The t-analogue of string functions for the affine Kac-Moody algebras

by

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Sachin S. Sharma

Abstract

We study Lusztig's t-analogue of weight multiplicities associated to the irreducible integrable highest weight modules of affine Kac-Moody algebras. First, for the level one representation of twisted affine Kac-Moody algebras, we obtain an explicit closed form expression for the corresponding t-string function using constant term identities of Macdonald and Cherednik. The closed form involves the generalised exponents of the graded pieces of the twisted affine algebra, considered as modules for the underlying finite dimensional simple Lie algebra. This extends previous work on level 1 t-string functions for the untwisted simply-laced affine Kac-Moody algebras. Next, for the Lie algebra $A_1^{(1)}$, we give a basis for the weight spaces of its basic representation, which is compatible with the affine Brylinski-Kostant filtration defined by Slofstra. Using this basis we give an alternative derivation of the expression for the t-string function of the basic representation. Finally, we obtain explicit formula for the t-string function of irreducible integrable highest weight $A_1^{(1)}$-modules of all levels. This is generalisation of a theorem of a Kac and Peterson.

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Contents

Chapter 1

Introduction

1.1 Basic t-string function for twisted affine Kac-Moody algebras

Let $\mathfrak{g}$ be an affine Kac-Moody algebra of rank $l+1$ ($l \geq 1$). Let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta_+}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}))$ be its root space decomposition, where Δ_+ is the set of positive roots, and let mult $\alpha := \dim(\mathfrak{g}_\alpha)$ be the multiplicity of α. Let $\overset{\circ}{\mathfrak{g}}$ denote its underlying finite dimensional simple Lie algebra of rank l. For a dominant integral weight λ, let $L(\lambda)$ denote the irreducible $\mathfrak{g}$-module of highest weight λ.

In this thesis we will first consider the basic representation $L(\Lambda_0)$ of $\mathfrak{g}$. Here Λ_0 denotes the fundamental weight corresponding to the extended node of the Dynkin diagram of $\mathfrak{g}$. The Kostant partition function of $\mathfrak{g}$ is defined by

$$\mathcal{P}(\beta) := [e[-\beta]] \prod_{\alpha\in\Delta_+} \frac{1}{(1-e[-\alpha])^{m_\alpha}}$$

where $[e[\alpha]]f$ denotes the coefficient of $e[\alpha]$ appearing in the expression of f, where $e[\alpha] = e^\alpha$ denote an element of the group algebra $\mathbb{C}[\mathfrak{h}^*]$ and $\beta \in \mathfrak{h}^*$. Note that $\mathcal{P}(\beta)$ is nothing but the number of partitions of β in to sum of positive roots, where each root is counted with its multiplicity. A t-analogue of the Kostant partition function is defined as

$$\mathcal{P}(\beta;t) := [e[-\beta]] \prod_{\alpha\in\Delta_+} \frac{1}{(1-te[-\alpha])^{m_\alpha}} .$$

We recall that the Kostant partition function is used to calculate the dimension of the weight spaces of the $\mathfrak{g}$-module $L(\lambda)$ by the formula

$$\dim(L(\lambda)_\mu) = \sum_{w\in W} \epsilon(w)\mathcal{P}(w(\lambda+\rho)-(\mu+\rho))$$

where W is the Weyl group of $\mathfrak{g}$ and ϵ is its sign character . Using the t-Kostant partition function we define a t-analogue of weight multiplicity :

$$K_\mu^\lambda(t) := \sum_{w\in W} \epsilon(w)\mathcal{P}(w(\lambda+\rho)-(\mu+\rho);t) \ .$$

$K_\mu^\lambda(t)$ is also called the affine Kostka-Foulkes polynomial.

To understand the structure of the module $L(\lambda)$ one studies the string function

$$a_\mu^\lambda(q) := \sum_{k\geq 0} \dim(L(\lambda)_{\mu-k\delta})\ q^k \ .$$

These are generating functions of weight multiplicities along δ-strings through dominant maximal weights μ, where δ is the null root of $\mathfrak{g}$. A t-analogue of string function or t-string function is defined as

$$a_\mu^\lambda(t;q) := \sum_{k\geq 0} K_{\mu-k\delta}^\lambda(t) q^k$$

In [10], it was shown that the $a_\mu^\lambda(t;q)$ are closely related to the constant term identities arising in the theory of Macdonald polynomials. An explicit formula for $a_{\Lambda_0}^{\Lambda_0}(t;q)$ of the basic representation for the untwisted simply laced algebras is obtained in [39] by using Cherednik's Macdonald Mehta constant term identity .

Theorem 1.1.1 ([39])**.** *Let $\mathfrak{g}$ be one of the simply laced untwisted affine Lie algebras $A_\ell^{(1)}, D_\ell^{(1)}, E_\ell^{(1)}$. Then*

$$a_{\Lambda_0}^{\Lambda_0}(t;q) = \prod_{i=1}^{\ell}\prod_{n=1}^{\infty}(1-t^{e_i+1}q^n)^{-1}$$

where $e_i\,(1\leq i\leq \ell)$ are the exponents of the underlying finite dimensional simple Lie algebra ($=A_\ell, D_\ell, E_\ell$ respectively).

In chapter 3 of this thesis, we extend above result for the twisted affine Lie algebras [35]. We recall the term generalised exponents, as it appears in the statement of our theorem. Let $\mathfrak{m}$ be a finite dimensional simple Lie algebra and $V = V(\lambda)$ be the irreducible finite dimensional $\mathfrak{m}$-module with highest weight λ. Fix a triangular decomposition $\mathfrak{m} = N_-\oplus\mathfrak{h}\oplus N_+$, and let $E\in N_+$ be a principal nilpotent element i.e., $E=\sum_{i=1}^{l} c_iE_i$ where $c_i\in\mathbb{C}-\{0\}$ and $E_1, E_2, ..., E_l$ are the Chevally generators. Let V_0 denote the zero weight space of V. Define the Brylinski-Kostant filtration of V_0 by $\mathcal{F}^{(0)}(V_0)\subseteq\mathcal{F}^{(1)}(V_0)\subseteq\mathcal{F}^{(2)}(V_0)\subseteq ...$, where $\mathcal{F}^{(p)}(V_0) := \ker(E^{p+1})\cap V_0$. Then the generalised exponents of V are the elements of the multiset $\mathbb{E}$(V) defined via the following relation:

$$\sum_{p\geq 0}\dim(\mathcal{F}^{(p)}(V_0)/\mathcal{F}^{(p-1)}(V_0))\, t^p = \sum_{k\in\mathbb{E}(V)} t^k.$$

The exponents of the finite dimensional simple Lie algebra $\mathfrak{m}$ are nothing but the generalised exponents of the adjoint representation of $\mathfrak{m}$.

Now, let $\mathfrak{g}$ be a twisted affine Lie algebra of type $X_N^{(r)}$, where $X_N = A_N, D_N, E_N$ and $r = 2$ or 3. Let $\mathfrak{m}$ denote the finite dimensional simple Lie algebra with Dynkin diagram X_N. Let σ be the diagram automorphism of $\mathfrak{m}$ of order r. Let $\mathfrak{m} = \bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}} \mathfrak{m}_j$ be the eigenspace decomposition of $\mathfrak{m}$ with respect to σ. Then $\mathfrak{m}_0$ is a finite dimensional simple Lie algebra and $\mathfrak{m}_j$ for $j \neq 0$ are irreducible $\mathfrak{m}_0$-modules. We let $\mathbb{E}_n$ denote the multiset of as generalised exponents of the $\mathfrak{m}_0$-module $\mathfrak{m}_n$. The main theorem of chapter 3 is the following:

Theorem 1.1.2. *Let $\mathfrak{g}$ be a twisted affine Lie algebra. The t-string function of the basic representation of $\mathfrak{g}$ is given by*

$$a_{\Lambda_0}^{\Lambda_0}(t;q) = \prod_{n=1}^{\infty} \prod_{e \in \mathbb{E}_n} (1 - t^{e+1} q^n)^{-1}$$

The proof of this theorem for $\mathfrak{g} \neq A_{2l}^{(2)}$ follows by using Cherednik's computation of Macdonald-Mehta type constant term, and a combinatorial characterisation of the generalised exponents. For $\mathfrak{g} = A_{2l}^{(2)}$, we use the Macdonalds constant term identity for the non-reduced affine root system of type $(C_n^\vee, C_n)$ to derive a Macdonald-Mehta identity for $\mathfrak{g}$.

1.2 Affine Brylinski-Kostant filtration on the basic representation of $A_1^{(1)}$

Consider the affine Lie algebra $A_1^{(1)}$. Let us consider its basic representation $L(\Lambda_0)$. By theorem 1.1.1 we have

$$a_{\Lambda_0}^{\Lambda_0}(t;q) = \sum_{n \geq 0} K_{\Lambda_0 - n\delta}^{\Lambda_0}(t) q^n = \prod_{k=1}^{\infty} (1 - t^2 q^k)^{-1}$$

By comparing the coefficients of the above expression, we get the formula for the affine Kostka-Foulkes polynomial which is,

$$K_{\Lambda_0 - k\delta}^{\Lambda_0}(t) = \sum_{\pi \vdash k} t^{2\,(\#\pi)}$$

where $\#\pi$ is the number of parts in the partition π of k.

Slofstra [37] showed that the affine Kostka-Foulkes polynomial $K_\mu^\lambda(t)$, where λ and μ are dominant, is equal to the Poincaré series of the associated graded space of an affine version of the Brylinski-Kostant filtration. By this, Slofstra extended Brylinski's result (which is for the finite dimensional simple Lie algebras) to affine Kac-Moody algebras.

For finite dimensional simple Lie algebras, the Brylinski-Kostant filtration uses a principal nilpotent element. Slofstra shows that in the affine case, a principal nilpotent element is not sufficient to define the filtration, but one has to use the positive part of a principal Heisenberg algebra.

In chapter 4 of this thesis, we give a basis for the space $L(\Lambda_0)_{\Lambda_0-n\delta}$ for $n \geq 0$, which is compatible with respect to the affine Brylinski-Kostant filtration for $\mathfrak{g} = A_1^{(1)}$. Using Slofstra's theorem this gives an alternative derivation for the expression of $a_{\Lambda_0}^{\Lambda_0}(t;q)$. Let $\mathfrak{g} = A_1^{(1)} = \mathbb{C}[z,z^{-1}]\otimes\mathfrak{sl}_2\oplus\mathbb{C}K\oplus\mathbb{C}d$, where K is the central element and d is a derivation. Let $\mathfrak{g} = \mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$ be a triangular decomposition of $\mathfrak{g}$. Let e, f and h be the usual basis of $\mathfrak{sl}_2$. For odd integers j, define $B_j := e \otimes z^{\frac{j-1}{2}} + f \otimes z^{\frac{j+1}{2}}$ and for non-zero even integers j define $H_j := h \otimes z^{\frac{j}{2}}$, and $H_0 := h \otimes 1 - \frac{1}{2}K$.

We recall that for a principal nilpotent element E' of an affine Lie algebra, we have the algebra $\mathfrak{s}_{E'} := \{x \in \mathfrak{g} : [x, E'] \in Z(\mathfrak{g})\}$, called as a principal Heisenberg subalgebra of $\mathfrak{g}$. Recall that the homogeneous Heisenberg subalgebra is defined as $\mathcal{H} := \bigoplus_{\substack{n\in\mathbb{Z}\\ n\neq 0}} z^n \otimes \mathfrak{h} \ \oplus \mathbb{C}K$, where $\mathfrak{h} = \mathbb{C}h$. Note that span of B_j (j odd) and K spans a principal Heisenberg subalgebra and span of H_j (j even) is the homogeneous Heisenberg subalgebra for $A_1^{(1)}$ [22]. We denote them by $\mathfrak{s}$ and $\mathcal{H}$ respectively (where $\mathfrak{s} = \mathfrak{s}_{E'}$ for $E' = e \otimes 1 + f \otimes z$). Let $L(\lambda)$ be the highest weight module of $\mathfrak{g}$ corresponding to the dominant integral weight λ. The affine Brylinski-Kostant filtration with respect to the principal Heisenberg algebra $\mathfrak{s}$ is given by $\mathcal{F}^j(L(\lambda)_\mu) = \{v \in L(\lambda)_\mu : x^{j+1}v = 0 \; \forall\, x \in \mathfrak{s}\cap\mathfrak{n}_+\}$. Let $P_\mu^\lambda(t)$ denote the Poincaré series of the associated graded space of $L(\lambda)_\mu$ i.e., $P_\mu^\lambda(t) = \sum_{i\geq 0} \dim\,(\mathcal{F}^i L(\lambda)_\mu/\mathcal{F}^{i-1}L(\lambda)_\mu)\; t^i$. We now state the following theorem due to Slofstra [37].

Theorem 1.2.1 **(Slofstra).** *Let $L(\lambda)$ be an integrable highest weight representation of an affine Kac-Moody algebra $\mathfrak{g}$, where λ is dominant integral weight. If μ is a dominant weight of $L(\lambda)$, then*

$$P_\mu^\lambda(t) = K_\mu^\lambda(t)\ .$$

It is a well known fact that $\mathfrak{s}$ acts irreducibly on $L(\Lambda_0)$. So by the standard representation theory of the Heisenberg Lie algebra we have $L(\Lambda_0) = \mathbb{C}[x_1, x_3, x_5, ...]$ and for all odd $j > 0$ B_j acts as operator $\frac{j}{2}\frac{\partial}{\partial x_j}$ and B_{-j} acts as $2x_j$. Using these facts, we see that $\mathcal{F}^j(L(\lambda)_\mu) = \{p \in L(\lambda)_\mu : \text{udeg}(p) \leq j\}$, where udeg$(p)$ denotes usual degree of polynomial p. On the other hand, the homogeneous Heisenberg algebra $\mathcal{H}$ does not act irreducibly on $L(\Lambda_0)$ and we have $U(\mathcal{H})v_{\Lambda_0} = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} L(\Lambda_0)_{\Lambda_0-n\delta} =: V$.

To state our theorem we need to recall the Sugawara operators. For $\mathfrak{g}$, they are given by

$$T_n = \sum_{m\in\mathbb{Z}} \left[e(-m)f(m+n) + f(-m)e(m+n) + h(-m)\frac{h}{2}(m+n)\right] \;\; \forall\, n \neq 0$$

and

$$T_0 = e(0)f(0) + f(0)e(0) + \frac{h(0)^2}{2} + 2\sum_{n=1}^{\infty}\left[e(-n)f(n) + f(-n)e(n) + h(-n)\frac{h(n)}{2}\right]$$

where $x(m)$ denotes $x \otimes z^n$ and (e, f, h) and $(f, e, \frac{h}{2})$ are the dual bases of each other. These operators lie in the restricted completion $U_c(\mathfrak{g}')$ of the universal enveloping algebra $U(\mathfrak{g}')$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ [15].

Now let $L_n^{\overset{\circ}{\mathfrak{g}}} := \frac{1}{6}T_n$, Similarly for the Lie algebra $\tilde{L}(\mathfrak{h})$, we define for $n \neq 0$

$$L_n^{\overset{\circ}{\mathfrak{h}}} := \frac{1}{2}\sum_{m\in\mathbb{Z}} h(m)h(n-m)$$

and

$$L_0^{\overset{\circ}{\mathfrak{h}}} := \frac{h(0)^2}{4} + \sum_{n=1}^{\infty} h(-n)\frac{h(n)}{2} \ .$$

Now consider the homogeneous Heisenberg algebra $\mathcal{H}$ generated by $H_{2j} =: H(j)$ for $n \neq 0$ and $H_0 := H(0) = h_0 - \frac{1}{2}K$. Note that $h(n) = H_{2n} =: H(n)$ for $n \neq 0$. Define the Virasoro operators with respect to $\mathcal{H}$ by

$$L_n^{\mathcal{H}} := \frac{1}{2}\sum_{m\in\mathbb{Z}} H(n)\frac{H(n-m)}{2} \quad \text{for } n \neq 0$$

$$L_0^{\mathcal{H}} := \frac{H(0)^2}{4} + \sum_{n=1}^{\infty} H(-n)\frac{H(n)}{2}$$

Let for a polynomial $f \in \mathcal{F}^{(j)}L(\Lambda_0)_{\Lambda_0 - s\delta}$, $\overline{f}$ denote image of f in the quotient space $\mathcal{F}^{(j)}(L(\Lambda_0)_{\Lambda_0-s\delta})/\mathcal{F}^{(j-1)}(L(\Lambda_0)_{\Lambda_0-s\delta})$. Now we are in position to state the main theorem of chapter 4 :

Theorem 1.2.2. *Let $\mathfrak{g} = A_1^{(1)}$. Let $r > 0$ and $k \geq 0$. Then a basis for the quotient space $\mathcal{F}^{(2k)}(L(\Lambda_0)_{\Lambda_0-r\delta})/\mathcal{F}^{(2k-1)}(L(\Lambda_0)_{\Lambda_0-r\delta})$ is given by the set*

$$\{\overline{L_{n_1}^{\mathcal{H}} L_{n_2}^{\mathcal{H}} \ldots L_{n_k}^{\mathcal{H}} v_{\Lambda_0}} : 0 > n_k \geq n_{k-1} \geq \cdots \geq n_1 \text{ and } n_1 + n_2 + \cdots + n_k = -2r\}$$

From the above theorem, we deduce that $\dim(\mathcal{F}^{2k}(L(\Lambda_0)_{\Lambda_0-r\delta})/\mathcal{F}^{2k-1}(L(\Lambda_0)_{\Lambda_0-r\delta})) = P(r,k)$, the number of partitions of r in to k parts. We also note the following easy fact:

$$\dim \mathcal{F}^{(2k-1)}(L(\Lambda_0)_{\Lambda_0-r\delta})/\mathcal{F}^{(2k-2)}(L(\Lambda_0)_{\Lambda_0-r\delta}) = 0 \ .$$

Using this observation and Slofstra's theorem, we have the following corollary to theorem 1.2.2:

Corollary 1.2.3. *Let $\mathfrak{g} = A_1^{(1)}$. Let $L(\Lambda_0)$ be its basic representation. Then*

$$a_{\Lambda_0}^{\Lambda_0}(t;q) = \prod_{n=1}^{\infty}(1 - t^2q^n)^{-1} \ .$$

Proof of the theorem 1.2.2 uses the facts that $L(\Lambda_0)$ is the highest weight module for the coset Virasoro operator $L_n^{\overset{\circ}{\mathfrak{g}},\overset{\circ}{\mathfrak{h}}} = L_n^{\overset{\circ}{\mathfrak{g}}} - L_n^{\overset{\circ}{\mathfrak{h}}}$ and $L_n^{\overset{\circ}{\mathfrak{g}},\overset{\circ}{\mathfrak{h}}}$ commutes with $\tilde{L}(\mathfrak{h})$. By these facts it follows that $L_n^{\overset{\circ}{\mathfrak{g}}} v_{\Lambda_0} = L_n^{\overset{\circ}{\mathfrak{h}}} v_{\Lambda_0}$ and $L_n^{\overset{\circ}{\mathfrak{g}}} = L_n^{\overset{\circ}{\mathfrak{h}}}$ on V. Now it can be easily proved that $\tilde{L}_n^{\mathcal{H}}$ is of usual degree 2 for all $n < 0$ and the result follows by induction.

1.3 The t-string functions for $A_1^{(1)}$

Let $\mathfrak{g} = A_1^{(1)} = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{sl}_2 \oplus \mathbb{C}K \oplus \mathbb{C}d$. Let $L(\Lambda)$ be an irreducible highest weight module of level $m \geq 1$. Kac and Peterson [18], have proved that the string functions corresponding to the $\mathfrak{g}$-module $L(\Lambda)$ are related to the Hecke indefinite modular forms. In chapter 5 of this thesis we give a t-analogue of this result, i.e, we give an explicit formula for all t-string functions of the $\mathfrak{g}$-module $L(\Lambda)$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, let $\Delta \subseteq \mathfrak{h}^*$ be the root system of $\mathfrak{g}$. Let Δ_+ be the set of positive roots and $\{\alpha_0, \alpha_1\}$ be the set of simple roots of $\mathfrak{g}$. Let Q and P denote the root and weight lattices of $\mathfrak{g}$ and let P_+ be the set of dominant integral weights. Let $\mathring{\mathfrak{g}} = \mathfrak{sl}_2$ and $\mathring{Q}$ and $\mathring{P}$ be the root and weight lattices of $\mathring{\mathfrak{g}}$. Let W be the Weyl group of $\mathfrak{g}$ generated by reflections r_{α_0} and r_{α_1}, which we denote by r_0 and r_1. Let $\mathring{W}$ be the subgroup generated by r_1. Recall that $W = \mathring{W} \ltimes \mathring{Q}$.

Let $\mathcal{P}(\beta)$, $\mathcal{P}(\beta; t)$ denote the Kostant partition function and t-Kostant partition function of $\mathfrak{g}$. Let $\Lambda \in P_+$ be of level $m \geq 1$. Let $L(\Lambda)$ denote the corresponding irreducible highest weight representation of $\mathfrak{g}$. Let λ be a maximal dominant weight of $L(\Lambda)$. Define the string function $c_\lambda^\Lambda(q) := q^{s_\Lambda(\lambda)} \sum_{s \geq 0} \mathrm{mult}_\Lambda(\lambda - s\delta)\, q^s$, where $q = e^{2\pi i \tau}$, $\tau \in$ upper half plane and $s_\Lambda(\lambda) = n_0(\Lambda - \lambda) + \frac{|\bar{\Lambda}+\bar{\rho}|^2}{2(m+2)} - \frac{|\bar{\lambda}|^2}{2m} - \frac{1}{8}$, where $\bar{\alpha}$ denotes the image of α under the projection from $\mathfrak{h}^*$ to $\mathring{\mathfrak{h}}^*$, n_0 is the function on Q defined by $n_0(a_0\alpha_0 + a_1\alpha_1) := a_0\ \forall\ a_0, a_1 \in \mathbb{Z}$, and $\mathrm{mult}_\Lambda(\cdot)$ denotes the dimension of the corresponding weight space of $L(\Lambda)$. Note that $c_\lambda^\Lambda(q) = q^{s_\Lambda(\lambda)} a_\lambda^\Lambda(q)$.

Let $\rho \in \mathfrak{h}^*$ be a Weyl vector, which satisfies the relation $(\rho, \alpha_i) = 1$ for $i = 0, 1$. Let us consider a shifted action of W on Q by $w.\alpha := w(\alpha + \rho) - \rho$, which induces an action of W on functions on Q by $(w.f)(\alpha) := f(w^{-1}.\alpha)$.

Let us briefly recall the term Hecke modular form. Let U be a two dimensional real vector space. M be a full rank lattice in U and let B' be an indefinite symmetric form on U such that $B'(\nu, \nu)$ is an even nonzero integer for all nonzero $\nu \in M$. Let $M^* := \{\nu' \in U : B'(\nu, \nu') \in \mathbb{Z}\ \ \forall\ \nu \in M\}$. Let G_0 be the subgroup of the identity component of the orthogonal group of (U, B') preserving M and fixing M^*/M. Fix a factorisation $B'(\nu, \nu) = l_1(\nu) l_2(\nu)$, where l_1 and l_2 are real-linear, and set $\mathrm{sign}(\nu) := \mathrm{sign}\ l_1(\nu)$ for $l_1(\nu) \neq 0$. For $\mu \in M^*$, set

$$\theta_{M,\mu} := \sum_{\substack{\nu \in M+\mu \\ B'(\nu,\nu)>0 \\ \nu \bmod G_0}} \mathrm{sign}(\nu) e^{\pi i \tau B'(\nu,\nu)} .$$

The $\theta_{M,\mu}$ is called a Hecke indefinite modular form.

Let $M := \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_1$ and $U := \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_1$. We identify M with $\mathbb{Z}^2$ and U with $\mathbb{R}^2$, and let $B(x, y) := 2(m+2)x^2 - 2my^2$ be the quadratic form corresponding to an indefinite symmetric bilinear form B'. Note that $B \neq 0$ on $M - \{0\}$. The dual lattice of M with respect to B is $M^* = \frac{1}{2(m+2)}\mathbb{Z} \oplus \frac{1}{2m}\mathbb{Z}$. Let a be the element of the identity component $SO_0(U)$ of the orthogonal group of (U, B) given by $a(x, y) = ((m+1)x + my, (m+2)x + (m+1)y)$. Then a generates the subgroup G_0' of $SO_0(U)$ preserving M, and a^2 generates the subgroup G_0 of G_0' fixing M^*/M. Let us define an element $J : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $J(x,y) := (-x, y)$; we note that J normalises G_0. Define the group $G :=< J > \ltimes G_0$ and $G' :=< J > \ltimes G_0'$. Let $U^+ := \{u \in U : B(u) > 0\}$. Then it is easy to see that $F_0 := \{(x,y) \in \mathbb{R}^2 : -|x| < y \leq |x|\}$ is a fundamental domain for G_0' on U^+, and $F_0 \bigcup aF_0$ is a fundamental domain for G_0 on U^+. Set $F := \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x \text{ or } 0 > y > x\}$. Then clearly $F_0 = F \cup J(F)$ and F is a fundamental domain for G' on U^+. We now state Kac-Peterson's theorem [18].

Theorem 1.3.1 (Kac-Peterson). *Let $\mathfrak{g}$ be of type $A_1^{(1)}$. Let $\Lambda \in P_+$, $\Lambda(K) = m$, and let $\lambda \in P_+$ be a maximal weight of $L(\Lambda)$. Then $\eta(\tau)^3 c_\lambda^\Lambda(\tau) = \theta_{M,(\tilde{A};\tilde{B})}$ is a Hecke indefinite modular form, where $\tilde{A} := (m+2)^{-1}(\bar{\Lambda} + \bar{\rho})$ and $\tilde{B} := m^{-1}\bar{\lambda}$, and $\eta(\tau)$ is the Dedekind eta function.*

Let us recall the constant term map $\mathrm{ct}(\cdot)$, which defined on formal sums $\sum_{\alpha \in Q} c_\lambda e[\lambda]$ by $\mathrm{ct}(\sum_{\alpha\in Q} c_\lambda e[\lambda]) := \sum_{n \in \mathbb{Z}} c_{n\delta} e[n\delta]$. We define $\xi_t := \dfrac{1}{\prod_{n\geq 1}(1-tq^n)(1-tq^n e[-\alpha_1])(1-tq^n e[\alpha_1])}$ and the Poisson kernel $P_t := \sum_{n\in\mathbb{Z}} t^{|n|} e[n\alpha_1]$. The following is our main theorem of chapter 5, and gives an expression for all t-string functions of $A_1^{(1)}$.

Theorem 1.3.2. *Let $\mathfrak{g}$ be of type $A_1^{(1)}$. Let $\Lambda \in P_+$, $\Lambda(K) = m \geq 1$, and let $\lambda \in P_+$ be a maximal dominant weight of $L(\Lambda)$. Then*

$$c_\lambda^\Lambda(t;q) = \mathrm{ct}(\xi_t \, P_t \, q^{\frac{1}{8}} \, t^{-2\tilde{B}} \, H_t)$$

where

$$H_t = \sum_{\substack{(x,y) \equiv (\tilde{A},\tilde{B}) \mod \mathbb{Z}^2 \\ B(x,y)>0 \\ (x,y) \bmod \mathbb{Z}^2}} \mathrm{sign}(x) \, q^{\frac{1}{2}B(x,y)} \, t^{2\bar{y}} \, e[((m+2)\bar{x} - m\bar{y} - \frac{1}{2})\alpha_1]$$

where $(\bar{x}, \bar{y})$ is the unique element in $F \cap G'(x,y)$.

We note that the form of H_t resembles that of a theta function.

Proof of the theorem 1.3.2 closely follows that of theorem 1.3.1, by expressing $\mathcal{P}(\beta;t)$ in terms of a simpler function $\mathcal{P}'(\beta;t) := (1 + tr_1).\mathcal{P}(\beta;t)$.

Chapter 2

Kac-Moody algebras

Let us consider an $n \times n$ complex matrix $A = (a_{ij})_{i,j=1}^n$. The matrix A is called a generalised Cartan matrix if it satisfies the following conditions:

(C 1) $a_{ii} = 2$ for $i = 1, ..., n$;

(C 2) a_{ij} are nonpositive integers for $i \neq j$;

(C 3) $a_{ij} = 0$ implies $a_{ji} = 0$.

2.1 Realisation of a matrix

A realisation of an $n \times n$ complex matrix A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_n^\vee\} \subset \mathfrak{h}$ are indexed subsets in $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, satisfying the following conditions:

1. Both sets Π and $\Pi^\vee$ are linearly independent;
2. $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij} \ \ (i, j = 1, 2, \cdots, n)$;
3. $n - \text{rank}(A) = \dim \mathfrak{h} - n$.

Two realisations $(\mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ are called *isomorphic* if there exists a vector space isomorphism $\phi : \mathfrak{h} \to \mathfrak{h}_1$ such that $\phi(\Pi^\vee) = \Pi_1^\vee$ and $\phi^*(\Pi_1) = \Pi$.

2.1.1 Auxiliary Lie algebras

Definition 2.1.1. *Let $A = (a_{ij})$ be an $n \times n$ matrix over $\mathbb{C}$, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of A. Define an auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$ with generators $e_i, f_i (i = 1, \cdots, n)$ and $\mathfrak{h}$, and the* *following defining relations:*

$$\left.\begin{aligned}
[e_i, f_j] &= \delta_{ij}\alpha_i^\vee \quad (i,j = 1,\cdots,n)\\
[h, h'] &= 0 \quad (h, h' \in \mathfrak{h}),\\
[h, e_i] &= \langle \alpha_i, h\rangle e_i\\
[h, f_i] &= -\langle \alpha_i, h\rangle f_i \quad (i,j = 1,\cdots,n; h \in \mathfrak{h})
\end{aligned}\right\} \tag{2.1.1}$$

Denote by $\tilde{\mathfrak{n}}_+$(resp.$\tilde{\mathfrak{n}}_-$) the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $e_1, e_2, \cdots, e_n$ (resp.$f_1, \cdots, f_n$). Now we state the following result for $\tilde{\mathfrak{g}}(A)$:

Theorem 2.1.2 ([15]). *1.* $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$ (direct sum of vector spaces)

2. $\tilde{\mathfrak{n}}_+$(resp. $\tilde{\mathfrak{n}}_-$) *is freely generated by* $e_1, \cdots, e_n$(resp. $f_1, \cdots, f_n$)

3. Among the ideals of $\tilde{\mathfrak{g}}(A)$ *intersecting* $\mathfrak{h}$ *trivially, there exists a unique maximal ideal* $\boldsymbol{\tau}$. *Furthermore,*

$$\boldsymbol{\tau} = (\boldsymbol{\tau} \cap \tilde{\mathfrak{n}}_-) \oplus (\boldsymbol{\tau} \cap \tilde{\mathfrak{n}}_-) \quad \text{(direct sum of ideals)}$$

2.1.2 Kac-Moody Lie algebras

For a given complex $n \times n$ matrix A, let $\tilde{\mathfrak{g}}(A)$ be the Lie algebra on the generators $e_i, f_i (i = 1, \cdots, n)$ and $\mathfrak{h}$, and defining relations equation 2.1.1. By theorem 2.1.2 the natural map $\mathfrak{h} \to \tilde{\mathfrak{g}}(A)$ is an imbedding. Let $\boldsymbol{\tau}$ be the maximal ideal of $\tilde{\mathfrak{g}}(A)$, which intersects $\mathfrak{h}$ trivially. Set

$$\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\boldsymbol{\tau}$$

The matrix A is called as the *Cartan matrix* of $\mathfrak{g}(A)$, and n is called as the *rank* of $\mathfrak{g}(A)$. Now we are in a position to define Kac-Moody algebras.

Definition 2.1.3. *The Lie algebra* $\mathfrak{g}(A)$ *whose Cartan matrix is a generalised Cartan matrix is called a Kac-Moody algebra.*

2.1.3 Symmetrisable Kac-Moody algebra

A $n \times n$ matrix $A = (a_{ij})$ is called *symmetrisable* if there exists an invertible diagonal matrix $D = \text{diag}(\epsilon_1, ..., \epsilon_n)$ and a symmetric matrix $B = (b_{ij})$ such that $A = DB$.

Definition 2.1.4. *Let* $A = (a_{ij})$ *be a symmetrisable generalised Cartan matrix. Then the Kac-Moody algebra associated with the matrix* A *is called a symmetrisable Kac-Moody algebra.*

For a symmetrisable Kac-Moody algebra $\mathfrak{g}(A)$ there exists a nondegenerate symmetric bilinear $\mathbb{C}$-valued form $(.,.)$ on $\mathfrak{g}(A)$ which is invariant i.e., $([x,y],z) = (x,[y,z])$ for all $x, y, z \in \mathfrak{g}(A)$.

2.1.4 Properties of Kac-Moody algebras

Theorem 2.1.5. *Let $\mathfrak{g}$ be a Kac-Moody algebra. Then,*

1. $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, *where* $\mathfrak{n}_+$, $\mathfrak{n}_-$ *are the Lie algebras generated by* e_i *and* f_i $(i = 1, \cdots, n)$ *respectively.*

2. $\mathfrak{h}$ *acts diagonalisably on* $\mathfrak{g}$ *i.e.,*

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \quad \forall h \in \mathfrak{h}\}$.

The subspace $\mathfrak{h}$ is called the Cartan subalgebra and $\mathfrak{g}_\alpha$ is said to be the root space of α. An element $\alpha \in \mathfrak{h}^*$ is called a root of $\mathfrak{g}$ if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. Let mult $\alpha :=$ dim $\mathfrak{g}_\alpha$; this is called *multiplicity* of α. Let Δ denote the set of all roots of $\mathfrak{g}$, then $\Delta = \Delta_+ \cup \Delta_-$(a disjoint union), where Δ_+, Δ_- are the sets of positive roots and negative roots respectively. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be the simple roots of $\mathfrak{g}$, $\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_n^\vee$ be the simple co-roots of $\mathfrak{g}$ and $Q := \sum_{i=1}^n \mathbb{Z}\alpha_i$ be the root lattice of $\mathfrak{g}$. Let $Q_+ := \sum_{i=1}^n \mathbb{Z}_+\alpha_i$. Introduce a partial ordering $\leq$ on $\mathfrak{h}^*$ by setting $\mu \leq \lambda$ if $\lambda - \mu \in Q_+$. Let $P := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}(i = 1, \cdots, n)\}$; this is called the weight lattice of $\mathfrak{g}$. Let $P_+ := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \geq 0 (i = 1, \cdots, n)\}$ and are called dominant weight lattices of $\mathfrak{g}$.

2.1.5 Weyl group of Kac- Moody algebras

Now we introduce an important group associated with Kac-Moody algebras, called the Weyl group. Let A be a $n \times n$ generalised Cartan matrix and let $\mathfrak{g}(A)$ be the associated Kac-Moody algebra. For each $i = 1, 2, ..., n$, define the fundamental reflection r_i by

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$$

for $\lambda \in \mathfrak{h}^*$.

The subgroup W of $GL(\mathfrak{h}^*)$ generated by all fundamental reflections is called the Weyl group of $\mathfrak{g}(A)$.

2.1.6 Classification of Kac-Moody algebras

The following theorem classifies the types of Kac-Moody algebra. For a real column vector $u = {}^t(u_1, \cdots, u_n)$ we write $u > 0$ if all $u_i > 0$, and $u \geq 0$ if all $u_i \geq 0$.

Theorem 2.1.6 ([15])**.** *Let A be a $n \times n$ generalised Cartan matrix. Then one and only one of the following three possibilities hold for both A and ${}^tA$:*

1. *(Finite type) There exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$;*

2. (Affine type) There exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$;

3. (Indefinite type) there exists $u > 0$ such that $Av < 0$; $Av \geq 0$, $v \geq 0$ imply $v = 0$.

2.2 Affine Kac-Moody algebras

The Kac-Moody algebra is called as affine Kac-Moody algebra if for the corresponding generalised Cartan matrix there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$;.
The affine Kac-Moody algebras are categorised in two parts:
1) Untwisted affine Kac-Moody algebra
2) Twisted affine Kac-Moody algebra.

2.3 Untwisted affine Kac-Moody algebra

The untwisted affine Kac-Moody algebra can be realised as a central extension of a loop algebra over a finite dimensional simple Lie algebra. Let $\mathfrak{m}$ be a finite dimensional simple Lie algebra. The corresponding affine Lie algebra is denoted by $\hat{\mathcal{L}}(\mathfrak{m}) = \mathbb{C}[z, z^{-1}] \oplus \mathfrak{m} \oplus \mathbb{C}\, K \oplus \mathbb{C}\, d$, where K is the center and d is a derivation [15].
The untwisted affine Lie algebra is of two types,
1) Simply laced
2) Non-simply laced .

Definition 2.3.1. *An untwisted affine Lie algebra is called simply laced if all the simple roots of the Lie algebra have same root length.*

Definition 2.3.2. *An untwisted affine Lie algebra is called non-simply laced if the simple roots of the Lie algebra have more than one root length.*

2.4 The twisted affine Kac-Moody algebras

Twisted affine Kac-Moody algebras are realised as fixed point subalgebras of untwisted affine Lie algebras under finite groups of automorphisms. Let $\mathfrak{m}$ be the finite dimensional simple Lie algebra with Dynkin diagram X_N where $X = A, D, E$ [15], and let σ be the corresponding Dynkin diagram automorphism of order $r \in \{2, 3\}$. Let $\hat{\mathcal{L}}(\mathfrak{m}) := \mathbb{C}[z, z^{-1}] \otimes \mathfrak{m} \oplus \mathbb{C}K \oplus \mathbb{C}d$, where K is the central element and d is the degree derivation. We extend σ to an automorphism $\tilde{\sigma}$ of $\hat{\mathcal{L}}(\mathfrak{m})$ by $K \mapsto K$, $d \mapsto d$ and $z^j \otimes x \mapsto \exp(\frac{-2\pi i j}{r}) z^j \otimes \sigma(x)$ for $j \in \mathbb{Z}, x \in \mathfrak{m}$. The fixed point set of $\tilde{\sigma}$ is the affine Lie algebra $\mathfrak{g} = \mathfrak{g}(X_N^{(r)})$.

The darkened node in the Dynkin diagrams of affine Kac-Moody algebras (figure 2.1 and figure 2.2) denotes the zeroth node. We note that, if we remove that node from the diagram,

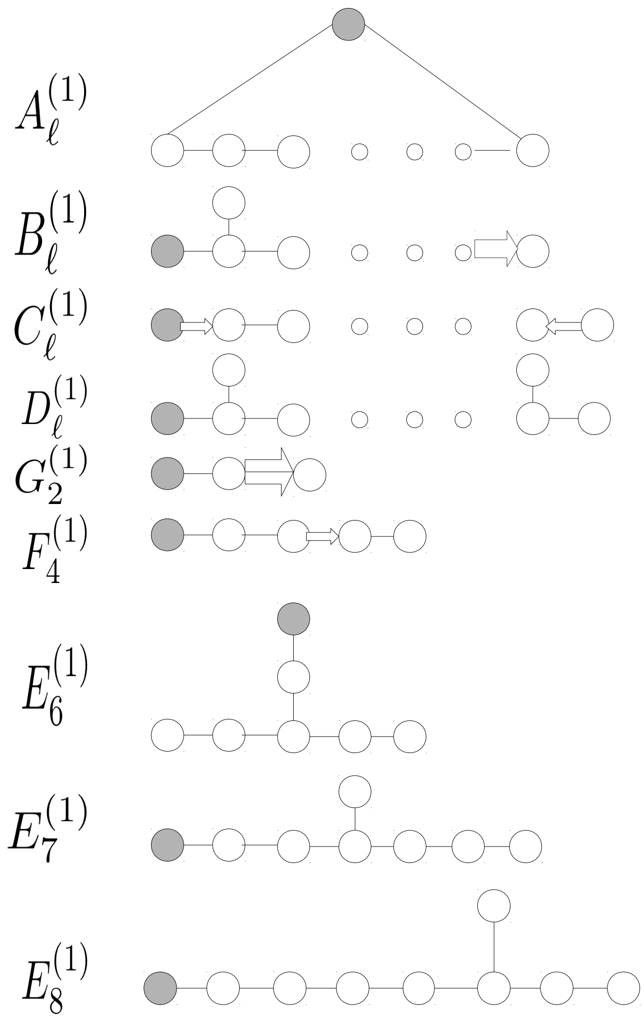

Figure 2.1: Dynkin diagrams of untwisted affine Lie algebras.

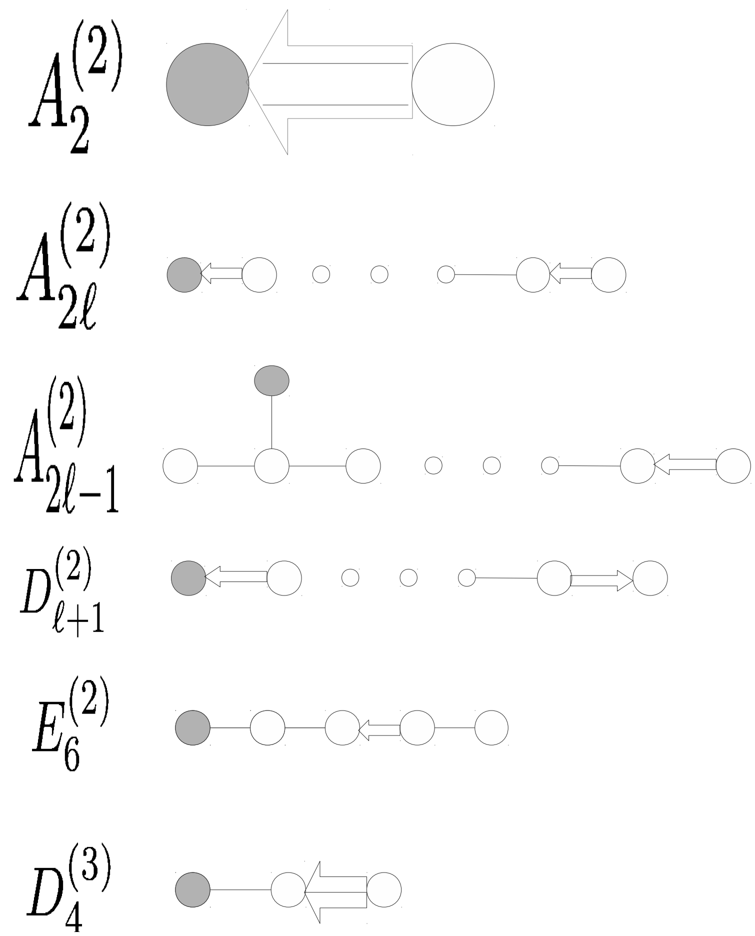

Figure 2.2: Dynkin diagrams of twisted affine Lie algebras.

the resulting diagrams are the Dynkin diagram of a finite dimensional simple algebras. We denote this finite dimensional simple Lie algebra by $\mathring{\mathfrak{g}}(A)$.

2.5 Weyl group of affine Kac-Moody algebra

Let $\mathfrak{g}(A)$ be an affine Kac-Moody algebra. Let W be the Weyl group of $\mathfrak{g}(A)$. Let $\mathring{W}$ denote the Weyl group of the finite dimensional algebra of $\mathring{\mathfrak{g}}(A)$. Let $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ denote a real Cartan subalgebra of $\mathring{\mathfrak{g}}(A)$. Let θ denote the highest root of the finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}(A)$ and $\theta^\vee$ be its co-root. Let us define an important lattice $M \subseteq \mathring{\mathfrak{h}}^*_{\mathbb{R}}$ defined by $M := \nu(\mathbb{Z}(\mathring{W}.\theta^\vee))$, where ν is map such that $\nu : \mathfrak{h} \to \mathfrak{h}^*$, $\nu(h) = (h, \cdot)$. We define a faithful action of M on $\mathfrak{h}^*$ by an endomorphism t_α for each $\alpha \in M$ as follows:

$$t_\alpha(\lambda) := \lambda + \langle \lambda, K \rangle \alpha - ((\lambda, \alpha) + \frac{1}{2}|\alpha|^2 \langle \lambda, K \rangle)\delta$$

where δ is the null root of $\mathfrak{g}(A)$. The group generated by t_α, $\alpha \in M$ is an abelian group denoted by T. Now, we are in position to state the proposition which gives the relation between W and $\mathring{W}$.

Proposition 2.5.1 ([15]). $W = \mathring{W} \ltimes T$

2.6 Representation theory of Kac-Moody algebra

A $\mathfrak{g}(A)$-module V is called $\mathfrak{h}$-diagonalisable if it admits a weight space decomposition $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V : h(v) = \lambda(h)v \ \ \forall\ h \in \mathfrak{h}\}$. A nonzero vector of V_λ is called a weight vector of weight λ. Let $P(V) := \{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$ denote the set of all weights of V. For $\lambda \in \mathfrak{h}^*$, denote $D(\lambda) := \{\mu \in \mathfrak{h}^* : \mu \leq \lambda\}$.

One studies mainly the representations of a Kac-Moody algebra from the category called as category $\mathcal{O}$. We define its objects as follows.

Definition 2.6.1. A $\mathfrak{g}(A)$*-module* V *is said to be in category* $\mathcal{O}$ *if*

1. *It is* $\mathfrak{h}$*-diagonalisable with finite dimensional weight spaces.*
2. *There exists a finite number of elements* $\lambda_1, \lambda_2, ..., \lambda_m \in \mathfrak{h}^*$ *such that* $P(V) \subseteq \cup_{i=1}^m D(\lambda_i)$

The morphisms in $\mathcal{O}$ are homomorphisms of $\mathfrak{g}(A)$-modules.

2.6.1 Highest weight modules

Important examples of modules from the category $\mathcal{O}$ are highest weight modules.

Definition 2.6.2. A $\mathfrak{g}(A)$*-module* V *is called as a highest weight module with highest weight* $\Lambda \in \mathfrak{h}^*$ *if there exists a nonzero vector* v_Λ *such that*

1. $\mathfrak{n}_+(v_\Lambda) = 0$; $h(v_\Lambda) = \Lambda(h)v_\Lambda\ \forall\, h \in \mathfrak{h}$; *and*

2. $U(\mathfrak{g}(A))(v_\Lambda) = V$*, where* $U(\mathfrak{g}(A))$ *is the universal enveloping algebra of* $\mathfrak{g}(A)$*.*

Remark 2.6.3. We note that by the condition 1) in above definition, condition 2) can be replaced by $U(\mathfrak{n}_-))(v_\Lambda) = V$. So we have $V = \oplus_{\lambda \leq \Lambda} V_\lambda$; $V_\Lambda = \mathbb{C}v_\Lambda$; $\dim(V_\lambda) < \infty$. Therefore, a highest weight module is an object of category $\mathcal{O}$.

Now, we define an important class of highest weight modules called as Verma modules.

Definition 2.6.4. A $\mathfrak{g}(A)$*-module* $M(\Lambda)$ *with highest weight* Λ *is called a Verma module if every* $\mathfrak{g}(A)$*-module with highest weight* Λ *is a quotient of* $M(\Lambda)$*.*

The following proposition justifies the importance of Verma modules.

Proposition 2.6.5 ([15])**.**

1. *For every* $\Lambda \in \mathfrak{h}^*$ *there exists a unique up to isomorphism Verma module* $M(\Lambda)$*.*

2. *Viewed as a* $U(\mathfrak{n}_-)$*-module,* $M(\Lambda)$ *is a free module of rank 1 generated by the highest weight vector.*

3. $M(\Lambda)$ *contains a unique proper maximal submodule* $M'(\Lambda)$*.*

It follows from 3) of above proposition that for $\Lambda \in \mathfrak{h}^*$, there is a unique irreducible module of highest weight Λ which we denote by $L(\Lambda) := M(\Lambda)/M'(\Lambda)$. For $\Lambda \in \mathfrak{h}^*$, the $\mathfrak{g}(A)$-modules $L(\Lambda)$ exhaust all irreducible modules of the category $\mathcal{O}$.

2.6.2 Integrable modules

Definition 2.6.6. A $\mathfrak{g}(A)$*-module* V *is called as integrable if the following holds:*

- *It is* $\mathfrak{h}$*-diagonalisable with finite dimensional weight spaces.*

- *The Chevalley generators* e_i *and* f_i $i = 1, ..., n$ *are locally nilpotent on* V*.*

We will restrict our attention to the category of integrable modules in category $\mathcal{O}$ denoted as $\mathcal{O}^{\text{int}}(\mathfrak{g})$, where $\mathfrak{g}$ is a Kac-Moody algebra. Let $L(\Lambda)$ be the irreducible highest weight module with highest weight Λ. Then one asks an obvious question, when is the $\mathfrak{g}$-module $L(\Lambda)$ integrable? The next proposition gives an answer to this question.

Proposition 2.6.7 ([15])**.** *The* $\mathfrak{g}(A)$*-module* $L(\Lambda)$ *is integrable if and only if* $\Lambda \in P_+$*.*

2.7 Character of a representation

Consider the algebra $\mathcal{E}$, whose elements are the series of the form $\sum_{\lambda \in \mathfrak{h}^*} c_\lambda e(\lambda)$, where $c_\lambda \in \mathbb{C}$ and $c_\lambda = 0$ for λ outside the union of finite number of sets $D(\mu)$ and $e(\lambda)$ are the elements of group algebra $\mathbb{C}[\mathfrak{h}^*]$. Now, we define character of a module V from the category $\mathcal{O}$ as follows.

Definition 2.7.1. *Let V be a module from the category $\mathcal{O}$ and let $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ be its weight space decomposition. We define formal character of V by*

$$\operatorname{ch} V := \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e(\lambda)$$

By the definition it is clear that $\operatorname{ch} V \in \mathcal{E}$.

Let $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = 1$ for $i = 1, \cdots, n$. Now we are in a position to state the fundamental result of the representation theory of Kac-Moody algebras.

2.7.1 Weyl-Kac character formula

Theorem 2.7.2. *Let $\mathfrak{g}(A)$ be a symmetrisable Kac-Moody algebra, and let $L(\Lambda)$ be the irreducible $\mathfrak{g}(A)$-module with highest weight $\Lambda \in P_+$. Then*

$$\operatorname{ch} L(\Lambda) = \frac{\sum_{w \in W} \epsilon(w) e(w(\Lambda + \rho) - \rho)}{\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\operatorname{mult} \alpha}}$$

Next, let us define the Kostant partition function, which will be used quite often in the upcoming part of this thesis.

2.7.2 Kostant partition function

Definition 2.7.3. *Let $\mathfrak{g}$ be a Kac-Moody algebra. Let $\mathfrak{h}, \mathfrak{h}^*, Q$ be its Cartan subalgebra, dual of Cartan subalgebra and root lattice respectively. The Kostant partition function $\mathcal{P}$ defined on $\mathfrak{h}^*$ by*

$$\mathcal{P}(\beta) := [e(-\beta)] \prod_{\alpha \in \Delta_+} \frac{1}{(1 - e(-\alpha))^{\operatorname{mult} \alpha}}$$

where $[e(\alpha)]f$ denotes the coefficient of $e(\alpha)$ in the expression for f.

Note that, $\mathcal{P}(\beta) = 0$ unless $\beta \in Q_+$. For $\beta \in Q_+$, $\mathcal{P}(\beta)$ is nothing but the number of partitions of β into a sum of positive roots, where each root is counted with its multiplicity. The next proposition shows the importance of the Kostant partition function as it is used in calculating the weight multiplicities of the $\mathfrak{g}$-module $L(\Lambda)$.

Proposition. *Consider an irreducible $\mathfrak{g}$-module $L(\Lambda)$. Let $L(\Lambda) = \bigoplus_{\mu \leq \Lambda} L(\Lambda)_\mu$ be its weight space decomposition. Then*

$$\dim(L(\Lambda)_\mu) = \sum_{w \in W} \epsilon(w) \mathcal{P}(w(\Lambda + \rho) - (\mu + \rho))$$

2.8 String function for $L(\Lambda)$

Let $\mathfrak{g}(A)$ be an affine Lie algebra. Recall from [15] that $\delta := \sum_{i=0}^{n} a_i\alpha_i$ called as the null root of $\mathfrak{g}(A)$, where the a_i are the labels of Dynkin diagram of $\mathfrak{g}(A)$. To study the weight system $P(\Lambda)$ of an integrable module $L(\Lambda)$ of $\mathfrak{g}(A)$ one classifies the weights which are sort of "maximal".

Definition 2.8.1. A *weight* $\lambda \in P(\Lambda)$ *is called maximal if* $\lambda + \delta \notin P(\Lambda)$*.*

Let $\max(\Lambda)$ denote the set of all maximal weights of $L(\Lambda)$. Next proposition shows the importance of the maximal weights.

Proposition. *Let* $L(\Lambda)$ *be an integrable module of positive level* k *over an affine Lie algebra. Then*

$$P(\Lambda) = \bigsqcup_{\lambda\in\max(\Lambda)} \{\lambda - n\delta : n \in \mathbb{Z}_+\}$$

Proof. See [15]. □

To understand the structure of the module $L(\Lambda)$, one studies the generating function

$$a_\lambda^\Lambda(q) = \sum_{n=0}^{\infty} \dim(L(\Lambda)_{\lambda-n\delta})q^n$$

where $\lambda \in \max(\Lambda)$. We will call this a string function (deviating mildly from the terminology of [15]).

Chapter 3

The t-analogue of the basic string function for twisted affine Lie algebras

The Kostant partition function can be used to determine the weight multiplicities associated to irreducible representations of Kac-Moody algebras. Its t-analogue was used by Lusztig to define a t-analogue of weight multiplicity. In this chapter we study Lusztig's t-weight multiplicities associated to the level one representation of twisted affine Kac-Moody algebras. We derive a closed form expression for the corresponding t-string function using constant term identities of Macdonald and Cherednik. We describe how generalised exponents of certain representations of the underlying finite dimensional simple Lie algebra enter the picture.

The results of this chapter have appeared in [35]. Throughout this chapter, we will assume that $\mathfrak{g}(A)$ is an affine Kac-Moody algebra. The other notations are as in Chapter 2.

3.1 Basic representation of $\mathfrak{g}(A)$

Let $\lambda \in P_+$ such that $\langle \lambda, K \rangle = 1$. Then the $\mathfrak{g}(A)$-module $L(\lambda)$ is called as a level one representation of $\mathfrak{g}(A)$. Let us consider an element $\Lambda_0 \in \mathfrak{h}^*$ defined by $\Lambda_0(\alpha_i^\vee) = \delta_{io}$ for $i = 0, 1, 2, ..., l$ and $\langle \Lambda_0, d \rangle = 1$. Note that for an affine Kac-Moody algebra $\mathfrak{g}$, Λ_0 is nothing but a fundamental weight corresponding to the zeroth node of its Dynkin diagram. The level one representation $L(\Lambda_0)$ is called as the basic representation of $\mathfrak{g}(A)$. Among the irreducible modules in $\mathcal{O}^{\text{int}}(\mathfrak{g}(A))$, the basic representation $L(\Lambda_0)$ can be singled out for the unique and important role it plays in the theory. It is the simplest non-trivial representation in $\mathcal{O}^{\text{int}}(\mathfrak{g}(A))$, and has many explicit realisations in terms of vertex operators [8, 16, 17]. If $\mathfrak{g}$ is an untwisted simply-laced affine Lie algebra, or $\mathfrak{g}$ is twisted, i.e., $\mathfrak{g} = X_N^{(r)}$ for $X_N = A_N, D_N, E_N$ and $r = 1, 2$ or 3, then all the level one simple modules of $\mathfrak{g}$ in $\mathcal{O}^{\text{int}}(\mathfrak{g}(A))$ can be obtained from $L(\Lambda_0)$ by the action of Dynkin diagram automorphisms of $\mathfrak{g}$, and tensoring with one dimensional $\mathfrak{g}$-modules.

One of the most important results for the basic representation was proved by Kac and Peterson [18], where they gave a closed form expression for the string function $a_{\Lambda_0}^{\Lambda_0}(q)$ of the basic representation which we state in the following proposition.

Proposition 3.1.1 ([18]). *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$, where X= A,D or E and $r = 1, 2, 3$, then*

$$a_{\Lambda_0}^{\Lambda_0}(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-\text{mult}\, n\delta} \tag{3.1.1}$$

Now we are heading towards the definition of the t-string function for which we need to define Lusztig's t-weight multiplicity (or affine Kostka Foulkes polynomial), which we do in the next section.

3.2 Lusztig's t-analogue of weight multiplicity

Recall from chapter 1 that the Kostant partition function is defined by

$$\mathcal{P}(\beta) := [e(-\beta)] \prod_{\alpha\in\Delta_+} \frac{1}{(1 - e(-\alpha))^{\text{mult }\alpha}}$$

Recall again from chapter 1 that, using the Kostant partition function, the weight multiplicity of the $\mathfrak{g}(A)$-module $L(\lambda)$ is given by

$$\dim(L(\lambda)_\mu) = \sum_{w\in W} \epsilon(w)\mathcal{P}(w(\lambda + \rho) - (\mu + \rho))$$

We define a t-analogue of the Kostant partition function by

$$\mathcal{P}(\beta; t) := [e(-\beta)] \prod_{\alpha\in\Delta_+} \frac{1}{(1 - te(-\alpha))^{\text{mult }\alpha}}$$

Note that $\mathcal{P}(\beta; 1) = \mathcal{P}(\beta)$.

Lusztig's t-analogue of weight multiplicity is defined as follows

$$K_\mu^\lambda(t) := \sum_{w\in W} \epsilon(w)\mathcal{P}(w(\lambda + \rho) - (\mu + \rho); t)$$

Now, using t-analogue of weight multiplicity we define a t-analogue of string function.

3.3 t-string function

As we have seen in the chapter 2 the string function for the $\mathfrak{g}(A)$- module $L(\lambda)$ is given by

$$a_\mu^\lambda(q) = \sum_{n=0}^{\infty} \dim(L(\lambda)_{\mu-n\delta})q^n$$

A t-analogue of a string function is obtained by replacing weight multiplicity by t-weight multiplicity as follows:

$$a_{\mu}^{\lambda}(t;q) := \sum_{n=0}^{\infty} K_{\mu-n\delta}^{\lambda}(t)q^n$$

We call these as the t-string functions. It was shown in [10] and [39] that the $a_{\mu}^{\lambda}(t;q)$ are closely related to the constant term identities arising in the theory of Macdonald polynomials. These papers gives the explicit formula for $a_{\Lambda_0}^{\Lambda_0}(t;q)$ of the basic representation for the untwisted simply laced affine algebras using Cherednik's Macdonald Mehta constant term identity. We state the explicit formula in the following theorem.

Theorem 3.3.1 ([10],[39]). *Let $\mathfrak{g}$ be one of the simply laced untwisted affine Lie algebras $A_{\ell}^{(1)}, D_{\ell}^{(1)}, E_{\ell}^{(1)}$. Then*

$$a_{\Lambda_0}^{\Lambda_0}(t;q) = \prod_{i=1}^{\ell}\prod_{n=1}^{\infty}(1-t^{e_i+1}q^n)^{-1} \tag{3.3.2}$$

where e_i $(1 \leq i \leq \ell)$ are the exponents of the underlying finite dimensional simple Lie algebra $(= A_{\ell}, D_{\ell}, E_{\ell}$ respectively).

3.4 Generalised exponents

Let $\mathfrak{m}$ be the finite dimensional simple Lie algebra and $V = V(\lambda)$ be the irreducible finite dimensional $\mathfrak{m}$ -module with highest weight λ. Fix a triangular decomposition $N_+ \oplus H \oplus N_-$, and let $E \in N_+$ be a principal nilpotent element i.e., $E = \sum_{i=1}^{\ell} c_i E_i$, where $c_i \in \mathbb{C} - \{0\}$ and $E_1, E_2, ..., E_{\ell}$ are the Chevalley generators. Let V_0 denote the zero weight space. Define the Brylinski-Kostant filtration of V_0 $\mathcal{F}^{(-1)}(V_0) \subseteq \mathcal{F}^{(0)}(V_0) \subseteq \mathcal{F}^{(1)}(V_0) \subseteq \cdots$, where $\mathcal{F}^{(p)}(V_0) := \ker(E^{p+1}) \cap V_0$. Then the generalised exponents of V are the elements of the multiset $\mathbb{E}$(V) defined via the following relation:

$$\sum_{p\geq 0} \dim(\mathcal{F}^{(p+1)}(V_0)/\mathcal{F}^{(p)}(V_0))\, t^p = \sum_{k\in\mathbb{E}(V)} t^k.$$

The exponents of a finite dimensional simple Lie algebra $\mathfrak{m}$ are nothing but the generalised exponents of the adjoint representation of $\mathfrak{m}$ i.e., V $= V(\theta)$, where θ is the highest root of $\mathfrak{m}$.

As our result is about the twisted affine Lie algebras, we recall some facts about them. Let $\mathfrak{g}$ be a twisted affine algebra of type $X_N^{(r)}$; here X_N is a simply laced $(A-D-E)$ Dynkin diagram of finite type with a diagram automorphism σ of order r ($r = 2$ or 3). Let $\mathfrak{m}$ denote the finite dimensional simple Lie algebra with Dynkin diagram X_N and let σ also denote the corresponding automorphism of $\mathfrak{m}$. For each $k \in \mathbb{Z}$, let $\mathfrak{m}_k \subset \mathfrak{m}$ be the eigenspace of σ for the eigenvalue $\exp(\frac{2\pi k i}{r})$ (so $\mathfrak{m}_k = \mathfrak{m}_{k+r}$). Since σ acts diagonalisably on $\mathfrak{m}$, we have a $\mathbb{Z}/r\mathbb{Z}$

gradation:

$$\mathfrak{m} = \bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}} \mathfrak{m}_j$$

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{m}$, let $\mathfrak{h}_j := \mathfrak{h} \cap \mathfrak{m}_j$ for all $j \in \mathbb{Z}$. We collect together the important facts about the above decomposition [15].

Proposition 3.4.1. *With notation as above, we have*

1. $\mathfrak{m}_0$ *is a simple Lie algebra and* $\mathfrak{m}_j$ *is an irreducible* $\mathfrak{m}_0$*-module* $\forall\ j$.
2. $\mathfrak{m}_1 \cong \mathfrak{m}_{-1}$ *as* $\mathfrak{m}_0$*-modules.*
3. $\mathfrak{h}_0$ *is Cartan subalgebra of* $\mathfrak{m}_0$ *and its centraliser in* $\mathfrak{m}$ *is* $\mathfrak{h}$.
4. *If* $\mathfrak{g}$ *is not of type* $A_{2l}^{(2)}$*, then* $\mathfrak{m}_0$ *and* $\mathring{\mathfrak{g}}$ *are isomorphic. Further the highest weight of the* $\mathfrak{m}_0$*-module* $\mathfrak{m}_1$ *is the dominant short root* θ_s *of* $\mathfrak{m}_0$.
5. *If* $\mathfrak{g}$ *is of type* $A_{2l}^{(2)}$*, then* $\mathfrak{m}_0$ *is of type* B_l*, while* $\mathring{\mathfrak{g}}$ *is of type* C_l*. Further, the highest weight of* $\mathfrak{m}_1$ *is* $2\theta_s$*, where* θ_s *is the dominant short root of* $\mathfrak{m}_0$.

Proof. See [[2],[15]]. □

We denote $l := \operatorname{rank} \mathfrak{m}_0$, $m :=$ the number of short simple roots of $\mathfrak{m}_0$ and let θ_l (resp.θ_s) be the dominant long (resp. short) root of $\mathfrak{m}_0$.

Proposition 3.4.2. *Let* $\mathfrak{g}$ *be a twisted affine algebra of type* $X_N^{(r)} \neq A_{2l}^{(2)}$*. Consider the action of the cyclic group generated by the automorphism* σ*, on the nodes of the Dynkin diagram of* X_N*. This has the following properties.*

1. *Each orbit has cardinality 1 or* r.
2. *The number of orbits equals* l.
3. *The number of orbits of cardinality* r *is equal to* m.
4. *Thus,* $m = \frac{N-l}{r-1}$.

Proof. See [[2],[15]]. □

We have a natural $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ with $\mathfrak{g}_0 = \mathfrak{m}_0 + \mathbb{C}K + \mathbb{C}d$ and $\mathfrak{g}_j = z^j \otimes \mathfrak{m}_j$ for $j \neq 0$. We observe that for $j \neq 0$, $\mathfrak{g}_j \cong \mathfrak{m}_j$ is an irreducible $\mathfrak{m}_0$-module and that $\mathfrak{g}_j \cong \mathfrak{g}_k$ when $j \equiv k \pmod r$, $j, k \neq 0$. Let $\mathbb{E}_n$ denote the multiset of *generalised exponents* of the $\mathfrak{m}_0$-module $\mathfrak{m}_n$ for $n \in \mathbb{Z}$.

Now we are in position to state our main theorem of this chapter.

$\mathfrak{g}$	$\mathbb{E}_0$	$\mathbb{E}_1 = \mathbb{E}_{-1}$
$A_{2l}^{(2)}$ $(l \geq 1)$	$1, 3, 5, \cdots, 2l-1$	$2, 4, 6, \cdots, 2l$
$A_{2l-1}^{(2)}$ $(l \geq 3)$	$1, 3, 5, \cdots, 2l-1$	$2, 4, 6, \cdots, 2l-2$
$D_{l+1}^{(2)}$ $(l \geq 2)$	$1, 3, 5, \cdots, 2l-1$	l
$E_6^{(2)}$	$1, 5, 7, 11$	$4, 8$
$D_4^{(3)}$	$1, 5$	3

Table 3.1: $\mathbb{E}_n$ for the twisted affines $\mathfrak{g} = X_N^{(r)}$ ($\mathbb{E}_{n+r} = \mathbb{E}_n$ for all n).

Theorem 3.4.3. *Let $\mathfrak{g}$ be a twisted affine algebra. The t-string function of the basic representation of $\mathfrak{g}$ is given by*

$$a_{\Lambda_0}^{\Lambda_0}(t;q) = \prod_{n=1}^{\infty} \prod_{e \in \mathbb{E}_n} (1 - t^{e+1} q^n)^{-1}$$

Remark 3.4.4. When $\mathfrak{g}$ is an untwisted simply-laced affine, this result was proved in [39]. In this case, the $\mathfrak{m}_0$-modules $\mathfrak{m}_j$ are all isomorphic to the adjoint representation of $\mathfrak{m}_0$. Thus $\mathbb{E}_n = \mathbb{E}(\mathfrak{m}_0)$, the set of exponents of $\mathfrak{m}_0$ for all n, and we recover theorem 3.3.1.

Remark 3.4.5. The cardinality of $\mathbb{E}_n$ is the dimension of the zero weight space of $\mathfrak{g}_n$. From proposition 3.4.1, it follows that $|\mathbb{E}_n| = \dim(z^n \otimes \mathfrak{h}_n)$. Since $z^n \otimes \mathfrak{h}_n$ is the root space of $\mathfrak{g}$ corresponding to the imaginary root $n\delta$, we deduce that $|\mathbb{E}_n| = \text{mult}(n\delta)$. Thus, this expression is a t-deformation of the expression for the basic string function 3.1.1.

Remark 3.4.6. From the explicit description of the Chevalley generators of $\mathfrak{m}_0$ in terms of those of $\mathfrak{m}$ [15], it is clear that a principal nilpotent element of $\mathfrak{m}_0$ is also a principal nilpotent of $\mathfrak{m}$. This observation, together with proposition 3.4.1 implies the following equality of multisets:

$$\mathbb{E}(\mathfrak{m}) = \bigsqcup_{j=1}^{r} \mathbb{E}_j$$

where the left hand side is the multiset of exponents of the Lie algebra $\mathfrak{m}$, i.e., the generalised exponents of its adjoint representation. Further, since $\mathfrak{m}_r = \mathfrak{m}_0$, we have $\mathbb{E}_r = \mathbb{E}(\mathfrak{m}_0)$. Thus, the sets $\mathbb{E}(\mathfrak{m})$ and $\mathbb{E}(\mathfrak{m}_0)$ determine the $\mathbb{E}_n$ for all n; this is clear for $r = 2$, while for $r = 3$ it follows from the further fact that $\mathbb{E}_1 = \mathbb{E}_2$. Table 3.1 lists the $\mathbb{E}_n$ for all twisted affine algebras.

3.4.1 Corollary of theorem 3.4.3

We derive an interesting corollary of theorem 3.4.3 . If $\mathfrak{g}$ is an affine Kac-Moody algebra of rank $l+1$, and e_i, f_i $(i = 0, \cdots, l)$ are the Chevalley generators, the principal Heisenberg subalgebra

$\mathfrak{s}$ of $\mathfrak{g}$ is defined to be

$$\mathfrak{s} := \{x \in \mathfrak{g} : [x, \sum_{i=0}^{l} e_i] \in \mathbb{C}K\}$$

where K is the central element of $\mathfrak{g}$. The principal gradation of $\mathfrak{g}$ induces a gradation $\mathfrak{s} = \oplus_{j \in \mathbb{C}} \mathfrak{s}_j$. If $\mathfrak{g}$ is an untwisted simply-laced or twisted affine algebra, the basic representation $L(\Lambda_0)$, as an $\mathfrak{s}$-module, is irreducible. The *exponents* of the affine algebra $\mathfrak{g}$ are the elements of the (infinite) multiset $\mathbb{E}(\mathfrak{g})$ of nonzero integers in which each j occurs $\dim \mathfrak{s}_j$ times. Let $\mathbb{E}^+(\mathfrak{g}) := \mathbb{E}(\mathfrak{g}) \cap \mathbb{E}_{>0}$ denote the positive exponents of $\mathfrak{g}$. The following lemma relates the multisets $\mathbb{E}^+(\mathfrak{g})$ and $\mathbb{E}_n$.

Lemma 3.4.7. *Let $\mathfrak{g}$ be a twisted affine algebra or an untwisted simply-laced affine algebra of type $X_N^{(r)}$, with Coxeter number h. Then*

$$\mathbb{E}^+(\mathfrak{g}) = \{e + hn : n \geq 0, e \in \mathbb{E}_n\}$$

Proof. Follows easily from [[15], chapter. 14] and table 3.1. □

We deduce the following nice formula for the specialisation of the t-string function $a_{\Lambda_0}^{\Lambda_0}(t; q)$ at $t \mapsto q, q \mapsto q^h$.

Corollary 3.4.8. *Let $\mathfrak{g}$ be a twisted affine algebra or an untwisted simply-laced affine algebra, with Coxeter number h. Let $\mathring{\mathfrak{g}}$ be its underlying finite dimensional simple Lie algebra. Then*

$$a_{\Lambda_0}^{\Lambda_0}(q; q^h) = \frac{\prod_{\bar{e} \in \mathbb{E}(\mathring{\mathfrak{g}})} (1 - q^{\bar{e}+1})}{\prod_{e \in \mathbb{E}^+(\mathfrak{g})} (1 - q^{e+1})}$$

where $\mathbb{E}(\mathring{\mathfrak{g}})$ is the (finite) multiset of exponents of $\mathring{\mathfrak{g}}$.

Proof. Applying the specialisation $t \mapsto q, q \mapsto q^h$ to theorem 3.4.3, and using lemma 3.4.7, we obtain the desired equation but with $\mathfrak{m}_0$ in place of $\mathring{\mathfrak{g}}$. Proposition 3.4.1 implies that $\mathfrak{m}_0$ and $\mathring{\mathfrak{g}}$ are either isomorphic or dual. Since dual algebras have the same exponents, the result follows in all cases. □

We will prove theorem 3.4.3 in two parts. In the first part we will prove the result for $\mathfrak{g} \neq A_{2l}^{(2)}$. In the second we complete the result by proving it for $\mathfrak{g} = A_{2l}^{(2)}$. First, we need to define some terms which are the crucial ingredients of the proof.

Given a Kac- Moody algebra $\mathfrak{g}$ of finite or affine type, we let $\Delta(\mathfrak{g}),\ \Delta_+(\mathfrak{g}),\ \Delta^{re}(\mathfrak{g}),\ \Delta^{im}(\mathfrak{g})$ denote the sets of roots, positive roots, real and imaginary roots respectively. Let $\Delta_+^{re}(\mathfrak{g}) := \Delta^{re}(\mathfrak{g}) \cap \Delta_+(\mathfrak{g})$ and $\Delta_+^{im}(\mathfrak{g}) := \Delta^{im}(\mathfrak{g}) \cap \Delta_+(\mathfrak{g})$.

Definition 3.4.9. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra. The Cherednik kernel $\hat{\boldsymbol{\mu}}$ of $\mathfrak{g}$ is the product*

$$\hat{\boldsymbol{\mu}} := \prod_{\alpha \in \Delta_+^{re}(\mathfrak{g})} \frac{1 - e^{-\alpha}}{1 - te^{-\alpha}} \tag{3.4.3}$$

If $\mathfrak{g}$ is affine, the corresponding product over the imaginary positive roots denoted by

$$\hat{\boldsymbol{\mu}}^{im} := \prod_{\alpha \in \Delta_+^{im}(g)} \left(\frac{1 - e^{-\alpha}}{1 - te^{-\alpha}} \right)^{(\text{mult } \alpha)} = \prod_{n \geq 1} \left(\frac{1 - q^n}{1 - tq^n} \right)^{(\text{mult } n\delta)} \tag{3.4.4}$$

where we let $q := e^{-\delta}$ throughout.

Recall from the chapter 2 that the formal character of the representation $L(\lambda)$, where λ is a dominant integral weight is given by $\operatorname{ch} L(\lambda) = \sum_\nu \dim(L(\lambda)_\nu)\, e^\nu$, we will denote this from now on by χ_λ. We have the following proposition for the basic representation $L(\Lambda_0)$.

Proposition 3.4.10. *Let $\mathfrak{g}$ be a twisted affine algebra. Then the formal character of $L(\Lambda_0)$ is :*

$$e^{-\Lambda_0} \chi_{\Lambda_0} = a_{\Lambda_0}^{\Lambda_0}(1; q) \Theta$$

where $\Theta := \sum_{\alpha \in M} e^\alpha q^{\frac{\langle \alpha, \alpha \rangle}{2}}$ is the theta function of the lattice M (defined in chapter 2, § 2.5) .

Proof. See [15] □

Definition 3.4.11. *Given a formal sum $\xi = \sum_{\alpha \in Q} c_\alpha e^\alpha$, the constant term of ξ denoted by $\operatorname{ct}(\xi)$ is defined as $\operatorname{ct}(\xi) := \sum_{n \in \mathbb{Z}} c_{n\delta} e^{n\delta}$.*

We will use the following simple fact from [39] to compute the t-string functions $a_\mu^\lambda(t; q)$:

$$a_\mu^\lambda(t; q) = \hat{\boldsymbol{\mu}}^{im} \operatorname{ct}(e^{-\mu} \chi_\lambda \hat{\boldsymbol{\mu}}) \tag{3.4.5}$$

where μ is a maximal dominant weight of $L(\lambda)$. Putting the above facts together, we obtain the following lemma.

Lemma 3.4.12. *Let $\mathfrak{g}$ be a twisted affine algebra. Then*

1. *The t-string function of the basic representation of $\mathfrak{g}$ is given by*

$$a_{\Lambda_0}^{\Lambda_0}(t; q) = a_{\Lambda_0}^{\Lambda_0}(1; q) \hat{\boldsymbol{\mu}}^{im} \operatorname{ct}(\hat{\boldsymbol{\mu}} \Theta)$$

2. *Further we have*

$$a_{\Lambda_0}^{\Lambda_0}(1; q) \hat{\boldsymbol{\mu}}^{im} = \prod_{n \geq 1} (1 - tq^n)^{-\text{mult}\,(n\delta)}$$

Proof. Part 1 of lemma follows from proposition 3.4.10 and equation 3.4.5. Part 2 is clear from proposition 3.1.1 and equation 3.4.4. □

From the above lemma we see that we only need to calculate $\operatorname{ct}(\hat{\boldsymbol{\mu}} \Theta)$ to get an expression for $a_{\Lambda_0}^{\Lambda_0}(t; q)$.

3.5 Proof of theorem 3.4.3 for $\mathfrak{g} \neq A_{2l}^{(2)}$

Throughout this section, we take $\mathfrak{g}$ to be a twisted affine algebra, $\mathfrak{g} \neq A_{2l}^{(2)}$. Let $\langle \cdot, \cdot \rangle$ denote the normalised invariant form of $\mathfrak{g}$ [15]. We then have $\langle \alpha, \alpha \rangle = 2$ for all short real roots of $\mathfrak{g}$. We recall that the *height* of a root α (written $\operatorname{ht}\alpha$) is the sum of the coefficients obtained when α is written as a linear combination of simple roots. The following result is a special case of Cherednik's *difference Macdonald-Mehta* constant term identity [6].

Proposition 3.5.1. *Let $\mathfrak{g}$ be a twisted affine algebra, $\mathfrak{g} \neq A_{2l}^{(2)}$. Let $\mathring{\mathfrak{g}}$ be the underlying finite dimensional simple Lie algebra and let $\langle \cdot, \cdot \rangle$ denote the normalised invariant form of $\mathfrak{g}$. Then we have*

$$ct(\hat{\boldsymbol{\mu}}\Theta) = \prod_{\Delta_+(\mathfrak{g})} \prod_{j=1}^{\infty} \frac{1 - t^{\operatorname{ht}\alpha} q^{\frac{\langle \alpha,\alpha \rangle}{2} j}}{1 - t^{\operatorname{ht}\alpha + 1} q^{\frac{\langle \alpha,\alpha \rangle}{2} j}} \tag{3.5.6}$$

Proof. The positive real roots of $\mathfrak{g}$ are given by $\Delta_+^{re}(\mathfrak{g}) = \{\beta + \frac{\langle \beta,\beta \rangle}{2} j\delta : j \geq 1, \beta \in \Delta(\mathring{\mathfrak{g}})\} \cup \Delta_+(\mathring{\mathfrak{g}})$. Thus the Cherednik kernel of $\mathfrak{g}$ becomes

$$\hat{\boldsymbol{\mu}} = \prod_{\beta \in \Delta_+(\mathring{\mathfrak{g}})} \prod_{j \geq 0} \frac{(1 - e^{-\beta} q^{j\langle \beta,\beta \rangle/2})(1 - e^{\beta} q^{(j+1)\langle \beta,\beta \rangle/2})}{(1 - te^{-\beta} q^{j\langle \beta,\beta \rangle/2})(1 - te^{\beta} q^{(j+1)\langle \beta,\beta \rangle/2})} \tag{3.5.7}$$

Applying [6, theorem 5.3] with R chosen to be the coroot system of $\mathring{\mathfrak{g}}$ yields equation 3.5.6. $\square$

To simplify notation, we let $(a_1, a_2, \cdots, a_p; x)_\infty := \prod_{i=1}^{p} \prod_{n=0}^{\infty} (1 - a_i x^n)$. Let us now separate the contributions of long and short roots in equation 3.5.6 . Define

$$K_s(q) \quad (\text{resp. } K_l(q)) := \prod_{\substack{\alpha \in \Delta_+(\mathring{\mathfrak{g}}) \\ \alpha \text{ short} \\ (\text{resp. long})}} \frac{(t^{\operatorname{ht}\alpha} q; q)_\infty}{(t^{\operatorname{ht}\alpha+1} q; q)_\infty}$$

Since $\langle \alpha, \alpha \rangle / 2$ is 1 (resp. r) if α is short (resp. long), proposition 3.5.1 implies

$$\operatorname{ct}(\hat{\boldsymbol{\mu}}\Theta) = K_s(q) K_l(q^r)$$

Now, for each $k \geq 1$, let n_k (resp. $n_k(s)$) denote the number of positive roots (resp. short positive roots) of $\mathring{\mathfrak{g}}$ of height k. This gives

$$K(q) := K_s(q) K_l(q) = \frac{{(tq; q)^l}_\infty}{\prod_{p \geq 1} (t^{p+1} q; q)_\infty^{n_p - n_{p+1}}}$$

where l is the number of simple roots of $\mathring{\mathfrak{g}}$. Similarly,

$$K_s(q) = \frac{(tq; q)^m}{\prod_{p \geq 1} (t^{p+1} q; q)^{n_p(s) - n_{p+1}(s)}}$$

where m is the number of short simple roots of $\mathring{\mathfrak{g}}$. We recall the following classical result (see, for example, [12]) relating the n_k and $n_k(s)$ to generalised exponents of certain representations of $\mathring{\mathfrak{g}}$.

Proposition ([12]). *With notation as above, $n_p - n_{p+1}$ is the number of times p occurs as an exponent of $\mathring{\mathfrak{g}}$ (i.e., as a generalised exponent of the adjoint representation $V(\theta_l)$). Similarly, $n_p(s) - n_{p+1}(s)$ is the number of times p occurs as a generalised exponent of the representation $V(\theta_s)$ of $\mathring{\mathfrak{g}}$.*

Now, rewriting $K_s(q)K_l(q^r) = K(q^r)\frac{K_s(q)}{K_s(q^r)}$ and using propositions 3.4.1 and 3.5, we get

$$\mathrm{ct}(\hat{\boldsymbol{\mu}}\Theta) = (tq^r;q^r)_\infty^l \frac{(tq;q)_\infty^m}{(tq^r;q^r)_\infty^m} \prod_{p\in\mathbb{E}_0} \frac{1}{(t^{p+1}q^r;q^r)_\infty} \prod_{p\in\mathbb{E}_1} \frac{(t^{p+1}q^r;q^r)_\infty}{(t^{p+1}q;q)_\infty}$$

We now observe that $l = \mathrm{mult} j\delta$ for $j \equiv 0 \pmod r$ and $m = \frac{N-l}{r-1} = \mathrm{mult} j\delta$ for $j \not\equiv 0 \pmod r$. Thus, the above equation can be rewritten as :

$$\mathrm{ct}(\hat{\boldsymbol{\mu}}\Theta) = \prod_{n\geq 1}(1-tq^n)^{\mathrm{mult}(n\delta)} \prod_{n\geq 1}\prod_{e\in\mathbb{E}_n} \frac{1}{1-t^{e+1}q^n}$$

Lemma 3.4.12 now completes the proof of theorem 3.4.3 for all twisted affine algebras $\mathfrak{g} \neq A_{2l}^{(2)}$.

3.6 $\mathfrak{g} = A_{2l}^{(2)}$

3.6.1 Proof of theorem 3.4.3 for $\mathfrak{g} = A_{2l}^{(2)}$

In this section, we consider the case $\mathfrak{g} = A_{2l}^{(2)}$. The underlying finite dimensional simple Lie algebra is $\mathring{\mathfrak{g}} = C_l$. Let $\mathring{\Delta}$ be the set of roots of $\mathring{\mathfrak{g}}$. Letting $\langle\cdot,\cdot\rangle$ denote the standard inner product in $\mathbb{R}^l$ and $\epsilon_i\,(1 \leq i \leq l)$ be the standard orthonormal basis, we can take $\mathring{\Delta} = \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq l\} \cup \{\pm 2\epsilon_i : 1 \leq i \leq l\}$. We observe that the coroot lattice M of $\mathring{\mathfrak{g}}$ is just $M = \oplus_{i=1}^l \mathbb{Z}\epsilon_i$. The set of real roots of $\mathfrak{g}$ is given by $\Delta^{re} = S_1 \cup S_2 \cup S_4$ where $S_1 = \{\frac{1}{2}(\alpha + (2n-1)\delta) : \alpha \in \mathring{\Delta}_l\}$, $S_2 = \{\alpha + n\delta : \alpha \in \mathring{\Delta}_s\}$ and $S_4 = \{\alpha + 2n\delta : \alpha \in \mathring{\Delta}_l\}$, where δ is the null root of $\mathring{\mathfrak{g}}$ and $\mathring{\Delta}_l$ (resp. $\mathring{\Delta}_s$) denotes the set of long (resp. short) roots in $\mathring{\Delta}$. The elements of S_n have norm n $(n = 1, 2, 4)$, and each S_n is invariant under the Weyl group W of $\mathring{\mathfrak{g}}$. Let $\hat{\boldsymbol{\mu}}$ denote the Cherednik kernel of $A_{2l}^{(2)}$, given by equation 3.4.3 .

Now enlarge Δ^{re} by defining:

$$\Phi := \bigcup_{i=1}^{5} \Phi_i \text{ where } \Phi_1 := (1/2)S_4,\ \Phi_2 := S_4,\ \Phi_3 := S_1,\ \Phi_4 := 2S_1, \Phi_5 := S_2$$

The set Φ is the non-reduced irreducible affine root system of type $(C_l^\vee, C_l)$ in the classification of Macdonald [30]. Observe that Φ is W-invariant, with each Φ_i being a W-orbit.

Following the notation of Macdonald [30], define $R_1^+ := \{\epsilon_1, \cdots, \epsilon_l\}$ and $R_2^+ := \{\epsilon_i \pm \epsilon_j : 1 \le i < j \le l\}$. We now let k_i $(1 \le i \le 5)$ be arbitrary parameters, and let $(u_1, u_2, u_3, u_4) = (q^{k_1}, -q^{k_2}, q^{k_3+\frac{1}{2}}, -q^{k_4+\frac{1}{2}})$ and $(u_1', u_2', u_3', u_4') = (qu_1, qu_2, u_3, u_4)$. The Cherednik kernel $\boldsymbol{\Delta}$ (with parameters k_i) for the non-reduced affine root system Φ then becomes $\boldsymbol{\Delta} := \boldsymbol{\Delta}^{(1)}\boldsymbol{\Delta}^{(2)}$ where

$$\boldsymbol{\Delta}^{(1)} = \prod_{\alpha \in R_1^+} \frac{(e^{-2\alpha}, qe^{2\alpha}; q)_\infty}{\prod_{i=1}^4 (u_i e^{-\alpha}; q)_\infty (u_i' e^{\alpha}; q)_\infty}$$

$$\boldsymbol{\Delta}^{(2)} = \prod_{\alpha \in R_2^+} \frac{(e^{-\alpha}, qe^{\alpha}; q)_\infty}{(q^{k_5} e^{-\alpha}, q^{k_5+1} e^{\alpha}; q)_\infty}$$

[30]. The following lemma relates the kernels $\boldsymbol{\Delta}$ and $\hat{\boldsymbol{\mu}}$.

Lemma 3.6.1. *Define* $t := q^{k_5}$*, and let the parameters* k_i *satisfy the relations* $k_3 = k_5 = 2k_1 = 2k_2$*. We then have:*

1. *If* $k_4 = 0$*, then* $\boldsymbol{\Delta} = \hat{\boldsymbol{\mu}}$*.*
2. *If* $k_4 \to \infty$ *(i.e.,* $q^{k_4} \to 0$*), then* $\boldsymbol{\Delta} \to \dfrac{\hat{\boldsymbol{\mu}}\,\Theta_M}{(q;q)^l}$ *where* $M := \oplus_{i=1}^l \mathbb{Z}\epsilon_i$ *and* $\Theta_M := \sum_{\alpha \in M} e^{\alpha} q^{\langle \alpha, \alpha \rangle/2}$ *is its theta function.*

Proof. The first statement is easy; in fact one can recover the Cherednik kernels of all twisted affines (and all non-reduced affine root systems) by appropriate specialisation of $\boldsymbol{\Delta}$ [30]. To prove (2), we observe that for the given choice of parameters, one gets

$$\boldsymbol{\Delta} = \hat{\boldsymbol{\mu}} \prod_{i=1}^{l} (-q^{1/2} e^{\epsilon_i}, -q^{1/2} e^{-\epsilon_i}; q)_\infty$$

Now, by the Jacobi triple product identity, we have

$$(-q^{1/2} e^{\epsilon_i}, -q^{1/2} e^{-\epsilon_i}; q)_\infty = (q;q)^{-1} \sum_{n \in \mathbb{Z}} q^{n^2/2} e^{n\epsilon_i} = (q;q)_\infty^{-1} \, \Theta_{\mathbb{Z}\epsilon_i}$$

Since the theta function Θ_M of the rectangular lattice $M = \oplus_{i=1}^l \mathbb{Z}\epsilon_i$ is just the product $\prod_{i=1}^l \Theta_{\mathbb{Z}\epsilon_i}$, the result follows. □

To obtain the explicit form of the Cherednik-Macdonald-Mehta constant term identity for $A_{2l}^{(2)}$ (i.e., an expression for $ct(\hat{\boldsymbol{\mu}}\,\Theta_M)$) it thus only remains to specialise the well-known formula for $\mathrm{ct}(\boldsymbol{\Delta})$ [[30], equation 5.8.20] at $k_3 = k_5 = 2k_1 = 2k_2$ and $k_4 \to \infty$ (and letting $q^{k_5} =: t$).

Proposition 3.6.2. *For* $\overset{\circ}{\mathfrak{g}} = A_{2l}^{(2)}$*, we have:*

$$\mathrm{ct}(\hat{\boldsymbol{\mu}}\,\Theta_M) = \frac{(tq;q)_\infty^l}{(t^2q^2,t^4q^2,\cdots,t^{2l}q^2;q^2)_\infty(t^3q,t^5q,\cdots,t^{2l+1}q;q^2)_\infty}$$

Proof. We start with some notations (see [30]): $t_at_{2a}^{\frac{1}{2}} = q^{k_a}$ for $a \in \Phi$ such that $2a \in \Phi$ and $t_{2a}^{\frac{1}{2}} = q^{k_{2a}}$, where $k_a = k_i$ for $a \in \Phi_i$ for $1 \le i \le 5$ and $t_a = 1$ if $a \notin \Phi$. Let $k'_1 = \frac{1}{2}(k_1+k_2+k_3+k_4)$, $k'_2 = \frac{1}{2}(k_1+k_2-k_3-k_4)$, $k'_3 = \frac{1}{2}(k_1-k_2+k_3-k_4)$, $k'_4 = \frac{1}{2}(k_1-k_2-k_3+k_4)$. $B := \{a \in \Phi : a = b + k\delta; b \in \overset{\circ}{\Delta}_-\}$.

Now $\mathrm{ct}(\boldsymbol{\Delta})$ is given by the formula:

$$\mathrm{ct}(\boldsymbol{\Delta}) = (\boldsymbol{\Delta}^-_{s',k'}\,\boldsymbol{\Delta}^-_{s',-k'})(-\rho_{k'}) \text{ by [30] [equation 5.8.20]} \tag{3.6.8}$$

$$\text{where } \boldsymbol{\Delta}^-_{s',k'} := \prod_{\substack{a\in\Phi^+\\ Da<0}} \boldsymbol{\Delta}_{a,k'} \text{ by [30] [equation 5.3.11]} \tag{3.6.9}$$

$$\text{and } \boldsymbol{\Delta}_{a,k'} := \frac{1-q^{k'_{2a}}e^a}{1-q^{k'_a}e^a} \text{ [30] [equation 5.1.2]} \tag{3.6.10}$$

$$\text{and } \rho'_k := \sum_{i=1}^{l}(k'_1+(l-i)k_5)\epsilon_i \text{ [30] [§1.5]} \tag{3.6.11}$$

The set $B = \{-\epsilon_i + n\delta : n \ge 1, 1 \le i \le l\} \cup \{-2\epsilon_i + 2n\delta : n \ge 1, 1 \le i \le l\} \cup \{-\epsilon_i + (n+\frac{1}{2})\delta : n \ge 0, 1 \le i \le l\} \cup \{-2\epsilon_i + (2n+1)\delta : n \ge 0, 1 \le i \le l\} \cup \{-(\epsilon_i+\epsilon_j) + n\delta : n \ge 1, 1 \le i < j \le l\} \cup \{-(\epsilon_i-\epsilon_j) + n\delta : n \ge 1, 1 \le i < j \le l\}$.

Now we calculate the right hand side of the equation 3.6.8:
Case 1): Let $a = -\epsilon_i + n\delta \in B \cap \Phi_1^+$
then

$$\boldsymbol{\Delta}_{a,k'}\,\boldsymbol{\Delta}_{2a,k'} = \frac{1-e^{2a}}{(1-q^{k'(a)}e^a)(1+q^{k'(2a)}e^a)} \text{ by [30] [equation 5.1.2]}$$

Here $k'(a) = k'_1$ since $a \in \Phi_1^+$ and $k'(2a) = k'_2$ therefore

$$\boldsymbol{\Delta}_{a,k'}\,\boldsymbol{\Delta}_{2a,k'}(-\rho_{k'}) = \frac{1-e^{2a}}{(1-q^{k'_1}e^a)(1+q^{k'_2}e^a)}\left(-\sum_{i=1}^{l}(k'_1+(l-i)k_5)\epsilon_i\right)$$

Now $e^{\langle a,-\rho_{k'}\rangle} = q^{k'_1+(l-i)k_5}q^n$
Since $k'_1 \to \infty$ as $k_4 \to \infty$, the term $\left(\frac{1-e^{2a}}{1-q^{k'_1}e^a}\right)(-\rho_{k'}) \to 1$ and

$$\left(\frac{1}{1-q^{k'_1}e^a}\right)(-\rho_{k'}) = \frac{1}{1+q^{k'_2}q^{k'_1+(l-i)k_5}q^n}$$

But as $k'_1 + k'_2 = k_1 + k_2$ we get

$$\boldsymbol{\Delta}_{a,k'}\,\boldsymbol{\Delta}_{2a,k'}(-\rho_{k'}) = \frac{1}{1+q^{k_1+k_2+(l-i)k_5}q^n}$$

Case 2): Let $a = -\epsilon_i + n\delta \in B \cap \Phi_1^+$
Then

$$\boldsymbol{\Delta}_{a,-k'}\,\boldsymbol{\Delta}_{2a,-k'} = \frac{1-e^{2a}}{(1-q^{-k'(a)}e^a)(1+q^{-k'(2a)}e^a)}$$

The numerator → to 1 as $k_1' \to \infty$.
The first term in denominator is

$$\begin{aligned} 1 - q^{-k_1'}e^{-\langle a,-\rho_{k'}\rangle} &= 1 - q^{-k_1'}q^{k_1'+(l-i)k_5}q^n \\ &= 1 - q^{(l-i)k_5}q^n. \end{aligned}$$

Second term in denominator is

$$1 + q^{-k_2'}q^{k_1'+(l-i)k_5}q^n = 1 + q^{(k_1'-k_2')}q^{(l-i)k_5}q^n$$

But $k_2' = \frac{1}{2}(k_1+k_2-k_3-k_4)$. Therefore $k_1'-k_2' = k_3+k_4 \to \infty$. So second term of denominator tends to 1. Therefore

$$(\boldsymbol{\Delta}_{a,-k'}\,\boldsymbol{\Delta}_{2a,-k'})(-\rho_{k'}) = \frac{1}{(1-q^{(l-i)k_5}q^n)}$$

Case 3): Let $a = -\epsilon_i + (n+\frac{1}{2})\delta \in B \cap \Phi_3^+$.
Then

$$\boldsymbol{\Delta}_{a,k'}\,\boldsymbol{\Delta}_{2a,k'}(-\rho_{k'}) = \frac{1-e^{2a}}{(1-q^{k_3'}e^a)(1+q^{k_4'}e^a)}(-\rho_{k'})$$

Now since $k_1' \to \infty$ and $k_3'+k_1' = k_1+k_3$ and $k_4'+k_1' = k_1+k_4 \to \infty$, we get

$$\boldsymbol{\Delta}_{a,k'}\,\boldsymbol{\Delta}_{2a,k'}(-\rho_{k'}) = \frac{1}{1-q^{k_1+k_3+(l-i)k_5}q^{n+\frac{1}{2}}}$$

Case 4): Let $a = -\epsilon_i + (n+\frac{1}{2})\delta \in B \cap \Phi_3^+$.
Then

$$\boldsymbol{\Delta}_{a,-k'}\,\boldsymbol{\Delta}_{2a,-k'}(-\rho_{k'}) = \frac{1-e^{2a}}{(1-q^{-k_3}e^a)(1+q^{-k_4}e^a)}(-\rho_{k'})$$

Now using $k_1'-k_3' = k_2+k_4 \to \infty$ and $k_1'-k_4' = k_2+k_3$ we get,

$$\boldsymbol{\Delta}_{a,k'}\,\boldsymbol{\Delta}_{2a,k'}(-\rho_{k'}) = \frac{1}{1+q^{k_2+k_3+(l-i)k_5}q^{n+\frac{1}{2}}}$$

Case 5): Let $-(\epsilon_i+\epsilon_j)+n\delta \in B \in \Phi_5^+$

$$\boldsymbol{\Delta}_{a,k'}\,\boldsymbol{\Delta}_{a,-k'}(-\rho_{k'}) = \frac{1-e^{2a}}{(1-q^{k_5}e^a)(1+q^{-k_5}e^a)}(-\rho_{k'})$$

As $e^{\langle a,-\rho_{k'}\rangle} = q^{2k_1'+(l-i+l-j)k_5+n} \to 0$ and $q^{k_5'+2k_1'} \to 0$, we get

$$\boldsymbol{\Delta}_{a,k'}\,\boldsymbol{\Delta}_{a,-k'}(-\rho_{k'}) = 1$$

Case 6): Let $-(\epsilon_i - \epsilon_j) + n\delta \in B \in \Phi_5^+$
Then

$$\begin{aligned}
\boldsymbol{\Delta}_{a,k'}(-\rho_{k'}) &= \frac{1-e^a}{1-q^{k'_5}e^a}(-\rho_{k'}) \\
&= \frac{1-q^n q^{(j-i)k_5}}{1-q^n q^{(j-i+1)k_5}} \\
\text{similarly } \ \boldsymbol{\Delta}_{a,-k'}(-\rho_{k'}) &= \frac{1-q^n q^{(j-i)k_5}}{1-q^n q^{(j-i-1)k_5}} \\
\text{So } \ \boldsymbol{\Delta}_{a,k'}\,\boldsymbol{\Delta}_{a,-k'}(-\rho_{k'}) &= \frac{(1-q^n q^{(j-i)k_5})^2}{(1-q^n q^{(j-i+1)k_5})(1-q^n q^{(j-i-1)k_5})}
\end{aligned}$$

Now we write total contribution of every case.
Case 1:

$$\prod_{n\geq 1}\prod_{i=1}^{l}\frac{1}{1+q^{(l-i+1)k_5+n}} = \frac{1}{\prod_{j=1}^{l}(-q^{jk_5+1};q)_\infty}$$

Case 2:

$$\prod_{n\geq 1}\prod_{i=1}^{l}\frac{1}{1-q^{(l-i)k_5+n}} = \frac{1}{\prod_{j=1}^{l-1}(q^{jk_5+1};q)_\infty(q;q)_\infty}$$

Cases 3 and 4:

$$\left(\prod_{n\geq 0}\prod_{i=1}^{l}\frac{1}{1-q^{k_3+(l-i+\frac{1}{2})k_5+n+\frac{1}{2}}}\right)\cdot\left(\prod_{n\geq 0}\prod_{i=1}^{l}\frac{1}{1+q^{k_3+(l-i+\frac{1}{2})k_5+n+\frac{1}{2}}}\right) = \frac{1}{\prod_{\substack{j=1\\ j\text{ odd}}}^{2l-1}(q^{2k_3+jk_5+1};q^2)_\infty}$$

Case 5:

$$1$$

Case 6:

$$\prod_{n\geq 1}\prod_{1\leq i<j\leq l}\frac{(1-q^n q^{(j-i)k_5})^2}{(1-q^n q^{(j-i+1)k_5})(1-q^n q^{(j-i-1)k_5})} = \prod_{1\leq i<j\leq l}\frac{(q^{(j-i)k_5+1};q)_\infty^2}{(q^{(j-i+1)k_5+1};q)_\infty(q^{(j-i-1)k_5+1};q)_\infty}$$

Consider the product in case 6. Denoting $u = q^{k_5}$, since $\epsilon_i - \epsilon_j, 1 \leq i <\leq l$ are the positive roots of A_{l-1}, we can write this product as

$$\prod_{\alpha\in\Delta^+_{\mathrm{re}}(A_{l-1})}\frac{(u^{\mathrm{ht}(\alpha)}q;q)_\infty^2}{(u^{\mathrm{ht}(\alpha)+1}q+1;q)_\infty(u^{\mathrm{ht}(\alpha)-1}q;q)_\infty}$$

where $\mathrm{ht}(\epsilon_{ij}) = j - i$. But for A_{l-1}, by height configuration, the above equation becomes

$$\frac{(uq;q)_\infty^l}{(q;q)_\infty^{l-1}(u^l q;q)_\infty} = \frac{(q^{k_5+1};q)_\infty^l}{(q;q)_\infty^{l-1}(q^{l(k_5)+1};q)_\infty}$$

Therefore the expression for $\mathrm{ct}(\boldsymbol{\Delta})$ becomes

$$\mathrm{ct}(\boldsymbol{\Delta}) = \frac{(q^{k_5+1};q)^l_\infty}{(q;q)^l_\infty} \frac{1}{\prod_{j=1}^l (q^{2j(k_5)+2};q^2)_\infty} \prod_{\substack{j=1 \\ j \text{ odd}}}^{2l-1} \frac{1}{(q^{2k_3+jk_5+1};q^2)_\infty} \tag{3.6.12}$$

Now letting $t = q^{k_5}$ and $t = q^{k_3}$ in above equation and using lemma 3.6.1 (part 2) we get

$$\mathrm{ct}(\hat{\boldsymbol{\mu}}\,\Theta_M) = (tq;q)^l_\infty \prod_{\substack{j=1 \\ j \text{ even}}}^{2l} \frac{1}{(t^j q^2;q^2)_\infty} \prod_{\substack{j=1 \\ j\text{odd}}} \frac{1}{(t^{j+2}q;q^2)_\infty} \tag{3.6.13}$$

which is the desired identity. □

Now by lemma 3.4.12 we have

$$\begin{aligned} a^{\Lambda_0}_{\Lambda_0}(q;t) &= \mathrm{ct}(e^{-\Lambda_0}\,\hat{\boldsymbol{\mu}}\,\mathrm{ch}L(\Lambda_0)) \\ &= \hat{\boldsymbol{\mu}}^{\mathrm{im}} a^{\Lambda_0}_{\Lambda_0}(q,1)\mathrm{ct}(\hat{\boldsymbol{\mu}}\,\Theta_M) \end{aligned}$$

so by lemma 3.4.12 part 2 we have

$$a^{\Lambda_0}_{\Lambda_0}(q;t) = \frac{1}{(tq;q)^l_\infty}\mathrm{ct}(\Theta_M).$$

So using equation 3.6.13 we get

$$a^{\Lambda_0}_{\Lambda_0}(q;t) = \prod_{\substack{j=1 \\ j \text{ even}}}^{2l} \frac{1}{(t^j q^2;q^2)_\infty} \prod_{\substack{j=1 \\ j\text{odd}}} \frac{1}{(t^{j+2}q;q^2)_\infty}$$

which completes proof of theorem 3.4.3 for $A^{(2)}_{2l}$ case.

3.6.2 Two variable generalisation

One can prove a slightly more general, two-variable version of theorem 3.4.3 for $\mathfrak{g} = A^{(2)}_{2l}$. To state this, let s,t be indeterminates, and define the two-variable Kostant partition function $\mathcal{P}(\beta;\,s,t)$ to be the coefficient of e^β in the product $\prod_{\alpha\in\Delta_+(\mathfrak{g})} (1-u_\alpha\,e^\alpha)^{-\mathrm{mult}\alpha}$ where $u_\alpha := s$ if α is a real root of norm 1 (=shortest root length) and $u_\alpha := t$ for all other roots (i.e., imaginary roots, and real roots of norms 2 and 4). For a dominant integral weight λ of $A^{(2)}_{2l}$, and a maximal dominant weight μ of $L(\lambda)$, define the two variable Kostka-Foulkes polynomial

$$K_{\lambda\mu}(s,t) := \sum_{w\in W} \epsilon(w)\,\mathcal{P}(w(\lambda+\rho)-(\mu+\rho);\,s,t)$$

and let the corresponding (s,t)-string function for the basic representation be $a^{\Lambda_0}_{\Lambda_0}(s,t,q) := \sum_{p\geq 0} K_{\Lambda_0,\Lambda_0-p\delta}(s,t)\,q^p$. The following is the two variable version of theorem 3.4.3.

Proposition 3.6.3. *For* $\mathfrak{g} = A_{2l}^{(2)}$,

$$a_{\Lambda_0}^{\Lambda_0}(s,t,q) = \prod_{\substack{j=1 \\ j \text{ even}}}^{2l} (t^j q^2; q^2)_\infty^{-1} \prod_{\substack{j=1 \\ j \text{ odd}}}^{2l} (s^2 t^j q; q^2)_\infty^{-1}$$

Proof. The proof is along the exact same lines as that of proposition 3.6.2, but now with parameters chosen differently. We choose $k_5 = 2k_1 = 2k_2$, $k_4 \to \infty$, but leave k_3 as a free parameter. We then take $t := q^{k_5}$ and $s := q^{k_3}$. The remaining details are easily checked. □

Corollary 3.6.4. $K_{\Lambda_0, \Lambda_0 - p\delta}(s,t) \in \mathbb{Z}_{\geq 0}[s,t]$ *for all* $p \geq 0$.

Finally, we remark that it would be of interest to find a more natural explanation for the positivity result of the above corollary (or more generally, for $K_{\lambda\mu}(s,t)$) in terms of a Brylinski-Kostant type filtration, as is known for the usual (one variable) affine Kostka-Foulkes polynomials [37]. We also note that the two variable Kostka-Foulkes polynomials can be defined for all twisted affines (in fact, for any affine root system with more than one root length) and in more than one way (corresponding to different choices of the u_α in the definition). But it appears, from preliminary calculations, that only $A_{2l}^{(2)}$ (with the given choice of u_α) exhibits the positivity property of corollary 3.6.4.

Chapter 4

Affine Brylinski-Kostant filtration on the basic representation of $A_1^{(1)}$

Consider the affine Lie algebra $A_1^{(1)}$. Let us consider its basic representation $L(\Lambda_0)$. By theorem 3.3.1 we have

$$a_{\Lambda_0}^{\Lambda_0}(t;q) = \sum_{n\geq 0} K_{\Lambda_0-n\delta}^{\Lambda_0}(t)q^n = \prod_{k=1}^{\infty}(1-t^2q^k)^{-1}$$

By comparing the coefficients of the above expression, we get a formula for the Kostka-Foulkes polynomial which is,

$$K_{\Lambda_0-n\delta}^{\Lambda_0}(t) = \sum_{\pi\vdash n} t^{2\,(\#\pi)}$$

where $\#\pi$ is the number of parts in the partition π of n.

Slofstra [37] shows that the affine Kostka-Foulkes polynomial $K_\mu^\lambda(t)$, where λ and μ are dominant, is equal to the Poincaré series of the associated graded space of affine Brylinski-Kostant filtration on $L(\lambda)_\mu$. This extends Brylinski's result [1] (which is for the finite dimensional simple Lie algebras) to the affine Kac-Moody algebras.

In this chapter, for $\mathfrak{g} = A_1^{(1)}$, we give a basis for the space $L(\Lambda_0)_{\Lambda_0-n\delta}$ for $n \geq 0$, which is compatible with the affine Brylinski-Kostant filtration. Using Slofstra's theorem this gives an alternative derivation of the expression for $a_{\Lambda_0}^{\Lambda_0}(t;q)$.

4.1 Heisenberg algebra

Definition 4.1.1. *The Heisenberg algebra is the complex Lie algebra with a basis* $\{a_n, n \in \mathbb{Z}\} \cup \{\hbar\}$*, and commutation relations*

$$[\hbar, a_n] = 0 \ \ \forall\, n \in \mathbb{Z}$$

$$[a_n, a_m] = m\delta_{m,-n}\hbar \ \forall\, m, n \in \mathbb{Z}$$

4.1.1 Homogeneous and principal Heisenberg subalgebras of affine Kac-Moody algebras

Let $\mathfrak{g}$ be an affine Kac-Moody algebra. Let e_i and f_i for $i = 0, 1, 2, ..., l$ be the Chevalley generators. Let δ be its null root and $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$. Recall that $\mathfrak{g}_{n\delta}$ denotes the root space of $\mathfrak{g}$ with corresponding to the root $n\delta$ for $n \neq 0$. Let K be the central element of $\mathfrak{g}$.

The homogeneous Heisenberg algebra $\mathcal{H}$ of $\mathfrak{g}$ is defined as

$$\mathcal{H} = \mathbb{C}K + \sum_{\substack{s\in\mathbb{Z}\\ s\neq 0}} \mathfrak{g}_{s\delta}$$

A principal nilpotent element of the affine Kac-Moody algebra is an element of the form $\sum_{i=0}^{l} c_i e_i$ where $c_i \in \mathbb{C} - \{0\}$. For a principal nilpotent element E' of $\mathfrak{g}$, the Lie subalgebra defined by

$$\mathfrak{s}_{E'} = \{x \in \mathfrak{g} : [x, E'] \in \mathbb{C}K\}$$

is the principal Heisenberg algebras of $\mathfrak{g}$.

4.2 The affine Brylinski-Kostant filtration

Let $\mathfrak{m}$ be a finite dimensional simple Lie algebra and $V = V(\lambda)$ be the irreducible finite dimensional $\mathfrak{m}$ -module with highest weight λ. Let $\mathfrak{m} = N_+ \oplus H \oplus N_-$ be a triangular decomposition and let $E \in N_+$ be a principal nilpotent element of $\mathfrak{m}$. Let $V(\lambda)_\mu$ be a weight space of $V(\lambda)$. Recall from chapter 3 § 3.4 that the Brylinski-Kostant filtration for $V(\lambda)_\mu$ with respect to the principal nilpotent element E is given by

$\mathcal{F}^{(-1)}(V(\lambda)_\mu) \subseteq \mathcal{F}^{(0)}(V(\lambda)_\mu) \subseteq \mathcal{F}^{(1)}(V(\lambda)_\mu) \subseteq \cdots$, where $\mathcal{F}^{(p)}(V(\lambda)_\mu) := \ker(E^{p+1}) \cap V(\lambda)_\mu$. The Poincaré series (polynomial) of the associated graded space of this filtration is defined as

$$P^\lambda_\mu(t) := \sum_{i\geq 0} \dim\,(\mathcal{F}^{(i)}V(\lambda)_\mu / \mathcal{F}^{(i-1)}V(\lambda)_\mu)\, t^i$$

where $\mathcal{F}^{(-1)}V(\lambda)_\mu := 0$.

Brylinski [1] proved that, for λ and μ dominant, $P^\lambda_\mu(t)$ is equal to the Kostka-Foulkes polynomial $K^\lambda_\mu(t)$. Slofstra [37] extends Brylinski's result to the affine Kac-Moody algebras. He shows that in the affine case, a principal nilpotent element is not sufficient to define the filtration, but one has to use the positive part of a principal Heisenberg subalgebra.

Let $\mathfrak{g}$ be an affine Kac-Moody algebra. Let $\mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be its triangular decomposition. Let $L(\lambda)$ be a highest weight irreducible integrable $\mathfrak{g}$-module, and μ be a weight. Let $\mathfrak{s}$ be a principal Heisenberg subalgebra of $\mathfrak{g}$. Then the affine Brylinski-Kostant filtration for the weight space $L(\lambda)_\mu$ is defined as [37]

$$\mathcal{F}^{(-1)}(L(\lambda)_\mu) \subseteq \mathcal{F}^{(0)}(L(\lambda)_\mu) \subseteq \mathcal{F}^{(1)}((L(\lambda)_\mu) \subseteq \cdots$$

where

$$\mathcal{F}^{(p)}(L(\lambda)_\mu) := \{v \in L(\lambda)_\mu : x^{p+1}v = 0 \; \forall \; x \in \mathfrak{s} \cap \mathfrak{n}_+\}.$$

Now we are in a position to state Slofstra's theorem.

Theorem 4.2.1 (Slofstra, [37])**.** *Let $L(\lambda)$ be an integrable highest weight representation of an affine Kac-Moody algebra $\mathfrak{g}$, where λ is a dominant integral weight. If μ is a dominant weight of $L(\lambda)$, then*

$$P^\lambda_\mu(t) = K^\lambda_\mu(t)\ .$$

4.2.1 $\mathfrak{g} = A_1^{(1)}$

We now consider the affine Lie algebra $A_1^{(1)}$. Let h, e, f be the usual basis of $\mathfrak{sl}_2$. Let $L(\Lambda_0)$ be its basic representation. Following Lepowsky-Wilson [22], for an odd integer j, define

$$B_j := e \otimes t^{\frac{j-1}{2}} + f \otimes t^{\frac{j+1}{2}}$$

and

$$C_j := -e \otimes t^{\frac{j-1}{2}} + f \otimes t^{\frac{j+1}{2}}.$$

For non-zero even integers j define

$$H_j := h \otimes t^{\frac{j}{2}}$$

and

$$H_0 := h \otimes 1 - \frac{1}{2}K.$$

The B_j, C_j and H_j satisfy the following bracket relations [22]:

$$[B_j, B_k] = j\delta_{j,-k}K \tag{4.2.1}$$

$$[C_j, C_k] = -j\delta_{j,-k}K \tag{4.2.2}$$

$$[H_j, H_k] = j\delta_{j,-k}K \tag{4.2.3}$$

$$[B_j, H_k] = 2C_{j+k} \tag{4.2.4}$$

$$[C_j, H_k] = 2B_{j+k} \tag{4.2.5}$$

$$[B_j, C_k] = 2H_{j+k} \tag{4.2.6}$$

Let $\mathfrak{s} := \text{span}(\{B_j : j \text{ odd}\} \cup \{K\})$ and $\mathcal{H} := \text{span}(\{H_j : j \text{ even}\} \cup \{K\})$. Then $\mathfrak{s}$ and $\mathcal{H}$ are the principal and homogeneous Heisenberg algebra of $\mathfrak{g}$ respectively. Note that here $\mathfrak{s} = \mathfrak{s}_{E'}$ for $E' = e \otimes 1 + f \otimes t$.

It is a well known fact that for the basic representation $L(\Lambda_0)$ of an affine Lie algebra, the principal Heisenberg algebra acts irreducibly. On the other hand the homogeneous Heisenberg algebra does not act irreducibly on $L(\Lambda_0)$ [15].

As $\mathfrak{s}$ acts irreducibly on $L(\Lambda_0)$, by the standard theory of the Heisenberg algebra we have $L(\Lambda_0) = \mathbb{C}[x_1, x_3, x_5, ...]$ and where for all odd $j > 0$ B_j acts as the operator $\frac{j}{2}\frac{\partial}{\partial x_j}$ and B_{-j} acts as $2T_{x_j}$, where the operator T_x denotes left multiplication by x.

Now, $\mathfrak{s} \cap \mathfrak{n}_+ =$ span of$\{B_j : j \text{ odd } > 0\}$. So the affine Brylinski-Kostant filtration with respect to the principal Heisenberg algebra $\mathfrak{s}$ is given by $\mathcal{F}^{(p)}(L(\lambda)_\mu) = \{v \in L(\lambda)_\mu : x^{p+1}v = 0 \,\forall\, x \in \mathfrak{s} \cap \mathfrak{n}_+\}$ or equivalently

$$\mathcal{F}^{(p)}(L(\lambda)_\mu) = \{v \in L(\lambda)_\mu : B_{i_1}B_{i_2}\cdots B_{i_{p+1}}v = 0 \,\forall\, i_1, i_2, ..., i_{p+1}(\text{odd}) > 0\}.$$

Since for all odd $j > 0$, B_j acts as operator $\frac{j}{2}\frac{\partial}{\partial x_j}$, we see that if $f \in \mathcal{F}^{(p)}(L(\lambda)_\mu)$, then $B_{i_1}B_{i_2}\cdots B_{i_{p+1}}f = 0 \,\forall\, i_1, i_2, ..., i_{p+1} > 0$. This implies that the usual degree of f in variables $x_1, x_3, x_5, \cdots$ is atmost p. Therefore we have

$$\mathcal{F}^{(p)}(L(\lambda)_\mu) = \{f \in (L(\lambda)_\mu) : \text{udeg}(f) \leq p\}$$

where udeg(p) denotes the usual degree of polynomial p. Using this we see that the quotient space

$$\mathcal{F}^{(p)}(L(\lambda)_\mu)/\mathcal{F}^{(p-1)}(L(\lambda)_\mu) = \{f \in L(\lambda)_\mu : \text{udeg}(f) = p\}.$$

To state our main theorem, we need to recall the Sugawara operators.

4.3 Sugawara operators

Let $\mathfrak{g}$ be an affine Kac-Moody algebra. For simplicity, we assume that $\mathfrak{g}$ is an untwisted affine Lie algebra. Let $\mathring{\mathfrak{g}}$ be its underlying finite dimensional simple Lie algebra. Let u_i and u^i be dual bases of $\mathring{\mathfrak{g}}$ with respect to Killing form i.e., $(u_i, u^j) = \delta_{ij}$. Let $u^{(n)}$ denote $t^n \otimes u$ where $n \in \mathbb{Z}$, $u \in \mathring{\mathfrak{g}}$. The Sugawara operators $T_n (n \in \mathbb{Z})$ of $\mathfrak{g}$ is defined as

$$T_0 = \sum_i u_i u^i + 2\sum_{n=1}^{\infty}\sum_i u_i^{(-n)}u^{i(n)}$$

$$T_n = \sum_{m\in\mathbb{Z}}\sum_i u_i^{(-m)}u^{i(m+n)} \text{ for } n \neq 0$$

Similarly for the algebra $\tilde{L}(\mathring{\mathfrak{h}}) = \mathbb{C}[z,^{-1}] \otimes \mathring{\mathfrak{h}} \oplus \mathbb{C}K$, where $\mathring{\mathfrak{h}}$ is Cartan subalgebra of $\mathring{\mathfrak{g}}$, we define the Sugawara operators T_n $(n \in \mathbb{Z})$ as follows. Let h_i and h^i be dual bases of $\mathring{\mathfrak{h}}$ and as above

$h^{(n)}$ denote $t^n \otimes h$ for an integer n. Then

$$T_0 = \sum_i h_i h^i + 2 \sum_{n=1}^{\infty} \sum_i h_i^{(-n)} h^{i(n)}$$

$$T_n = \sum_{m \in \mathbb{Z}} \sum_i h_i^{(-m)} h^{i(m+n)} \quad \text{for} \quad n \neq 0$$

Let $L(\lambda)$ be a highest weight representation with level of λ equal to $k \in \mathbb{Z}_{\geq 0}$. Define $L_n^{\overset{\circ}{\mathfrak{g}}} := \frac{1}{2(k+h^\vee)} T_n$ and $c(k) := \frac{k(\dim(\overset{\circ}{\mathfrak{g}}))}{k+h^\vee}$ where T_n are Sugawara operators obtained from $\mathfrak{g}$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Similarly we define $L_n^{\overset{\circ}{\mathfrak{h}}} := \frac{1}{2k} T_n$ and $\overset{\circ}{c}(\overset{\circ}{k}) := \dim(\overset{\circ}{\mathfrak{h}})$ where T_n are Sugawara operators obtained from $\tilde{L}(\overset{\circ}{\mathfrak{h}})$.

We recall the term Virasoro algebra.

Definition 4.3.1. *The Virasoro algebra is the complex vector space with basis* $\{d_n : n \in \mathbb{Z}\} \cup \{c\}$ *and commutation relations*

$$[d_m, c] = 0,$$

$$[d_m, d_n] = (m-n) d_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} c \ .$$

Let us define $L_n^{\overset{\circ}{\mathfrak{g}}, \overset{\circ}{\mathfrak{h}}} := L_n^{\overset{\circ}{\mathfrak{g}}} - L_n^{\overset{\circ}{\mathfrak{h}}}$. We state the following proposition from [15]

Proposition 4.3.2. *1.* $L_n^{\overset{\circ}{\mathfrak{g}}, \overset{\circ}{\mathfrak{h}}}$ *commutes with* $\tilde{L}(\overset{\circ}{\mathfrak{h}})$ *as operators on* $L(\lambda)$.

2. The map $d_n \to L_n^{\overset{\circ}{\mathfrak{g}}, \overset{\circ}{\mathfrak{h}}}, c \to c(k) - \overset{\circ}{c}(\overset{\circ}{k})$ *defines a representation of Virasoro algebra on* $L(\lambda)$.

The action of the operator $L_n^{\overset{\circ}{\mathfrak{g}}, \overset{\circ}{\mathfrak{h}}}$ on $L(\lambda)$ is called the coset-Vir action and $L(\lambda)$ is called coset-Vir module with respect to this action.

Now consider the module $L(\Lambda_0)$. In the following proposition we recall some of the facts from [15].

Proposition 4.3.3. *1. The coset Vir action is trivial on the highest weight vector* v_{Λ_0} *of* $L(\Lambda_0)$, *i.e.,* $L_n^{\overset{\circ}{\mathfrak{g}}, \overset{\circ}{\mathfrak{h}}} v_{\Lambda_0} = (L_n^{\overset{\circ}{\mathfrak{g}}} - L_n^{\overset{\circ}{\mathfrak{h}}}) v_{\Lambda_0} = 0$. *Therefore* $L_n^{\overset{\circ}{\mathfrak{g}}} v_{\Lambda_0} = L_n^{\overset{\circ}{\mathfrak{h}}} v_{\Lambda_0}$.

2. The coset Vir action on $L(\Lambda_0)$ *commutes with the action of* $\tilde{L}(\overset{\circ}{\mathfrak{h}})$. *Therefore* $[L_n^{\overset{\circ}{\mathfrak{g}}} - L_n^{\overset{\circ}{\mathfrak{h}}}, x] = 0 \ \forall \ x \in \tilde{L}(\overset{\circ}{\mathfrak{h}})$

3. Let $V := \oplus_{n \geq 0} L(\Lambda_0)_{\Lambda_0 - n\delta} = U(\tilde{L}(\overset{\circ}{\mathfrak{h}})) v_{\Lambda_0}$. *By (1) and (2) above we see that the coset Vir action is trivial on* V, *i.e., the Virasoro operators obtained from* $\mathfrak{g}$ *and the Virasoro operators obtained from* $\tilde{L}(\overset{\circ}{\mathfrak{h}})$ *act in the same way on* V.

4.3.1 Sugawara operators for $A_1^{(1)}$

Let $\mathfrak{g} = A_1^{(1)}$, and let (e, f, h) and $(f, e, \frac{h}{2})$ be dual bases for $\mathfrak{sl}_2$. Then the Sugawara operators for the $A_1^{(1)}$-module $L(\Lambda_0)$ are given by

$$L_n^{\mathring{\mathfrak{g}}} = \frac{1}{6}\sum_{m\in\mathbb{Z}}\left[e(-m)f(m+n) + f(-m)e(m+n) + h(-m)\frac{h}{2}(m+n)\right] \quad \forall n \neq 0$$

and

$$L_0^{\mathring{\mathfrak{g}}} = \frac{1}{6}\left(e(0)f(0) + f(0)e(0) + \frac{h(0)^2}{2} + 2\sum_{n=1}^{\infty}\left[e(-n)f(n) + f(-n)e(n) + h(-n)\frac{h(n)}{2}\right]\right)$$

where $x(m)$ denotes $x \otimes t^m$. Similarly for the Lie algebra $\tilde{L}(\mathring{\mathfrak{h}})$, we have for $n \neq 0$

$$L_n^{\mathring{\mathfrak{h}}} := \frac{1}{2}\sum_{m\in\mathbb{Z}} h(m)\frac{h(n-m)}{2}$$

and

$$L_0^{\mathring{\mathfrak{h}}} := \frac{h(0)^2}{4} + \sum_{n=1}^{\infty} h(-n)\frac{h(n)}{2}$$

Now consider the homogeneous Heisenberg algebra $\mathcal{H}$ generated by $H_{2j} =: H(j)$ for $n \neq 0$ and $H_0 := H(0) = h_0 - \frac{1}{2}K$. Note that $h(n) = H_{2n} =: H(n)$ for $n \neq 0$. Define the Virasoro operators with respect to $\mathcal{H}$ by

$$L_n^{\mathcal{H}} := \frac{1}{2}\sum_{m\in\mathbb{Z}} H(n)\frac{H(n-m)}{2} \quad \text{for } n \neq 0$$

$$L_0^{\mathcal{H}} := \frac{H(0)^2}{4} + \sum_{n=1}^{\infty} H(-n)\frac{H(n)}{2}$$

Then we see that $L_n^{\mathcal{H}} = L_n^{\mathring{\mathfrak{h}}} - \frac{h(n)}{4}$ for $n \neq 0$. So define $\tilde{L}_n^{\mathring{\mathfrak{g}}} := L_n^{\mathring{\mathfrak{g}}} - \frac{h(n)}{4}$ for $n \neq 0$. From proposition 4.3.3 part 3 we note that $\tilde{L}_n^{\mathring{\mathfrak{g}}} = L_n^{\mathcal{H}}$ for $n \neq 0$ on V. Let for a polynomial $f \in \mathcal{F}^{(j)}L(\Lambda_0)_{\Lambda_0 - s\delta}$, $\overline{f}$ denote image of f in the quotient $\mathcal{F}^{(j)}(L(\Lambda_0)_{\Lambda_0-s\delta})/\mathcal{F}^{(j-1)}(L(\Lambda_0)_{\Lambda_0-s\delta})$.

Now we are in a position to state our main theorem :

Theorem 4.3.4. *Let $\mathfrak{g} = A_1^{(1)}$. Let $r > 0$ and $k \geq 0$. Then a basis for the quotient space $\mathcal{F}^{(2k)}(L(\Lambda_0)_{\Lambda_0-r\delta})/\mathcal{F}^{(2k-1)}(L(\Lambda_0)_{\Lambda_0-r\delta})$ is given by the set*

$$\{\overline{L_{n_1}^{\mathcal{H}}L_{n_2}^{\mathcal{H}}...L_{n_k}^{\mathcal{H}}v_{\Lambda_0}} : 0 > n_k \geq n_{k-1} \geq \cdots \geq n_1 \text{ and } n_1 + n_2 + \cdots + n_k = -2r\}$$

From above theorem, we deduce that

$$\dim \mathcal{F}^{(2k)}(L(\Lambda_0)_{\Lambda_0-r\delta})/\mathcal{F}^{(2k-1)}(L(\Lambda_0)_{\Lambda_0-r\delta}) = P(r,k)$$

i.e., the number of partitions of r in k parts. We also note that

$$\dim \mathcal{F}^{(2k-1)}(L(\Lambda_0)_{\Lambda_0-r\delta})/\mathcal{F}^{(2k-2)}(L(\Lambda_0)_{\Lambda_0-r\delta}) = 0 \; . \qquad (4.3.7)$$

To see this let us define the *depth* for the weights of module $L(\Lambda_0)$.

Definition 4.3.5. *Let $\mathfrak{g}$ be a Kac -Moody algebra and let V be a highest weight module of $\mathfrak{g}$. Let $\lambda \in P(V)$ be such that $\Lambda - \lambda = \sum_{i=1}^{n} a_i\alpha_i$. Then depth of λ denoted as dep(λ) $:= \sum_{i=1}^{n} a_i$.*

By this we have a gradation $L(\Lambda) = \oplus_{d\in\mathbb{Z}} L(\Lambda)_d$ defined by $L(\Lambda)_d = \oplus_{dep(\nu)=d} L(\Lambda)_\nu$. Now consider the case where $\mathfrak{g} = A_1^{(1)}$ and $L(\lambda) = L(\Lambda_0)$. Then $\text{dep}(\Lambda_0 - r\delta) = 2r$ and we note that $B_j(L(\Lambda_0)_d) \subset L(\Lambda_0)_{d-j}$ for an odd integer j. To prove equation 4.3.7, we need to show that there is no nonzero odd degree polynomial in $(L(\Lambda_0)_{\Lambda_0-r\delta}$.

Now assume contrary. Let there exist $0 \neq \overline{f} \in \mathcal{F}^{(2k-1)}(L(\Lambda_0)_{\Lambda_0-r\delta})/\mathcal{F}^{(2k-2)}(L(\Lambda_0)_{\Lambda_0-r\delta})$ for some k, i.e., f has usual degree $2k-1$. So there exists $j_1, \cdots, j_{2k-1}$ such that $B_{j_1} \cdots B_{j_{2k-1}} f =\neq 0 \in \mathbb{C}$. So the nonzero element $B_{j_1} \cdots B_{j_{2k-1}} f \in L(\Lambda_0)_{\Lambda_0}$ and Λ_0 has depth 0. But the element $B_{j_1} \cdots B_{j_{2k-1}} f$ lies in the space which has depth $2r - (j_1 + \cdots + j_{2k-1})$ which is an odd integer and we get the desired contradiction. Using this observation and Slofstra's result, we have the following corollary:

Corollary 4.3.6. *Let $\mathfrak{g} = A_1^{(1)}$. Let $L(\Lambda_0)$ be its basic representation. Then*

$$a_{\Lambda_0}^{\Lambda_0}(t;q) = \prod_{n=1}^{\infty} (1 - t^2 q^n)^{-1}$$

Proof. As

$$\dim \mathcal{F}^{2k}(L(\Lambda_0)_{\Lambda_0-r\delta})/\mathcal{F}^{2k-1}(L(\Lambda_0)_{\Lambda_0-r\delta}) = P(-r,k)$$

So by theorem 4.2.1 and by equation 4.3.7, we have

$$K_{\Lambda_0-n\delta}^{\Lambda_0}(t) = \sum_{\pi\vdash n} t^{2\,(\#\pi)}$$

where $\#\pi$ is the number of parts in the partition π of n and the result follows. □

4.4 Proof of theorem 4.3.4

The first step to prove the theorem is to prove following proposition.

Proposition 4.4.1. *Let $\mathfrak{g} = A_1^{(1)}$. Then $\tilde{L}_{n_1}^{\mathring{\mathfrak{g}}} \tilde{L}_{n_2}^{\mathring{\mathfrak{g}}} ... \tilde{L}_{n_k}^{\mathring{\mathfrak{g}}} v_{\Lambda_0} \in \mathcal{F}^{(2k)}(L(\Lambda_0)_{\Lambda_0-r\delta})$ where $n_i < 0$ for $1 \leq i \leq k$ and $n_1 + n_2 + \cdots + n_k = -2r$.*

Proof. The key step is to prove that for $n < 0$, $\tilde{L}_n^{\mathring{\mathfrak{g}}}(v_{\Lambda_0}) \in \mathcal{F}^{(2)}(L(\Lambda_0)_{\Lambda_0+n\delta})$. As $\tilde{L}_n^{\mathring{\mathfrak{g}}} = L_n^{\mathcal{H}}$, it is clear that $\tilde{L}_n^{\mathring{\mathfrak{g}}}(v_{\Lambda_0}) \in L(\Lambda_0)_{\Lambda_0+n\delta}$. So for j odd, let us compute the bracket $[B_j, \tilde{L}_n^{\mathring{\mathfrak{g}}}]$:

$$\begin{aligned}
[B_j, \tilde{L}_n^{\mathring{\mathfrak{g}}}] &= [e \otimes t^{\frac{j-1}{2}}, L_n^{\mathring{\mathfrak{g}}}] + [f \otimes t^{\frac{j+1}{2}}, L_n^{\mathring{\mathfrak{g}}}] - [B_j, \frac{h(n)}{4}] \\
&= (\frac{j-1}{2})(e \otimes t^{\frac{j-1}{2}+n}) + (\frac{j+1}{2})(f \otimes t^{\frac{j+1}{2}+n}) - \frac{1}{2} C_{j+2n} \text{ by [15, § 12.8], , and by equation 4.2.4} \\
&= \frac{j}{2} B_{j+2n} + \frac{1}{2} C_{j+2n} - \frac{1}{2} C_{j+2n} \\
&= \frac{j}{2} B_{j+2n}
\end{aligned}$$

Now consider for positive odd j

$$B_j \cdot \tilde{L}_n^{\mathring{\mathfrak{g}}} \cdot v_{\Lambda_0} = \tilde{L}_n^{\mathring{\mathfrak{g}}} \cdot B_j \cdot v_{\Lambda_0} + [B_j, \tilde{L}_n^{\mathring{\mathfrak{g}}}] v_{\Lambda_0}$$

As for $j > 0$, $B_j \cdot v_{\Lambda_0} = 0$, we see that $B_j \cdot \tilde{L}_n^{\mathring{\mathfrak{g}}} \cdot v_{\Lambda_0} = [B_j, \tilde{L}_n^{\mathring{\mathfrak{g}}}] v_{\Lambda_0}$ from commutation relations of B_j's, for positive odd j_1 and j_2, we have

$$B_{j_1} B_{j_2} \tilde{L}_n^{\mathring{\mathfrak{g}}} v_{\Lambda_0} = (\frac{j_2}{2})\, B_{j_1} B_{j_1+2n} v_{\Lambda_0}$$

So

$$B_{j_1} B_{j_2} \tilde{L}_n^{\mathring{\mathfrak{g}}} v_{\Lambda_0} = [B_{j_1}, B_{j_2+2n}] v_{\Lambda_0}$$

Therefore for odd $j_1, j_2, j_3 > 0$, we have $B_{j_1} B_{j_2} B_{j_3} \tilde{L}_n^{\mathring{\mathfrak{g}}} v_{\Lambda_0} = 0$ or equivalently $B^3 \tilde{L}_n^{\mathring{\mathfrak{g}}} v_{\Lambda_0} = 0$ $\forall\, B \in \mathfrak{s} \cap \mathfrak{n}_+$. This proves that for $\forall\, n < 0$, $\tilde{L}_n^{\mathring{\mathfrak{g}}} \in \mathcal{F}^{(2)}(L(\Lambda_0)_{\Lambda_0+n\delta})$.

Now consider

$$B^p(\tilde{L}_{n_1}^{\mathring{\mathfrak{g}}} \tilde{L}_{n_2}^{\mathring{\mathfrak{g}}} ... \tilde{L}_{n_k}^{\mathring{\mathfrak{g}}}) v_{\Lambda_0} = \sum_{p_1+p_2+...+p_k=p} \left(\left((\mathrm{ad}B)^{p_1} \tilde{L}_{n_1}^{\mathring{\mathfrak{g}}} \right) \left((\mathrm{ad}B)^{p_2} \tilde{L}_{n_2}^{\mathring{\mathfrak{g}}} \right) \cdots \left((\mathrm{ad}B)^{p_k} \tilde{L}_{n_k}^{\mathring{\mathfrak{g}}} \right) \right) .v_{\Lambda_0}$$

We see that if $p \geq 2k+1$ then at least one of p_i's should be greater than or equal to 3. Therefore the above expression is zero and proof is complete. □

Next step is to prove following proposition.

Proposition 4.4.2. *Let* $\mathfrak{g} = A_1^{(1)}$. *Then* $\overline{L_{n_1}^{\mathcal{H}} L_{n_2}^{\mathcal{H}} ... L_{n_k}^{\mathcal{H}} v_{\Lambda_0}} \in \mathcal{F}^{(2k)}(L(\Lambda_0)_{\Lambda_0 - r\delta}) / \mathcal{F}^{(2k-1)}(L(\Lambda_0)_{\Lambda_0 - r\delta})$ *is nonzero (i.e.,* $\mathrm{udeg}(\overline{L_{n_1}^{\mathcal{H}} L_{n_2}^{\mathcal{H}} ... L_{n_k}^{\mathcal{H}} v_{\Lambda_0}}) = 2k$*) where* $n_1, n_2, \cdots, n_k \in \mathbb{Z}_{<0}$ *and* $n_1 + n_2 + \cdots + n_k = -2r$.

Proof. As

$$[B_j, \tilde{L}_n^{\mathring{\mathfrak{g}}}] = \frac{j}{2} B_{j+2n}.$$

So we see that for $n < 0$

$$B_{j_1} B_{j_2} \tilde{L}_n^{\mathring{\mathfrak{g}}} v_{\Lambda_0} = (\frac{j_2}{2})\, B_{j_1} B_{j_1+2n} v_{\Lambda_0}$$

$$B_{j_1}B_{j_2}\tilde{L}_n^{\mathring{\mathfrak{g}}}v_{\Lambda_0} = [B_{j_1}, B_{j_2+2n}]v_{\Lambda_0}$$
$$\Rightarrow B_{j_1}B_{j_2}\tilde{L}_n^{\mathring{\mathfrak{g}}}v_{\Lambda_0} \neq 0 \text{ if } j_1 + j_2 = -2n. \tag{4.4.8}$$

As $B^3\tilde{L}_n^{\mathring{\mathfrak{g}}}v_{\Lambda_0} = 0, \forall B \in \mathfrak{s} \cap \mathfrak{n}_+$, the term $B_{j_1}B_{j_2}...B_{j_{2k-1}}B_{j_{2k}}\tilde{L}_{n_1}^{\mathring{\mathfrak{g}}}\tilde{L}_{n_2}^{\mathring{\mathfrak{g}}}...\tilde{L}_{n_k}^{\mathring{\mathfrak{g}}}v_{\Lambda_0}$ has the following expression

$$\left(\sum_{\sigma \in S_{2k}} [B_{j_{\sigma(1)}}, [B_{j_{\sigma(2)}}, \tilde{L}_{n_1}^{\mathring{\mathfrak{g}}}]][B_{j_{\sigma(3)}}, [B_{j_{\sigma(4)}}, \tilde{L}_{n_2}^{\mathring{\mathfrak{g}}}]]...[B_{j_{\sigma(2k-1)}}, [B_{j_{\sigma(2k)}}, \tilde{L}_{n_k}^{\mathring{\mathfrak{g}}}]]\right) v_{\Lambda_0}$$

where S_n denotes the permutation group on n letters. Therefore by the equation 4.4.8 we have $B_{j_1}B_{j_2}...B_{j_{2k-1}}B_{j_{2k}}\tilde{L}_{n_1}^{\mathring{\mathfrak{g}}}\tilde{L}_{n_2}^{\mathring{\mathfrak{g}}}...\tilde{L}_{n_k}^{\mathring{\mathfrak{g}}}v_{\Lambda_0} \neq 0$ iff $\exists$ a $\sigma \in S_{2k}$ such that $j_{\sigma(1)} + j_{\sigma(2)} = -2n_1, j_{\sigma(3)} + j_{\sigma(4)} = -2n_2, ..., j_{\sigma(2k-1)} + j_{\sigma(2k)} = -2n_k$. But for $\overline{L_{n_1}^{\mathcal{H}}L_{n_2}^{\mathcal{H}}...L_{n_k}^{\mathcal{H}}v_{\Lambda_0}}$ with given condition on n_i's clearly there is a choice of j_i's for which there exist a $\sigma \in S_{2k}$, which satisfies the desired condition. □

Now as mentioned before we have $L(\Lambda_0) = \mathbb{C}[x_1, x_3, x_5, ...]$. We embed the quotient space $\mathcal{F}^{(2k)}(L(\Lambda_0)_{\Lambda_0 - r\delta})/\mathcal{F}^{(2k-1)}(L(\Lambda_0))_{\Lambda_0 - r\delta}$ into the space of homogeneous polynomials in $x_1, x_3, x_5...$ of usual degree $2k$.

Note that $\tilde{L}_{n_1}^{\mathring{\mathfrak{g}}}\tilde{L}_{n_2}^{\mathring{\mathfrak{g}}}...\tilde{L}_{n_k}^{\mathring{\mathfrak{g}}}v_{\Lambda_0} \in \mathcal{F}^{(2k)}(L(\Lambda_0)_{\Lambda_0 - r\delta})/\mathcal{F}^{(2k-1)}(L(\Lambda_0)_{\Lambda_0 - r\delta})$ is a linear combination of monomials $x_{j_1}x_{j_2}...x_{j_{2k}}$ satisfying the property that $\exists$ a $\sigma \in S_{2k}$ such that $j_{\sigma(1)} + j_{\sigma(2)} = -2n_1, j_{\sigma(3)} + j_{\sigma(4)} = -2n_2, ..., j_{\sigma(2k-1)} + j_{\sigma(2k)} = -2n_k$.

Given a partition $\pi := n_k \geq n_{k-1} \geq \cdots \geq n_1$ of $-r$ with negative parts define $m(\pi) := x_{-2n_1-1}x_{-2n_2-1}\cdots x_{-2n_k-1}x_1^k$. Observe that $m(\pi) = m(\pi')$ implies $\pi = \pi'$. Let denote $\tilde{L}_{n_1}^{\mathring{\mathfrak{g}}}\tilde{L}_{n_2}^{\mathring{\mathfrak{g}}}...\tilde{L}_{n_k}^{\mathring{\mathfrak{g}}}v_{\Lambda_0} =: \tilde{L}_{\pi}^{\mathring{\mathfrak{g}}}v_{\Lambda_0}$ and for a monomial $f = x_{i_1}x_{i_2}x_{i_3}...x_{i_q}$ we define its dual by $B_f := B_{i_1}B_{i_2}B_{i_3}...B_{i_q} \in U(\mathfrak{s})$. Note that for a nonzero monomial f, $B_f(f) = c(f)$ is a nonzero element of $\mathbb{C}$. Now, let us define the order '$\prec$' on the monomials. For $i_1 \geq i_2 \geq \cdots \geq i_l$ and $j_1 \geq j_2 \geq \cdots \geq j_l$ a monomial $f = x_{i_1}x_{i_2}\cdots x_{i_l} \prec g = x_{j_1}x_{j_2}\cdots x_{j_l}$ iff there exists m where $1 \leq m \leq l$ such that $i_k = j_k\ \forall k < m$ and $i_m < j_m$. Let us state the following important lemma.

Lemma 4.4.3. $\overline{\tilde{L}_{\pi}^{\mathring{\mathfrak{g}}}v_{\Lambda_0}} = k(\pi)m(\pi)$ *+ a linear combination of monomials* ξ *of udeg 2k such that* $\xi \prec m(\pi)$ *, further* $k(\pi) \neq 0$.

We prove theorem 4.3.4 modulo the above lemma. Let $P(-r, k) = m$ and $\pi_1, \pi_2, ..., \pi_m$ be all the partitions of r into k parts such that $\pi_m \prec \pi_{m-1} \prec \cdots \prec \pi_1$. Assume that

$$c_1\overline{\tilde{L}_{\pi_1}v_{\Lambda_0}} + c_2\overline{\tilde{L}_{\pi_2}v_{\Lambda_0}} + \cdots + c_m\overline{\tilde{L}_{\pi_m}v_{\Lambda_0}} = 0$$

Then applying $B_{m(\pi_1)}$ to above equation, and using lemma 4.4.3 we have $c_1k(\pi_1)c(m(\pi_1)) = 0$ where $c(m(\pi_1)) \neq 0$ and $k(\pi_1) \neq 0$. This implies that $c_1 = 0$. Similarly applying $B_{m(\pi_2)}$ to the equation

$$c_2\overline{\tilde{L}_{\pi_2}v_{\Lambda_0}} + ... + c_m\overline{\tilde{L}_{\pi_m}v_{\Lambda_0}} = 0$$

we get $c_2 = 0$. Continuing in the same way we get $c_i = 0$ for $1 \leq i \leq m$. Now we prove the lemma 4.4.3

The monomials appearing in $\tilde{L}_{\pi}^{\mathring{\mathfrak{g}}} v_{\Lambda_0}$ are $x_{i_1} x_{i_2} ... x_{i_{2k}}$ such that $i_{\sigma(1)} + i_{\sigma(2)} = -2n_1, i_{\sigma(3)} + i_{\sigma(4)} = -2n_2, ..., i_{\sigma(2k-1)} + i_{\sigma(2k)} = -2n_k$. Let assume for contrary, let $\xi = x_{j_1} x_{j_2} ... x_{j_{2k}}$ where $j_1 \geq j_2 \geq ... \geq j_{2k} \geq 1$ such that $m(\pi) \prec \xi$ and ξ appears in $\tilde{L}_{\pi}^{\mathring{\mathfrak{g}}} v_{\Lambda_0}$. This implies that $j_1 \geq -2n_1 - 1$. If $j_1 > -2n_1 - 1$, then $j_1 + 1 > -2n_1$ which is not possible. So $j_1 = -2n_1 - 1$. By continuing this way we get $j_2 = -2n_2 - 1, ..., j_k = -2n_k - 1, j_{k+1} = 1, ..., j_{2k} = 1$. This is the desired contradiction.

Chapter 5

The t-string functions of $A_1^{(1)}$

Let $\mathfrak{g} = A_1^{(1)} = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{sl}_2 \oplus \mathbb{C}K \oplus \mathbb{C}d$. Let $\Lambda \in P_+$ be of level $m \geq 1$ (i.e., $\langle \Lambda, K \rangle = m$). Let $L(\Lambda)$ be the corresponding irreducible highest weight module. Kac and Peterson [18], have explicitly computed the string functions corresponding to the $\mathfrak{g}$-module $L(\Lambda)$, and shown them to be related to the product of Dedekind eta function and the Hecke indefinite modular forms. In this chapter, we give a t-analogue of this result, i.e., we give a formula for all t-string functions of the $\mathfrak{g}$-module $L(\Lambda)$.

For $\mathfrak{g} = A_1^{(1)}$, let $\mathfrak{h}$ be its Cartan subalgebra and let $\Delta \subseteq \mathfrak{h}^*$ be the set of roots. Let Δ_+ be the set of positive roots and $\{\alpha_0, \alpha_1\}$ be the set of simple roots of $\mathfrak{g}$. Let $Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1$ and P be the root and weight lattices of $\mathfrak{g}$. Let $\mathring{\mathfrak{g}} = \mathfrak{sl}_2$ and $\mathring{Q} = \mathbb{Z}\alpha_1$ and $\mathring{P} = \mathbb{Z}\frac{\alpha_1}{2}$ be the root and weight lattices of $\mathring{\mathfrak{g}}$. Let W be the Weyl group of $\mathfrak{g}$ generated by reflections r_{α_0} and r_{α_1}, which we denote by r_0 and r_1. Let $\mathring{W}$ be the subgroup generated by r_1.

Let $\rho \in \mathfrak{h}^*$ be a Weyl vector, which satisfies the relation $(\rho, \alpha_i) = 1$ for $i = 0, 1$. Define $\sigma \in GL(\mathfrak{h}^*)$ such that $\sigma(\alpha_0) = \alpha_1, \sigma(\alpha_1) = \alpha_0, \sigma(\rho) = \rho$. Note that σ leaves Q and Q + ρ invariant and normalises W, since $\sigma r_i \sigma^{-1} = r_{\sigma(\alpha_i)}$. Consider the group $W_\sigma := <\sigma> \ltimes W$. Note that $\Delta, Q, Q+\rho$ are invariant under W_σ. Let us consider a shifted action of W_σ on Q by $w.\alpha = w(\alpha+\rho)-\rho$. This induces an action of W_σ on functions on Q by $(w.f)(\alpha) = f(w^{-1}.\alpha)$, so an action of $\mathbb{C}[W_\sigma]$ on functions on Q. We note that $W_\sigma \simeq \mathring{W} \ltimes \mathring{P}$, the extended affine Weyl group.

Recall from chapter 2 that the Kostant partition function and t-Kostant partition function $\mathcal{P}(\beta)$ and $\mathcal{P}(\beta; t)$ are given by

$$\mathcal{P}(\beta) := [e^{-\beta}] \prod_{\alpha \in \Delta_+} \frac{1}{(1-e^{-\alpha})^{m_\alpha}}$$

and

$$\mathcal{P}(\beta; t) := [e^{-\beta}] \prod_{\alpha \in \Delta_+} \frac{1}{(1-te^{-\alpha})^{m_\alpha}} \quad .$$

We note that $\mathcal{P}(\beta), \mathcal{P}(\beta;t)$ are elements of $\mathcal{E}_t$, the set of all series of the form $\sum_{\lambda\in\mathfrak{h}^*} B_\lambda(t)\, e^\lambda$ $B_\lambda(t) \in \mathbb{C}[[t]]$ and $B_\lambda = 0$ outside the union of a finite number of sets of the form $D(\mu)$, $\mu \in \mathfrak{h}^*$.

Let $\Lambda \in P_+$ be of level $m \geq 1$. Let $L(\Lambda)$ denote the corresponding irreducible highest weight representation of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$, define the string function

$$c_\lambda^\Lambda(q) := q^{s_\Lambda(\lambda)} \sum_{s\geq 0} \mathrm{mult}_\Lambda(\lambda - s\delta)\, q^s \tag{5.0.1}$$

where

$$s_\Lambda(\lambda) = n_0(\Lambda - \lambda) + \frac{|\bar{\Lambda} + \bar{\rho}|^2}{2(m+2)} - \frac{|\bar{\lambda}|^2}{2m} - \frac{1}{8}\,. \tag{5.0.2}$$

Here, $\bar{\alpha}$ denotes the image of α under the projection from $\mathfrak{h}^*$ to $\overset{\circ}{\mathfrak{h}}{}^*$ and n_0 is the function on Q defined by $n_0(a_0\alpha_0 + a_1\alpha_1) := a_0 \;\; \forall\, a_0, a_1 \in \mathbb{Z}$. Note that for $\lambda \in \max(\Lambda)$ (chapter 2 § 2.8) $c_\lambda^\Lambda(q) = q^{s_\Lambda(\lambda)} a_\lambda^\Lambda(q)$.

5.1 Weyl group of $A_1^{(1)}$

Recall from chapter 2 that for an affine Lie algebra $\mathfrak{g}$ the Weyl group W is given by $W = \overset{\circ}{W} \ltimes T$, where T is the group consisting of the endomorphisms

$$t_\alpha(\lambda) = \lambda + \langle\lambda, K\rangle\alpha - ((\lambda,\alpha) + \frac{1}{2}|\alpha|^2\langle\lambda, K\rangle)\delta \tag{5.1.3}$$

where $\alpha \in M$ and $M = \nu(\mathbb{Z}(\overset{\circ}{W}.\theta^\vee))$. For $\mathfrak{g} = A_1^{(1)}$, $M = \overset{\circ}{Q}$ and so $T = \{t_{n\alpha_1} : n \in \mathbb{Z}\}$. So it follows that W is generated by r_1 and t_{α_1} . Recall that the extended affine Weyl group is given by $\widehat{W} = \overset{\circ}{W} \ltimes \widehat{T}$, where $\widehat{T}$ is a group consisting of the endomorphisms t_α where $\alpha \in \overset{\circ}{P}$. So $\widehat{W}$ is a group generated by r_1 and $t_{\frac{\alpha_1}{2}}$. Let us denote $\boldsymbol{\tau} := t_{\frac{\alpha_1}{2}}$. Then we have $\widehat{W} = \{\boldsymbol{\tau}^n, r_1\boldsymbol{\tau}^n : n \in \mathbb{Z}\}$ and $W = \{\boldsymbol{\tau}^{2n}, r_1\boldsymbol{\tau}^{2n} : n \in \mathbb{Z}\}$. Now we make a useful observation: $\boldsymbol{\tau} = \sigma r_1$. To prove this, let α_1, δ and ρ be a basis for $\mathfrak{h}^*$. We will show that $\boldsymbol{\tau}$ and σr_1 are equal on the chosen basis. For

$$\boldsymbol{\tau}(\alpha_1) = t_{\frac{\alpha_1}{2}}(\alpha_1) = \alpha_1 - \delta = -\alpha_0 = \sigma r_1(\alpha_1) \text{ by equation 5.1.3}$$

and

$$\boldsymbol{\tau}(\delta) = \delta = \sigma r_1(\delta)$$

and

$$\boldsymbol{\tau}(\rho) = \rho + \alpha_1 - \delta = \rho - \alpha_0$$

but

$$\sigma r_1(\rho) = \sigma(\rho - \alpha_1) = \sigma(\rho) - \alpha_0 = \rho - \alpha_0\,.$$

Now we briefly recall the term Hecke modular form.

5.2 Hecke indefinite modular form

We follow [18] closely. Let U be a two dimensional real vector space, M be a full rank lattice in U and let B' be an indefinite symmetric form on U such that $B'(\nu, \nu)$ is an even nonzero integer for all nonzero $\nu \in M$. Let $M^* := \{\nu' \in U : B'(\nu, \nu') \in \mathbb{Z} \;\; \forall\, \nu \in M\}$. Let G_0 be the subgroup of the identity component of the orthogonal group of (U, B') preserving M and fixing M^*/M pointwise. Fix a factorisation $B'(\nu, \nu) = l_1(\nu)l_2(\nu)$, where l_1 and l_2 are real-linear, and set $\text{sign}(\nu) := \text{sign}\, l_1(\nu)$ for $l_1(\nu) \neq 0$. For $\mu \in M^*$, set

$$\theta_{M,\mu} := \sum_{\substack{\nu \in L+\mu \\ B'(\nu,\nu)>0 \\ \nu \bmod G_0}} \text{sign}(\nu) q^{\frac{1}{2}B'(\nu,\nu)}.$$

The $\theta_{M,\mu}$ is called a Hecke indefinite modular form. It is a cusp form of weight 1 (with $q = e^{2\pi i \tau}$, where $\tau \in$ upper half plane). For more details see [18].

Let $U := \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_1$ and $M := \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_1$. We identify M with $\mathbb{Z}^2$ and U with $\mathbb{R}^2$, let $m \geq 1$ and define $B(x, y) := 2(m + 2)x^2 - 2my^2$ be a quadratic form on U. Let the corresponding indefinite symmetric bilinear form be B'. Note that $B \neq 0$ on $M - \{0\}$. The dual lattice of M with respect to B is $M^* = \frac{1}{2(m+2)}\mathbb{Z} \oplus \frac{1}{2m}\mathbb{Z}$. Let a be the element of the identity component $SO_0(U)$ of the orthogonal group of (U, B) given by $a(x, y) = ((m+1)x+my, (m+2)x+(m+1)y)$. Then a generates the subgroup G_0' of $SO_0(U)$ preserving M, and a^2 generates the subgroup G_0 of G_0' fixing M^*/M pointwise. Let us define an element $J : U \longrightarrow U$ by $J(x, y) := (-x, y)$; we note that J normalises G_0. Define the group $G := < J > \ltimes G_0$ and $G' := < J > \ltimes G_0'$. Let $U^+ := \{u \in U : B(u) > 0\}$. Then it is easy to see that $F_0 := \{(x, y) \in \mathbb{R}^2 : -|x| < y \leq |x|\}$ is a fundamental domain for G_0' on U^+, and $F_0 \bigcup aF_0$ is a fundamental domain for G_0 on U^+. Set $F := \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \text{ or } 0 > y > x\}$. Then clearly $F_0 = F \cup J(F)$ and F is a fundamental domain for G' on U^+.

Let $\Lambda \in P_+$ and $\Lambda(K) = m \geq 1$. Let $L(\Lambda)$ denote the irreducible $\mathfrak{g}$-module with highest weight Λ. Let λ be the maximal dominant weight of $L(\Lambda)$ such that $\lambda(\alpha_0{}^\vee) = N_0$ and $\lambda(\alpha_1{}^\vee) = N_1$. As λ is maximal dominant we have $\Lambda - \lambda = l\alpha_i$ where $i = 0$ or 1 and $l \in \mathbb{Z}_{\geq 0}$. But it is easy to see from definitions that $c^\Lambda_\lambda(q) = c^{\sigma(\Lambda)}_{\sigma(\lambda)}(q)$, so we assume without loss of generality that $\Lambda - \lambda = l\alpha_1$. Now we state a theorem of Kac and Peterson.

Theorem 5.2.1 (Kac-Peterson). *Let $\mathfrak{g}$ be of type $A_1^{(1)}$. Let $\Lambda \in P_+$, $\Lambda(K) = m$, and let $\lambda \in P_+$ be a maximal weight of $L(\Lambda)$. Then*

$$\eta(q)^3 c^\Lambda_\lambda(q) = \theta_{M,(\tilde{A};\tilde{B})}$$

is a Hecke indefinite modular form, where $\tilde{A}\alpha_1 = (m + 2)^{-1}(\bar{\Lambda} + \bar{\rho})$ and $\tilde{B}\alpha_1 = m^{-1}\bar{\lambda}$, and η is the Dedekind eta function.

5.3 $\mathcal{P}'$ and its t-analogue

Our aim in this chapter is to derive an expression for $c_\lambda^\Lambda(t;q)$. In the proof of the theorem 5.2.1 Kac and Peterson use a function $\mathcal{P}'$ defined on Q by $\mathcal{P}'(\beta) := (1+r_1).\mathcal{P}(\beta)$. The key fact about $\mathcal{P}'$ is it is easier to calculate than $\mathcal{P}$.

Definition 5.3.1. *A t-analogue of function $\mathcal{P}'$ is defined by $\mathcal{P}'(\beta;t) := (1+tr_1).\mathcal{P}(\beta;t)$ i.e., $\mathcal{P}'(\beta;t) = \mathcal{P}(\beta;t) + t\mathcal{P}(r_1.\beta;t)$ for all $\beta \in Q$*

Let us define

$$\mathbf{P}_\beta(t;q) := \sum_{n\geq 0} \mathcal{P}(\beta + n\delta)q^n$$

and

$$\mathbf{P}'_\beta(t;q) := \sum_{n\geq 0} \mathcal{P}'(\beta + n\delta)q^n.$$

We recall the constant term map $\mathrm{ct}(\cdot)$ from chapter 3, which is defined on the formal sums $\sum_{\alpha\in Q} c_\lambda e^\lambda$ by $\mathrm{ct}(\sum_{\alpha\in Q} c_\lambda e^\lambda) := \sum_{n\in\mathbb{Z}} c_{n\delta}e^{n\delta}$. We will let $q = e^{-\delta}$ throughout this chapter.

Recall the fact that Δ_+ for $A_1^{(1)}$ is given by $\Delta_+ = \{n\alpha_0+n\alpha_1, n\alpha_0+(n-1)\alpha_1, (n-1)\alpha_0 + n\alpha_1 : n \in \mathbb{Z}_{>0}\}$. As $Q = \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1 = \{b\alpha_1 + n\delta : b, n \in \mathbb{Z}\}$, so $Q_+ \subsetneq \{b\alpha_1 + n\delta : b \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}\}$; for $\beta \in Q$ we write $\beta = b(\beta)\alpha_1 + d(\beta)\delta$, where $b(\beta), d(\beta) \in \mathbb{Z}$. We also note that for $\beta \notin Q_+$ $\mathcal{P}(\beta;t) = 0$. Using this, we make a useful observation: For $\beta \in Q$ such that $\beta = b(\beta)\alpha_1 + d(\beta)\delta$ and $d(\beta) \leq 0$, we have

$$\mathbf{P}_\beta(t;q) = q^{-d(\beta)}\,\mathbf{P}_{b(\beta)\alpha_1}(t;q). \tag{5.3.4}$$

and

$$\mathbf{P}'_\beta(t;q) = q^{-d(\beta)}\,\mathbf{P}'_{b(\beta)\alpha_1}(t;q). \tag{5.3.5}$$

Let us define

$$\Gamma_t := \prod_{\alpha\in\Delta_+} \frac{1}{1-te^{-\alpha}} = \sum_{\beta\in Q^+} \mathcal{P}(\beta;t)e^{-\beta}$$

Note that $\Gamma_t \in \mathcal{E}_t$. Thus we can write

$$\Gamma_t = \frac{\xi_t}{(1-te^{-\alpha_1})}$$

where

$$\xi_t = \frac{1}{\prod_{n\geq 1}(1-tq^n)(1-tq^ne^{-\alpha_1})(1-tq^ne^{\alpha_1})}\ .$$

5.4 Expressions for $\mathbf{P}_{b\alpha_1}$ and $\mathbf{P}'_{b\alpha_1}$

In this section we give an expression for $\mathbf{P}_{b\alpha_1}(t;q)$ and $\mathbf{P}'_{b\alpha_1}(t;q)$ in terms of the constant term map. For $\xi = \sum_{\lambda\in\mathfrak{h}^*} c_\lambda(t)e^\lambda$, define $\bar{\xi} := \sum_{\lambda\in\mathfrak{h}^*} c_\lambda(t)e^{r_1(\lambda)}$. Note that $\mathrm{ct}(\xi) = \mathrm{ct}(\bar{\xi})$

By definition $\Gamma_t = \sum_{\beta\in Q^+} \mathcal{P}(\beta;t)e^{-\beta}$, as $Q_+ \subsetneq \{b\alpha_1 + n\delta : b \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}\}$ we write

$$\Gamma_t = \sum_{b\in\mathbb{Z}, n\in\mathbb{Z}_{\geq 0}} \mathcal{P}(b\alpha_1 + n\delta;t)e^{-b\alpha_1}e^{-n\delta} \tag{5.4.6}$$

$$= \sum_{b\in\mathbb{Z}} e^{-b\alpha_1}(\sum_{n\geq 0} \mathcal{P}(b\alpha_1 + n\delta;t)q^n) \tag{5.4.7}$$

$$= \sum_{b\in\mathbb{Z}} e^{-b\alpha_1}\mathbf{P}_{b\alpha_1}(t;q) \tag{5.4.8}$$

$$\text{so} \quad \mathrm{ct}(\Gamma_t) = \mathbf{P}_0(t;q) = \sum_{j\geq 0} \mathcal{P}(j\delta;t)q^j \tag{5.4.9}$$

$$\text{therefore} \quad \mathrm{ct}(\Gamma_t e^{b\alpha_1}) = \mathbf{P}_{b\alpha_1}(t;q)\ . \tag{5.4.10}$$

We write for $\beta \in Q$, $\mathbf{P}_\beta = \mathbf{P}_\beta(t;q)$ and $\mathbf{P}'_\beta = \mathbf{P}'_\beta(t;q)$. Now consider

$$\mathcal{P}'(\beta;t) = \mathcal{P}(\beta;t) + t\mathcal{P}(r_1.\beta;t) \tag{5.4.11}$$

Using equation 5.4.11 one gets

$$\mathbf{P}'_{b\alpha_1} = \mathbf{P}_{b\alpha_1} + t\mathbf{P}_{(-b-1)\alpha_1} \tag{5.4.12}$$

$$= \mathrm{ct}(\Gamma_t\, e^{b\alpha_1}) + t\,\mathrm{ct}(\Gamma_t\, e^{(-b-1)\alpha_1}) \tag{5.4.13}$$

$$= \mathrm{ct}(\Gamma_t\, e^{b\alpha_1}) + t\,\mathrm{ct}(\overline{\Gamma_t}\, e^{(b+1)\alpha_1}) \tag{5.4.14}$$

$$= \mathrm{ct}\left(\xi_t e^{b\alpha_1}\left(\frac{1}{1-te^{-\alpha_1}} + \frac{te^{\alpha_1}}{1-te^{\alpha_1}}\right)\right) \tag{5.4.15}$$

$$= \mathrm{ct}(e^{b\alpha_1}\xi_t P_t) \tag{5.4.16}$$

$$\text{where,}\ P_t = \frac{1}{1-te^{-\alpha_1}} + \frac{te^{\alpha_1}}{1-te^{\alpha_1}} = \sum_{n\in\mathbb{Z}} t^{|n|}e^{n\alpha_1} \tag{5.4.17}$$

$$= \sum_{n\in\mathbb{Z}} t^{|n|}e^{n\alpha_1}\ \text{(the classical Poisson kernel of the unit disc).} \tag{5.4.18}$$

We have proved the following lemma:

Lemma 5.4.1. *For $b \in \mathbb{Z}$, we have*

1. $\mathbf{P}_{b\alpha_1} = \mathrm{ct}(e^{b\alpha_1}\,\Gamma_t)$
2. $\mathbf{P}'_{b\alpha_1} = \mathrm{ct}(e^{b\alpha_1}\xi_t P_t)$, *where* $P_t = \sum_{n\in\mathbb{Z}} t^{|n|}e^{n\alpha_1}$

5.4.1 An expression for $\mathcal{P}(\beta;t)$

Recall that $\sigma \in GL(\mathfrak{h}^*)$ is defined as $\sigma(\alpha_0)=\alpha_1, \sigma(\alpha_1)=\alpha_0, \sigma(\rho)=\rho$. Note that

$$\mathcal{P}(\beta;t) = \mathcal{P}(\sigma.\beta;t) \tag{5.4.19}$$

$$\text{hence } \mathcal{P}(\beta;t) = \mathcal{P}'(\beta;t) - t\,\mathcal{P}(r_1.\beta;t) \tag{5.4.20}$$

$$\mathcal{P}(r_1.\beta;t) = \mathcal{P}(\sigma r_1.\beta;t) = \mathcal{P}'(\sigma r_1.\beta;t) - t\mathcal{P}(r_1.\sigma r_1.\beta;t) \tag{5.4.21}$$

$$\Rightarrow \mathcal{P}(\sigma r_1.\beta;t) = \mathcal{P}'(\sigma r_1.\beta;t) - t\mathcal{P}((\sigma r_1)^2.\beta;t) \tag{5.4.22}$$

Continuing this procedure we get

$$\mathcal{P}(\beta;t) = \sum_{j\geq 0}(-1)^j\, t^j \mathcal{P}'(\boldsymbol{\tau}^j\cdot\beta;t), \text{ where } \boldsymbol{\tau}=\sigma r_1 \tag{5.4.23}$$

but

$$\boldsymbol{\tau}^j\cdot\beta = \boldsymbol{\tau}^j(\beta+\rho)-\rho = \beta + s\alpha_1 - (\frac{s}{2}(\beta,\alpha_1)+\frac{s(s+1)}{2})\delta$$

as $\mathcal{P}(\beta;t)=0$ for $\beta\notin Q_+$, we note that the sum in equation 5.4.23 is actually a finite sum. Similarly if we replace β by $r_1.\beta$ in the equation 5.4.20, we get

$$\mathcal{P}(r_1.\beta;t) = \mathcal{P}'(r_1.\beta;t) - t\,\mathcal{P}(\beta;t) \tag{5.4.24}$$

$$\Rightarrow \mathcal{P}(\beta;t) = \frac{1}{t}\left[\mathcal{P}'(r_1.\beta;t) - \mathcal{P}(\sigma r_1.\beta;t)\right] . \tag{5.4.25}$$

Using the same procedure as above we get an another expression for $\mathcal{P}(\beta;t)$ in terms of $\mathcal{P}'(\beta;t)$:

$$\mathcal{P}(\beta;t) = \sum_{j\geq 0}(-1)^j\, t^{-(j+1)}\mathcal{P}'(r_1\cdot\boldsymbol{\tau}^j\cdot\beta;t) \tag{5.4.26}$$

by the same reasoning as before, we note that the sum in equation 5.4.26 is also a finite sum. It is easy to see that $r_1\boldsymbol{\tau}^j = \boldsymbol{\tau}^{-j}{r_1}^{-1}$. Using this in equation 5.4.26 we get

$$\mathcal{P}(\beta;t) = \sum_{j\geq 0}(-1)^j t^{-(j+1)}\mathcal{P}'(\boldsymbol{\tau}^{-j}\cdot {r_1}^{-1}\cdot\beta;t) \tag{5.4.27}$$

$$\mathcal{P}(\beta;t) = \mathcal{P}(\sigma.\beta;t) \tag{5.4.28}$$

$$\Rightarrow \mathcal{P}(\beta;t) = \sum_{j\geq 0}(-1)^j t^{-(j+1)}\mathcal{P}'(\boldsymbol{\tau}^{-j}\cdot {r_1}^{-1}\cdot\sigma^{-1}\beta;t) \tag{5.4.29}$$

$$\Rightarrow \mathcal{P}(\beta;t) = \sum_{j\geq 0}(-1)^j t^{-(j+1)}\mathcal{P}'(\boldsymbol{\tau}^{-(j+1)}\beta;t) \tag{5.4.30}$$

$$\text{letting } s=j+1 \text{ we get} \tag{5.4.31}$$

$$\mathcal{P}(\beta;t) = -\sum_{s<0}(-1)^s\, t^s\,\mathcal{P}'(\boldsymbol{\tau}^s\cdot\beta;t) . \tag{5.4.32}$$

We have proved the following lemma:

Lemma 5.4.2. *Let* $\beta \in Q$ *then*

1. $\mathcal{P}(\beta;t) = \sum_{j\geq 0}(-1)^j\, t^j \mathcal{P}'(\boldsymbol{\tau}^j \cdot \beta; t)$

2. $\mathcal{P}(\beta;t) = -\sum_{j<0}(-1)^j t^j \mathcal{P}'(\boldsymbol{\tau}^j \cdot \beta; t)$.

$$\text{So we have } \mathcal{P}(\beta;t) = \sum_{r\geq 0}(-1)^r\, t^r\, \mathcal{P}'(\boldsymbol{\tau}^r \cdot \beta; t) = -\sum_{r<0}(-1)^r t^r \mathcal{P}'(\boldsymbol{\tau}^r \cdot \beta; t)\ . \tag{5.4.33}$$

We have proved the following corollary:

Corollary 5.4.3. *For* $\beta \in Q$

$$\sum_{r\in\mathbb{Z}}(-1)^r\, t^r \mathcal{P}'(\boldsymbol{\tau}^r \cdot \beta; t) = 0$$

Lemma 5.4.2 and corollary 5.4.3 are t-version of [[18],Lemma 5.8].

5.5 An expression for $\mathbf{P}_{b\alpha_1}$

In this section we give an expression for $\mathbf{P}_{b\alpha_1}(t;q)$ in terms of $\mathbf{P}'_{b\alpha_1}(t;q)$.

$$\text{Consider } \mathbf{P}_\beta(t;q) = \sum_{n\geq 0}\mathcal{P}(\beta+n\delta)q^n \tag{5.5.34}$$

$$= \sum_{n\geq 0}\sum_{s\geq 0}(-1)^s t^s \mathcal{P}'(\boldsymbol{\tau}^s.(\beta+n\delta); t) \tag{5.5.35}$$

$$\text{but } \boldsymbol{\tau}^s \cdot (\beta+n\delta) = \boldsymbol{\tau}^s(\beta+n\delta+\rho) - \rho \tag{5.5.36}$$

$$= \boldsymbol{\tau}^s(\beta+\rho) - \rho + n\delta = \boldsymbol{\tau}^s \cdot \beta + n\delta \tag{5.5.37}$$

$$\text{so for any } \beta \in Q, \quad \mathbf{P}_\beta = \sum_{s\geq 0}(-1)^s\, t^s\, \mathbf{P}'_{\boldsymbol{\tau}^s\cdot\beta} = -\sum_{s<0}(-1)^s\, t^s\, \mathbf{P}'_{\boldsymbol{\tau}^s\cdot\beta} \tag{5.5.38}$$

$$\text{but } \boldsymbol{\tau}^s(\beta+\rho) - \rho + n\delta = \beta + s\alpha_1 - (\frac{s}{2}(\beta,\alpha_1) + \frac{s(s+1)}{2})\delta + n\delta \tag{5.5.39}$$

$$\text{so } \mathbf{P}_\beta(t;q) = \sum_{s\geq 0}(-1)^s\, t^s\, \mathbf{P}'_{\beta+s\alpha_1-(\frac{s}{2}(\beta,\alpha_1)+\frac{s(s+1)}{2})\delta} \tag{5.5.40}$$

$$\text{for } \beta = b\alpha_1, \boldsymbol{\tau}^s(b\alpha_1+n\delta) = (b+s)\alpha_1 - (sb + \frac{s(s+1)}{2})\delta + n\delta\ . \tag{5.5.41}$$

We see that from equations 5.3.4 and 5.3.5, for $r \in \mathbb{Z}_{\geq 0}$ we have

$$\mathbf{P}_{r\alpha_1}(q;t) = \sum_{s\geq 0}(-1)^s\, t^s\, \mathbf{P}'_{(r+s)\alpha_1} q^{sr+\frac{s(s+1)}{2}} = \sum_{s\geq 0}(-1)^s t^s \mathbf{P}'_{b(\boldsymbol{\tau}^s\cdot(r\alpha_1))} q^{-d(\boldsymbol{\tau}^s\cdot r\alpha_1)} \tag{5.5.42}$$

Similarly for $r < 0 \in \mathbb{Z}$, we use the expression 5.4.32 of $\mathcal{P}(\beta; t)$ to get

$$\mathbf{P}_{r\alpha_1}(q;t) = -\sum_{s<0}(-1)^s\, t^s\, \mathbf{P}'_{(r+s)\alpha_1} q^{sr+\frac{s(s+1)}{2}} = -\sum_{s<0}(-1)^s t^s \mathbf{P}'_{b(\boldsymbol{\tau}^s\cdot(r\alpha_1))} q^{-d(\boldsymbol{\tau}^s\cdot r\alpha_1)} \tag{5.5.43}$$

We have proved the following lemma:

Lemma 5.5.1. *1. For $b \geq 0$, we have*

$$\mathbf{P}_{b\alpha_1}(q;t) = \sum_{s\geq 0}(-1)^s\, t^s\, q^{sb+\frac{s(s+1)}{2}}\, \mathbf{P}'_{(b+s)\alpha_1}$$

2. For $b < 0$, we have

$$\mathbf{P}_{b\alpha_1}(q;t) = -\sum_{s<0}(-1)^s\, t^s\, q^{sb+\frac{s(s+1)}{2}} \mathbf{P}'_{(b+s)\alpha_1}\ .$$

Thus for any $b \in \mathbb{Z}$, we have

$$\mathbf{P}_{b\alpha_1}(q;t) = \sum_{s\in\mathbb{Z}}(-1)^s\, I(b,s)\, t^s\, q^{sb+\frac{s(s+1)}{2}}\, \mathbf{P}'_{(b+s)\alpha_1} \tag{5.5.44}$$

where for $b \geq 0$

$$I(b,j) = \begin{cases} 1 & j \geq 0 \\ 0 & j < 0 \end{cases}$$

and for $b < 0$

$$I(b,j) = \begin{cases} 0 & j \geq 0 \\ -1 & j < 0 \end{cases}$$

5.6 An expression for $c^\Lambda_\lambda(t;q)$

By the definition

$$c^\Lambda_\lambda(q) := q^{s_\Lambda(\lambda)} \sum_{j\geq 0} \mathrm{mult}_\Lambda(\lambda - j\delta)\, q^j$$

we note that for $\lambda \in \max(\Lambda)$ we have

$$c^\Lambda_\lambda(q) := q^{s_\Lambda(\lambda)} a^\Lambda_\lambda(q)\ .$$

Definition 5.6.1. *We define*

$$c^\Lambda_\lambda(t;q) := q^{s_\Lambda(\lambda)} a^\Lambda_\lambda(t;q) = q^{s_\Lambda(\lambda)} \sum_{n\geq 0} K^\Lambda_{\lambda-n\delta}(t)\, q^n$$

$$\text{So } c_\lambda^\Lambda(t;q) = q^{s_\Lambda(\lambda)} \sum_{w\in W}\sum_{n\geq 0}(-1)^{l(w)}\mathcal{P}((w(\Lambda+\rho)-(\lambda-n\delta+\rho));t) \tag{5.6.45}$$

$$= q^{s_\Lambda(\lambda)} \sum_{w\in W}(-1)^{l(w)}\mathbf{P}_{w(\Lambda+\rho)-(\lambda+\rho)} \tag{5.6.46}$$

We have for $w=\boldsymbol{\tau}^{2n}$

$$w(\Lambda+\rho)-(\lambda+\rho) = \Lambda-\lambda+(m+2)n\alpha_1-[n(N_1+1)+n^2(m+2)]\delta \tag{5.6.47}$$

For $w=\boldsymbol{\tau}^{2n}r_1$

$$w(\Lambda+\rho)-(\lambda+\rho) = \Lambda-\lambda+((m+2)n-(N_1+1))\alpha_1-[n(N_1+1)+n^2(m+2)]\delta \tag{5.6.48}$$

Let us write $w(\Lambda+\rho)-(\lambda+\rho)=b(w)\alpha_1+d(w)\delta$, then in both cases we see that $d(w)\leq 0$; For $n\geq 0$ it is trivial and for $n<0$ it follows from the fact that $N_1=\langle\lambda,\alpha_1^\vee\rangle\leq m=\langle\lambda,K\rangle$. So by 5.3.4 we have

$$c_\lambda^\Lambda(t;q) = q^{s_\Lambda(\lambda)} \sum_{w\in W}(-1)^{l(w)}\mathbf{P}_{b(w)\alpha_1}q^{-d(w)} \tag{5.6.49}$$

Now in order to use Lemma 5.5.1 we see that for $w=\boldsymbol{\tau}^{2n}, b(w)\geq 0$ for $n\geq 0$ and $b(w)<0$ for $n<0$, and $w=\boldsymbol{\tau}^{2n}r_1, b(w)>0$ for $n>0$ and $b(w)<0$ for $n\leq 0$. So using Lemma 5.5.1 in 5.6.49 we get

$$c_\lambda^\Lambda(t;q) = q^{s_\Lambda(\lambda)} \sum_{(w,j)\in W\times\mathbb{Z}}(-1)^{l(w)}(-1)^j\, I(b(w),j)\, q^{jb(w)+\frac{j(j+1)}{2}-d(w)}\,\mathcal{P}'_{b(w)+j} \tag{5.6.50}$$

Recall that the qudratic form $B(x,y):=2(m+2)x^2-2my^2$ and the corresponding indefinite symmetric bilinear form is B'. Now we state a generalised version of a lemma due to Kac-Peterson [18]:

Lemma 5.6.2.

$$-n_0(t_\nu(w(\Lambda+\rho)-\lambda)-\rho) = \frac{1}{2}B\left(\nu+\frac{\overline{w(\Lambda+\rho)}}{m+2}, \nu+\frac{\bar\lambda}{m}\right)-\left(\frac{|\overline{\Lambda+\rho}|^2}{2(m+2)}-\frac{|\bar\lambda|^2}{2m}\right)-n_0(\Lambda-\lambda) \tag{5.6.51}$$

where $\nu\in M$ and $w\in W$.

Proof. Case 1 : Let $w=t_{n\alpha_1}$. We rewrite the left-hand side of equation 5.6.51 as

$$n_0((\Lambda+\rho)-t_{\nu+n\alpha_1}(\Lambda+\rho))-n_0(\lambda-t_\nu(\lambda))-n_0(\Lambda-\lambda)$$

which is equal to

$$(\Lambda+\rho,\nu+n\alpha_1)+\frac{1}{2}|\nu+n\alpha_1|^2\langle\Lambda+\rho,K\rangle+(\lambda,\nu)+\frac{1}{2}|\nu|^2\langle\lambda,K\rangle-n_0(\Lambda-\lambda) \tag{5.6.52}$$

On the other hand the right hand side of equation 5.6.51 equal to

$$\frac{1}{2}B\left(\nu+n\alpha_1+\frac{\overline{\Lambda+\rho}}{m+2},\nu+\frac{\bar{\lambda}}{m}\right)-\left(\frac{|\overline{\Lambda+\rho}|^2}{2(m+2)}-\frac{|\bar{\lambda}|^2}{2m}\right)-n_0(\Lambda-\lambda)$$

which becomes by definition of B

$$\frac{1}{2}\left[(m+2)\left(\nu+n\alpha_1+\frac{\overline{\Lambda+\rho}}{m+2},\nu+n\alpha_1+\frac{\overline{\Lambda+\rho}}{m+2}\right)+m\left(\nu+\frac{\bar{\lambda}}{m},\nu+\frac{\bar{\lambda}}{m}\right)\right]-\left(\frac{|\overline{\Lambda+\rho}|^2}{2(m+2)}-\frac{|\bar{\lambda}|^2}{2m}\right)-n_0(\Lambda-\lambda)$$

which is equal to

$$(\overline{\Lambda+\rho},\nu+n\alpha_1)+\frac{1}{2}|\nu+n\alpha_1|^2\langle\Lambda+\rho,K\rangle+(\overline{\lambda},\nu)+\frac{1}{2}|\nu|^2\langle\lambda,K\rangle-n_0\ . \tag{5.6.53}$$

But as for $\nu\in\mathbb{Z}\alpha_1$ and for $f\in\mathfrak{h}^*$ we have $\langle f,\nu\rangle=\langle\overline{f},\nu\rangle$ since $\overline{f}=\frac{\langle f,\alpha_1^\vee\rangle}{2}\alpha_1$. So equations 5.6.52 and 5.6.53 are the same and we are done in this case.
Case 2: Proof for $w=r_1t_{n\alpha_1}$ follows similarly by using the fact that $B(-\nu,\nu')=B(\nu,\nu')$. □

But

$$-n_0(t_{\frac{j}{2}\alpha_1}(w(\Lambda+\rho)-\lambda)-\rho)=-d(w)+jb(w)+\frac{j(j+1)}{2}\quad\forall\, w\in W \tag{5.6.54}$$

Using 5.0.2, 5.6.51 and 5.6.54 in 5.6.49 we get

$$c^\Lambda_\lambda(t;q)=q^{\frac{1}{8}}\sum_{(w,j)\in W\times\mathbb{Z}}(-1)^{l(w)}(-1)^j\,I(b(w),j)\,\mathcal{P}'_{(b(w)+j)\alpha_1}\,q^{\frac{1}{2}B(\frac{j}{2}\alpha_1+\frac{\overline{w(\Lambda+\rho)}}{m+2},\frac{j}{2}\alpha_1+\frac{\bar{\lambda}}{m})}\ . \tag{5.6.55}$$

Now we define a map $\phi_1: W\times\mathbb{Z}\longrightarrow\mathbb{R}\alpha_1\oplus\mathbb{R}\alpha_1$ as follows

$$(w,j)\longrightarrow\left(\frac{j}{2}\alpha_1+\frac{\overline{w(\Lambda+\rho)}}{m+2},\frac{j}{2}\alpha_1+\frac{\bar{\lambda}}{m}\right)$$

Let $\tilde{A}:=\frac{(\Lambda+\rho,\alpha_1)}{2(m+2)}$ and $\tilde{B}:=\frac{(\lambda,\alpha_1)}{2m}$. We will identify $\mathbb{R}\alpha_1\oplus\mathbb{R}\alpha_1$ with $\mathbb{R}^2$ and $\mathbb{Z}\alpha_1\oplus\mathbb{Z}\alpha_1$ with $\mathbb{Z}^2$. We state and prove some properties of ϕ_1.

Proposition 5.6.3. *1. ϕ_1 is 1-1.*

2. $Im(\phi_1)\subset M^*$.

3. $Im(\phi_1)=\amalg_{i=1}^4 L_i$ where $L_1=(\tilde{A},\tilde{B})+\mathbb{Z}^2, L_2=(\tilde{A}+\frac{1}{2},\tilde{B}+\frac{1}{2})+\mathbb{Z}^2$, $L_3=(-\tilde{A},\tilde{B})+\mathbb{Z}^2$ and $L_4=(-\tilde{A}-\frac{1}{2},\tilde{B}+\frac{1}{2})+\mathbb{Z}^2$ respectively.

Proof. For 1 assume contrary: $\phi_1(w_1,j_1)=\phi_1(w_2,j_2)$ for some $w_1,w_2\in W$ and $j_1,j_2\in\mathbb{Z}$. Let us consider case 1) $w_1=t_{n\alpha_1}$ and $w_2=t_{s\alpha_1}$. Then

$$\left(\frac{j_1}{2}\alpha_1+\frac{\overline{t_{n\alpha_1}(\Lambda+\rho)}}{m+2},\frac{j_1}{2}\alpha_1+\frac{\bar{\lambda}}{m}\right)=\left(\frac{j_2}{2}\alpha_1+\frac{\overline{t_{s\alpha_1}(\Lambda+\rho)}}{m+2},\frac{j_2}{2}\alpha_1+\frac{\bar{\lambda}}{m}\right)$$

This implies that $j_1 = j_2$ and

$$\frac{\overline{t_{n\alpha_1(\Lambda+\rho)}}}{m+2} = \frac{\overline{t_{m\alpha_1(\Lambda+\rho)}}}{m+2}$$

$$\frac{\overline{t_{n\alpha_1(\Lambda+\rho)}}}{m+2} = \frac{(\Lambda+\rho,\alpha_1)}{2(m+2)} + n = \frac{(\Lambda+\rho,\alpha_1)}{2(m+2)} + s = \frac{\overline{t_{s\alpha_1(\Lambda+\rho)}}}{m+2}$$

so $s = n$, therefore $w_1 = w_2$. Case 2 : $w_1 = t_{n\alpha_1} r_1$ and $w_2 = t_{s\alpha_1} r_1$ as in case 1) we get $j_1 = j_2$ and

$$\frac{\overline{t_{n\alpha_1} r_1(\Lambda+\rho)}}{m+2} = \frac{\overline{t_{s\alpha_1} r_1(\Lambda+\rho)}}{m+2}$$

So we get

$$\frac{\overline{t_{n\alpha_1} r_1(\Lambda+\rho)}}{m+2} = -\frac{(\Lambda+\rho,\alpha_1)}{2(m+2)} + n = -\frac{(\Lambda+\rho,\alpha_1)}{2(m+2)} + s = \frac{\overline{t_{s\alpha_1} s(\Lambda+\rho)}}{m+2}$$

So we get $s = n$ and $w_1 = w_2$.
Case 3 : $w_1 = t_{n\alpha_1} r_1$ and $w_2 = t_{s\alpha_1}$.
As in previous cases we get

$$\frac{\overline{t_{n\alpha_1} s(\Lambda+\rho)}}{m+2} = \frac{\overline{t_{s\alpha_1(\Lambda+\rho)}}}{m+2}$$

But

$$\frac{\overline{t_{n\alpha_1} r_1(\Lambda+\rho)}}{m+2} = \frac{\overline{r_1 t_{-n\alpha_1}(\Lambda+\rho)}}{m+2} = -\frac{(\Lambda+\rho,\alpha_1)}{2(m+2)} - n = \frac{(\Lambda+\rho,\alpha_1)}{2(m+2)} + s = \frac{\overline{t_{s\alpha_1(\Lambda+\rho)}}}{m+2}$$

So we get

$$-2\frac{(\Lambda+\rho,\alpha_1)}{2(m+2)} = s - n$$

which is impossible.
2 follows from the definition of ϕ_1 and M^*.
For 3, consider $\phi_1(W\times\mathbb{Z}) = \phi_1(T\times 2\mathbb{Z}) \cup \phi_1(T\times(2\mathbb{Z}+1)) \cup \phi_1(Tr_1\times 2\mathbb{Z}) \cup \phi_1(Tr_1\times(2\mathbb{Z}+1))$. Now by the calculation in 1 we see that $\phi_1(T\times 2\mathbb{Z}) = L_1$, $\phi_1(T\times(2\mathbb{Z}+1)) = L_2$, $\phi_1(Tr_1\times 2\mathbb{Z}) = L_3$ and $\phi_1(Tr_1\times(2\mathbb{Z}+1)) = L_4$. □

Now we will be interested in the set of points of image ϕ_1 for which $I(b(w), j) \neq 0$. As $\Lambda - \lambda = l\alpha_1$, for $l \geq 0$. If $l > 0$, we see that $\frac{1}{2} > \tilde{A} \geq \tilde{B}$. For

$$\tilde{A} - \tilde{B} = \frac{\langle \Lambda+\rho, \alpha_1^\vee\rangle}{2(m+2)} - \frac{\langle \lambda, \alpha_1^\vee\rangle}{2m}$$

$$= \frac{m(2l) + m - 2\langle\lambda,\alpha_1^\vee\rangle}{2(m+2)(m)} > 0$$

Now for $l = 0$ we have $\Lambda = \lambda$ then we have $\tilde{A} < \tilde{B}$ if $\Lambda(\alpha_0^\vee) < \Lambda(\alpha_1^\vee)$. In this case we consider $\tilde{\Lambda} := \sigma(\Lambda)$. So $\tilde{B} < \tilde{A}$ implies that $A' := \frac{\overline{\sigma(\Lambda)+\rho}}{(m+2)} < B' := \frac{\overline{\sigma(\Lambda)}}{m}$. But as $c^\Lambda_\Lambda(t;q) = c^{\sigma(\Lambda)}_{\sigma(\Lambda)}(t;q)$, without loss of generality we will assume that $\frac{1}{2} > \tilde{A} \geq \tilde{B}$.

Recall that for $b(w) \geq 0$

$$I(b(w), j) = \begin{cases} 1 & j \geq 0 \\ 0 & j < 0 \end{cases}$$

and for $b(w) < 0$

$$I(b(w), j) = \begin{cases} 0 & j \geq 0 \\ -1 & j < 0 \end{cases}$$

For $b(w) \geq 0$

case 1) $w = \boldsymbol{\tau}^{2n} = t_{n\alpha_1}$: As $b(w) \geq 0$ we see that $n \geq 0$. So

$$(x, y) := \phi_1(w, j) = (\frac{j_1}{2}\alpha_1 + \frac{\overline{t_{n\alpha_1}(\Lambda + \rho)}}{m+2}, \frac{j_1}{2}\alpha_1 + \frac{\bar{\lambda}}{m})$$

$$\frac{\overline{t_{n\alpha_1}(\Lambda + \rho)}}{m+2} = n + \tilde{A}$$

$$\Rightarrow x, y \geq 0, x - y = \frac{\overline{t_{n\alpha_1}(\Lambda + \rho)}}{m+2} - \frac{\bar{\lambda}}{m}$$

$$\Rightarrow x - y = \tilde{A} - \tilde{B} + n \geq 0.$$

Case 2) $w = \boldsymbol{\tau}^{2n} r_1 = t_{n\alpha_1} r_1$: In this case we have $n > 0$, so

$$(x, y) := \phi_1(w, j) = (\frac{j_1}{2}\alpha_1 + \frac{\overline{t_{n\alpha_1} r_1(\Lambda + \rho)}}{m+2}, \frac{j_1}{2}\alpha_1 + \frac{\bar{\lambda}}{m})$$

$$\frac{\overline{t_{n\alpha_1} r_1(\Lambda + \rho)}}{m+2} = n - \tilde{A} > 0$$

$$\Rightarrow x, y \geq 0, x - y = \frac{\overline{t_{n\alpha_1} r_1(\Lambda + \rho)}}{m+2} - \frac{\bar{\lambda}}{m}$$

$$\Rightarrow x - y = n - (\tilde{A} + \tilde{B}) \geq 0.$$

For $b(w) < 0$

case 1) $w = \boldsymbol{\tau}^{2n} = t_{n\alpha_1}$: For $b(w) < 0$, in this case $n < 0$. So

$$(x, y) := \phi_1(w, j) = (\frac{j_1}{2}\alpha_1 + \frac{\overline{t_{n\alpha_1}(\Lambda + \rho)}}{m+2}, \frac{j_1}{2}\alpha_1 + \frac{\bar{\lambda}}{m})$$

$$\frac{\overline{t_{n\alpha_1}(\Lambda + \rho)}}{m+2} = n + \tilde{A}$$

$$\Rightarrow x, y < 0, x - y = \frac{\overline{t_{n\alpha_1}(\Lambda + \rho)}}{m+2} - \frac{\bar{\lambda}}{m}$$

$$\Rightarrow x - y = \tilde{A} - \tilde{B} + n < 0.$$

Case 2) $w = \boldsymbol{\tau}^{2n} r_1 = t_{n\alpha_1} r_1$: In this case we have $n \leq 0$, so

$$(x, y) := \phi_1(w, j) = (\frac{j_1}{2}\alpha_1 + \frac{\overline{t_{n\alpha_1} r_1(\Lambda + \rho)}}{m+2}, \frac{j_1}{2}\alpha_1 + \frac{\bar{\lambda}}{m})$$

$$\frac{\overline{t_{n\alpha_1} r_1(\Lambda+\rho)}}{m+2} = n - \tilde{A} > 0$$
$$\Rightarrow x, y < 0, x - y = \frac{\overline{t_{n\alpha_1} r_1(\Lambda+\rho)}}{m+2} - \frac{\bar{\lambda}}{m}$$
$$\Rightarrow x - y = n - (\tilde{A} + \tilde{B}) < 0.$$

We have proved the following lemma.

Lemma 5.6.4. *Let* $D := \{(w, j) \in W \times \mathbb{Z} : I(b(w), j) \neq 0\}$, *then*

$$\phi_1(D) = F \cap \cup_{i=1}^{4} L_i$$

We have by above lemma

$$\phi_1(D) = \{(x, y) \in F : (x, y) \equiv (\tilde{A}, \tilde{B}) \text{ or } \equiv (\tilde{A}+\frac{1}{2}, \tilde{B}+\frac{1}{2}) \text{ or } \equiv (-\tilde{A}, \tilde{B}) \text{ or } \equiv (-\tilde{A}-\frac{1}{2}, \tilde{B}+\frac{1}{2}) \mod \mathbb{Z}^2\}$$

which can be written as

$$\{(x, y) \in F : (x, y) \equiv (\tilde{A}, \tilde{B}) \text{ or } a(x, y) \equiv (\tilde{A}, \tilde{B}) \text{ or } J(x, y) \equiv (\tilde{A}, \tilde{B}) \text{ or } Ja(x, y) \equiv (\tilde{A}, \tilde{B}) \mod \mathbb{Z}^2\} \tag{5.6.56}$$

As F is a fundamental domain for G' on U^+, we define a map $\phi_2 : U^+ \twoheadrightarrow F$ by $\phi_2(\xi) = \xi'$ where ξ' is the unique element in $G'(\xi) \cap F$. Recall that $F_0 \cup aF_0$ is a fundamental domain for G_0 on U^+, we see that the set in 5.6.56 is the image of the set

$$\{(x, y) \in F_0 : (x, y) \equiv (\tilde{A}, \tilde{B}) \text{ or } a(x, y) \equiv (\tilde{A}, \tilde{B})\}$$

i.e.,

$$D \xrightarrow{\phi_1} F \xleftarrow{\phi_2} (U^+ \cap (\tilde{A}, \tilde{B}) + \mathbb{Z}^2) \mod G_0$$

and

$$\phi_1(D) = \phi_2((U^+ \cap (\tilde{A}, \tilde{B}) + \mathbb{Z}^2) \mod G_0$$

Let for a function $\phi : V \to \mathbb{R}^2$, where V is a two dimensional vector space, $\operatorname{sign} \phi(x_1, x_2)$ denote the sign of the first component of $\phi(x_1, x_2)$. Now we prove the following lemma.

Lemma 5.6.5. *For* $(w, j) \in D$,

$$I(b(w), j)(-1)^j \epsilon(w) = \operatorname{sign}(\phi_2^{-1}\phi_1(w, j)) .$$

Here $\operatorname{sign}(\phi_2^{-1}\phi_1(w, j))$ *is independent of the representive chosen in its* G_0 *orbit.*

Proof. We will prove the lemma case by case: Case 1) $w = t_{n\alpha_1}$. Let $n \geq 0$. In this case we note that $I(w, j) = 1$ if $j \geq 0$ and 0 otherwise. So we have $I(w, j)(-1)^j \varepsilon(w) = 1$ if j is even and -1 if j is odd. Now

$$\phi_1(x, y) = (\frac{j_1}{2}\alpha_1 + \frac{\overline{t_{n\alpha_1(\Lambda+\rho)}}}{m+2}, \frac{j_1}{2}\alpha_1 + \frac{\bar{\lambda}}{m})$$

By calculation in above proposition we see that

$$\phi_1(w,j) = (\frac{j}{2} + \frac{(\Lambda+\rho,\alpha_1)}{2(m+2)} + n, \frac{j}{2} + \frac{\bar{\lambda}}{m})$$

We note that in this case sign$(\phi_1(w,j))$is always positive So if j is even then $\phi_1(w,j) \equiv (\tilde{A},\tilde{B})$. In this case $\phi_1(w,j) = {\phi_2}^{-1}\phi_1(w,j)$; so both sides are positive. If j is odd positive integer, we have sign of $\phi_1(w,j)$ is positive but $\phi_1(w,j) \equiv (\frac{1}{2}+\tilde{A}, \frac{1}{2}+\tilde{B})$ so $\phi_2^{-1}\phi_1(w,j) = Ja\phi_1(w,j)$ which has negative sign.

For $n < 0$, $I(b(w),j) = -1$ and $j < 0$ then as above we see that for even j, $I(b(w),j)(-1)^j\epsilon(w) = -1$ and sign $\phi_1(w,j)$ is negative. As $\phi_1(w,j) = {\phi_2}^{-1}\phi_1(w,j)$ we are done in this case. For j odd negative, $I(b(w),j)(-1)^j\epsilon(w) = 1$, and sign $\phi_1(w,j)$ is negative. But as $\phi_1(w,j) \equiv (\frac{1}{2}+\tilde{A}, \frac{1}{2}+\tilde{B})$, so $\phi_2^{-1}\phi_1(w,j) = Ja\phi_1(w,j)$ has a positive sign.
Case 2) $w = t_{n\alpha_1}r_1$: Let $n > 0$. In this case $I(b(w),j) = 1$ and $j \geq 0$ and $\epsilon(w) = -1$. So for even j, $I(b(w),j)(-1)^j\epsilon(w) = -1$ and

$$\begin{aligned}\phi_1(w,j) &= (\frac{j_1}{2}\alpha_1 + \frac{\overline{t_{n\alpha_1 r_1(\Lambda+\rho)}}}{m+2}, \frac{j_1}{2}\alpha_1 + \frac{\bar{\lambda}}{m}) \\ &= (\frac{j}{2} + n - \tilde{A}, \frac{j}{2} + \tilde{B}).\end{aligned}$$

So we see that sign$(\phi_1(w,j))$ is positive but as $\phi_1(w,j) \equiv (-\tilde{A},\tilde{B}) \bmod \mathbb{Z}^2$ we have $\phi_2^{-1}\phi_1(w,j) = J\phi_1(w,j)$ which has negative sign. For positive odd integer j, $I(b(w),j)(-1)^j\epsilon(w) = 1$ and $\phi_1(w,j) \equiv (\frac{1}{2}-\tilde{A}, \frac{1}{2}+\tilde{B}) \bmod \mathbb{Z}^2$, so $\phi_2^{-1}\phi_1(w,j) = a^{-1}\phi_1(w,j)$ and as a^{-1} does not change the sign of ϕ_1, we are through in this case.

Now let $n \leq 0$, then $I(b(w),j) = -1$ and $j < 0$. Let j be even then $I(b(w),j)(-1)^j\epsilon(w) = 1$. Then sign $\phi_1(w,j)$ is negative and $\phi_1(w,j) \equiv (-\tilde{A},\tilde{B}) \bmod \mathbb{Z}^2$ we have $\phi_2^{-1}\phi_1(w,j) = J\phi_1(w,j)$ which has positive sign. Now let j be odd then $I(b(w),j)(-1)^j\epsilon(w) = -1$. We have sign $\phi_1(w,j)$ is negative and which is same as $\phi_2^{-1}\phi_1(w,j) = a^{-1}\phi_1(w,j)$. □

Now for $(w,j) \in W \times \mathbb{Z}$ consider the term $(b(w)+j)\alpha_1$ and let $(x,y) = \phi_1(w,j)$. We claim that

$$(b(w)+j)\alpha_1 = (m+2)x - my - \frac{1}{2}\alpha_1 \tag{5.6.57}$$

but

$$\begin{aligned}(m+2)x - my - \frac{1}{2} &= (m+2)(\frac{j}{2} + \frac{(\Lambda+\rho,\alpha_1)}{2(m+2)} + n, \frac{j}{2}) - m\frac{(\lambda,\alpha_1)}{2m} - \frac{1}{2}\alpha_1 \\ &= j + n(m+2) + \frac{\Lambda-\lambda}{2}\alpha_1 \\ &= j + n(m+2) + \Lambda - \lambda = (b(w)+j)\alpha_1\end{aligned}$$

the last equality follows by the equations 5.6.47 and 5.6.48.

Collecting all pieces together we have proved our main theorem:

Theorem 5.6.6. *Let $\mathfrak{g}$ be of type $A_1^{(1)}$. Let $\Lambda \in P_+$, $\Lambda(K) = m \geq 1$, and let $\lambda \in P_+$ be a maximal dominant weight of $L(\Lambda)$. Then*

$$c_\lambda^\Lambda(t;q) = \mathrm{ct}(\xi_t \, P_t \, q^{\frac{1}{8}} \, t^{-2B} \, H_t)$$

where

$$H_t = \sum_{\substack{(x,y) \equiv (\tilde{A},\tilde{B}) \mod \mathbb{Z}^2 \\ B(x,y)>0 \\ (x,y) \bmod G_0}} \mathrm{sign}(x) \, q^{\frac{1}{2}B(x,y)} \, t^{2\bar{y}} \, e^{((m+2)\bar{x} - m\bar{y} - \frac{1}{2})\alpha_1}$$

where $(\bar{x}, \bar{y})$ is the unique element in $F \cap G'(x,y)$.

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