be constant along the tangent space. As we will see, this is necessary in order to reduce the field equations to attractor equations (which are algebraic equations for all practical purposes). The constancy of ${c_{bc}}^a$ ensure the regularity of the resultant geometry and together with constant A^I_a, they ensure that the field strengths,

$$F_{ab} = e^\mu_a e^\nu_b(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu)A_c = {c_{ab}}^c A_c \; , \tag{6.2.3}$$

are constant at the attractor points, which is expected for an attractor behaviour.

6.2.1 Equations of Motion

We now analyse the equations of motion of the bosonic fields in $\mathcal{N} = 2, d = 5$ gauged supergravity. We first derive the gauge field and tensor field equations. Subsequently we discuss the Einstein's equations followed by the equation of motion for the scalars and quarternions which leads to the attractor potential.

Gauge Fields and Tensor fields

The Lagrangian for the $\mathcal{N} = 2, d = 5$ theory contains tensor fields and a Chern Simons term which contribute to the gauge field equation of motion given by,

$$\begin{aligned}\partial_\mu(\hat{e} a_{I\tilde{J}} \mathcal{H}^{\tilde{J}\mu\nu}) = &- \frac{1}{2\sqrt{6}} C_{I\tilde{J}\tilde{K}} \epsilon^{\nu\mu\rho\sigma\tau} \mathcal{H}^{\tilde{J}}_{\mu\rho} \mathcal{H}^{\tilde{K}}_{\sigma\tau} \\ &+ g\hat{e}[g_{XY} K^X_I \mathcal{D}^\nu q^Y + g_{\tilde{x}\tilde{y}} K^{\tilde{x}}_I \mathcal{D}^\nu \phi^{\tilde{y}}] \, .\end{aligned} \tag{6.2.4}$$

We have used the Bianchi identity $d(*F) = 0$, the symmetry of $C_{\tilde{I}\tilde{J}\tilde{K}}$ [1] and $\Omega_{MN}\Lambda^N_{IP} = \frac{2}{\sqrt{6}} C_{PMI}$ [18] for simplification. One can then use the following relations for antisymmet-

[1] As mentioned earlier in (5.3.8), the only non vanishing components are C_{IJK} and C_{IMN}, which can be used to write $C_{IJK}F^J F^K + C_{IMN}B^M B^N = C_{I\tilde{J}\tilde{K}}\mathcal{H}^{\tilde{J}}\mathcal{H}^{\tilde{K}}$.

ric tensors,

$$\partial_\mu(\hat{e}G^{\mu\nu}) = \hat{e}\nabla_\mu G^{\mu\nu} \ , \ \partial_\mu\epsilon^{\mu\nu\rho\sigma\tau} = -\frac{\partial_\mu\hat{e}}{\hat{e}}\epsilon^{\mu\nu\rho\sigma\tau} \ , \tag{6.2.5}$$

and change to tangent space. Since the scalars, gauge fields, field strengths and tensor fields are constant at the attractor points (6.2.2) the derivatives drop out and we get,

$$\partial_a F^{abI} = 0, \ \partial_a A^{bI} = 0, \ \partial_a B^M_{bc} = 0, \ \partial_a \phi^{\tilde{z}} = 0, \ \partial_a q^Z = 0 \ . \tag{6.2.6}$$

In tangent space the gauge field equation can be written as,

$$\begin{aligned}\hat{e}\ a_{I\tilde{J}}[\omega_{a,}{}^a{}_c\mathcal{H}^{cb\tilde{J}} + \omega_{a,}{}^b{}_c\mathcal{H}^{ac\tilde{J}}] = &- \frac{1}{2\sqrt{6}}C_{I\tilde{J}\tilde{K}}\epsilon^{bacde}\mathcal{H}^{\tilde{J}}_{ac}\mathcal{H}^{\tilde{K}}_{de} \\ &+ g^2\hat{e}[g_{XY}K^X_I K^Y_J + g_{\tilde{x}\tilde{y}}K^{\tilde{x}}_I K^{\tilde{y}}_J]A^{Jb} \ .\end{aligned} \tag{6.2.7}$$

The spin connection is expressed in terms of anholonomy coefficients in the absence of torsion (A.0.4). Hence, constant anholonomy implies constant spin connection. Thus (6.2.7) is an algebraic equation at the attractor point.

Similarly the tensor field equation can be worked out as,

$$\frac{1}{g}\epsilon^{\mu\nu\rho\sigma\tau}\Omega_{MP}\mathcal{D}_\rho B^M_{\mu\nu} + \hat{e}a_{\tilde{I}P}\mathcal{H}^{\tilde{I}\sigma\tau} = 0 \ . \tag{6.2.8}$$

Note that the Bianchi identity for the B-field, $d(B^M_{\mu\nu}) \neq 0$ in general [18]. So we will not be able to use it for simplification. Converting the above to tangent space, we get,

$$\frac{1}{g}\epsilon^{abcde}[c_{ac}{}^f B^M_{fb} + gA^I_c\Lambda^M_{IN}B^N_{ab}]\Omega_{MP} + \hat{e}a_{\tilde{I}P}\mathcal{H}^{\tilde{I}de} = 0 \ . \tag{6.2.9}$$

As in the previous case, the equation of motion (6.2.9) reduced to an algebraic equation at the attractor point.

Einstein equation

The Einstein equation for the Lagrangian (5.3.9) at the attractor point is given by,

$$R_{ab} - \frac{1}{2} R \eta_{ab} = T_{ab}^{attr} \,. \tag{6.2.10}$$

At the attractor point, the Riemann tensor is a function of spin connections, which are in turn expressed in terms of the constant anholonomy coefficients (A.0.3). This also applies to the Ricci tensor and the scalar curvature. As a consequence, the left hand side of the Einstein equation is algebraic in $c_{ab}{}^{c}$. The stress energy tensor at the attractor points is given by,

$$T_{ab}^{attr} = \mathcal{V}_{attr}(\phi, q)\eta_{ab} - \left[a_{\tilde{I}\tilde{J}} \mathcal{H}_{ac}^{\tilde{I}} \mathcal{H}_{b}{}^{c\tilde{J}} + g^2 [g_{XY} K_I^X K_J^Y + g_{\tilde{x}\tilde{y}} K_I^{\tilde{x}} K_J^{\tilde{y}}] A_a^I A_b^J \right] . \tag{6.2.11}$$

As one can see, the energy momentum tensor is a function of constant scalars, gauge fields and field strengths at the attractor points and hence the Einstein equation reduces to an algebraic equation at the attractor points. Note the appearance of the attractor potential $\mathcal{V}_{attr}(\phi, q)$ which is defined in (6.3.2). Later, we show that the $\mathcal{V}_{attr}(\phi, q)$ follows from the scalar field equations of motion and can be constructed from generalised fermion shifts of the supersymmetry transformations.

Scalar and Quaternions

The equation of motion for the scalars $\phi^{\tilde{x}}$ in the vector and tensor multiplet is given by,

$$\begin{aligned} \hat{e}^{-1}\partial_\mu[\hat{e}\, g_{\tilde{z}\tilde{y}} \mathcal{D}^\mu \phi^{\tilde{y}}] - \frac{1}{2}\frac{\partial g_{\tilde{x}\tilde{y}}}{\partial \phi^{\tilde{z}}} \mathcal{D}_\mu \phi^{\tilde{x}} \mathcal{D}^\mu \phi^{\tilde{y}} - g A_\mu^I g_{\tilde{x}\tilde{y}} \frac{\partial K_I^{\tilde{x}}}{\partial \phi^{\tilde{z}}} \mathcal{D}^\mu \phi^{\tilde{y}} \\ - \frac{1}{4}\frac{\partial a_{\tilde{I}\tilde{J}}}{\partial \phi^{\tilde{z}}} \mathcal{H}_{\mu\nu}^{\tilde{I}} \mathcal{H}^{\tilde{J}\mu\nu} - \frac{\partial \mathcal{V}(\phi, q)}{\partial \phi^{\tilde{z}}} = 0 \,. \end{aligned} \tag{6.2.12}$$

Using the ansatz (6.2.2), it can be shown that the above scalar field equation reduces to the following form in the tangent space,

$$\frac{\partial}{\partial\phi^{\tilde{z}}}\left[\mathcal{V}(\phi,q)+\frac{1}{2}g^2\, g_{\tilde{x}\tilde{y}}K_I^{\tilde{x}}K_J^{\tilde{y}}A^{Ia}A_a^J+\frac{1}{4}a_{\tilde{I}\tilde{J}}\mathcal{H}_{ab}^{\tilde{I}}\mathcal{H}^{\tilde{J}ab}\right]=0\,. \tag{6.2.13}$$

For the quaternion q^Z, the equation of motion reads as,

$$\hat{e}^{-1}\partial_\mu[\hat{e}\; g_{ZY}\mathcal{D}^\mu q^Y]-\frac{1}{2}\frac{\partial g_{XY}}{\partial q^Z}\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y-gA_\mu^I g_{XY}\frac{\partial K_I^X}{\partial q^Z}\mathcal{D}^\mu q^Y \\ -\frac{\partial\mathcal{V}(\phi,q)}{\partial q^Z}=0\,. \tag{6.2.14}$$

Using the ansatz (6.2.2), the quaternion equation of motion (6.2.14) in tangent space reduces to,

$$\frac{\partial}{\partial q^Z}\left[\mathcal{V}(\phi,q)+\frac{1}{2}g^2 g_{XY}K_I^X K_J^Y A^{aI}A_a^J\right]=0\,. \tag{6.2.15}$$

As one can see from (6.2.13) and (6.2.15), the equation of motion for the scalars at the attractor point reduces to an extremization condition on a potential.

6.3 Attractor Potential

We define our attractor potential to be the one which gives rise to the attractor values of the scalars and quaternions upon extremization,

$$\frac{\partial\mathcal{V}_{attr}(\phi,A)}{\partial\phi}=0\,. \tag{6.3.1}$$

Thus, observing the equations of motion for the scalars (6.2.13) and the quaternions (6.2.15) the attractor potential for the $\mathcal{N}=2, d=5$ gauged supergravity can be constructed to have the form,

$$\mathcal{V}_{attr}(\phi,q)=\left[\mathcal{V}(\phi,q)+\frac{1}{2}g^2[\, g_{\tilde{x}\tilde{y}}K_I^{\tilde{x}}K_J^{\tilde{y}}+g_{XY}K_I^X K_J^Y]A^{Ia}A_a^J+\frac{1}{4}a_{\tilde{I}\tilde{J}}\mathcal{H}_{ab}^{\tilde{I}}\mathcal{H}^{\tilde{J}ab}\right]. \tag{6.3.2}$$

Note the similarity of the attractor potential (6.3.2) with the one obtained in [14] for $\mathcal{N} = 2, d = 4$ gauged supergravity. This is expected since both the theories have the same supersymmetries and the quaternionic structure. The difference is in the reality of the scalar fields and the presence of tensor fields, which contribute to the attractor potential. Thus (6.3.2) obeys both (6.2.13) and (6.2.15). Note that, this exact form of the attractor potential (6.3.2) also appears in the Einstein equation. Now, we show that the potential can be constructed from fermion shifts defined at the attractor points.

In chapter 5, we saw that gauging introduces additional terms in the Lagrangian that depend on the gauge coupling and for supersymmetry to be preserved the supersymmetry transformations have to be modified accordingly. These additional terms in the supersymmetry transformations can be incorporated in terms of what are called as the fermion shifts, which are usually defined as the non-derivative scalar dependent bosonic terms in the supersymmetry transformations of the fermions in the theory (see for eg [12]).

The notion of fermionic shifts was generalised by [14]. The shifts at the attractor points and included terms that depend on constant gauge fields and field strengths. It was shown that the attractor potential can be written as squares of the generalised fermion shifts. In our case the generalised fermion shifts contain terms that depend on constant tensor fields as well. We will use a notation similar to that of [12] for defining the generalised fermion shifts.

The supersymmetry transformations (5.3.13), take the following form at the attractor points defined by (6.2.2),

$$\begin{aligned}
\delta\psi_{ai} &= \sqrt{6}D_a\epsilon_i + (\Sigma_{i|j})^{bc}(\gamma_{abc} - 4\eta_{ab}\gamma_c)\epsilon^j + \gamma_a S_{ij}\epsilon^j \,, \\
\delta\lambda_i^{\tilde{a}} &= \Sigma^{\tilde{a}}{}_{i|j}\epsilon^j + (\Sigma^{\tilde{a}}{}_{i|j})^a\gamma_a\epsilon^j + (\Sigma^{\tilde{a}}{}_{i|j})^{ab}\gamma_{ab}\epsilon^j \,, \\
\delta\zeta^A &= (\Sigma^A{}_{|j})\epsilon^j + (\Sigma^A{}_{|j})^a\gamma_a\epsilon^j \,,
\end{aligned} \tag{6.3.3}$$

where, the gravitino mass matrix and shifts are given by,

$$
\begin{aligned}
\Sigma^{\tilde{a}}{}_{i|j} &= g_R P^{\tilde{a}}{}_{ij} - g W^{\tilde{a}} \epsilon_{ij} \quad , \quad (\Sigma^{A}{}_{|j}) = g \mathcal{N}_j{}^{A} \quad , S_{ij} = i g_R P_{ij} \,, \\
(\Sigma^{\tilde{a}}{}_{i|j})^{a} &= \frac{i}{2} g f^{\tilde{a}}_{\tilde{x}} K^{\tilde{x}}_I A^{Ia} \epsilon_{ij} \quad , \quad (\Sigma^{A}{}_{|j})^{a} = -\frac{i}{2} g f^{A}_{jX} K^{X}_I A^{aI} \quad , \\
(\Sigma^{\tilde{a}}{}_{i|j})^{ab} &= -\frac{1}{4} h^{\tilde{a}}_{\tilde{I}} \mathcal{H}^{\tilde{I}ab} \epsilon_{ij} \quad , \quad (\Sigma_{i|j})^{bc} = -\frac{i}{4} h_{\tilde{I}} \mathcal{H}^{bc\tilde{I}} \epsilon_{ij} \quad ,
\end{aligned} \tag{6.3.4}
$$

Using the relations (5.2.4),(5.2.7), and (5.2.25) the attractor potential (6.3.2) can be written in terms of the shifts (6.3.4) and their complex conjugates as follows,

$$
\begin{aligned}
-\mathcal{V}_{attr} \frac{\epsilon^{l}{}_{k}}{4} = \bar{S}^{i}{}_{k} S_{i}{}^{l} - \epsilon^{lj} \Big\{ & [(\overline{\Sigma^{A}{}_{|k}})(\Sigma_{A|j}) + \frac{1}{2} (\overline{\Sigma^{\tilde{a}i}{}_{|k}})(\Sigma^{\tilde{a}}{}_{i|j})] \\
& + [(\overline{\Sigma^{A}{}_{|k}})_a (\Sigma_{A|j})^a + \frac{1}{2} (\overline{\Sigma^{\tilde{a}i}{}_{|k}})_a (\Sigma^{\tilde{a}}{}_{i|j})^a] \\
& + [(\overline{\Sigma^{i}{}_{|k}})_{ab} (\Sigma_{i|j})^{ab} + (\overline{\Sigma^{\tilde{a}i}{}_{|k}})_{ab} (\Sigma^{\tilde{a}}{}_{i|j})^{ab}] \Big\} \,.
\end{aligned} \tag{6.3.5}
$$

This relation for the attractor potential is similar to the one obtained in [14] for $\mathcal{N} = 2, d = 4$ gauged supergravity. In fact, it seems that such a result could be derived for any gauged supergravity in arbitrary dimension for an attractor ansatz similar to (6.2.2).

The form of the attractor potential also makes evident the condition for the attractor to respect maximal supersymmetry. For example, the integrability condition from the gravitino supersymmetry transformation is given by,

$$
\begin{aligned}
\frac{1}{4} R_{ae}{}^{cd} \gamma_{cd} \epsilon_i = & \frac{1}{\sqrt{6}} (\Sigma_{i|j})^{fc} [\omega_{a,}{}^{b}{}_{f} (M_{ebc} - M_{ecb}) - \omega_{e,}{}^{b}{}_{f} (M_{abc} - M_{acb})] \chi^j \\
& + \frac{1}{6} \Big\{ [(\Sigma_{i|j})^{bc} M_{abc} + \gamma_a S_{ij}][(\Sigma_{k|l})^{gh} M_{egh} + \gamma_e S_{kl}] \\
& - [(\Sigma_{i|j})^{bc} M_{ebc} + \gamma_e S_{ij}][(\Sigma_{k|l})^{gh} M_{agh} + \gamma_a S_{kl}] \Big\} \epsilon^{jk} \epsilon^{l} \,,
\end{aligned} \tag{6.3.6}
$$

where $M_{abc} = \gamma_{abc} - 4\eta_{ab}\gamma_c$. We further simplify (6.3.6) by expanding and writing in terms of the linearly independent matrix basis $\{I, \gamma_a, \gamma_{ab}\}$ in five dimensions (see Appendix B).

We get the constraints from the coefficients of γ_{ab} terms,

$$\frac{1}{4}R_{ae}{}^{cd}\gamma_{cd}\epsilon_i = -\frac{2i}{\sqrt{6}}(\Sigma_{i|j})_f{}^{[c}\omega_{[a,}{}^{b]f}\epsilon_{e]bcgh}\gamma^{gh}\epsilon^j$$
$$+\frac{1}{6}\Big\{(\Sigma_{i|j})^{bc}(\Sigma_{k|l})^{gh}(-36\eta_{ef}\eta_{gp}\eta_{hq}\delta_{[ab}{}^{[fp}\gamma_{c]}{}^{q]}+32\eta_{ab}\eta_{eg}\gamma_{ch})$$
$$+16(\Sigma_{i|j})^{bc}S_{kl}\eta_{b[a}\gamma_{e]c}+2S_{ij}S_{kl}\gamma_{ae}\Big\}\epsilon^{jk}\epsilon^l\ , \tag{6.3.7}$$

and the and γ_a terms,

$$-\frac{16}{\sqrt{6}}(\Sigma_{i|j})_f{}^{[c}\omega_{[a,}{}^{b]f}\eta_{e]b}\gamma_c\epsilon^j+\frac{1}{6}\Big\{16i(\Sigma_{i|j})^{bc}(\Sigma_{k|l})^{gh}\eta_{g[a}\epsilon_{e]hbcp}\gamma^p$$
$$+4i(\Sigma_{i|j})^{bc}S_{kl}\epsilon_{aebcp}\gamma^p\Big\}\epsilon^{jk}\epsilon^l=0\ . \tag{6.3.8}$$

For maximal supersymmetry each of these independent terms should vanish identically. This can be easily achieved when all the fermionic shifts (6.3.4) vanish. The above equations reduce to,

$$\frac{1}{4}R_{ae}{}^{cd}\gamma_{cd}\epsilon_i=\frac{1}{3}S_{ij}S_{kl}\gamma_{ae}\epsilon^{jk}\epsilon^l\ . \tag{6.3.9}$$

Which can be further simplified using $\delta_{ij}\delta_{kl}\epsilon^{jk}\epsilon^l=-\epsilon_i$ as,

$$\Big(\frac{1}{4}R_{ae}{}^{cd}\gamma_{cd}-\frac{1}{3}g_R^2h^Ih^JV_IV_J\gamma_{ae}\Big)\epsilon_i=0\ . \tag{6.3.10}$$

The above is just the vacuum Einstein equation in $\mathcal{N}=2, d=5$ gauged supergravity for arbitrary ϵ_i. The bosonic term in the right hand side is the cosmological constant, and the above equation implies Einstein equation in the absence of matter. There are no algebraic constraints on the spinors from the supersymmetry transformations of $\lambda_i^{\tilde{a}}$ and ζ^A when all the fermionic shifts vanish. In such a scenario, AdS_5 is a unique maximally supersymmetric ground state of this theory [13, 157]. There could also be solutions such as BPS domain walls, which preserve maximal supersymmetry. For example, In the absence of tensor multiplets, one has $W^{\tilde{a}}=0$. In addition, when the vector fields vanish $\Sigma^{\tilde{a}}{}_{i|j}$ and $\Sigma^A{}_{|j}$ are the only non-vanishing fermionic shifts in (6.3.4). Maximal supersymmetry requires that there should be no algebraic constraints on the spinors, therefore the following terms must vanish

$$P^{\tilde{a}}{}_{ij} = h^{\tilde{a}I} P_{Iij} = 0\,, \qquad \mathcal{N}_j{}^A = \frac{\sqrt{6}}{4} h^I K_I^X f_X^{Ai} = 0\,. \tag{6.3.11}$$

The above equations lead to the attractor conditions derived in [54] for domain wall solutions that interpolate between AdS vacua. Such planar domain wall solutions characterised by constant anholonomy coefficients are non-trivial examples of supersymmetric generalised attractors.

For non-supersymmetric attractors or attractors that preserve a part of the supersymmetry there are non-vanishing shifts. Hence there will be constraints on the spinors, as a result of which one will either have some amount of supersymmetry preserved (which is expected at least for Lifshitz solutions [15,16]) or none at all. In the cases where one deals only with vector multiplets, the integrability conditions on the Killing spinors have been worked out in [158] and the constraints imply that one gets either $1/2$ BPS or $1/4$ BPS solutions.

6.4 Bianchi Attractors in gauged supergravity with one vector multiplet

In this section, we show that some of the simple Bianchi type metrics can be realised from simple $d = 5$ gauged supergravity models. Our objective here is not to be exhaustive regarding the possibilities, as this has already been considered in [7]. We will take some of the explicit examples discussed in chapter 4 and show that they can be obtained from the gauged supergravity model discussed in chapter 5.4. We are motivated by the observation that most of the Bianchi attractors constructed in [7] are sourced by massive gauge fields. In gauged supergravity one does not have explicit massive gauge fields, as these would break supersymmetry. Nevertheless, expanding the scalar kinetic term (see (5.3.9)) one gets terms like ,

$$g^2 g_{\tilde{x}\tilde{y}} K_I^{\tilde{x}} K_J^{\tilde{y}} A_a^I A^{Ja} \,, \tag{6.4.1}$$

that are proportional to square of the gauge field. These terms appear due to the presence of covariant derivatives which appear due to the gauging. Since the scalars are constant at the attractor point, the coefficients of these terms act like a mass for the gauge field. Hence, one can expect to realise the Bianchi attractors from specific truncations of gauged supergravity models. Firstly, the vaccum AdS_5 solution is given by,

$$\begin{aligned}
& ds^2 = L^2\Big[-\hat{r}^2 d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2)\Big] \,, \\
& \phi_c^2 = 0 \,, \quad \phi_c^3 = 0 \,, \quad \phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}} \,, \quad \Lambda = -6 g_R^2 V_1^2 (\phi_c^1)^2 \,, \\
& V_0 V_1 > 0 \,, \quad 32\frac{g_R^2}{g^2} V_0^2 \leq 1 \,, \quad L^2 = -\frac{6}{\Lambda} \,,
\end{aligned} \tag{6.4.2}$$

where Λ is the cosmological constant. The value of the cosmological constant comes from the potential of the gauged supergravity model discussed in chapter 5.4.

6.4.1 Bianchi type I: Lifshitz solution

We now look for Lifshitz like solutions within the model described in §5.4. We take the metric ansatz to be of the form given in (4.4.10),

$$ds^2 = L^2\Big[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2\Big((\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2\Big)\Big] \,, \tag{6.4.3}$$

where the invariant one forms (4.5.4) are,

$$\omega^1 = d\hat{x} \,, \quad \omega^2 = d\hat{y} \,, \quad \omega^3 = d\hat{z} \,. \tag{6.4.4}$$

We will solve the equations of motion derived in §6.2.1 for the Lifshitz metric in this model. The Lifshitz solutions considered in the literature are often sourced by massive time like gauge fields [15, 16]. We assume that the $SO(2)$ gauge field [2] has only time like component given by,

$$A^{0t} = e^{\hat{t}}_{\bar{0}} A^{0\bar{0}} = \frac{\hat{r}^{-u_0}}{L} A^{0\bar{0}} \,. \tag{6.4.5}$$

We do not make any assumptions on the other gauge field A^1. We take it to be of the general form $A^{1\mu} = e^{\mu}_{a} A^{1a}$. As a further simplification we also assume that all the tensor field components vanish. This need not be true for a more general theory with different gauging or a different metric ansatz. However, we find that the Lifshitz like ansatz does not admit any consistent solution with non-vanishing tensor fields within the model considered. We will explain the reason for this towards the end of the section.

As before, we will work in tangent space. The undetermined parameters are,

$$A^{0\bar{0}}\,, A^{1\bar{0}}\,, A^{1\bar{1}}\,, A^{1\bar{2}}\,, A^{1\bar{3}}\,, A^{1\bar{4}}\,, u_0\,, L\,, \tag{6.4.6}$$

where u_0 is the exponent and L is the size of the spacetime. These are to be determined in terms of the gauge couplings g, g_R and the free parameters V_0, V_1 which are constrained by (5.4.22).

The equations (6.2.4) for the gauge fields A^0 and A^1 evaluated at the critical point (5.4.22) read as follows:

$$\begin{aligned}
\hat{e} A^{0\bar{0}}(g^2 L^2 - u_0 (\phi^1_c)^8) &= 0\,, \\
\hat{e} A^{1\bar{0}} u_0 &= 0\,, \\
\hat{e} A^{1\bar{2}}(2 + u_0) &= 0\,, \\
\hat{e} A^{1\bar{3}}(2 + u_0) &= 0\,, \\
\hat{e} A^{1\bar{4}}(2 + u_0) &= 0\,,
\end{aligned} \tag{6.4.7}$$

[2]We have used he notation A^{Ia} earlier with I labelling the vectors and a the tangent space index. In component form, to avoid confusion we denote the tangent space indices with with an overbar, i.e $A^{0\bar{0}}$

whereas the off-diagonal components of the Einstein field equations are,

$$
\begin{aligned}
A^{1\bar{0}}A^{1\bar{2}}u_0 &= 0 \,, \\
A^{1\bar{0}}A^{1\bar{3}}u_0 &= 0 \,, \\
A^{1\bar{0}}A^{1\bar{4}}u_0 &= 0 \,, \\
A^{1\bar{2}}A^{1\bar{3}} &= 0 \,, \\
A^{1\bar{2}}A^{1\bar{4}} &= 0 \,, \\
A^{1\bar{3}}A^{1\bar{4}} &= 0 \,.
\end{aligned} \tag{6.4.8}
$$

The gauge field equations of motion (6.4.7) imply that $A^{1\bar{0}} = 0$ for a non-zero u_0. The off-diagonal Einstein equations imply that any two of the three components $A^{1\bar{2}}, A^{1\bar{3}}, A^{1\bar{4}}$ must vanish. If we take say $A^{1\bar{3}} = A^{1\bar{4}} = 0$, then the gauge field equation for $A^{1\bar{2}}$ give $u_0 = -2$ which is inconsistent with the equation for $A^{0\bar{0}}$. Hence, we set all three of them to zero. Note that this still leaves $A^{1\bar{1}}$ unfixed[3]. With these simplifications, the diagonal $(\hat{t}\hat{t}, \hat{r}\hat{r}, \hat{x}\hat{x})$ components of the Einstein equation are,

$$12(\phi_c^1)^4 + (A^{0\bar{0}})^2(3g^2L^2 + {u_0}^2(\phi_c^1)^8) - 24L^2g_R^2V_0^2 = 0 \,, \tag{6.4.9}$$

$$-6(1+u_0)(\phi_c^1)^4 + (A^{0\bar{0}})^2(3g^2L^2 - u_0^2(\phi_c^1)^8) + 24L^2g_R^2V_0^2 = 0 \,, \tag{6.4.10}$$

$$-2(3+2u_0+u_0^2)(\phi_c^1)^4 + (A^{0\bar{0}})^2(3g^2L^2 + u_0^2(\phi_c^1)^8) + 24L^2g_R^2V_0^2 = 0 \,. \tag{6.4.11}$$

The $(\hat{y}\hat{y}, \hat{z}\hat{z})$ components give the same equations as the $\hat{x}\hat{x}$ one. Subtracting (6.4.10) and (6.4.11), $A^{0\bar{0}}$ can be determined as,

$$A^{0\bar{0}} = \sqrt{\frac{u_0 - 1}{u_0}}\frac{1}{(\phi_c^1)^2} \,, \tag{6.4.12}$$

where we have chosen the positive sign for $A^{0\bar{0}}$. The values of L and u can be determined

[3]For all the Bianchi classes the Field strengths do not depend upon $A^{1\bar{1}}$, so this component can enter only through an $A^{1a}A^1_a$ term or the Chern-Simons term. The former does not happen here as A^{1a} is not used to gauge the symmetries of the scalar manifold. The latter does not occur since topological terms do not contribute in this case.

from the gauge field and scalar field equations. Substituting $A^{1\bar{2}} = A^{1\bar{3}} = A^{1\bar{4}} = 0$, the gauge field equations (6.4.7) reduce to,

$$g^2L^2 - u_0(\phi_c^1)^8 = 0\,. \tag{6.4.13}$$

The scalar field equations (6.2.12) evaluated at the attractor point (5.4.22), must vanish and this gives the relation,

$$3g^2L^2 - u_0^2(\phi_c^1)^8 = 0\,. \tag{6.4.14}$$

The two equations (6.4.13) and (6.4.14) can be solved to get,

$$u_0 = 3\,, \quad L = \sqrt{3}\frac{(\phi_c^1)^4}{g}\,. \tag{6.4.15}$$

Substituting the values of (6.4.15), (6.4.12) in (6.4.9), one gets the following constraint that relates the free parameters V_0, V_1 to the ratio of the couplings g and g_R.

$$\frac{1}{3(\phi_c^1)^4} = \frac{g_R^2}{g^2}V_0^2\,. \tag{6.4.16}$$

Let us summarise the solution,

$$\begin{aligned}
&ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2)\Big]\,,\\
&u_0 = 3, \quad L = \sqrt{3}\frac{(\phi_c^1)^4}{g}\,, \quad A^{0\bar{0}} = \sqrt{\frac{2}{3}}\frac{1}{(\phi_c^1)^2}\,,\\
&\phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}}\,, \quad V_0V_1 > 0\,, \quad \frac{32}{3(\phi_c^1)^4} \leq 1\,.
\end{aligned} \tag{6.4.17}$$

The attractor potential for the above solution is given by,

$$\mathcal{V}_{attr}(\phi_c^1) = -\left[\frac{(A^{0\bar{0}})^2}{2(\phi_c^1)^4}\left(3g^2 + u^2\frac{(\phi_c^1)^8}{L^2}\right) - \mathcal{V}_{AdS}\right]\,, \tag{6.4.18}$$

where $\mathcal{V}_{AdS} = -6g_R^2(\phi_c^1)^2V_1^2$ is the cosmological constant.

The attractor potential can be written in terms of the fermionic shifts (6.3.4) defined earlier. The shifts $(\Sigma^{A}_{|j})^{a}, \Sigma^{A}_{|j}$ vanish since there are no hypermultiplets in the theory. The shift $\Sigma^{\tilde{a}}_{i|j}$ vanishes due to the choice (5.4.22)[4]. The remaining shifts are non-vanishing at the attractor points. Since some of the shifts are non-vanishing, the solution (6.4.17) preserves only a part of the supersymmetry.

6.4.2 Bianchi II

In this subsection we give the Bianchi type II solution arising (4.5.11) from gauged supergravity. We recollect the metric,

$$ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2(u_1+u_3)}(\omega^1)^2 + \hat{r}^{2u_3}(\omega^2)^2 + \hat{r}^{2u_1}(\omega^3)^2\Big]\,, \tag{6.4.19}$$

where the invariant one forms (4.5.8) are,

$$\omega^1 = d\hat{y} - \hat{x}d\hat{z}\,, \quad \omega^2 = d\hat{z}\,, \quad \omega^3 = d\hat{x}\,, \tag{6.4.20}$$

The analysis is entirely parallel to the previous section. As before, we consider only the time like component for the $SO(2)$ gauge field (6.4.5) to be non-vanishing and set all the tensor fields to be zero. We also find that the off-diagonal Einstein equations for all cases imply,

$$A^{1\bar{0}} = A^{1\bar{2}} = A^{1\bar{3}} = A^{1\bar{4}} = 0\,, \tag{6.4.21}$$

The diagonal components of the Einstein equations $(\hat{t}\hat{t}, \hat{r}\hat{r}, \hat{x}\hat{x}, \hat{y}\hat{y}, \hat{z}\hat{z})$ are as follows. The $(\hat{t}\hat{t})$ component of the Einstein equation is,

$$(1 + 12u_3^2 + 20u_3u_1 + 12u_1^2)(\phi_c^1)^4 + 2(A^{0\bar{0}})^2(3g^2L^2 + u_0^2(\phi_c^1)^8) - 48L^2g_R^2V_0^2 = 0\,. \tag{6.4.22}$$

[4] $P^{\tilde{a}}_{ij}, W^{\tilde{a}}$ vanish for this choice [19].

The $(\hat{r}\hat{r})$ component of the Einstein equation is,

$$(1 + 4u_3^2 + 12u_3u_1 + 4u_1^2 + 8u_0u_3 + 8u_0u_1)(\phi_c^1)^4 \\ - (A^{0\bar{0}})^2(6g^2L^2 - 2u_0^2(\phi_c^1)^8) - 48L^2g_R^2V_0^2 = 0\,, \tag{6.4.23}$$

The $(\hat{x}\hat{x})$ component of the Einstein equation is,

$$(-1 + 12u_3^2 + 12u_3u_1 + 4u_1^2 + 8u_0u_3 + 4u_0u_1 + 4u^2)(\phi_c^1)^4 \\ - 2(A^{0\bar{0}})^2(3g^2L^2 + u_0^2(\phi_c^1)^8) - 48L^2g_R^2V_0^2 = 0\,, \tag{6.4.24}$$

The $(\hat{y}\hat{y})$ component of the Einstein equation is,

$$(3 + 4u_3^2 + 4u_1^2 + 4u_0u_1 + 4u_3u_1 + 4u_0u_3 + 4u_0^2)(\phi_c^1)^4 \\ - 2(A^{0\bar{0}})^2(3g^2L^2 + u_0^2(\phi_c^1)^8) - 48L^2g_R^2V_0^2 = 0\,, \tag{6.4.25}$$

The $(\hat{z}\hat{z})$ component of the Einstein equation is,

$$(-1 + 4u_3^2 + 12u_1^2 + 8u_0u_1 + 4u^2 + 12u_3u_1 + 4u_0u_3) \\ - 2(A^{0\bar{0}})^2(3g^2L^2 + u_0^2(\phi_c^1)^8) - 48L^2g_R^2V_0^2 = 0\,, \tag{6.4.26}$$

The gauge field equation for $A^{0\bar{0}}$ is,

$$\hat{e}A^{0\bar{0}}(3g^2L^2 - 2u_0(u_3 + u_1)(\phi_c^1)^8 = 0\,. \tag{6.4.27}$$

The scalar field equation is,

$$3g^2L^2 - u_0^2(\phi_c^1)^8 = 0\,. \tag{6.4.28}$$

The $\hat{x}\hat{x}$ and $\hat{z}\hat{z}$ components of the Einstein equations together with the gauge field and the scalar field equations give,

$$u_0 = \sqrt{3}\frac{gL}{(\phi_c^1)^4}\,, \quad u_3 = u_1 = \frac{u_0}{4}\,. \tag{6.4.29}$$

Substituting this into the Einstein equations, the $(\hat{t}\hat{t}, \hat{r}\hat{r}, \hat{x}\hat{x}, \hat{y}\hat{y})$[5] equations are given by,

$$\begin{aligned}
12(A^{0\bar{0}})^2 g^2 L^2 + \frac{33g^2L^2}{4(\phi_c^1)^4} + (\phi_c^1)^4 - 48L^2 g_R^2 V_0^2 &= 0\,,\\
\frac{-63g^2L^2}{4(\phi_c^1)^4} - (\phi_c^1)^4 + 48L^2 g_R^2 V_0^2 &= 0\,,\\
12(A^{0\bar{0}})^2 g^2 L^2 - \frac{105g^2L^2}{4(\phi_c^1)^4} + (\phi_c^1)^4 + 48L^2 g_R^2 V_0^2 &= 0\,,\\
12(A^{0\bar{0}})^2 g^2 L^2 - \frac{81g^2L^2}{4(\phi_c^1)^4} - 3(\phi_c^1)^4 + 48L^2 g_R^2 V_0^2 &= 0\,,
\end{aligned} \tag{6.4.30}$$

which can be solved to get,

$$A^{0\bar{0}} = \sqrt{\frac{5}{8}}\frac{1}{(\phi_c^1)^2}\,, \quad L = \sqrt{\frac{2}{3}}\frac{(\phi_c^1)^4}{g}\,, \tag{6.4.31}$$

with the constraint,

$$\frac{23}{2(\phi_c^1)^4} = 32\frac{g_R^2}{g^2}V_0^2\,. \tag{6.4.32}$$

We summarise the Bianchi II solution as,

$$\begin{aligned}
ds^2 = L^2\Big[&-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_1}d\hat{x}^2 + \hat{r}^{2(u_3+u_1)}d\hat{y}^2\\
&- 2\hat{x}\hat{r}^{2(u_3+u_1)}d\hat{y}d\hat{z} + [\hat{r}^{2(u_3+u_1)}\hat{x}^2 + \hat{r}^{2u_3}]d\hat{z}^2\Big]\,,\\
u_0 = \sqrt{2}\,, \quad & u_3 = u_1 = \frac{1}{2\sqrt{2}}\,, \quad L = \sqrt{\frac{2}{3}}\frac{(\phi_c^1)^4}{g}\,, \quad A^{0\bar{0}} = \sqrt{\frac{5}{8}}\frac{1}{(\phi_c^1)^2}\,,\\
\phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}}\,, \quad & V_0V_1 > 0\,, \quad \frac{23}{2(\phi_c^1)^4} \leq 1\,,
\end{aligned} \tag{6.4.33}$$

[5]The $\hat{z}\hat{z}$ equation is same as the $\hat{x}\hat{x}$ equation.

where we have substituted the invariant one forms (4.5.8) in (4.5.11) and written the metric explicitly.

6.4.3 Bianchi VI

In this subsection, we give the Bianchi type VI solution (4.5.15) arising from gauged supergravity. We recollect the metric,

$$ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_2}(\omega^1)^2 + \hat{r}^{2u_3}(\omega^2)^2 + (\omega^3)^2\Big]. \tag{6.4.34}$$

where the invariant one forms are,

$$\omega^1 = e^{-\hat{x}}d\hat{y}\,, \quad \omega^2 = e^{-h\hat{x}}d\hat{z}\,, \quad \omega^3 = d\hat{x}\,. \tag{6.4.35}$$

The analysis is again entirely parallel to the previous sections. As before, we consider only the time like component for the $SO(2)$ gauge field (6.4.5) to be non-vanishing and set all the tensor fields to be zero. We also find that the off-diagonal Einstein equations for all cases imply,

$$A^{1\bar{0}} = A^{1\bar{2}} = A^{1\bar{3}} = A^{1\bar{4}} = 0\,, \tag{6.4.36}$$

For this case, the off-diagonal $(\hat{r}\hat{x})$ Einstein equation gives the condition,

$$u_2 = -u_3 h\,, \tag{6.4.37}$$

The rest of the Einstein equations $(\hat{t}\hat{t}, \hat{r}\hat{r}, \hat{x}\hat{x}, \hat{y}\hat{y}, \hat{z}\hat{z})$ are as follows. The $(\hat{t}\hat{t})$ component of the Einstein equation is,

$$2(1 + h + h^2 + u_3^2(1 - h + h^2))(\phi_c^1)^4 + (A^{0\bar{0}})^2(3g^2L^2 + u_0^2(\phi_c^1)^8) - 24L^2g_R^2V_0^2 = 0\,. \tag{6.4.38}$$

The $(\hat{r}\hat{r})$ component of the Einstein equation is,

$$2(1 + u_0u_3 + h - u_3(u_3 + u_0)h + h^2)(\phi_c^1)^4 - (A^{0\bar{0}})^2(3g^2L^2 - u_0^2(\phi_c^1)^8) - 24L^2g_R^2V_0^2 = 0\,. \tag{6.4.39}$$

The $(\hat{x}\hat{x})$ component of the Einstein equation is,

$$\begin{aligned}&2(u_3^2 + u_0u_3 + u^2 + h - u_3(u_3 + u_0)h + u_3^2h^2)(\phi_c^1)^4\\ &\qquad - (A^{0\bar{0}})^2(3g^2L^2 + u_0^2(\phi_c^1)^8) - 24L^2g_R^2V_0^2 = 0\,.\end{aligned} \tag{6.4.40}$$

The $(\hat{y}\hat{y})$ component of the Einstein equation is,

$$2(u_3^2 + u_0u_3 + u^2 + h^2)(\phi_c^1)^4 - (A^{0\bar{0}})^2(3g^2L^2 + u_0^2(\phi_c^1)^8) - 24L^2g_R^2V_0^2 = 0\,. \tag{6.4.41}$$

The $(\hat{z}\hat{z})$ component of the Einstein equation is,

$$2(1 + u_0^2 - u_0u_3h + u_3^2h^2)(\phi_c^1)^4 - (A^{0\bar{0}})^2(3g^2L^2 + u_0^2(\phi_c^1)^8) - 24L^2g_R^2V_0^2 = 0\,. \tag{6.4.42}$$

The gauge field equation for $A^{0\bar{0}}$ is given by,

$$3g^2L^2 + u_0u_3(-1 + h)(\phi_c^1)^8 = 0\,. \tag{6.4.43}$$

The scalar field equation reduces to,

$$3g^2L^2 - u_0^2(\phi_c^1)^8 = 0\,. \tag{6.4.44}$$

The two equations obtained above can be solved (assuming $h \neq 1$) to get,

$$u_0 = \sqrt{3}\frac{gL}{(\phi_c^1)^4}\,, \quad u_3 = \frac{u_0}{1-h}\,. \tag{6.4.45}$$

The remaining Einstein equations are all not independent and can be solved to get,

$$L = \frac{(\phi_c^1)^4}{\sqrt{6}g}|h-1| \,, \quad A^{0\bar{0}} = \sqrt{\frac{-2h}{(-1+h)^2}}\frac{1}{(\phi_c^1)^2} \,. \tag{6.4.46}$$

The gauge field solution implies that $h < 0$ for $A^{0\bar{0}}$ to be real. The constraint on the free parameters is given by,

$$\frac{8(3-h+h^2)}{(\phi_c^1)^4(h-1)^2} = 32\frac{g_R^2}{g^2}V_0^2 \,. \tag{6.4.47}$$

We summarise the solution as,

$$\begin{aligned}
&ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + d\hat{x}^2 + e^{-2\hat{x}}\hat{r}^{2u_2}d\hat{y}^2 + e^{-2h\hat{x}}\hat{r}^{2u_3}d\hat{z}^2\Big] \,, \\
&u_0 = \frac{1}{\sqrt{2}}(1-h) \,, \quad u_2 = -\frac{1}{\sqrt{2}}h \,, \quad u_3 = \frac{1}{\sqrt{2}} \,, \quad L = \frac{(\phi_1^c)^4}{\sqrt{6}g}(1-h) \,, \\
&A^{0\bar{0}} = \sqrt{\frac{-2h}{(-1+h)^2}}\frac{1}{(\phi_c^1)^2} \,, \quad h < 0 \,, \quad h \neq 0, 1 \,, \\
&\phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}} \,, \quad V_0 V_1 > 0 \,, \quad \frac{8(3-h+3h^2)}{(\phi_c^1)^4(-1+h)^2} \leq 1 \,,
\end{aligned} \tag{6.4.48}$$

where we have substituted the invariant one forms (4.5.12) in (4.5.15) and written it explicitly.

As one can see from the above equations, we require $h < 0$ for the gauge field to be real which agrees with [7]. In deriving this particular solution, we also required in addition $h \neq 0, 1$. These two cases correspond to the Bianchi type III and type V metrics which can be realised as limiting cases of type VI. The type V metric is obtained in the $h \to 1$ limit of the type VI metric. In [7] it was found that the solution exists in the massless limit. In this case the equivalent of a mass less limit would be to take $g \to 0$ as $h \to 1$. Even though the length of the space time can be kept finite the time component of the gauge field blows up in this limit. Thus in this model we cannot obtain the type V solution in this manner. A similar issue occurs for the $h \to 0$ limit for the type III metric. In this case, the gauge field vanishes. In both situations one cannot take either V_1 or V_0 to zero, as this would jeopardise the gauging procedure. In summary, the type V and type III metrics do not seem to be valid attractors of the gauged supergravity considered here. However, they may still be solutions to some generic supergravity that belongs to the same class. For example, the type VII metric requires two massive gauge fields to start with, therefore one has to start from a supergravity model that uses two gauge fields to gauge the symmetries of the scalar manifold.

The attractor potential for the Bianchi II and Bianchi VI considered here is the same as (6.4.18) with the values of the parameters and constraints specific to each case. The solutions are determined by the parameters g, V_0 and V_1 together with the constraints. In this section, we have given a general idea of how to get such metrics from a simple gauged supergravity model via the generalised attractor ansatz (6.2.2). The other Bianchi classes may be realised in a similar way from more generic gauged supergravities.

6.5 Bianchi attractors in $U(1)_R$ gauged supergravity

In this section, we construct a special subclass of the Bianchi metrics whose symmetry groups factorise into a direct product form given by (4.5.35) [21]. We consider a truncated version of the gauged supergravity model discussed earlier in §5.4 with just $U(1)_R$ gauging. There is no gauging of the symmetries of the scalar manifold and hence there are no tensors as well. The field content of the reduced model is given by,

$$\{e_\mu^a\,, \psi_\mu^i\,, A_\mu^I\,, \lambda^{i\tilde{a}}\,, \phi^1\}\,, \tag{6.5.1}$$

with $I = 0, 1$ and $I = 0$ corresponds to the graviphoton as before. The field ϕ^1 is the scalar in the single vector multiplet. The gauge field combination used for the $U(1)_R$ gauging is same as before. The potential of the $U(1)_R$ gauged supergravity contains only terms proportional to g_R^2 as there is no gauging of the symmetries of scalar manifold and can be obtained by setting $\phi^2 = \phi^3 = 0$ in (5.4.19),

$$\mathcal{V}(\phi^1) = -2g_R^2\Big[\frac{2\sqrt{2}V_0V_1}{\phi^1} + (\phi^1)^2V_1^2\Big] . \qquad (6.5.2)$$

We show the embedding of the $Lif_{u_0}(2) \times M_I$ solution, which has the form,

$$ds^2 = L^2\Big[- \hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2\Big] . \qquad (6.5.3)$$

When $u_0 = 1$ we have the familiar $AdS_2 \times \mathbb{R}^3$ solution. We consider the gauge field ansatz as before,

$$A^{I\hat{t}} = e^{\hat{t}}_{\bar{0}}A^{I\bar{0}} = \frac{1}{L\hat{r}}A^{I\bar{0}} . \qquad (6.5.4)$$

For the $U(1)_R$ gauged supergravity the field equations at the attractor point for the gauge field, scalar field and Einstein equation are read off by setting $g = 0$ in the corresponding field equations found for the general gauging considered in [17]. The gauge field equation has the form,

$$a_{IJ}[\omega_{a,}{}^{a}{}_{c}F^{cbJ} + \omega_{a,}{}^{b}{}_{c}F^{acJ}] = 0 , \qquad (6.5.5)$$

and is identically satisfied for the gauge field ansatz considered above. The scalar field equation is given by,

$$\frac{\partial}{\partial\phi^1}\Big[\mathcal{V}_{attr}(\phi^1, A^{1\bar{0}}, A^{0\bar{0}})\Big] = 0, \quad \mathcal{V}_{attr}(\phi^1, A^{1\bar{0}}, A^{0\bar{0}}) = \mathcal{V}(\phi^1) + \frac{1}{4}a_{IJ}F^I_{ab}F^{Jab} . \qquad (6.5.6)$$

At the critical point $\phi^1_c = (\sqrt{2}\frac{V_0}{V_1})^{\frac{1}{3}}$, it relates the parameters V_0 and V_1 to the charges,

$$(A^{1\bar{0}})^2 - 2(A^{0\bar{0}})^2(\phi^1_c)^6 = 0 . \qquad (6.5.7)$$

The Einstein's equations are,

$$R_{ab} - \frac{1}{2}R\eta_{ab} = T^{attr}_{ab} , \qquad (6.5.8)$$

where,

$$T^{attr}_{ab} = \mathcal{V}_{attr}(\phi^1, A^{1\bar{0}}, A^{0\bar{0}})\eta_{ab} - a_{IJ}F^I_{ac}F_b{}^{cJ} \ . \tag{6.5.9}$$

At the attractor point, there are only two independent equations in the above set,

$$\begin{aligned} 3(A^{0\bar{0}})^2 u_0^2 (\phi^1_c)^5 - 12\sqrt{2} L^2 g_R^2 V_0 V_1 &= 0 \ , \\ u_0^2 \phi^1_c(-2 + 3(A^{0\bar{0}})^2(\phi^1_c)^4) + 12\sqrt{2} L^2 g_R^2 V_0 V_1 &= 0 \ , \end{aligned} \tag{6.5.10}$$

Where we have used (6.5.7) for simplification. This can be solved to get,

$$\begin{aligned} L^2 &= -\frac{u_0^2}{2\Lambda}, \quad \Lambda = -6 g_R^2 V_1^2 (\phi^1_c)^2 \ , \\ A^{0\bar{0}} &= \frac{1}{\sqrt{3}(\phi^1_c)^2}, \quad A^{1\bar{0}} = \sqrt{\frac{2}{3}}\phi^1_c \ , \end{aligned} \tag{6.5.11}$$

where Λ is the AdS cosmological constant. Note that the Einstein equation does not place additional constraints on any of the gauged supergravity parameters V_0, V_1, g_R, unlike in the other Bianchi cases considered here. We summarise the solution,

$$\begin{aligned} & ds^2 = L^2\Big[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2 \Big] \ , \\ & A^{0\hat{t}} = \frac{1}{L\hat{r}} A^{0\bar{0}} \ , \quad A^{1\hat{t}} = \frac{1}{L\hat{r}} A^{1\bar{0}} \ , \quad \frac{A^{0\bar{0}}}{A^{1\bar{0}}} = \frac{1}{2}\frac{V_1}{V_0} \ , \quad L^2 = -\frac{u_0^2}{2\Lambda} \ , \\ & \Lambda = -6 g_R^2 V_1^2 (\phi^1_c)^2 \ , \quad \phi^1_c = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}} \ , \quad V_0 V_1 > 0 \ . \end{aligned} \tag{6.5.12}$$

Note that the solution exists for any $u_0 > 0$ and in particular, when $u_0 = 1$ we get the familiar $AdS_2 \times \mathbb{R}^3$ solution.

The $Lif_{u_0}(2) \times M_{II}$ metric has the form,

$$ds^2 = L^2\Big[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + d\hat{x}^2 + d\hat{y}^2 - 2\hat{x}d\hat{y}d\hat{z} + (\hat{x}^2+1)d\hat{z}^2 \Big] . \tag{6.5.13}$$

We consider the same gauge field ansatz as for the previous case (6.5.4). As earlier, the gauge field equations are identically satisfied. The attractor equations are again same as (6.5.7) at the critical point. There are three independent Einstein equations given by,

$$
\begin{aligned}
\phi_c^1 + 6(A^{0\bar{0}})^2(\phi_c^1)^5 - 24\sqrt{2}L^2 g_R^2 V_0 V_1 &= 0\,, \\
\phi_c^1 - 4u_0^2\phi_c^1 + 4(A^{0\bar{0}})^2 u_0^2(\phi_c^1)^5 + 24\sqrt{2}L^2 g_R^2 V_0 V_1 &= 0\,, \\
-3\phi_c^1 - 4u_0^2\phi_c^1 + 6(A^{0\bar{0}})^2 u_0^2(\phi_c^1)^5 + 24\sqrt{2}L^2 g_R^2 V_0 V_1 &= 0\,,
\end{aligned}
\tag{6.5.14}
$$

where we have again used (6.5.7) for simplification. This set of algebraic equations can be solved to get,

$$
\begin{aligned}
A^{0\bar{0}} &= \frac{\sqrt{2}}{u_0(\phi_c^1)^2}\,, \quad A^{1\bar{0}} = \frac{2\phi_c^1}{u_0}\,, \\
u_0 &= \sqrt{\frac{11}{2}}\,, \quad L^2 = -\frac{13}{4\Lambda}\,,
\end{aligned}
\tag{6.5.15}
$$

where Λ is the AdS cosmological constant. We summarise the $Lif_{u_0}(2) \times M_{II}$ as,

$$
\begin{aligned}
& ds^2 = L^2\Big[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + d\hat{x}^2 + d\hat{y}^2 - 2\hat{x}\,d\hat{y}\,d\hat{z} + (\hat{x}^2+1)d\hat{z}^2 \Big]\,, \\
& A^{0\hat{t}} = \frac{1}{L\hat{r}}A^{0\bar{0}}\,, \quad A^{1\hat{t}} = \frac{1}{L\hat{r}}A^{1\bar{0}}\,, \quad \frac{A^{0\bar{0}}}{A^{1\bar{0}}} = \frac{1}{2}\frac{V_1}{V_0}\,, \quad u_0 = \sqrt{\frac{11}{2}}\,, \\
& L^2 = -\frac{13}{4\Lambda}\,, \quad \Lambda = -6g_R^2 V_1^2(\phi_c^1)^2\,, \quad \phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}}\,, \quad V_0V_1 > 0\,.
\end{aligned}
\tag{6.5.16}
$$

6.6 Summary

We studied the generalised attractors in $\mathcal{N} = 2, d = 5$ gauged supergravity defined by constant anholonomy, constant gauge fields, constant tensor fields and constant scalars at the attractor points. We showed that all the equations of motion become algebraic at the attractor points. We constructed the attractor potential from the scalar field equations and showed that it can be written independently from squares of the bosonic terms in the fermion supersymmetry transformations. We analysed the Killing spinor integrability conditions and determined the conditions for maximal supersymmetry. We showed that some of the simplest Bianchi attractors sourced by massive gauge fields can be realised from gauged supergravity models as generalised attractor solutions. In particular, we constructed a Lifshitz solution, a Bianchi type II and a Bianchi type VI solution from a simple gauged supergravity model with gauging of symmetries of scalar manifold and R symmetry. We also constructed solutions of the type $Lif_{u_0} \times M$ in $U(1)_R$ gauged supergravity.

We now conclude this section with a few comments. Let us first note that the Chern-Simons term had no contribution whatsoever for any of these solutions. In particular as observed in [15], topological terms vanish for the Lifshitz like solution sourced by a time-like gauge field. Remember that the field strengths are written in terms of the anholonomy coefficients. For the Lifshitz like solution and in general for any Bianchi type I metric the non-vanishing anholonomy coefficients are ${c_{01}}^0, {c_{21}}^2, {c_{31}}^3, {c_{41}}^4$. Due to this the Chern-Simons term $\epsilon^{bacde} {c_{ba}}^f {c_{cd}}^g A_f A_g A_e$ vanishes. For similar reasons the structure constants of the Bianchi classes [118] imply that there can be no support from the Chern-Simons term for any of the Bianchi type metrics which are sourced by time-like (or space-like) gauge fields. Note that for metrics with homogeneous directions greater than three, the possible symmetry groups are given by the classification of real Lie algebras (see, for example, table I of [120]). In such cases, the topological terms could have an effect on the solution.

Another important point to discuss here is the absence of tensor fields. In the literature there are known anisotropic Lifshitz solutions sourced by massive two forms [125]. However, it is not possible to realise such solutions within gauged supergravity. Unlike in [125], the kinetic terms for the tensor fields in gauged supergravity have a toplogical origin (5.3.9). In fact, the kinetic term for the tensor field comes from the Chern-Simons term in the original ungauged supergravity. Therefore, we do not expect the tensor fields in the theory to contribute to Lifshitz like metrics. In the supergravity model under consideration we have verified that the tensor fields do not contribute to the other Bianchi type metrics. This is in accordance with the results of [7] where such metrics were supported only by the gauge fields.

Before concluding, we would like to caution about the usage of the term attractor in this context. The attractor mechanism originally studied in [4] was in the context of supersymmetry preserving black hole configurations. As it has been subsequently realised, the critical points of the black hole effective potential may not be supersymmetric in general [94]. A detailed analysis of stability of non-supersymmetric black holes in asymptotic Minkowski space carried out in [115] suggests that the stable attractors corresponds to the absolute minima of the effective black hole potential. This condition slightly differs for black holes in (Anti)-deSitter spaces. We investigate the stability of Bianchi attractors in gauged supergravity in the next chapter.

Chapter 7

Stability of Bianchi attractors in gauged supergravity

7.1 Introduction

In chapter 4, we discussed the classification of homogeneous but anisotropic extremal black brane horizons known as Bianchi attractors. In chapter 6, we studied generalised attractors in gauged supergravity and constructed some explicit examples of such Bianchi attractors. One of the important issues being investigated currently is the stability of such Lorentz violating geometries [159–164]. Instabilities due to scalar field fluctuations were found to exist in a class of charged black brane geometries [165, 166]. Presence of such instabilities in these solutions plays a crucial role because they indicate that the geometry might get corrected in the deep infrared [161]. Though the stability analysis has been carried out in a number of examples, a common recipe to figure out whether certain geometry has any instability is still lacking.

In chapter 3, we discussed the attractor mechanism which has been studied quite extensively in the context of extremal black holes in Minkowski space with near horizon geometry $AdS_2 \times S^2$. The study of a similar mechanism for generalised attractors has not yet been explored thoroughly for the new class of Lorentz violating geometries arising as gravity duals of condensed matter systems. Especially, it is not at all obvious which among these entire class of new attractor geometries are stable and can survive in the deep infrared. Since a number of such geometries can be embedded in gauged supergravity, where the scalar couplings and potential term are determined by symmetry, it is natural to ask whether these gauged supergravity attractors are stable.

In this chapter, we analyse the stability of electrically charged Bianchi attractors in gauged supergravity. For attractors which asymptote to Minkowski space the conditions for stability is well understood [95]. In such cases the attractor values of the scalar fields must correspond to an absolute minimum of the black hole potential. We discussed this in chapter 3. In this chapter we derive the analogous condition for the generalised attractors in gauged supergravity. The main reference for this chapter is [21].

We consider the fluctuations of the scalar fields about their attractor value. We take the fluctuation to be of the form,

$$\phi_c + \epsilon\delta\phi(r,t)\,, \tag{7.1.1}$$

where t denotes the time, r is the radial direction, ϕ_c are the attractor values of the scalars and $\delta\phi$ is the perturbation with $\epsilon < 1$. We have taken the fluctuation to not depend on the (x, y, z) directions to respect the Bianchi type symmetries along these directions. Besides, we are primarily interested in the radial behavior of the fluctuation as one approaches the horizon. We also assume that the black brane metric can be expanded about the near horizon geometries as follows,

$$\tilde{g}_{\mu\nu} \sim g_{\mu\nu}(r - r_h) + \epsilon\, g^1_{\mu\nu}(r - r_h) + O(\epsilon^2) + \dots\,, \tag{7.1.2}$$

where $g_{\mu\nu}$ is the near horizon metric given by the Bianchi type geometries. The higher order terms like $g^1_{\mu\nu}$ are due to the back reaction of the scalar field fluctuations on the attractor geometry.

We study the stress energy tensor in gauged supergravity and expand it in first order of scalar fluctuations. We find that the stress energy tensor in gauged supergravity depends on the scalar fluctuations even at first order perturbation due to non-trivial interaction terms in the theory. If there is a large backreaction due to scalar fluctuations, the geometry would significantly differ from the attractor geometry indicating an instability. Therefore, stable attractor geometries are those where the scalar fluctuations die out as one approaches the horizon.

We then study the scalar field equations with the fluctuations at first order, determine the general solution and the conditions under which these fluctuations can exist. These conditions are such that the generalised attractor geometries must exist at critical points which are maxima of the attractor potential. We then derive conditions for stability of the Bianchi attractors in gauged supergravity by studying the near horizon behaviour of the scalar fluctuations and demanding regularity. In particular, we find that this severely restricts the general form of these metrics.[1] We find that metrics which factorise as

$$ds^2 = L^2\left[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \eta_{ij}\omega^i \otimes \omega^j\right], \tag{7.1.3}$$

are stable under scalar fluctuations about the attractor value. The parameter u_0 must be positive in order to have a regular horizon. In particular, when $u_0 = 1$ we get an AdS_2 factor and the symmetry is enhanced to $SO(2,1)\times M$. This factorisation is reminiscent of extremal black holes in four dimensions where the near horizon geometries factorise as $AdS_2 \times S^2$. We briefly mentioned such a class of metrics in chapter 4 (eq (4.5.35)) with scale invariance only along the $\hat{r}, \hat{t}$ directions. In the previous chapter, we constructed explicit examples of such metrics from $U(1)_R$ gauged supergravity in §6.5.

The chapter is organised as follows. In §7.2 we expand the stress energy tensor under

[1] In deriving this result, we make certain technical assumption on the killing vectors used in gauging, as well as on the nature of the critical points giving rise to the attractor geometry which will be discussed in due course.

scalar fluctuations about the attractor value and discuss the backreaction. We then derive the general solutions for the scalar fluctuations and describe the conditions under which these fluctuations exist in §7.3. Following this we study the near horizon behaviour of the fluctuations and derive stability conditions for the Bianchi attractors and discuss the constraints on the metric in §7.4. We conclude and summarise our results in §7.5.

7.2 Backreaction at first order

In this section, we analyse the stress energy tensor in gauged supergravity under scalar fluctuations. The stress energy tensor calculated from (5.3.9) takes the form,

$$\begin{aligned} T_{\mu\nu} =& g_{\mu\nu}\Big[\frac{1}{4}a_{IJ}F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{2}g_{xy}\mathcal{D}_\mu\phi^x\mathcal{D}^\mu\phi^y + \mathcal{V}(\phi)\Big] \\ & - \Big[a_{IJ}F^I_{\mu\lambda}F^{J\ \lambda}_{\nu} + g_{xy}\mathcal{D}_\mu\phi^x\mathcal{D}_\nu\phi^y\Big]. \end{aligned} \tag{7.2.1}$$

We now expand the stress energy tensor (7.2.1) upto first order in ϵ under the scalar perturbations (7.1.1) to get,

$$\begin{aligned} T_{\mu\nu}(\phi_c + \delta\phi) =& g_{\mu\nu}\Big[\frac{1}{4}\Big(a_{IJ}|_{\phi_c} + \frac{\partial a_{IJ}}{\partial\phi^z}\Big|_{\phi_c}\delta\phi^z\Big)F^I_{\lambda\sigma}F^{J\lambda\sigma} + g\Big(g_{xy}K^x_I\Big)\Big|_{\phi_c}A^I_\lambda\partial^\lambda(\delta\phi^y) \\ & + \frac{1}{2}g^2A^I_\lambda A^{\lambda J}\Big(K_{IJ}|_{\phi_c} + \frac{\partial K_{IJ}}{\partial\phi^z}\Big|_{\phi_c}\delta\phi^z\Big) + \Big(\mathcal{V}(\phi_c) + \frac{\partial\mathcal{V}}{\partial\phi^z}\Big|_{\phi_c}\delta\phi^z\Big)\Big] \\ & - \Big[\Big(a_{IJ}|_{\phi_c} + \frac{\partial a_{IJ}}{\partial\phi^z}\Big|_{\phi_c}\delta\phi^z\Big)F^I_{\mu\lambda}F^{J\ \lambda}_{\nu} + g\Big(g_{xy}K^x_I\Big)\Big|_{\phi_c}\Big(A^I_\mu\partial_\nu\delta\phi^y \\ & + A^I_\nu\partial_\mu\delta\phi^y\Big) + \Big(K_{IJ}|_{\phi_c} + \frac{\partial K_{IJ}}{\partial\phi^z}\Big|_{\phi_c}\delta\phi^z\Big)g^2A^I_\mu A^J_\nu\Big], \end{aligned} \tag{7.2.2}$$

where we have defined $K_{IJ} = K_I^x K_J^y g_{xy}$. The above equation can be further simplified and written as,

$$
\begin{aligned}
T_{\mu\nu}(\phi_c + \delta\phi) =& T^{attr}_{\mu\nu}|_{\phi_c} + gK_{yI}|_{\phi_c}\Big(A^{\lambda I}\partial_\lambda(\delta\phi^y)g_{\mu\nu} - A^I_\mu \partial_\nu(\delta\phi^y) - A^I_\nu \partial_\mu(\delta\phi^y)\Big) \\
&+ \left[\frac{\partial a_{IJ}}{\partial\phi^z}\bigg|_{\phi_c}\left(\frac{1}{4}g_{\mu\nu}F^{\lambda\sigma I}F^J_{\lambda\sigma} - F^I_{\mu\lambda}F^{J\ \lambda}_{\nu}\right)\right. \\
&\left.+ g^2\frac{\partial K_{IJ}}{\partial\phi^z}\bigg|_{\phi_c}\left(\frac{1}{2}g_{\mu\nu}A^I_\lambda A^{\lambda J} - A^I_\mu A^J_\nu\right) + \frac{\partial\mathcal{V}}{\partial\phi^z}\bigg|_{\phi_c}\right]\delta\phi^z\,. \qquad (7.2.3)
\end{aligned}
$$

where,

$$
T^{attr}_{\mu\nu}|_{\phi_c} = \mathcal{V}_{attr}(\phi_c)g_{\mu\nu} - \left[a_{IJ}|_{\phi_c}F^I_{\mu\lambda}F_\nu^{\ \lambda\tilde{J}} + g^2K_{IJ}|_{\phi_c}A^I_\mu A^J_\nu\right]. \qquad (7.2.4)
$$

The attractor equations (6.3.1) can be used for further simplification to get,

$$
\begin{aligned}
T_{\mu\nu}(\phi_c + \delta\phi) =& T^{attr}_{\mu\nu}|_{\phi_c} + gK_{yI}|_{\phi_c}\Big(A^{\lambda I}\partial_\lambda(\delta\phi^y)g_{\mu\nu} - A^I_\mu \partial_\nu(\delta\phi^y) - A^I_\nu \partial_\mu(\delta\phi^y)\Big) \\
&- \left[\frac{\partial a_{IJ}}{\partial\phi^z}\bigg|_{\phi_c}F^I_{\mu\lambda}F^{J\ \lambda}_{\nu} + g^2\frac{\partial K_{IJ}}{\partial\phi^z}\bigg|_{\phi_c}A^I_\mu A^J_\nu\right]\delta\phi^z\,. \qquad (7.2.5)
\end{aligned}
$$

It is already clear that for general perturbations of the scalar field, there is backreaction at first order even after using the attractor equations. In particular this requires the fluctuations and their derivatives to be well behaved as one approaches the horizon. Any divergent fluctuation would cause infinite backreaction and deviation from the attractor geometry indicating an instability. Taking the trace of (7.2.5) we get,

$$
\begin{aligned}
T_\mu^{\ \mu}(\phi_c + \delta\phi) =& T^{attr\mu}_{\mu}|_{\phi_c} + (d-2)gK_{yI}|_{\phi_c}A^{\lambda I}\partial_\lambda(\delta\phi^y) \\
&- \left[\frac{\partial a_{IJ}}{\partial\phi^z}\bigg|_{\phi_c}F^I_{\mu\nu}F^{J\mu\nu} + g^2\frac{\partial K_{IJ}}{\partial\phi^z}\bigg|_{\phi_c}A^I_\mu A^{J\mu}\right]\delta\phi^z\,, \qquad (7.2.6)
\end{aligned}
$$

where d is the space time dimension. Once again we can use the attractor equations (6.3.1) to simplify, and the Einstein equations take the form,

$$\begin{aligned} R\frac{(2-d)}{2} =& T_{\mu}^{attr\mu}|_{\phi_c} + (d-2)gK_{yI}|_{\phi_c}A^{\lambda I}\partial_{\lambda}(\delta\phi^{y}) \\ &+ \left[g^2\frac{\partial K_{IJ}}{\partial\phi^z}\bigg|_{\phi_c}A_{\mu}^{I}A^{J\mu} + 4\frac{\partial\mathcal{V}}{\partial\phi^z}\bigg|_{\phi_c}\right]\delta\phi^z \,. \end{aligned} \tag{7.2.7}$$

Suppose if the critical points of the attractor potential are also simultaneous critical points of the gauged supergravity scalar potential (as was the case with all the examples discussed in chapter 6), we see that the terms relevant for the backreaction are proportional to g:

$$R\frac{(2-d)}{2} = T_{\mu}^{attr\mu}|_{\phi_c} + (d-2)gK_{yI}|_{\phi_c}A^{\lambda I}\partial_{\lambda}(\delta\phi^{y}) + g^2\frac{\partial K_{IJ}}{\partial\phi^z}\bigg|_{\phi_c}A_{\mu}^{I}A^{J\mu}\delta\phi^z \,, \tag{7.2.8}$$

Thus, for gauging of R symmetry, $g = 0$ and hence the backreaction is absent:

$$R\frac{(2-d)}{2} = T_{\mu}^{attr\mu}|_{\phi_c} \,. \tag{7.2.9}$$

(See §6.5 for some examples of generalised attractor in gauged supergravity with just R symmetry gauging). However, in gauged supergravity with a generic gauging of symmetries of the scalar manifold, the equation depends on the first order fluctuations in the scalar fields. Thus, the generalised attractor geometries in gauged supergravity with a generic gauging can get backreacted by fluctuations of scalar fields. It then follows that the relevant boundary conditions to have stable attractors should be such that the fluctuations and derivatives of fluctuations vanish as one approaches the horizon.

7.3 Scalar fluctuations

In this section, we will analyse the scalar fluctuations in detail using the equation of motion for the scalar fields. The field equation (6.2.12) can be rewritten as:

$$\hat{e}^{-1}\partial_\mu[\hat{e}\, g_{zy}\mathcal{D}^\mu\phi^y] - \frac{1}{2}\frac{\partial g_{xy}}{\partial\phi^z}\nabla_\mu\phi^x\nabla^\mu\phi^y - g\frac{\partial K_{Iy}}{\partial\phi^z}A^I_\mu\nabla^\mu\phi^y - \frac{\partial\mathcal{V}_{attr}}{\partial\phi^z} = 0\,. \tag{7.3.1}$$

We will now expand the scalar fields about their attractor values and keep terms of $O(\epsilon)$ to get:

$$\begin{aligned} g_{zy}|_{\phi_c}\nabla_\mu\nabla^\mu\delta\phi^y - \frac{\partial^2\mathcal{V}_{attr}}{\partial\phi^z\partial\phi^y}\bigg|_{\phi_c}\delta\phi^y + g\bigg[\frac{\partial K_{Iz}}{\partial\phi^y} - \frac{\partial K_{Iy}}{\partial\phi^z}\bigg]\bigg|_{\phi_c}A^{\mu I}\nabla_\mu\delta\phi^y \\ + g\bigg[K_{Iz}|_{\phi_c} + \frac{\partial K_{Iz}}{\partial\phi^y}\bigg|_{\phi_c}\delta\phi^y\bigg]\nabla_\mu A^{\mu I} = 0\,. \end{aligned} \tag{7.3.2}$$

Here the covariant derivative ∇_μ is taken with respect to the zeroth order metrics which represent the near horizon Bianchi geometries. Note that the higher order metric terms which are undetermined are not required at $O(\epsilon)$. We choose the gauge condition $\nabla_\mu A^{\mu I} = 0$ to eliminate the last term. Finally we get,

$$\nabla_\mu\nabla^\mu\delta\phi^x - g^{zx}\frac{\partial^2\mathcal{V}_{attr}}{\partial\phi^z\partial\phi^y}\bigg|_{\phi_c}\delta\phi^y + 2g\,(g^{zx}\tilde{\nabla}_y K_{Iz})|_{\phi_c}A^{\mu I}\nabla_\mu\delta\phi^y = 0\,, \tag{7.3.3}$$

where $\tilde{\nabla}$ is the covariant derivative with respect to the metric on the scalar manifold g_{xy}. The Laplacian operator can be written as,

$$\nabla_\mu\nabla^\mu = g^{\hat{r}\hat{r}}\partial^2_{\hat{r}} + g^{\hat{t}\hat{t}}\partial^2_{\hat{t}} + (g^{\hat{r}\hat{r}}\frac{\partial_{\hat{r}}\hat{e}}{\hat{e}} + \partial_{\hat{r}}g^{\hat{r}\hat{r}})\partial_{\hat{r}}\,, \tag{7.3.4}$$

since the scalar fluctuations depend only on the radial and time co-ordinates.

Before substituting the details, we would like to make some comments on the co-ordinate system used for writing the Bianchi attractor geometries. In [7], the horizon for the Bianchi metrics was located at $r = -\infty$, where as in chapters 4 and 6 we have chosen the co-ordinate $\hat{r} = e^r$ such that the horizon lies at $\hat{r} = 0$ instead. As can be seen from the general form of the Bianchi metrics (4.5.3), the constants u_0, u_i must be positive in order to have a regular horizon. Thus one can see that the general form of the determinant is,

$$\hat{e} = \sqrt{-detg_{\mu\nu}} \sim L^5 \hat{r}^m f(x,y,z)\,, \tag{7.3.5}$$

where $m = -1 + \sum_l c_l u_l$, u_l are the various exponents and c_l is a positive number with $c_0 = 1$ for all Bianchi attractors. For example, in the Bianchi II case (see (6.4.33)) $m = -1 + u_0 + 2(u_1 + u_3)$. We can also see that,

$$g^{\hat{r}\hat{r}} = \frac{\hat{r}^2}{L^2}, \quad g^{\hat{t}\hat{t}} = -\frac{1}{L^2 \hat{r}^{2u_0}}\,, \tag{7.3.6}$$

for all Bianchi attractors. Using the above data, the Laplacian (7.3.4) can be expressed as,

$$\nabla_\mu \nabla^\mu = \frac{1}{L^2}\left[\hat{r}^2 \partial_{\hat{r}}^2 + (m+2)\hat{r}\partial_{\hat{r}} - \frac{1}{\hat{r}^{2u_0}}\partial_{\hat{t}}^2\right]. \tag{7.3.7}$$

Substituting (7.3.7) in (7.3.3) and using the ansatz (6.4.5) for A^I_μ we get,

$$\left[\hat{r}^2 \partial_{\hat{r}}^2 + (m+2)\hat{r}\partial_{\hat{r}} - \frac{1}{\hat{r}^{2u_0}}\partial_{\hat{t}}^2\right]\delta\phi^x - M^x_y|_{\phi_c}\delta\phi^y + N^x_y|_{\phi_c}\frac{1}{\hat{r}^{u_0}}\partial_{\hat{t}}\delta\phi^y = 0\,, \tag{7.3.8}$$

where,

$$M^x_y|_{\phi_c} = L^2 g^{zx}\frac{\partial^2 \mathcal{V}_{attr}}{\partial\phi^z \partial\phi^y}\bigg|_{\phi_c}, \quad N^x_y|_{\phi_c} = 2gLA^{I\bar{0}}(g^{zx}\tilde{\nabla}_y K_{Iz})|_{\phi_c}\,. \tag{7.3.9}$$

The metric on the moduli space g_{xy} is chosen to be positive definite and the nature of the critical point is given by the sign of the double derivative of the attractor potential. We further assume that $M^x_y|_{\phi_c}$ is diagonal so that,

$$M^x_y|_{\phi_c}\delta\phi^y = \lambda\delta\phi^x\,. \tag{7.3.10}$$

The term N^x_y can be non-zero in general, but vanishes trivially for the gauged supergravity model where we found some examples Bianchi attractors (see §(5.4)). There is only one Killing vector (5.4.14) that generates the $SO(2)$ isometry on the scalar manifold, and the critical point is such that $\phi^2_c = \phi^3_c = 0$. Therefore one is left with just the $\tilde{\nabla}_x K_{Ix}$ component which vanishes due to the Killing vector equation on the manifold.[2]

Thus, the scalar fluctuation equation (7.3.3) has the final form,

$$\left[\hat{r}^2\partial^2_{\hat{r}} + (m+2)\hat{r}\partial_{\hat{r}} - \frac{1}{\hat{r}^{2u_0}}\partial^2_{\hat{t}} - \lambda\right]\delta\phi^x = 0\,. \tag{7.3.11}$$

The above equation admits a simple solution when the fluctuations $\delta\phi^x$ are time independent. In this case, we have ,

$$\delta\phi^x = C_1 r^{\left(\sqrt{4\lambda+(1+m)^2}-(1+m)\right)/2} + C_2 r^{\left(-\sqrt{4\lambda+(1+m)^2}-(1+m)\right)/2}\,. \tag{7.3.12}$$

Thus, one of the modes vanishes as $r\to 0$ provided λ is positive and it is possible to get stable attractors upon setting $C_2 = 0$. However, all the explicit examples we discussed in chapter 6, do not admit a critical point with $\lambda > 0$. Thus, such fluctuations are unstable.

Now we turn to the case of time dependent fluctuations. Since the equation for $\delta\phi^x$ is separable, we try the ansatz $\delta\phi(\hat{r},\hat{t}) = f(\hat{r})e^{ik\hat{t}}$ (with k real) to get the Bessel equation:

$$\left[\hat{r}^2\partial^2_{\hat{r}} + (m+2)\hat{r}\partial_{\hat{r}} + (\frac{k^2}{\hat{r}^{2u_0}} - \lambda)\right]f(\hat{r}) = 0\,. \tag{7.3.13}$$

The general solutions for this equation are given by the standard Bessel functions (see, for example, [167], page 932):

$$f(X) = \left(\frac{X}{2}\right)^{\nu_0}\left[C_1\Gamma(1-\nu_\lambda)J_{-\nu_\lambda}(X) + C_2\Gamma(1+\nu_\lambda)J_{\nu_\lambda}(X)\right], \tag{7.3.14}$$

[2]Here, the single surviving component of the Killing vector is along the direction of ϕ^1 on the scalar manifold.

where,

$$X = \frac{k}{u_0 \hat{r}^{u_0}}, \quad \nu_\lambda = \frac{\sqrt{(1+m)^2 + 4\lambda}}{2u_0}, \quad \nu_0 = \frac{(1+m)}{2u_0} \;, \tag{7.3.15}$$

C_1 and C_2 are arbitrary constants, and the Bessel functions are,

$$\begin{aligned} J_{\nu_\lambda}(X) &= \left(\frac{X}{2}\right)^{\nu_\lambda} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu_\lambda+1)} \left(\frac{X}{2}\right)^{2j} \;, \\ J_{-\nu_\lambda}(X) &= \left(\frac{X}{2}\right)^{-\nu_\lambda} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j-\nu_\lambda+1)} \left(\frac{X}{2}\right)^{2j} \;. \end{aligned} \tag{7.3.16}$$

The power series representation is valid in the small X or equivalently, in the large r regime. We can rewrite the solution in terms of the Hankel functions,

$$\begin{aligned} J_{\nu_\lambda}(X) &= \frac{1}{2}(H^1_{\nu_\lambda}(X) + H^2_{\nu_\lambda}(X)) \;, \\ J_{-\nu_\lambda}(X) &= \frac{1}{2}(H^1_{\nu_\lambda}(X)e^{i\nu_\lambda\pi} + H^2_{\nu_\lambda}(X)e^{-i\nu_\lambda\pi}) \;, \end{aligned} \tag{7.3.17}$$

to get,

$$\begin{aligned} f(X) = \left(\frac{X}{2}\right)^{\nu_0} \Big[& C_1 H^1_{\nu_\lambda}(X)[\Gamma(1-\nu_\lambda)e^{i\nu_\lambda\pi} + \Gamma(1+\nu_\lambda)] \\ & + C_2 H^2_{\nu_\lambda}(X)[\Gamma(1-\nu_\lambda)e^{-i\nu_\lambda\pi} + \Gamma(1+\nu_\lambda)] \Big] \;. \end{aligned} \tag{7.3.18}$$

As one can see from above equation, there is already a restriction on ν_λ from the Gamma function that appears in the general solution. First let us consider the case ν_λ real, then we have the condition,

$$\nu_\lambda = \frac{\sqrt{(1+m)^2 + 4\lambda}}{2u_0} = \frac{\sqrt{(\sum_l c_l u_l)^2 + 4\lambda}}{2u_0} \leq 1 \;, \tag{7.3.19}$$

for,

$$-\frac{(\sum_l c_l u_l)^2}{4} \leq \lambda < 0 \;. \tag{7.3.20}$$

Note that only negative λ can satisfy (7.3.19).[3] Since $c_l > 0$ and all the u_l have to be positive for the existence of a regular horizon, we conclude that λ has to be negative. Remember that the sign of λ is provided by the double derivative of the attractor potential eqs. (7.3.9,7.3.10). This implies that the critical points correspond to maxima of the attractor potential. For the case of imaginary ν_λ we have,

$$\lambda < -\frac{(\sum_l c_l u_l)^2}{4} \,, \tag{7.3.21}$$

and hence, even in this case the critical points correspond to a maxima of the attractor potential. Thus we have determined the general solution for the scalar fluctuation (7.3.18) and and we find that they are well behaved at large distance provided they satisfy the conditions (7.3.20,7.3.21). This may be useful for the study of attractor flow equations for black holes in AdS.

7.4 Stable Bianchi attractors

In this section, we will analyse the stability of the Bianchi attractors by studying the behaviour of the solution in the $r \to 0$ limit. We are interested in the question which class of the Bianchi attractors can be stable attractor geometries in gauged supergravity. This can be answered by looking at the near horizon behaviour of the scalar fluctuations (7.3.18). From our analysis of the stress energy tensor in gauged supergravity (7.2.8), we find that there is dependence on the fluctuations and their derivatives at first order perturbation. Hence, we only require that the fluctuations do not blow up near the horizon as that would backreact strongly and deviate from the geometry. This requirement places some constraints on the form of the metric itself as we explain in the rest of the section.

Both the solutions in (7.3.18) are given in terms of the Hankel functions, the behaviour near the horizon can be determined by considering the asymptotic expansions of the Han-

[3] We do not consider $\lambda = 0$ as that would leave the nature of the critical point undetermined.

kel functions. Remember that the horizon for the Bianchi metrics (4.5.3) is located at $\hat{r} = 0$. The form of the solution (7.3.18) makes it convenient to use the asymptotic expansions of the Hankel functions, since from (7.3.15) $X \to \infty$ as $\hat{r} \to 0$. The asymptotic expansions are given by,

$$H^1_{\nu_\lambda}(X) \sim \sqrt{\frac{2}{\pi X}} e^{i(X - \frac{\pi}{2}(\nu_\lambda + \frac{1}{2}))} ,$$
$$H^2_{\nu_\lambda}(X) \sim \sqrt{\frac{2}{\pi X}} e^{-i(X - \frac{\pi}{2}(\nu_\lambda + \frac{1}{2}))} . \tag{7.4.1}$$

Substituting (7.4.1) in (7.3.18) we determine the behaviour of the fluctuation near the horizon as,

$$f(X) \sim \left(\frac{X}{2}\right)^{\nu_0 - \frac{1}{2}} \sqrt{\frac{1}{\pi}} \Big[C_1 e^{i(X - \frac{\pi}{2}(\nu_\lambda + \frac{1}{2}))} [\Gamma(1 - \nu_\lambda) e^{i\nu_\lambda \pi} + \Gamma(1 + \nu_\lambda)]$$
$$+ C_2 e^{-i(X - \frac{\pi}{2}(\nu_\lambda + \frac{1}{2}))} [\Gamma(1 - \nu_\lambda) e^{-i\nu_\lambda \pi} + \Gamma(1 + \nu_\lambda)] \Big] . \tag{7.4.2}$$

Since $X \sim \frac{1}{\hat{r}^{u_0}}$ and $u_0 > 0$, there is a leading divergent term as $\hat{r} \to 0$ unless,

$$\frac{1 - 2\nu_0}{2} \geq 0 , \tag{7.4.3}$$

which can be rewritten as,

$$\nu_0 = \frac{(1 + m)}{2u_0} = \frac{\sum_l c_l u_l}{2u_0} \leq \frac{1}{2} . \tag{7.4.4}$$

Since $c_0 = 1$, this implies,

$$\sum_{l, l \neq 0} c_l u_l \leq 0 , \tag{7.4.5}$$

which can never be satisfied without some of the exponents u_l being negative. Since we require a regular horizon, all the exponents have to be positive. Thus the only possibility for which eq. (7.4.5) can be satisfied is,

$$u_0 \neq 0, \quad u_l = 0 \ \ \forall \, l \neq 0 . \tag{7.4.6}$$

The conditions on λ (7.3.20),(7.3.21) for the general solution (7.3.18) to exist can now be written as,

$$-\frac{u_0^2}{4} \leq \lambda < 0\,, \tag{7.4.7}$$

for real ν_λ and,

$$\lambda < -\frac{u_0^2}{4}\,, \tag{7.4.8}$$

for imaginary ν_λ. To summarise, Bianchi attractors are stable against scalar fluctuations about the attractor value for the class of metrics which satisfy the condition (7.4.6).

The condition (7.4.6), is highly restrictive on the form of the Bianchi metrics. In particular it follows from (7.4.6) that $\nu_0 = \frac{1}{2}$ for any $u_0 > 0$ and the scalar fluctuations (7.4.2) do not diverge near the horizon.[4] In particular this restricts the metrics (4.5.3) to be of the form,

$$ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \eta_{ij}\omega^i \otimes \omega^j\Big]\,. \tag{7.4.9}$$

It is very interesting to note that the symmetry group of this metric form factorises into a direct product of the $(1 + 1)$ dimensional Lifshitz group and a group in the Bianchi classification. This is similar to what happens for example in four dimensional extremal black holes where the near horizon geometry factorises as $AdS_2 \times S^2$.

The simplest non-trivial example of this class is the $Lif_{u_0}(2) \times M_I$ solution,

$$ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2)\Big]\,, \tag{7.4.10}$$

one obtains the $AdS_2 \times \mathbb{R}^3$ solution when $u_0 = 1$. Another less trivial example is the $Lif_{u_0}(2) \times M_{II}$ solution,

$$ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (d\hat{x}^2 + d\hat{y}^2 - 2\hat{x}d\hat{y}d\hat{z} + (\hat{x}^2 + 1)d\hat{z}^2)\Big]\,. \tag{7.4.11}$$

[4]Note that there are still oscillatory terms in the fluctuation.

Geometry	λ	u_0	$u_l, l \neq 0$	Stability
Lifshitz	-34	3	1	no
Bianchi II	$-\frac{22}{3}$	$\sqrt{2}$	$u_1 = u_3 = \frac{1}{2\sqrt{2}}$	no
Bianchi VI $h < 0$	$-1 + \frac{14h}{3} - h^2$	$\frac{1}{\sqrt{2}}(1-h)$	$u_2 = -\frac{1}{\sqrt{2}}h, u_3 = \frac{1}{\sqrt{2}}$	no
$Lif_{u_0}(2) \times M_I$	$-\frac{5u_0^2}{3}$	any $u_0 > 0$	0	yes
$AdS_2 \times M_I$	$-\frac{5}{3}$	1	0	yes
$Lif_{u_0}(2) \times M_{II}$	$-\frac{61}{6}$	$\sqrt{\frac{11}{2}}$	0	yes
$Lif_{u_0}(2) \times M*$	$\lambda < 0$	any $u_0 > 0$	0	yes

Table 7.1: *Bianchi attractor geometries in gauged supergravity, nature of critical points and stability. The first three entries are for the solutions found in [17] and discussed in §6.4. The next three entries are generalised attractors in $U(1)_R$ gauged supergravity discussed in §6.5. The last entry with the * is the most general possible Bianchi attractor geometry (7.4.9) that satisfies our stability criteria.*

We have constructed the $Lif_{u_0}(2)\times M_I$ for any $u_0 > 0$ and a $Lif_{u_0}(2)\times M_{II}$ in a simple $U(1)_R$ gauged supergravity theory with one vector multiplet. These solutions were discussed in §6.5. It can be seen from Table (7.1), that these solutions satisfy our stability criteria (7.4.6) and hence are examples of stable Bianchi attractors in gauged supergravity.

The examples we constructed earlier in [17] (discussed in §6.4) all have $\lambda < 0$ and exist at maxima of the attractor potential. Therefore the condition (7.3.19) allows scalar fluctuations about the attractor values. However as one can see from table (7.1) all the metrics have some $u_l \neq 0$ for $l \neq 0$ and do not satisfy our stability condition(7.4.6). Hence the radial fluctuation of the scalar field diverges near the horizon for all these metrics. To complicate matters further, as one can see from (7.2.8) the fluctuations and their derivatives backreact on the geometry strongly. Thus there would be significant deviation of the geometry even at the first order and we conclude that these geometries are unstable attractors in the theory. These results are summarised in Table (7.1).

7.5 Summary

In this chapter , we have studied the stability of Bianchi attractors in gauged supergravity by considering scalar fluctuations about the attractor value. In general, the stress energy tensor in a generic gauged supergravity depends on the scalar fluctuations and their derivatives even at first order perturbation. Therefore, it is important that the scalar fluctuations are well behaved near the horizon. In particular, if there is a large backreaction then the geometry would deviate from the attractor geometry. Hence the fluctuations must vanish as one approaches the horizon for the attractor geometry to be stable.

We analysed the scalar fluctuation equations and found that the fluctuations can exist in general when the attractor geometries in consideration exist at critical points which, in the present case, correspond to maxima of the attractor potential. By demanding that the fluctuations vanish as one approaches the horizon we determined the conditions of stability for the metric. We found that the Bianchi attractors are stable if the metric factorises as,

$$ds^2 = L^2\left(-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2}\right) + L^2\left(\eta_{ij}\omega^i \otimes \omega^j\right) \,, \tag{7.5.1}$$

which is a subclass of the Bianchi attractors discussed in chapter 4. We have referred to this class of metrics as $Lif_{u_0}(2) \times M$, where M refers to three dimensional manifolds invariant under the nine groups given by the Bianchi classification. As stated before, these solutions exist for critical points which are maxima of the attractor potential and they satisfy all the conditions of stability. It would be interesting to explore whether this is a generic feature of attractors in gauged supergravity or an artifact of the gauged supergravity models considered.

Chapter 8

Conclusion

In this thesis, we studied the microscopic and macroscopic descriptions of black holes in string theory. In the microscopic side, we studied the counting of a class of states called twisted BPS states in a supersymmetric theory. In the macroscopic side, we studied the attractor mechanism in gauged supergravity. We first summarise the contents of the thesis, and then highlight the main results and open questions.

In Chapter 2, we studied the microscopic state counting in string theory [3]. We computed the generating functions for a class of $1/2$ BPS states called twisted BPS states in CHL orbifold theories, when the twists do not commute with the orbifold group. The generating functions turn out be ratios of the theta functions for the orbifold group and the twist generating group. The orbifold partition function counts the states which are invariant under the orbifolding group. The twists count states invariant under the twist generating group within the states invariant under the orbifolding group. So, it is natural to expect that the number of twisted states in orbifold theory should be lesser than untwisted states. We verified this expectation by computing the asymptotic expansion of the degeneracy. One direction where this work may be extended is to do an analogous computation for $1/4$ BPS states. It is also possible that the twists may provide a controlled way to break supersymmetry and help extend the counting problem to situations with reduced supersymmetry.

In the macroscopic side, we studied the attractor mechanism in gauged supergravity. We covered the necessary back ground materials in attractor mechanism in supergravity in chapter 3. In chapter 4, we studied AdS black holes and homogeneous extremal black brane horizons known as Bianchi attractors [7,53]. These metrics are of the general form,

$$ds^2 = L^2\Big[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2(u_i+u_j)}\eta_{ij}\omega^i \otimes \omega^j\Big]\,. \tag{8.0.1}$$

and display homogeneous symmetries. In addition, they also exhibit scale invariance in all directions. We were interested in embedding these metrics as attractor solutions in gauged supergravity.

We covered the necessary background material on $\mathcal{N} = 2, d = 5$ gauged supergravity in chapter 5. We saw that gauging the symmetries of the scalar manifold gives rise to interesting structure such as a potential for the scalar fields. We also saw that gauging the R symmetry of the theory leads to terms in the potential which can support AdS vacuum. We highlighted these features through a simple example of a gauged supergravity theory with one vector multiplet.

In chapter 6, we discussed the generalised attractors in $\mathcal{N} = 2, d = 5$ gauged supergravity characterised by constant anholonomy coefficients [17]. The generalised attractor points are obtained by solving the field equations when all the bosonic fields of the theory become constants in tangent space. The field equations at the attractor points are algebraic, and the moduli are determined in terms of the charges as expected for attractor solutions. We constructed the attractor potential from the scalar field equations and showed that it can be written independently from squares of the bosonic terms in the fermion supersymmetry transformations. We constructed some explicit examples of homogeneous extremal black brane horizons studied in chapter 4, as generalised attractor solutions in gauged supergravity.

The generalised attractor procedure relied on extremization of an attractor potential and not on supersymmetry. From the discussions in chapter 3, we see that this method is general and can include non-supersymmetric attractors. In chapter 7 we investigated the stability of the Bianchi attractors in gauged supergravity. We considered the scalar perturbations about the critical value and performed a fluctuation analysis in the field equations. Since the stress energy tensor depended on the scalar fluctuations even at first order, we demanded that the fluctuation be regular. We then analysed the scalar field equations and solved for the fluctuations. We obtained the conditions under which the fluctuations are well defined and demanded regularity at the horizon to determine the conditions for stability.

The main results of the study of attractor mechanism in gauged supergravity are,

- We have extended the study of generalised attractors in $\mathcal{N} = 2, d = 4$ gauged supergravity [14] to $\mathcal{N} = 2, d = 5$ gauged supergravity in [17].
  - The field equations become algebraic at the attractor points. The moduli are determined as functions of the charges by extremising an attractor potential.
  - The attractor potential can be constructed independently from fermionic shifts in gauged supergravity.
  - The homogeneous extremal near horizon geometries known as the Bianchi attractors [7, 53] are generalised attractor solutions in gauged supergravity.
  - We constructed explicit examples of Bianchi I, Bianchi II and Bianchi VI type solutions as generalised attractors in a simple gauged supergravity theory with one vector multiplet.
- We have studied the stability of the Bianchi attractor solutions in gauged supergravity under scalar fluctuations about the critical value [21].
  - The stress energy tensor in gauged supergravity (with a generic gauging of the symmetries of the scalar manifold) depends on scalar fluctuations even at first

order perturbation.

- Fluctuations which do not have regular behaviour near the horizon will backreact strongly leading to significant deviation from the attractor geometry. This indicates an instability.
- The scalar fluctuations are well defined when the critical point in consideration is a maxima of the attractor potential.
- Regularity of the fluctuations near the horizon require the near horizon geometry to factorise as $Lif_{u_0} \times M$,

$$ds^2 = L^2 \left(-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} \right) + L^2 \left(\eta_{ij} \omega^i \otimes \omega^j \right) \ , \tag{8.0.2}$$

  where $M = M_I, M_{II} \ldots M_{IX}$ are the homogeneous subspaces invariant under the Bianchi type symmetries.

We now discuss some of the implications of our results. Our study of generalised attractors indicates that there are several possible end points for an attractor flow in five dimensional gauged supergravity. Even in the simple example of the gauged supergravity with one vector multiplet, the Bianchi I, II and VI solutions all exist for the same value of the critical point. It will be interesting to study this further and see if there is a preferred end point. We have answered this partially by the stability analysis which indicates that the Bianchi type metrics which factorise as (8.0.2) represent stable end points. This factorisation is reminiscent of the fact that near horizon geometry of extremal black holes in four dimensions factorise as $AdS_2 \times S^2$. It would also be very interesting to see if the results of the stability analysis are model independent.

We have studied the generalised attractors by extremising an attractor potential. This method is more generic and describes non-supersymmetric attractor points as well. The construction of the attractor potential from fermionic shifts in the supersymmetry transformation indicates that supersymmetry may have a very important role to play in this construction. We hope to explore this in future.

In the gauged supergravity model we considered, the critical points of the attractor potential coincide with the critical point of the scalar potential in gauged supergravity. This is very similar to the situation in ungauged supergravity where the critical points of the effective potential coincide with the critical points of the central charge. It would be interesting to see if the potential of gauged supergravity is related to the central charge of the theory. This may be related to the issue regarding the fermionic shifts and we hope to explore this in future.

Another implication from the study of generalised attractors is the similarity in the description of generalised attractors in the $\mathcal{N} = 2, d = 4$ theory [14] and the $\mathcal{N} = 2, d = 5$ theory. In ungauged supergravity a large class of BPS solutions in four and five dimensions are related to each other [168, 169]. It would be interesting to see if this connection extends to gauged supergravity.

One of the most interesting problems in the study of attractor mechanism in gauged supergravity is the construction of the flow equations. These equations require a full analytical or numerical black hole solution interpolating between an asymptotic AdS geometry and a near horizon geometry. To construct the flow equations for generalised attractors, it is necessary to construct black brane solutions interpolating between the Bianchi type geometries and the AdS geometry. This will help prove the attractor mechanism for black branes in gauged supergravity. We hope to address this in future.

There are classes of near horizon geometries which are more general than the ones considered in the thesis. Typical examples are metrics which are conformal to the Lifshitz metric, which belongs to the Bianchi I class. These family of metrics exhibit different scaling symmetries, hyperscale violation and occur as gravity duals in studies of doped matter in AdS/CMT [170–173]. Such solutions have been studied systematically in ungauged supergravity [174]. It has been shown that hyperscale violating metrics can arise upon dimensional reduction of some null deformations of the AdS factors that appear in the near horizon geometry of various extremal brane configurations in string theory [175]. More recently, hyperscale violating metrics conformal to Lifshitz have also been constructed in gauged supergravity [176]. In [53], examples of Bianchi attractors that exhibit hyperscale violation have been constructed in Einstein-Maxwell-Dilation theories. It would be interesting to develop a technique similar to the generalised attractor approach to systematically obtain Bianchi attractors which exhibit hyperscale violation in gauged supergravity. Our previous attempts to obtain such metrics as generalised attractors point that the assumptions of constant anholonomy and constant fields in tangent space need to be relaxed suitably. Furthermore, the supersymmetry of such hyperscale violating metrics need to be studied and perhaps this will shed some light on modification of the generalised attractor assumptions.

Another important issue is string embedding. In general, the embedding of Bianchi attractors in gauged supergravity does not imply string embedding. For instance, It appears that the gauged supergravity models that we have considered [18, 19] are not embeddible in string theory [1]. Perhaps, one way to approach this problem is to look for low dimensional gauged supergravity models with known string embeddings, construct the Bianchi attractors in these theories and then attempt a ten dimensional lift. Perhaps, it is also possible to approach this question top-down from flux compactifications [9]. Gauged supergravities can arise as low energy effective theories from flux compactifications. The various flux parameters associated in a given compactification can be grouped in a tensorial form on which the duality action is manifest. This tensorial form is called an embedding tensor which encodes all the various possible gaugings of the supergravity [10]. This may help narrow down possible models where one could then look for generalised attractor solutions.

The stability condition (7.4.6), is similar to the condition for an isotropic universe in the AdS-Kasner metric. It implies that the near horizon geometries must be free from any

[1]We would like to thank Prof Marco Zagermann for informing us about this issue and for several helpful discussions.

anisotropies with respect to the radial direction. This manifests as a loss of scale invariance in the spatial directions. However, the homogeneous symmetries in the spatial directions and scale invariance along the radial, time directions are preserved. The stability analysis predicts that the generalised attractors which are stable under scalar fluctuations about the attractor value, factor into a direct product form, have homogeneous symmetries and are isotropic with respect to a radial flow. In this respect, the stability conditions certainly narrows down the possible IR candidate geometries. However, one should exercise caution as some model dependent information has gone into the final stages of this calculation.

Appendix A

Tangent space and constant anholonomy

In this chapter, we summarise our convention for tangent space and some definitions of anholonomy coefficients used throughout the thesis. Greek indices $\mu, \nu, \ldots$ denote space time indices with $\mu = 0, 1, \ldots, 4$ and $g_{\mu\nu}$ is the space time metric. Latin indices $a, b, \ldots$ denote tangent space indices with $a = 0, 1 \ldots, 4$. The tangent space metric has the signature $\eta_{ab} = \{-, +, +, +, +\}$. The vielbeins $e^a_\mu(x)$ are related to the space time metric by,

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta^{ab}. \tag{A.0.1}$$

We define the one form $e^a \equiv e^a_\mu dx^\mu$ and its dual $\tilde{e}_a \equiv e^\mu_a \partial_\mu$. The anholonomy coefficients are defined as Lie brackets of the duals $\tilde{e}_a$,

$$[\tilde{e}_a, \tilde{e}_b] \equiv c_{ab}{}^c \tilde{e}_c; \quad c_{ab}{}^c = e^\mu_a e^\nu_b (\partial_\nu e^c_\mu - \partial_\mu e^c_\nu) \tag{A.0.2}$$

The tangent space curvature can be written in terms of the anholonomy coefficients and the spin connection,

$$R_{abc}{}^{d} = \partial_a \omega_{bc}{}^{d} - \partial_b \omega_{ac}{}^{d} - \omega_{ac}{}^{e} \omega_{be}{}^{d} + \omega_{bc}{}^{e} \omega_{ae}{}^{d} - c_{ab}{}^{e} \omega_{ec}{}^{d}. \tag{A.0.3}$$

In the absence of torsion the spin connection and anholonomy coefficients are related by,

$$\omega_{a,bc} = \frac{1}{2}[c_{ab,c} - c_{ac,b} - c_{bc,a}], \tag{A.0.4}$$

where $\omega_{a,bc} = -\omega_{a,cb}$ and $c_{ab,c} = -c_{ba,c}$. The comma is used to indicate the antisymmetric indices. It follows that when one takes constant $c_{ab}{}^{c}$, the derivatives in (A.0.3) vanish and the Riemann tensor is a function of the constant anholonomy coefficients. For tangent space covariant derivatives acting on a spinor χ_α we use the convention,

$$D_a(\omega)\chi_\alpha = \partial_a \chi_\alpha - \frac{1}{4}\omega_a{}^{bc}\gamma_{bc}\chi_\alpha, \tag{A.0.5}$$

where α is the spinor index. The action of the covariant derivative on vectors is given by,

$$D_a(\omega)V^b = \partial_a V^b + \omega_{a,}{}^{b}{}_{c} V^c. \tag{A.0.6}$$

Appendix B

Gamma matrices and Spinors in five dimension

The Clifford algebra in 5 space-time dimensions is,

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}. \tag{B.0.1}$$

The Dirac matrices in five dimensions are given by [1],

$$\begin{aligned}
\gamma^0 &= -i\sigma_2 \otimes \sigma_3 \\
\gamma^1 &= -\sigma_1 \otimes \sigma_3 \\
\gamma^2 &= I_2 \otimes \sigma_1 \\
\gamma^3 &= I_2 \otimes \sigma_2 \\
\gamma^4 &= -i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma_3 \otimes \sigma_3
\end{aligned} \tag{B.0.2}$$

where $\sigma_i, i = 1, 2, 3$ are the usual Pauli matrices and I_2 is the two dimensional unit matrix. The charge conjugation matrix C has the property $C^t = -C = C^{-1}$ and,

$$C\gamma^a C^{-1} = (\gamma^a)^t, \tag{B.0.3}$$

where $C = B\gamma^0$, with $B = \gamma^3$ such that $B^*B = -1$. The spinor indices which are usually suppressed in most places are raised and lowered by $C_{\alpha\beta}$ using the NW-SE convention. Expressions such as $\bar{\psi}\psi$ and $\bar{\psi}\gamma^a\psi$ are understood as,

$$\bar{\psi}\psi = \bar{\psi}^\alpha\psi_\alpha, \quad \bar{\psi}\gamma^a\psi = \bar{\psi}^\alpha(\gamma^a)_\alpha^{\ \beta}\psi_\beta. \tag{B.0.4}$$

In addition, the spinors in the theory carry an $SU(2)$ index which is raised and lowered using ϵ_{ij},

$$X^j = \epsilon^{ji}X_i, \quad X_j = X^i\epsilon_{ij}, \tag{B.0.5}$$

with $\epsilon_{12} = \epsilon^{12} = 1$. With these conventions the mixed ϵ tensors are antisymmetric

$$\epsilon^{jk}\epsilon_{ki} = \epsilon^j_{\ i} = -\delta^j_i = -\epsilon_i^{\ j}. \tag{B.0.6}$$

Spinors in $d = 5$ satisfy a symplectic majorana condition. To apply this condition one needs $B^*B = -1$, even number of Dirac spinors $\psi_i, i = 1, \ldots, 2n$ and an antisymmetric real matrix Ω_{ij} with $\Omega^2 = -1_{2n}$. The symplectic majorana condition on a generic spinor reads as,

$$\psi_i^* = \Omega_{ij}B\psi_j, \tag{B.0.7}$$

or equivalently as,

$$\bar{\psi}^i \equiv (\psi_i^*)^t\gamma^0 = (\psi^i)^tC. \tag{B.0.8}$$

For $\mathcal{N} = 2$ supersymmetry $i = 1, 2$, and using $\Omega_{ij} = \epsilon_{ij}$ (B.0.7) reads as,

$$\psi_1^* = \gamma^3\psi_2. \tag{B.0.9}$$

Note that this condition does not reduce the degrees of freedom as compared to a single unconstrained Dirac spinor. This is because one needs at least a pair of Dirac spinors to apply the symplectic majorana condition (B.0.7). However, the action of the R-symmetry is manifest with this condition. Let us start with a pair of generic Dirac spinors in five dimensions,

$$\epsilon_1 = \begin{pmatrix} \epsilon_{11R} + i\epsilon_{11I} \\ \epsilon_{12R} + i\epsilon_{12I} \\ \epsilon_{13R} + i\epsilon_{13I} \\ \epsilon_{14R} + i\epsilon_{14I} \end{pmatrix}, \epsilon_2 = \begin{pmatrix} \epsilon_{21R} + i\epsilon_{21I} \\ \epsilon_{22R} + i\epsilon_{22I} \\ \epsilon_{23R} + i\epsilon_{23I} \\ \epsilon_{24R} + i\epsilon_{24I} \end{pmatrix}, \tag{B.0.10}$$

where all the components are real valued constants. Using (B.0.7) one finds that,

$$\begin{aligned} \epsilon_{21R} &= -\epsilon_{13I} \quad , \quad \epsilon_{21I} = -\epsilon_{13R} \\ \epsilon_{22R} &= -\epsilon_{14I} \quad , \quad \epsilon_{22I} = -\epsilon_{14R} \\ \epsilon_{23R} &= \epsilon_{11I} \quad , \quad \epsilon_{23I} = \epsilon_{11R} \\ \epsilon_{24R} &= \epsilon_{12I} \quad , \quad \epsilon_{24I} = \epsilon_{12R}. \end{aligned} \tag{B.0.11}$$

Therefore,

$$\epsilon_1 = \begin{pmatrix} \epsilon_{11R} + i\epsilon_{11I} \\ \epsilon_{12R} + i\epsilon_{12I} \\ \epsilon_{13R} + i\epsilon_{13I} \\ \epsilon_{14R} + i\epsilon_{14I} \end{pmatrix}, \epsilon_2 = \begin{pmatrix} -\epsilon_{13I} - i\epsilon_{13R} \\ -\epsilon_{14I} - i\epsilon_{14R} \\ \epsilon_{11I} + i\epsilon_{11R} \\ \epsilon_{12I} + i\epsilon_{12R} \end{pmatrix}. \tag{B.0.12}$$

As one can see, there are 8 independent real components, which is same as the number of degrees of freedom of a single unconstrained Dirac spinor. The minimum supersymmetry that one can have in five dimensions is then $\mathcal{N} = 2$ and thus the R symmetry group of the Poincaré superalgebra is $USp(2)_R \simeq SU(2)_R$. The advantage of using the symplectic majorana condition is that the action of the $SU(2)_R$ symmetry is manifest. For example, rewriting the symplectic majorana spinors in two component notation one sees that,

$$\epsilon_i = \begin{pmatrix} i\epsilon_{ij}\lambda_j \\ \lambda_i^* \end{pmatrix}, \tag{B.0.13}$$

where,

$$\lambda_1 = \begin{pmatrix} \epsilon_{13R} - i\epsilon_{13I} \\ \epsilon_{14R} - i\epsilon_{14I} \end{pmatrix}, \lambda_2 = \begin{pmatrix} \epsilon_{11I} - i\epsilon_{11R} \\ \epsilon_{12I} - i\epsilon_{12R} \end{pmatrix}. \tag{B.0.14}$$

We do not require the two component notation for our purposes, we will use (B.0.12). Antisymmetrisation is done with the following convention,

$$\gamma_{a_1 a_2 \ldots a_n} = \gamma_{[a_1 a_2 \ldots a_n]} = \frac{1}{n!} \sum_{\sigma \in P_n} Sign(\sigma) \gamma_{a_{\sigma(1)}} \gamma_{a_{\sigma(2)}} \ldots \gamma_{a_{\sigma(n)}}. \tag{B.0.15}$$

In $d = 5$ only I, γ_a, γ_{ab} form an independent set, other matrices are related by the general identity for $d = 2k + 3$,

$$\gamma^{\mu_1 \mu_2 \ldots \mu_s} = \frac{-i^{-k+s(s-1)}}{(d-s)!} \epsilon^{\mu_1 \mu_2 \ldots \mu_s} \gamma_{\mu_{s+1} \ldots \mu_d}. \tag{B.0.16}$$

We also list some useful identities involving various dirac matrices [29],

$$\begin{aligned}
[\gamma_a, \gamma_b] =& 2\gamma_{ab}, \\
[\gamma_h, \gamma_{abc}] =& 2\gamma_{habc}, \\
[\gamma_{abc}, \gamma_{egh}] =& \eta_{ef} \eta_{gp} \eta_{hk} (2\gamma_{abc}{}^{fpk} - 36 \delta_{[ab}{}^{[fp} \gamma_{c]}{}^{k]}).
\end{aligned} \tag{B.0.17}$$

Appendix C

Origin of tensor multiplets in gauged supergravity

A novel feature of the gauged supergravity in five dimensions is the entry of tensor multiplets upon gauging. Once the gauge group $K \subset G$ is identified, if one chooses to gauge the $n_V + 1$ vector fields A^I , one is left with $n_T = dim(G) - n_V$ vector fields A^M charged under K. These n_T gauge fields are dualised into antisymmetric tensor fields $B^M_{\mu\nu}$ and give rise to the tensor multiplets in the theory. It is important to note that there is no need for a tensor multiplet in the ungauged supergravity (5.2.1) since the vectors and tensors are equivalent by the duality relation [151],

$$\partial_{[\mu}A_{\nu]} = \epsilon_{\mu\nu}{}^{\lambda\rho\sigma}\partial_\lambda B_{\rho\sigma}. \tag{C.0.1}$$

This is however not true when the tensors carry massive degrees of freedom. The "self-duality" condition for a massive tensor field in five dimensions is given by,

$$B_{\mu\nu} = \frac{i}{3!m}\epsilon_{\mu\nu\lambda\rho\sigma}H^{\lambda\rho\sigma}, \tag{C.0.2}$$

where H is the three form field strength of B and m is a mass parameter. In fact, the condition (C.0.2) follows from the generalisation of the Proca Lagrangian for tensor fields,

$$\mathcal{L}_{proca} = B^{*\mu\nu}B_{\mu\nu} - \frac{i}{3!m}\epsilon_{\mu\nu\lambda\rho\sigma}B^{*\mu\nu}H^{\lambda\rho\sigma}. \tag{C.0.3}$$

One can compare the above Lagrangian with (5.3.9) and see that in the gauged supergravity the tensor fields appear exactly as in the Proca Lagrangian, except that there are covariant derivatives that appear due to the gauging. The presence of the i also implies that these tensor fields are complex. In this discussion we will consider the tensor fields to be decomposed of real and imaginary parts and hence the index M is always even.

We now explain briefly how the vectors A^M_μ lose the degrees of freedom to tensors $B^M_{\mu\nu}$ via a Higgs type mechanism. Remember that the A^M_μ are those vector fields which are neither adjoint nor singlets under K, hence they cannot describe Yang-Mills gauge fields. In particular, they describe massive degrees of freedom. The field strength $F^M_{\mu\nu}$ is replaced by the combination,

$$B^M_{\mu\nu} = F^M_{\mu\nu} + b^M_{\mu\nu}, \tag{C.0.4}$$

where $b^M_{\mu\nu}$ is an antisymmetric tensor invariant under the gauge transformation for a tensor field,

$$\delta b^M_{\mu\nu} = \partial_{[\mu}\Lambda^M_{\nu]}. \tag{C.0.5}$$

This immediately implies that the $A^M_{\mu\nu}$ transform as,

$$\delta A^M_\mu = -\Lambda^M_\mu, \tag{C.0.6}$$

so that (C.0.4) is gauge invariant. The above form of the gauge transformation also follows from the closure of the supersymmetry algebra [13]. Using this gauge transformation one can always choose $\Lambda^M = A^M$ and get $B^M_{\mu\nu} = b^M_{\mu\nu}$. Thus the massive degrees of freedom of A^M_μ are absorbed by $b^M_{\mu\nu}$ by a Higgs mechanism. It follows that the tensor fields $B^M_{\mu\nu}$ describe massive degrees of freedom.

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