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IMSc Report No. 113

MATRIX MODELS, RANDOM SURFACES

AND

2-D GRAVITY

(November 19 - 23 1990)

The Institute of Mathematical Sciences
Madras 600 113

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Foreword

About a little over an year ago during one of the faculty meetings proposal for holding a series of Mini Workshops as an ongoing activity of the Institute was mooted. The idea behind the workshops was to gather together the talent available in the country for a somewhat informal but well focused discussions on specific topics of current interest. The proposal was welcomed by the faculty and I am happy to note that in the past year three such workshops have been held. The participants responses have also been very encouraging.

These workshops typically have had a few invited review talks as well as presentation of recent work by some participants. We felt that the material discussed during the workshops may be made available to a wider set of people especially students. So we decided to have the proceedings published as Institute of Mathematical Sciences reports.

This report is the first in the series of reports to follow. I hope that these reports do in fact serve the intended purpose.

Director,
The Institute of Mathematical Sciences,
Madras.

R. Ramachandran

Preface

We are happy to bring out this report containing proceedings of the "**Mini Workshop on Matrix Models, Random Surfaces and 2-D Gravity**" held during November last year. We must admit immediately that we also feel a bit apologetic about the time it has taken to prepare the report.

A few words about the workshop. This was first in the series of Workshops held at our Institute. The organizing committee consisted of G. Date, T.R. Govindarajan, N.D. Hari Dass (Convenor) and R. Ramachandran. We had a total of 42 participants with 23 from outside Madras. The workshop was held for five days. The workshop consisted of Review talks on the central theme as well as a few seminars during which original contributions were presented. Typically afternoons were used for detailed discussions on the talks held during mornings. Thanks to the then newly acquired Guest House of the Institute, the participants also had all evenings and nights(!) for discussions. There was a fair amount of student participation. We are happy to note that the response of the participants was very positive.

Unfortunately, for reasons beyond our control, this report does not contain the manuscripts of all the talks. We have tried to remedy this deficiency by including a list of references for those talks for which we did not receive manuscript. We would like to thank *T. Jayaraman* for his help in this regard.

The articles are organised as review talks first followed by presented papers. The later papers are in a loose sense further away from the central theme of the workshop.

This report is produced in LaTeX. The help rendered by *P. Majumdar* and *Suresh Govindarajan* in this regard was invaluable. We have tried to minimise typographical errors. For the errors still remaining we express our faith in our intelligent readers!

We would like to take this opportunity to acknowledge the excellent support we received from the lecturers and the participants of the workshop, our colleagues and the students at the Institute, the administrative staff, the Guest House staff, the Library staff, the accounts section during the workshop. We also acknowledge the secretarial assistance rendered by *Ms. Indira* and *Ms. Usha Nandhini*. We thank each one for their support. It is a pleasure to thank N.D. Hari Dass and R. Ramachandran for their active support and encouragement.

G. Date
December 12, 1991 **T.R. Govindarajan**

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RANDOM SURFACES

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Introduction

These lectures were delivered as an introduction to Random Surface theory at the Workshop held on Random Surfaces, Matrix Models and $2-D$ gravity. I wanted to present both the Continuum picture as well as the dynamically triangulated random surface pictures. These notes are heavily based on Polyakov's book on "Gauge Fields and Strings" (Harwood Academic Publications) for the first third of the material. The second part is a discussion of the Knizhnik-Polyakov-Zamolodchikov (KPZ) relation as interpreted by Distler and Kawai (DK) and KPZ themselves. The last part is based on the recent works of Ambjorn, Boulatov and Kazakov. No pretense is made in this review at citing all the relevant literature.

I.Continuum Theories

Two important questions one would like to understand in the context of random surfaces are i) characterisation of such surfaces and ii) making a meaningful sum over such surfaces with appropriate 'measures' or weight factors. Let us begin by asking ourselves analogous questions about paths as many of the conceptual ingredients needed for an understanding of the surface problem can already be found in the simpler case of paths.

Thus, consider a sum over all paths $P_{xx'}$ connecting the points (x,x') represented by

$$G(x,x') = \sum_{P_{xx'}} e^{-S(P_{xx'})} \tag{1}$$

Where S is a functional of the paths. Usually one takes for S a functional that has a coordinate invariant description. In addition to coordinate invariance, S could enjoy other, invariances which will have bearing on the problem of constructing the sum. The simplest invariant characteristic of a path is its length $L(P_{xx'})$ i.e.

$$S(P_{xx'}) = m_0 L(P_{xx'}) \tag{2}$$

Our task now is to give a meaning to the 'sum' in (1). One way of doing this is to approximate $\mathbf{R}^d$ by a lattice whence the number of paths connecting (x,x') become finite and the sum can easily be carried out. The continuum sum is then defined as the limit in which the lattice spacing a is taken to zero. Doing this in the true spirit of lattice field theories requires tuning $m(a)$ carefully. Now let us see how the sum over paths can be made meaningful in a manifestly continuum approach.

The action (2), written explicitly, has the form

$$S = m_o \int_0^1 d\tau \left[\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} \right]^{1/2} \tag{3}$$

where τ parametrises the path such that

$$x^\mu(0) = x^\mu; \qquad x^\mu(1) = x'^\mu \tag{4}$$

The precise parametrisation of the curve is obviously without any significance. This is reflected in the invariance of the 'action' S under the reparametrisations

$$x_\mu(\tau) \to x_\mu(f(\tau)) \tag{5}$$

where the function $f(\tau)$ satisfies

$$f(0) = 0 \qquad f(1) = 1 \qquad \frac{df(\tau)}{d\tau} > 0 \tag{6}$$

the last of these conditions ensures that $\dot{f}(\tau)$ does not vanish anywhere in the range (0,1) and consequently every point on the curve will have an unique image under the reparametrisation map.

The implication of the reparametrisation invariance for the sum over paths is that the measure should not count a path and its parametrisation as distinct i.e. it should count the paths modulo reparametrisations. The sum (1) can be written as

$$G(x, x') = \int \frac{DX(\tau)}{Df(\tau)} e^{-m_o \int_0^1 \sqrt{\dot{x}^2} d\tau} \tag{7}$$

where $\frac{DX(\tau)}{Df(\tau)}$ symbolises the measure over the coset space (A) : space of all $x_\mu(\tau)$/space of all reparametrisations.

Following Polyakov let us introduce a metric tensor $h(\tau)$ as follows :

$$G(x, x') = \int \frac{Dh(\tau)}{Df(\tau)} e^{-m_o \int_0^1 \sqrt{h(\tau)} d\tau} \int Dx(\tau) \delta(\dot{x}^2 - h) \tag{8}$$

The second integral can be represented as

$$\begin{aligned} K(x, x', h) &= \int Dx(\tau) \delta(\dot{x}^2 - h) \\ &= \int D\lambda(\tau) e^{\int_0^1 \lambda(\tau) h(\tau) d\tau} \int DX(\tau) e^{-\int_0^1 \lambda(\tau) \dot{x}^2(\tau) d\tau} \end{aligned} \tag{9}$$

The action

$$S' = \int_0^1 \lambda(\tau)h(\tau)d\tau - \int_0^1 \lambda(\tau)\dot{x}^2(\tau)d\tau \tag{10}$$

is invariant under :

$$\begin{array}{lcl} d\tau & \rightarrow & df\,[\frac{df(\tau)}{d\tau}]^{-1} \\ x(\tau) & \rightarrow & x(f(\tau)) \\ \lambda(\tau) & \rightarrow & \lambda(f(\tau))\,[\frac{df(\tau)}{d\tau}]^{-1} \\ h(\tau) & \rightarrow & h(f(\tau))\,[\frac{df(\tau)}{d\tau}]^{2} \end{array} \tag{11}$$

The evaluation of $K(x,x',h)$ becomes standard if we work with the 'propertime' t defined by

$$dt = \sqrt{h}d\tau$$

i.e.

$$t = \int_0^\tau h^{1/2}(\tau)d\tau \tag{12}$$

with $T = t(1)$ the invariant length of the path. Indeed

$$K(x,x',T) = T^{-D/2}e^{\frac{-(x'-x)^2}{T}(\alpha)+<\alpha>T} \tag{13}$$

where $<\alpha>$ denotes an invariant 'cut off'. Now we have to evaluate

$$G(x,x') = \int \frac{Dh(\tau)}{Df(\tau)}e^{-m_0\int_0^1 h^{1/2}(\tau)d\tau}K(x,x',h) \tag{14}$$

If we can show that all the matrics $h(\tau)$ can be obtained by diffeomorphisms $f(\tau)$ of a standard metric parametrised by $\{t_i\}$ it would follow that

$$dh(\tau) = \prod dt_i Df(\tau) \bullet Jacobian \tag{15}$$

Thus finding the measure over the coset space (A) is equivalent to finding the Jacobian. Just as the Jacobian required for invariant integrations over coordinates is given by $\sqrt{g}$, the Jacobian in (15) is given by the determinant of the metric on the space of metrics.

The reparametrisation invariant local distance $\| \delta h \|$ between the metrics $h(\tau)$ and $h(\tau)+\delta h(\tau)$ is

$$\| \delta h \|^2 = \int_0^1 d\tau(\delta h(\tau))^2 h^{-3/2}(\tau) \tag{16}$$

Proof :

$$\int_0^1 d\tau' \delta h'(\tau')^2 h'^{-3/2}(\tau)$$
$$= \int_0^1 d\tau' . \delta h^2(\tau) . f'^{-4} h^{-3/2}(\tau) f'^3 = \int_0^1 d\tau \delta h^2(\tau) h^{-3/2}(\tau)$$

The special feature of one dimensions is that by a coordinate transformation the metric can be made uniform i.e. we can go to a parametrisation τ' such that

$$h'(\tau') = \text{ const} \tag{17}$$

But $\int_0^1 d\tau' \sqrt{h'(\tau')}$ is an invariant T. Thus

$$h'(\tau') = T^2 \tag{18}$$

The transformation law of $h(\tau)$ implies that

$$h(\tau) = (\frac{df}{d\tau})^2 T^2 \tag{19}$$

Where f is the reparametrisation that takes $h(\tau)$ into the 'standard form'. Consequently,

$$Dh(\tau) = Df(\tau) dT \bullet Jacobian \tag{20}$$

Now consider, for a generic geometry, two metrics $\tilde{h}_{ab}$ and h_{ab} related by a diffcomorphism i.e.

$$\tilde{h} = (h)^f$$

or

$$\tilde{h}_{ab}(\xi) = \frac{\partial f^c}{\partial \xi^a}(\xi) \frac{\partial f^d}{\partial \xi^b}(\xi) h_{cd}(f(\xi)) \tag{21}$$

The variation in $\tilde{h}$ due to small variations in f and h is given by

$$\delta \tilde{h}_{ab} = \frac{\partial f^c}{\partial \xi^a} \frac{\partial f^d}{\partial \xi^b} \delta h_{cd} + ((h)^{f+\delta f} - (h)^f)_{ab}$$
$$= \frac{\partial f^c}{\partial \xi^a} \frac{\partial f^d}{\partial \xi^b} \delta h_{cd} + (\{(h)^{1+\delta f . f^{-1}} - h\}^f)_{ab} \tag{22}$$

Calling $\delta f^a . f^{-1} = \omega^a(\xi)$ one finds

$$\delta \tilde{h}_{ab} = (\frac{\partial f^c}{\partial \xi^a} \frac{\partial f^d}{\partial \xi^b} (\delta h_{cd}) + \nabla_c \omega_d + \nabla_d \omega_c) f(\xi) \tag{23}$$

In our one dimensional case at hand

$$(\delta h) = (2T\delta T) \tag{24}$$

and hence

$$\delta\tilde{h} = 2(\frac{df}{d\tau})^2(T\delta T + \frac{d\omega}{df}) \tag{25}$$

and

$$\| \delta\tilde{h} \| = 4\int_0^1 d\tau T^{-3}(\frac{df}{d\tau})^{-3}(\frac{df}{d\tau})^4(T\delta T + \frac{d\omega}{df})^2 \tag{26}$$

$$\text{Now } \delta f(1) = \delta f(0) = 0 \Rightarrow \omega(1) = \omega(0) = 0.$$

Therefore

$$\| \delta h \|^2 = \frac{4(\delta T)^2}{T} + \frac{4}{T_3}\int_0^1 df(\frac{d\omega}{df})^2(f) \tag{27}$$

Note the absence of terms that mix modular variation with diff. variation.

Now consider the measure and the metric in the space of diffeomorphisms. The left and right invariant metric is given by

$$\| \delta f \|^2 = \int h^{1/2} h_{ab}\omega^a\omega^b d^n\xi \tag{28}$$

$$\text{where } \omega^a = \delta f^a . f^{-1}$$

To see invariance under right multiplication consider

$$\begin{gathered} f' = f \bullet \alpha \\ \text{Therefore} f'^{-1} = \alpha^{-1}.f^{-1} \qquad \delta f' = \delta f \bullet \alpha \\ \text{Hence } \omega' = \delta f \bullet \alpha \bullet \alpha^{-1} \bullet f^{-1} = \omega \end{gathered} \tag{29}$$

For left multiplication

$$\begin{aligned} f'^a &= \beta^a(f(\xi)) = \beta \bullet f \\ \delta f'^a &= \frac{\partial \beta^a}{\partial f^b}(f)\delta f^b \\ \delta f'^a . f^{-1} &= \frac{\partial \beta^a(\beta^{-1})}{\partial(\beta^{-1})^b}\delta f^b(f^{-1}(\beta^{-1})) \\ \omega'^a &\rightarrow \frac{\partial \beta^a(\beta^{-1})}{\partial(\beta^{-1})^b}\omega^b(\beta^{-1}) \end{aligned} \tag{30}$$

Hence w^a transforms like a covariant vector. If h_{ab} is transformed appropriately $\| \delta f \|^2$ is invariant.

Remark : It is completely analogous to the left and right invariant killing metrics for finite dimensional groups where

$$\| \delta f \|^2 = tr\omega^2, \omega = \delta f . f^{-1} \tag{31}$$

For the one dimensional case we had

$$\| \delta f \|^2 = T^{-3} \int_0^1 \dot{\omega}^2 df \qquad w = \delta d \tag{32}$$

$$\text{Rescaling} \qquad \omega = T\epsilon, \qquad t = Tf$$

$$\| \delta h \|^2 = \frac{(\delta T)^2}{T} + \int^T dt \dot{\epsilon}^2(t) \tag{33}$$

Hence the measure is

$$Dh(\tau) = \frac{dt}{\sqrt{T}}.Df(\tau)Det^{1/2}\left(-\frac{d^2}{dt^2}\right) \tag{34}$$

and consequently

$$\frac{Dh(\tau)}{Df(\tau)} = \frac{dT}{\sqrt{T}} det^{1/2}\left(-\frac{d^2}{dt^2}\right) \tag{35}$$

Regularising the determinant :

Formally one has

$$\log \ \text{Det} \ A = tr \ln A = -tr \int^\infty d\tau \frac{e^{-\tau A}}{\tau} \tag{36}$$

This can be regulated invariantly by

$$\log \ \text{Det}_R A = -\int_{\epsilon^2}^\infty \frac{d\tau}{\tau} \sum_n e^{-\lambda n \tau} \tag{37}$$

where λ_n are the eigenvalues of $\frac{-d^2}{dt^2}$ (the eigenfunctions satisfy $\omega(T) = \omega(0) = 0$).

These are

$$\lambda_n = \frac{\pi^2 n^2}{T^2} \qquad n = 1, \ldots . \tag{38}$$

Note that the boundary conditions preclude any zero modes.

Now

$$\sum_1^\infty e^{-\frac{\pi^2 n^2 \tau}{T^2}} = \frac{1}{2}\sum_{-\infty}^{+\infty} e^{-\frac{\pi^2 n^2 \tau}{T^2}} - \frac{1}{2} = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-\frac{\pi^2 x^2 \tau}{T^2}} dx - \frac{1}{2} + \mathcal{O}(e^{-c/\tau}) \tag{39}$$

$$= \frac{1}{2}\sqrt{\frac{\pi T^2}{\pi^2 \tau}} - \frac{1}{2} + \mathcal{O}(e^{-c/\tau})$$
$$= \frac{1}{2}\frac{T}{\sqrt{\pi\tau}} - \frac{1}{2} + \mathcal{O}(e^{-c/\tau}) .$$

This causes log det A to diverge because the integrals $\int \frac{d\tau}{\tau^{3/2}}$ and $\int \frac{d\tau}{\tau}$ blow up near $\tau = 0$ cutting off the integration at $\tau = \epsilon$ provides an invariant way of regulating log det A. The required integral is

$$\int_{\epsilon^2}^\infty \frac{d\tau}{\tau}\sum_1^\infty e^{-\frac{n^2\pi^2\tau}{T^2}}$$

Change variables to $\tau \to x = \frac{\tau}{T^2}$

$$\int_{\frac{\epsilon^2}{T^2}}^\infty \frac{dx}{x}\sum_1^\infty e^{-n^2\pi^2 x} \tag{40}$$

(Zero modes! None because of fixed boundaries)

Separate the range (artificially) into $\int_{\epsilon^2/T^2}^1 + \int_1^\infty$; the latter integral is well defined and independent of T.

$$\int_{\epsilon^2/T^2}^1 \frac{dx}{x}\sum_1^\infty e^{-n^2\pi^2 x} + \text{ Const}$$
$$= \int_{\epsilon^2/T^2}^1 \frac{dx}{x}\left(\frac{1}{2\sqrt{\pi x}} - \frac{1}{2}\right) + \text{ Const}$$

Thus

$$\lim_{\epsilon\to 0} -\log\det\left(-\frac{d^2}{dt^2}\right) \sim \frac{T}{\sqrt{\pi}\epsilon} - \log\frac{T}{\epsilon} \tag{41}$$

Therefore

$$\frac{Dh(\tau)}{Df(\tau)} = \frac{dT}{\sqrt{T}} e^{\frac{-T}{2\epsilon\sqrt{\pi}}} \sqrt{T}\left(\frac{\text{Const}}{\sqrt{2}}\right) \tag{42}$$

with the result

$$G(x,x') = (\frac{Const}{\sqrt{\epsilon}})\int_0^\infty dTe^{-(m_o - Const)T}.T^{-D/2}e^{\frac{-(x-x')^2}{\epsilon T}}$$

Denoting $\epsilon^{-1}(m_o - m_{cr})$ by μ,

$$G(x,x') = \text{Const} \int_0^\infty dTe^{-\mu\epsilon T}T^{-D/2}e^{\frac{-(x-x')^2}{\epsilon T}} \sim \int \frac{d^D\rho}{\rho^2+\mu}e^{i\rho.(x-x')} \tag{43}$$

The continuum limit is possible only if m_o is tuned to m_{cr} keeping μ fixed. The lattice picture is

$$G(x,x') = \sum_{paths} n(L)e^{-m_0 L} \tag{44}$$

But $n(L) \sim C^L$ as $L \to \infty$. For lnC $< m_o$, the dominant paths have lengths of the order of lattice spacing and for lnC $> m_o$ the sum diverges. If m_o is tuned to lnC, the typical lengths $\sim | \frac{1}{m_o - \ln C} | >> 1$ and continuum limit exists.

Now let us generalise these discussions to the problem of random surfaces. Let us start with the properties of random hypersurfaces in arbitrary dimensions and then specialise to the case of two dimensions.

Two Dimensional Random Surfaces

Now let us consider two dimensional surfaces with possibly a boundary $C(S)$ i.e. the coordinates of the points on the boundary in the embedding space are given by

$$x^\mu(\xi(s)) = C^\mu(s) \tag{45}$$

As before we are interested in computing the Jacobian

$$\frac{Dh_{ab}(\xi)}{Df(\xi)} \tag{46}$$

where $f(\xi)$ are the diffeomorphisms. These are restricted by the requirement that the boundary points are not moved i.e.

$$f(\xi(s)) = \xi(s) \tag{47}$$

Then the results are invariant under reparametrisations of the boundary.

In the one dimensional path example we used diffeomorphisms to bring the 'metric' into the 'standard' form where it was constant everywhere. In the 2-dimensional case this is not possible because the metric h_{ab} has 3 independent components while the number of possible diffeomorphism is only 2. Then one may hope to bring h_{ab} into the 'standard' form

$$h_{ab} = e^{\phi(\xi)}\delta_{ab} \tag{48}$$

such that every metric could be parametrised by ϕ and the 2 diffeomorphisms that would bring it to the standard form. Locally this is always possible but let us examine this question from the global point of view. If it is possible to bring every 2-metric to this form, we must have

$$g_{ab}(\xi) = (e^{\phi}\delta_{ab})^{f} = e^{\phi(f(\xi))}\frac{\partial f^c}{\partial \xi^a}\frac{\partial f^c}{\partial \xi^c} \tag{49}$$

Consider a small variation δg_{ab} of g_{ab} :

$$\delta g_{ab} = (\delta\phi h_{ab} + \nabla_a\omega_b + \nabla_b\omega_a)^f \tag{50}$$

$$\text{where } \omega_a = \delta f_a(f^{-1}(\xi)) \tag{51}$$

If for any δg_{ab}, one could find nonsingular $\delta\phi$ and ω_a, itwill indeed be possible to bring g_{ab} into h_{ab} globally. Calling
$\delta g_{ab}^{f^{-1}} = \gamma_{ab}$ one has

$$\delta\phi(\xi)h_{ab} + \omega_{a;b} + \omega_{b;a} = \gamma_{ab} \tag{52}$$

where ; indicates covariant differentiation w.r.t. the metric h_{ab}. Then

$$\begin{aligned} \omega_{a;b} + \omega_{b;a} - h_{ab}\omega^c_{c:} &= \gamma_{ab} - \frac{1}{2}h_{ab}\gamma^c_c \\ &= (L\omega)_{ab} \end{aligned} \tag{53}$$

L maps vector fields into traceless symmetric tensors. The conjugate operator L^+ maps traceless sym. tensors into vector fields

$$\text{i.e} \qquad (L^+ f_{ab}) = -\nabla^a f_{ab} \qquad f^a_a = 0 \tag{54}$$

It is obvious that if L has zero modes, the soln of (53) is not unique. If on the other hand L^+ has zero modes i.e. there exist f_{ab} s.t

$$L^+ f = 0 \tag{55}$$

we see that

$$(f, L\omega) = (L^+ f, \omega) = (f, \gamma) \tag{56}$$

and hence for γ s.t $(f,\gamma) \neq 0$, there is an obstruction to solving

$$Lw = \gamma$$

However, if there are no zero modes of L^+, the general solution is

$$\omega = \frac{1}{L^+L}(L^+\gamma) + \sum_\alpha C_\alpha \omega_{\alpha,o} \tag{57}$$

where $\omega_{\alpha,o}$ are the zero modes of L and L^+L is restricted to its nonzero eigenvalues only. Fortunately it is possible to count the number of zero modes $N_0(L)$ and $N_0(L^+)$. Let us consider manifolds without boundary.

The first point is that nonzero eigenvalues of L^+L and LL^+ are paired:

$$\text{Let } L^+L\varphi = \epsilon^2\varphi \qquad \epsilon \neq 0 \tag{58}$$

if we call

$$L\varphi = \epsilon\chi$$

it follows that

$$L^+\chi = \epsilon\varphi \text{ and } \qquad LL^+\chi = \epsilon L\varphi = \epsilon^2\chi$$

But there is no such pairing required for the zero modes of these operators. Then

$$N_o(L) - N_o(L^+) = Tr(e^{-tL^+L} - e^{-tLL^+}) \tag{59}$$

This is strictly valid only if zero norm states have been excluded.

Evaluation of LHS can therefore be done in the limit $t \to 0$. For flat space

$$\begin{aligned}(L^+L\omega)_a &= -\nabla^b(\nabla_a\omega_b + \Delta_b\omega_a - h_{ab}\nabla^c\omega_c)\\ &= -\partial^2\omega_a\\ tre^{-tL^+L} &= 2\int \frac{d^2\rho}{(2\pi)^2}e^{-\rho^2 t} = \frac{1}{2\pi t}\end{aligned} \tag{60}$$

Likewise

$$\begin{aligned}(LL^+f)_{ab} &= -\nabla_a(L^+f)_b + \nabla_b(L^+f)_a - h_{ab}\nabla^c(L^+f)_c\\ &= -\nabla_a\nabla^c f_{cb} - \nabla_b\nabla^c f_{ca} + h_{ab}\nabla^c\nabla^d f_{dc}\\ &= (\delta_{ab}\partial^c\partial^d - \delta^d_{da}\partial_b\partial^c - \delta^d_b\partial_a\partial^c)f_{dc}\end{aligned}$$

In terms of the Fourier components of f_{ab}, one has

$$\begin{aligned}&-(\delta_{ab}k^ck^d - \delta^d_a k_b k^c - \delta^d_b k_a k^c)f_{dc}(k) = \lambda^2 f_{ab}(k)\\ &\text{i.e. } \lambda^2 f_{ab} = k_b(k^c f_{ac}) + k_a(k^c f^a_{bc}) - \delta_{ab}(k^c k^d f_{ca})\end{aligned}$$

ting

$$f_{ac}(k) = k_a k_c f_1(k^2) + \delta_{ac} f_2(k^2)$$

ets

$$2f_2 = -k^2 f_1.$$

$$f_{ac} = f_2(\delta_{ac} - 2\frac{k_a k_c}{k^2})$$

$$\lambda^2 f_2(\delta_{ab} - 2\frac{k_a k_b}{k^2}) = -k_b k_a f_2 - k_b k_a f_2 + \delta_{ab} f_2 k^2$$

$$\lambda^2 = \mathrm{k}^2$$

hus the leading behaviour of e^{-tL^+L} and e^{-tLL^+} are the same. On general grounds one pects

$$< \xi | e^{-tL^+L} | \xi >= \frac{1}{2\pi t}\{1 + t a_1 R(\xi) + \ldots\}$$

$$< \xi | e^{-tLL^+} \xi >= \frac{1}{2\pi t}\{1 + t a_2 R(\xi) + ..\} \tag{61}$$

where $R(\xi)$ is the scalar curvature of the two dimensional surface. This yields the important relation

$$N_o(L) - N_o(L^+) = 2(a_1 - a_2)\chi. \tag{62}$$

with χ the Euler character stic of the surface. To determine the coefficient $(a_1 - a_2)$ it therefore suffices to know $N_o(L)$ and $N_o(L^+)$ for any particular surface.

Consider Riemann Sphere for which $\chi = 2$. The zero modes of L count conformal killing vectors and for Riemann spheres there are 3 complex zero modes corresponding to the 3 $SL(2,c)$ transformations. See Polyakov for the Proof that $N_o(L^+) = 0$. Thus

$$(a_1 - a_2) = +\frac{3}{2}$$

$$\text{and} \qquad N_o(L) - N_o(L^+) = 3\chi = 6 - 6g \tag{63}$$

In fact, one can go further; $g = 1$ it is possible to relate, by diffeomorphisms, all metrics to a flat metric. In this case $N_o(L) = N_o(L^+) = 2$. For $g > 2, \chi$ is negative and one can consider a metric with constant negative curvature. Now

$$(L^+L\omega)_a = -\nabla^b(\nabla_a\omega_b + \nabla_b\omega_a - h_{ab}\nabla^c\omega_c)$$
$$= -\nabla^2\omega_a + [\nabla_a, \nabla^b]\omega_b$$
$$= -(\nabla^2 + \frac{1}{2}R)\omega_a \tag{64}$$

Therefore,

$$(L\omega, L\omega) = (\nabla_b\omega_a, \nabla_b\omega_a) - \frac{R}{2}(\omega_a, \omega_a) \qquad > 0 \tag{65}$$

Hence

$$L\omega \neq 0 \tag{66}$$
$$N_o(L) = 0$$

Therefore, $N_o(L^+) = 6g - 6$. All this means is that by diffeomorphisms the best that can be achieved is to bring the metric into the form

$$h_{ab}(\xi) = e^{\phi(\xi)}h^o_{ab}(\xi; \tau_1, \ldots, \tau_{6g-6}) \tag{67}$$

$\tau's$ are called moduli. As before we have to start with the norm in the space of all metrics

$$||\delta h||^2 = \int \sqrt{h}\delta h_{ab}\delta h_{a'b'}(h^{aa'}h^{bb'} + ch^{ab}h^{a'b'}) \tag{68}$$

where c is non negative. Now

$$h_{ab} = e^{\phi}h^o_{ab} \tag{69}$$

and

$$\delta h_{ab} = (\delta\varphi + \nabla^a\epsilon_a)h_{ab} + (L\epsilon)_a \tag{70}$$

Hence

$$\| \delta h \|^2 = \int e^{\varphi}(\delta\varphi + \nabla^a\epsilon_a)^2(2 + 4c)d^2\xi$$
$$+(L\epsilon, L\epsilon) \tag{71}$$

yielding

$$Dh = d\tau D\phi.D\epsilon.det^{1/2}L^+L \tag{72}$$

The origin of the Liouville action is in det L^+L. With these preliminaries, the object of interest becomes

$$Z = \int \frac{Dg}{\mathrm{Vol}Diff} D_g x e^{-S_M(x;g) - \frac{\mu_0}{2\pi}\int d^2\xi\sqrt{g}} \tag{73}$$

Here the μ_o term is just the weight factor depending on the area of surface. It should be recalled that the genesis of this relation is as follows :

$$G(C(S) = \int \frac{Dx(\xi)}{Df(\xi)} e^{-m_o^2\int\sqrt{\det\|\partial_a x\partial_b x\|}d^2\xi}$$

$$= \int \frac{Dh_{ab}(\xi)}{Df(\xi)} e^{-m_o^2\int h^{1/2}d^2\xi} \int Dx(\xi)\delta(\partial_a x\partial_b x - h_{ab}) \tag{74}$$

The latter integral is represented through the Lagrange multiplier to yield

$$K[c,h] = \int D\lambda_{ab} e^{\int\sqrt{h}\lambda^{ab}h_{ab}d^2\xi} \int Dx e^{\int\sqrt{h}\lambda^{ab}\partial_a x.\partial_b x}$$

$$= e^{\mu\int\sqrt{h}d^2\xi} \int Dx e^{-\int\sqrt{h}\lambda^{ab}\partial_a x.\partial_b x d^2\xi} \tag{75}$$

Polyakov derives (75) based on his conjectures on the "freezing" of Lagrange multipliers. This gives

$$G(C(S)) = \int \frac{Dg}{\mathrm{Vol}diff} Dx e^{\frac{-\mu_0}{2\pi}\int\sqrt{h} - S_M(\mathrm{X,g})} \tag{76}$$

where

$$S_M(x,g) = \frac{1}{2}\int g^{1/2}g^{ab}\partial_a x\partial_b x \tag{77}$$

At this stage there are several approaches to the problem. The pioneering work of Polyakov string $2-D$ quantum gravity in the light cone gauge is one such. One of the most important outgrowths from that development is the celebrated Knizhnik-Polyakov-Zamolodchikov formula stating the manner in which conformal dimensions of the $2-D$ CFT get transformed due to gravitational dressing. Another approach is the one advocated by Distler and Kawai based on the conformal gauge. As the derivation of the KPZ formula in this approach is somewhat more transparent, we shall henceforth follow the DK approahc. It is also possible to analyse these issued in the Hamiltonian formulation. The manifestly gauge invariant formulation has not yet succeeded in deriving the KPZ relation Isler and Pena have however shown that in the light cone gauge Hamiltonian formulation it is possible to get the KPZ relation.

The string exponents :

Let us follow the analyses of Distler and Kawai to determine the so called string exponents. First let us consider

$$I = \int D_g x e^{-\frac{1}{2}\int g^{1/2} g^{ab}\partial_a x^\mu \partial_b x^\mu} \tag{78}$$

There is a dependence on the metric in the measure for integration over x_μ arising out of the requirement that this functional integral be regulated maintaining general covariance. To display this more explicitly note that I can be rewritten as

$$I = \int D_g x e^{\frac{1}{2}\int x^\mu . \Delta x^\mu g^{1/2}} \tag{79}$$

where Δ is the general covariant Laplacian acting on functions. Let the complete set of eigenfunctions and eigenvalues of Δ be given by

$$\Delta \phi_m(\xi) = \lambda_m \phi_m(\xi) \tag{80}$$

the eigenfunctions $\phi_m(\xi)$ are normalised by

$$\int g^{1/2} \phi_m(\xi)\phi_n(\xi) = \delta_{mn} \tag{81}$$

Because of this coordinate invariant way of normalising the eigenfunctions there is an implicit dependence of DX on g. In the conformal gauge

$$g_{ab} = e^\phi \delta_{ab} \tag{82}$$

the action carries no explicit dependence on g. Now

$$I \sim det^{-1/2}(-\Delta) = I(g) \tag{83}$$

$$det(-\Delta) = \int_{\epsilon^2}^{\infty} \frac{dt}{t} \sum e^{-\lambda_n t} \tag{84}$$

the action

$$S = \frac{1}{2}\int g^{1/2} g^{ab}\partial_a x^\mu \partial_b x^\mu \tag{85}$$

also has another invariance namely, invariance under

$$x^\mu \rightarrow x^\mu \; ; \; g_{ab}(x) \rightarrow e^{\lambda(x)} g_{ab}(x) \tag{86}$$

But I, while invariant under diffeomorphisms, is not invariant under Weyl scaling. In fact

$$I(e^\sigma g) = \exp[\frac{d}{48\pi} S_L(\sigma; g)].I(g) \tag{87}$$

where, d is the dimensionality of the target space $\{x\}$ and

$$S_L(\sigma; g) = \int g^{1/2}\{g^{ab}\frac{1}{2}\partial_a\sigma\partial_b\sigma + R\sigma + \mu e^\sigma\} \tag{88}$$

This can be restated as the impossibility of finding a measure that is both diff and weyl invariant. If it is diff invariant, it changes under Weyl transformations as

$$D_{e^\sigma g}x = D_g x.\exp[\frac{d}{48\pi}S_L(\sigma;g)] \tag{89}$$

Likewise

$$\frac{Dg}{Df} = [d\tau]D_g\phi_o det^{1/2}L^+L \tag{90}$$

(See eqn. 73)
and it is convenient to rewrite the determinant as the functional integral

$$\int D_g b D_g c e^{S_{gh}(b,c,g)} \tag{91}$$

over grassmann functions b,c. The measures $D_g b$, $D_g c$ are chosen to be diff invariant. This implies

$$S_{gh}(b,c,g) \tag{92}$$

is diff invariant. It also turns out to be Weyl invariant i.e.

$$S_{gh}(b,c,e^\sigma g) = S_{gh}(b,c,g) \tag{93}$$

but the measures $D_g b, D_g c$ are not Weyl invariant. In fact,

$$D_{e^\sigma g}bD_{e^\sigma g}c = e^{-\frac{26}{48\pi}S_L(\sigma;g)D_g b D_g c} \tag{94}$$

The partition function now becomes

$$Z = \int [d\tau]D_g\phi_o D_g b D_g c D_g X e^{-S_M - S_{gh} + \frac{\mu_0}{2\pi}\int g^{1/2}} \tag{95}$$

Now the problem is that the measure $D_g\phi_o$ is induced by the norm

$$||\delta\phi_0||^2 = \int d^2\xi e^{\phi_o}\delta\phi_o^2 g^{1/2} \tag{96}$$

which has a complicated dependence on ϕ_o. It is better to seek a variable ϕ such that the norm

$$||\delta\phi||^2 = \int d^2\xi g^{1/2}\delta\phi^2(\xi) \tag{97}$$

is diff invariant. Noting

$$g = e^{\phi_o}\hat{g}$$

we write

$$D_g\phi_o D_g b D_g c D_g x = D_{\hat{g}}\phi D_{\hat{g}} b D_{\hat{g}} c D_{\hat{g}} x e^{-S(\phi,\hat{g})} \tag{98}$$

Distler and Kawai make an ansatz for $S(\phi,\hat{g})$ of the form :

$$S(\phi,\hat{g}) = c_1\int d^2\xi(\hat{g})^{1/2}\hat{g}^{ab}\partial_a\phi\partial_b\phi + c_2\int d^2\xi(\hat{g})^{1/2}\hat{R}\phi$$
$$+c_3\int d^2\xi(\hat{g})^{1/2}e^{\phi} \tag{99}$$

Changing the scale of ϕ to make the kinetic term have the standard appearance one has

$$S(\phi,\hat{g}) = \frac{1}{8\pi}\int d^2\xi(\hat{g})^{1/2}\{\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi - Q\hat{R}\phi + 4\mu_1 e^{\alpha\phi}\} \tag{100}$$

The original metric is given by

$$g = \hat{g}e^{\alpha\phi} \tag{101}$$

DK choose the bare cosmological constant to cancel μ_1. The original theory only depended on g and not on ĝ. Thus shifting both ϕ and $\hat{g}$ s.t. $\hat{g}e^{\alpha\phi}$ is unchanged should leave everything unchanged i.e. for,

$$\hat{g}\to e^{\sigma}\hat{g} : \phi\to\phi-\sigma/\alpha\ , \tag{102}$$

$$D_{e^{\sigma}\hat{g}}(\phi-\frac{\sigma}{\alpha})D_{e^{\sigma}\hat{g}}bD_{e^{\sigma}\hat{g}}cD_{e^{\sigma}\hat{g}}xe^{-S(\phi-\sigma/\alpha;\hat{g}e^{\sigma})}$$
$$= D_{\hat{g}}\phi D_{\hat{g}}bD_{\hat{g}}cD_{\hat{g}}xe^{-S(\phi,\hat{g})}\ . \tag{103}$$

Now

$$S_M(x;g) = S_M(x;\hat{g}) \qquad and$$
$$S_{gh}(x;g) = S_{gh}(b,c;\hat{g}) \tag{104}$$

calling

$$\int D_{\hat{g}}\phi e^{-S(\phi,\hat{g})} = e^{-F(\hat{g})} \tag{105}$$

One gets

$$e^{-F(\hat{g})}D_{\hat{g}}bD_{\hat{g}}cD_{\hat{g}}xe^{-S_M(x,\hat{g})} - S_{gh}(b,c,\hat{g}) =$$
$$e^{-F(\hat{g}e^{\sigma})}D_{e^{\sigma}\hat{g}}bD_{e^{\sigma}\hat{g}}cD_{e^{\sigma}\hat{g}}xe^{-S_M(x,e^{\sigma}\hat{g})-S_{gh}(b,c,e^{\sigma}\hat{g})} \tag{106}$$

This is a weaker statement than the one stated by DK. At this stage one could view the problem as one with the fields b, c, x, ϕ propogating in a fixed background $\hat{\mathbf{g}}$. (106) is weaker than the one in DK on two counts. DK write the analogous eqn at the level of the measure itself and secondly they have to invoke that

$$D_{e^\sigma \hat{g}}\phi = D_{e^\sigma \hat{g}}(\phi - \sigma/\alpha) \tag{107}$$

Viewed as a problem of four quantum fields coupled to $\hat{g}$, (106) may be interpreted as the vanishing of the central charge of the 'matter' system. The central charge of the ϕ system, when the cosmological constant μ, is tuned to zero is governed by the Lagrangian

$$\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi \; - \; Q\hat{R}\phi$$

and is easily evaluated on noting that the stress tensor of the system is

$$T_{zz} = -\frac{1}{2}(\partial\phi\partial\phi + Q\partial^2\phi) \tag{108}$$

and therefore

$$c = 1 + 3Q^2. \tag{109}$$

Thus the vanishing of the total central charge implies,

$$1 + 3Q^2 + d - 26 = 0 \text{ i.e. } Q = \sqrt{\frac{25-d}{3}}$$

DK justified their ansatz on the basis that general covariance restricts $S(\phi, \hat{g})$ to be of the form given by (99). But it is clear that general covaraince does not rule out terms of the type

$$\lambda_1 \int d^2\xi(\hat{g})^{1/2} f(\phi)\hat{g}^{ab}\partial_a\phi\partial_b\phi + \lambda_2 \int d^2(\hat{g})^{1/2} K(\phi)\hat{R} \tag{110}$$

where f and g are arbitrary functions. On requiring that $f(\phi)$ transform as a one dimensional metric, the function $K(\phi)$ can be transformed to ϕ. The central charge is unaffected if $f(\phi)$ is of the form

$$f(\phi) = 1 + P(\phi) \tag{111}$$

where $P(\phi)$ is some polynomial s.t $P(0) \;=\; 0$. But any polynomial $P(\phi)$ spoils the DK argument for the scaling behaviour for fixed area partition functions to be given below.

It would be desirable to see if the DK ansatz (99) could be derived from first principles. Let us examine a few possibilities.

Again let us consider the case where the cosmological constant term is absent. Then it is plausible to treat $e^{\phi o}$ as a scalar if we interpret it as the Weyl scaling factor between two metrics

$$g_1 = e^{\phi o} g_2 \tag{112}$$

and if both g_1 and g_2 transform like metrics, $e^{\phi o}$ will have to transform as a scalar (see 67). On the otherhand, if a metric g has been parametrised as

$$g = e^{\phi} \hat{g} \tag{113}$$

where $\hat{g}$ is fixed, ϕ certainly does not transform as a scalar. It should be carefully examined to see if in (73) ϕ is indeed to be interpreted in the first sense. The second interpretation amounts to gauge fixing and then one cannot even talk about the behaviour of ϕ under general coordinate transformations. In fact only transformations under the residual general coordinate transformations maintaining the gauge condition can be used to characterise the behaviour of ϕ.

If, however, $\phi(\phi_0$ in (90)) can be treated as a scalar, one would expect

$$D_{ge^{\sigma}} \phi_0 = D_g \phi_o e^{(1/48\pi) S_L(\sigma; g)} \tag{114}$$

Functionally extending this to $\sigma = \phi_0$

$$D_g \phi_o = D_{\hat{g}} \phi_o e^{1/48\pi S_L(\phi_o; \hat{g})} \tag{115}$$

The above equation has been derived in a slightly different way by Mavromatos and Miramontes. They consider the change of variables

$$2e^{\phi_o/2} = \phi \tag{116}$$

and effectively compute the Jacobian for this transformation and show that it is $\exp(\frac{1}{48\pi} S_L)$. Clearly, all the derivations need to be examined critically. Once (115) is accepted,

$$D_g x D_g b D_g c D_g \phi_0 = D_{\hat{g}} x D_{\hat{g}} b D_{\hat{g}} c D_{\hat{g}} \phi_0 e^{\frac{d-25}{48\pi} \int \sqrt{\hat{g}} [\hat{g}^{ab} \partial_a \phi_0 \partial_b \phi_0 + \hat{R} \phi_0]} \tag{117}$$

Performing a scaling

$$\left(\frac{25-d}{3}\right)^{1/2} \phi = \phi' \tag{118}$$

the exponential factor becomes

$$-\frac{1}{8\pi}\int(\hat{g})^{1/2}\left[(\hat{g})^{ab}\partial_a\phi_o\partial_b\phi_o+\sqrt{\frac{25-d}{3}}\,\phi_0\right] \tag{119}$$

This automatically fixes Q of DK at $(\frac{25-d}{3})^{1/2}$ and the cancellation of the central charge need not be derived by demanding special properties of the measure. Infact, even the scaling (118) is not needed to see the cancellation of the total central change. If a Lagrangian $\mathcal{L}$ is scaled by a, i.e.

$$\mathcal{L}\to a\mathcal{L},$$

the stress tensor also scale like

$$T\to aT$$

but the two point function (propagator) scales as a^{-1}. Thus the central charge from the quadratic part of T scales as

$$<TT>\to a^2\frac{1}{\alpha^2}<TT>$$

i.e it is unchanged. But the central charge from the linear term scales as

$$<TT>\to a^2.\frac{1}{a}<TT>$$

i.e. it scales as a. In the unscaled version, the quadratic part contributes one unit to the central charge and the linear term 25-d. The total being $26-d$ which exactly cancels the $d-26$ of matter + ghost (in the conformal gauge).

The conformal wt of $e^{q\phi}$is also easily calculated

$$\Delta_o(e^{q\phi})=-\frac{1}{2}q(Q+q) \tag{120}$$

The area functional $A = \int d^2\xi(\hat{g})^{1/2}e^{\alpha\phi}$ where $e^{\alpha\phi}$ must be a wt$(1,1)$ object implying

$$\alpha_{\pm}=-\frac{1}{2\sqrt{3}}[\sqrt{25-d}\mp\sqrt{1-d}] \tag{121}$$

$d\to\infty$ can be calculated perturbatively and only

$$\alpha_{-}=-\frac{1}{2\sqrt{3}}[\sqrt{25-d}-\sqrt{1-d}]$$

matches it. It is at this stage that one notices the sickness when $1 < d < 25$. In this range α-has both a real part and an an imaginary part rendering A meaningless.

For $d > 25$, both Q and α are purely imaginary suggesting the use of $i\phi$ as the time coordinate for $d+1$ dimensional theory.

Now let us consider fixed Area partition function $Z(A)$ defined by

$$Z(A) = \int D\phi Dx e^{-S} \delta(\int e^{\alpha\phi} (\hat{g})^{1/2} - A) \tag{122}$$

Consider the shift

$$\phi \rightarrow \phi + \rho/\alpha \tag{123}$$

with ρ, a constant. Assuming the shift invariance of $D\phi, DK$ obtain

$$Z(A) = \int D_{\hat{g}}\phi D_{\hat{g}}x D_{\hat{g}}b D_{\hat{g}}c \delta(e^{\rho} \int (\hat{g})^{1/2} e^{\alpha\phi} - A)$$
$$e^{-S_g(\phi+\rho/\alpha) - S_M(x;\hat{g}) - S_{gh}(x;\hat{g})} \tag{124}$$

But

$$S_{\hat{g}}(\phi + \rho/\alpha) = S_{\hat{g}}(\phi) - \frac{Q\rho}{\alpha}\frac{1}{8\pi}\int (\hat{g})^{1/2}\hat{R}d^2\xi \tag{125}$$

$$= S_{\hat{g}}(\phi) - \frac{Q\rho}{\alpha}(1-h) \tag{126}$$

Hence

$$Z(A) = e^{(\frac{Q}{\alpha}(1-h)-1)\rho} Z(e^{-\rho}A) \tag{127}$$

Note that these arguments would break down if there were a cosmological constant.

In the DK treatment this comes out as an exact scaling law valid for all areas while one expects such a scaling only asymptotically ! The solution of (127) is

$$Z(A) = KA^{(1-h)Q/\alpha - 1} \tag{128}$$

The string susceptibility is, by definition,

$$Z(A) \sim A^{-\gamma} \tag{129}$$

Hence

$$\gamma(h) = (1-h)\frac{(d-25) - \sqrt{(25-d)(1-d)}}{12} + 2 \tag{130}$$

This is also the KPZ result, when $h = 0$.

Before commenting on the DK derivation let us contrast it with KPZ derivation of this result.

The KPZ derivation is based on quantising $2-d$ gravity in the light cone gauge defined by

$$h_{--} = f_{--}(x) \qquad h_{+-} = f_{+-}(x) \tag{131}$$

where $f_{--}(x)$ and $f_{+-}(x)$ are fixed functions and not dynamical degrees of freedom. The expectation value of any gauge invariant operator F must be independent of f_{--} and f_{+-} i.e.

$$\left.\frac{\delta Z_{\mathcal{F}}}{\delta f_{--}}\right|_{f_{--}=0} = \left.\frac{\delta Z_{\mathcal{F}}}{\delta f_{+-}}\right|_{f_{+-}=1} = 0 \tag{132}$$

where $Z_{\mathcal{F}=0}$ is 1.

This implies that

$$T_{++}^{\text{tot}} = T_{++}(h) + T_{++}^{\text{matter}} \tag{133}$$

is weakly zero i.e. matrix elements of this operator between arbitrary physical states vanish

KPZ interpret T_{++}^{tot} to vanish strongly yielding

$$C^{\text{tot}} = d - 28 + \frac{3K}{K+2} - 6K = 0 \tag{134}$$

where K is the coefficient of the Liouville action (the ghost contribution in the light cone gauge is -28).

In the absence of qauntum Liouville dynamics

$$K_o = \frac{d-26}{6} \tag{135}$$

Asymptotically as $d \to -\infty$ one can see that K_o satisfies (134). The next in the chain of arguments used by KPZ is more involved.

Consider fields ϕ transforming under the change of x coordinates as

$$\delta\phi = \epsilon_+ \partial_- \phi + \lambda(\partial_- \epsilon_+)\phi \tag{136}$$

The ward identity relevant for this is

$$\begin{aligned} &\partial_+ < T_{--}(z)\phi(x_1)\ldots\phi(x_N) > \\ &= \sum_i \delta^2(z-x_i) < \phi(x_1)\ldots\delta\phi(x_i)\ldots\phi(x_N) > \end{aligned} \tag{137}$$

Polyakov under the conjecture that (137) quantum mechanically has the same form but with parameters renormalised, proves that this implies a differential equation

$$f\frac{\partial}{\partial_{x_i^+}} < \phi(x_1)\ldots\phi(x_N) >= \sum_{j\neq i}\frac{l_i^a(\lambda)l_j^a(\lambda)}{(x_i^+ - x_j^+)} < \phi_-(x_1)\ldots\phi_-(x_N) > \tag{138}$$

This can be shown to imply

$$\lambda - \lambda^{(o)} = -\frac{\lambda(1-\lambda)}{K+2} \tag{139}$$

where $\lambda^{(0)}$ is the weight in the absence of $2-d$ gravitational effects. KPZ had conjectured that $\lambda^{(0)} = 0$ yields the string susceptibility for the genus zero case. Thus we see that the two critical assumptions in the KPZ analysis are :

(i)

$$T_{++}^{tot} = 0 \qquad \text{strongly} \tag{140}$$

(ii) The classical equations

$$\frac{\partial}{\partial x_i^+}\phi = h_{++}\partial_-\phi_- + \lambda^{(o)}\delta_- h_{++}\phi$$

gets modified to

$$f\partial_+\phi =: h_{++}\partial_-\phi \; : \; + \; \lambda : \partial_- h_{++}\phi : \tag{141}$$

To what extent these conclusions are inevitable should be investigated further. In a Hamiltonian analysis of this problem we find ambiguities in quantisation in such a way that for pure gravity problem the KPZ solution is one of the allowed possibilities.

II. Numerical Simulation of Dynamically Triangulated Random Surfaces

One of the first attempts at studying random surfaces by discretising them was to take a hypercubic lattice and take the action to be the area. The continuum limit of such a theory was found to be trivial in the sense that the string tension became infinite at the critical point. The surfaces degenerate into branched polymers.

The next attempt was to sum over triangulated surfaces with the Nambu-Goto action but fixed triangulations. Again the partition function was found to be ill defined with the surfaces degenerating into spikes.

Currently the most popular candidate for discretised surfaces is the so called DTRS where the triangulations change randomly. This is also the basis for making the connection to Matrix Models. Claims which seem to indicate that only the so called Arithmetic surfaces are triangulatable this way not withstanding, the regularised version of the sum over random surfaces embedded in $\mathbf{R}^d$ is defined by

$$Z(\beta) = \sum_{T\in\mathcal{F}} \frac{1}{S_T}\rho(T) \int \prod_i dx_i \delta(\sum_{i\in T} \frac{x_i}{|T|}).e^{-\beta\sum_{<ij>}(x_i-x_j)^2} \tag{1}$$

Here $\mathcal{F}$ represents the set of non singular triangulations. S_T is the symmetry factor for the triangulation and $\rho(T)$ is a weight factor that will be discussed shortly. The rigid motion due to translation has been frozen with the δ-function. The continuum version of (1) is the polyakov action

$$Z = \int DgDxe^{-T\int(g)^{1/2}g^{\alpha\beta}\partial_\alpha x\partial_\beta x} \tag{2}$$

The factor $\rho(T)$ is generically chosen to be of the type

$$\rho(T) = e^{\alpha\sum_i \log n_i - \sum c_l \sum_i (\frac{n_i-6}{n_i})^{l+1} n_i} \tag{3}$$

The first factor corresponds to $(g)^{\alpha/2}$ factors that should be present in the measure e.g. Fujikawa ... The other terms are counter terms of the type $\int g^{1/2}R$ etc. It is clear that both D_gX and $Dg_{\mu\nu}$contribute to α. The contribution from Dg arises as follows. The distance in metric space is given by

$$\| \delta g_{\mu\nu} \|^2 = \int \delta g_{\mu\nu}(x)\delta g_{\lambda\rho}(y)(g)^{1/2}(g^{\mu\lambda}g^{\nu\rho} + g^{\mu\rho}g^{\nu\lambda} + \gamma g^{\mu\nu}g^{\lambda\rho})\delta(x-y) \tag{4}$$

This yields

$$\det G^{\mu\nu\lambda\rho} = \prod_x g^{\frac{(D-4)(D+1)}{4}}(x) \tag{5}$$

This quantity is well defined only when a regularisation prescription is given. The precise choice of the regularisation is unimportant as long as all the relevant ward identities are satsified. But the measure factor itself could depend on the regularisation scheme. Since DTRS is also a regularisation scheme α for this choice should be determined. α values for some other approaches adopted in the literature are e.g. in the Hamiltonian approach Faddeev and Popov get

$$\det G = \prod_x g(x)^{-5/2} \qquad for \qquad d = 4 \tag{6}$$

while conformally invariant measures yield in $d = 4$

$$\det G = \prod_x g(x)^2 \tag{7}$$

But even in these cases it is clear that a regularisation prescription has to be specified. Till then (6) and (7) are empty of content (I thank Peter Nieuwenhuizen for explaining this to me).
The matrix models seem to be compatible only with $\rho(T) = 1$. It is expected that $\rho(T)$ represents an irrelevant modification but this should be established properly. One can study a larger class of problems by looking at

$$G(\gamma_1 \dots \gamma_n) = \sum_{T \epsilon \mathcal{F}} \rho(T) \int \prod_{i \in T/\partial T} \dots . \tag{8}$$

where the loops γ_i form the boundary. As is the case in Lattice Gauge theory, no gauge fixing for reparametrisation invariance is needed.

The observables of interest, for example, are the mass gap $m(\beta)$, susceptibility $\chi(\beta)$ etc. These are defined by first considering Green's functions with all loops γ_i contracted to points x_i. Then

$$G_\beta(x_1, x_2) \;\sim\; e^{-m(\beta)|x_1 - x_2|} \; as \; |x_1 - x_2| \;\rightarrow\; \infty \tag{9}$$

The critical exponent associated with this is

$$m(\beta) \;\sim\; (\beta - \beta_c)^\nu \;\; as \;\; \beta \rightarrow \beta_c\,! \tag{10}$$

The susceptibility is defined by

$$\chi(\beta) = \int dx G_\beta(x, y) \tag{11}$$

with the associated susceptibility exponent

$$\chi(\beta) \;\sim\; (\beta - \beta_c)^{-\gamma} \tag{12}$$

The one point function $G_\beta(\chi_0)$ scales as

$$G_\beta(\chi_o) \sim (\beta - \beta_c)^{-\gamma-1} \tag{13}$$

will be explained in the other lectures, the discretised surface can also be studied rough the Dual graphs which for triangulations are ϕ^3 graphs. The equivalent of the rtition function is

$$G = \sum_{Q^3 \in \mathcal{F}} \int \prod_{t \in \phi^3} dk_t e^{-\beta' \sum_{<tt'>}(k_t - k_i)^2} \tag{14}$$

he numerical simulations can be done by using either of these representations. The rocedure involves the following steps:

1) Select the topology
2) Introduce a seed discretisation
3) Select an updating procedure that walks through the space of triangulations
4) Make measurements.

Let us first illustrate the method for the original triangulation, for genus zero case. The seed triangulation is a tetrahedron (fig1).

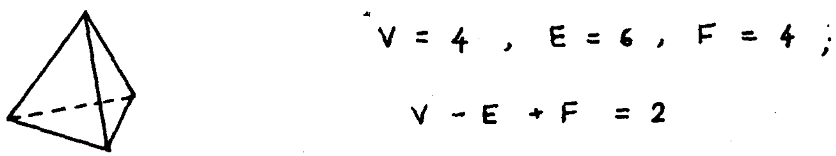

fig.1

Other configurations like

$$V = 2 \qquad E = 2 \qquad F = 2; V = 1 \qquad E = 1 \qquad F = 2$$

are not considered to be 'proper triangulation'. These are called self energy and tadpole graphs (fig2).

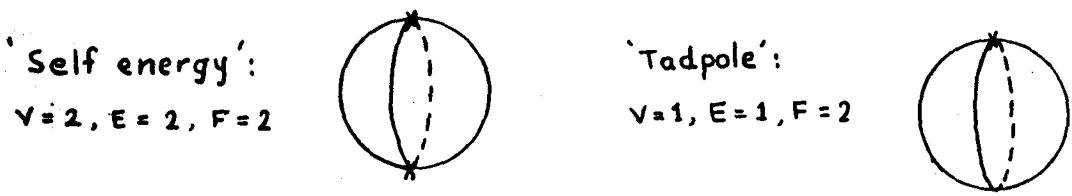

fig.2

The updating procedure should not generate these. Checking this is in fact a major burden on computer time.

In the updating two distinct steps are necessary. If one wants to work with the Grand canonical ensemble, the updating should in corporate change of the number of vertices. Even if one is working with the microcanonical ensemble, such a step is used to generate valid triangulations from the seed. The procedure to add a new point is as follows.

Open out one of the triangle and connect the new vertex to the left over vertex of the Δ . This way $\delta V = 1, \delta E = 3, \delta F = 2$ not changing the Euler characterstic. The reverse process of removing a point is accomplished by reducing a tetrahedron to one of its base triangles (fig3).

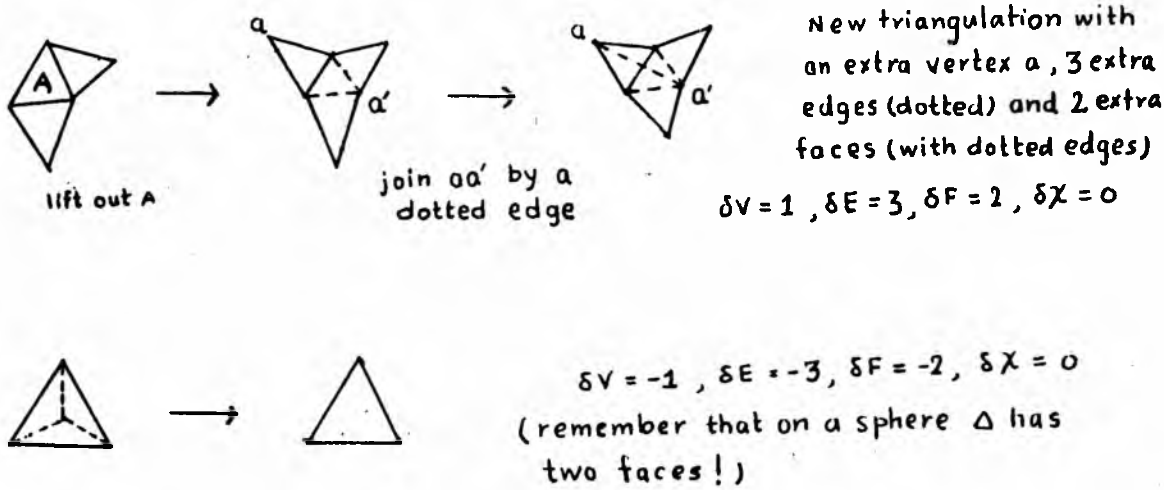

fig.3

By repeated use of this procedure a valid triangulation with any desired number of vertices can be generated. Once that is done, we need an updating technique that samples through all the triangulations of this class. This can be done in a variety of ways. One of the simplest and most elegant is the flip-algorithm.

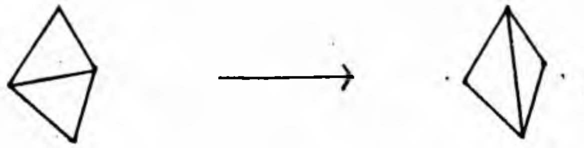

fig.4

other alternatives are: the 'insert', 'delete' routines

fig.5

In both methods the surface has to be checked to see that no self energy or tadpoles are created. Now let us review the corresponding methods for ϕ^3 graphs.

The way to add points or remove them can again be done in a variety of ways. A popular method is contraction of 3 loops and reverse

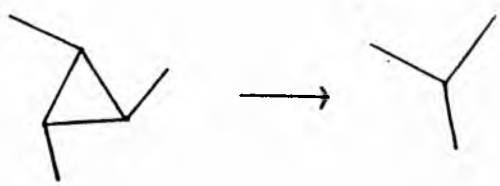

fig.6

The reverse of this operation is

fig.7

The updates that keep the number of vertices fixed are done according to

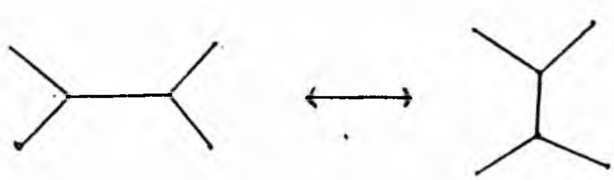

fig.8

This is the dual analog of flipping.

Details : The problem is to determine the partition function

$$G_\beta \sim \sum_N Z_N e^{-\hat{\beta}N}$$

$$Z_N \sim N^{\gamma-2} e^{\tilde{\beta}_c N}$$

where N is the number of faces.

Determination of Z_N directly, while possible in principle is not the one best suited for Monte Carlo Methods as one will have to sample all the configurations. A cleverer method exists.

Let S_N denote the set of configurations with N vertices and let $V_+(x \to y)$ be the transition prob. for $x \epsilon S_{N-2}$ to be changed to $y \epsilon S_N$ and likewise let $V_-(y \to x)$ be the transformation $S_N \to S_{N-2}$. Let $V_\pm$ be chosen so as to satisfy, detailed balance i.e.

$$W_x^{(N-2)} V_+(x \to y) = W_y^{(N)} V_-(y \to x)$$

where W's are the corresponding statistical weights. By definition

$$\sum_{x \in S_{N-2}} W_x^{N-2} = Z_{N-2} \ ; \ \sum_{y \in S_N} W_y^N = Z_N \ .$$

Now consider

$$W_x^{N-2} \sum_{z \in S_N} V_+(x \to z) = \sum_{z \in S_N} W_z^N V_-(z \to x)$$

and

$$\begin{aligned} Z^{N-2} &= \sum_{x \in S_{N-2}} \sum_{z \in S_N} W_z^N \frac{V_-(z \to x)}{\sum_{z' \in S_N} V_+(x - z')} \\ &= \sum_{z \in S_N} W_z^N \sum_{x \in S_{N-2}} \frac{V_-(z \to x)}{\sum_{z' \in S_N} V_+(x - z')} \\ &= Z^N \langle \sum_{x \in S_{N-2}} \frac{V_-(z \to x)}{\sum_{z' \in S_N} V_+(x - z')} \rangle_N \ . \end{aligned}$$

Thus the ratio of $\frac{Z^{N-2}}{Z_N}$ can be expressed as the average of a certain operator, the averages being performed only over states of S_N.

Now a general state in $S_N$is described by

$$y = \{G_N; k_1, \ldots.k_N\}$$

where G_N denotes the ϕ^3 graph with N vertices and $k_1, \ldots k_N$ are the d-dimensional coordinates of the vertices. Let P_0, P_1, P_2 the coordinates of a 3-loop. Let us randomly choose

P_0 to be the point to which the 3-loop is contracted.

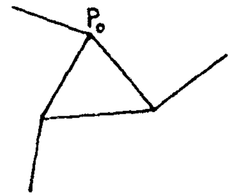

fig.9

Let the graph so obtained be called $G_{N-2}$and let the coordinates of the neighbouring points to P_0 be q_0, q_1, q_2 ; clearly q_i are also neighbours of P_i in G_N. Then

$$W_y^N \sim \exp -\frac{1}{2}[\sum_{<i,j>k_i \neq p} (k_i - k_j) + \sum_{i=0}^{2}(p_i - q_i)^2 + \sum_{i<j}(p_i - p_j)^2]$$

$$W_x^{N-2} \sim \exp -\frac{1}{2}[\sum_{<i,j>k_i \neq p} (k_i - k_j) + \sum_{i=0}^{2}(p_o - q_i)^2]$$

choose $V_-(y \rightarrow x) = 1$ so that

$$V_+(x \rightarrow y) = \exp -\frac{1}{2}[\sum_{o}^{2}(p_i - q_i)^2 - (q_i - p_o)^2 + \sum_{i<j}(p_i - p_j)^2]$$

Now

$$\sum_{x \in s_N} V_+(x \rightarrow z) = \sum_{G_N} 8^{-d/2} \exp -\frac{1}{16}(q_1 - q_2)^2 + \frac{1}{4}(q_1 - p_0)^2$$

$$+\frac{1}{4}(q_1 - p_0)^2$$

$$\mathrm{O}(y) = \sum_{G_{N-2}} \frac{8^{d/2}}{\sum_{G_N} \exp \ldots}$$

It may appear that for each choice of $x \epsilon S_{N-2}$ one has to perform a global sum. But only one global quantity needs to be evaluated once and for the rest the changs are local. Details can be found in Boulatov and Kazakov (NBI-HE-88-42).

The expected behaviour of Z_{2N-2}/Z_{2N} is :

$$\log\left(\frac{Z_{2N-2}}{Z_{2N}} \right) = \mu_c + \frac{2-\gamma}{N} + \frac{c_2}{N^2} + ..$$

the parameter μ_1 can be made to vanish by tuning the bare "cosmological constant". The fit of the data for $d = 1$ yields $\gamma = -0.35$ instead of 0 as given by KPZ relation for $d = 1, h = 0$. There is a correction to the above behaviour which may necessitate going to very large N for an accurate determination of γ. This is suggested most strongly by the matrix model studies and has been confirmed by Das and Jevicki in an effective action formulation of string field theory. These matrix models have to assume a "massive" propagator of the form

$$\prod_{<i,j>} \frac{1}{(k_i - k_j)^2 + 1}$$

in place of the Gaussian propagator. The Matrix models then yield

$$Z_{2N} \sim \frac{N^{\tau-2}}{\log N} (e^{\mu_c})^N + ..$$

and consequently

$$\log \frac{Z_{2N-2}}{Z_{2N}} \sim \mu_c + \frac{2-\gamma}{N} + \frac{1}{N \log N} + ..$$

one sees that the logarthmic corrections can appreciably affect an accurate determination of γ.

When corrected for this, the $d = 1$ data indicate

$$\gamma = -0.15 \pm 0.1.$$

A value considerably closer to the KPZ value of zero.

Clearly the most interesting question to answer from the numerical simulations is what happends for $d > 1$. Since the partition sum is well defined as every stage in the numerical simulations, there is no question of the string susceptibility exponent turning complex. On the other hand the scaling behaviour may not allow the continuum limit to be taken. But the most exciting situation would be when the numerical simulations point to a continuum theory clearly different from the Polyakov formulation of noncritical strings.

Non Perturbative Two dimensional Gravity

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Matrix Models

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WKB ANALYSIS OF THE STRING EQUATION

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Abstract

We give zero curvature formulation of the string equation for pure gravity. Using the method of isomonodromy deformation, we explore the possibility of unique solution consistent with the boundary condition arising from perturbation theory, supplemented by the fact that there are no poles on the real axis.

Recent developments in the study of random surfaces and two-dimensional gravity [1] have brought forth a number of interesting questions of both mathematical and physical nature. One of the remarkable achievements has been a non-perturbative formulation of strings in $d \leq 1$ [Ref.2-4]. The string susceptibility is described as the solution of a non-linear differential equation, often referred to as the string equation of motion. The study of the correlation functions of these theories reveals the close connection of this theory to the KdV hierarchy and its generalizations. The string equations of motion are described by commutation relations between the differential operators of the KdV hierarchy.

One of the interesting questions that arises however, is the nature of the boundary conditions on the string equations of motion. While some of these are fixed by the requirement of matching perturbation theory, the rest can be fixed by the location of the movable poles of the solutions of the string equations of motion. It was subsequently shown that the presence of poles on the real axis for the string susceptibility was inconsistent with the Schwinger-Dyson equations of the theory [5]. It was also shown that the boundary conditions can be fixed by specifying the behaviour at $x \to \infty$ for the $k =$ odd one-matrix models. In these cases numerical work suggests that the string susceptibility has no poles on the real axis [6].

Here, we shall study a different method of understanding the boundary conditions starting from the commutation relation description of the string equation of motion. We show a straight-forward connection between the differential operators described by Douglas [7] and the theory of monodromy-preserving deformations of ordinary differential equations [8,9]. It turns out that the boundary conditions may be equivalently described by the behaviour of the theory for large eigenvalues in the scaling limit. There is also a neat relation with the KdV structure in the theory. Though the method has implications for the study of renormalization group flows between the different one-matrix models, most of our work is confined to the study of the $k = 2$ matrix model. After completing this work, we became aware of a preprint by Moore [10] where, among other things he also addressed the question of uniqueness of the solution of the string equation using the framework of monodromy preserving deformation. The ideas presented in here overlap to a great extent with his work [10] and for ease of presentation we have appropriated some of his ideas.

1.ZERO CURVATURE METHOD AND ISOMONODROMY DEFORMATION

Our starting point is the observation by Douglas [7] that the string equation of motion is given by the relation

$$[P, Q] = 1$$

where P and Q are operators corresponding to $d/d\lambda$ and λ respectively. In the double scaling limit they are differential operators. For pure gravity the operators are

$$Q \equiv D^2 - u \tag{1}$$

$$P \equiv \frac{1}{2}D^3 - \frac{3}{4}uD - \frac{3}{8}u' \tag{2}$$

where $D \equiv \frac{\partial}{\partial x}$. If ψ is the wave-function of the Schrodinger operator Q we can write

$$\mathrm{Q}\psi = \lambda\psi \tag{3}$$

$$\mathrm{P}\psi = \frac{d\psi}{d\lambda} \tag{4}$$

These equations may now be converted to matrix form by defining

$$\Psi \equiv \begin{pmatrix} \frac{\partial\psi}{\partial x} \\ \psi \end{pmatrix}.$$

This gives us the following system of matrix equations

$$\frac{d\Psi}{d\lambda} = A(x, \lambda)\Psi \tag{5}$$

where

$$A(x, \lambda) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\lambda^2}{2} + \begin{pmatrix} 0 & \frac{u}{2} \\ 1 & 0 \end{pmatrix} \frac{\lambda}{2} + \begin{pmatrix} \frac{-u'}{8} & \frac{u^2}{8} + x \\ \frac{-u}{4} & \frac{u}{8} \end{pmatrix} \right) \tag{6}$$

and

$$\frac{\partial \psi}{\partial x} = B(x, \lambda)\psi \tag{7}$$

where

$$B(x, \lambda) = \begin{pmatrix} 0 & \lambda + u \\ 1 & 0 \end{pmatrix} \tag{8}$$

Note that equation (6) may be obtained from eq.(2) by using the Schrodinger equation. The string equation follows naturally as a zero-curvature condition

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial \lambda} + [A, B] = 0 \tag{9}$$

For purposes of further analysis, it is necessary to 'shear' eq.(6). The shearing transformation is such that it diagonalises the highest power of λ in the matrix $A(x, \lambda)$, -i.e. for $\lambda = \xi^2$

$$\psi(x, \lambda) = \sqrt{\xi} \begin{pmatrix} 1 & 1 \\ \frac{1}{\xi} & -\frac{1}{\xi} \end{pmatrix} Y(x, \lambda) \tag{10}$$

so that

$$\frac{dY}{d\xi} = A(x, \xi)Y \tag{11}$$

where

$$A(x, \xi) = [(\xi^4 + \frac{u^2}{8} + x)\sigma_3 - (\frac{u}{2}\xi^2 + \frac{u^2}{8} + x)i\sigma_2 - (\frac{u'}{4}\xi + \frac{1}{2\xi})\sigma_1] \tag{12}$$

Equation (7) may also be similarly transformed. This method can obviously be extended to all string equations with arbitrary k. Written in this system form, these equations may be recognised to precisely correspond to the form of monodromy-preserving deformations of ordinary differential equations with an irregular singular point at infinity. In particular, these equations are similar to those written down by Jimbo and Miwa [8] for the case of Painleve-I. Here we explain the basic idea and apply it to the case at hand.

The basic idea is as follows. The fundamental equation is

$$\frac{d\psi}{d\lambda} = A(x, \lambda)\Psi \tag{13}$$

referred to as the 'equation in λ'. However, in practice we will use the equation in ξ as the equation in λ. The solutions of this equation namely Y provide a set of monodromy data. The second equation(7) deforms the solution, by varying the parameter x while keeping the monodromy data fixed.

To define the monodromy data we first write down the asymptotic solution as $\xi \to \infty$,

$$Y \approx \xi \to \infty(I + \frac{\hat{Y}_1}{\xi} + \frac{\hat{Y}_2}{\xi_2} + \ldots) \exp T(\lambda) \tag{14}$$

where

$$T(\lambda) = \frac{4}{5}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\xi^5 + \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} x\xi - \frac{1}{2} ln\xi \tag{15}$$

and

$$\hat{Y}_1 = \begin{pmatrix} -H_I & \\ & H_I \end{pmatrix} \quad \text{and} \quad \hat{Y}_2 = \begin{pmatrix} & u \\ u & \end{pmatrix} \tag{16}$$

with $H_I = \frac{1}{2}u'^2 - (2u^3 + xu)$.

This asymptotic solution is not valid in all angular sectors at ∞. The growing and decaying solutions interchange roles in neighbouring sectors. In general, the decaying solution in one angular sector can be analytically continued into the neighbouring sector to a growing solution. The growing solution needs, however, an addition of a decaying part in order to reproduce asymptotically a decaying solution [8].

We may define angular sectors at ∞, separated by lines where $Re\xi^5 = 0$ and hence the decaying and growing solutions interchange their roles. These lines (anti-Stokes lines) separate angular sectors $A_{j+1} = \frac{(2j-1)\pi}{10} l\theta < \frac{(2j+1)\pi}{10}$, j =1 to 10. The solution in each sector is labelled as Y_j and $Y_{j+1} = Y_j S_j$ where S_j is

$$\begin{pmatrix} 1 & 0 \\ S_j & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & S_j \\ 0 & 1 \end{pmatrix} \tag{17}$$

for j odd and even respectively. This is referred to as the Stokes phenomenon and the matrices the Stokes matrices. There is also a fixed square-root branch in the solution at ∞ and a square-root branch at zero. The solutions at zero and ∞ in each angular sector may be related by a connection matrix C

$$Y = \Phi C \quad with \quad C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad and \quad \alpha\delta - \beta\gamma = 1 \tag{18}$$

where Φ is the asymptotic solution at zero. The collection of monodromy data which is to be held fixed under deformation is given by the set

$$S_j \ \ j = 1, \ldots, 10; \ \ \alpha, \beta, \gamma, \delta \quad with \quad \alpha\delta - \beta\gamma = 1 \tag{19}$$

This large collection of data is not independent and the symmetries of the equation reduce the total number. We can check that if $Y(x, \xi)$ is a solution so is $MY(x, -\xi)$ where

$M = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Thus $S_j = S_{j+5}$ and the number of Stokes parameters is reduced by half. We can also compute the monodromy matrix at 0 and ∞.

$$\Phi(x, \xi e^{2\pi i}) = \Phi(x, \xi) M_o \quad and \qquad\qquad Y_1(\xi) = Y_{10}(\xi e^{2\pi i}) S_{10} M_\infty \tag{20}$$

There, using the connection matrix, we can write the following constraint

$$\prod_j S_j M_\infty = C^{-1} M_o^{-1} C. \tag{21}$$

However this constraint is not always true. A sharper constraint is obtained by the following considerations. Under the $\xi \to \xi e^{\pi i}$ we can see by direct inspection of the asymptotic solution that

$$Y_6^{(1)}(\xi, x) \sim M Y_1^{(2)}(\xi e^{\pi i}, x) \quad where \quad M = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{22}$$

The asymptotic solution at 0 gives us

$$\Phi(\xi e^{\pi i}) = \Phi(\xi) J. \tag{23}$$

Now,

$$Y(\xi e^{\pi i}) = \Phi(\xi e^{\pi i}) A = \Phi J A \tag{24}$$

$$Y_1(\xi) S_5 S_4 S_3 S_2 S_1 = Y_6(\xi) \tag{25}$$

But

$$Y_6(\xi) = \left(Y_1^{(2)}(\xi e^{\pi i}), Y_1^{(1)}(\xi e^{\pi i})\right) \tag{26}$$

Putting all these together we get

$$S_5 S_4 S_3 S_2 S_1 = M A^{-1} J A. \tag{27}$$

Thus with these relations we see that only two of the Stokes parameters are independent. Of course, a proper proof is needed that the second equation indeed generates deformations that do not change the monodromy data. We asume that such a proof can be provided following the lines indicated by Jimbo and Miwa [8]. Since the two independent Stokes parameters are invariants of the deformation flow, they may be considered as providing the boundary conditions on the Painleve-type equations.

2.THE INVERSE AND DIRECT PROBLEM OF MONODROMY THEORY

The inverse problem consists in determining what $u(x)$ is obtained by a specification of the monodromy data. This means that we are solving the initial value problem of the string equations. However, in general this problem is quite complicated and the number of attempts to solve for similar equations like the Painleve-II have not led to much progress. The most recent attempt has been to reduce the problem to that of solving a Riemann-Hilbert problem. However, in the case of Painleve-II even this method is useful only in special cases and that too in conjunction with the techniques of the direct problem. The inverse problem for the Painleve-I and its generalizations appears to be open in the mathematical literature. Hence, we will discuss here only the direct problem in any substantial detail.

The direct problem consists of determining the Stokes data that characterises different solutions of the string equations of motion. The method of choice here is the WKB method. From the string equations we can obtain appropriate asymptotics for different solutions at $x \to \pm\infty$. Since the equation in λ becomes an equation with x as a large parameter, it is natural to apply the WKB method. In general, for the string equations we know that perturbation theory fixes only the boundary conditions at $x \to \infty$. Thus the Stokes parameters would not be fully determined. However, in the odd k one-matrix models we could use the suggestion of BMP for the boundary conditions at ∞ to fix the Stokes parameters completely [11]. It is much harder however, to relate this result to the location or indeed the non-existence of the poles of $u(x)$ on the real axis.

3.THE WKB METHOD

It is easy to show that for $x \to -\infty$ to leading order $u(x) \approx \sqrt{\frac{-8x}{3}}$. Using the rescaled variable $\xi \to \eta(\frac{-8x}{3})^{1/4}$, and substituting this in eq.(12) we obtain

$$\frac{dY}{d\eta} = \tau\left[\left(\eta^4 - \frac{1}{4}\right)\sigma_3 - \left(\frac{\eta^2}{2} - \frac{1}{4}\right)i\sigma_2 - \frac{1}{2\eta T}\left(\frac{2\eta^2}{3} + 1\right)\sigma_1 + O\left(\tau^{-2}\right)\right] \tag{28}$$

We must first transform the equation such that the $O(1)$ terms on the right-hand side are diagonal, giving an equation of the form

$$\frac{d\tilde{Y}}{d\eta} = \tau\mu\sigma_3\tilde{Y} + \tau F\tilde{Y} \tag{29}$$

where μ is diagonal and order 0 in τ while F is of order $1/\tau$. The turning points are then given by the zeroes of μ. Thus, we compute the leading WKB expression

$$Y_{WKB} \approx T(\eta)\exp\left(\int_{\eta_c}^{\eta}\mu d\eta\right) - diag\left(T^{-1}\frac{dT}{\eta}\right). \tag{30}$$

In our case the corresponding calculations give the following :

$$\mu = \sqrt{\left(n^4 - \frac{1}{4}\right)^2 - \frac{1}{4}\left(\eta^2 - \frac{1}{2}\right)^2 + \frac{1}{4\eta^2 T\tau^2}\left(\frac{2\eta^2}{3} + 1\right)^2} \tag{31}$$

and

$$T(\eta) = \begin{pmatrix} 1 & \frac{-\frac{1}{2\eta\tau}(\frac{2\eta^2}{3}+1)+(\frac{\eta^2}{2}-\frac{1}{4})}{\mu+(\eta^4-\frac{1}{4})} \\ \frac{\mu-(\eta^4-\frac{1}{4})}{\frac{1}{2\eta\tau}(\frac{2\eta^2}{3}+1)-(\frac{\eta^2}{2}-\frac{1}{4})} & 1 \end{pmatrix} \tag{32}$$

The next step is to find the domains of $Re \int_{\eta_c}^{\eta} \mu$ =+ve and -ve. We therefore plot the lines of $Re \int_{\eta_c}^{\eta} \mu = 0$. These are given by the curves in u and v which are the real and imaginary parts of η. We note, that asymptotically, the lines tend to the anti-Stokes lines at ∞ as already described.

In the neighbourhood of the real root of μ, we expand μ in powers of $\eta - \eta_0$. Defining $\zeta_0 = \tau^{1/2}(\eta - \eta_0)$, we get the equation

$$\frac{dY_0}{d\zeta_0} = [(A\sigma_3 + B\sigma_2)\zeta_0 + O(\tau^{-1/2})]Y_0. \tag{33}$$

This equation can be diagonalised and its solution is asymptotic to

$$Y_0 = T' exp\left[\sqrt{A^2+B^2}\sigma_3\frac{\zeta_0^2}{2}\right] \tag{34}$$

where, T' diagonalises the matrix $A\sigma_3 + B\sigma_2$. Now, using the formal asymptotic solution, the WKB solution and the solution near the turning point we determine the connection matrices relating the WKB solution to the other two solutions. The connection matrices are,

$$C = \lim_{\eta\to\infty} exp\left[\tau\int_{\eta_k}^{\eta}\mu(\eta')d\eta' - \frac{\eta^5}{5} - x\eta\right] \tag{35}$$

and

$$N = exp\left[\tau\int_{\zeta_k}^{\zeta}\mu(\zeta')d\zeta' - \sqrt{A^2+B^2}\frac{\zeta^2}{2}\right]. \tag{36}$$

Following the procedure of ref.[12] we can write down the Stokes matrices as,

$$S_k = C_{k+1}^{-1}N_{k+1}^{-1}N_kC_k. \tag{37}$$

Determination of all the Stokes matrices using this method and then imposing the constraints derived earlier is a long and tedious calculation [13]. Here, instead we will follow the approach of Moore [10]. The theorem that he proved is as follows. Suppose that at a turning point four Stokes lines join to form three open regions, then the Stokes matrix for the middle region is trivial. Using this theorem and the fact that the reality condition on u relates the Stokes matrices symmetric about the $Im\eta$ axis with each other, we determine the Stokes parameters for the case at hand. From fig. 1 and using the theorem it is easy to see that the Stokes parameter s_5 vanishes. Using this information in eq.(21) and (27) we can eliminate one of the two independent Stokes parameters, which implies that there is a one parameter family of solutions consistent with the asymptotic behaviour, -i.e., a one parameter family of solution consistent with perturbation theory.

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D = 1 SUPERSYMMETRIC MATRIX MODEL

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In recent times it has become possible to study string theories in low dimensions by discretizing the world sheet and establishing a connection to matrix models. The connection to random matrix theory led to summing the series to all orders (to arbitrary genus surfaces) in the critical limit. This has been achieved in the case of $d = 1$ which is considered as one dimensional matter coupled to gravity in 2-dimensions. The reduction of this model to N fermions in one dimensional quantum mechanics can be exactly solved in the critical limit and expressions can be given for the partition function. Supersymmetric version can also given and specific potentials can be analysed. We consider the arbitrary order critical limit of Supersymmetric model in this lecture.

First we consider the non perturbative expansion for $D = 1$ string/matrix model.

The representation of sum over random surfaces in terms of matrix models is well known.

$$Z_N(\beta) = \int D^{N^2}\phi e^{-\beta \int dt Tr[\frac{1}{2}\dot{\phi}^2 + U(\phi)]}$$

In the critical limit $\frac{N}{\beta} \equiv \rightarrow g_c$ perturbation series diverges and this coincides with the continuum definition.

The problem of finding the Groundstate energy $\ln Z(\beta)$ can be obtained from the equivalent problem of finding the ground state of N fermions in a potential $U(x)$:

$$H = -\frac{1}{2\beta^2}\frac{d^2}{dx^2} + U(x).$$

If the energy levels of such a Hamiltonian are βe_i then ground state energy of the N fermion problem.

$$E_{gs} = \beta \sum^N e_i$$

Density of states

$$\rho(e) = \frac{1}{\beta}\sum \delta(e_n - e)$$

$g = \int^{\mu_F} \rho(e)$ de : μ_F : Fermi level.

$$E = \beta^2 \int \rho(e) e \text{ de}$$

The appropriate sign that we are in the continuum limit is that ρ is singular function of the coupling constant.

This is obtained by adjusting the maximum of the potential as it approaches Fermi level from above.

$$\text{Let } \Delta = g_c - g$$

For finding the critical behaviour it is easier to find the derivatives of g and E.

$$\frac{\partial g}{\partial \mu_F} = -\rho(\mu_F); \frac{\partial E}{\partial \mu} = -\beta^2 \mu_F \rho(\mu_F)$$

For solving them on Sphere one can use leading order WKB approx.

$$\left(-\frac{\hbar^2}{-2m}\frac{d^2}{dx^2} + V - E\right)\psi = 0.$$

$$\psi = e^{i\hbar \int \frac{\hbar^n}{i^n} S_n(x) dx}$$

then

$$S_o = \sqrt{2m(E - V)}$$

and

$$S'_{n-1} = -\sum S_{n-m} S_m$$

The exact quantisation condition would be.

$$\int S_o + \sum_2^{\infty} \int \frac{\hbar^n}{i^n} S_n dx = 2\pi n\hbar.$$

This condition amounts to

$$\sqrt{2m}\int_{x_1}^{x_2} \sqrt{E - V} dx - \frac{\hbar^2}{24\sqrt{2m}}\frac{d}{dE}\int_{x_1}^{x_2} \frac{V''}{\sqrt{E - V}} dx +$$
$$\frac{\hbar^4}{2880(2m)^{\frac{3}{2}}}\frac{d^3}{dE^3}\int_{x_1}^{x_2} \frac{7(V'')^2 - 5V'V''}{\sqrt{E - V}} dx -$$

$$\frac{\hbar^6}{725760(2m)^{\frac{5}{2}}}\left[\frac{d^4}{dE^4}\int_{x_1}^{x_2}\frac{216(V''')^2}{\sqrt{E-V}}dx+\right.$$
$$\left.\frac{d^5}{dE^5}\int_{x_1}^{x_2}\frac{93(V'')^3-224V'V''V'''+35(V')^2V''''}{\sqrt{E-V}}dx\right] = n\pi\hbar \tag{1}$$

Let us take the case of Sphere :

$$\int\sqrt{2m(E-V)} = n\pi\hbar$$

$$\int\frac{1}{\sqrt{2(E-V)}} \sim \rho$$

$$\mu_c-\mu_F=\mu$$

$$\rho \sim \int\frac{dx}{(x_c-x)} \sim -\frac{1}{\pi}ln(x_c-x) = -\frac{1}{2\pi}ln\mu$$

$$E\approx-\frac{N^2\Delta^2}{\ln\Delta} \qquad \frac{\partial g}{\partial\mu}=\ln\mu$$

$$\Delta g=\mu\ln\mu$$

$$\Delta\approx\mu\ln\mu$$

Crucial result required is singularity behaviour of ρ :
For higher order result : Let $V\approx\tilde{V}-2x^2$.

$$E-\tilde{V}\approx-2x^2\approx\mu.$$

$$-\frac{1}{24\beta^2}\frac{d^2}{d\mu^2}\left(\int\frac{-4dx}{\sqrt{2(E-V)}}\right)\approx-\frac{1}{-6\beta^2}\frac{d^2}{d\mu^2}\left(-\ \tfrac{1}{2}\ \ln\ \ \mu\right)$$

$$\approx\frac{1}{12\beta^2\mu^2}.$$

$$\frac{1}{2880\beta^42}\frac{d^4}{d\mu^4}\left(\int\ \ \frac{7.16dx}{\sqrt{2(E-V)}}\right)\approx\frac{7.16}{2880\beta^4}\frac{d^4}{d\mu^4}\left(-\ \tfrac{1}{2}\ \ln\ \ \mu\right)$$

$$\approx\frac{7}{120\beta^4\mu^4}.$$

$$-\frac{93(-64)}{725760\beta^64}\frac{d^6}{d\mu^6}\{-\frac{1}{2}\ln\mu\}$$

$$\approx -\frac{31}{252\beta^6\mu^6}.$$

$$\rho(\mu) = \frac{1}{2\pi}\left[-\ln\mu + \frac{1}{6\beta^2\mu^2} + \frac{7}{60\beta^4\mu^4} + \frac{31}{126\beta^6\mu^6} + ..\right]$$

The same thing can be obtained from density of states of harmonic oscillator analytically continued to imaginary frequencies

$$\rho = \frac{1}{2\pi} Im \sum \frac{1}{(n+\frac{1}{2})\hbar\omega - E}$$

$$\approx -\frac{1}{2\pi}\ln\mu + \frac{1}{\pi\beta\mu} Im \sum \frac{1}{1 - i(\frac{2n+1}{\beta\mu})}.$$

$$\approx \frac{1}{2\pi}\left[-\ln\mu + \sum_1^\infty (2^{2m-1} \quad - \quad 1)\frac{|B_{2m}|}{m}\frac{1}{(\beta\mu)^{2m}}\right]$$

SUSY GENERALIZATION :

Supersurface as a mapping from $2D$ surface to Superspace :-

X = Superposition : triangulate the $2D$ surface and each Δ corresponds a point in the superspace. Each super triangulation is fixed by the topology of the $2D$ surface. We assign weight as a product of the relative super distance of all the contiguous triangles, then we have supersymmetry of the target space and use the free propagator of the superfield. We start from,

$$S = \int dt d\theta d\bar{\theta}\,(-\phi D^2\phi + W(\phi))$$

Marinari & Parisi Super potential

$$W'(\phi) = \frac{1}{2}\phi^2 - \frac{\lambda}{3\sqrt{N}}\phi^3.$$

$$\phi \equiv X + \theta\psi + \bar{\theta}\bar{\psi} + A\theta\bar{\theta}$$

$$X, \psi, \bar{\psi}, A \text{ are } NxN \text{ matrices.}$$

$$S = \int dt Tr\left[-X\frac{d^2X}{dt^2} + F^2 + \bar{\psi}\frac{d\psi}{dt}\right] + \bar{\psi}\frac{dF}{dX}\psi$$

$$F = X - \frac{\lambda}{\sqrt{N}}X^2.$$

One can integrate out ψ one gets Witten Super symmetric quantum mechanics : For the case

$$N=1 \qquad H = p^2 + F^2 - \sigma_3 F' = \begin{pmatrix} H_B & \\ & H_F \end{pmatrix}$$

$$H_{B,F} \equiv p^2 + F^2 \mp F'$$

For general N.

$$H \equiv Tr\left(p^2 + F^2 - \frac{dF}{dX} + 2\bar{a}\frac{dF}{dX}a\right).$$

For purely Bosonic sector

$$H_B = Tr\left(p^2 + F^2 - \frac{dF}{dX}\right).$$

Critical properties of this Hamiltonian will be required

$$V(X) = x^2(1-\lambda x)^2 - 1 + 2\lambda x$$

For low λ this V has two minima and one maximum.
At the critical point

$$\lambda_c = \frac{1}{6\sqrt{3}}.$$

But,this coresponds to $m=3$ critical point. For further work let us put proper factors of β in the Hamiltonian.

$$H_B = p^2 + F^2 - \frac{1}{\beta}F'.$$

N "fermions" fill up all levels upto Fermi level and in the large $N \to \infty, \rho$ becomes dense and maximum of the potential just touches Fermi level.

$$\rho = \frac{1}{\beta}\sum \delta(E_n - E); g = \frac{N}{\beta}; E_{gs} = \beta^2 \int E\rho \,\mathrm{dE}$$

$\mu = V_{\max} - E_F$. As $\mu \to 0$. Critical behaviour is obtained.

To zeroeth order in WKB.

$$\rho = \frac{1}{\pi}\int \frac{dx}{\sqrt{E-V}} \approx -\frac{1}{2\pi}\ln\mu$$

(one has to take into account β dependence of V and turning points.)
The same behaviour in Susy case. We obtain the higher order results in a Susy invariant way.

Expanding around the maximum of F^2

$$H = p^2 + a^2x^2 + \ldots. - \frac{1}{\beta}(a + \ldots.)$$

In the sealing limit our procedure leads to a Supersymmetric harmonic oscillator analytically continued to imaginary frequency.

$$\rho = \frac{1}{\pi} Re \sum \frac{1}{2n + i\beta\mu} = -\frac{1}{2\pi}\log\mu - \frac{1}{2\pi}$$

$$\rho(\mu) = -\frac{1}{2\pi}\left[\ln\mu + \frac{1}{3\beta^2\mu^2} + \frac{2}{15\beta^4\mu^4} + \frac{16}{63\beta^6\mu^6} + \ldots\right]$$

To do this higher order WKB. We modify WKB to suit Susy i.e. SWKB.

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V - E\right]\psi = 0.$$

$V = f^2 - \hbar f'$ - integrand is f^2 instead of V

- turning points $E - f^2 = 0$ instead of $E - V = 0$.

Many of the potentials can be solved in one order of calculation which will require infinite order in WKB. But for us the interesting point is $E^B_{n+1} = E^F_n$ is maintained in every order.

The SWKB series upto $\hbar^4$ is :

$$\sqrt{2m}\int_a^b \sqrt{E - f^2}dx - \frac{\hbar^2 E}{6\sqrt{2m}}\frac{d^2}{dE^2}\int_a^b \frac{(f')^2}{\sqrt{E - f^2}}dx +$$
$$\frac{\hbar^4}{720(2m)^{\frac{5}{2}}}\left[\frac{d^2}{dE^2}\int_a^b \frac{30f'f'''}{\sqrt{E - f^2}}dx + \right.$$
$$\left.\frac{d^3}{dE^3}\int_a^b \frac{-8(f')^4 - 31f(f')^2f'' + 7f^2(f'')^2 - 5f^2f'f'''}{\sqrt{E - f^2}}dx\right] +$$
$$\frac{\hbar^6}{90720(2m)^{\frac{5}{2}}}\left[\frac{d^3}{dE^3}\int_a^b \frac{378(f''')^2}{\sqrt{E - f^2}}dx + \right.$$
$$\frac{d^4}{dE^4}\int_a^b \frac{-2160f'f''f''' + 1674(f')^2(f'')^2 - 108f^2(f''')^2}{\sqrt{E - f^2}}dx +$$
$$\frac{d^5}{dE^5}\int_a^b \frac{1}{\sqrt{E - f^2}}\left[96(f')^6 - 1119(f')^4f'' + 729f^2(f')^2(f'')^2 + \right.$$
$$\left.\left. 399f^2(f')^3f''' - 93f^3(f'')^3 + 224f^3f'f''f''' - 35f^3(f')^2f''''\right]dx\right] = N\pi\hbar$$

Four our case $f = i\sqrt{2}x \& f' = i\sqrt{2}$. To get $\rho(\mu)$ one must differentiate N w.r.t.E.

$$\rho_o = \frac{\sqrt{2}}{2\pi}\int \frac{dx}{\sqrt{2x^2-\mu}} \approx -\frac{1}{2\pi}\ln\mu$$

$$\rho_1 = \rho_1' + \rho"_1$$

$$\rho' = -\frac{\mu}{6\beta^2}\frac{d^3}{d\mu^3}\int \frac{(-2)dx}{\sqrt{2(-\mu+2x^2)}} = -\frac{1}{3\mu^2\beta^2}$$

$$\rho"_1 = \frac{-1}{6}\frac{d^2}{d\mu^2}\int \frac{(-2)dx}{\sqrt{2(-\mu+2x)}} = -\frac{1}{6\mu^2\beta^2}$$

$$\rho_1 = -\frac{1}{6\mu^2\beta^2}$$

$$\rho_2 = -\frac{1}{15}\beta^4\mu^4\left[\frac{-8.4}{7202\beta^4}\frac{d^4}{d\mu^4}\left(-\ \frac{1}{2}\ \ln\ \ \mu\right)\right]$$

$$\rho_3 = -\frac{8}{63\beta^6\mu^6}\left[\frac{96(-8)}{90720x4}\frac{d^6}{d\mu^6}\{\frac{-1}{2}ln\mu\}\right]$$

$$\rho = -\frac{1}{2\pi}\left[ln\mu + \frac{1}{3(\beta\mu)^2} + \frac{1}{15(\beta\mu)^4} + \frac{16}{63\beta^6\mu^6} + \ldots\right]$$

$$= -\frac{1}{2\pi}Re\psi\left(\tfrac{i\beta\mu}{2}\right) - \log\frac{\beta}{2}$$

$$= -\frac{1}{2\pi}\ln\mu - \frac{1}{2\pi}\sum\frac{|B_{2k}|}{2k}\left(\frac{2}{\beta\mu}\right)^{2k}.$$

Exactly similar to $d = 1$ Bosonic String.
Similar Logarithmic violation for genus '0' & '1'.

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FOUR MATRIX MODEL IN KdV APPROACH

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Nonperturbative string equations are found explicitly for the three-state Potts model coupled to two dimensional gravity. These string equations are used to derive the scaling behaviour of several correlation functions on the sphere and it is shown that they agree with the calculations of the continuum theory. Relationship with W-gravity has been pointed out.

I INTRODUCTION

There has been a great deal of interest recently in the study of Conformal Field Theory[1](CFT) models coupled to the two dimensional quantum gravity both in the context of large N matrix model[2] and in the continuum[3,4] conformal field theory approach. A major discovery in the matrix model approach was made by three groups [5] simultaneously towards the end of 1989. They were able to obtain the nonperturbative equations satisfied the specific heat (second derivative of the free energy) for the pure gravity from a singel hermitian large N matrix model in a double scaling limit. Subsequently, it was shown by Douglas[6], in a beautiful paper, that the string equations for any minimal CFT coupled to two dimensional quantum gravity can be wirtten as a universal equation in the Lax operator formalism. The string equation that are derived for the Ising [7,8,10] and the Tricritical Ising [9] models by solving the two and three matrix models in a double scaling limit coincide with the universal equations written down by Douglas.

In this article, we present the string equations for the 3-state Potts model explicitly in the Lax operator approach and make connection with W_N-gravity theories. Reader can look into reference 15 for details. We find [15] that the string equations for the 3-states Potts model are a system of four coupled differential equations and are parametrized by three constants. The 'critical' 3-states Potts model corresponds to the case when all these three constants are set equal to zero and this is equivalent to the case of vanishing external

[1]Talk presented by Alok Kumar

fields. It will be interesting to show that there are special values of these constants where 3-state Potts model makes phase transition to differnt Conformal Field Theories. The rest of the paper is arranged as follows : In section II we present the essential features of the 3-state Potts model and its scaling properties in the presence of two dimensional quantum gravity. Section III deals with the Lax operator formalism and its application in deriving the string equations. Then we present such equations for the 3-state Potts model. In section IV, the scaling properties of correlation functions are obtained from the results of Sec.III and these are compared with the scaling properties derived from the CFT approach of section II. Concluding remarks and discussions are presented in section V.

II THE 3-STATE POTTS MODEL

The 3-states Potts model is the $m = 5$ model in the minimal unitary series of Friedan, Qiu and Shenker [11]. The mth minimal model has the central charge

$$c = 1 - \frac{6}{m(m+1)} \tag{2.1}$$

Therefore, this model has central charge $c = 4/5$. The conformal weights of the primary fields are given by,

$$h_{(p,q)} = \frac{[(m+1)P - mq]^2 - 1}{4m(m+1)} \tag{2.2}$$

where, $p \leq m - 1$ and $q \leq$ m. The allowed conformal weights are given in table 1. Fusion rules allowed by coformal symmetry has been derived in ref.1. In CFT. the correlation functions among the primary fields on the sphere can be computed either by solving the differential equations obtained from the conformal Ward identities or alternatively, by writing down the primary fields as vertex operatos in the Feigen-Fuchs constructions [12]. The 3-states Potts model coupled to the 2-dimensional quantum gravity can be studied in the continuum approach either in the light cone gauge [3] or in the conformal gauge [4]. As a result of the coupling of the two dimensional quantum gravity to the primary fields, they get gravitaionally dressed. The correlation functions exhibit scaling behaviour as a function of area or its conjugate, the cosmological constant. The area scaling behaviour of the partition function and the one-point function are given respectively as [4],

$$Z(A) \sim A^{\Gamma-3} \tag{2.3a}$$

$$F_{\phi}(A) \sim A^{1-\Delta} \tag{2.3b}$$

where string susceptibility exponent Γ for genus h and the gravitational scaling dimension Δ of a primary field Φ are given by

$$\Gamma(h) = (1-h)\frac{d - 25 - [(25-d)(1-d)]^{1/2}}{12} + 2 \tag{2.4a}$$

and

$$\Delta = \frac{\pm[1-d+24\Delta o]^{1/2} - [1-d]^{1/2}}{\sqrt{(-25-d)} - \sqrt{(1-d)}} \tag{2.4b}$$

For the 3-state Potts model, the string susceptibility exponent $\Gamma(0)$ on sphere is -1/5. The gravitational scaling dimensions of various primary fields are also listed in table 1. It is interesting to note that $\Gamma(0)$ for pure W_N-gravity theory is equal to $-1/N$[16]. In particular the string susceptibility exponent for pure W_5-gravity is same as that of the three states Potts model. For this reason it is conjectured that $(p = N, q = N+1)$ conformal field theory coupled to the two dimensional gravity is equivalent to pure W_N-gravity theory.

III THE LAX OPERATOR APPROACH

In this section we write down the fundamental differntial equations satisfied by specific heat in the 3-states Potts model explicitly using the Lax pair approach. It was argued by Douglas [6] that the differntial equations satisfied by the specific heat in (p,q) conformal field theory coupled to 2-dimensional quantum gravity can be obtained as a universal equation,

$$\left[L, \left(Q^{q/p}\right)_+\right] = 1 \tag{3.1}$$

where L is a p the order self adjoint differntial operator

$$L = D^p + (u_{p-2}D^{p-2} + D^{p-2}u_{p-2}) + \ldots + (u_1 D + u_1 D) + u_o \tag{3.2}$$

and Q, the p th root of L is defined by

$$Q = D - r_1 D^{-1} + r_2 D^{-2} + \ldots \tag{3.3}$$

where the coefficients $r_i's$ of Q are obtained from the consistency condition

$$Q^p = L \tag{3.4}$$

The suffix '+' in eq.(3.1) implies that only the positive powers of D in the operator $Q^{q/p}$ has been kept. Equation (3.1) in the large N matrix model is interpreted as the canonical commutation relation between matrix variable M and its conjugate in a double scaling limit.

The differential equations for the specific heat derived from the Lax operator approach for (2,3), (3,4) and (4,5) as well as many $(2,p)$ models has been shown to be identical to the ones derived from the matrix moel approach by using the orthogonal polynomial methods in the double scaling limit. An explicit solution of the 3-matrix model has been obtained recently in ref.9. To our knowledge such as analysis has not been carried out for four or higher number of matrices. However, as stated in the introduction, we work directly in the double scaling limit of the theory using the Lax operator approach and keep a complete study of the 4-matrix model using orthogonal polynomials as a future project.

The 3-States Potts Model

In this section we apply the Lax operator formalism to the (5,6) model. The Lax operator takes the form

$$L = D^5 + (u_3 D^3 + D^3 u_3) + (u_2 D^2 + D^2 u_2) + (u_1 D + D u_1) + u_o \tag{3.5}$$

We can rewrite the above equation as

$$\begin{aligned} L = D^5 + 2u_3 D^3 \quad + \quad & (2u_2 + 3u')D^2 + (2u_1 + 2u_2' + 3u_3'')D + \\ & (u_o + u_1' + u_2'' + u_3''' \qquad (3.6) \end{aligned}$$

u_3, with dimension 2, plays the role of the specific heat in the double scaling limit. u_2, u_1, and u_o respectively have weights of 3,4 and 5. Their presence is related to the fact that this model can be given an alternative interpretation as a model of pure W_5-gravity theory. In order to get the string equations explicitly from the universality condition, eq.(3.1), we define

$$Q \equiv L^{1/5} \equiv D + \sum_{p=1}^{\infty} (-1)^p r_p D^{-p}$$

By comparing the coefficients of Q^5 with L in eq.(3.6) one gets

$$r_1 = -\frac{2}{5} u_3 \tag{3.8a}$$

$$r = \frac{1}{5}(2u_2 - u_3') \tag{3.8b}$$

$$r_3 = -\frac{1}{5}\left[u_3'' - 2u_2' - \frac{8}{5}u_3^2 + 2u_1\right] \tag{3.8c}$$

$$r_4 = -\frac{1}{25}\left[u_o - 3u_1' + u_2'' - u_3''' + \frac{24}{5}u_3 u_3' - \frac{16}{5}u_2 u_3\right] \tag{3.8d}$$

The other $r_i's$ can be determined in terms of $r_1 \dots r_4$. Given the expressions for L, or alternatively Q^5 and Q^6_+ one can write down the commutator of L and Q^6_+ explicitly as,

$$\left[L, Q^6_+\right] = f_{-8}D^8 + f_{-7}D^7 + f_{-6}D^6 + f_{-5}D^5 + f_{-4}D^4 + f_{-3}D^3 f_{-2}D^2 + f_{-1}D + f_o \tag{3.9}$$

We find

$$f_{-8} = f_{-7} = f_{-6} = f_{-5} = f_{-4} = 0 \tag{3.10}$$

identically. The universality condition, eq. (3.1), implies

$$f_{-3} \;=\; f_{-2} \;=\; f_{-1} \;=\; 0 \text{ and } f_o \;=\; 1$$

and gives the string equations.

IV THE SCALING PROPERTIES

The string equations in the spherical limit are

$$r_1 r_4 + r_2 r_3 - 4r_1^2 r_2 = K_1 \tag{4.1a}$$

$$-2r_2 r_4 + 4r_1 r_2^2 + r_1^4 - r_3^2 = K_2 \tag{4.1b}$$

$$2r_1^3 r_2 - r_1 r_2 r_3 - r_1^2 r_4 + r_3 r_4 = K_3 \tag{4.1c}$$

$$30r_1 r_2 r_4 - 50r_1^2 r_2^2 - 12r_1^5 - 10r_2^2 r_3 10 r_1 r_3^2 + 20r_1^3 r_3 - 5r_4^2 = x \tag{4.1d}$$

The n-point correlation function involving the operators corresponding to the three integration constants K_1, K_2 and K_3 are given by

$$F_{\phi_1 .. \phi_N} = \left[\frac{1}{u_3}\left(\frac{\partial u_3}{\partial K_1 \ldots \partial K_N}\right)\right]_{K_1=K_2=K3=0}$$

$$= \left[\frac{1}{r_1}\left(\frac{\partial r_1}{\partial K_1 \ldots \partial K_N}\right)\right]_{K_1=K_2=K3=0} \tag{4.2}$$

We can derive the following scaling properties of one and two point functions using eq.(4.1)

$$\left[\frac{1}{r_1}\frac{\partial r_1}{\partial K_1}\right]_{K_i=0} \sim x^{-7/10} \tag{4.3a}$$

$$\left[\frac{1}{r_1}\frac{\partial r_1}{\partial K_2}\right]_{K_i=0} \sim x^{-4/5} \tag{4.3b}$$

$$\left[\frac{1}{r_1}\frac{\partial r_1}{\partial K_3}\right]_{K_i=0} \sim x^{-4/5} \tag{4.3c}$$

and

$$\left[\frac{1}{r_1}\frac{\partial^2 r_1}{\partial K_1^2}\right]_{K_i=0} \sim x^{-7/5} \tag{4.4a}$$

$$\left[\frac{1}{r_1}\frac{\partial^2 r_1}{\partial K_2^2}\right]_{K_i=0} \sim x^{-8/5} \tag{4.4b}$$

$$\left[\frac{1}{r_1}\frac{\partial^2 r_1}{\partial K_3^2}\right]_{K_i=0} \sim x^{-9/5} \tag{4.4c}$$

The scaling behaviours (4.3) and (4.4) are in agreement with the KPZ results with $\Delta_o = 1/40, 1/15$ and $1/8$. In order to derive the scaling behaviour of other operators one has to add to the string equations (4.1) the KdV flow operators (5,4), (5,3), (5,2) besides the irelevant operators. We skip this aspect here.

V CONCLUSIONS

To conclude, we have found the nonperturbative string equations for the 3-state Potts model coupled to the two dimensional gravity. It will be interesting to further analyze these equations. For example, it was shown in CFT by Dotsenko[13] that out of the 10 operators of table 1, 6 of them form a close algebra. It will be interesting to know whether this conlcusion is carried over to the large N matrix model also. Recently it has been pointed out[14] that the phase space of a generalized 3-matrix model also contains the 3-state Potts model. It will be interesting to derive the nonperturbative string equations for this case and compare them with our results.

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TABLE I

(P,Q)	Δ_O	$\Delta(\equiv \Delta_+)$	$\frac{\beta}{\alpha} = 1 - \Delta$
(1,1)	0	0	1
(2,1)	2/5	3/5	2/5
(3,1)	7/5	6/5	-1/5
(4,1)	3	9/5	-4/5
(1,2)	1/8	3/10	7/10
(2,2)	1/40	1/10	9/10
(3,2)	21/40	7/10	3/10
(4,2)	13/8	13/10	-3/10
(1,3)	2/3	4/5	1/5
(2,3)	1/15	1/5	4/5

Table I: Conformal weights (Δ_o) and gravitational scaling dimensions (Δ) for the primary fields of the 3-state Potts model.

On the Geometrical Origin of the Fermionic Gauge Symmetry in Green Schwarz Siegel Superstrings

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Abstract

The fermionic κ symmetry which invariably occurs in strings and membranes with ten dimensional manifest spacetime supersymmetry is traced directly to the supersymmetry algebra (as modified recently by Green), using the techniques of Hamiltonian reduction applied to the constrained gauged version of the Wess Zumino Witten model obtained by Green from the Chern Simons gauge theory corresponding to this superalgebra. We outline a recipe whereby the remaining first class constraints of Siegel's modified superstring may be similarly obtained.

Introduction

Green Schwarz superstrings have been of some interest during the last two years following some developments towards manifestly Lorentz covariant quantization of these theories.[1] Despite these successes, however, the complete covariant quantization of these models in accord with BRST invariance is yet to be worked out [2]. The problem, as is well-known, stems from the difficulty in applying straightforwardly the Faddeev-Popov method to covariantly fix the fermionic gauge symmetry present in these theories, because this gauge symmetry is *infinitely reducible*, hence requiring an infinite number of Faddeev-Popov ghosts. The novelty in the approaches of ref.s [1] consisted in application of the Batalin-Vilkovisky scheme developed for reducible gauge theories, together with a clever (albeit tentative) gauge choice for the fermionic string coordinates. However, the physical spectrum (BRST cohomology) thus obtained did not coincide with that obtained by light cone gauge quantization. The inconsistency seemed to be avoidable in variants of the theory, at the particle level, proposed by Siegel and collaborators [2] which do not possess second class constraints unlike the original Green Schwarz superstrings; the former thus contain gauge connections (Lagrange multiplier fields) for each of the classical gauge generators (first class constraints), in contrast to the latter where there is no genuine gauge field for κ symmetry. While the covariant quantization of the so-called first ilk superparticle has been completed,[8] the corresponding for the superstring is yet to be worked out. Even so, the geometrical origin of the various two dimensional gauge symmetries vis-a-vis the target space supersymmetry is not well understood.

The first attempt to place the theory on a geometrical footing was that of Henneaux and Mezincescu [3] who formulated the type II Green Schwarz superstring as a Wess Zumino Witten model on a coset (super)manifold at the classical level. Some technical features of this construction were addressed and clarified by Milewski.[3] However, none of these works focussed on Siegel's modification of the Green Schwarz formulation which, as we have stated above might be more amenable to covariant quantization. Of the various approaches in the literature [3] which attempt to provide a geometrical foundation to this class of string theories, the work of Green is the first that formulates the theory in terms of a three dimensional Chern Simons gauge theory whose gauge potentials take values in a supersymmetric extension of the spacetime translation algebra. In this work, Green follows path-breaking work by Witten [4] to derive the action of the Green Schwarz Siegel superstring as a Wess Zumino Witten model whose currents enclose the loop superalgebra corresponding to the modified supertranslation algebra. Given this loop superalgebra, it is interesting to ask whether a Hamiltonian reduction [7] of it is indeed possible, and if so, what induced superalgebra would then result from such a reduction.[2]. It is this question that we wish to investigate in this article. However, rather than following the original Hamiltonian procedure of Drinfeld and Sokolov [7], we adopt the somewhat more intuitive approach given by Bershadsky and Ooguri [6] for $SL(N, R)$ loop algebras and also their quantum (Kač-Moody) generalizations, using constrained gauged Wess Zumino Witten models. The choice of the constraint to be imposed on the supertranslation currents is a trifle subtle, since the absence of a Cartan-Weyl structure of this superalgebra prevents us from projecting out currents with values in a Borel subalgebra as has been done in [7]. Nevertheless, our somewhat arbitrarily chosen constraint does yield the correct expressions for the generator of 2d reparametrizations and more importantly, the generator of κ symmetry, as shown by comparison with those obtained by Siegel by the Sugawara construction, albeit in the light cone gauge.

The paper is outlined as follows : we begin with a review of Green's work in the next section which also serves to establish our notation. This is followed by an attempt to formulate a classical gauged Wess Zumino Witten model following Elitzur et.al. [4] appropriate to Green's modified supertranslation algebra. Next after a brief review of the approach of ref. [6] for the $SL(2, R)$ current algebra, we present our procedure and results. We end with a discussion of the inadequacy of the seminal supertranslation algebra for this program to yield the full covariant expressions of Siegel's first class constraints, and suggest an improved approach that might be more successful.

---

[2]Recall that the Hamiltonian reduction of the $SL(2, R)$ loop algebra leads to the Virasoro algebra as a residual symmetry [7, 6]

Supertranslation Algebra and Wess Zumino Witten model

Green bases his construction on the following modified supertranslation algebra,[3]

$$
\begin{aligned}
[P_a\ ,\ P_b] = \{Q_\alpha\ ,\ K^\beta\} &= \{K^\alpha\ ,\ K^\beta\} = 0 \\
[P_a\ ,\ Q_\beta] = 2i(\sigma_a)_{\beta\gamma}K^\gamma\ ,\ \{Q_\alpha\ ,\ Q_\beta\} &= 2(\sigma\cdot P)_{\alpha\beta}\ ,
\end{aligned}
\tag{1}
$$

where, K^α is like a fermionic central extension; the algebra (1) has the non-degenerate Killing metric

$$
Tr(P_aP_b) = -\tfrac{1}{2}i\eta_{ab}\ ,\ Tr(Q_\alpha K^\beta) = -\tfrac{1}{2}\delta^\beta_\alpha\ . \tag{2}
$$

Observe that unlike the usual supersymmetry algebra, the supersymmetry generators Q_α in (1) are not tanslationally invariant; this is necesary for the non-degeneracy of the Killing metric. [3] Also, one can perhaps interpret the generator K as a *cubic* Casimir operator, although for our purpose we shall continue to to treat it as an independent generator following ref. [3].

One now defines a one form $\mathcal{A}$ on a base threefold which is a solid cylinder (with axis identified with the time direction) and taking values in the Lie superalgebra (1); this one form admits the expansion

$$
\mathcal{A} = (i\Pi^a_i P_a + \psi^\alpha_i Q^\alpha + K^\alpha\zeta_{\alpha i})dx^i\ ,\ i = 1,2,3. \tag{3}
$$

Starting with the Chern Simons action

$$
S_{CS} = \frac{k}{4\pi}Tr\int_M(\mathcal{A}d\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3)\ , \tag{4}
$$

and following Witten and Elitzur et. al. [4], one obtains for the partition function of the theory [3]

$$
\mathcal{Z} = \int \mathcal{D}\tilde{\mathcal{A}}\delta(\tilde{\mathcal{F}})\ exp\frac{ik}{4\pi}\int_M \tilde{\mathcal{A}}d\tilde{\mathcal{A}}\ , \tag{5}
$$

where, we have decomposed the one form $\mathcal{A}$ as $\mathcal{A} = \mathcal{A}_0 d\tau + \tilde{\mathcal{A}}_i d\sigma^i\ , i = 1,2,$ and performed the functional integral over $\mathcal{A}_0$. We have also chosen the boundary condition $\mathcal{A}_0|_{\partial M} = 0$. The 'transverse' field strength $\tilde{\mathcal{F}}$ is defined as

$$
\tilde{\mathcal{F}} \equiv \tilde{d}\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^2\ ,
$$

and satisfies the constraint $\tilde{\mathcal{F}} = 0$ which can be solved for the transverse gauge one form $\tilde{\mathcal{A}} = -\tilde{d}\tilde{U}\tilde{U}^{-1}$, where $\tilde{U}(\sigma^i,\tau)$ takes values in the supergroup obtained by exponentiation of the superalgebra (1). Substituting the solution for $\tilde{\mathcal{A}}$ into (5), the partition function is rewritten

$$
\mathcal{Z} = \int \mathcal{D}U\ expikS^+_{WZ}(U)\ , \tag{6}
$$

where $U(\sigma,\tau)$ is the restriction of $\tilde{U}$ to the boundary ∂M of the threefold M, i.e., the surface $R\otimes S^1$ of the cylinder, and

$$S^+_{WZ} = \frac{1}{4\pi}Tr\int_{\partial M} U^{-1}\partial_\sigma U U^{-1}\partial_\tau U + \frac{1}{12\pi}Tr\int_M (\tilde{U}^{-1}d\tilde{U})^3 . \tag{7}$$

Observe that S_{WZ} above is invariant under

$$U(\sigma,\tau) \rightarrow U(\sigma.\tau) \rightarrow V_1(\sigma)UV_2(\tau)$$

of which only the $V_2(\tau)$ constitute gauge transformations which leave the spectrum invariant.[4] This is because the Chern Simons action (4) is invariant only under those local transformations which reduce to the identity on the boundary ∂M. Clearly $V_1(\sigma)$ (with σ the azimuthal angle on the surface of the cylinder) does not become the identity on ∂M, and is therefore not a gauge transformation despite being local. Further the WZ current components $\mathcal{J} \equiv \Pi_a(\sigma), \psi^\alpha(\sigma), \zeta_\alpha(\sigma)$ are seen to be independent of τ; the model is thus chiral in this sense. The corresponding Poisson bracket current algebra is given by

$$\begin{aligned} [\Pi_a(\sigma)\ ,\ P_b(\sigma')] &= i\eta_{ab}\delta'(\sigma'-\sigma) \\ \{\psi^\alpha(\sigma)\ ,\ \zeta_\beta(\sigma')\} &= i\delta^\alpha_\beta\delta'(\sigma'-\sigma) \\ \{\zeta_\alpha(\sigma)\ ,\ \zeta_\beta(\sigma')\} &= 2(\sigma\cdot\Pi)_{\alpha\beta}\delta(\sigma-\sigma') \\ [\zeta_\alpha(\sigma)\ ,\ \Pi^b(\sigma')] &= 2i(\sigma^c)_{\alpha\beta}\psi^\beta\delta(\sigma-\sigma') \\ rest &= 0 . \end{aligned} \tag{8}$$

The algebra is realized in terms of commuting and anticommuting 2d scalar fields which parametrize the supergroup manifold which in this case is simply $D=10, N=1$ superspace augmented by extra Majorana-Weyl fermionic coordinates $\bar{\theta}_\alpha$ which are infinitesimally translated by the action of the generators K. This realization is given by [3]

$$\begin{aligned} U &= expZ\ ,\ Z \equiv iP_aX^a(\sigma,\tau) + \theta^\alpha(\sigma,\tau)Q_\alpha + K^\alpha\bar{\theta}_\alpha(\sigma,\tau) \\ \Pi^a &= \partial X^a - i\theta(\sigma)^a\partial\theta \\ \psi^\alpha &= \partial\theta^\alpha \\ \zeta_\alpha &= \partial\bar{\theta}_\alpha + 2(\sigma\cdot\Pi)_{\alpha\beta}\theta^\beta - \tfrac{2}{3}(\sigma_a\theta)_\alpha(\theta\sigma^a\partial\theta) . \end{aligned} \tag{9}$$

With this realization, it is not difficult to show that the WZ action (7) is given by

$$S_{WZ} = S_{GS} + \frac{1}{8\pi}\int_{\partial M}\psi^\alpha\zeta_\alpha , \tag{10}$$

where, S_{GS} is the usual Green Schwarz action in a conformal type gauge, and the second term corresponds to Siegel's modification.[5] In a general gauge this action is to be augmented by terms involving 2d gauge fields multiplying the gauge generators (first class constraints). These constraints were written down by Siegel [5] using a Sugawara type construction

$$
\begin{aligned}
\mathcal{T} &= \tfrac{1}{2}(\Pi^2 + \psi^\alpha \zeta_\alpha) \, , \; \mathcal{S}^\alpha = (\sigma \cdot \Pi)^{\alpha\beta}\zeta_\beta \\
\mathcal{W}^a &= \zeta_\alpha (\sigma^a)^{\alpha\beta} \partial \zeta_\beta \, , \; \mathcal{V}_{\alpha\beta} = \zeta_{[\alpha}\zeta_{\beta]} \; .
\end{aligned}
\tag{11}
$$

These generators can be shown to form a closed Poisson bracket algebra. Note, however, the structure of the algebra is not unique because of linear relations like $\zeta_\alpha \mathcal{S}^\alpha = (\sigma \cdot \Pi)^{\alpha\beta}\mathcal{V}_{\alpha\beta}$ etc. This reducibility is precisely the one alluded to earlier. But a more striking feature is the fact that among many more possible bilinears of the component currents $\mathcal{J}$, only the four in (11) appear to form a closed algebra. The origin of this is obscured by the somewhat unmotivated nature of the Sugawara construction. We shall attempt to trace the origin of this curious property in the sequel.

Constrained Gauged Wess Zumino Model and Gauge Fixing

We now proceed to gauge the Wess Zumino model derived earlier. To this end we ascribe a non-vanishing boundary field configuration $\bar{\mathcal{A}}_0(\sigma, \tau)$ to the timelike component; in order that this does not affect the stationarization of the action, one must add to (4) a surface term

$$
S_b = -\frac{1}{4\pi} Tr \int_{\partial M} \tilde{\mathcal{A}} \bar{\mathcal{A}}_0 \, , \tag{12}
$$

and retain the surface term that appears as a result in the 2+1 decomposition. Proceeding as before we get, upon solving the constraint $\tilde{\mathcal{F}} = 0$ in the modified action

$$
S_g(U, \bar{\mathcal{A}}_0) = kS^{+}_{WZ} - \frac{k}{2\pi} Tr \int_{\partial M} \mathcal{J} \bar{\mathcal{A}}_0 d\sigma d\tau \quad , \tag{13}
$$

where, $\mathcal{J} \equiv -\partial U U^{-1}$. In this action, the KM current $\mathcal{J}$ has been coupled to an 'external' gauge field whose component in the angular direction $\bar{\mathcal{A}}_\sigma$ has been gauge fixed to zero. We have not been able to derive the complete gauge invariant gauged WZ model whose action is known [4] to be

$$
\begin{aligned}
S'_g = S^{+}_{WZ}(U) &+ \frac{k}{2\pi} \int_{\partial M} [\partial U U^{-1} \bar{\mathcal{A}}_0 - \bar{\mathcal{A}}_\sigma U^{-1} \bar{\partial} U \\
&- \bar{\mathcal{A}}_\sigma U^{-1} \bar{\mathcal{A}}_0 U + \bar{\mathcal{A}}_\sigma \bar{\mathcal{A}}_0] \, ,
\end{aligned}
\tag{14}
$$

and is invariant under

$$
U \rightarrow gUg^{-1} \, , \; \mathcal{A} \rightarrow \mathcal{A}^g = g\mathcal{A}g^{-1} + dgg^{-1} \; .
$$

This theory will be the one that we shall use in the sequel, since, if we were to start from eq. (13) and define a partition function by integrating over both U and the boundary field configuration $\bar{\mathcal{A}}_0$, the WZ currents $\mathcal{J}$ will be constrained to vanish upon performing the (trivial) integral over $\bar{\mathcal{A}}_0$. We would prefer instead to be able to impose a non-trivial constraint on the currents.

To elucidate our approach, we consider first the simpler case of gauged $SL(2,R)$ WZ model.[6] It is not difficult to obtain the following *constrained* action from (14) above for this model,

$$S_{eff} = kS_{WZ} - \frac{k}{2\pi}\int_{\partial M}(\mathcal{J}^- - 1)\bar{\mathcal{A}}_0^+ , \tag{15}$$

where, the + and - refer to $SL(2,R)$ components in the adjoint representation. Clearly, performing the integral over $\bar{\mathcal{A}}_0^+$ would enforce the constraint $\mathcal{J}^- = 1$.

Observe now that the action (15) is invariant under

$$U \rightarrow gU \ , \ \bar{\mathcal{A}}_0^+ \rightarrow \bar{\mathcal{A}}_0^+ + Tr(t^+\partial g g^{-1}) , \tag{16}$$

with $g = expt^-\epsilon^+(\sigma,\tau)$. These transformations thus leave the constraint invariant. However, the other components of the current do change non-trivially under the gauge transformations (16),

$$\mathcal{J}^+ \rightarrow \mathcal{J}^+ + \partial\epsilon^+ + \epsilon^+\mathcal{J}^3 \ , \ \mathcal{J}^3 \rightarrow \mathcal{J}^3 - \epsilon^+, \tag{17}$$

where, $\epsilon^+(\sigma,\tau)$ is an arbitrary local gauge parameter.

For the $SL(2,R)$ case, the subgroup generated by t^- is called (one of) the Borel subgroup(s), defined as follows. For any $g \in SL(2,R)$,

$$g = g_- g_d g_+ , \tag{18}$$

where, $g_{\pm,d}$ are respectively upper, lower, diagonal $2 \otimes 2$ matrices; the former two are sometimes referred to as Stoke's matrices. Thus, the constraint $\mathcal{J}^- = 1$ corresponds to constraining an element of the Borel subalgebra. However, the gauge invariance (17) allows us to make the gauge choice $\epsilon^+ = \mathcal{J}^3$, thus gauging $\mathcal{J}^3$ away; however, in this gauge,

$$\mathcal{J}^+ \rightarrow \tilde{\mathcal{J}}^+ = \mathcal{J}^+ + \partial\mathcal{J}^3 + (\mathcal{J}^3)^2 . \tag{18}$$

Using the original $SL(2,R)_k$ classical current algebra, it is easy to show that [6]

$$[\mathcal{J}^+(\sigma) \ , \ \mathcal{J}^+(\sigma')] = [\mathcal{J}^+(\sigma) + \mathcal{J}^+(\sigma')]\delta'(\sigma'-\sigma) + c_k\delta'''(\sigma'-\sigma) , \tag{19}$$

where, we have dropped the tildes for simplicity. As is well known, this is the Virasoro algebra, and $\mathcal{J}^+$ plays the role of the modified energy momentum tensor, which is basically a deformation of the Sugawara tensor to account for the constraint we have imposed. We mention in passing that the above technique of reducing the full Kać-Moody symmetry corresponding to the $SL(2,R)$ Lie algebra, to the Virasoro algebra, by gauge fixing the residual gauge invariance of the constrained WZ model, is called Hamiltonian reduction and used extensively in 2d integrable field theories [7].

Hamiltonian Reduction of the Supertranslation Loop Algebra

Before applying the above technique to the case of the WZW model of eq. (10), we note that what was actually constrained was an element of the Borel algebra, that is to say, an element generated by the nilpotent ladder operator t^-. In the case of the WZW model corresponding to the superalgebra (1), unfortunately, the superalgebra does not have a unique Cartan-Weyl basis, since the bosonic subalgebra, viz., the translation algebra is Abelian. Thus there is no straightforward analogue of the ladder operators of classical Lie algebras or even their Z_2 gradings. A complete resolution of this problem is beyond the scope of this paper, although we shall indicate a possible approach at the end. For the present, we circumvent the impasse by arbitrarily (at least at this stage) constraining some of the components of the WZ (super-)currents. We next attempt to find the subgroup of transformations that preserve these constraints, and then fix this residual symmetry as in the case of $SL(2,R)$.

Recall that, in accord with standard lore [6], $\mathcal{J}$ admits the expansion

$$\mathcal{J}(\sigma) = iP^{(+}\Pi^{-)} + iP^i\Pi^i + \tfrac{1}{2}\psi^{(+}Q^{-)} + \tfrac{1}{2}K^{(+}\zeta^{-)} , \tag{20}$$

where, we have used a spacetime lightcone notation, with the Majorana-Weyl fermions being projected out as usual : $\psi^{\pm} \equiv \sigma^{\pm}\psi$ etc. Let us also designate the components of $\bar{\mathcal{A}}_0$ as $A^a, \xi^\alpha, \eta_\alpha$. We choose the constraints $\Pi^+ = a^+$, $\psi^+ = 0$. The effective action (analogue of eq. (15)) is then given by

$$\begin{aligned} S_{eff} = S_{WZ}(U)|_{lc} - \frac{k}{2\pi}\int_{\partial M}&[(\Pi^+ - a^+)A^- - \Pi^-A^+ + \Pi^iA^i \\ &+ \psi^{(+}\eta^{-)} + \xi^{(+}\zeta^{-)}] . \end{aligned} \tag{21}$$

If we let $U \to VU$, then, $\mathcal{J} \to \tilde{\mathcal{J}} = \partial VV^{-1} + V\mathcal{J}V^{-1}$. Now, for $V = I + \chi^-Q^+$, we have

$$\begin{aligned} \tilde{\mathcal{J}}|_{constr} = &\ iP^+(\Pi^- - i\tfrac{1}{2}\chi^-\sigma^+\psi^-) + iP^-a^+ \\ &+ \tfrac{1}{2}(\psi^- + \tfrac{1}{2}\partial\chi^-)Q^+ - iP^iA^i + \tfrac{1}{2}K^+(\zeta^- \\ &- \sigma^iA^i\chi^-) + \tfrac{1}{2}K^-(\zeta^+ - \sigma^+\chi^-a^+) , \end{aligned} \tag{22}$$

where, χ is a Majorana Weyl fermionic gauge parameter. Eq. (22) clearly shows that the gauge transformations given by V above leave the constraints invariant.

Proceeding as in the $SL(2,R)$ case, we now fix gauge by choosing the gauge parameter χ to satisfy the equation $\sigma^+\chi^- a^+ = \zeta^+$, which has the solution $\chi^- = \frac{\sigma^-\zeta^+}{2a^+}$. This gauges away the component ζ^+ just as earlier in the $SL(2,R)$ case we chose ϵ to gauge away $\mathcal{J}^3$. In this gauge,

$$\Pi^- \rightarrow \tilde{\Pi}^- = \Pi^- - \frac{1}{2}\frac{\sigma^-\zeta^+}{2a^+}\sigma^+\psi^- \ . \tag{23}$$

We observe that eq. (23) gives part of the Virasoro generator $\mathcal{T}$ of eq. (11) expressed in the light cone gauge. Similarly, the above gauge choice yields the component

$$\tilde{\zeta}^- = \frac{1}{a^+}(a + \zeta^- - \sigma^i\Pi^i\sigma^-\zeta^+) \ , \tag{24}$$

which resembles the κ generator S^α of eq. (11) in the light cone gauge. Thus, parts of the 2d gauge generators (first class constraints) seem to appear with some modification thereby indicating the applicability of this mathematical technique to this class of WZW models.

Discussion

The emergence of the Virasoro constraint and the generator of the κ symmetry in consonance with the results of [5] appears to reveal intriguing relationships of a geometrical nature, since the technique employed here is firmly grounded in the geometry of coadjoint orbits of the supertranslation group. Indeed it is the first application of this technique to the spacetime supersymmetry algebra to the best of our knowledge. However, there are some inadequacies in the present approach that we must now remark on.

First of all, the choice of the constraint had to be made somewhat arbitrarily for reasons that have already been discussed. Secondly, despite the use of this new technique, we have not been able to reproduce the complete covariant forms of the first class constraints, and so the derivation of the full gauge algebra is still beyond us. Finally, we have not been able to determine the deformations of the various gauge generators; this is necessary in order to fix the central charges of the super-Virasoro-Siegel algebra.[5] All these shortcomings are of course not surprising, given that the superalgebra (1) does not have the usual Cartan-Weyl basis, and hence, the Borel subalgebra(s) of classical (super-)algebras,[7] as already mentioned. In conclusion, we now proceed to indicate what might constitute the elements of an improved framework.

The extended supertranslation algebra (1) is actually the coset algebra

$$ISO(9,1|16)_k \ / \ SO(9,1);$$

on the other hand, the algebra $ISO(9,1|16)$ has at least the Cartan structure of the $D = 10$ Lorentz group $SO(9,1)$. This algebra is simply the augmentation of (1) by the usual Lorentz commutators, together with

$$
\begin{aligned}
[Q_\alpha \ , \ M_{bc}] &= \tfrac{1}{2}(\sigma_{bc})_\alpha{}^\beta Q_\beta \\
[K^\alpha \ , \ M_{bc}] &= \tfrac{1}{2}(\sigma_{bc})^\alpha{}_\beta K^\beta \ .
\end{aligned}
\tag{25}
$$

The problem one faces right away is the parametrization of the supergroup manifold corresponding to the algebra (25). If one naively defines a group element $g \equiv expZ'$ with $Z' \equiv Z + iM_{ab}Y^{ab}$, where Z is the string supercoordinate of eq. (9), then one is stuck with the antisymmetric coordinate Y^{ab}, which is undesirable since there are no known string theories with *antisymmetric* string coordinates. A better alternative is to think of Y^{ab} as a composite 2d field, given in terms of the spinorial coordinates as $Y^{ab} \sim \theta(\sigma^{ab})\bar{\theta}$. One can write down a 'minimal' extension of the loop (super)algebra consistent with the Jacobi identities

$$
\begin{aligned}
[\Pi^a(\sigma) \ , \ \Pi^b(\sigma')] &= \eta^{ab}\delta'(\sigma' - \sigma) \ + \ B^{ab}(\sigma)\delta(\sigma' - \sigma) \\
\{\psi^\alpha(\sigma) \ , \ \zeta_\beta(\sigma')\} &= \delta^\alpha_\beta\delta'(\sigma' - \sigma) \ + \ (\sigma^{ab})^\alpha_\beta\delta(\sigma' - \sigma)B_{ab}(\sigma') \\
rest &= as \ \ before \ \ ,
\end{aligned}
\tag{26}
$$

where, B^{ab} is the current component along M_{ab}.

The gauged WZW model appropriate to this loop algebra has not been investigated yet. But if it exists, it will be a prime candidate for the correct string theory with manifest spacetime supersymmetry, since it shall have a full covariant set of first class constraints. Furthermore, these constraints shall be directly linked to the spacetime supersymmetry loop algebra. All these exciting possibilities may of course only be realized after a better understanding is reached of the techniques used in other aspects of string theory, and also after the manifestly supersymmetric string is better understood. It is hoped that the present work is a preliminary step in that direction.

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FUNCTIONAL MEASURE IN TOPOLOGICAL FIELD THEORIES

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There has been a considerable interest in the topological field theories in last two years. Earliest discussion of a topological field theory is due to Schwarz - abelian Chern - Simons in 3d. First non-abelian topological field theoriès were discovered by Witten about two years ago - he related them to Donaldson polynomials in $4d$ and Jones polynomials in 3d. There is also interest in $2d$ or $3d$ topological field theories of gravity.

Generically, topological field theories provide field theoretic methods of categorizing the topological properties of the space-time manifold on which these field theories live.

Topological field theories have no dynamical degrees of freedom - the whole content of these theories is described by some global discrete topological degrees of freedom, which are characterized by the vacuum expectation values of socalled **Topological Invariants.** An intersting fact : Topological field theories exhibit a BRST type anticommuting (nilpotent) symmetry which may be a genuine BRST invaraince associated with fixing of some gauge symmetries of the system.

Within the context of path integral quantization of the topological field theories, there have been some issues related to the functional measure which till recently were not sorted out.

* * * * *

What do we mean by a topological field theory ?

These theories are independent of the metric of the space-time manifold on which these theories are defined. That, energy momentum tensor, $T_{\mu\nu} = 0$. Quantum mechanical topological invariance means that $\frac{\delta Z}{\delta g_{\mu\nu}} = 0$. Furthermore the expectation value of a metric independent operator W is also metric independent

$$\frac{\delta < W >}{\delta g_{\mu\nu}} = 0.$$

Even if classical $T_{\mu\nu} = 0$, at quantum level, for topological invariance to hold, we need to establish that $\frac{\delta Z}{\delta g_{\mu\nu}} = 0$ and $\frac{\delta < W >}{\delta g_{\mu\nu}} = 0$. It is possible that there may be anomalies which would prohibit this. It is here the role of functional measure becomes important in the quantum formulation of these theories.

Fujikawa has recently proposed a general coordinate invariant measure in a different context. This measure generically introduces metric dependences and hence the need of a pro of metric independence of Z and $< W >$.

Consider a general coordinate invariant field theory in arbitrary dimensions whose quantum action can be written as

$$S[\Phi_r, g_{\mu\nu}] = S_o[\Phi_r]\delta V[\Phi_r, g_{\mu\nu}]$$

$\Phi_r r = 1, 2, \cdots$. all the fields, matter, gauge, ghosts, auxiliaries etc. S_o does not involve ghosts, auxiliaries. An example, Chern Simous action. S_o is metric independent $\delta S_o = 0, \delta^2 = 0, \delta$ a nilpotent BRST variation. V is a local functional of all the fields and also depends on the metric, $g_{\mu\nu}$.

Topological fields theories fall into two categories :

i) $S_o \neq 0$: Schwarz type
(example Chern Simons)

ii) $S_o = 0$: Witten type

Now partition function

$$Z = \int d\hat{\mu}[\Phi_r] \exp\{-S[\Phi_r, g_{\mu\nu}]\}$$

Fujikawa measure :

$$d\hat{\mu}[\Phi] = \prod_x \prod_r d\hat{\Phi}_r(x)$$

$$\hat{\Phi}_r(x) = g^{\alpha_r}(x)\Phi_r(x) \qquad g = \det g_{\mu\nu}.$$

For example, α for a scalar field is $\frac{1}{4}$.

Consider with usual functional measure :

$$\int \left[\prod_x d\phi(x)\right] e^{-\int d^dx\sqrt{g}\phi^2(x)} = [detg^{-1/4}]$$

This is not general coordinate invariant, even though the expression in the exponential is. Measure $\prod_x d\phi(x)$ is not general coordinate invariant.

$$\int \prod_x d(g^{1/4}\phi(x))e^{\int d^dx}\sqrt{g}\phi(x)^2 = \int \prod_x d\hat{\phi}(x)e^{-\int d^dx\hat{\phi}(x)^2} = 1.$$

(general coordinate invariance)

Hence measure

$$\prod d(g^{1/4}\phi(x)) \equiv \prod d\hat{\phi}(x)$$

is general coordinate invariant.
For vectors

$$
\begin{aligned}
A_\mu &= e^a_\mu A_a \\
\prod_{x,a} d\hat{A}_a(x) &= \prod_{x,a} d(g^{1/4} A_a(x)) \\
&= \prod d(g^{1/4} C^\mu_a A_\mu(x)) \\
&= \prod_{x/\mu} g^{-1/2} d(g^{\frac{d-2}{4d}} A_\mu(x)) \\
&= \prod d\hat{A}_\mu(x) \\
\hat{A}_\mu &= g^\alpha A_\mu, \alpha = \frac{d-2}{4d}
\end{aligned}
$$

Similarly for a contravariant vector $\alpha = \frac{d+2}{4d}$.
For a covariant tensor of rank r,

$$
\begin{aligned}
\prod_{\mu,x} d\hat{A}_{\mu_1 \cdots \mu_r} &= \prod d(g^\alpha A_{\mu_1 \cdots \mu_r}) \\
\alpha &= \frac{d-2r}{4d} \\
\hat{Z} &= \int \prod_{x,r} [d\hat{\Phi}_r(x)] exp\{-S_o[g^{-\alpha}\hat{\Phi}_r] + \delta_Q V[g^\alpha_r \hat{\Phi}_r, g_{\mu\nu}]\}
\end{aligned}
$$

Two problems :
(i) S_o developes metric dependence
(ii) BRST transformations of $\hat{\Phi}$ also contain metric

$$
[\frac{\delta}{\delta_{g\mu\nu}}, \delta] \neq 0.
$$

A way to circumvent these problems :

$$
\begin{aligned}
\prod_{x,r} [d\hat{\Phi}_r(x)] &= \prod_{x,r} d[g^{\alpha_r} \Phi_r(x)] \\
&= \prod_{x,r} g^{\alpha_r \sigma_r d_r} d\Phi_r(x)
\end{aligned}
$$

$\sigma_r = +1 or -1$ for commuting or anticommuting Φ_r d_r = dimension of Φ_r. Thus general coordinate invariant measure

$$
\begin{aligned}
d\hat{\mu} = \prod d\hat{\Phi}_r &= \prod g(x)^K \prod_{x,r} d\Phi_r(x) \\
K &= \sum_r \sigma_r d_r \alpha_r
\end{aligned}
$$

K is an index characterizing the field content of the theory. Suppose $K = 0$,

$$\begin{aligned}
d\hat{\mu} = \prod_{r,x} d\hat{\Phi}_r &= \prod_{x,r} d\Phi_r \\
\hat{Z} &= \int \prod_{r,x} d\Phi_r(x) \exp[-S_o[\Phi_r] + \delta V[\Phi_r, g_{\mu\nu}]] \\
\frac{\delta Z}{\delta g_{\mu\nu}} &= \int \prod_{x,r} d\Phi_r(x)(exp[-S]) \frac{\delta}{\delta g_{\mu\nu}} \delta_Q V \\
&= \int \prod_{x,r} d\Phi_r(x) e^{-S} \delta_Q (\frac{\delta}{\delta g_{\mu\nu}} V) \\
&= \int \prod_{x,r} d\Phi_r(x) \delta_Q (e^{-S} \frac{\delta}{\delta g_{\mu\nu}} V) = 0
\end{aligned}$$

by BRST invariance.

$$\frac{\delta}{\delta g_{\mu\nu}} \delta_{BRST} V = \delta_{BRST} \frac{\delta}{\delta g_{\mu\nu}} V \quad [+\delta_{\mathrm{BRST}} W_{\mu\nu}]$$

For BRST invariant, metric independent W

$$\begin{aligned}
\frac{\delta < W >}{\delta g_{\mu\nu}} &= \frac{\delta}{\delta g_{\mu\nu}} \int \prod_{x,r} d\Phi_r(x) W e^{-S_o + \delta_Q V} \\
&= \int \prod_{x,r} d\Phi_r(x) W e^{-S_o + \delta_Q V} \delta_Q (\frac{\delta}{\delta g_{\mu\nu}} V) \\
&= \int \prod_{x,r} d\Phi_r(x) \delta_Q (W e^{-S_o + \delta_Q V} \frac{\delta}{\delta g_{\mu\nu}} V) = 0
\end{aligned}$$

by BRST invariance.

Thus for theories with index $K = \Sigma \sigma_r \alpha_r d_r = 0$, the topological invariance of the path integral is immediate.

But one subtlity:

$$\begin{aligned}
d\hat{\mu} = \prod_{x,r} d\hat{\Phi}_r &= \prod_{x,r} g^{\alpha_r \sigma_r d_r} d\Phi_r(x) \\
&= \prod_{x,r} e^{\alpha_r \sigma_r d_r lng} d\Phi_r(x) \\
&= \exp[\Sigma \alpha_r d_r \sigma_r tr_{\Phi_r} lng]_x \prod_{x,r} d\Phi_r(x) \\
\text{“}Index\text{”} \ \hat{K} &= \Sigma d_r \sigma_r \alpha_r tr_{\Phi_r} lng
\end{aligned}$$

We require

$$\hat{K}_{\rm reg} = \lim_{t \longrightarrow 0} \sum_r dr \sigma_r \alpha_r t_r \ln g(x) e^{-tM_{\Phi r}} = 0$$

Specific Models:
(I)Witten type model:
Basic field is a group valued connection

$$A_\mu = A_\mu^a T^a$$

with shift symmetry $\delta A_\mu = \theta_\mu(x), \theta$ commuting transformation parameter and action $S_o = 0$.
Fix gauge : introduce an auxiliary commuting tensor field $G_{\mu\nu}$ and F.P. anticommuting ghosts $\psi_\mu, \chi_{\mu\nu}$. The BRST transformations

$$\begin{aligned}
\delta A_\mu = \psi_\mu, \delta\psi_\mu = 0 \quad &, \quad \delta G_{\mu\nu} = 0, \delta\chi_{\mu\nu} = G_{\mu\nu} \\
\delta^2 \; &= \; 0 \quad offshell \\
S_{\rm g.f} = \frac{1}{e^2}\int d^4x\sqrt{g}tr[(F_{\mu\nu} - \tilde{F}_{\mu\nu})G^{\mu\nu} \; &- \; (D_\mu\psi_\nu - D_\nu\psi_\mu - \frac{\epsilon_{\mu\nu}^{\alpha\beta}}{\sqrt{g}}D_\alpha\psi_\beta)\chi^{\mu\nu}] \\
&= \; \delta V, \\
\tilde{F}_{\mu\nu} \; &= \; \frac{1}{2}\frac{\epsilon_{\mu\nu\alpha\beta}}{\sqrt{g}}g^{\alpha\alpha'}g^{\beta\beta'}F^{\alpha'\beta'} \\
V \; &= \; \frac{1}{e^2}\int d^4x\sqrt{g}tr[(F_{\mu\nu} - \tilde{F}_{\mu\nu})\chi^{\mu\nu}]
\end{aligned}$$

Further invariances

$$\begin{aligned}
\delta\psi_\mu = D_\mu \in, \qquad \delta G_{\mu\nu} \; &= \; \{\in, \chi_{\mu\nu}\}, \in \text{ anticommuting} \\
\delta A_\mu = -D_\mu\theta, \qquad \delta G_{\mu\nu} \; &= \; [\theta, G_{\mu\nu}], \qquad \delta\psi_{\mu\nu} = [\theta, \chi_{\mu\nu}] \\
\delta\psi_\mu = [\theta, \psi_\mu], \qquad &\theta \text{ commuting parameter}
\end{aligned}$$

Fix both these symmetries : Introduce commuting auxiliary field B, and anticommuting F.P. ghosts C,$\bar{C}$. And, anticommuting auxiliary field η and commuting F.P. ghosts ϕ, λ :

$$\begin{aligned}
S \; &= \; S_{\rm g.f} + S'_{\rm g.f} \\
S'_{g.f} \; &= \; \frac{1}{e^2}\int d^4x\sqrt{g}tr[A^\mu\partial_\mu B + (D_\mu C - \psi_\mu)\partial_\mu\bar{C} \\
&+ \; \psi^\mu\partial_\mu\eta + (D_\mu\phi - \{C, \psi_\mu\})\partial_\mu\lambda]
\end{aligned}$$

which is BRST invariant.
Combined BRST transformation :

$$\delta A_\mu = \psi_\mu - D_\mu C \qquad \delta\psi_\mu = D_\mu\phi - \{C,\psi_\mu\}$$
$$\delta G_{\mu\nu} = [C, G_{\mu\nu}] - [\phi,\chi_{\mu\nu}] \quad , \quad \delta\chi_{\mu\nu} = G_{\mu\nu} - \{C,\chi_{\mu\nu}\}$$
$$\delta\bar{C} = B \qquad \delta C = CC + \varphi$$
$$\delta\eta = 0 \qquad \delta\lambda = \eta \qquad \delta\psi = [C,\varphi]$$
$$\delta^2 = 0 \qquad offshell.$$
$$S = \delta V$$
$$V = \frac{1}{e^2}\int d^4x\sqrt{g}tr[(F_{\mu\nu} - \tilde{F}_{\mu\nu})\chi^{\mu\nu} + \psi^\mu\partial_\mu\lambda + A^\mu\partial_\mu\bar{C}]$$
$$d\hat{\mu} = \prod_{x,r} d\hat{\Phi}_r = \prod g^K \prod_{x,r} d\Phi_r(x)$$
$$K = \Sigma\sigma_r\alpha_r d_r$$
$$= (d_A\alpha_A - d_\psi\alpha_\psi) + (d_G\alpha_G - d_\chi\alpha_\chi) + d_B\alpha_B - d_c\alpha_c - d_{\bar{C}}\alpha_{\bar{C}}$$
$$-d_\eta\alpha_\eta + d_\phi\alpha_\phi + d_\lambda\alpha_\lambda \equiv 0$$

Naive counting leads to index $K = 0$. But we need

$$\hat{K}_{reg} = \lim_{t\to 0}\sum_r tr_r[d_r\sigma_r\alpha_r \ln g e^{-tM_{\Phi_r}}]$$

Expansion around background:
Extrema of action [1] are given lby self-dual fields

$$F_{\mu\nu} - \tilde{F}_{\mu\nu} = 0$$

with $\psi_\mu, G_{\mu\nu}, \chi_{\mu\nu}$ all zero.

$$A_\mu = A^I_\mu + e a_\mu$$

$\psi_\mu, G_{\mu\nu}, \chi_{\mu\nu}$ are also fluctuations and hence replace them by $e\psi_\mu, eG_{\mu\nu}, e\chi_{\mu\nu}$ to reflect this fact explicitly.

$$S[A^I] = 0$$
$$S = \int d^4x\sqrt{g}tr[(D_\mu a_\nu - D_\nu a_\mu - \frac{\epsilon^{\alpha\beta}_{\mu\nu}}{\sqrt{g}}D_\alpha a_\beta)G^{\mu\nu}$$
$$-(D_\mu\psi_\nu - D_\nu\psi_\mu - \frac{\epsilon^{\alpha\beta}_{\mu\nu}}{\sqrt{g}}D_\alpha a_\beta)\chi^{\mu\nu}]$$
$$+\int d^4x\sqrt{g}tr[a^\mu D_\mu B - D_\mu C D^\mu\bar{C} + \psi^\mu D_\mu\eta + (D_\mu\varphi - e\{C,\psi_\mu\})D^\mu\lambda$$

$$-e(D_\mu C - \psi_\mu)[\psi^\mu, \lambda] - e(D_\mu C - \psi_\mu)[a^\mu, \bar{C}]] = \delta V$$

$$V = \int d^4x\sqrt{g}tr[(D_\mu a_\nu - D_\nu a_\mu - \frac{\epsilon^{\alpha\beta}_{\mu\nu}}{\sqrt{g}} D_\alpha a_\beta)\chi^{\mu\nu} + a^\mu D_\mu \bar{C} + \psi^\mu D_\mu \lambda]$$

$$D_\mu = D^I_\mu + e[a^\mu, \alpha] \qquad D^I_\mu = \partial_\mu + [A^I_\mu, *]$$

BRST Transformations:

$$\delta A^I_\mu = 0 \qquad \delta a_\mu = \psi_\mu - D_\mu C \qquad \delta\psi_\mu = D_\mu\varphi - e\{C, \psi_\mu\}$$

$$\delta G_{\mu\nu} = e[C, G_{\mu\nu}] - e[\varphi, \chi_{\mu\nu}], \delta\chi_{\mu\nu} = G_{\mu\nu} - e\{C, \chi_{\mu\nu}\}$$

$$\delta\bar{C} = B \qquad \delta C = \text{ eCC } + \varphi, \qquad \delta B = 0$$

$$\delta\eta = -B \qquad \delta\lambda = \eta - \bar{C} \qquad \delta\varphi = e[C, \varphi]$$

$$\delta^2 \equiv 0$$

$$\eta - -\eta + \bar{C}$$

Eigenvalue problems associated with the quadratic part of the action :

$$\begin{aligned}
D^I_\mu a_\nu - D^I_\nu a_\mu - \frac{\epsilon_{\mu\nu\alpha\beta}}{\sqrt{g}} D^I_\alpha a_\beta &= \omega G_{\mu\nu} \\
-2g^{\mu\nu} D^I_\mu G_{\nu\alpha} + D^I_\alpha B &= \omega a_\alpha \\
-g^{\mu\nu} D^I_\mu a_\nu &= \omega B \\
D^I_\mu\psi_\nu - D^I_\nu\psi_\mu - \frac{\epsilon^{\alpha\beta}_{\mu\nu}}{\sqrt{g}} D^I_\alpha\psi_\beta &= \omega\chi_{\mu\nu} - 2g^{\mu\nu} D^I_\mu\chi_{\nu\alpha} + D^I_\alpha\eta = \omega\psi_\alpha \\
-g^{\mu\nu} D^I_\mu\psi_\nu &= \omega\eta \\
-g^{\mu\nu} D_\mu D_\nu \begin{pmatrix}\lambda \\ \varphi\end{pmatrix} &= \omega^2 \begin{pmatrix}\lambda \\ \varphi\end{pmatrix} \\
-g^{\mu\nu} D_\mu D_\nu \begin{pmatrix}C \\ \bar{C}\end{pmatrix} &= \omega^2 \begin{pmatrix}C \\ \bar{C}\end{pmatrix}
\end{aligned}$$

$$\begin{array}{llll}
[I]' & (M_a)a = \omega^2 a & [\text{II}]' & (M_\psi)\psi = \omega^2\psi \\
 & (M_G)G = \omega^2 G & & (M_\chi)\chi = \omega^2\chi \\
 & (M_B)B = \omega^2 B & & (M_\eta)\eta = \omega^2\eta
\end{array}$$

Notice $M_a \equiv M_\psi, M_G = M_\chi, M_B = M_\eta M_\lambda = M_\phi = M_C = M_{\bar{C}}$

$$d\hat{\mu} = \prod_{r,x} d\hat{\Phi}_r(x) = e^{\hat{K}}_r \prod_{r,x} d\Phi_r(x)$$

$$\hat{K} = \lim_{t\to 0} \sum \alpha_r \sigma_r d_r tr \ln g(x) e^{-tM_{\Phi r}} = 0.$$

[II] Topological gravity in $2d$

Basic field two-bein $e_{a\mu}$(or metric $g_{\mu\nu}$) subject to shift symm. Fix this symmetry by introducing auxiliary field $G_{\mu\nu}$ and F.P.ghosts $\psi_{a\mu}$ and $\chi_{\mu\nu}$.

$$\delta e_{a\mu} = \frac{1}{2}\psi_{a\mu}, \qquad \delta\psi_{a\mu} = 0, \qquad \delta G_{\mu\nu} = 0, \qquad \delta\chi_{\mu\nu} = G_{\mu\nu}$$

From these

$$\omega_\mu \equiv \omega_\mu^{12} : \delta\omega_\mu = -e_\alpha^a \in^{ab} D_\alpha \delta e_{b\mu} = -\frac{1}{2} e_\alpha^a \in^{ab} D_\alpha \psi_{b\mu} \equiv \hat{\Psi}_\mu$$

$$S_{\text{g.f}} = \int d^2 x e[R_{\mu\nu} G^{\mu\nu} - (D_\mu \hat{\Psi}_\nu - D_\nu \hat{\Psi}_\mu)\chi^{\mu\nu}] = \delta V$$

$$V = \int d^2 x e[R_{\mu\nu}\chi^{\mu\nu}], R_{\mu\nu} \equiv \delta_\mu \omega_\nu - \delta_\nu \omega_\mu$$

Expand around background

$$e_\mu^a = e_\mu^{Ba} + \hat{e}_\mu^a, \text{ or } g_{\mu\nu} = g_{\mu\nu}^B + \hat{g}_{\mu\nu}$$

$$S_{\text{g.f}} = \int d^2 x e[(D_\mu \hat{\omega}_\nu - D_\nu \hat{\omega}_\mu) G^{\mu\nu} - (D_\mu \hat{\Psi}_\nu - D_\nu \hat{\Psi}_\mu)\chi^{\mu\nu}]$$

$\hat{\omega}_\mu = \hat{\omega}_\mu(\hat{e})$ covariant derivatives are with background fields.
Further invariances:

$$(i) \text{ g.c.} \qquad \delta g_{\mu\nu}^B = 0, \qquad \delta\hat{g}_{\mu\nu} = -(D_\mu \theta_\nu + D_\nu \theta_\mu - \hat{g}_{\mu\nu} \text{ D}.\theta)$$

θ - commuting

$$(ii) \delta\psi_{\mu\nu} = D_\mu \in_\nu + D_\nu \in_\mu - \hat{g}_{\mu\nu} \text{ D}. \in, \in \text{ anticommuting}$$

B.R.S.T.(i) introduces fields $h_{\mu\nu}, C_\mu, b_{\mu\nu}$
(ii) $\quad \eta_{\mu\nu}, \gamma_\mu, \beta_{\mu\nu}$.

$$\delta g_{\mu\nu}^B = 0, \qquad \delta g_{\mu\nu} = \psi_{\mu\nu} - (D_\mu C_\nu + D_\nu C_\mu - \hat{g}_{\mu\nu} \text{ D.C})$$

$$\delta h_{\mu\nu} = 0, \qquad \delta C_\mu = 0, \qquad \delta b_{\mu\nu} = h_{\mu\nu}$$

$$\delta\psi_{\mu\nu} = (D_\mu \gamma_\nu + D_\nu \gamma_\mu - \hat{g}_{\mu\nu} \text{ D}.\gamma)$$

$$\delta\eta_{\mu\nu} = 0, \qquad \delta\gamma_\mu = 0, \qquad \delta\beta_{\mu\nu} = \eta_{\mu\nu}$$

$$\delta\chi_{\mu\nu} = G_{\mu\nu} - \delta_\mu C^\alpha \chi_{\alpha\nu} - \partial_\nu C^\alpha \chi_{\mu\alpha} - C^\alpha \partial_\alpha \chi_{\mu\nu}$$

$$\delta G_{\mu\nu} = -\partial_\mu C^\alpha G_{\alpha\nu} - \partial_\nu C^\alpha G_{\mu\alpha} - C^\alpha \partial_\alpha G_{\mu\nu}$$

$$S = \delta V$$

$$V = \int d^2x\sqrt{g}[(D_\mu \hat{W}_\nu - D_\nu W_\mu)\chi^{\mu\nu} + \hat{g}_{\mu\nu} b^{\mu\nu} + \psi_{\mu\nu}\beta^{\mu\nu}]$$

Field content

Commuting	*anticommuting*
$\hat{g}_{\mu\nu}$	$\psi_{\mu\nu}$
$G_{\mu\nu}$	$\chi_{\mu\nu}$
$h_{\mu\nu}$	$\eta_{\mu\nu}$
$r_\mu, \eta_{\mu\nu}$	$C_\mu, b_{\mu\nu}$

Bose-Fermi pairing and same action and hence the same regulators e^{-tM_Φ} within a pair.

$$\hat{K} = \sum_r \alpha_r \sigma_r d_r tr \ln g(x) e^{-\mathrm{tH}_{\Phi r}} = 0$$

$$d\hat{\mu} = \prod_{x,r} d\hat{\Phi}_r = \prod_{x,r} d\Phi_r(x).$$

Conformal anomaly is also absent. Contribution due to $(\hat{g}, h, b, c)$ cancels against that from $(\psi, \eta, \beta, \gamma)$.

[III] 3 – dim Chern-Simons System

$$S = \frac{\bar{e}K}{2}\int d^3x \epsilon^{\mu\nu\lambda} Tr(A_\mu \partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda)$$

$$+\frac{\bar{e}}{2}\int d^3x\sqrt{g}Tr(g^{\mu\nu}A_\mu\partial_\nu B + g^{\mu\nu}D_\mu C\partial_\nu \bar{C})$$

$$= S_o + \delta V$$

$$V = \frac{\bar{e}}{2}\int d^3x\sqrt{g}Tr(g^{\mu\nu}A_\mu\partial_\nu\bar{C})$$

BRST Transformations :

$$\delta A_\mu = -D_\mu C \qquad \delta C = CC, \qquad d\bar{C} = B, \qquad \delta B = 0$$

$$\delta^2 \equiv 0 \qquad \delta S = 0$$

$$K = \sum \sigma_r d_r \alpha_r$$

Field content :$(A_\mu, B, C, \bar{C})$

$$\alpha_r = (\frac{1}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$

$$\sigma_r = (1,1,-1,-1)$$
$$d_r = dim G(3,1,1,1)$$
$$\alpha_A = \frac{d-2}{4d}$$
$$K = 0$$

More careful evaluation

$$\hat{K}_{\text{reg}} = \lim_{t\to 0} \sum \sigma_r d_r \alpha_r tr_r \ln g(x) e^{-tM_{\Phi r}}$$

Background expansion :

$$A_\mu = A_\mu^B + e a_\mu$$
$$F_{\mu\nu}(A_\mu^B) = 0$$

ithin a pair.

$$S = \int d^3x \epsilon^{\mu\nu\lambda} Tr(a_\mu D_\nu a_\lambda) + \int d^3x \sqrt{g} tr[g^{\mu\nu} a_\mu D_\nu B + g^{\mu\nu} D_\mu C D_\nu \bar{C}]$$

Background gauge fixing

$$= S_o + \delta_Q V$$

against that

$$V = \int d^3x \sqrt{g} tr(g^{\mu\nu} a_\mu D_\nu \bar{C})$$

BRST :

$$\delta A_\mu^B = 0, \delta a_\mu = -D_\mu C, \delta C = eCC, \delta\bar{C} = B, \delta B = 0$$
$$D_\mu = \partial_\mu + [A_\mu^B, *] + e[a_\mu, *] + [\Gamma_\mu, *]$$

Eigenvalue problem associated with the bilinear part of the action :

$$[\mathrm{I}] \qquad \frac{\epsilon^{\mu\nu\lambda}}{\sqrt{g}} D_\nu^B a_\lambda + g^{\mu\nu} D_\nu^B B = \lambda g^{\mu\nu} a_\nu$$

$$-g^{\mu\nu} D_\mu^B a_\nu = \lambda B$$

$$[\mathrm{II} \qquad g^{\mu\nu} D_\mu^B D_\nu^B \begin{pmatrix} C \\ \bar{C} \end{pmatrix} = \lambda^2 \begin{pmatrix} C \\ C \end{pmatrix}$$

Multiply by D_μ^B :

$$-g^{\mu\nu} D_\mu^B D_\nu^B B = \lambda^2 B$$

Multiply $\frac{\epsilon^{\mu\nu\beta}}{\sqrt{g}} D_\alpha^B$:

$$(M_a) a = \lambda^2 a$$

For every eigenfunction a^λ, there corresponds an eigenfunction B^λ, with same eigenvalue, except for a possible zero eigenvalue eigenfunction. For R^3, there are no zero eigenvalues.

$$\hat{K} = \lim_{t\to 0} \sum_r \sigma_r d_r \alpha_r tr \ln g(x) e^{-tM_{\Phi,r}}$$

$$= (\dim G)(\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4}) tr g(x) e^{-tD^2} = 0$$

Thus again

$$d\hat{\mu} = \Pi d\hat{\Phi}_r = \Pi d\Phi_r$$

and hence $\frac{\delta}{\delta g^{\mu\nu}}$ can be taken across the measure. Furthermore $\frac{\delta}{\delta g^{\mu\nu}}$ commutes with δ_{BRST} on the fields $A_\mu, C, \bar{C}$,B. (not so on $\hat{A}_\mu, \hat{C}, \hat{\bar{C}}, \hat{B}$).
All the examples we have studied $\delta^2 = 0$. Gauge fixing is done in such a way so that $\delta^2 = 0$ off shell. We could gauge fix differently, e.g., for Chern-Simons case

$$S = S_{CS} + \int d^3x \sqrt{g} Tr[\frac{1}{2\alpha}(g^{\mu\nu}\nabla_\mu A_\nu)^2 + g^{\mu\nu} D_\mu \bar{C} \partial_\nu C]$$

$$\delta A_\mu = -D_\mu C, \delta C = CC, \qquad \delta\bar{C} = \frac{1}{\alpha} g^{\mu\nu} \nabla_\mu A_\nu$$

No auxiliary field B. Here $\delta^2 = 0$ only on shell. Further more $S_{\mathrm{g.f}} \neq \delta$V. Thus our proof does not go through in this gauge fixing.
However, this does not mean that this version is not topological invariant. Only the proof would be involved. After all two formulations differ only in the choice of gauge fixing.
[IV] Topological field theories with antisymmetric tensor gauge fields: (Schwarz type, Abelian)

$$S_o = \int_{M_{2n+1}} A^{(n)} F^{(n+1)}$$

$$A^{(n)} = A_{\mu_1} \ldots {}_{\mu_{u_n}} dx^{\mu_1} \wedge dx^{\mu_2} \ldots \wedge dx^{\mu_n}$$

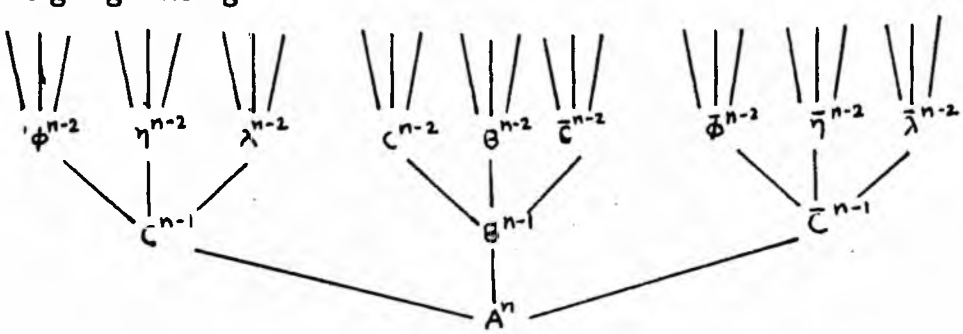
$$F^{(n+1)} = dA^{(n)}$$

$n = 1$ is Abelian Chern-Simons.
Gauge fixing
Introduce commuting auxiliary $(n-1)$ form $B^{(n-1)}$ and two F.P. anticommuting ghosts, $(n-1)$ forms $C^{(n-1)}, \bar{C}^{(n-1)}$.

$$\delta A^{(n)} = -dC^{(n-1)}, \delta\bar{C}^{(n-1)} = B^{(n-1)}, \delta C^{(n-1)} = 0, \delta B^{(n-1)} = 0$$

$$S'_{\rm g.f} = \int dx^{2n+1}\sqrt{g}[dB^{(n-1)} * A^{(n)} + dC^{(n-1)} * d\bar{C}^{(n-1)}]$$
$$= \delta V$$
$$V = \int dx^{2n+1}\sqrt{g}[d\bar{C}^{(n-1)} * A^{(n)}]$$
$$\alpha_A = \frac{d-2n}{4d} = \frac{1}{4(2n+1)}$$
$$\alpha_B = \alpha_C = \alpha_{\bar{C}} = d - \frac{2(n-1)}{4d} = \frac{3}{4(2n+1)}.$$

Tree of gauge fixing



(i) For every commuting tensor gauge field of rank j gauge fixing involves two anticommuting ghost fields and one commuting auxiliary field, all of rank $j-1$.
(ii) For every anticommuting tensor gauge field of rank j, gauge fixing involves two commuting ghosts and one anticommuting auxiliary field, all of rank $j-1$.

$$K = \sum_r \sigma_r \alpha_r d_r$$
$$= K_{B(n)} + K_{F(n-1)} + K_{B(n-2)} + \cdots\cdots$$

At each level it is the contribution of only one commuting or anticommuting field that survices. Furthermore, they alternate.

$$K = K_{B(n)} + K_{F(n-1)} + K_{B(n-1)} + K_{F(n-2)} + \cdots$$
$$= \frac{2n!}{4(n+1)!n!} - \sum_{j=1}^{[\frac{n+1}{2}]} \frac{(4j-1)}{(n+2j)!(n-2j+1)} + \sum_{j=1}^{[n/2]} \frac{(4j+1)}{(n+2j+1)!(n-2j)} = 0$$

Perhaps $\hat{K} = 0$ also.

Conclusion :

In topological field theories, it is possible to fix gauges in such a way so that the resulting field content is such that the general coordinate invariant measure suggested by Fujikawa

$$d\hat{\mu} = \prod_{x,r} d\hat{\Phi}_r(x), \qquad \hat{\Phi}_r(x) = g^{\alpha_r}\Phi_r(x)$$

is the same as the ordinary measure

$$d\mu = \prod_{x,r} d\Phi_r(x)$$

For this to be true

$$\hat{K} = \lim_{t\to 0} \sum \sigma_r \alpha_r d_r tr_{\Phi_r} \ln g e^{-tM_\Phi} = 0$$

This happens, if the gauge fixings are so done that $\delta^2_{\mathrm{BRST}} = 0$ off shell. The action is written as $S = S_o + \delta V$. Then the topological invariance of such theories follows immediately. Perhaps, the structure is such that $K = 0$ is due to the fact that there are no dynamical degrees of freedom for topological field theories - regular fields cancelling the ghosts.

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In Lighter Vein

On the last day of the workshop an opinion poll was conducted by *G. Rajasekharan*. We reproduce here the result of the poll.

What is the probability that matrix models, random surfaces and 2-d gravity will throw light on string theory?

10%	[illegible]
100%	S. [illegible]
50%	Ashoke Sen
$0 \leq P \leq 1$	N. D. Hari Dass (Healthy sign is that it is not negative)
100%	R. Kaul
25%	[illegible]
0%	[illegible]
20%	P. Majumdar
[illegible]%	P. P. [illegible]
100%	Jayaraman (LIGHT ALREADY THROWN).
99%	[illegible] [illegible]
50%	A. Khan
40%	D. P. Jatkar
70%	S. Mahapatra
20%	R. P. Malik
$e^{-N\delta J}$	A. M. Sengupta
SOME	Debashis Ghoshal

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